Fourth conference of the International Network for Didactic Research in University Mathematics

INDRUM 2022

October 19-22

Hannover

Leibniz University Hannover

PROCEEDINGS

http://indrum2022.sciencesconf.org/

INDRUM 2022 is an ERME topic conference

erme european society for research in mathematics education

INDRUM

khd m kompetenzzentrum hochschuldidaktik mathematik

Leibniz Universität Hannover

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INDRUM2022 was an ERME Topic Conference.

ERME Topic Conferences (ETC) are conferences organised on a specific research theme or themes related to the work of ERME as presented in associated working groups at CERME conferences. Their aim is to extend the work of the group or groups in specific directions with clear value to the mathematics education research community.

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INDRUM2022 Editorial

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INDRUM2022 was the fourth conference of the International Network for Didactic Research in University Mathematics. Initiated by an international team of researchers in didactics of mathematics in 2014, INDRUM aims at contributing to the development of research in didactics of mathematics at all levels of tertiary education, with a particular concern for the development of new researchers in the field and the dialogue with mathematicians. After three very successful conferences in 2016 (Montpellier, France), 2018 (Kristiansand, Norway), and 2020 (Bizerte, Tunisia, online) the INDRUM Network Scientific Committee (INSC) decided to continue the cycle of biennial conferences with a fourth INDRUM conference to be held in Hannover, Germany, on October 19th-22nd 2022.

The INSC nominated the INDRUM2022 International Programme Committee (IPC) and the Local Organising Committee (LOC), with an intersection to facilitate the coordination of both committees. The IPC was composed of María Trigueros (Ciudad de México, México) Chair; Berta Barquero (Barcelona, Spain) Co-chair; Rolf Biehler (Paderborn, Germany); Marianna Bosch (Barcelona, Spain); Laura Branchetti (Milan, Italy); Viviane Durand-Guerrier (Montpellier, France); Alejandro González-Martín (Montreal, Canada); Thomas Hausberger (Montpellier, France); Reinhard Hochmuth (Hannover, Germany); Barbara Jaworsky (Loughborough, United Kingdom); Rafael Martínez–Planell (Mayagüez, Puerto Rico); Chris Rasmussen (San Diego, United States). The LOC was composed of Reinhard Hochmuth (Hannover, Germany) Chair; Christine Bessenrodt (Hannover), Rolf Biehler (Paderborn), Andreas Eichler (Kassel), Sarah Khellaf (Hannover), Michael Liebendörfer (Paderborn), Jana Peters (Hannover), Johanna Ruge (Hannover), Hanh Vothi (Hannover) and Johannes Wildt (Bielefeld).

The first announcement, published in September 2021, communicated the structure of the conference. Similarly to the two previous INDRUM conferences, themes to be addressed at INDRUM2022 covered teacher and student practices and the teaching and learning of specific mathematical topics at undergraduate and postgraduate level, as well as across disciplines. Accepted scientific contributions were to be discussed in four to six thematic working groups (six hours each) after their presentation in two parallel sessions of short communications (two sessions of 1h 30min). The programme also included a poster exhibition and a workshop for early-career researchers. It was announced that Andreas Eichler (University of Kassel, Germany) had accepted to give the plenary lecture and that an expert panel discussion on tertiary education on "Innovation in teaching at the university based on research in mathematics education" chaired by Rafael Martinez-Planell (Universidad de Puerto Rico at Mayagüez) was in preparation. Although the primary language of the conference was English, the linguistic characteristics of the host country were considered, as in previous INDRUM conferences. Therefore, authors were offered the opportunity to write and present a paper in German, provided that the presenter considered how to address the conference audience in its linguistic diversity through slides or a handout in English. Besides, INDRUM2022 was the fourth INDRUM conference to be accepted as a Topic Conference by the European Society for Research in Mathematics Education (ERME).

The second announcement was published in October 2021 with further details on the submission. Authors were provided with a list of 15 keywords or topics as a means to classify their submission (using two keywords from the list and three optional additional keywords) and also to help us in the subsequent process of allocating papers to different working groups after the review process.

In response to the call, 76 papers and 20 posters were received. The review process was organised by the chair and co-chair according to principles that were discussed among the IPC. Thus, each paper was reviewed by a member of the INSC and by an author of another paper; posters were reviewed by the chair or the co-chair and by an author of another poster. Final decisions, in cases where both reviewers had diverging opinions, were made after discussion among the IPC. At the end of the reviewing process, 56 papers and 20 posters were accepted for presentation at the conference. Authors of rejected papers that fell within the scope of the conference were offered the opportunity to resubmit their contribution as a poster. This last step increased the number of accepted posters to 39 in total.

Given the number of accepted contributions and the keywords provided by the authors, it was considered possible and appropriate to organise six balanced thematic working groups (TWG). The allocation of papers and posters was proposed by the chair and co-chair, and approved by the IPC. The appointment of TWG co-leaders from among the INSC members was made with a view to geographical diversity, gender balance, and the involvement of colleagues who had not previously or recently served as leaders. We were grateful that the appointed INSC members were able to accept our invitation.

The third announcement was published in September 2022 with the following list of groups (TWG) and names of co-leaders:

TWG1: *Transition to, across and from university mathematics.* Chairs: Thomas Hausberger (France), Heidi Strømskag (Norway)

TWG2: *Teaching and learning of analysis and calculus.* Chairs: Erik Hanke (Germany), Rafael Martínez-Planell (Puerto Rico)

TWG3: *Teaching and learning of linear and abstract algebra, logic, reasoning and proof.* Chairs: Viviane Durand-Guerrier (France), Melih Turgut (Turkey-Norway)

TWG4: *Teaching and learning of mathematics for engineers, and other disciplines.* Chairs: Ignasi Florensa (Spain), Ghislaine Gueudet (France),

TWG5: *Teacher education in university.* Chairs: Marianna Bosch (Spain), Carl Winsløw (Denmark)

TWG6: *Students' practices and assessment.* Chairs: Nicolas Grenier-Boley (France), Frank Feudel (Germany)

The third announcement also included the names of the panel chair and panelists, who were appointed by the IPC from among the conference participants on the basis of their expertise in the topic of the panel. Rafael Martinez-Planell (Universidad de Puerto Rico in Mayagüez, Puerto Rico) accepted to chair the panel on "Innovation in teaching at the university based on research in mathematics education" with Ignasi Florensa (Universidad Salesiana de Sarrià-Universitat de Barcelona, Spain), Max Hoffmann (Paderborn University, Germany), Avenilde Romo (CINVESTAV, México) and Michelle Zandieh (University of Arizona, USA) as speakers. Finally, Elena Nardi (University of East Anglia, United Kingdom) and Megan Wawro (Virginia Tech, USA) prepared a workshop for INDRUM early career researchers on "Starting to write journal articles" for INDRUM early career researchers based on two of their published papers. The purpose of the workshop was to share experiences and stimulate discussion on what constituted the challenges –and ways to overcome them– of preparing a manuscript for submission to a mathematics education research journal, with a particular focus on university mathematics education.

Table 1. Participants and countries

Country	Participants
Canada	4
Croatia	6
Denmark	2
Finland	3
France	12
Germany	39
Greece	2
Hungary	1
Italy	6
Israel	4
Japan	6
Mexico	3
Norway	6
Puerto Rico	1
Spain	7
Sweden	5
The Netherlands	4
Tunisia	1
United Kingdom	2
United States	4

The third announcement included the conference timetable and the conference pre-proceedings. In parallel, the LOC was getting ready to welcome delegates in Hannover, Germany.

118 participants from 20 countries registered for the INDRUM2022 conference (see Table 1). The opening and closing sessions were lively thanks to the work of the LOC. Work in the TWG was also lively with interesting exchanges and interaction between delegates. The TWG's leaders encouraged rich interaction among participants. They also prepared a summary which was presented at the closing session.

Papers and posters appear in these Proceedings in a version chosen by the participants, following the optional possibility to upload a final version of their paper.

Last but not least, we would like to thank the IPC, chaired by Maria Trigueros and Berta Barquero, and the LOC, chaired by Reinhard Hochmuth and his entire team (especially Mrs. Krampe), for their tireless work over many months to organise the conference. We are also grateful for the support provided by Jana Peters for her work on both the pre-proceedings and these proceedings. Many thanks to all TWG co-leaders for your support and help in making INDRUM2022 a fruitful and enjoyable experience for the participants.

FOLLOW-UP

The INDRUM2022 closing ceremony was the occasion to announce some additional interesting news. We are delighted to announce that authors of an accepted contribution (paper or poster) in the INDRUM2022 proceedings will be offered the opportunity to publish an expanded, updated or reworked version of their contribution to match the requirements of a special issue in IJMEST (International Journal of Mathematical Education in Science and Technology) Special Issue guest-edited by the INDRUM2022 Chair, co-Chair and LOC Chair. We invite papers of 15-20 pages, written in English, with the aim of publishing approximately ten papers among the best research represented in the INDRUM2022 Proceedings. While we aim to reflect the thematic richness of the INDRUM2022 programme, we will not commit to a strict representation of the conference structure. We particularly welcome proposals that substantially elaborate and expand the content of the INDRUM2022 submissions.

The deadlines for this Special Issues have been fixed as follows: April 23rd, 2023: deadline to submit papers; July 28th, 2023: decision letters sent to authors; September 29th, 2023: deadline for revised manuscripts; December 5th, 2023: final decisions and February 2024 Publication. The official call for contributions has been sent to the authors of INDRUM2022 accepted contributions through the INDRUM mailing list. The Call for papers is available at the INDRUM2022 webpage (https://indrum2022.sciencesconf.org/); as well as in the IJMEST website: https://bit.ly/INDRUM2022_Conference.

Finally, we are delighted to spread the news that INDRUM2024 will be held in Barcelona, Spain in June 2024 from 10th to 14th June 2024. The final dates will be decided in September 2023. The local Chair is Ignasi Florensa, with Alejandro González Martín (Canada) as Chair of the IPC and Ghislaine Gueudet (France) as co-Chair. The INDRUM2024 website https://indrum2024.sciencesconf.org/, which is currently under construction, will open with updated information as soon as possible.

We now invite you to carry on reading this volume and we hope that the promise of its contents will encourage you to consider joining or continuing to be part of the ambitious and stimulating INDRUM network.

Plenary talk

Developing digital networks for learning and teaching mathematics in introductory courses

Andreas Eichler[1], Michael Liebendörfer[2], Mirko Schürmann[2] and Franziska Sommerlade[1]

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In recent years, there has been increased development of digital tools designed to improve teaching and learning in university mathematics. Such tools can now better accommodate student heterogeneity, different learning speeds, or individual feedback. However, embedding digital elements into coherent teaching approaches is challenging. In particular, digital tools are rarely developed and used in a multi-university system to actively involve students and faculties from different departments. We report on the ongoing LLV.HD project that addresses these challenges. Our main goal is to develop a coherent system of digital elements that facilitates the teaching and learning of mathematics at different universities and in different courses, and that meets the needs of the specific courses, faculty, and students at the different universities. We outline the main goals of the project, and the challenges in developing shared digital tools and establishing the aforesaid networks.

Keywords: Digital elements, instructional videos, STACK.

INTRODUCTION

An overarching goal for all levels of education in the digital age is to accelerate digital change in teaching and learning (e.g. European Commission, 2020). The Covid-19 crisis, in particular, emphasised the need to increase efforts to develop approaches for digital teaching and learning (Carrillo & Flores, 2020; Sánchez Ruiz et al., 2021). Fostering participation, collaboration or communication are crucial demands for developing approaches of digital teaching and learning (e.g. European Commission, 2020), avoiding isolated application and instead establishing coherent teaching and learning concepts across universities (KMK, 2016). These demands form the basis for the project LLV.HD (*Lehr-Lern-Verbünde in mathematikhaltigen Studiengängen – hochschul-übergreifend und digital,* translated as "Teaching-learning alliances in mathematics-containing degree programmes - cross-university and digital"; www.uni-kassel.de/go/llv-hd). LLV.HD is a project within the Competence Centre for Higher Education Mathematics (khdm; www.khdm.de/en/).

The main goal for LLV.HD is to develop digital tools for teaching and learning mathematics that fit the specific demands of university teachers in different mathematical study programmes of at least two universities. The study programmes are BA mathematics, engineering, economics and mathematics teacher education. Developing digital tools, not only for one specific course but for at least two courses at different universities, is more complicated than one might think. Therefore, this project results in an extensive need for communication. For example, LLV.HD depends on the collaboration between mathematicians and mathematics educators, lecturers, developers and students, lecturers in different study programmes or lecturers and students of different universities. Collaborating with students and lecturers as well as integrating the needs of both stakeholders in the development of those tools calls for participation as one characteristic of digital teaching and learning.

In this paper, we refer to different challenges and questions that shape the process of collaboration, communication and participation in LLV.HD during the development of shared digital tools for teaching and learning mathematics at university level. Thus, in the following section, we highlight seven central questions. These questions should not be viewed as typical research questions but can rather be seen as central aspects that arose during the agile development of our project. In the following section, we highlight seven central aspects of our project by addressing these seven questions:

1. Which circumstances have to be considered when trying to develop digital materials that should be integrated into mathematics courses of different universities and different disciplines?
2. What may be an appropriate way to organise a project aiming to promote collaboration between lecturers, developers and students from different study programmes and universities?
3. How can the development of common digital elements for mathematics courses in different universities and study programmes be organised?
4. What are the requirements for LLV.HD concerning students' needs and lecturers' beliefs?
5. What are the pre-conditions for developing a common mathematical language and notation?
6. What are the challenges concerning the collaboration of lecturers and students?
7. How can we support practice courses based on students' needs?

CIRCUMSTANCES FOR THE DEVELOPMENT OF LLV.HD (ASPECT 1)

In Germany, about 40% of all students are enrolled in degree programmes that include some higher mathematics (Statistisches Bundesamt, 2016). This percentage does not include statistics courses. Related study programmes are, for example, the undergraduate degrees in mathematics, teacher education for mathematics teachers, computer sciences, engineering, economics or sciences. Traditionally, most mathematics courses follow a common structure consisting of a lecture of two or four hours a week, a practice course partly in small groups of students supervised by student assistants, and students' weekly assignments (Liebendörfer, 2018).

In Germany, the transition from school to university is often difficult, with similar issues to those exemplarily described by Gueudet (2008). For example, students experience the organisation of learning in universities and particularly in mathematics courses as challenging (cf. also Fleischer, 2019; Guzmán et al., 1998). As a possible consequence, large-scale studies show that study programmes with a strong mathematical component have drop-out rates ranging from 40% to 50% (e.g. Heublein

& Schmelzer, 2018). The problem of high drop-out numbers in mathematical study programmes is also discussed internationally (e.g. Troelsen et al., 2014; Wolter et al., 2014).

Although there are a variety of reasons for students dropping out, research findings suggest that problems with performance and motivation in the first years of a study programme are particularly responsible (e.g. Fleischer et al., 2019). One possible reason for a lack of motivation could be that students experience a discrepancy between their own needs and the reality at universities or the standards of their university teachers (Geisler & Rolka, 2020; Eichler & Isaev, 2022).

As a consequence of the situation in higher education mathematics, there is a strong focus on measures to improve students' motivation and achievement, particularly in the first years of a related study programme. The approach in the LLV.HD project is to develop digital tools or digital elements as a basis to potentially improve students' motivation, participation and achievement. By doing so, LLV.HD follows the recommendations of various research reports which reveal that digital tools seem to have the potential to support mathematics students' learning (Kerres, 2018). For example, instructional videos have the potential to improve understanding and to facilitate learning of mathematics (Bersch et al., 2020; Kay, 2012). Audience response systems foster participation and commitment of students in mathematics courses (Schmidt, 2011; Kempen, 2021), quizzes also support mathematical learning (Martins, 2018) and digital exercises such as STACK tasks facilitate self-regulated learning and can provide individual feedback as for a contribution to successful learning (Sangwin, 2013; Speer & Eichler, 2022).

Such digital tools seem to provide an appropriate basis to help students with their difficulties in mathematics courses in the transition from school to university. Nevertheless, despite the overwhelming number of digital elements available on internet, these tools are hardly used yet as part of regular teaching, and their implementation faces some challenges. For example, there is a huge number of slightly structured resources, such as instructional videos in Youtube channels, repositories with STACK tasks (e.g., https//db.ak-mathe-digital.de) and collections of audience response resources (Quibeldey-Cirkel, 2018). However, it is very time consuming to find a digital element in these resources that fit the demands of a specific mathematics course.

As a consequence, state institutions in Germany (particularly) try to establish repositories of structured materials for teaching and learning mathematics (e.g. https://www.vhb.org; https://www.dh.nrw/). Lecturers from different universities are asked to contribute with digital resources for these repositories. However, these resources are developed for specific teaching or learning trajectories in specific courses and thus, it might be difficult to integrate related materials into other courses.

As a consequence of these circumstances, the following main aims for the development of digital tools in our project LLV.HD state that the digital tools in LLV.HD have

- to fit the aims of at least two mathematics courses at different universities,
- to be meaningful for mathematics courses in different study programmes (for example, teacher education and economics)
- to be integrated into the structure of a coherent teaching and learning environment
- be at least partly based on students' needs.

ORGANISATION OF LLV.HD (ASPECTS 2 AND 3)

The project LLV.HD is located at the universities of Paderborn and Kassel and refers to four study programmes: undergraduate mathematics, engineering mathematics, economics and teacher education. In each of the four study programmes, teams of both universities including mathematicians and mathematics educators work together. Furthermore, different individuals at both universities manage specific tasks including the coordination, the development of STACK tasks and instructional videos, the setup of a digital drop-in centre, and the transfer, evaluation and media pedagogy.

Referring to organisation of the content, an early decision was made to concentrate on introductory analysis courses that are common for all study programmes. In the teacher education programme, both the introductory analysis course and a mathematics education course concerning teaching analysis in school are regarded.

The main structural element in LLV.HD is a digital network. A digital network in LLV.HD is shown in Fig. 1 and consists of two analysis courses at the two universities in the different study programmes including lectures, practice courses and students' homework.

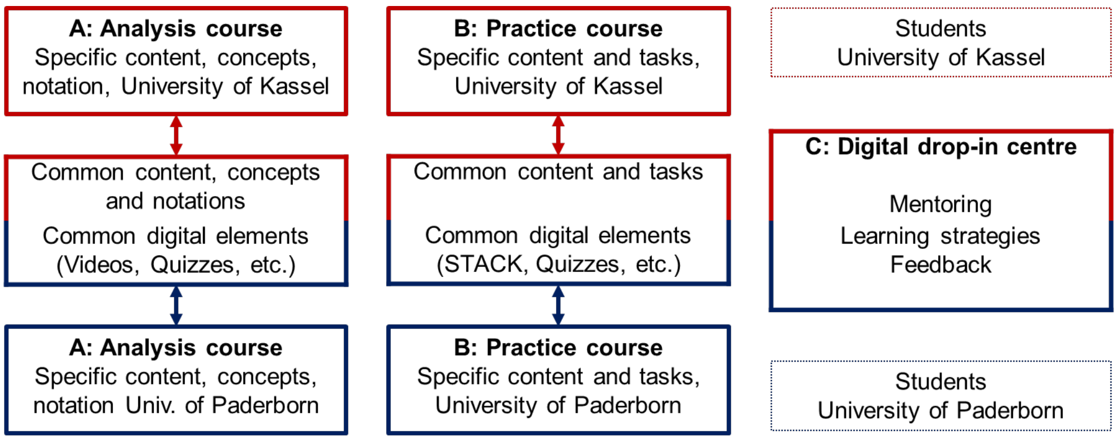


Figure 1: Digital network in LLV.HD

Each course has themes or topics that are specific to one university. Nevertheless, we expected that two parallel courses share common themes and subjects which could provide the basis for developing common digital elements (e.g. instructional videos, quizzes for the lectures or digital tasks for the practice courses). Discovering common themes or subjects forms the basis of the networking idea. Only if a theme or subject is taught at both universities is it qualified for developing corresponding and content-related digital elements. Student participation and commitment should be improved by digital drop-in centres that offer mentoring, provide support concerning learning strategies and organise students' feedback on digital tools or requirements for digital tools.

In advance, a preliminary image of a possible outcome of LLBV.HD after three years, consisting of a structure of digital elements, is shown in Fig. 2: There are different topics in an analysis lecture such as sequences, convergence etc. that might be common to different mathematics courses of different universities and study programmes. Within these topics, specific content might be common such as definitions, theorems, proofs, applications or examples. For common content, it is possible to develop common digital elements. However, we expected that there would be topics and content for which it is not possible to identify common digital elements. The elements could be appropriate in a specific manner for the different study programmes. Thus, lecturers or students of undergraduate mathematics may use mainly one certain element, while other lecturers or students may focus on other elements. However, some elements are not developed particularly for interdisciplinary use but are accessible for all study programmes. Ideally, a digital element is meaningful for lecturers and students of different universities and different study programmes.

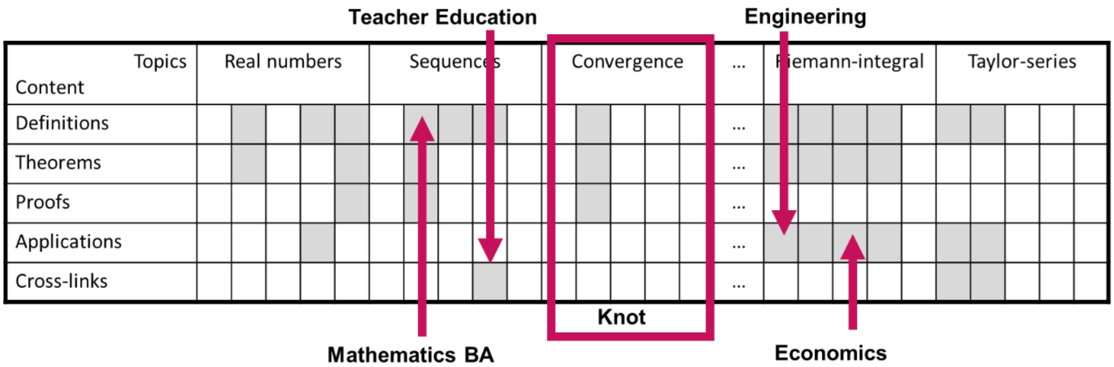


Figure 2: Possible matrix of digital elements in LLV.HD

The main structural elements of LLV.HD are knots that are defined by a theme or topic, such as convergence. Besides the topics, these knots comprise three further levels or dimensions: (1) the study programme, (2) the kind of a digital element, such as instructional video or quizzes and (3) the content, such as definitions or examples.

The initial ideas for LLV.HD are not final. The project follows an agile approach to development. That means that it is possible and desirable to adjust, reconsider or develop goals of the project within the process. In this agile development, a principle goal of LLV.HD is to integrate every group of individuals who participate in analysis courses such as students, student teachers (tutors), professors or lecturers (cf. Fig. 3).

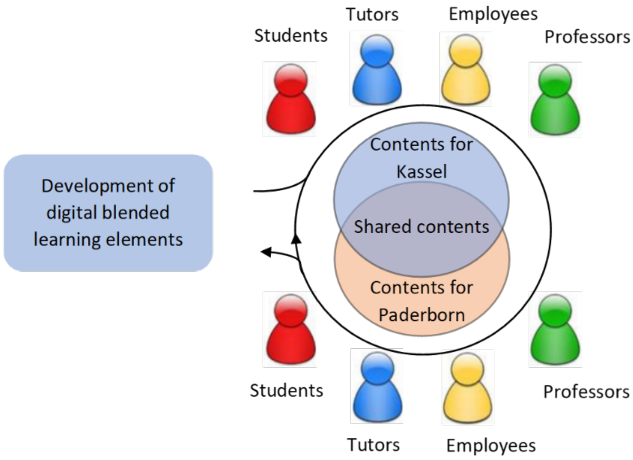


Figure 3: Students, tutors and lecturers as agents in the agile development in LLV.HD

STUDENTS' NEEDS AND LECTURERS' BELIEFS (ASPECT 4)

To gain knowledge about students' needs and lecturers' beliefs as a fundamental aspect for developing common digital tools, data were collected in both universities and the different study programmes. In this paper, we refer to the data of four study programmes at the two universities. Data were collected in 2021 through an online survey, and resulted in answers provided by 168 students from analysis and engineering courses at both universities. Further data were collected by interviews with groups of students from student councils and lecturers. We here present some statements from six student interviews and eight lecture interviews. A detailed report on the results of this multi-perspective analysis of students' needs is in preparation and will be published separately.

First, the interviews with the students of the student councils confirmed published research findings (e.g., Gueudet, 2008). These students reported the excessive demands with which mathematics students are faced at the beginning of their study programmes. For example, one student mentioned:

Student: "Just an incredible amount of material per week with incredibly difficult tasks."

As a main challenge, another student referred to the difficulties with the mathematical language.

Student: "All these new notations are overwhelming for Bachelor freshmen."

Finally, a third student referred to unfulfilled expectations (cf. Eichler & Isaev, 2022; Geisler & Rolka, 2021) caused by the gap between school mathematics and university mathematics:

Student: "There are different ideas about mathematics. This applies above all to students in the first semesters, because mathematics at university has a completely different approach from that of school lessons. Many students come with different expectations of studying and are then faced with a different reality."

A lecturer in the economics courses agreed, in some points, with the students' views. For example, he/she stated that, particularly at the beginning of study programmes, the organisation of learning and the amount of new content is demanding:

Lecturer: "The content of the lecture has been overloaded up to now […] And that was, I think, too much for the students."

Another lecturer stated that students' challenges were caused by language difficulties, referring also to native speakers:

Lecturer: "Students can hardly speak German, cannot articulate themselves correctly, especially in writing. Students have a problem with language in general and with the language of mathematics."

In contrast to the students, a lecturer also referred to students' lack of knowledge when entering universities:

Lecturer: "Students do not know elementary arithmetic rules such as fractions, term transformations, bracketing, dissolving parentheses."

The latter belief of a lecturer refers to a controversial debate in Germany, where stakeholders of schools and universities discuss what can be learned in school, and what has to be learned in school, as prerequisite for a potentially successful study of mathematics.

The students of the student council further commented on possible effects of digital tools for facilitating mathematical learning for students in the first year of their studies. For example, one student mentioned:

Student: "I think that learning videos can help a lot; calculus, especially, is not only heard from pure mathematicians, but also from teacher trainees, computer science students, who perhaps find it more difficult to work on something like this with a script."

In this student's opinion, digital tools can particularly help students with difficulties. In this regard, it is noteworthy that students in a student council (in Germany) often run their study programme successfully.

Another student mentioned that digital tools are not helpful per se, but must be integrated within an overarching and meaningful structure:

Student: "So I don't think that a pure 'Here you have 20 videos on every topic. If you have a question, you can take a look at it' is the best way to go, but rather to integrate them into the learning path or to consider where it might make sense to refer to a video and perhaps only make it available at certain points in time."

Again, the lecturers agreed to the considerations made by students with regard to the facilitating effect of digital tools. Thus, one lecturer agreed that digital tools such as videos can be of support especially for students with difficulties:

> Lecturer: "One should first consider for whom one wants to offer digital support; probably best for underperformers who have difficulty passing the exam."

Another lecturer agreed that a digital element is not, in itself, helpful for a specific course with its specific purposes:

> Lecturer: "An overview of definitions could help, but every lecturer defines something differently, so there are problems of transferability between semesters and universities."

Additionally, we gained knowledge about students' needs from a bigger sample of 169 students from analysis and engineering courses. These students, amongst others, were asked which digital tools they are actually using to receive support for mathematics courses. The majority of students referred exclusively to instructional videos. The most popular Youtube channels were "Mathe-Peter" (math-peter) or "Daniel Jung". Both channels are German-speaking and focus on several topics of school mathematics and introductory courses at universities. However, these kinds of instructional videos are criticised for some reasons. The videos, for example, provide mostly isolated explanations, they include some errors, and they often use only a vague mathematical language although language seems to be crucial for beginner mathematics students (Bersch et al., 2020). Also, these videos often refer to procedural aspects of mathematics and seldom explain connections of concepts as a basis of conceptual knowledge (Bersch et al., 2020). In the survey, students also mentioned that they desire videos addressing corresponding examples and counterexamples of terms, explanations of sentence formulations or strategies for mathematical problem solving, thus addressing conceptual knowledge.

DEVELOPING A COMMON MATHEMATICAL LANGUAGE (ASPECT 5)

Mathematical language is a crucial topic of teaching and learning mathematics at university level (Berger, 2004; Körtling & Eichler, 2022; Morgan, 2005). Accordingly, mathematical language issues have been found to be a key aspect in lecturers' and students' beliefs. Moreover, language is also crucial for the development of digital networks in LLV.HD, because common digital elements must include a common language for different courses. Thus, the main question regards the pre-conditions for developing a common mathematical language and notation.

Before the project started, based on a search in different textbooks (e.g. Forster, 2016) and recommendations for the use of language (e.g. Alcock, 2013; Gillman, 1987; Halmos, 1970), different uses of mathematical language and different notations were identified as a main obstacle for developing common digital elements. For example, Lew and Mejía-Ramos (2019) reported differences in language preferences of different mathematicians that potentially hinder the development of common digital tools.

As a first step in LLV.HD, we investigated differences in the notation or the use of mathematical language by analysing the lecture notes of analysis lecturers of the two universities and the four study programmes, since common notations are a prerequisite for common digital elements. As a first result and against our expectations, only slight differences in the use of mathematical language were found. Table 1 shows some examples concerning the topics sequences, convergence and differentiability.

Table 1: Differences in mathematical notations at two universities

Topic	Mathematics (University of Kassel)	Mathematics (University of Paderborn)	Economics (University of Kassel)
Sequences	… function $\varphi\colon N \to A$ with $a_n = \varphi(n)$	… mapping $a\colon N \to X$ with $a_n = a(n)$	… function $f\colon N \to R$ with $a_n = f(n)$
Convergence	It exists $N \epsilon N$ ...	It exists $n_0 \epsilon N$ ...	It exists $N \epsilon N$ ... (partly only as a heuristic)
Differentiability	m exists with $g(x) = g(c) + m(x-c) + r(x)$ and $\frac{r(x)}{x-c} = 0$ and … $\frac{g(x)-g(c)}{x-c}$ exists. … $\frac{g(x+h)-f(x)}{h}$ exists.	m exists with $f(x) = f(x_0) + m(x-x_0) + r(x)$ and $\frac{r(x)}{x-x_0} = 0$ and … $\frac{f(x)-f(x_0)}{x-x_0}$ exists. … $\frac{f(x+h)-f(x)}{h}$ exists.	… $\frac{\Delta y}{\Delta x}(x) = \frac{f(x+\Delta x)-f(x)}{\Delta x}$

Differences in the definition of sequences, for example, refer to the name of the object, i.e., function or mapping, the symbol for the function or mapping (φ, a, f) and the codomain $(A, X, \mathbb{R})$. In the definition of convergence, we only found a difference concerning one symbol (N vs. n_0). Differences in defining the derivative refer to the number of equivalent definitions. However, common definitions only differ in the use of specific names of variables.

To conclude, there are slight differences in naming variables or objects such as functions. Nevertheless, some slight differences in comprehension were identified: codomain, for example, is any metric space in one course and a specific metric space in another course. In one course, three equivalent definitions are introduced, in another course only one. Finally, in the economics course some mathematical objects are only used with a vague heuristic description instead of a rigorous definition. We believe that all these differences can be addressed well in common digital elements, for example, by giving additional information to raise awareness of a different variable use and naming according to different study programmes.

CHALLENGES FOR COLLABORATION: THE CASE OF INSTRUCTIONAL VIDEOS (ASPECT 6)

The development of instructional videos can reveal possible challenges in the collaboration of lecturers and students. We provide an example of such a challenge, regarding a video that focuses on linear approximation as the basis for defining differentiability. In this video – Figure 4 shows a screenshot of this video – two people, an expert and a novice, develop an understanding in a conversation. The video focuses partly on a visual illustration of ideas of a linear approximation and partly on a symbolic representation of this linear approximation. The whole video lasts about 3.5 minutes.

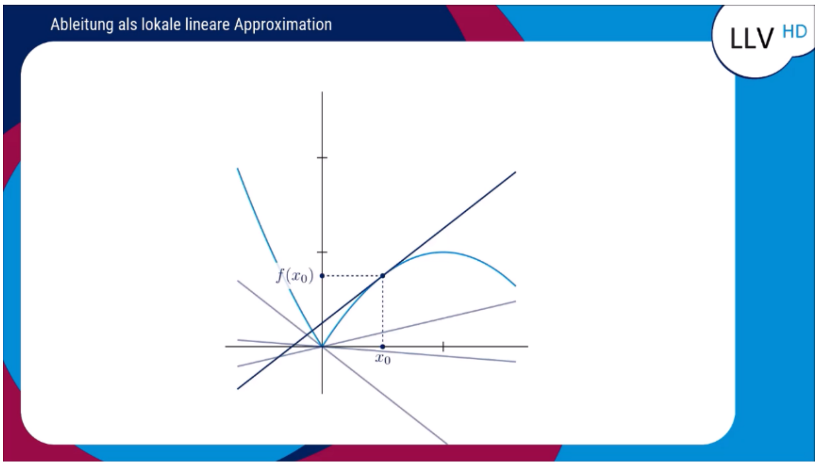


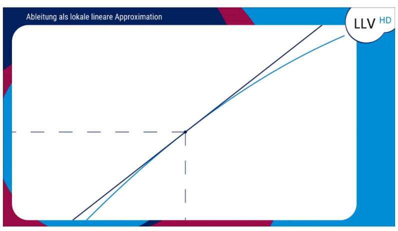
Figure 4: Screenshot of an instructional video addressing linear approximation as the basis for defining differentiability

During the video development, we made an observation that resulted in a new aspect of video development. The observation was that lecturers and students evaluated the same video differently using different perspectives: whereas the lecturers focused on the logic of a mathematical theme, the students focused on their learning in a sense of increased understanding that may extend beyond mathematical logic and include visualisations, for example. It emerged that mathematical rigour and the psychology of learning value some elements in precisely opposite ways. We may more generally experience this tension when reflecting on mathematics and mathematics education as scientific disciplines.

Another disagreement between lecturers' demands and students' needs regarded the length of the video and, subsequently, the number of topics discussed in the video. For lecturers, an instructional video such as the one illustrated in Fig. 4 provides too many topics. Lecturers claimed their explanations to be sufficient and therefore preferred illustrating their explanations. In contrast, students desired further explanations as an alternative and supplementary support to explanations from their own lecturers.

As a consequence, a new development in LLV.HD was to create a tool that allows lecturers to control the explanation used when introducing visualisations. We therefore use specific parts (for example, the animations) of a video and adapt these to a specific situation. Whereas the full video is provided for students, lecturers are able to use parts of the video and adapt these parts to their own requirements. The related tool is created using python code. Fig. 5 illustrates three steps of this tool:

- Step 1 refers to a part of the linear-approximation video illustrated in Fig. 4. After showing the graph of the function with the tangent, in the video a zoom to the intersection point of the graph and the tangent is given (Fig. 5, left side). This animation is selected as an isolated digital element for the following steps.
- Step 2 shows the python code (Fig. 5, middle, without comments), where the function term was changed from $-(x-1)^2+1$ to $sin(x)$ and, also, the scale of the axes, the point of observation and the colour of the graph were changed. A user may indicate these settings at the very beginning of the code.
- Step 3 includes the compilation process of the modified python script resulting in a modified animation of which the starting image is shown in Fig. 5 (right side).




```
# function and derivative
function = lambda x : np.sin(x)
derivative = lambda x : np.cos(x)

# axes and x_0
x_axis = [-0.2, 4]
y_axis = [-1.1, 1.1]
x_0 = 1.5

# colors
color_funcrion = `red`
color_tangent = `#00205b`
```

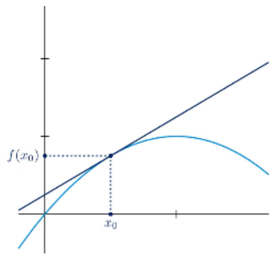


Figure 5: Illustration of an animation tool for selected sequences of an instructional video

SUPPORT OF PRACTICE COURSES (ASPECT 7)

The final challenge we present from the development of digital elements in the project LLV.HD refers to STACK tasks. STACK represents one solution to the question of how practice courses could be supported to better address students' needs.

STACK is the abbreviation of "System for Teaching and Assessment using Computer algebra Kernel" and was developed by Chris Sangwin (Sangwin, 2013). It uses the computer algebra system Maxima in the backend, which evaluates inputs algebraically.

We consider the following task: “Find the limit of the sequence $a_n = \frac{(5n+4)}{40n}$ for $n \to \infty$”.

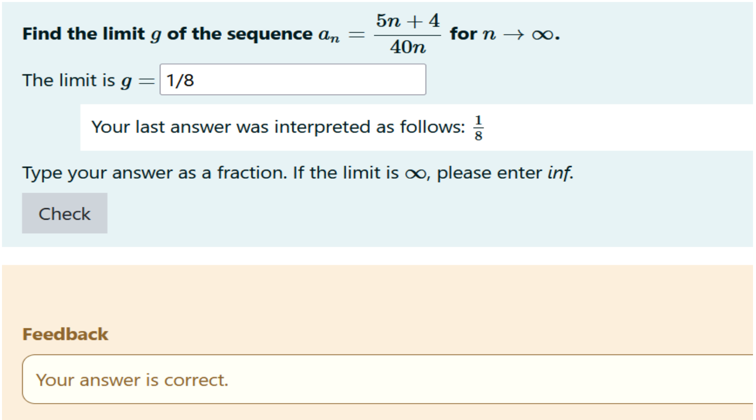


Figure 6: Screenshot of a STACK task

The answer can be entered into the white field below the sequence. Any input by a student needs to fit the form of a specific number representing the limit. Below this field, STACK interprets the input, in this case identifying the input as the fraction $\frac{1}{8}$. Below the interpretation, there are hints for a user, referring to possible inputs. STACK is especially noted for its ability to implement feedback. In this case, a simple and evaluative feedback, stating simply “correct answer”, can be seen in the orange field (Narciss, 2013).

Evaluating an input algebraically means that every expression that equals $\frac{1}{8}$ is interpreted as a correct solution of the tasks. Thus, it is possible to describe $\frac{1}{8}$ via the limit for $n \to \infty$ of the arbitrary term $\frac{n^2}{n^3} + \frac{3}{24}$. STACK also allows a randomisation of task parameters (Sangwin, 2013). In the case of the sequence in Fig. 6, the numbers in the term of the sequence were randomised. After restarting the task, a modified sequence appears, for example $a_n = \frac{(8n+5)}{40n}$. Not only the numbers, but also the exponents of the variables, can be randomised in this task. Thus, with randomisation, one task represents a class of similar tasks. Another very important feature of STACK is that if, for example, typical errors of students are known, STACK allows individualised feedback using potential response trees (Sangwin, 2013). Moreover, STACK allows tasks with graphical elements and also graphical feedback.
From research, it is known that users – for example, prospective teachers developing STACK tasks in a seminar – are enthusiastic when introduced to STACK tasks. The enthusiasm decreases only to some extent, and beliefs about STACK become more differentiated, when these prospective teachers use STACK tasks with school students (Speer & Eichler, 2022). However, a goal of LLV.HD is to gain knowledge about students' evaluation of the use of STACK tasks in different study programs and different scenarios within an analysis course. A further ongoing goal is to include

common STACK tasks in different mathematics courses and have the opportunity to analyse possible inputs, including errors of students in different universities and different study programs. For the students' acceptance of STACK tasks, we use the tool STACKrate (Lache & Meißner, 2022). This allows you to simply evaluate a task with a Likert-like scale after finishing a task (see Fig. 7). In this case, the student is asked to rate the task concerning task difficulty, usability, processing and feedback, allocating from one to five stars.

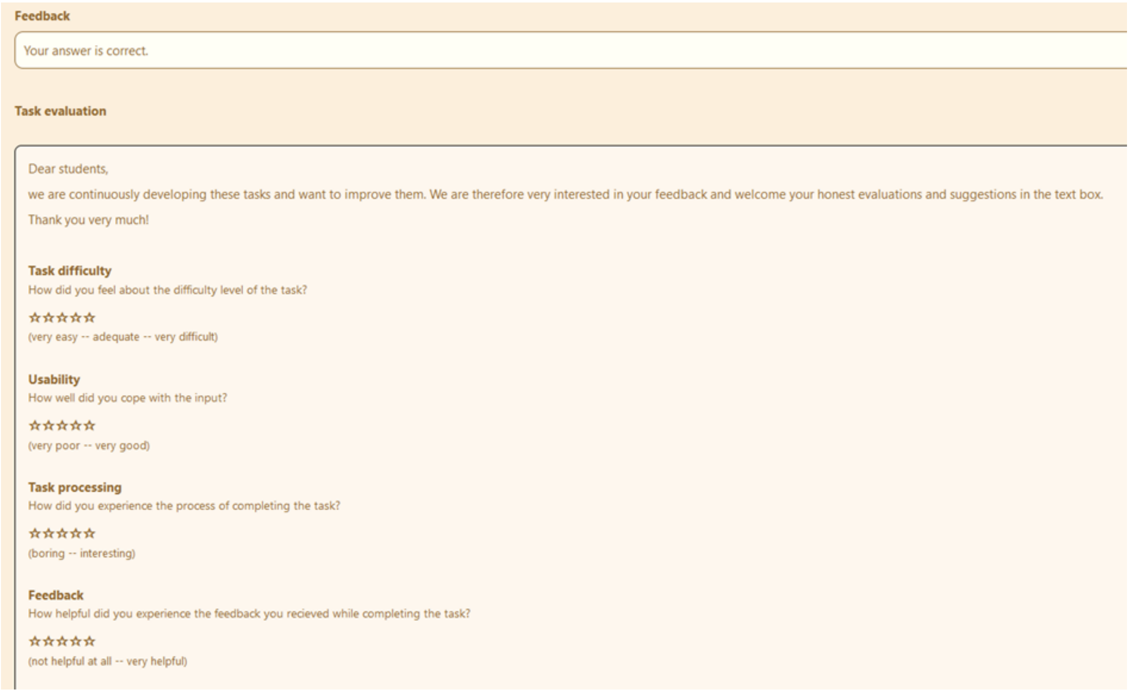


Figure 7: Screenshot of STACKrate as a brief evaluation at the end of a STACK task

With STACKrate, it is possible to collect data about STACK tasks that allows users to match performance in a task and students' acceptance.

Although STACK tasks offer a lot of possibilities to enrich self-directed learning of students, STACK may result in some problems. For example, we observed that the majority of tasks are procedural tasks. Conceptual tasks are rare and the development of substantial feedback for conceptual tasks is a complex challenge. A further problem is that conceptual tasks must allow an algebraic expression as an input, or something that could be interpreted as an algebraic expression. A possible example for a conceptual task concerning sequences is given in Figure 8, where a sequence must be found that satisfies designated limits.

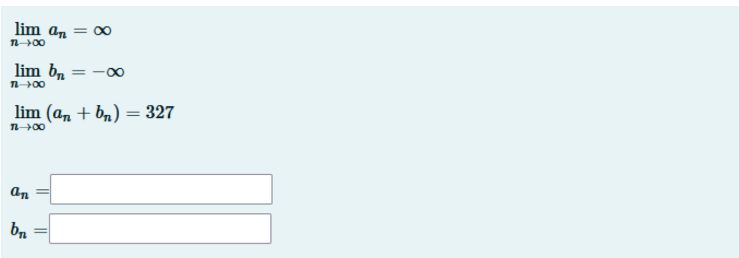


Figure 8: Example of a task triggering conceptual knowledge

It is also possible to pose more complex tasks referring to conceptual knowledge focusing, for example, on proofs using the input type "equivalence reasoning". However, in this case, potential response trees that allow individual feedback can display an increased complexity, resulting in correct feedback for every possible input.

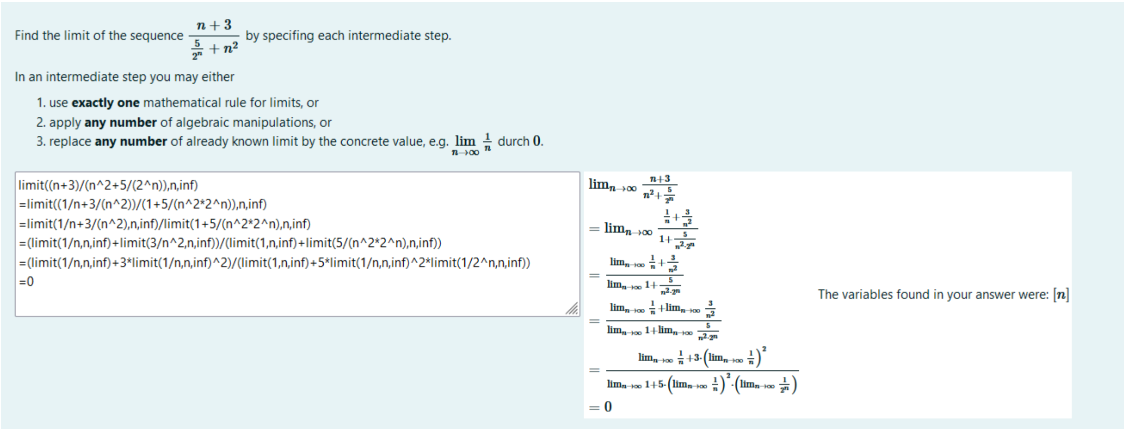


Figure 9: Complex task using the input type "equivalence reasoning"

CONCLUDING REMARKS

The main aim of the project is to develop digital elements that satisfy the demands and needs of at least two analysis courses at two universities. Such an aim results in some challenges and open questions that were addressed in this paper.

Actually, it seems possible to detect themes in introductory mathematics courses of different universities and different study programmes that enable us to develop common digital elements. This finding is the basis for all the work done up to now and the work that should be accomplished in the near future. Moreover, differences in notation and approaches are, so far, smaller than expected. This is a crucial result, since we can therefore develop digital elements based on common notations used in different analysis courses, different universities and study programmes. If agreement on a common notation is not possible or slight differences about the notation emerge, remarks about notation differences are included in those digital elements either directly in the element itself or given as further information about it on the platform of our project. By doing so, the project aims to provide an interdisciplinary overview of notation use and facilitate students' ability to adapt to different notation systems.

There are, nevertheless, challenges for collaboration and communication between lecturers and students. For example, the perspectives of mathematics lecturers and mathematics students are fundamentally different. Concerning lecturers and students, LLV.HD observed opposing evaluations between rigorous mathematics and the psychology of learning mathematics, which proved similar to the differences between mathematics and mathematics education. As a consequence, digital elements that use the same notation for the same course must be different for lecturers and students.

Students do not favour digital elements per se, but favour digital elements that are connected to their mathematics courses. Thus, a half-structured matrix of digital elements that was planned in LLV.HD could fit the needs of the students at the universities that are engaged in LLV.HD. However, it is still an open question whether students embrace further digital elements such as podcasts or quizzes since, so far, we have collected feedback only on instructional videos.

Furthermore, a lesson has been learned from the videos and STACK tasks: There is a need for digital elements that address conceptual knowledge instead of exclusively focusing on procedural knowledge.

Finally, following an agile project development shifted the goal of LLV.HD from producing a huge number of digital elements to developing elements that satisfy the demands and needs of lecturers and students from different universities and study programmes and, thus, could be integrated into digital networks as the central idea of LLV.HD. These digital elements do not replicate the concepts of analogue teaching, but require new conceptions raising, to some extent, fundamental questions of mathematics education.

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Panel

Innovations in university teaching based on mathematic education research

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We report on a variety of innovative projects that are at different stages of development and implementation. We start by presenting a project still in development to help address Klein's second discontinuity problem, that is, the perception of pre-college teachers that the advanced mathematics courses they took at the university are of little use in the practice of their profession. Then we briefly discuss the study and research paths (SRP). This is the proposal from the Anthropological Theory of the Didactic (ATD) to foment a move from the prevailing paradigm of visiting works to that of questioning the world. This is followed by the discussion of an online course for in-service teachers, designed to help them experience, adapt, and class-test a modeling intervention, as well as reflect on institutional issues that might constrain the future application of modeling in their teaching. We end with a discussion of a project based on the idea of guided reinvention, to design and study the implementation of inquiry-oriented linear algebra.

Keywords: Study and research paths, Klein's second discontinuity, modeling, inquiry-based mathematics education, linear algebra.

INTRODUCTION

What do we mean by innovation in university teaching? Century and Cassata (2016) define innovations as "programs, interventions, technologies, processes, approaches, methods, strategies, or policies that involve a change for the individual end-users enacting them." We add that innovation should help students learn a particular mathematical content better than traditional teaching, and the innovations considered in the panel must be based on mathematics education research. The first part of this definition underscores that the innovation does not have to be new to the field at large; rather, the practice should require that users change what they are doing, so what is emphasized is that the practice of interest is different from current practice. This way of viewing innovation stresses concern for change in teaching practices beyond the classroom of the individual researcher.

With this in mind, we have chosen four projects that propose teaching innovations at the university level and that are at different stages of implementation: one project, the Geometry Capstone course for pre-service teachers, has so far been implemented several semesters by the researcher in his own classroom; another project, the design and implementation of *study and research paths* (SRP), that has been implemented in different classrooms by different instructors and universities but that still is not widely adopted; a project dealing with an online MS program in mathematics education for in-service teachers that includes a modeling component that has been fully implemented in a university, and a project for an inquiry-oriented linear algebra course, that aims to attain national dissemination.

The following sections discuss each of the four highlighted innovations. Each is presented by the respective panelist Max Hoffmann, Ignasi Florensa, Avenilde Romo Vázquez, and Michelle Zandieh.

A CAPSTONE COURSE "GEOMETRY FOR STUDENT TEACHERS" AT PADERBORN UNIVERSITY

In this section, we present an innovation that we implemented in the context of a course named Geometry for Student Teachers at Paderborn University in Germany. The course is scheduled in the curriculum for upper secondary math teachers in the third year of study. Like other German universities, the subject-related part of this study program consists of courses on academic mathematics and on didactics of mathematics. While student teachers attend most of their mathematics courses jointly with mathematics major students, this course is taken exclusively by student teachers.

Innovation goals

In the project SiMpLe-Geo we develop and study innovations to increase professional orientation in the course Geometry for Student Teachers. In this way, we want to counteract the second discontinuity in teacher education. The course concept's theory-based development and initial research are part of the Ph.D. thesis of Hoffmann (2022). In addition, various other publications have been produced as part of the project (e.g., Biehler & Hoffmann, 2022; Hoffmann & Biehler, 2022), from which some text elements have been taken verbatim for this overview.

As a basis for the course concept, we have worked out the following three design principles for academic math courses for student teachers with a particular focus on professional orientation:

1. *Orientation to the scientific systematics of mathematics*: The course aims to treat an area of academic mathematics in a systematic and structured way. The course follows the usual scientific standards of mathematics. These can be prototypically described by the three steps: definition - theorem - proof. The necessary level of detail in the argumentation must be adapted to the students' level of knowledge and experience.
2. *Orientation to the math-specific presentation- and communication methods*: The study of mathematics uses methods common in scientific practice for gaining and exchanging knowledge. Accordingly, the three-step process described in 1. is supplemented by other elements, e.g., examples and non-examples, heuristics, and historical backgrounds.

3. *Implicit professional orientation*: The professional orientation should be considered in every decision to be made in the context of the course conception (e.g., selection of content). The basic credo should be: In any conceptual decision with several similarly suitable options, the one that can best be related to the future teaching profession should be chosen.
4. *Explicit professional orientation*: At appropriate points of the course, activities in which the mathematical knowledge and skills acquired are explicitly used functionally as a disposition for acting in profession-oriented situations. This use must also be explicitly reflected upon.

Overview of the innovations

We take a holistic approach to implementing professional orientation in the course, using innovation at both the content level and the level of teaching/learning methods.

Content structure of the course

A significant part of the course deals with axiomatic plane geometry. The careful selection of the axiom system represents an important aspect of implementing professional orientation. We use one based on the work of Iversen (1992), in which neutral plane geometry is built upon metric spaces and later is supplemented by the parallel axiom. Two major advantages of this approach are that it is productively interconnected with the fundamental analysis and linear algebra courses the students already have taken (e.g., we use the real numbers right from the beginning) and the fact that many definitions and proofs can be didactically reduced for school geometry.

Interface weeks

Interface-Weeks are one of the two main innovations on the level of teaching/learning methods. The idea is, to shift the course focus from a mathematical theory to discussing and reflecting on connections between the academic mathematics learned and the aspired profession. Therefore, lectures, exercise groups, and home assignments are designed according to the principle of *explicit professional orientation* and differ substantially from the other weeks. As the main focus of the interface weeks, we have chosen the central geometric concepts of *congruence* and *symmetry*. For both topics, first, essential characteristics of their rigorous mathematical treatment are detached from the particular axiomatic approach of the lecture. We do this by explicating so-called *interface aspects*, which result from inductive subject-specific-didactical analyses (Biehler & Hoffmann, 2022; Hoffmann & Biehler, 2022). Using such *interface aspects*, typical approaches to the concepts (e.g., from textbooks) are discussed from a professional perspective. In addition, various focal points of the instructional treatment of these concepts will be located from a mathematics perspective, with special consideration given to intellectual honesty. In the exercise groups, students work on corresponding, discussion-oriented tasks and collaboratively use the mathematical knowledge they have learned in contexts relevant to their profession. The homework consists exclusively of tasks for the *interface-ePortfolio*.

Interface-ePortfolio

The course-accompanying *interface-ePortfolio* is the second main innovation at the level of teaching/learning methods. In this learning activity, we combined the idea of a course-accompanying ePortfolio (see, e.g., the project *dikopost* (Siebenhaar et al., 2013)) with the use of profession-oriented tasks, so-called *interface-tasks* (e.g., Bauer, 2013). The use of this innovation is organized in such a way that in some weeks, ePortfolio-tasks replace some of the ordinary homework tasks. In addition, the students got feedback on their work from a student tutor. Those suggestions for improvement could be used for optional revision. The ePortfolios are technically realized so that only the student tutor can see the students' real names; the lecturer can only see them in pseudonymized form. This was done to keep the *interface-ePortfolio* as an ungraded learning opportunity with a high amount of (honest) reflection.

We used four different task formats to work on the ePortfolio:

- *Competence Grids for Self-Assessment*: Using these grids, students must self-assess their competencies in mathematical backgrounds of school geometry concepts and theorems and their skills in dealing with math-containing job-tasks. This activity is used at the beginning and the end of the semester, which allows the students to reflect on their competence growth during the course.
- *Interface Tasks*: In these tasks, students use their mathematical knowledge and skills as dispositions to look at and analyze profession-oriented situations (e.g., a real or fictional student contribution or a textbook page).
- *Reflection Tasks*: In the context of their *interface-ePorfolio*, students have to work on reflection tasks on different levels. This includes activities in which students reflect on how the competencies acquired in the course influence their work on interface tasks, occasions for reflection on their prior knowledge of the central geometric concepts, activities that generally refer to which sense students see geometry as relevant content in school mathematics, and self-perceptions about the ability to teach geometry.
- *Fact-Sheets for Geometric Mappings*: During the semester, students study different geometric mappings (orthographic projections, reflections, rotations, central dilations, reflections at circles) and their properties. In this fourth type of task, students summarize their stepwise growing knowledge of those geometric mappings in a pre-structured way. This consists of a formal definition, an explanation of all possible variants of the formalization (e.g., as a term or as a matrix), and a detailed written example calculation.

Current interim status of the project

We have already taught the course according to this concept four times and researched and further developed it within a design research approach. Initial results show that students substantially contribute to overcoming the second discontinuity (related to plane geometry), at least from a subjective perspective. We are currently evaluating further data to gain insights into the objective impact.

STUDY AND RESEARCH PATHS: THE ATD PROPOSAL

SRPs: an ATD-founded device

Study and research paths (SRP) are inquiry-based teaching formats framed in the Anthropological Theory of the Didactic (ATD). SRPs are long teaching and learning processes, lasting from some 8-10 sessions (2h) to an entire course, that start with the consideration of an open generating question that student(s) address under the guidance of the teacher(s). Describing and analyzing the SRP proposal cannot be done without explicitly mentioning some of the ATD principles and theoretical developments that are in the inner heart of the proposal.

The first aspect that is undetachable from the SRP proposal is the notion of didactic paradigm. SRPs are conceived as didactic devices fostering a shift in the prevailing didactic paradigm in our societies, from the *paradigm of visiting works* (PVW) to the *paradigm of questioning the world* (PQW) (Chevallard, 2015). Teaching and learning processes in undergraduate mathematics courses are particularly experiencing this crisis of the old PVW, where content organizations are presented for students to "visit" them, which contrasts with the emergence of the PQW, where the study of questions becomes the center of the study process. The implementation of an SRP is a way to analyze the conditions needed for the paradigm shift. Diverse experiences at the undergraduate level show relevant results of this evolution (e.g., Barquero et al., 2018; Florensa et al., 2018a).

A second point that is necessary to consider when describing the SRP proposal is its link to the didactic engineering methodology (Barquero & Bosch, 2015; García et al., 2019). In other words, SRPs are also research artifacts allowing the research community to generate empirical material to conduct didactic and epistemological analyses. In fact, the strong relationship between the conception of the knowledge to be taught and the didactic phenomena emerging in the teaching and learning processes assumed as a founding principle of the ATD, turns SRPs into key elements of didactic research. An illustrative example of the role of the SRPs as artifacts allowing researchers to modify and study specific didactic phenomena are the works of Berta Barquero when describing, analyzing, and modifying the phenomenon of "applicationism" in mathematics courses in applied sciences degrees (Barquero, Bosch, & Gascón, 2014). An important aspect of SRPs is this twofold nature: a teaching proposal and a research artifact that inevitably fosters a change on the activity existing in school (or university) institutions.

The institutional approach of the ATD is the third aspect that defines the SRP proposal. On the one hand, as mentioned before, SRPs have the capacity to modify teaching and activating activities in school institutions. On the other hand, the implementation and viability of SRPs are undetachable from the study of the institutional *ecology*, that is, the set of conditions needed and restrictions hindering their viability in different institutions. The ecology of SRPs has been studied in some research works at the university level (Barquero, Bosch, & Gascón, 2013; Barquero, 2018) using the scale of levels of didactic codeterminacy (Chevallard, 2015). It allows researchers to identify restrictions that appear outside the level of the classroom and, at the same time, make explicit the changes that the change of paradigm (and in particular the SRP implementation) would cause in the organization of subjects and contents.

Finally, the study and analysis of SRPs cannot be detached from the notion of Herbartian schema. The Herbartian schema is a model of inquiry processes. It considers didactic system as formed around a question (and not a specific work as usual in school institutions):

$$[S(X;Y;Q_0) \rightarrow M] \hookrightarrow A^{\blacktriangledown}$$

In this context, the group of students X with the help of a group of teachers Y must provide an answer to Q_0: $A^{\blacktriangledown}$. The process of inquiry of Q_0 leads the community of study (X, Y) to meet different pre-existing answers $A^{\Diamond}$, derived questions Q_i, other works W_n needed to interpret $A^{\Diamond}$, and empirical data D_j. The set of these elements constitutes the *milieu* M of the inquiry:

$$M = \{A_1^{\Diamond}, A_2^{\Diamond}, \dots, A_m^{\Diamond}, W_{m+1}, \dots W_n, Q_{n+1}, \dots Q_p, D_{p+1}, \dots D_q\}$$

The Herbartian schema pinpoints the fact that putting the questions at the center of the inquiry process fosters the transition between didactic paradigms but not in terms of "substitution": the works and pre-existent answers are still relevant and are studied. However, its new role is subordinated to the generation of an answer $A^{\blacktriangledown}$ to a question Q_0, which remains in the heart of the study process. This capacity of SRPs to enable moments of study of previously existing works and moments of research during the same inquiry process contrasts with other proposals where the study activity is not present. The Herbartian schema highlights another commonality in the different SRPs implementations: the responsibility of enriching the milieu during the inquiry is shared by X and Y. While in other proposals the teachers often assume the role of enrichers and validators, in an SRP the evolution of the process is taken by the whole community of study.

SRPs: from the first implementations to the transposition to lecturers

From the first implementations of SRP at undergraduate level in 2005 with the thesis of Berta Barquero, the way SRPs have been integrated and their role has very much evolved. We can describe this evolution in terms of integration to the courses, role of the teacher or inquiry guide, domains of intervention, and dissemination to teachers.

Regarding the integration to the courses, the first SRPs were implemented as *modelling workshops* running in parallel to the mathematics courses. This evolution is closely related to the ecological conditions in the institutions where the SRPs were implemented. According to Barquero et al. (2021), the different settings require different levels of change in the previous organization. This flexibility in the SRP organization facilitates its implementation in very different institutions with different pedagogical conditions and constraints.

Another relevant aspect is the SRP's teacher or guide (*Y*, in the didactic system). In the first experiences, researchers were those in charge of the design and management of the SRPs. This situation led to very fragile environments. In other words, the first SRPs lasted while the researcher kept the position of guide of the study. Once the researcher left the institution, the SPRs tended to disappear or significantly reduce their time dedication.

The first SRPs implemented at the undergraduate level were implemented in mathematics courses for applied sciences and in business administration degrees. However, this past decade, SRPs have spread in different domains. One of the first domains that adopted SRPs outside mathematics education was mechanical engineering (with subjects such as elasticity and strength of materials) (Florensa et al., 2018a, Bartolomé et al., 2019) and applied statistics (Markulin et al., 2021). In the last two years SRPs have also been adopted in Chemical and ICT courses for engineers and accounting courses in Business Administration degrees.

This spreading of SRPs cannot be understood without two factors that foster SRP dissemination as a research-based teaching innovation device. First, is the explicit training of university teachers (Florensa et al., 2018b). The implementation of diverse teacher development courses has enabled teachers to start collaborating with researchers to design and implement SRPs, overcoming the fragility of the researcher-teacher positions concentrated in a sole person.

Second, the diffusion of SRPs has been done in parallel with the transposition of different tools and devices that have helped both teachers and students to deal with the new organization and conception of knowledge around questions. The incorporation of questions-answers maps, logbooks or weekly reports and final reports addressed to the receiver of the answer seem to facilitate the inquiry management and assessment.

A final aspect that remains open is the (inquiry) contract that needs to be established around the generating question. What characteristics does it need to fulfill? Even if there is still a lot of research to do in this field, some of our last analyses seem to indicate that the existence of an external instance receiving the answer to the generating question facilitates the implementation of a rich inquiry process and a shared assumption of responsibilities within the community of study.

MATHEMATICAL MODELLING COURSES IN AN ONLINE PROFESSIONAL DEVELOPMENT PROGRAM

In 2000, Mexico's National Polytechnic Institute created a master's program for in-service mathematics teachers in the virtual modality. The groups formed could include teachers from different educational levels: secondary school, high school, and university, and from different geographical locations in Mexico and Latin America. This heterogeneity made it necessary to design courses that could contribute to the analysis, innovation, and regulation of diverse teaching practices. In 2010, some courses were designed to focus on designing mathematical modelling activities specifically for the study of non-mathematical contexts, such as engineering. One objective was to offer tools to aid in designing didactic proposals for training non-specialists at the university level. In the framework of these courses, professors implemented mathematical modelling activities that related math to other disciplines and encouraged reflection on the minimum conditions necessary for integrating mathematical modelling into teaching. Some examples of these courses and the work carried out by the teachers will illustrate this professional development proposal, its scope, and its limitations.

A Mexican professional development program for in-service mathematics teachers: ProME

Currently in Mexico, there is training for future teachers for elementary and secondary school, but no specific training for high school and university mathematics teachers. Most mathematics teachers and professors at these levels are mathematicians, engineers, or professionals with a four-year undergraduate degree in an area with a specific mathematical-scientific orientation who have a vocation and interest in teaching. Many in-service teachers feel a significant need for specific training. Several master's programs have been created in Mexico to meet the professional and didactic needs of high school and university mathematics teachers and professors. These are two-year programs that include several courses and the elaboration of a master's thesis. Most are offered at universities in the in-person modality. Some are full-time and research oriented. Most students in those programs have scholarships. Other programs are part-time and oriented more towards professional development. However, teachers who live far from universities cannot register in these programs. For this reason, the program for the professional development of mathematics teachers (ProME) was created in 2000 at the National Polytechnic Institute in the online modality with two goals, one academic, the other social:

Academic: To introduce groups of mathematics teachers into the practices, theories, and languages of Mathematics Education by connecting research with practice.

Social: To modify, as far as possible, the scenario of social exclusion that many in-service teachers experience because the opportunities for training in Mathematics Education do not provide them with any space.

The Study and Research Path: a theoretical tool for analyzing ProME's educational model

In general, the courses in this master's program can be analyzed by considering a didactical system composed of students (X), educators (Y), and courses (Q):

- *The teacher-students* (X) are in-service mathematics teachers and professors from Mexico and other Latin American countries with diverse professional backgrounds, and different teaching experiences who were working at distinct educational levels: secondary school, high school, and university.

- *The educators* (*Y*) are researchers in Mathematics Education with experience as math teachers.
- *Courses/SRP-TE* (*Q*). Three types of courses are offered: theoretical, theoretical-practical, and seminars. The first focus on specific theoretical frameworks. The second analyze elements of research in Mathematics Education in relation to teaching practices in mathematics, while the seminars guide the students in writing up their theses.

The design of the theoretical-practical courses identified two types of questions:

- Professional questions that arise in practice, such as how to integrate technology into mathematics teaching and how to design mathematical modelling activities.
- Research questions analyzed in the context of math education, such as how to identify the nature of obstacles –didactical, epistemological, etc.– in teaching mathematics.

In other words, we identify objects of study and outcomes of Mathematics Education related to professional issues that math teachers and professors may not be aware of.

Mathematical modelling courses

There are two kinds of mathematics modelling courses, discussed here as a Study and Research Path for teacher education (SRP-TE). They were designed after 2013. The generating questions that motivated these courses were Q_0-*TE* (professional questions):

- How can a learning process related to *mathematical modelling* be *analyzed, adapted, developed*, and *integrated* into our teaching practice?
- How can *long-term learning processes* based on modelling *be sustained institutionally*? What *difficulties* need to be overcome? What didactic tools are needed? What new questions arise and how can they be addressed?

In general, these generating questions are integrated using the methodology proposed by Ruiz-Olarría (2015) and adapted to the online modality by our team of educators (see, for example, Barquero et al. (2018)). The strategy developed has four steps:

1. Allow teachers to experience an SRP like mathematicians or apprentice mathematicians.
2. Analyze the SRP using didactic tools:
   - Mathematical analysis (reference epistemological model)
   - Didactic analysis: changes in didactics (and pedagogy) contracts, dialectic media-milieu, questions and answers, etc.
   - Ecology and sustainability of the SRP: institutional conditions

3. Adapt the SRP experienced (in step 1) so it can be implemented with a given group of students
4. Implement an a posteriori analysis of the SRP experienced with their students.

The small difference between the two types of SRP-TE is the way in which the SRP proposed in step 1 is designed. For the first type, the design of the SRP does not require an analysis of a non-mathematical context, but in the second type this is necessary. The first SRP-TE proposed, for example, analyzing and solving 'Forecasting sales for *Desigual* (a Spanish fashion brand)'. An epistemological dimension is considered by addressing several questions, such as What is modelling? How can the modelling process be described? and What is inquiry? These SRP-TE have been implemented in several editions by a large team of educators from Mexico and Spain to make the institutional conditions that drive –or constrain– the integration of mathematical modelling activities in the classroom visible to math teachers and professors (see Barquero et al., 2018; Romo et al., 2016). The second type of SRP-TE integrates the SRP that originated in non-mathematical contexts; for example, the Blind Source Separation method (BSS) used in acoustics, geophysics, and biosignal analysis. The BSS is an exciting method as it constitutes a case of inverse modelling that makes it possible to separate mixes without knowing the components or how they were mixed. One of the algorithms involved is based on the matrix model, $Ax=b$ (Vázquez et al., 2016). Using this approach, Camilo Ramírez designed an SRP in his Ph.D. thesis–in progress– that was implemented in an SRP-TE, as discussed below.

An example of a mathematical modelling SRP-TE: the case of the BSS method

The SRP-TE lasted four weeks (September 28-October 23, 2020) and was composed of three activities. Six members of the group (two secondary school teachers, two high school teachers, two university professors) and three educators participated (an engineer-researcher who was an expert in the BSS method and two researchers in Mathematics Education). In activity 1, two teams of students develop an SRP using the BSS method and then analyzed the process followed to answer the generating question: what is the mathematical technique that makes it possible to separate a mixture of sounds? The main media for this activity was an online resource that showed three different mixes of the same sounds. The mixes differed in terms of the distance between the sources (sound instrument) and the observations (recorders). Various elements were provided to analyze these mixes, including a geometric representation of the sources (sounds) and observations (recordings) and the hearing and tabular representations. In addition, we proposed identifying the derived questions and their answers to analyze the modelling process followed in this activity.

Activity 2 consisted in adapting the SRP developed in Activity 1 so that it could be implemented with students in an online modality (due to the conditions of the Covid-19 pandemic). To this end, three elements were given: 1) a BSS-praxeology ; 2) a school BSS-praxeology obtained from a didactic transposition performed on the BSS praxeology; and 3) an SRP designed for first-year university students that included four activities and elements to integrate a milieu: two free online resources (designed by one of the educators) that allowed them to listen to two mixtures of pure tones and explore different geometrical configurations between the sources and observations, such that they could identify the distance between them, which represented the coefficients of the system of linear equations; that is, a mathematical model of the mixtures. The other variables considered were frequency and amplitude. The student-teachers could use two of these four activities in their adapted SRP and had to modify the other two activities. Likewise, they had to perform an *a priori* analysis that showed the questions and answers that the students proposed. Activity 3 consisted in carrying out the *a posteriori* analysis. One of the most exciting adaptations of the SRP was made by a university professor with a background in engineering who adapted it for a group of volunteer high school students who had begun their first year of university. He modified the online resource proposed in Activity 1 and proposed quadratic signals and several activities to study three variables– distance, frequency, and amplitude– and the relations among them. His analysis of the students' activities showed the elements of the milieu associated with his SRP and affirmed that managing the SRP had proven to be: "Students had a clear difficulty in identifying that the modelling of the system is performed through a system of equations. Here, a series of activities that ask for different configurations to lead to the conclusion is probably required because giving them freedom to modify the scenario [online resource] was ineffective during implementation."

The other adaptions revealed the need to modify Activity 2 to analyze the milieu more deeply and determine how it can be extended or adapted with respect to the characteristics of the math class where the SRP will be implemented. Despite these issues, the student-teachers recognized that the SRP made it possible to perform a modelling activity in math class that allowed them to resolve challenging tasks.

PROJECT IOLA: INQUIRY ORIENTED LINEAR ALGEBRA

The Inquiry-Oriented Linear Algebra (IOLA) curriculum has been developed over the past 15 years and is continuing to evolve. The materials have been developed based on a set of design principles taken from Realistic Mathematics Education (RME; Freudenthal, 1991; Gravemeijer, 2020) and the design process is implemented through a series of teaching experiments and other mechanisms as described by our design research spiral (Wawro et al., 2022). The project began with a National Science Foundation (NSF) grant on student learning during which the initial tasks were developed (Rasmussen & Zandieh, 2007). This work continued with a grant focused specifically on the IOLA curriculum (Wawro, Zandieh, & Rasmussen, 2013). An additional grant (still in progress) is extending the IOLA materials (Wawro, Zandieh, Andrews-Larson, & Plaxco, 2019).

By the end of the 2013-2018 grant period, we had completed three Units, each with teacher support materials posted to our IOLA website (http://iola.math.vt.edu; Wawro, Zandieh et al., 2013). Each unit consists of a series of activities on a specific topic that typically takes 3-5 class periods. Figure 1 lists the units developed for the 2013 grant as well as the units that we are developing currently as part of the 2019 grant. The title of each unit refers to the experientially real setting (Gravemeijer & Doorman, 1999) in which the task sequence takes place. Below the title is a short description of the mathematical emphasis of the unit.

Created during previous grant	Created during the current grant
Unit 1: Magic Carpet Ride Span and linear independence	
	Unit 2: Meal Plans Solutions to systems of linear equations
Unit 3: Italicizing N Matrices as linear transformations	
	Unit 4: Distortion Determinants
	Unit 5: Hallways Subspaces
Unit 6: Blue to Black Change of basis, diagonalization, and eigentheory	
	Unit 7: Mail Delivery Least squares approximation and projection

Figure 1. IOLA curriculum units completed and under development.

We begin Unit 1 with vectors because we see vectors themselves, linear combinations of vectors, and vector equations as the most fundamental aspects of a beginning linear algebra course. We have also found that starting with a travel metaphor for exploring initial vector ideas works well as a starting point for students (Wawro et al., 2012). Research on the completed units and initial results regarding the new units can be found in various publications: Unit 2 (Smith et al., 2022), Unit 3 (Andrews-Larson et al., 2017), Unit 4 (Wawro et al., in press), Unit 5 (Andrews-Larson et al., 2021), Unit 6 (Zandieh et al., 2017; Plaxco et al., 2018), Unit 7 (Lee et al., 2022).

How do we design the units?

The development of all seven units has followed a design research cycle (Cobb et al., 2003) in which we engaged students with the activities, documented this process and used the results of this research to rewrite or refine the activities. In developing the four new units we have been particularly intentional in following the design research spiral shown in Figure 2 (Wawro et al., in press).

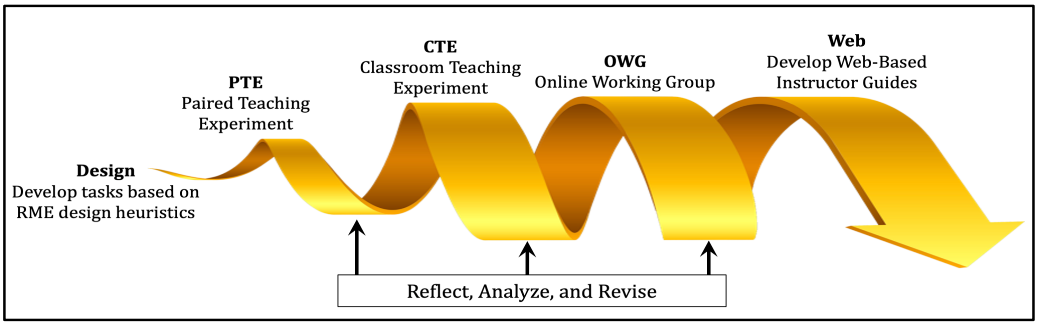

Figure 2. Design Research Spiral as shown in Wawro et al. (in press).

The initial task design is developed by a subgroup of the project team with other project team members working through the tasks in front of the development team to give initial feedback. The developers then conduct a PTE, paired teaching experiment, (similar to Steffe & Thompson, 2000) that allows for a detailed focus on the progression of the reasoning of the two students as they work through the tasks. We typically use pairs of students instead of individual students as this allows for students to learn from interacting with each other's ideas, much like we intend students to engage in a classroom setting. An analysis of the PTE allows for refinement that feeds into the CTE, classroom teaching experiment (Cobb, 2000).

The CTE is conducted by an IOLA project team member in an introductory level university linear algebra course that is part of his or her regular teaching load. Data is collected about how the students interacted with the activity in class, the role of the teacher in the classroom and student written work regarding the task. This data is analyzed and revisions to the unit are completed in preparation for the (OWG) online working group.

For the purpose of the development of the four new units, our online groups were designed to be composed of experienced IOLA instructors, i.e., instructors who had used the initial three units multiple times in the classroom. This included project team members but was intended to focus on getting feedback from outside of the project team. These groups met once per week for 6-8 weeks to prepare to implement the new unit, discuss reactions, questions, and feedback during the implementation, and then finally to reflect back on student and instructor interaction with the task sequence and how the unit may be improved.

Design Heuristics

The units are designed using three RME heuristics: didactical phenomenology, emergent models, and guided reinvention (Gravemeijer & Terwel, 2000; Gravemeijer, 2020). *Didactical phenomenology* is a way of determining a context that is well suited for the learning of a particular set of mathematical ideas. The context should be experientially real for the students; in other words, it is a setting that the students can immediately interact with and engage in. Given an appropriate task, students organize and structure aspects of that context in ways that create the mathematical ideas intended by the curriculum designers. The *emergent models heuristic* highlights how instructional designers can support students in transitioning from less formal to more formal ways of reasoning with and about these mathematical ideas.

Gravemeijer (1999) elaborated the development of emergent models as a progression through four levels of activity: situational activity, referential activity, general activity, and formal activity. Wawro, Rasmussen, et al. (2013) describe the transition across these levels of activity that occurs in Unit 1 of the IOLA curriculum. Students start working with two modes of transportation (a magic carpet and a hoverboard) each given by vectors in two dimensions. Initial exploration about what locations can be reached lead students to create ideas that the instructor can label as span, with further tasks (in three dimensions) leading to a formal definition of linear independence, and theorems about when a set of vectors will be linearly independent or dependent. Student initial activity in the task setting is organized or mathematized by the students in ways that the instructor can notate in terms of standard mathematical definitions and theorems. The process of students reinventing these ideas through their organizing activity, combined with the role of the instructor in guiding this process, is called *guided reinvention*.

Role of the Instructor

Given the important role of guided reinvention in the RME design heuristics, it is necessary to reflect on what instructional strategies can be implemented to support students in this process. Our project, IOLA, is called inquiry-oriented because we believe in both the importance of student inquiry into mathematical ideas and the importance of instructor inquiry into students' emerging mathematics (Rasmussen & Kwon, 2007). Johnson et al. (2015) created the TIMES (Teaching Inquiry-Oriented Mathematics: Establishing Supports) project to provide instructors with opportunities to implement inquiry-oriented curricula. As part of that process, they studied what is involved in inquiry-oriented instruction (IOI).

Kuster et al. (2017) characterize inquiry-oriented instruction around "four instructional principles: generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation," (p. 14). The instructor *generates students' ways of reasoning* by engaging them in goal-oriented activity with their classmates, usually in small group work. As student reasoning is generated, the instructor finds ways to *build on student contributions* with the goal of guiding students toward a reinvention of the mathematical ideas. To *develop a shared understanding* across students, the instructor acts as a broker between small groups and between small groups and the whole class (Rasmussen et al., 2009). The instructor also acts as a broker between the local classroom community and the broader mathematics community by helping the students *connect* their emerging mathematics *to standard mathematical language and notation*.

More specific ways of accomplishing these principles include what Rasmussen and Marrongelle (2006) refer to as pedagogical content tools. *Generative alternatives* are examples given by the instructor to elicit student reactions to possible alternative solutions or strategies. A transformational record is a way of notating student thinking that a student agrees captures their idea, but that the instructor knows is also a steppingstone to the standard mathematical notation. In these ways, an instructor may support guided reinvention by encouraging students to make explicit their ways of reasoning and by building on these through a transformational record toward a shared understanding that uses standard mathematical language and notation.

Implementation of IOLA

There are various ways that IOLA is being currently implemented in classrooms in varied instructional settings. Most recently there were over 700 accounts on the IOLA website (http://iola.math.vt.edu). Of course, not all accounts represent people who teach with the materials. In 2018 we conducted a survey of the then 328 faculty with accounts and found that of the 94 who responded to the survey 61 (65%) had adopted and integrated at least some of the existing IOLA materials in their classrooms. In addition, there is anecdotal evidence that some instructors who do not have accounts on the website use versions of the materials adopted from published sources like journal articles.

Over the years we have provided a variety of types of support to instructors who would like to use the IOLA materials. For account holders, the website has the full set of activities (for the initial three units) as well as instructor resources and examples of student thinking when using the materials. We have written articles for researchers and practitioners highlighting the progression of student thinking possible with the tasks as well as papers that explore the role of the instructor (e.g., Andrews-Larson et al., 2017; Zandieh et al., 2017). We have presented at research conferences and have provided workshops for instructors. Of particular note, the TIMES project (Johnson et al., 2015) recruited and worked with instructors using three inquiry-oriented curriculum materials, including IOLA. They leveraged the web-based instructional support materials provided by IOLA, provided summer workshops, and instituted Online Working Groups (OWG) that met to discuss implementation on a weekly basis. These OWG functioned to allow instructors new to IOLA, and perhaps new to any inquiry-oriented instruction, to have a place to get feedback, support, and exchange ideas with other instructors as they implemented something new to them.

To summarize, over the past 15 years the IOLA project has benefitted from a growing network of researchers and instructors contributing to this work. The project is centered around principles for curriculum design (RME) and research-based feedback on the design process (design research spiral). Implementation strategies include online instruction support materials as well as workshops and OWGs to aid instructors in implementing inquiry-oriented instruction (IOI).

CONCLUSION

The projects presented in the panel offer different views of inquiry in mathematics education. These projects can be positioned in different places on the continuum from open to directed inquiry; The more radical and open proposal is that of the SRPs. It can be expected to face institutional constraints in its quest to challenge the didactical paradigm that is prevalent at universities. This is followed by the inquiry fostered by the online modeling projects for in-service teachers, which can also be viewed as a special type of SRP (for teacher education). The openness of these modeling projects varies depending on the type of problem and the resources made available to students. Then, the guided reinvention of project IOLA may be thought to be within the paradigm of visiting works as it does not depart from a standard curriculum while following its instructional principle of connecting students' productions and ways of thinking to standard mathematical language and notation. The more directed modality of inquiry is that of the geometry capstone course for pre-service teachers. Students here inquire while working on interface tasks to relate the advanced viewpoint of the geometry course to their future careers. Nevertheless, it is in large part a lecture-based course that follows the definition-theorem-proof format.

Klein's second discontinuity problem, study and research paths, modeling, and inquiry-based mathematics education are all well-known approaches in the mathematics education community. They are actively researched, and the implementation and dissemination of their different proposals present a challenge. The projects discussed in the panel propose different ways to attend to this challenge. In the SRPs, this is the focus of their research; the modeling projects for in-service teachers include their adaptation and implementation at different educational levels thus providing a rich ground for the study of institutional constraints as well as for reflection on what it may take to implement modeling in these different contexts; and project IOLA facilitates its dissemination with their web page, articles, workshops for instructors, and online working groups. The geometry for pre-service teachers' project is in its development phase and can only start to envision what its approach will be to implementation and dissemination, a challenge we all share in the mathematics education community.

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TWG1: Transition to, across and from university mathematics

TWG1: Transitions to, across and from University Mathematics

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INTRODUCTION

A specific thematic working group dedicated to the topic of transitions first appeared at the INDRUM conference in 2018. To attend to the growing number of papers, the programme committee introduced this theme—in line with increasing research on transitions in mathematics education research (Gueudet et al., 2016)—as a new TWG transversal to mathematical domains, alongside students' and teachers' practices. Although it disappeared in 2020, it continued to thrive through a dedicated chapter in the ERME volume on INDRUM research from the first two conferences (Hochmuth et al., 2021). The INDRUM2020 keyword "transition to and across university mathematics" was then modified in the INDRUM2022 call for papers to encompass a larger spectrum of transitions and TWG1 was named accordingly.

In fact, the school to university transition (Klein's first discontinuity) is still dominant: it is the focus of 5/8 papers and 3/4 posters which were assigned to our group. Dually, 2 papers and 1 poster deal with the transition from university to secondary education (Klein's second discontinuity). A single paper considers the "across university" transition (with a focus on the teaching of a mathematical concept throughout the Bachelor in the Abstract Algebra track). To complete the perspective, papers assigned to other TWGs but mentioning transitions as a keyword shall also be counted; hence there are 2 additional papers on the school-university transition in TWG3 (focusing on proof), 2 in TWG6 (on students' learning), as well as 3 papers on Klein's second discontinuity attached to the new TWG5 on teacher education. It is worth noting that papers which investigate the case of engineering students do not use the lens of transitions, so that the transition from university to the workplace remains under-researched except in the context of pre-service teacher education.

Altogether, the theme of transitions overlaps with several TWGs and the core of the idea of transition that grants the unity to our TWG is still an open research question. Moreover, various facets of transitions may be studied using a diversity of theoretical/methodological frameworks. In what follows, we restrict our account to the 8 papers and 1 poster which have been presented and discussed during the group sessions, hence the figures in parentheses. We thus note the following facets: epistemological (7), cognitive (2), affective-emotional (1), socio-cultural, institutional (7); and the frameworks used: the Anthropological Theory of the Didactic, ATD (6), Commognition (1), concept image/concept definition (1), person-environment fit (1), and mathematical content analysis (3).

As Hochmuth et al. (2021) already pointed out, a large number of authors use the institutional perspective of ATD to study transitions, which led us to group those papers in the first parallel presentation session. By contrast, a diversity of perspectives (facets of transitions and theoretical/methodological tools) were offered in the second presentation session. After an in-depth discussion of each paper, discussions opened up to examine the topic of transitions in the light of all the papers and finally envisage opportunities for collaborations and avenues for further research. We will begin with an account of the contributions and then highlight some of the main points raised during our discussions. We conclude with a few ideas on the topic of transitions that may inspire future research.

HIGHLIGHTS FROM THE CONTRIBUTIONS

We asked authors to produce a highlight of their research in the form of a question/problem and its answer. In this section, we use these highlights—which were communicated in the group report at the conference—as a means to summarize striking features of the contributions and introduce readers to these works.

The school-university transition

Sarah Khellaf and Jana Peters raise the following questions: In what way can praxeological analysis inform the creation of study materials for first-year mathematics (teacher) students, that aim to make apparent to them differences between the institutions of school mathematics and university mathematics? What type of empirical questions about the implementation of these tasks could be asked and answered in the framework of ATD? As an answer, a task-design rationale has been explained in the paper. The reference model discussed can be used to identify ‘unusual’ (personal) praxeologies in student solutions. These can be compared with known dominant epistemological models from school and university, to generate hypotheses about their possible origin.

Tobias Mai and Rolf Biehler put the following problem in the foreground: School textbooks tend to introduce vectors as a mixture of the notions of *n*-tuples, translations, and sets of arrows—there is a need to explicitly and mathematically work out and integrate these settings in order to analyse and untangle interwoven approaches in school textbooks. As an answer, in the reference model presented in the paper, all three approaches to vectors are explained and finally discussed regarding their isomorphy. In the end, the most ostensive (illustrative) approach via arrows turns out to be the most complex approach of the three.

Jelena Pleština and Željka Milin Šipuš ask: How do polynomial-related praxeologies develop and differentiate through secondary school and a first-year bachelor programme in mathematics? In secondary school textbooks, the algebraic and analytical approaches to polynomials induce two disjoint praxeological organizations. In a first-year bachelor programme, specific and reduced praxis blocks align with the general logos blocks of praxeologies whose object of knowledge is the notion of polynomial. As a consequence, the relation first-year undergraduate students have to formal polynomials is marked by almost empty logos blocks.

Sarah Schlüter and Michael Liebendörfer explore: Which strategies do students use to cope with difficulties in "borderline cases" when their concept image seems to contradict the definition? Even if students apply the definition correctly, they do not trust the formal argumentation and tend to rely on intuition and their concept image. In addition to strategies based on informal reasoning, they manage to argue on a meta-level themselves, for instance by using transfers to similar "borderline cases".

Katharina Kirsten and Gilbert Greefrath ask: What are the characteristics of university students who choose on-campus or distance learning courses? Students with weaker connections to mathematics (e.g., in terms of self-efficacy and final math grade) and a higher digital readiness are more likely to choose a distance learning course. By contrast, students with strong math prerequisites tend to choose an on-campus course. Learning types based on self-regulation and peer learning do not play a significant role in course decision—at least in preparatory courses.

Finally, the poster by Ana Katalenić, Aleksandra Čižmešija and Željka Milin Šipuš tackles the following question: How can the discourses on asymptotes change and develop in the transition from upper secondary to university education? As a result, discourses can develop from colloquial narratives supported by iconic representation through working on techniques of evaluating function values and finding asymptotes, towards the formal definition using distance between points on a curve and the line and expressions with a function limit.

Other transitions

Thomas Hausberger and Julie Jovignot investigate: How can students' difficulties in acquiring a structural sense be understood in terms of institutional gaps in the Abstract Algebra track throughout the bachelor programme in mathematics? As a result, the study of structuralist levels of structuralist praxeologies and the values of their didactic variables in relation to the dialectic of contextualisation and decontextualisation points towards a huge gap at the 3rd year of the bachelor programme in France. It seems to be reinforced by the compartmentalisation of knowledge in small teaching units that hinders the vitality of the dialectic.

Heidi Strømskag and Yves Chevallard examine: What transformations has the notion of concavity of functions undergone during the didactic transposition process from the knowledge taught at university to the knowledge to be taught in upper secondary school? Praxeological analyses of a university textbook and a Grade 12 textbook show that while in the university presentation, the graphical notion of concavity is mathematised, in the school presentation, it remains non-mathematised: concavity is to be seen on the graph of the function—where the theorem proved at university becomes now a mathematically unfounded definition of concavity.

Finally, Max Hoffmann and Rolf Biehler study the following question: What prior knowledge do student teachers have on the geometric concept of congruence before taking a geometry course at university? As a fact, this multi-faceted concept is treated rather “one-dimensionally” in German schools. Taking at university the resulting pre-formal and superficial prior knowledge not into account to focus on formal aspects is likely to perpetuate Klein’s second discontinuity. There is a risk that prior mathematical knowledge from school will coexist with the academic mathematics learned, rather than being studied, corrected, and updated.

HIGHLIGHTS FROM THE DISCUSSIONS

Common themes emerging

Definitions in mathematics were debated in relation to their role in acquiring concepts, solving problems, and proving theorems. The notion of borderline/challenging cases was treated as such examples play an important role in complementing an incomplete predominant concept image. Polynomials in school mathematics, in abstract algebra, and in analysis appeared as examples in the considerations. Another aspect that came up was that of theorems and examples used as definitions in school mathematics, most notably in textbooks—possibly with the intention of making the knowledge at stake available to a larger group of students—, a transformation that likely simplifies and distorts the mathematical knowledge.

Attention was also given to the various transitions that occur in the education of teachers and that teacher education should address. Studying mathematics in view of teaching it requires developing other, new relations to mathematics compared to relations a mathematics student must develop. The *topos* changes, for instance, a mathematics teachers will have to choose examples and design tasks related to particular mathematics content in order to create opportunities for others to study it.

Theoretical frameworks and methodologies

The concept of *praxeology*—an analytic tool provided by ATD to model any human activity in terms of *praxis* (the type of tasks and the technique to solve them) and *logos* (the way to explain the technique and the theory to justify the explanation)—was used in six papers. Praxeological analysis was discussed on a general basis and linked to the notion of *reference epistemological model* (REM) to be used, for example, in didactic design, trying to remedy ruptures identified in dominant epistemological models, as well as overcoming didactic phenomena caused by such dominant models.

In ATD, Klein’s double discontinuity can be expressed in terms of transpositive processes: When one goes from a level n to a level $n + p$ in a curriculum (e.g., from secondary school to university), one generally faces an increasing rate of *mathematisation*, and conversely, in the opposite direction, there is generally a *demathematisation* of the mathematical content. This was discussed and related to the formalization developed by Winsløw and Grønbæk (2014).

On methodology, the problem of standardized methods for elaboration of REMs was raised and related to three dimensions of the questioning of any object: its structure, its functioning, and its utility. Networking of theories, ATD and Commognition or ATD and Stoffdidaktik (subject matter didactics), was mentioned as a promising research methodology to cross perspectives and promote collaborations but not really discussed in depth due to lack of time. Finally, an understanding was reached that when communicating research to non-specialists of ATD (especially in oral presentations), it is appropriate to avoid excessive formalism.

CONCLUSION

With a focus on transitions, researchers are aiming at the investigation of didactical phenomena in terms of continuities/discontinuities/ruptures. They may be pursuing different goals: their endeavour may be to identify difficulties related to epistemological/cognitive/institutional discontinuities, to suggest ways to smoothen ruptures or assess existing measures (to respond to institutional and societal demands), to contribute to teacher education (since most researchers are teacher educators), to refine theoretical constructs (such as models of transitions), or to study the effects of the didactic transposition.

Avenues for further research are wide. At the level of the school-university transition, collaboration among researchers should entitle a shift from small-scale local studies (centred on a concept or a single institutional context) to wider perspectives and contexts, including comparative or longitudinal studies. Research on ruptures across university studies, in particular towards advanced mathematics, is still rare. With the intensification of research on Klein’s second discontinuity, we expect reports on curricular innovation to account for strategies developed to tackle institutional constraints and to provide means to cooperate with mathematicians. Finally, transitions from university to the workplace for other careers than teachers (e.g., engineers) need also greater attention. INDRUM looks forward to receiving contributions in these directions at the next conference.

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Student Teachers' Knowledge of Congruence before a University Course on Geometry

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We present a qualitative study on student teachers' knowledge on the geometric concept "congruence". The study is part of a design research project in which we develop and study a 6^th^-semester geometry course for upper secondary student teachers with a particular focus on profession orientation. Based on theoretical analysis of how congruence is introduced and used in textbooks, we use qualitative content analysis to evaluate student teachers' answers on an ePortfolio-task where they have to complete definitional sentences for congruence of different types of plane figures.

Keywords: Teaching and learning of specific topics in university mathematics; Transition to, across and from university mathematics; Student Teachers, Geometry, Congruence, Double Discontinuity.

INTRODUCTION

As an innovation to overcome the *second discontinuity* in teacher education (cf. Klein 1967, p. 1), we developed, implemented, and researched a geometry capstone course for upper secondary student teachers (Hoffmann & Biehler, 2020; Hoffmann, 2022). Our course concept's two essential design elements were replacing two semester weeks with so-called "interface weeks" (dealing with the topics congruence and symmetry) and implementing a semester-accompanying so-called "interface ePortfolio". Both design decisions intend to support a stronger and more explicit professional orientation: Explicit connections between the course content and the intended profession (teaching mathematics) are addressed in the interface weeks. The ePortfolio-activities allow students to practice typical mathematics-related professional job tasks (e.g., judge textbook materials or analyze and respond to student contributions, see Ball and Bass (2002, p. 11), Prediger (2013, p. 156)). In this way, situation-specific skills focusing on mathematical dispositions are to be promoted. Hence, the learning activities contribute to acquiring professional competence in the continuum model from Blömeke, Gustafsson, and Shavelson (2015).

We develop and study our interface activities within a design research approach (van den Akker et al., 2006), adapting the particular methodology of Prediger et al. (2012). For our project, this means following the cycle (Hoffmann & Biehler, 2020, p. 341): Specifying and structuring the interface topic (Step 1), (re)designing interface learning activities (Step 2), using, and studying interface activities (Step 3) and developing and refining (local) theories (Step 4). Specifying and structuring the interface topic (step 1) includes, in particular, analyzing the learner's perspective on the topic. Initially, this can only be formulated based on existing literature and theoretical considerations. We continuously refine the learner's perspective based on empirical data during the design research project. For this purpose, we use special ePortfolio activities. In this paper, we present results obtained on the interface topic of *congruence*. These provide interesting insights into the learning requirements of student teachers on this topic, even independently of our specific project, and are thus valuable both for research and university teaching practice. The study presented is part of the first author's PhD-project (Hoffmann, 2022, p. 212) and is embedded there in a larger theoretical framework. In the following, we focus on the presentation of relevant results.

THEORETICAL BACKGROUND TO THE CONCEPT OF CONGRUENCE

Before describing the research design, we will give an overview of the mathematical background knowledge of the concept of *congruence* and summarize our literature- and textbook review on congruence as a topic in secondary school.

Congruence from a Mathematical Perspective

From a mathematical perspective, there are several ways to formalize *congruence*. The decisive factors here are, first, the role isometries play in the theory building and, second, the generality of the congruence concept concerning the figures (subsets of the geometric space used) that can be handled with it. In our course, we build up the plain geometry on the definition of metric spaces. Thus the concept of *isometries* is naturally available as an essential tool for definitions and proofs. In this approach, congruence can be defined for any two figures F, G via the existence of an isometry φ with the property $\varphi(F) = G$. Instead of using a general concept of isometry (in the sense of an arbitrary mapping that leaves distances invariant), one can also use special isometries (without necessarily naming them as such) like just reflections or reflections, rotations, and translations. The *Three Reflections Theorem* provides the equivalence of the three approaches. Alternatively, congruence can also be defined entirely without the use of isometries. For example, Hilbert (1902) explains with axioms what congruence means for line segments and angles and then postulates the congruence theorem SAS. In a final step, the concept is extended to any finite set of points and thereby implicitly extended to any infinite set of points uniquely specified by a finite set of points. However, congruence for more complicated figures cannot be described this way (in contrast to the approach described first).

For teacher professionalization, it is necessary that important mathematical concepts are carefully embedded axiomatically but also considered detached from a specific axiomatic structure. In this sense, we refer to essential characteristics of mathematical concepts found through inductive subject-specific-didactical analyses as *interface aspects*. For *congruence*, we have worked out the following four interface aspects (refined compared to Hoffmann and Biehler (2020, p. 343)):

1. *The aspect of quantities with identical sizes:* Congruent figures match in several geometric quantities. This emphasizes the static-comparative character of *congruence*.

2. *The aspect of mapping*: This aspect describes a dynamic perspective on *congruence*: For every two congruent figures, a mapping (bijective isometry) exists, which transfers the figure into the other.
3. *The aspect of relation:* Congruence is an *equivalence relation* on the power set of the geometric space used. The statement delivers characteristics connected intuitively with the concept of congruence as a kind of "geometric equal sign". Furthermore, the congruence relation divides the plane figures into equivalence classes, so-called congruence classes.
4. *The aspect of classification:* This aspect brings together the previous aspects. The *aspect of relation* provides a disjunctive division of all figures into congruence classes. All figures of a congruence class correspond in different geometric sizes (the *aspect of quantities with identical sizes*) and can be transformed into each other in pairs by isometries (the *aspect of mapping*).

Congruence in German Mathematics School Classes

Congruence is a central concept in German lower secondary geometry teaching (Holland, 2007, p. 65). It appears mainly in the form of the congruence theorems for triangles. Those are used on the one hand as a theoretical background for construction problems (e.g., because the theorem SSS holds, all triangles constructed from three lengths are congruent) and, on the other hand, as a method for geometric reasoning (Weigand et al., 2014, p. 202). The main question underlying the construction of triangles is when a triangle can be uniquely (except for congruence) constructed from a subset of size specifications. Therefore, it is a question of which subset of size specifications a congruence class of triangles is already uniquely determined. Thus congruence is used in the sense of the *aspect of classification* described above. Geometric reasoning with congruence is about identifying congruent figures (usually triangles) and then taking advantage of the fact that the equality of corresponding quantities follows from congruence (*aspect of quantities with identical sizes*). In addition, the *aspect of classification* also plays an important role here, as it allows to deduce congruence of partial figures of the proof figure based on incomplete information.

In an analysis of several current relevant textbooks, we were able to determine that, in addition to the emphasis above on congruence theorems and triangles, there was always a preformal-illustrative definition of a general congruence concept. However, this did not play a role in the rest of the chapter in the sense that neither the congruence theorems were justified regarding the definition, nor were there tasks in which one had to work with the definition (details of this textbook analysis: Hoffmann, 2022, p. 170).

Overall, it can be stated that congruence in Germany is a topic of lower secondary geometry that usually does not play any further role in mathematics teaching at the upper secondary level. Also, at university, congruence is usually not a topic of the standard courses in teacher education. Thus, the prerequisites of student teachers result from the skills and abilities that are still available from geometry lessons they attended in lower secondary school. We have made the experience that this includes a conceptual idea of *congruence* and a rough remembrance of the congruence theorems SSS, SAS, and ASA in connection with construction tasks. Knowledge of a precise definition of congruence cannot be assumed.

RESEARCH QUESTION AND STUDY DESIGN

We present a qualitative study that contributes to answering the following research question, which we already outlined in the introduction: *What knowledge do German student teachers have on the geometric concept of congruence before taking a university course on geometry?*

The data basis are pseudonymized responses to an ePortfolio-activity that the students completed directly at the beginning of the semester. The students' knowledge about the concept of congruence was mainly acquired in the context of their mathematics classes in secondary school. Some students may have "refreshed" their knowledge in the meantime through internships or private tutoring, but congruence was not a compulsory component of other courses in their studies. Thus, when dealing with congruence in the course (both as mathematical content and as an interface topic), it cannot be assumed that students have sound prior knowledge, let alone common learning prerequisites. This is exactly where the following ePortfolio activity comes in:

> Please write down how you would complete the following definitions. Please do not look them up in a book or online, and please do not exchange ideas with your fellow students but answer as if you were asked these questions in a conversation. It is not necessarily about a "formal definition." Just write what is on your mind now: (1) Two triangles are congruent if …, (2) Two quadrilaterals are congruent if …, (3) Two circles are congruent if …, (4) Any two figures of the plane are congruent if ….

With the aim of better understanding what concepts and misconceptions students have at the beginning of the course, they are asked to complete congruence definitions for different geometric figures: The first subtask is about triangles which are the central objects for congruence considerations in mathematics teaching. Next, congruence is to be defined for quadrilaterals. These are much more complex from a congruence perspective (e.g., Laudano & Vincenzi, 2017). The third subtask is on defining congruence for circles, which is a step away from considering n-gons. Finally, the students should give a general congruence definition for arbitrary figures in the plane and thus think about a concept of congruence that is detached from the unique geometric properties of individual object classes.

The task is designed to make it as easy as possible for the students to complete. On the one hand, this shows up in that the task is not to write down a perfect, "formal definition" but to answer as spontaneously as possible. On the other hand, sentence starters are already given to encourage the students to answer based on their previous knowledge and not to research in books or online. In this way, the data collected should provide a sample as realistic as possible of the students' concepts of congruence.

However, this procedure has the disadvantage that the students are not explicitly asked to reflect on their answers' precision and mathematical content. At the end of this paper, we will discuss how the methodological design can be optimized for this purpose in subsequent studies. In addition to the usefulness for research, the task also supports students' professionalization as it provides a starting point for them to reflect on their learning progress.

We used the activity in the winter semester of 2019/2020 (cycle 2) and the summer semester of 2020 (cycle 3) after the heterogeneity of prior knowledge of the concept of congruence became very apparent in the practical experiences in the summer semester of 2019 course (cycle 1). A total of 44 students' texts are available for analysis (cycle 2: n = 12, cycle 3: n = 32). We analyzed them using the method of structuring qualitative content analysis (QCA) (Kuckartz, 2018, p. 97) to describe typical concepts and (mis)conceptions of the concept of congruence.

SELECTED RESULTS OF THE QUALITATIVE CONTENT ANALYSIS

To develop an initial category system, we did an a-priori-analysis based on the summarized theoretical background described above to develop hypotheses on the student's possible answers. As it is not possible to present the QCA in all its details within this paper, we have decided on the following structure: In the subsequent subsections, sorted by the subtasks of the ePortfolio activity, we first present a summary of the a-priori analysis and then selected results of the QCA. The fully documented content analysis (including the complete system of categories) can be found at Hoffmann (2022, p. 212).

Congruence of Triangles

As mentioned, triangles are the central class of objects for congruence considerations in school mathematics. In this context, the congruence theorems have much greater importance than working with the actual definition of congruence. For this reason, we assume that most students define the congruence of triangles via congruence theorems and, conversely, almost no one chooses a definition using isometries (e.g., via "rotate, reflect, translate" or via "cover one triangle exactly with the other"). In fact, from a purely mathematical perspective, there is nothing wrong with using one of the congruence theorems as a definition for the congruence of triangles since all congruence theorems are equivalence statements. This means: If one uses one of the congruence theorems as a definition, the others can be concluded from it. As mentioned above, Hilbert also argues in this way. In the context of this study, however, it is essential to note that it does not automatically follow from the use of a congruence theorem when completing the sentence in this subtask that the person is aware that all other congruence theorems can be deduced from the chosen one. Mainly when students write down several congruence theorems simultaneously, this tends to indicate that they do not have a concept of a concise definition for triangular congruence. In addition, it can also be expected that some students will complete the definition mathematically incorrectly.

The analysis of the student responses confirmed these hypotheses. As expected, most students used one or more of the congruence theorems to complete the sentence (SSS, 26 codings; SAS, 22 codings; ASA, 17 codings; SsA, 10 codings). Most students who mention a congruence theorem refer to several congruence theorems (24 codings). Six students use exactly one congruence theorem. Furthermore, five students correctly define the congruence of triangles by the correspondence of all side lengths and all angles. Eight Students define the congruence of triangles via "exact covering" and/or a colloquial formulation referring to explicit geometric mappings (e.g., "rotate, reflect, translate"). 15 students give incorrect statements about the congruence of triangles. It becomes apparent that many students know the congruence theorems in principle but not their exact prerequisites (e.g., the position of the given sizes in relation to each other. This leads to the formulation of false congruence theorems for triangles. The following quotes from Romy and Jason are examples of the two most frequent mistake patterns in this category. Jason mentions the incorrect congruence theorem AAA. The problem with Romy's completion is more subtle: "Two triangles are congruent if […] one side and two angles are equal." Romy obviously aims at the congruence theorem ASA but does not consider that the two angles and the side must each be in the same position to each other; without this precondition, the statement is incorrect.

In summary, it can be stated that a reference to the congruence theorems characterizes a large part of the students' answers. Many students make correct statements about the congruence of triangles, but often no distinction is made between a congruence theorem and the definition of congruence for triangles; in some cases, this cannot be decided with the available data.

Congruence of Quadrilaterals

The congruence of quadrilaterals is a topic that is treated at most as an excursus in school mathematics, but in no case systematically. Therefore, it is not to be assumed that the students have substantial prior knowledge here. There are at least two plausible strategies for completing the sentence: On the one hand, students could try to generalize a congruence theorem for triangles; on the other hand, students could try to use a general, quadrilateral-unspecific congruence definition. Due to the dominance of the congruence theorems in mathematics lessons, we assume that the first approach will dominate. Because finding congruence theorems for quadrilaterals is not trivial, correct and incorrect formulations are expected in the congruence statements for quadrilaterals generalized from triangle congruence theorems. In addition, one has to consider the role of convexity as an essential precondition for the validity of certain congruence theorems. However, since non-convex quadrilaterals do not play a designated role in mathematics teaching, we do not expect students to consider this case when working on the task. Accordingly, we evaluate the correctness of the statements made assuming that it is only about convex quadrilaterals.

Based on the analysis results, it can be stated that, as expected, convexity was not mentioned by any of the students. Overall, the picture is more heterogeneous than in the case of triangular congruence. This is consistent with the a priori analysis. The heterogeneity may be a consequence of the fact that the students cannot use prior knowledge here but have to make independent mathematical considerations. As suspected, many students make congruence statements for quadrilaterals analogous to the congruence theorems for triangles. The correct congruence theorems for quadrilaterals mentioned by the students are SASAS (7 codings), ASASA (4 codings), and SSSSD (1 coding). The most frequently coded correct category for the congruence of quadrilaterals is the equality of the sizes of all corresponding angles and sides (11 codings). This is similar to the described congruence sentences in that congruence is also defined here via the specification of matching sizes. The difference is that no attempt is made (as is usual for a congruence theorem) to describe congruence by a minimal subset of these quantities. Furthermore, six students chose a formulation that refers to isometries and would be transferable to other figures.

However, incorrect congruence statements for quadrilaterals were found most frequently (19 codings). The mistake patterns are more diverse than the statements concerning the congruence of triangles. The inadmissible generalization of the congruence theorem SSS to SSSS occurs several times. The other incorrect statements on the congruence of quadrilaterals cannot be summarised under a common description.

Overall, the analysis results confirm that most students have no systematic prior knowledge about the congruence of quadrilaterals and seem to make their own considerations when answering the task. In addition, many students try to generalize their knowledge about congruence theorems to triangles, which only some of the students succeed in doing.

Congruence of Circles

The question of the congruence of circles is almost certainly new to nearly all students. Assuming again that most students associate congruence with the congruence theorems for triangles, a logical transfer of the idea of specifying sizes is to define two circles as congruent if they have the same radius, diameter, or circumference, for example.

The results of the analysis support this hypothesis. Practically all students succeeded in making a congruence statement for circles. Almost exclusively, the equality of the radii was used as a criterion (40 codings). Some students (11 codings) list several properties simultaneously (e.g., radius, diameter, circumference). For these students, it is plausible to assume that defining congruence (as required in the task) is not distinguished from stating congruence theorems (which must be proved based on a definition). This problem has also been identified with triangles.

Congruence of two Arbitrary Figures

In completing the fourth sentence, depending on how congruence was treated in one's school lessons, aspects from all the approaches to the concept of congruence identified in the textbook analysis can occur. These include the idea of "exact covering," the approach of "reflect, rotate, translate," the formulation of "same shape and size," as well as a reference to geometric constructions. In addition, a further generalization of the congruence theorems in the sense of "equality of the sizes of all corresponding angles and sides" is conceivable.

The results of the analysis confirm the hypotheses in the sense that all different approaches could be identified: Congruence as "exact covering" (17 codings), congruence as mappable by "reflect, rotate, translate" (10 codings), and congruence as "same shape and size" (9 codings). The "equality of the sizes of all corresponding angles and sides" also be coded (11 codings). Apparently, the students whose explanations were coded in this category, understand by "arbitrary figures" only n-gons since this approach does not work for curvilinear bounded figures.

Some students refer to the existence of a transferring congruence mapping or isometry. This is striking from the perspective of the school textbook analyses because isometries play only a minor role in German mathematics teaching at school. Two possible explanations are that the students are attending the course for the second time or have learned the terms in the context of a non-obligatory seminar. However, not all students who recur abstractly to the existence of a certain mapping succeed in correctly specifying the necessary properties of this mapping. (3 codings). Instead of the isometry property, only affine linearity, bijectivity, or equality of aspect ratios is formulated as a requirement.

Finally, we would like to present a result of the analysis that was not included in the conception of the study but was so striking in the analysis of the students' texts that it is briefly described here. Even though the assignment did not explicitly ask for a "formal definition", the lack of language precision in the statements of many students is worth mentioning. Often, it is clear to a reader who already knows the subject what is meant, but many formulations are problematic as a satisfactory mathematical statement, even or in particular for school students, because they contain unnecessary redundancies, are ambiguous, and/or lack necessary conditions.

DISCUSSION AND OUTLOOK

The qualitative study presented in this paper aims to gain insights into student teachers' knowledge about the concept of congruence, which must be considered as prerequisites for teaching geometry in a university course. Taking this previous knowledge not into account and teaching just on a university mathematical level is likely to perpetuate what Klein described as the second discontinuity. Previous school mathematical knowledge is expected to coexist with university mathematical knowledge instead of correcting and updating it. For researching this, we analyzed student work on an ePortfolio activity. The results presented in this paper have to be interpreted considering the research methodology's limitations. First, the results are not generalizable but show a spectrum of possible student perspectives. Furthermore, the task is designed so that the students are asked to answer as spontaneously as possible and without external support. This procedure has the disadvantage that it is not possible to ascertain whether the students, at a second glance, may correct their own statements. In particular, the observed mention of different congruence statements for a figure raises the question of whether these students have clear ideas about the difference between a *definition* of a concept and *theorems* about that concept. It would make sense to try a more elaborate research design in a subsequent course cycle. One option is to send the students their own statements after a short time, e.g., via an activity in the e-learning system, with the task to look at them again, reflect on their function as a definition, and, if necessary, revise them. With a view to the professional orientation of the event, it would be helpful here to let the students develop and compare both a definition at the university level and a definition at the school level.

A key result of our study is the confirmation of the hypotheses made in the a priori analysis: The study results confirm the strong connection between the concept of congruence and the congruence theorems for triangles in students' minds. This leads to congruence statements being linked to the goal of specifying the smallest possible set of geometric quantities from which congruence can already be inferred. In terms of the interface aspects, congruence is thus mainly used in terms of the *aspect of quantities with identical sizes*, while the *aspect of mapping* hardly plays a role in the students' minds. The concept of congruence, characterized by the *aspect of quantities of identical size*, can also be seen in the students' formulations of quadrilaterals and circles and, to some extent, even in arbitrary figures. More general prior knowledge beyond the congruence theorems is mainly preformal and not very systematic.

The results on the student formulations of the congruence of quadrilaterals were particularly interesting. Here, significant heterogeneity in the students' prerequisites on congruence became apparent in the fact that some students gave very differentiated correct congruence theorems for quadrilaterals, while others simply transferred the general informal definition of figure congruence (roughly: two quadrilaterals are congruent if one of them exactly covers the other) and still others produced incorrect congruence theorems, which, however, provide rich discussion opportunities for a deeper study of quadrilateral congruence. That is why quadrilateral congruence seems particularly suitable for reflecting on previous school experiences of congruence and for initiating a productive discussion of the topic. A corresponding interface activity was used later in the lecture (Hoffmann & Biehler, 2020, p. 344).

For the design research on the interface topic *congruence*, which we will report in a different paper, this study provides results on several levels: First of all, the learner perspective could be sharpened to the extent that the theoretically founded hypotheses were confirmed, according to which most students do not bring a precise content knowledge to the concept of congruence and, in further considerations, build on the congruence theorems for triangles and try to apply them to other contexts in terms of the *aspect of quantities with identical size*. In addition, the non-optimal or incorrect student formulations, in particular, provide rich opportunities for the construction of further learning opportunities on the concept of congruence, for example, by using them as the basis for a critical group discussion or a comparative written analysis.

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Transitions in abstract algebra throughout the Bachelor: the concept of ideal in ring theory as a gateway to mathematical structuralism

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The concept of ideal, because of its role in the construction of structuralist algebra, is an important entry point for studying the teaching of this field. In this article, we will focus our attention on the management of transitions in abstract algebra (Hausberger, 2018). To do so, we will place ourselves in the framework of the Anthropological Theory of the Didactic (Chevallard, 2000) and provide salient results of praxeological analyses of our corpus. The latter is made up of the teaching material of three French post-secondary teachers. These analyses will allow us to study continuities and ruptures in the praxeologies of the abstract algebra track throughout the Bachelor in France, but also to shed light on the way in which the professors manage the transitions in the development of structuralist praxeologies.

Keywords: teaching and learning of linear and abstract algebra; transition to, across and from university mathematics; structuralist praxeologies; anthropological theory of the didactic; ideal in ring theory.

INTRODUCTION

The issue of transitions has received increasing attention in research on mathematics education, resulting in a recent topical ICME survey (Gueudet, diSessa, Kwon and Verschaffel, 2016). The state of the art of the literature reviewed in the survey underlines different facets (cognitive, epistemological, socio-cultural, institutional,...) of the transitions. The latter are investigated in terms of continuities, discontinuities, ruptures that occur at different transition points (e.g. from school to university) and research-based devices are proposed to accommodate them. Even more recently, Hochmuth, Broley and Nardi (2021) report on the works on this theme carried out within INDRUM.

This paper focuses on transitions across university mathematics courses. Such transitions have been mainly investigated in the context of the analysis path (loc. cit., p. 203-204). In particular, several studies adopt the institutional perspective offered by the Anthropological Theory of the Didactic (ATD; Chevallard and Bosch, 2020) and refer to a model of the calculus to analysis transition introduced by Winsløw (2006). In their concluding section, Hochmuth et al. (2021) raise the question of the situation for the other fields, and, in reference to Winsløw, "what does the observed jump from the first to the third stage of the model mean in a longer-term perspective, e.g. taking into account what students learn in more advanced mathematics studies?" (p. 210).

Our goal is to investigate these questions in the context of the abstract algebra path, with an epistemological and institutional lense. It is based on the perspective opened up by Hausberger (2018): the concrete to abstract transition identified by Winsløw in analysis is generalized in the form of the wider perspective of the teaching and learning of mathematical structuralism at large - in ATD terms, the development of structuralist praxeologies. A new three stage model has been proposed by Hausberger and applied to abstract algebra in a small-scale pioneering study. A larger-scale study, centered on second and third year post-secondary teaching practices in France and Switzerland, is carried out by Candy (2020a) in her PhD project with a focus on the concept of ideal in ring theory. This choice is motivated by the role played by the concept of ideal (Corry, 2004, p. 15): its central importance for the theory of abstract rings and, even more, for the "rise of structures" due to its strong interconnections with other algebraic concepts (fields, modules, groups, etc.). A model for such praxeologies taught at the second year of post-secondary studies in France has been presented at INDRUM2020 (Candy, 2020b).

This paper reports on the results of the PhD project that connect to the issue of transitions, in France. It addresses the following research questions: What continuities and ruptures can be observed in the praxeologies of abstract algebra that involve the concept of ideal, as they are taught throughout the Bachelor in France? How are transitions in the development of structuralist praxeologies handled by abstract algebra teachers? Both questions are related since our methodology is based on the analysis of teaching material provided by selected teachers, under the light of Hausberger's model. It also refers implicitly to didactic transposition processes (Chevallard, 2020), but the discussion of conditions and constraints that explain the observed states of equilibrium within institutions are out of the scope of the paper.

We begin by presenting our theoretical framework and the model, and then outline the methodology for analyzing the data. We illustrate the methodology through its application to selected excerpts from exercise sheets. Then, we discuss the results obtained in relation to the research questions, before concluding with the highlights of the study and prospects for further development.

THEORETICAL FRAMEWORK

Hochmuth et al (2021) highlight the following main features of ATD that justify its frequent use in research on transitions: the consideration of knowledge as living within *institutions*, the institutionalization of knowledge seen as the result of complex processes of *didactic transposition* subject to a set of conditions and constraints at various levels, and finally the central 4T-model of *praxeologies* (task, technique, technology, theory) that allows researchers to build reference models of the knowledge to be taught for application to teaching-learning phenomena. We direct the reader unfamiliar with ATD to the mathematics education encyclopedia (Chevallard, 2020) for an introduction to these notions and will focus the rest of our discourse on the additional tools, specific to the point of view of mathematical structuralism, that have been developed.

Structuralist praxeologies and their levels. The starting point is the consideration of mathematical structuralism as a methodology, which consists of reasoning in terms of classes of objects, relations between these classes and stability properties for operations on structures (Hausberger, 2018). The general view of structures thus allows particular properties of objects to be demonstrated by making them appear as consequences of more general facts (theorems about structures). Dually, generealizations are put to the test of objects, hence a *dialectical relationship between objects and structures*. In praxeological terms, the simplification produced lies in the passage from a praxeology $P=[T/?/?/\Theta_{\text{particular}}]$ where it is unclear which technique to apply, to a *structuralist praxeology* $P_s=[T^g/\tau/\theta/\Theta_{\text{structure}}]$ where, modulo generalization of the type of task (T^g), the theory of a given type of structure guides the mathematician in solving the problem. Furthermore, Hausberger (2018) distinguishes several levels of structuralist praxeologies: at *level 1*, structures act as a vocabulary and appear mainly through definitions (e.g., the type of tasks "prove that a ring A is a principal ideal domain (PID)" is solved by showing, by hand, that the definition is satisfied, i.e. that any ideal is monogeneous); at *level 2*, the technique used mobilizes general abstract results about structures (on our example, one shows the existence of an Euclidean algorithm, which invokes in the logos of the praxeology the structuralist theorem that any Euclidean ring is a PID).

Transitions in the development of structuralist praxeologies. Following Winsløw, Hausberger (2018, p. 89) proposes a three-phase model (Figure 1): while the first type of transition amounts to going from P to P_s, the second type leads to praxeologies whose entirety of praxis and logos lies in the abstract. To unfold our example, the student then encounters tasks like "show that a Noetherian integral domain such that any maximal ideal is principal is a PID".

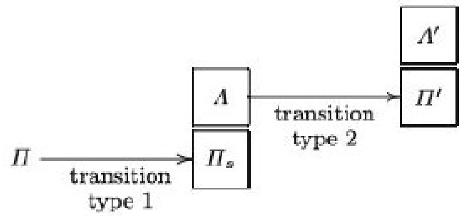


Figure 1: a model for transitions in the development of structuralist praxeologies

Contextualization and decontextualization of structuralist praxeologies in relation to the dialectic of objects and structures. In the example given, the ring A plays the role of a *didactic variable* of the type of tasks: the structuralist praxeology is thus *contextualized to domains of mathematical objects*, whose variation is crucial to lead - in fine - to a *decontextualization* (the ring A is defined *abstractly*). We will be particularly attentive to the choices made by university teachers in relation to these didactic variables which are essential to operate the objects-structures dialectic.

METHODOLOGY

In order to shed light on didactic choices that concern transitions in abstract algebra throughout the Bachelor, we conducted a case study of 9 university teachers considered representative of 5 teaching levels, in France and Switzerland (Candy, 2020a). In this article, we will rely on data from three teachers: MP1, EC2 and EC4. MP1 teaches mathematics in the second year of the Classes Préparatoires aux Grandes Ecoles Mathématiques-Physiques (CPGE-MP; these are classes reserved for the best students who are destined to enter the French Grandes Ecoles), EC2 teaches in the second year of the Bachelor's degree in a 7.5 ECTS course called “Linear Algebra” and EC4 teaches in the third year of the Bachelor's degree in a 5 ECTS course called “Elements of Ring and Field Theory”. EC4 teaches in the same university as EC2; moreover, some of the students of the third year of the Bachelor's degree come from a CPGE-MP. Thus, in this article, we can study the transitions through two possible curricula experienced by those third year Bachelor students.

In our study, we began by conducting an ecological analysis of the official sylabi in order to bring to light the places where the concept of ideal lives. Then, our praxeological analysis of the course documents (lecture notes and tutorial sheets) consisted in highlighting the praxeologies that mobilize the concept of the ideal within the exercise sheets of the corpus. When the exercises were not corrected, we used the correction of exercises of the same type of tasks present in the institution. We took care to link the exercises to the contents of the lectures, which allow to identify the global organization of the praxeologies (their unification by common technologies or theories, within themes or sectors of study) and to provide certain technological and theoretical elements that remain partially implicit in tutorials.

Finally, the structuralist level of the praxeologies has been carefully noted, as well as the choice of the didactic variables of contextualization of the structuralist praxeologies. The aim is to analyze the continuity and rupture that can be observed in relation to the two types of transitions described in the model, under the hypothesis that the type I (epistemological) transition would be situated at the level of the (institutional) transition between the second and third year of the Bachelor's degree, while the type II transition would be linked to that of the Bachelor's to the Master's degree.

PRAXEOLOGICAL ANALYSES

In this section, we illustrate our analytical tools on salient excerpts from the corpus while reporting on the main findings of our analyses. The discussion of the results in order to answer the research questions will be the subject of the next section.

CPGE-MP: the MP1 corpus

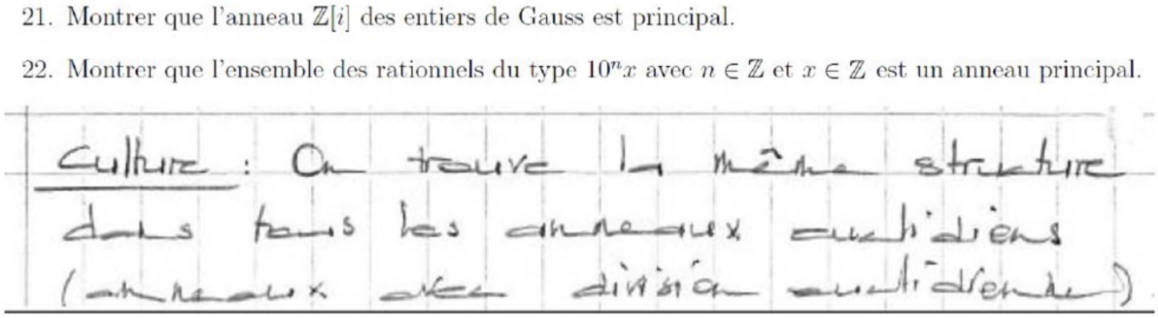
The analysis of the official program of CPGE-MP[1] allows to identify three main habitats of the concept of ideal. The latter is introduced by its general definition in the sector “common algebraic structures”, at the level of the theme “ideals of a commutative ring” where it is linked to the notion of ring homomorphism (as kernel) and to the notion of divisibility (inclusion of ideals), then illustrated in the case of the ring **Z**. It is then taken up again at the level of the theme “rings of one-dimensional polynomials” of the same sector, through the item “ideals of K[X]”. One can note that the program does not mention the principality property (of an ideal, of a ring) and that it does not underline the analogy between the arithmetic of **Z** and that of K[X] (which follows from the principality). Nevertheless, the implicit organization of the contents is based on this analogy. Finally, the concept of ideal is mentioned in the

1 https://cache.media.education.gouv.fr/file/special_1_MEN_ESR/42/4/MP-mathematiques_287424.pdf

theme “polynomials of an endomorphism, of a square matrix” of the sector “reduction of endomorphisms and square matrices” where the properties of ideals previously studied allow to justify the existence of the minimal polynomial.

The study of the objects at stake in the exercise sheets shows a diversity of contextualizations, although limited to numbers and polynomials. For the principal ideal domains (PID), the classical examples **Z** and K[X], quoted in the official program, are mainly worked on through their arithmetic (definition of gcd and lcm in terms of ideals) and the consequences in linear algebra of the principality of K[X]. The ring **Z**[X] is studied as a non-example of a PID. Finally, MP1 has chosen to introduce **Z**[i] (the Gaussian ring of integers, whose historical importance in the development of abstract algebra is well known) and the set **D** of decimal numbers to work on the principality of Euclidean rings on less classical examples.

The analysis of the tasks shows that the students' work is mainly situated at level 1 of the structuralist praxeologies. Thus the type of task T_1 (to demonstrate that a subset I of a ring A is an ideal), present in 2 occurrences, gives rise to a praxeology whose technology is based on the definition of an ideal and the type T_2 (to demonstrate that a given Euclidean ring A is a PID, 4 occurrences), in spite of the genericity of the technique, proceeds by hand from the definition of principality. It is at the level of a meta-discourse that the teacher underlines the analogy between the two contexts and the generality of the method, without going so far as to quote a structuralist theorem (figure 2). Indeed, the theorem in question is not on the syllabus; its status is that of a cultural element and the definition of a Euclidean ring is not formalized. Only one occurrence of T_1 leads to a level 2 structuralist praxeology where the technique uses the structuralist theorem characterizing ideals as kernels of ring homomorphisms.


21. Montrer que l'anneau $\mathbb{Z}[i]$ des entiers de Gauss est principal.

22. Montrer que l'ensemble des rationnels du type $10^n x$ avec $n \in \mathbb{Z}$ et $x \in \mathbb{Z}$ est un anneau principal.

Culture : On trouve la même structure dans tous les anneaux euclidiens (anneaux avec division euclidienne)

Figure 2: example of a structuralist theorem that remains implicit

The abstract tasks, 3 in total, are situated within the same exercise devoted to the notion of radical $\sqrt{I}$ of an ideal I of a ring A: it is proved to be an ideal (T_1) then appears the type of task T_7 (to prove properties of operations on ideals). This last type, introduced during the lectures on the gcd and lcm (addition and intersection of ideals), is carried out here in an abstract context, about a new operation whose behavior with respect to the two previous ones is studied (e.g. $\sqrt{I \cap J}=\sqrt{I} \cap \sqrt{J}$), to finally be contextualized to **Z** through the task of determining the radical of an ideal of **Z**. These tasks show a dialectic between contextualization and decontextualization, since the general formula may be used to reduce to computing the radical of prime ideals of **Z**. It is a local implementation of the dialectic between objects and structures in the sense of Hausberger (2018), but the notion of radical remains weakly motivated.

Second year of the Bachelor: the EC2 corpus

We notice that the concept of ideal does not appear explicitly in the official syllabus of EC2. However, the same niches as in the case of MPSI-MP are likely to be invested, since the arithmetic of polynomials and the reduction of endomorphisms are part of the study program. Although he starts his course with a chapter "small panorama of algebraic structures" (like MP1), EC2 chooses to introduce the ideal concept at the level of the theme "arithmetic of K[X]" of the sector "the algebra K[X]", which constitutes its main habitat, with as niche the principality of K[X] and the reformulation of the gcd (defined from the divisibility relation) in terms of ideals. Not surprisingly, these results are subsequently applied to the theme "polynomials of endomorphisms" of the sector "reduction of endomorphisms".

Of the 7 tasks on ideals contained in the tutorial sheets, only two are contextualized (to K[X], one to prove principality and the other to prove the existence of the lcm of two elements a and b of K[X], via the introduction of a generator of the ideal $(a)\cap(b)$). They appear as isolated tasks aimed at proving theoretical elements stated in the course. We identify a single proper praxeology in this corpus, generated by the type of task denoted previously T_1. Its 4 occurrences are all decontextualized: it consists in proving that the sum and intersection of ideals is still an ideal, starting with the case of principal ideals. This shows a deficit of the objects-structures dialectic. A last abstract task is given in connection with the reduction of endomorphisms: it is to prove that the kernel of a homomorphism of algebras is an ideal. It shall be noted that the definition of a ring homomorphism (and its intimate link with the notion of ideal) are not mentioned in the course.

Third year of the Bachelor: the EC4 corpus

The syllabus of EC4 is quite detailed: it includes both a large panel of concepts (ideal, ideal generated by a subset, quotient rings, prime/maximal ideal, PID/euclidean ring/unique factorization domain) and the study of their properties in a structuralist perspective (behavior of ideals under homomorphisms, isomorphism theorems), from which structuralist theorems result (e.g. characterization of prime and maximal ideals by quotient properties), but it also mentions specific examples that must be treated (**Z**, K[X], **Z**[X], **Z**[i]). We thus find the paradigmatic examples used by MP1 and the perspective of unification between the contexts of numbers and polynomials, which gives meaning to the abstract theorems. Unlike MP1, the learning goals are organized around the structures, it is no longer the numbers or the polynomials that are put to the fore.

EC4 reintroduces in his course all the basic notions related to ideals even if EC2 had already introduced some of them. The structure of the course allows to link the 12 exercises dealing with ideals to the 4 following sectors: "ideals and quotients", "polynomials and ideals", "prime and maximal ideals", "operations on ideals". Considering their complexity (we have identified 41 different tasks), we can only sketch the corresponding praxeologies and refer the reader to the thesis manuscript (Candy, 2020a, appendix A5, p. 400-410) for a detailed description.

Of these 41 tasks, 18 use a definition and 23 use a theorem about structures. This highlights EC4's didactic intention to reach a structuralist level 2. More precisely, the praxeologies mobilizing prime or maximal ideals (for example, the praxeology generated by T_{18} : show that an ideal *I* of a ring *A* is prime, figure 3) mostly use as technology the isomorphism theorems or the Chinese theorem; they thus work at a level 2. On the other hand, those concerning the principality of ideals (e.g. T_{13} : show that an ideal *I* of a ring *A* is principal) have as technology the definition of a principal ideal and are thus mostly done "by hand". Indeed, they are mostly situated in the sector "polynomials and ideals" and are contextualized to *K*[X]. We do not note any subsequent development of the praxeology at the structuralist level 2, when the principality of Euclidean rings is known.

7. Idéaux de $\mathbb{Z}[X]$. On travaille dans l'anneau $\mathbb{Z}[X]$.

1) Montrer que l'idéal (2) est premier mais pas maximal.
2) Montrer que l'idéal (X) est premier mais pas maximal.
3) Montrer que l'idéal $(2, X)$ n'est pas principal. Montrer que c'est un idéal maximal.
4) Montrer que l'idéal (X^2+1) est premier mais pas maximal. Reconnaissez-vous le quotient $\mathbb{Z}[X]/(X^2+1)$?

Figure 3: illustration of T18 in EC4's corpus

Of these 41 tasks, 10 are abstract tasks. Within the corpus, we find occurrences of task types that are first contextualized and then decontextualized in order to demonstrate a general result (but not a generalization of the former contextualized occurrences). This is the case, for example, of T_{18} , contextualized twice in **Z**[X] in exercise 7 (figure 3), before being posed to the case of the inverse image of any prime ideal by a ring homomorphism, and then to the case of an ideal *P* defined abstractly by a condition that may be interpreted as a rewriting, in the set-theoretic terms of ideals, of the definition of a prime ideal (figure 4).

8. Idéaux premiers. Soit A un anneau et P un idéal de A avec $P \neq A$. Montrer que P est premier si et seulement si pour tous I, J idéaux de A on a : $IJ \subset P \Rightarrow I \subset P$ ou $J \subset P$.

Figure 4: one occurrence of T_{18}, decontextualized

The corresponding praxeology is of structuralist level 1, unlike the one applied to **Z**[X]; we can regret that the usefulness of these results is not highlighted by new contextualizations, in the spirit of the dialectic between objects and structures. The other abstract tasks are related to the discussion of the consequences of the presence of invertible elements in an ideal, to the determination of the ideals of a Cartesian product of rings and its consequences for a quotient of such a product. These are therefore theoretical results which are intended to feed contextualized structuralist praxeologies of level 2. For example, the question of the determination of the ideals of $\mathbf{Z}^2$ is posed in application of the principality of **Z** and the results on the Cartesian product. A praxeology generated by T_{13} is thus obtained, at structuralist level 2.

DISCUSSION

	Summary of main characteristics	Type 1 transition	Type 2 transition
MP1	Praxeologies mainly contextualized and limited to the structuralist level 1; a relative diversity of contexts (**Z**, *K*[X], **Z**[i], **D**)	Spotted preparation (via meta discourse)	Absent
EC2	A single type of tasks dealt with at structuralist level 1 and in an abstract context; the objects-structures dialectic is extremely weak	Absent	Absent
EC4	Work at structuralist levels 1 and 2; decontextualized instances of praxeologies are introduced to establish structuralist properties of ideals and punctually serve to develop contextualized praxeologies of structuralist level 2	At the center of the course	Spotted preparation (implicit)

Table 1: summary of the main results

As suggested by the analysis of the syllabi, the type 1 transition does not appear as a learning goal in the CPGE-MP and second year Bachelor institutions under study. Accordingly, when tasks are contextualized to object domains, mainly *K*[X] and **Z,** the technique consists in applying the definitions of concepts without relating the properties at stake to general results. Thus, the structuralist praxeologies involved are all level 1 and the objects-structures dialectics remains largely invisible in these institutions.

Nevertheless, MP1 chose to introduce a relative diversity of examples of PIDs which are all Euclidean, and he uses meta discourse to allude to the underlying general principle (a structuralist theorem). This didactic gesture may be considered an intention to facilitate the type 1 transition. At university, EC2 introduces structures as a “vocabulary” and assigns abstract tasks on basic formal properties of ideals. The type 1 transition is therefore envisaged from a top-down perspective: although the course content is organized around domains of objects (polynomials, matrices, endomorphisms), concepts are introduced beforehand and taken as a given. This strategy may be questioned since it may be argued that the resulting level 1 structuralist praxeologies will tend to be weakly motivated.

It is from the third year of the Bachelor's degree onward that the type 1 transition appears as a real objective. We have seen, in the EC4 course, that the syllabus is organized around structures. Moreover, EC4 deploys praxeologies, in numerous contexts, which mobilize the concept of ideal at a structuralist level 2, notably around prime/maximal ideals and the isomorphism theorems or the Chinese theorem. The objects-structures dialectic is at play and assigned abstract tasks punctually serve to subsequently develop contextualized level 2 structuralist praxeologies.

Type 2 transition does not appear as a learning objective in the institutions under study. Abstract tasks are rare in CPGE-MP and restricted to the first basic properties of ideals in the second year of Bachelor. At the third year, EC4 assigns tasks to explore structuralist properties of ideals (e.g. behavior under inverse image, Cartesian product). Nevertheless, the main application is the enrichment of contextualized level 2 structuralist praxeologies. A body of abstract knowledge in which ideals play a major role (e.g. Elimination Theory, Algebraic Geometry) is not in the horizon. However, the scope of exercise 8 (figure 4) may be connected to Algebraic Geometry. The teacher probably had this connection in mind, but it remains invisible to the students. Thus, the type 2 transition is left to the Master's degree.

Our analyses are limited to the case of three professors chosen as representatives of their respective institutions. However, Candy (2020a) provides results on a larger corpus that support these analyses. In the context of this article, we can provide some initial answers to our research questions.

To answer the question of continuities and ruptures which can be observed in the abstract algebra track around the concept of ideal through the Bachelor in France, it seems important to us to recall that the students who take the course of EC4 could have first followed the course of EC2 or that of MP1. However, the treatment between EC2 and MP1 appears different. In both cases, the praxeologies involved are of structuralist level 1. But, if EC2 chooses to work on praxeologies in a decontextualized way, MP1 chooses to work on contextualized praxeologies and to accompany the type 1 transition by meta-discourse. Thus, students from the MP1 course could be better prepared for the upcoming transition since the structuralist perspective is pointed out as an horizon.

Structuralist praxeologies at the third year of Bachelor are either of level 1 or 2. However, discontinuities in the type 1 transition can be noted since structuralist praxeologies are most of the time elaborated from theoretical elements provided a priori, in a top-down perspective. The analogy between the arithmetic of numbers and polynomials, carefully developed by MP1 in a bottom-up perspective, remains a missed opportunity to develop a structuralist praxeology in the EC4 course, since all students do not share such a background. Finally, the transition of type 2 is not worked out; it would be necessary to analyze a corpus of teaching material at the Master's degree to measure the epistemological gap in praxeological terms.

As for the management of transitions by teachers, our study tends to show, for both type 1 and type 2 transition, that it is the objects-structures dialectics which is central in the management of transitions by teachers. For the type 1 transition, this can be done through meta-discourse which deals with the identification of a technique present in the structuralist level 1 praxeologies which could be generalized to a structure class in order to create a technological element of level 2 (figure 2, MP1). The type 1 transition can also be managed through the choice of variables contextualizing the tasks which favour the use of level 2 praxeologies and make level 1 praxeologies costly. This is the case, for example, of EC4 in the framework of prime and maximal ideals (figure 4). Finally, for the type 2 transition, we have seen that it can be punctually favored by a set of contextualization then contextualization as in figures 3 and 4 of the EC4 corpus.

GENERAL CONCLUSION AND PERSPECTIVES

This analysis allows us to highlight the objects-structures dialectic as a nodal point of the management of transitions in abstract algebra. The interplay of contextualizations and decontextualizations allows the passage from structuralist praxeologies of level 1 to 2 by recognition of a unifying technology on contextualized praxeologies (type 1 transition); then, decontextualization offers an opportunity to engage in abstract tasks with familiarity gained from contextualized praxeologies (type 2 transition).

The phenomena observed in this reduced corpus would benefit from being tested in the context of a larger corpus, as is the case with Candy (2020a), so as to be able to argue for the presence of dominant praxeological models within institutions. They are also to be related to the ecological study of the conditions and constraints that are exercised at the level of the different institutions, in order to shed light on the states of equilibrium reflected in these dominant models. Moreover, our results suggest that this equilibrium is likely to be unstable with respect to the management of transitions in structuralist algebra, and this instability should also be investigated.

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Design and analysis of an unusual curve sketching exercise for first year teacher students

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Abstract: This paper intends to show how certain theoretical elements and tools of the Anthropological Theory of the Didactic (ATD) can be used in the development of exercises that address specific mathematical difficulties of students at the transition from school to university and in a subsequent analysis of student solutions.

Keywords: ATD; REM; Herbartian schema; Teachers' and students' practices at university level; Transition to, across and from university mathematics.

INTRODUCTION

The switch from school mathematics to a more reflected use of mathematical practices at university is still considered a challenge for students at the transition from school to university (Liebendörfer, 2018, chap. 2.3). Teacher students, specifically, experience what is known in German mathematics education as "double discontinuity" (Winsløw, 2017). To address such transition challenges in an introductory course of mathematics education for first-year teacher students[1], we employ exercises and teaching materials which aim to examine specific aspects of the relationship between school mathematics and university mathematics. The materials are created on the basis of design principles which address professionalisation aims of teacher education (Ruge et al., 2019; Ruge et al., 2021, p. 250) and were elaborated for the creation of mathematical exercises using the notion of *praxeology* of the *ATD* (Khellaf et al., 2021).

This contribution focuses on one particular exercise that deals with the topic of *curve sketching*. Curve sketching is a standard topic in calculus courses in German schools and typically serves as application field for differentiation techniques. The exercise discussed in this contribution intends, on the one hand, to help students overcome certain mathematical difficulties connected with the topic of curve sketching (for an overview of typical difficulties see Roos, 2020, chap. 3). On the other hand, the exercise intends to bring to our students' attention specific limitations of typical school mathematical approaches to the topic. It is designed in such way that the application of standard solution strategies leads to a dead end. A solution can be found by directly applying the definitions of extremum and inflection point to the given graphs (Fig. 1). Our students typically experience difficulties in switching from solution strategies they learned in school to practices such as checking prerequisites of theorems and consulting definitions, which are more closely associated with university mathematics. When the usual criteria do not work, some students engage in a variety of 'unproductive' and

1 Course development happens as part of the BMBF-funded teacher education project *Leibniz-Prinzip* (cf. LSE, 2021)

‘unusual’ practices and many have unexpected difficulty perceiving definitions as a resource. As most students attend several mathematics lectures before encountering our exercise (e. g. Analysis I), this persistence of school approaches surprised us and incited us to investigate this didactical phenomenon.

In this paper, we intend to do two things:

1. We employ the Herbartian schema of the ATD to describe in detail the rationale behind the exercise’s design and successive modifications. This adds to previous work on exercise’s design that drew mainly on Critical Psychology and the ATD’s notion of praxeology (cf. Khellaf et al., 2021; Ruge et al., 2019).
2. We present a *reference epistemological model (REM)* that was created to represent a range of (according to the reference institution) ‘legitimate’, ‘plausible’ and ‘expected’ solutions to the exercise. This REM shall in the future serve as analytical tool for the analysis of student solutions to the exercise.

We finish by discussing potentially generalizable aspects of our investigation and observations that might be of interest to for example DBR or SoTL communities (inside and outside the field of mathematics education).

THEORETICAL BACKGROUND: ELEMENTS OF THE ANTHROPOLOGICAL THEORY OF THE DIDACTIC

The *Anthropological Theory of the Didactic (ATD)* (Chevallard, 1992, 2019) is a research programme for the study of human practices from an institutional perspective[2]. *Institution* in the sense of ATD means any form of legitimised social group. Any form of knowledge, and thus also actions in relation to this knowledge, is located in institutions and subject to institutional conditions and legitimisations. Institutional conditions do not merely represent external societal conditions but are constitutive for knowledge and the actions associated with it. In the ATD, knowledge is understood as human activity – legitimised, justified and explained within the institution – that includes not only practical aspects of “know-how” (i.e. *praxis*, task and technique) but also knowledge in the sense of “know-why” (i.e. *logos*, technology and theory). This is subsumed under the term *praxeology*.

The study of the dissemination of knowledge through institutions and among persons is at the heart of the ATD. One important tool to capture study processes is the Herbartian schema (cf. Bosch, 2019): $[S(X; Y; Q^{\heartsuit}) \rightarrow M=\{A^{\Diamond}_i, ..., Q_i, \ldots, W_i, \ldots, D_i, \ldots\}] \Rightarrow A^{\heartsuit}$. It consists of the Didactic System $S(X; Y; Q^{\heartsuit})$[3] around a question $Q^{\heartsuit}$ that is studied by X (one or more students) with the help of Y (one or more teachers) to find an answer $A^{\heartsuit}$. The study process takes place in a *didactic milieu M* consisting of

2 Due to limited space, we cannot really explain all relevant concepts of the ATD in detail and therefore have to refer the reader to the existing literature. Selecting a few works, we refer to Chevallard (2019) for an introduction to ATD and an overview. Lucas, Fonseca, Gascón and Schneider (2019) focus on the REM and the DEM as important concepts for research in ATD. For a concise introduction to the Herbartian schema we refer to Bosch (2019).
3 The $^{\heartsuit}$ is a metaphor used within the ATD to indicate that this question is “at the heart” of the study process. $A^{\heartsuit}$ is then the answer to the question $Q^{\heartsuit}$.

different types of objects: established answers $A^{\lozenge}_i$ to questions Q_i that come up in the study process and have to be deconstructed and reconstructed in order to arrive at $A^{\heartsuit}$ – this process is known as *question-answer-dialectic*; works W_i and data D_i, which help with or provide feedback on the study process. Another category for researching study processes is *media*. Media are all types of systems that issue a message or statements. This can be textbooks, other works, but also fellow students or teachers. The usefulness of a statement to arrive at $A^{\heartsuit}$ is evaluated against the milieu and fed back to X and Y. This is called the *media-milieu-dialectic*. Media-milieu- and question-answer-dialectics are analytical tools to describe the dynamics of study processes.

The institutional perspective of the ATD on study processes means that X and Y are institutional positions, $Q^{\heartsuit}$ and $A^{\heartsuit}$ as well as the milieu are part of institutionally legitimised knowledge. In each institution, e.g. school, there is a predominant way of describing and presenting the knowledge, i.e. the set of relevant questions, what answers, works and data are regarded as legitimate or adequate, in what way the knowledge is used etc. This is called the *dominant epistemological model (DEM)*. To study didactic phenomena of an institution, the ATD proposes to build a *reference epistemological model (REM)* (cf. Lucas et al., 2019). The REM can be seen as a phénoménotechnique (in the sense of Bachelard) with which didactic phenomena linked to the DEM can be produced and thus studied in the research process.

As mentioned in the introduction, we suspect that institutional differences between school mathematics and university mathematics might have played a role in the genesis of the phenomenon to be studied. Differences between the knowledge of different educational institutions and their implications for possible study processes are central concerns of the ATD. Therefore, ATD is particularly useful for us, as it makes possible a detailed description of practices typical for the involved institutions.

Give correct mathematical answers to the following questions and provide an acceptable mathematical justification.

1) How many inflection points does graph 1 have?
2) How many extrema does graph 2 have?

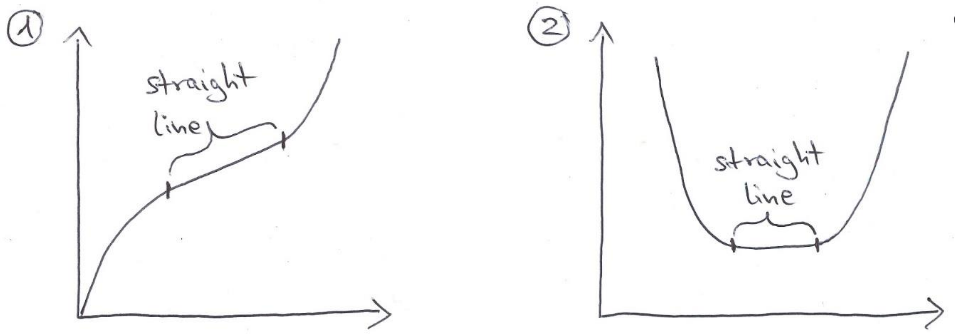

Fig. 1: Task instruction of the curve sketching exercise

DESIGN AND ANALYSIS OF THE EXERCISE

Task description[4]

Fig. 1 shows *the curve sketching exercise* given to students (cf. Ruge et al., 2021, section 14.2.2). Students are additionally given a chapter on curve sketching from a German mathematics textbook for upper secondary school (Freudigmann et al., 2012, pp. 38-67) as reference material. The chapter contains definitions, theorems, examples and exercises. Figures 2 and 3 show the definitions of *extremum* and *inflection point*.

> **Definition:** A function has a
>
> **local maximum $\boldsymbol{f(x_0)}$,** | **local minimum $\boldsymbol{f(x_0)}$,**
>
> at x_0, if there exists an interval I with $x_0 \in I$, so that for all $x \in I$:
>
> $\boldsymbol{f(x) \leq f(x_0)}$ | $\boldsymbol{f(x) \geq f(x_0)}$.
>
> In this case, the point $\left(x \middle| f(x_0)\right)$ [sic] is called
>
> **"high point"** of the graph | **"low point"** of the graph.

Fig. 2: Definition of *extremum* (Freudigmann et al., 2012, p.46, translation by authors)

> **Definition:** Let the function f be defined on an interval I, differentiable and let x_0 be an inner point of the interval.
>
> A point x_0 at which the graph of f changes from being a left-hand[5] curve to being a right-hand[4] curve or the other way around is called **inflection point** of f.
>
> The respective point $W\left(x_0 \middle| f(x_0)\right)$ is called **inflection point** of the respective graph.

Fig. 3: Definition of *inflection point* (Freudigmann et al., 2012, p.56, translation by authors)

Among the theorems are four which specify criteria for the existence of extremums and inflection points on certain differentiable functions; Fig. 4 shows one such theorem.

> **Theorem:** Let the function f be arbitrarily often differentiable on an interval I and let x_0 be an inner point of the interval.
>
> 1. If $f''(x_0) = 0$ and f'' has a sign change in the vicinity of x_0, then f has an inflection point at x_0.
> 2. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then f has an inflection point at x_0.

Fig. 4: Schoolbook theorem: criteria for inflection points (Freudigmann et al., 2012, p. 56, transl. by authors)

4 For a more detailed description of the 2020 implementation cf. https://platinum.uia.no/download/io3-teaching-units-for-student-inquiry/luh-teaching-units/ (last accessed 12.08.2022).

5 'left-hand curve' = strictly convex curve; 'right-hand curve' = strictly concave curve [the authors]

Questions 1) and 2) can be answered with the help of the definitions of extremum and inflection point that can be found in the schoolbook (Fig. 2 & Fig. 3). According to these definitions, there are no inflection points on graph 1 but infinitely many extremums on graph 2 (cf. Fig. 1).

Examples of 'peculiar' student responses to the task

In 2019, a student group presented an exercise solution and explained correctly why the schoolbook theorems do not yield useful information, but could still not answer questions 1) and 2) because they had not thought of looking into the definition as a viable course of action. In 2021 a student, who wanted to write an essay about the topic, tried to apply the strategy of approximating the functions pertaining to graphs 1 and 2 with step functions and to transfer knowledge about properties of the step functions onto the limit functions, before we directed him towards the definitions. Overall, we could observe in many students' approaches a peculiar absence of the strategy of consulting the (formal) definitions of the concepts in question, and a replacement of this strategy by a variety of 'unproductive' and 'unusual' strategies.

Description of the task's economy and ecology

In German *school mathematics*, the curve sketching exercise can be considered 'unusual' because exercises within the topic of curve sketching typically ask for the application of algorithmic procedures derived from schoolbook theorems which give criteria for the existence of extremums and inflection points. These algorithms constitute the institution's 'official answer' $A^{\lozenge}_{\text{School}}$ for virtually all schoolbook exercises which ask to find extremums or inflection points on functions, which are typically given in algebraic form. $A^{\lozenge}_{\text{School}}$ thus constitutes the *dominant epistemological model (DEM)* in the institution of school mathematics regarding the topic of curve sketching. In the curve sketching exercise, however, the standard procedures for determining extrema and inflection points fail: The schoolbook theorems are formulated as unidirectional conditional statements (cf. Fig. 4), but on the straight segments of graph sketches 1 and 2 (Fig. 1) the sufficient conditions do not hold.

In *university mathematics*, a standard technique to approach any mathematical problem is the verification of one's own knowledge of the involved definitions, which we will call $A^{\lozenge}_{\text{University}}$. In the case of the curve sketching exercise, making sensible use of the schoolbook definitions will lead to an answer to questions 1) and 2).

The curve sketching exercise was designed in such way, that the *media* provided strongly evoke the DEM of the institution of school mathematics: it is introduced by a fictional conversation between two school students and provides a schoolbook excerpt as material (W_{School}). $A^{\lozenge}_{\text{University}}$, on the other hand, becomes less associated with the task ($\rightarrow A^{\lozenge}_{\text{University}}$). One aim of evoking the DEM was to make our students experience its mathematical limitations and provoke subsequent questioning of the DEM.

The *milieu* actively provided was a page on an online learning platform that contained the media and some instructions. However, as the page is non-responsive and does not give feedback on student's activities (apart from providing the media), it is not included in the Herbartian schema. The teaching team Y offered to respond to questions at any time and would add further elements to the milieu upon request (in the form of replies). An 'invisible' but nonetheless crucial part of the milieu are *the results of an application of logical/mathematical reasoning* (D_{math}). $A^{♥}$ was not included in the milieu (e.g. in the form of a sample solution for self-check).

Question-answer-dialectic, chronogenesis and media-milieu-dialectic

Students are confronted with $Q^{♥}$ (cf. Fig.1) and many first try $A^{\Diamond}_{School}$ to solve it. A correct(!) application of *mathematical reasoning* (i. e. interaction with the milieu) should yield D_{math}, i.e. the 'feedback' that $A^{\Diamond}_{School}$ cannot answer $Q^{♥}$, which should give rise to new questions Q_i. This coincides with the broader intention of the curve sketching exercise to initiate a questioning of $A^{\Diamond}_{School}$ with regard to its "validity and limitations …, its adequacy to Q[♥, the authors], the adaptations required, etc." (Bosch, 2019, p. 4040).

The original curve sketching exercise can be depicted schematically as:[6]

$$[S(X;\ Y;\ Q^{♥}) \rightarrow M=\{A^{\Diamond}_{School},\ A^{\Diamond}_{University},\ W_{School},\ D_{math},\ Q_i,\ \dots\}] \Rightarrow A^{♥}$$

However, problems with the chronogenesis of the inquiry can arise and students can get stuck in a particular way: as the only part of the (initially provided) milieu that can give feedback on the correctness of students' answers is Y or D_{math}, which are both spatially and/or temporarily displaced (hence the grey font), it can happen that students end up believing, e.g. due to a logical error in the interpretation of the schoolbook theorems (taking unidirectional conditional statements for equivalences), that they have answered the question $Q^{♥}$ correctly even though they haven't. We suspect, that the DEM is being evoked too strongly and that the initial milieu doesn't provide enough feedback to enable (at least part of) our students to overcome and question the DEM.

As a consequence, in subsequent implementations of the online course, the milieu was enriched by adding new materials $(\dots)_i$ to the learning platform which consisted in fictional dialogues $W_{Student}$ between students who try to solve the curve sketching exercise but who run into the same dead ends as some actual students did and give an incorrect answer. The materials then pose the question Q_{DM}, why the displayed reasoning is incorrect, and include an official answer $A^{\Diamond}_{DM}$ to Q_{DM} by the institution that is our didactics course in the form of a sample explanation.

The modified curve sketching exercise can thus be schematically represented as:

$$[S(X;\ Y;\ Q^{♥}) \rightarrow M=\{A^{\Diamond}_{School},\ A^{\Diamond}_{University},\ (W_{Student},\ Q_{DM},\ A^{\Diamond}_{DM})_1,\ (\dots)_2,\ \dots,\ W_{School},\ D_{math},\ Q_i,\ \dots\}] \Rightarrow A^{♥}$$

6 The symbols stand for: X = students, Y = teaching team, $Q^{♥}$ = questions 1) and 2) from Fig.1, $A^{\Diamond}_{School}$ = algorithms for checking criteria for the existence of extremums and inflection points, $A^{\Diamond}_{University}$ = verification of one's own knowledge of the involved definitions, W_{School} = provided schoolbook excerpt, D_{math} = results of (correct) mathematical reasoning

THE REFERENCE EPISTEMOLOGICAL MODEL

In order to better understand the potential of the exercise to induce question-answer- and media-milieu-dialectics and to enable more in-depth analyses of students' solutions we started to work out a REM with reference to the *institution of mathematics education* constituted by our working group and through our courses.[7] The REM contains all praxeological elements which occur in a set of solutions to the curve sketching exercise that we created on the basis of W_{School} and our knowledge of mathematics (cf. D_{math}) and which were deemed the most 'obvious', 'plausible' and 'expected' (i.e. 'normal') solutions from the point of view of reference.[8] A visualisation of this set of solutions is given in the form of a flowchart (Fig. 5), which was inspired by the tool of *questions-answers map* (Bosch, 2019, p. 4041). The chart is not meant to be interpreted strictly chronologically: drawing a 'solution path' into the chart is supposed to indicate which praxeological elements occur and don't occur, but not necessarily at which stage of the solution they are employed (they might be employed several times at different stages in the solution). The detailed REM can be accessed by consulting tables 1, 2 and 3 in the supplementary material in combination with the flowchart. The tables specify *three regional praxeologies* that will now be explained.

Regional praxeology *Differentiation* (cf. Table 1)

We regard the differentiation of real functions as a praxeology in its own right, because it is taught in school as separate topic before curve sketching is introduced in later school years. In the context of our course, differentiation is also treated differently from curve sketching as our didactic contract demands that the schoolbook be cited when curve sketching theorems are used, while justifications for the application of differentiation techniques are not necessary. The praxeology *differentiation* includes all praxeological elements that were relevant in our solutions, even if some of them are likely not commonly taught in German schools but in introductory Analysis courses at university (e.g. θ(diff).1 and 2). In this sense, the resulting praxeology is specific to the institution of mathematics education, whose praxeological equipment intersects with that of both school and university mathematics (as well as that of other disciplines).

Regional praxeology *Curve Sketching* (cf. Table 2)

We consider the praxeologies *extrema (E)* and *inflection points (IP)* to be two subpraxeologies of the regional praxeology *curve sketching*. The two praxeologies possess largely similar logos-blocks (monotony is relevant for both; justifications of theorems are based on the same ideas) and are consolidated on page 67 of Freudigmann et al. (2012) into a sequence of activities commonly called "curve sketching". However, the book's presentation assigns the two praxeologies to two different subchapters with their own specific definitions, theorems, examples and tasks. Table 2

7 It is prominently characterised by the assessment criteria we apply in our exams and which we communicate to our students, by the knowledge and practices we teach in our courses, etc.
8 Naturally, included solutions had to be mathematically sound.

represents the praxeology extrema (E); the respective praxeology inflection points (IP) is analogous, but “E” is swapped with “IP” and other schoolbook theorems are relevant.

Regional praxeology *Solving Tasks* (cf. Table 3)

After we had coded all steps of each solution using the above praxeologies as coding manual[9], there remained passages in our solutions to which no praxeological elements had yet been assigned. For this reason, we defined an additional praxeology *solving tasks* that includes general strategies for solving tasks. These strategies are commonly introduced in school, applied in all subjects (not just mathematics), and are relevant for university mathematics as well, where they undergo a mathematical specialisation as stronger emphasis is placed on mathematical rigour (e.g. in mathematical case differentiations at university, the cases have to cover all logical possibilities and any two cases have to be disjunct from each other).

DISCUSSION

The design of the curve sketching exercise addresses two circumstances: First, it is a reaction to mathematical difficulties students typically experience at the transition from school to university.[10] By posing a mathematical problem which cannot be solved within a well-known DEM from school mathematics, we intend to initiate questioning and further analysis of this DEM. This idea is especially relevant in the education of teacher students, as this group can not only benefit from questioning and increasing their mathematical knowledge, but also from using didactic theory to analyse and reflect upon the teaching of the DEM in school, its goals and effects. Secondly, exercise modifications intend to mitigate teaching difficulties arising from the non-responsiveness of our digital learning environment and the lack of feedback by the milieu. These difficulties are in fact not only a problem of our specific learning platform but also, more generally, a feature of the type of task proposed: the idea for the task is based on the expectation (and experience) that many students are fixated on the DEM so strongly that they overlook solution strategies that are very elementary from the point of view of university mathematics. The potential of the exercise lies in its capacity to incentivise reflection on these issues, i. e. on biases in one’s own mathematical thinking and their possible origins. With regard to the learning platform, the question arises of how to provide feedback that is standardised but will nonetheless relate to individual mistakes to some degree in order to help students who are ‘stuck in the DEM’ and to encourage them to continue their investigation into $Q^{\heartsuit}$. Due to worries that the presence of a sample solution $A^{\heartsuit}$ might discourage further investigation and bar access to certain learning experiences, we did not provide $A^{\heartsuit}$ on the page of the learning platform that contained the exercise. The option of enriching the milieu with discursive material in the way explained, on the other hand, has worked well for us:

9 Coding is understood here in the sense of qualitative content analysis; the units that were coded were the elements of the 4-T-model, i. e. (the application of) techniques, reasons given for applying a technique, more rarely: intermediate questions/tasks.

10 In German mathematics education, such exercises are referred to as “Schnittstellenaufgaben”.

some students became aware of mistakes they had made themselves while all of them were provided an opportunity to analyse typical learner mistakes using didactic theory.

In order to create and refine an exercise that targets DEM bias in the described way, it is necessary to gather knowledge about typical student difficulties. Consultation of literature may guide towards promising mathematical topic areas and can generate ideas for potential exercises at the beginning of a design process. Once concrete exercises have been formulated, the potential of an exercise can be illuminated by creating a questions-answers map (our flowchart being a variation thereof). An initial questions-answers map functions as a reflection tool that gives an overview of the knowledge involved in various ways of dealing with the task and may generate (on the basis of knowledge about the DEM) a priori hypotheses about possible student difficulties. After a first trial of the exercise, materials that address mathematical difficulties with the task can be created on the basis of (naïve) observations of typical student mistakes and with the help of the map, which can facilitate a systematic covering of all difficulties of interest.

The questions-answers map can be refined in connection with the creation of a detailed REM, which has happened in our case. The REM does not only help deepen reflection on an exercise's potential and the involved institutional knowledge(s), but can be employed as analytical tool in ATD-based qualitative analyses of (various types of) data documenting student solutions or solution processes. In such analyses, it will serve as a normative reference against which deviations in solutions (or solution processes) can be systematically identified and characterised. This may be useful in cases where student difficulties are yet unknown or not documented sufficiently in existing research. We plan an exploratory investigation into our students' solution strategies on the basis of our REM in order to see if we can learn more about the types of difficulties that occur and how to address them in our teaching. (There is a possibility, too, that unexpected solution strategies which are viable in certain milieus will inform the development of further materials.) Additionally, our REM constitutes a novelty in that it mixes praxeological elements from university and school mathematics to create a representation of a praxeological organisation that stems from the institution of mathematics education. This may open up new possibilities of investigating questions of teacher education more broadly. E.g.: Can the necessity to use praxeological elements from university mathematics in tasks that 'could possibly occur in school' provide a profession-oriented raison-d'être for these elements in the study curriculum?

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On-campus vs distance learning in university preparatory courses: Which factors influence students' course decision?

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Mathematics preparatory courses intend to strengthen prospective students' content-related and affective prerequisites for studies involving mathematics. Since participation is voluntary, courses should be designed to appeal to students and encourage them to participate. In this context, modality seems to be a key factor, which is why various distance learning courses exist in addition to on-campus courses. However, research on these different course formats is scarce. In this contribution, we therefore investigate which affective and performance-related factors predict students' decision to choose on-campus or distance learning. Our results indicate that students' digital readiness cannot be overestimated, even after several pandemic years, and that mathematical self-efficacy is a relevant factor to be considered in course design.

Keywords: Transition to, across and from university mathematics, Novel approaches to teaching, Preparatory course, Emergency remote teaching, Person-environment fit.

INTRODUCTION

Entering a mathematics degree program involves a variety of changes that many first-year students perceive as obstacles. On the one hand, students from different schools come together, forming a heterogeneous group with different biographies, prior knowledge, and learning socialisations. On the other hand, the transition from school to university is accompanied by changes in the learning environment since the way mathematics is characterised and taught at university differs significantly from the methods at school (Rach & Heinze, 2017). In order to facilitate the transition process, many universities offer voluntary preparatory courses to prospective students. These courses intend to strengthen the students' content-related and affective study prerequisites and give insight into the characteristics of the new learning environment (Biehler et al., 2018). According to theories of person-environment fit (e.g. Swanson & Fouad, 1999), the extent to which such courses enable successful learning depends on how well the learners' individual characteristics fit the characteristics of the learning environment. A good fit of needs and offerings as well as prerequisites and demands supports academic achievement and the students' well-being, while an insufficient fit may lead to failure and demotivation. To increase the person-environment fit and encourage students to participate, there are, for example, preparatory courses with different contents, goals, and structures (Biehler et al., 2018). In addition, modality seems to be an important issue in the design of preparatory courses, as on-campus and distance learning are associated with different learning conditions. While some universities have been offering distance or blended learning courses for years (Derr, 2017; Greefrath et al., 2017), others have recently had experience with such formats in the COVID-19 pandemic (Büchele et al., 2021). However, the extent to which such distance learning courses are better suited to particular student groups than traditional on-campus courses has not yet been explored in detail. Assuming that students themselves choose the learning environment that best fits their own characteristics, this contribution compares the affective and performance-related characteristics of students in an on-campus and a distance learning course. The results give insight into university students' expectations regarding course requirements and provide design principles for preparatory courses depending on the course modality.

ON-CAMPUS, BLENDED AND DISTANCE LEARNING

Even before the pandemic, some universities offered blended and distance learning preparatory courses to their students (Biehler et al., 2018; Derr, 2017; Greefrath et al., 2017). While distance learning in this context is often implemented as an asynchronous online course with uploaded materials, blended learning courses also include single face-to-face sessions on campus. These types of preparatory courses are particularly common in student programs with a very heterogeneous student body, such as in STEM education. Because of the high level of independent learning, such courses allow prospective students to work at their own pace and intensity without being bound to the university location (Derr, 2017; Greefrath et al., 2017). However, compared to on-campus courses, distance learning is more demanding as it requires adequate equipment as well as additional skills in using it (Hong & Kim, 2018). In addition to the obvious, distance learning requires a higher degree of self-regulated learning and, therefore, addresses other learning styles (Artino & Stephens, 2009; Reinhold et al., 2021). Thus, against the background of person-environment fit, it can be assumed that distance learning appeals to students with certain personal characteristics more than others. This assumption is also supported by initial research on emergency remote teaching showing that pandemic distance learning is challenging for students in different ways (Händel et al., 2020; Kempen & Liebendörfer, 2021; Reinhold et al., 2021). Because courses carefully designed for distance learning differ to some extent from pandemic distance learning, we provide a brief overview about key aspects of person-environment fit for both perspectives.

Students' choice for on-campus and distance learning preparatory courses

Since some universities offer preparatory courses with different modalities, individual fit criteria can be derived from students' course decision. Analysing students' reasons for choosing a blended learning or an on-campus course, Biehler et al. (2011) revealed different extrinsic and intrinsic factors, with the latter being attributed greater importance. Extrinsic factors mainly arise from *demographic aspects of the person-environment fit* and contain restrictions imposed by students' living or working situation. Students who choose a blended or distance learning course appreciate the high degree of flexibility in terms of attendance and time (Biehler et al., 2011; Thompson & McDowell, 2019). From this perspective, distance learning courses mainly address students who live far away, work during normal course hours or have other commitments. Although older students are more likely to belong to one of these groups, Fischer (2014) reports no significant differences between students choosing a blended learning or an on-campus course in terms of age and year of graduation. Thus, external conditions do not appear to be sufficient for predicting the course decision.

Intrinsic factors include both *affective* and *performance-related aspects of person-environment fit*. Comparing students' prior academic performance, Fischer (2014) reported that students with stronger prerequisites, such as a better overall school performance (GPA) or a better math grade, tend to choose a blended learning course. Consistent with this, students in the blended learning course also have higher self-efficacy (Fischer, 2014), that is, they have higher confidence in their own mathematical abilities than their peers on campus (Bong & Skaalvik, 2003). In contrast, digital readiness in terms of computer or internet self-efficacy was similar in both groups, representing no sufficient reason for or against a blended learning course (Biehler et al., 2011). Thus, course decision seems to be influenced less by (self-reported) abilities for digital learning than by individual learning routines. While students in blended learning courses cite self-regulated learning and individual time management as important aspects of the learning environment, students in on-campus courses value personal contact and peer learning (Biehler et al., 2011). In summary, previous research indicates that students with a stronger connection to mathematics and a preference for independent learning are more likely to choose a distance or blended learning course. Students who tend to choose an on-campus course, on the other hand, have weaker prerequisites and value, above all, the social interaction.

Distance learning during COVID-19 pandemic

Due to the pandemic situation in 2020, regular on-campus courses had to be spontaneously converted to distance learning. Since emergency remote teaching is closely aligned with on-campus teaching, pandemic distance learning often contains many more synchronous elements than established distance learning courses. While lectures are provided in form of video recordings at many universities, tutorials can also be successfully offered synchronously via video conferencing (Mullen et al., 2022). In this context, distance learning courses may also include fixed schedules and video-based face-to-face interactions. Thus, pandemic distance learning as a learning environment has some specific characteristics that limit the transferability of previous findings. Rather, it is likely that pandemic distance learning places specific demands on students in terms of self-regulated learning, social interaction, and digital readiness. Thus, research on emergency remote teaching shows that students' ability to adapt to distance learning varies according to their individual characteristics (Händel et al., 2020; Kempen & Liebendörfer, 2021). In general, there is a positive relationship between attitudes toward online learning and coping with the new learning conditions (Reinhold et al., 2021). However, students with stronger motivational orientation toward mathematics express a higher need for face-to-face interaction and prefer traditional on-campus instruction. Strong motivational orientations in this study include, for example, a high mathematical self-concept (Reinhold et al., 2021), which is a more general construct than self-efficacy, but still relates to student self-assessment (Bong & Skaalvik, 2003). On the other hand, Büchele et al. (2021) report that preparatory course performance in 2020 was generally better than in previous years. However, it remains uncertain whether this observation is explained by a higher person-environment fit or by more opportunities for cheating in distance learning.

THE PRESENT STUDY

Since a sufficient person-environment fit is crucial for effective learning, this contribution aims to identify factors that influence university students' decision to take an on-campus or a distance learning preparatory course. The study is based on a preparatory course held at the University of Münster in September 2021. Due to pandemic constraints, this course was offered in two variants. In both variants, lectures were video-recorded and provided asynchronously on the university's learning platform. The daily tutorials, on the other hand, were offered synchronously, either on campus or via distance learning video conferencing. Both tutorial options were designed the same in terms of time slots, tutors, group size, and instruction. Since external factors have been shown to be less relevant in predicting course decision (Biehler et al., 2011), we focus on affective and performance-related factors in this study. Due to pandemic-driven digitalisation, we expect changes on both sides of person-environment fit, i.e. students becoming accustomed to digital, self-regulated learning and distance learning environments incorporating face-to-face-interaction. With this in mind, research is guided by the following questions:

RQ1: *Which individual affective and performance-related factors influence students' decision to take an on-campus or a distance learning preparatory course?*

Both affective and performance-related factors represent intrinsic factors and, therefore, are supposed to have a (strong) influence (Biehler et al., 2011). Since previous research findings are contradictory (Fischer, 2014; Reinhold et al., 2021), we will exploratively investigate the relationship between strong mathematical learning prerequisites and course decision. However, it is hypothesised that students who appreciate self-regulated learning and feel ready for digital learning are more likely to choose a distance learning course. Students who value personal contact and peer learning, on the other hand, are expected to enrol in an on-campus course.

RQ2: *How strong are the effects of performance-related factors on course decision in comparison to affective factors?*

Performance-related factors such as grades provide clear feedback on students' (mathematical) academic achievements. Affective factors, on the other hand, concern a persons' characteristics that may match or differ from the learning environment that is offered. Since all courses should equally promote mathematical learning, we expect the affective factors to have a stronger influence on the course decision.

MATERIAL AND METHODS

Sample

The preparatory course is designed for prospective mathematics teachers at primary and lower secondary level. These two degree programs contain a comparable proportion of mathematics. In this context, we investigated N = 159 students, of whom n = 133 were pursuing a primary teaching degree (127 female, 5 male, one did not specify) and n = 26 a secondary teaching degree (16 female, 10 male). Students could independently sign up for a course option, resulting in n = 71 students in the on-campus course and n = 88 in the distance learning course. On-campus learners and distance learners did neither differ significantly in their age (M_{OC} = 19.53, SD = 2.16 and M_{DL} = 19.39, SD = 2.01) nor their study program (80.3% and 86.4% primary level).

Instruments

To answer the research questions, we collected data on a total of six potentially predictive variables describing affective or performance-related aspects of person-environment fit (see Table 1 for an overview).

Predictor	Coding	Value	# Items	α	Mean (SD)
A1: Self-regulation		1 to 7	3	.82	4.78 (0.90)
A2: Peer learning		1 to 6	3	.68	4.44 (0.98)
A3: Self-efficacy		1 to 4	4	.82	2.56 (0.52)
A4: Digital readiness		1 to 4	7	.68	3.07 (0.42)
P1: Prior GPA	lower = better	1 to 4	1		1.91 (0.45)
P2: Final math. grade	higher = better	1 to 15	1		10,66 (2.36)

Table 1: Overview of the instruments and the sample's descriptive statistics

To measure *affective factors*, we made use of existing instruments that have proven successful in previous studies. Overall, we collected data on four variables in this domain, namely self-regulation, peer learning, self-efficacy, and digital readiness. Students' *self-regulated learning* was measured using items of the short version of the strategies of university students questionnaire (LIST-K; Klingsieck, 2018; sample item: "I change my learning technique when I encounter difficulties."). This scale is based on self-reports and gives insight into students' meta-cognitive learning strategies. To assess students' need for face-to-face interaction, we focus on students' engagement in peer learning as collaboration with fellow students is one of the most important aspects of the tutorials. *Peer learning* was measured using items from the LimST scale (Liebendörfer et al., 2021; sample item: "When I have a solution, I want to discuss it with fellow students."). Students' confidence in their own mathematical abilities was assessed in this study using the construct of *mathematical self-efficacy*. Self-efficacy has been shown to be a meaningful construct in the analysis of preparatory courses and is surveyed in this study using the scale from the WiGeMath project (Biehler et al., 2018; Hochmuth et al., 2018; sample item: "In math, I am confident that I can understand even the most difficult material."). Since working in a digital learning environment involves various actions, *digital readiness* includes different facets such as application usage or information sharing. In order to reflect the requirements of the digital learning environment as accurately as possible, we developed items for this scale ourselves, using the digital readiness for academic engagement scale as a basis (DRAE; Hong & Kim, 2018; sample item: “I find it easy to follow courses in a video conference.”). The reliabilities of the four scales range from acceptable to good (see Table 1), with digital readiness and peer learning scales having the lowest reliability (α = .68). Considering the broad scope of the construct, reliability can be considered satisfactory as both scales encompass different facets of digital or peer learning.

Performance-related factors were measured by asking students (1) about their school grade point average as an indicator of prior academic performance and (2) about their final grade in mathematics as an indicator of performance in school mathematics.

RESULTS

To answer the first research question, logistic regressions were conducted with the dichotomous criterion course decision (on-campus vs. distance learning) as the dependent variable (see Table 2). Model 1 includes affective factors as predictors, while Model 2 consists of performance-related predictors. For each model, correlations between predictor variables were low (r < .57), indicating that multicollinearity was not a confounding factor in the analysis. In examining studentised residuals, one outlier was isolated. There are no unusually high values of Cook’s distance, indicating that there are no influential cases.

	Model 1			Model 2			Model 3		
Predictor	b	SE	OR	b	SE	OR	b	SE	OR
Self-regulation	0.01	0.20					0.01	0.21	
Peer learning	-0.35	0.19					-0.36	0.19	
Self-efficacy	-1.16^{**}	0.37	0.31				-1.40^{***}	0.40	0.25
Digital readiness	0.75^{+}	0.44	2.12				0.77^{+}	0.45	2.17
Prior GPA				0.25	0.43		0.41	0.45	
Final math grade				0.03	0.08		0.16^{+}	0.10	1.18
Nagelkerke R^2	.14			.003			.16		

Table 2: Coefficients of the models (method: inclusion) predicting whether a student chooses an on-campus or a distance learning course; $^{***}p < .001$ $^{**}p < .01$ $^{+}p < .1$

Of the four variables entered into Model 1, only the students’ mathematical self-efficacy ($OR = 0.31, p = .002$) contributed significantly to predicting course decision. However, digital readiness ($OR = 2.12, p = .087$) also has a weakly significant impact. While self-efficacy is a negative predictor of choosing a distance learning course, digital readiness is positively related to this type of learning. The variables concerning learning strategies, i.e. self-regulation and peer learning, did not have additional impact on course decision. Compared to baseline, Model 1 has improved significantly by adding affective predictors (χ^2 $(4) = 16.42, p = .002$) and explains 14% of the variance (Nagelkerke R^2) in course decision. In Model 2, neither the final grade in mathematics nor the overall school performance proved to be a significant predictor of course decision. Accordingly, Model 2 is not statistically significant as a whole ($\chi^2(2) = 0.34, p = .842$) and does not contribute to the variance explained.

For the second research question, we compared the influence of affective and performance-related factors on course decision. Therefore, a logistic regression was conducted, including both affective and performance-related factors (see Table 2). Model 3 can explain 16% of the variance in course decision ($\chi^2(3) = 19{,}33$, $p = .004$), with the mathematical self-efficacy contributing the most to the explained variance ($r_{y,A3} = .25$, $r_{y,A4} = .1, r_{y,P2} = .01$). A one-point increase in self-efficacy is associated with a 75% decrease in the relative probability of choosing a distance learning course ($OR = 0.25, p < .001$). In Model 3, when controlling for affective characteristics, students' final grade in mathematics also becomes a weakly significant predictor ($OR = 1.18, p = .093$). Although this effect can be explained by collinear effects ($r = .34, p < .001$), it should be noted that self-efficacy is a negative predictor and final grade in mathematics is a positive predictor for choosing a distance learning course. Here we find that the relationship between self-efficacy and mathematics grade is substantially stronger for students in on-campus courses ($r = .47, p < .001$) than for students in distance learning courses ($r = .25, p = .022$).

DISCUSSION AND CONCLUSION

Preparatory courses have been offered as blended or distance learning courses for several years (Biehler et al., 2018; Derr, 2017; Fischer, 2014). However, synchronous distance formats have only become established through pandemic-related restrictions. To evaluate this type of distance learning for future teaching, we examined which types of students choose an on-campus or a distance learning format.

Based on theories of person-environment fit, the first research question (**RQ1**) asked for affective and performance-related characteristics that influence students' course decision. Unlike in previous studies (Biehler et al., 2011; Fischer, 2014), self-regulated learning and peer learning did not have an impact on course decision in our study. This unexpected finding may be related to pandemic-driven developments on both sides of the person-environment fit: On the one hand, video conferencing enables synchronous elements in distance learning that structure the learning process and allow students to collaborate online. On the other hand, first-year students in 2021 have already experienced distance learning in school and have developed new learning strategies for digital interaction. These personal and environmental developments may reduce the importance of self-regulated and peer learning as decision criteria. In contrast, students' digital readiness and self-efficacy proved to be relevant affective factors for course decision, replicating results from research on emergency remote teaching (Händel et al., 2020; Kempen & Liebendörfer, 2021; Reinhold et al., 2021).

With respect to the second research question (**RQ2**), a more concise picture of choice prediction emerges since affective and performance-related factors are included: In general, students are more likely to choose a distance learning course if they exhibit weak mathematics self-efficacy, have a higher grade in mathematics, and feel ready for digital learning. Self-efficacy showed the strongest impact, which is consistent with our hypothesis that affective factors should influence course decision more strongly than performance-related factors. In line with theories of person-environment fit, it is reasonable that distance learning offers students the opportunity to learn in a familiar, protected environment where feelings of embarrassment are less pronounced. Therefore, such a learning environment might be more attractive to students with weaker self-efficacy. This explanation is supported by research on pandemic distance learning (Büchele et al., 2021; Reinhold et al., 2021), but requires further investigation into how factors such as mathematical anxiety influence course decision. With regard to the performance-related factors, we could only report a slight preference of higher-performing students for distance learning. This is consistent with the results of previous studies, which found that higher-performing students prefer blended or online learning courses (Biehler et al., 2011; Greefrath et al., 2017). It is noteworthy, however, that high self-efficacy and good grades in mathematics are associated with different course decisions, contrary to what was described, for example, in Fischer (2014). On the one hand, this finding once again highlights the differences in the various digital learning environments. On the other hand, the relationship between self-efficacy and mathematics grade may be overlaid by other factors, especially for students in distance learning. For example, extrinsic demographic factors such as living situation or even current illness have not been included in this study, although they may influence the weighting of the decision criteria (Biehler et al., 2011).

In conclusion, our findings indicate that distance learning with synchronous elements represents a learning environment with its own characteristics and, therefore, attracts different students than asynchronous distance and on-campus learning. Because preparatory courses are voluntary and take place prior to the start of studies, distance learning options will still be relevant after the pandemic. Taking the relevant decision criteria into account, instructors can better tailor courses to the students' characteristics. For example, distance learning courses should pay more attention to students' self-efficacy, while on-campus courses could prepare more for digital learning.

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Insights into a reference model for school-based approaches to vectors

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In this paper, we present insights into the mathematical complexity of our reference model regarding different approaches to introducing vectors in school. We will discuss and compare approaches using n-tuples, arrow classes, or translations. While an approach via n-tuples is relatively simple, arrow classes turn out to be much more complex. To give a detailed example, we will discuss the proof that vector addition is commutative in terms of both approaches separately. This is part of the larger research interest to identify possible prerequisites regarding vectors that first-year students bring from school during their transition to university. Our detailed reference model is an essential foundation for further research regarding the transition issues of the vector topic.

Vectors, reference model, Anthropological Theory of the Didactic, Epistemological studies of mathematical topics, transition to university mathematics.

INTRODUCTION

The transition from school to university has been a broadly conceived issue (Hoyles et al., 2001). There are differences between these two institutions concerning formal notation, rigour, and abstractness of mathematics (Luk, 2005). This process of transition can be investigated from different perspectives. For example, one can study the mathematical beliefs of students and how they change (Geisler & Rolka, 2021) or focus on offers of support for the transition (Gallimore & Stewart, 2014). Another perspective is to dive more deeply into the different types of practices of mathematics at school and university. The Anthropological Theory of the Didactic (ATD) provides an adequate framework to investigate mathematical practices in detail and while being sensitive to institutions and institutional effects (Chevallard, 2019).

In many countries, students learn about vectors in school and bring their learned knowledge about vectors when they start studying at university. Students seem to come to university with a surprisingly diverse and very unsound understanding of vectors, consisting of incoherent elements of different views on vectors (Mai et al., 2017). This observation raised questions about the roots of these conceptions. Therefore, we are analysing school textbooks for mathematics as one possible source. For this goal, we use ATD, which suggests developing a so-called epistemological reference model as an analytical basis (Chevallard & Bosch, 2014). This paper will focus on insights into our reference model about approaches to vectors at school.

THEORETICAL FRAMEWORK

In the Anthropological Theory of Didactic (ATD), a core idea is that knowledge is always related to an institution to which individuals can belong (Chevallard, 2019).

Another key concept from the ATD, which describes that knowledge can be situated in different institutions, is the notion of the didactic transposition. According to the model of didactic transposition, four different kinds of knowledge exist: "scholarly knowledge," "knowledge to be taught," "taught knowledge," and "learned/available knowledge" (Chevallard & Bosch, 2014). From the model's perspective those four kinds of knowledge are part of the didactic transposition that knowledge undergoes before students learn it. Different institutions are involved in this process. For this paper, we consider mathematics at schools and university as vaguely defined institutions. Researchers working with ATD are encouraged to develop their own institutional point of view in a reference model, which can be understood as positioned outside the above mentioned process of didactic transposition (Barbé et al., 2005). Within the ATD, knowledge is conceptualised in terms of praxeologies. Praxeologies contain a praxis block and a logos block. The praxis block concerns problems (tasks) and activities to solve them (techniques). The logos block contains justifications (technologies) for why those activities do work and further justifications of these justifications themselves (theory).

A BRIEF OVERVIEW OF APPROACHES TO VECTORS AT A SCHOOL LEVEL

Tietze, Klika, and Wolpers (1982) summarize the following relevant interpretations of vectors at the school level exist: n-tuples, points, pointers/location vectors, arrow classes and translations. Filler, and Todorova (2012) state two mathematical approaches towards introducing vectors that are suitable and common for school level: n-tuples and arrow classes. Both mention the axiomatical approach and agree that it is rather unsuitable for introducing vectors to students in school. In German mathematics textbooks, we found a mixture of (partial) approaches using n-tuples, arrow classes, and translations to be most relevant (Brandt et al., 2014; Griesel et al., 2014; Körner et al., 2015).

RESEARCH INTEREST

Based on the discussion from the previous section, the research interest of this paper is to flesh out a reference model for the introduction of vectors at school level using three different approaches, namely vectors as n-tuples, as arrow classes and as translations. Through an explicit and detailed elaboration of such a reference model, focusing first on the formulation of the necessary definitions and theorems and the associated proofs, the reference model becomes logos centred. Praxis will be considered later, when the analysis of the school textbooks as artifacts of the knowledge to be taught are analysed.

Only some excerpts of the reference model can be presented below, because the original model would be too long to fit in here. Beyond this paper, the reference model is intended as a framework for analysing school textbooks regarding the topic of introducing vectors. In the end, this research will help to better understand what prior knowledge students bring to university as textbooks are an influential factor in developing students' knowledge at school (Valverde et al., 2002). In this paper, we show the readers that large parts of the mathematical background regarding the introductions of vectors (have to) remain implicit for students at school.

THE BASIC STRUCTURE OF THE REFERENCE MODEL

For the formulation of the reference model, we decided that three types of vector objects should be considered: n-tuples (here and in the following with real-valued components and $n = 2$ or $n = 3$), arrow classes and translations as mappings from the plane into the plane or respectively in 3d-space. All three types of objects appear in school textbooks and are referred to as vectors. Often textbooks present an incoherent mixture of these three approaches and textbooks also differ in how they combine them. Although arrow classes are uncommon for university mathematics, they are an important tool at school level for solving tasks in geometrical contexts.

In the reference model, a definition is given for each of the above-mentioned vector objects and operations with them. The defined structures are similar to vector spaces. However, this is not the implicit and more general approach of defining vectors in university mathematics as elements of vector spaces. The reference model includes further terms and operations relevant to school textbooks, including vector addition, multiplication by a scalar, the magnitude of a vector, opposite vectors, and location vectors. Our reference model is divided into two parts. First, we introduce each of the three approaches to vectors. Afterwards, we prove that these three approaches are isomorphic regarding the operations of addition, scalar multiplication and the assignments of “length” resp. “magnitude” (isometrical isomorphy).

Defining a vector and the related operations and concepts based on n-tuples does not cause any difficulties. Building on the definition of n-tuples, the further desired concepts and properties such as addition and scalar multiplication can be easily defined by using the properties of real numbers. The situation is different if vectors are to be introduced as arrow classes as we will show later on. At school level, this term is restricted to arrow classes consisting of arrows in the plane or space. The one-dimensional case is often neglected in school, although positive and negative numbers could be easily associated with one-dimensional vectors, as is done in some older textbooks.

In our reference model, the arrow concept is defined as a 2-tuple consisting of a starting point and an endpoint. For this definition, we assume Euclidian geometry with properties of points, lines, and planes without specifying an axiomatic system for these objects. In the beginning, a coordinate system is not necessarily needed for this approach. Later, it becomes relevant when a connection (isomorphy) to the other two approaches to defining vectors is established. Furthermore, the length of an arrow and the relation “is parallel to” for two arrows have to be introduced. These two concepts can easily be traced back to known geometrical facts. We intend to build an equivalence relation on the sets of arrows by regarding arrows that are parallel, have the same length, and have the same orientation as equivalent. It is surprisingly complicated to precisely define the intuitively simple concept of “having the same orientation” in geometric terms (see also below).

Once this is done, the relation of two arrows being equal in length, parallel to each other, and oriented the same way can be proved to be an equivalence relation of arrow classes. Subsequently, it is clear that each arrow is an element of an associated arrow class. Such arrow classes are called vectors. Within this approach, it is important also to define the zero vector. Therefore, another class of “arrows” by all those “degenerated” arrows of the kind $\overrightarrow{AA}$ has to be introduced. These arrows have to form an equivalence class of their own. However, the defining equivalence relation for non-degenerate arrows is not applicable (parallelism and orientation are difficult to define including these degenerated arrows). This mathematical difficulty is solvable but is related to erroneous or missing introductions of zero vectors in school textbooks based on vectors as arrow classes.

A well-known difficulty of textbook language and students’ conceptions of vectors is the missing symbolic distinction between arrows and arrow classes (vectors). Usually, both are written as $\overrightarrow{AB}$. To avoid symbolic confusion in our reference model, we use the notation with one arrow, e.g. $\overrightarrow{AB}$, to explicitly denote the arrow from A to B and the notation with two arrows above, e.g. $\overrightarrow{\overrightarrow{AB}}$, to refer to the class of arrows determined by the representing arrow $\overrightarrow{AB}$.

Finally, translations (as mappings) are introduced in the reference model as vector objects. Mathematically, these can be defined similarly to n-tuples or arrow classes. Our reference model introduces the $\mathbb{R}^n$ with usual operations. For $a \in \mathbb{R}^n$ a translation can be defined as a mapping $t: \mathbb{R}^n \to \mathbb{R}^n$ with $t(x) = a + x$ for all $x \in \mathbb{R}^n$. This way, they are introduced similarly to the n-tuple approach but as mappings. In this sense, they mainly differ from the n-tuple approach by a different notation and the explicit possibility to apply the translation to a point (given by its coordinates).

This section will present a definition for arrows of the same orientation and investigate the consequences of such a definition for the proof that vector addition is commutative. The first complication is that it is not directly possible to define the orientation as a property of an arrow and then assess whether two orientations are the same. We have to define a relation for two parallel arrows: “Arrow a has the same orientation as b if…”. This is a logical challenge for textbooks and students’ understanding. It is similar to defining whether two finite sets have the same size (existence of a bijective mapping) without determining their size before by counting.

The relation of two arrows “having the same orientation” is essential for defining arrow classes. For a theory about vectors as arrow classes, it is enough to define “orientation” only for arrows with the same length and parallel to each other. This reduces the possibilities for two given arrows to have same orientation or to have opposite orientations. From an ostensive point of view, this seems trivial to decide. However, the mathematical definition in the reference model has to be precise.

Definition of the same and opposite orientation of two parallel arrows with the same length

The following definition of two arrows having the same orientation is based on the suggestion of Filler (2011, p. 88). It is here supplemented with the case 3. for two identical arrows to guarantee that the relation of having the same orientation is a reflexive relation.

Let A, B, C, and D be points on a plane. Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two parallel arrows of the same length with $A \neq B$ and $C \neq D$. We say $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same orientation if and only if one of the following three statements is true:

1. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ do not lie on the same line and the lines AC and BD are parallel to each other (see figure 1).
2. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are on the same line, but both arrows are not identical. Additionally, either B and C lie on the line segment between A and D or A and D lie on the line segment between B and C (see figure 2).
3. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal ($A = C$ and $B = D$).

Otherwise, the arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have an opposite orientation.

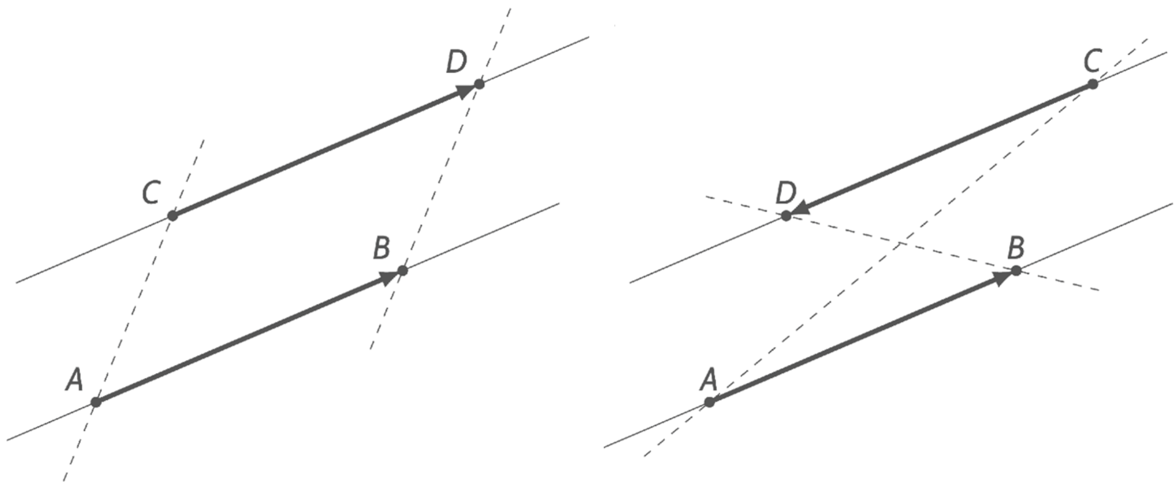

Figure 1: Two examples of two arrows with the same and with opposite orientation (the figure is similar to Filler, 2011, p. 88).

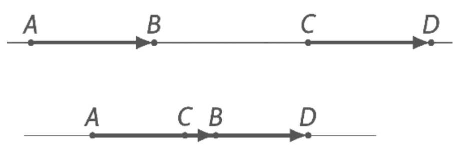

Figure 2: Illustration of two cases of the points B and C lying between the points A and D on a line.

Thus, it is not surprising that school textbooks known to us do not use this logically complicated definition and mostly rely on an intuitive assessment of the "same orientation". However, it is clear that a relational concept of "same orientation" is necessary to overcome an ostensive definition of "same orientation".

With this definition as preparation, we can turn to the proof of the commutativity of vector addition for 2-tuples and arrow classes in comparison.

Proof that vector addition is commutative for 2-tuples

Let $V_1 = \binom{a_1}{a_2}$ and $V_2 = \binom{b_1}{b_2}$ be vectors (in the sense of 2-tuples). Then $V_1 + V_2 = \binom{a_1}{a_2} + \binom{b_1}{b_2} = \binom{a_1 + b_1}{a_2 + b_2} = \binom{b_1 + a_1}{b_2 + a_2} = \binom{b_1}{b_2} + \binom{a_1}{a_2} = V_2 + V_1$ is true because of the commutativity of the addition of real numbers. Therefore, the addition of two vectors is also commutative.

In the world of tuples this proof is relatively easy and does, of course, work analogously for every n-tuple. Now, we turn our focus to arrow classes.

Definition of the addition of plane arrow classes

Let V_1 and V_2 be plane arrow classes (vectors). Select any arrow $\overrightarrow{AB} \in V_1$. Select the specific arrow from V_2 which has the start point B (its existence and uniqueness has to be shown before), which is the endpoint of the selected first arrow $\overrightarrow{AB}$. We call the second arrow's endpoint C and can refer to this arrow as $\overrightarrow{BC}$. Now we define $V_1 + V_2 = \overrightarrow{\overrightarrow{AB}} + \overrightarrow{\overrightarrow{BC}} := \overrightarrow{\overrightarrow{AC}}$. The resulting arrow $\overrightarrow{AC}$ represents the arrow class $\overrightarrow{\overrightarrow{AC}}$ which is the result of the addition. This concept of addition corresponds to common idea of attaching an arrow to the tip of the other arrow. In the elaboration of our reference model – but not here in this paper – we prove that this definition is well defined for an arbitrarily selected representative from the arrow class V_1. Again, showing the independence of the definition of the chosen representative is often neglected at school level. In principle, it is also needed when introducing fractions and their operations. That this neglection happens consistently at school level, might be done so because of the difficulty of the concept of equivalence classes.

Proof that vector addition is commutative for arrows on a plane

A challenge is that a proof needs to distinguish different configurations of vectors: being parallel or not, or including a zero vector. Also, we use the term "collinear" in a sense for parallel that is worked out in more detail in our reference model.

We start with V_1 and V_2 being non-collinear plane arrow classes (vectors). Let A, B and C be points with $A \neq B,\ B \neq C$, $V_1 = \overrightarrow{\overrightarrow{AB}}$ and $V_2 = \overrightarrow{\overrightarrow{BC}}$. By the definition of addition for arrow classes the following is true: $V_1 + V_2 = \overrightarrow{\overrightarrow{AB}} + \overrightarrow{\overrightarrow{BC}} = \overrightarrow{\overrightarrow{AC}}$. Thus, we are to show that $V_2 + V_1 = \overrightarrow{\overrightarrow{AC}}$. This can be achieved by proving that the arrows $\overrightarrow{AB}$ and $\overrightarrow{DC}$ (see figure 3) have the same length, are parallel to each other, and have the same orientation.

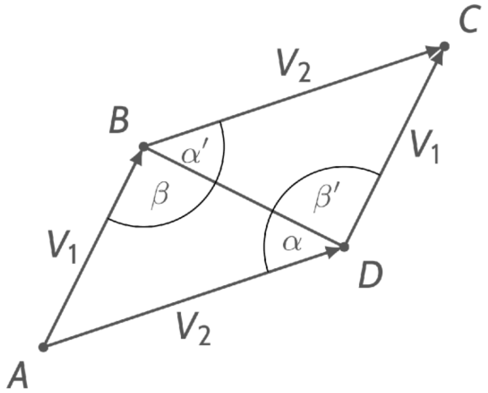

Figure 3: Illustration of the geometric configuration for the proof that the addition of arrow classes is commutative.

(a) $V_1 + V_2 = \overrightarrow{\overrightarrow{AB}} + \overrightarrow{\overrightarrow{BC}} = \overrightarrow{\overrightarrow{AC}}$ by definition.
(b) Choose the point D so that $\overrightarrow{AD} \in \overrightarrow{\overrightarrow{BC}}$ is true.
(c) $BC \parallel AD$ as $\overrightarrow{AD}, \overrightarrow{CD} \in \overrightarrow{\overrightarrow{BC}}$.
(d) Define $\alpha \coloneqq \angle BDA$ and $\alpha' \coloneqq \angle DBC$.
(e) It follows that $\alpha = \alpha'$ because of (c) and the Alternate Interior Angles Theorem.
(f) Further, $|\overline{BC}| = |\overline{AD}|$ holds as $\overrightarrow{AD}, \overrightarrow{BC} \in \overrightarrow{\overrightarrow{BC}}$.
(g) $\overline{BD}$ is a side of the triangles $\triangle ADB$ and $\triangle BDC$ each.
(h) The triangles $\triangle ADB$ and $\triangle BDC$ are congruent because of the Congruence Theorem SAS together with (e), (f) and (g).
(i) $|\overline{AB}| = |\overline{CD}|$ as the triangles $\triangle ADB$ and $\triangle BDC$ are congruent.
(j) Define $\beta \coloneqq \angle ABD$ and $\beta' \coloneqq \angle CDB$.
(k) It follows that $\beta = \beta'$, because (h) $\triangle ADB$ and $\triangle BDC$ are congruent and (i) $|\overline{AB}| = |\overline{CD}|$ (the adjacent sides have the same length).
(l) $AB \mid\mid CD$ because of the Converse Interior Angle Theorem together with (k) $\beta = \beta'$.
(m) $\overrightarrow{AB}$ and $\overrightarrow{DC}$ have the same orientation, because (l) $AB \mid\mid CD$, (i) $|\overline{AB}| = |\overline{CD}|$, and (c) $BC \parallel AD$.
(n) $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are elements of the same arrow class, because they are (l) parallel, (i) have the same length, and (m) they have the same orientation.

After this thorough argumentation, the proof is still not complete, yet. Since it was necessary to presume non-collinear arrow classes for the given argumentation above, we now turn our attention to the formerly skipped cases.

1. V_1 and V_2 are both the zero vector.
2. Either V_1 or V_2 is the zero vector.
3. V_1 and V_2 are parallel vectors.

Regarding 1. and 2. the proof cannot rely on the geometric situation as seen in figure 3. Luckily, it can easily be calculated that the addition of arrow classes in these specific cases is commutative. We turn now to case 3. which is more complex.

If V_1 and V_2 are collinear vectors, the definition of vector addition can be used to only consider arrows on the same line due to the independence of chosen representatives. Obviously, the sum leads to two parallel arrows and attaching them in any order leads to a line segment of the same length. Consequently, to prove that the addition, in this case, is commutative, we need to prove that the results of $V_1 + V_2$ and $V_2 + V_1$ have the same orientation. Same orientation for arrows on the same line was defined with the help of six different cases regarding the relative position of the two arrows to each other. Furthermore, it is possible that the considered arrows from V_1 and V_2 are not of the same length and contrary orientated (here, we use the term in common sense, since above we did not cover a definition for the case of arrows of different lengths). The proof has to consider every possible constellation. We will only give a representation, as seen in figure 4, of three exemplary constellations due to the limited space in this paper. In figure 4 one can see an arrow $\overrightarrow{AB}$ to which an arrow $\overrightarrow{BC}$ is attached. This represents $V_1 + V_2$. To the arrow $\overrightarrow{BC}$ is another arrow $\overrightarrow{CD}$ attached which is from V_1, because it has the same length as $\overrightarrow{AB}$. So, this is a representation of the commuted addition $V_2 + V_1$. For the proof it would further be necessary to argue in each possible case that $V_1 + V_2$ and $V_2 + V_1$ result in the same arrow class resp. both resulting arrows are representatives of the same arrow class.

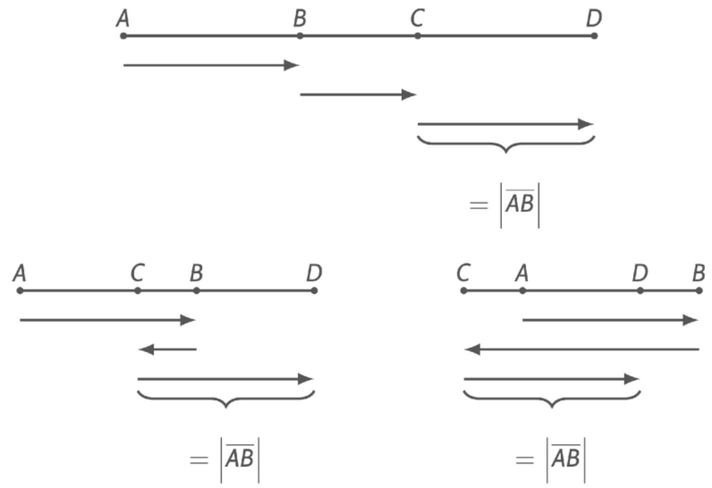

Figure 4: Exemplary constellations for the parallel case.

DISCUSSION

The elaboration of a reference model as mathematical background theory for the introduction of vectors using the usual school approaches via n-tuples, arrow classes, and translations has proven to be very insightful. The presented reference model constitutes a mathematical view that is neither the scholarly knowledge nor the knowledge to be taught at school. Because no task types or techniques are addressed in the reference model, the reference model can be compared to the logos block of a praxeology. It offers definitions and theorems with their proofs which constitutes a collection of technologies and their theory.

In comparison, the access to a vector object as an *n*-tuple is clearly shorter than the access via arrow classes. The latter approach to vectors is already in the preparations for the definition of the object "arrow class" clearly more complex in comparison to the *n*-tuple approach. Subsequently the study of a proof for the commutativity of vector addition shows in an example case that the difference in complexity stays relevant as the reference model progresses. Regarding school mathematics, this fact reveals that the most ostensive approach to vectors contains a high complexity while an algebraic approach is much more straight forward from a mathematical perspective. In mathematics textbooks, this complexity is partly condensed into single sentences like "In geometry, a vector can be described graphically by a set of mutually parallel, equally long and equally oriented arrows" (Brandt et al., 2014, translated by the authors). Without corresponding further explanations, vast parts of the associated logos remain implicit in such statements.

The investigation and elaboration of the presented reference model are by themselves insightful and reveal the mathematical structures connected to the different approaches. Nevertheless, it is only a first step. In the future, our research will use the reference model to analyse textbooks and the knowledge to be taught that comes along with them. By this analysis, there will undoubtedly be a gap between the given justifications in the textbook logos and further implicit justifications which do not surface in the school institutions but are made visible in our reference model. Having a tool to describe the mentioned gap better and will also help to reflect the knowledge on vectors that students might learn at school and, on the other side, to reflect on what beginners have to learn when they first engage with university mathematics.

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THE NOTION OF A POLYNOMIAL IN THE SECONDARY-TERTIARY TRANSITION

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The notion of a polynomial, as one of the most fundamental concepts in all areas of mathematics, has a prominent role in pre-tertiary and tertiary education. Due to the repeated encounter of students with this notion, it makes sense to study its role in the transition to university mathematics. In this paper, the secondary-tertiary transition of mathematics students at the University of Split concerning the notion of a polynomial is analysed using tools of the Anthropological Theory of the Didactic.

Keywords: Transition to, across and from university mathematics; Teaching and learning of linear and abstract algebra; Curricular and institutional issues concerning the teaching of mathematics at university level; Polynomial; Anthropological Theory of the Didactic.

THE POSITION OF POLYNOMIALS IN MATHEMATICS AND RESEARCH IN MATHEMATICAL EDUCATION

The notion of a polynomial is significant in almost every area of mathematics as a discipline. The short route through algebra shows a long and fundamental role of polynomials: from classical algebra problems such as solving algebraic equations, through Galois theory and ring theory in modern algebra and beyond. On the other hand, the notion of a polynomial includes polynomials as the simplest functions that are easy to evaluate. Therefore, polynomials play an important role in the analysis and theory of approximations. It is sufficient to mention the Taylor polynomial or the Weierstrass approximation theorem. An attempt to position polynomials in somewhat younger areas of mathematics than those previously mentioned leads to the same conclusion. The close connection of some areas of mathematics with computers, such as coding theory and cryptography, leads to the importance of polynomials over finite fields. As the notion of a polynomial can be expected to be significant for both pre-tertiary and tertiary education, it makes sense to observe a secondary-tertiary transition in terms of this notion. Expected repeated encounters with polynomials will be observed in the Croatian context, more precisely, through the secondary-tertiary transition that students of the first year of Mathematics at the Faculty of Science, University of Split are going through. The motivation to study this topic comes from the difficulties of first-year students observed through teaching courses in linear algebra. According to our knowledge, many studies mention polynomials, but most often as examples in a topic that includes polynomials. Although the cumulative effect of such studies contributes significantly to the knowledge about the realization of the notion of a polynomial in different phases of education, there is a lack of research dedicated exclusively to this notion at the tertiary level. Bolondi, Ferretti & Maffia's (2020) analysis of some Italian, Spanish, and North American high school textbooks came across different schemes for defining the notion of a polynomial. Sultan and Artzt (2011), in a book aimed at bridging the gap between mathematics that high school teachers learn at university and mathematics that they teach in high school, write the following:

> Before getting into a deep discussion of finding roots of polynomials, we review the definition of a polynomial. This is probably the most misunderstood word in secondary school mathematics. (p. 71)

Therefore, the question arises about the cause of the observed phenomena in secondary education and its possible effects on undergraduate mathematics education.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Secondary-tertiary transition is a common research topic in mathematics education (Vleeschouwer, 2010). As it includes institutional transition, the Anthropological Theory of the Didactic (ATD) is a suitable theoretical framework for the topic as it observes mathematical and didactic activities in the complexity of social institutions. Dependence of knowledge on the institution, which breaks the *illusion of transparency* of knowledge, is explained by the process of *didactic transposition* (Bosch & Gascón, 2006), which observes knowledge in four phases: *scholarly knowledge*, *knowledge to be taught* (curricula), *taught knowledge* by teachers in the classroom and *learned knowledge* by students. For an object of knowledge o in an institution I in which p is a position, the relation, in the notation $R_I(p,o)$, to o with respect to p in I can be observed (Bosch et. al., 2019). It remains to explain how o can be described by the tools of ATD. According to ATD, every human activity, including mathematical ones, can be described in terms of *praxeology* (Bosch & Gascón, 2006). Praxeology $[T/\tau/\theta/\Theta]$ consists of four components: type of tasks T; a *set of techniques* τ that can solve T; a *technology* θ that explains and justifies τ; and a *theory* Θ that is a formal argument and thus justifies θ. The ordered pair $[T/\tau]$ of *type of tasks* T and the *set of techniques* τ is called the *praxis* block of $[T/\tau/\theta/\Theta]$ and is associated with *know-how*; while the ordered pair $[\theta/\Theta]$ of *technology* θ and *theory* Θ is called the *logos* block of praxeology and is associated with *know-why* (Bosch et al., 2019). *Mathematical praxeologies* or *mathematical organizations* (MO) are classified as follows: *point praxeology* (generated by a single type of tasks), *local praxeology* (several point praxeologies with common technological discourse), *regional praxeology* (several local praxeologies with common theory). Every mathematical praxeology is "activated through the manipulation of ostensives" (Arzarello et al., 2008, p. 181), perceptual objects; whereby this manipulation is guided by *non-ostensives*, that is, concepts.

Before addressing research questions, let us introduce the necessary notations to apply the ATD tools to the topic of this paper. A relation of the high school student hs to the knowledge P of the notion of a polynomial at the end of the high school HS in Croatia will be denoted by $R_{HS}(hs,P)$. The relation of the student s_1 of the first year of the undergraduate study program Mathematics at the Faculty of Science, University of Split (designation U) to the P will be denoted by $R_U(s_1, P)$. The transition from secondary education HS to the first year of U is denoted by $R_{HS}(hs, P) \to R_U(s_1, P)$.

RQ1: How do polynomial-related praxeologies develop and differentiate through Croatian high schools? What can be said about $R_{HS}(hs, P)$?

RQ2: How do polynomial-related praxeologies develop and differentiate through the first year of U? What phenomena can be observed in the transition $R_{HS}(hs, P) \to R_U(s_1, P)$?

METHODOLOGY

The research methodology follows the phases of didactic transposition of the notion of a polynomial both in HS and U. Croatian high school mathematics curricula and textbooks are analysed to assess the prior knowledge of first-year mathematics students about polynomials, i.e., $R_{HS}(hs, P)$. Analogously, to assess $R_U(s_1, P)$, the study program, syllabi, exams and materials of first-year courses at U are analysed. This part provides information about *knowledge to be taught* in both institutions, HS and U, and the way in which the *taught* and *learned knowledge* are assessed is explained below. $R_{HS}(hs, P)$ are supplemented by the results of the questionnaire conducted among students at the beginning of the first semester of U in the academic year 2019/2020. Questionnaires were also conducted among students at the beginning of, and in the middle of the second semester to answer RQ2 more precisely. All questionnaires, with polynomials as the main topic, were conducted unannounced among mathematics students in attendance in classes of one of the first-year courses. All the above, supplemented by interviews with two high school mathematics teachers and seven university teachers at U, will enable the detection of the phenomenon of the transition $R_{HS}(hs, P) \to R_U(s_1, P)$.

KNOWLEDGE TO BE TAUGHT

High school curricula and textbooks

In this article, only the gymnasium[1] mathematics curricula, which were in force in Croatia from 1994 to 2019, will be presented[2]. The observed curricula prescribed the following contents: the notion of a polynomial, the algebra of polynomials, rational functions (first grade - age 15); the notion, graph and zero points of a second-degree polynomial, problems with extremes, quadratic inequalities, intersection of lines and parabolas (second grade - age 16). In addition to the previous contents, the curriculum of mathematical gymnasiums[3] prescribed additional and more advanced contents such

[1] In Croatia, a gymnasium is a type of high school (lasting four years) whose goal is to prepare students for further university education.

[2] The U is enrolled almost only by students who have completed some of the gymnasium programs, and the generation of students who finished high school according to the new curricula is still not enrolled.

[3] Gymnasiums with the maximum number of mathematics classes.

as reducible and irreducible polynomials, Fundamental Theorem of Algebra, the ring of polynomials of two variables. Polynomials are no longer explicitly mentioned anywhere in curricula, but a direct connection with the following contents is evident: linear equations and inequalities, linear and affine functions, the graph of a linear function (first grade); second-order curves (third grade - age 17); functions, derivation and integral (fourth grade - age 18). Although curricula seem to prescribe a functional approach to a polynomial, an analysis of high school textbook sets used in the last twenty years, shows that most authors introduce this notion in the first grade of high school in a section about algebraic expressions, with noticeable differences in discourse and its level. The observed textbooks for first grade differ in the following issues.

> In which way the notion of a variable is introduced? What does it mean? What is the relationship between notions of exponentiation, algebraic expression, and polynomial? Are operations between polynomials explained in some way, or are they introduced as rules? Is the algebra of polynomials related to the properties of real numbers? Has the notion of equality of polynomials been introduced, and if so, in what way? What is the null polynomial? Has a connection been established between the notion of a first-degree polynomial and the notion of a linear function? If so, in what way?

Comparing even the sets of textbooks of the same group of authors, a gradual decrease in the representation of logos blocks in textbooks of the observed period is noticeable. The changes in the observed period will be briefly illustrated through the example of the most used sets of textbooks intended for mathematical gymnasiums. The most used textbook from 2005 explicitly explains the difference between a constant ("predetermined number") and a variable ("any real number", "number not predetermined"). An algebraic expression is defined as "any expression that consists of variables and constants, obtained by four basic algebraic operations and parentheses", while a polynomial is defined as "an algebraic expression obtained only by addition and multiplication operations", and in the lesson on polynomial algebra is written:

> The general form of a polynomial of one variable is
>
> $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{1}$$
>
> The exponent n is called the degree of the polynomial. Coefficients of the polynomial are the real numbers $a_0, a_1, \dots, a_n$. The coefficient $a_n \neq 0$ is called the leading coefficient.

In the 2015 textbook of the same group of authors, it is commented that the constants are "special numbers", and the variables are "general numbers". An algebraic expression is defined as "any expression that consists of variables and constants related to basic algebraic operations", a monomial as "a product of a constant and a variable", a binomial as a "sum of monomials", and a polynomial as a "multi-member algebraic expression". The algebra of polynomials has been transferred to the second-grade textbook (when a polynomial is defined as a real function), and in the first-grade textbook is only briefly mentioned that "polynomials are calculated using known properties of arithmetic operations". Definitions of a polynomial in Croatian textbooks for first grade, but also in textbooks in other countries for students of similar age (Bolondi et al., 2020), are an obvious example of *didactic transposition*, i.e., adapting mathematical *scholarly knowledge* to *knowledge to be taught* in high school. For example, for the logos of the presented textbook from 2005, we certainly cannot say that it corresponds to the academic approach to a polynomial as a function, in the notation P_F; but it obviously does not define a polynomial as a completely algebraic object, in the notation P_A, because x in (1) is not a formal variable (indeterminate, placeholder) but x is an arbitrary real number.

When looking at praxeologies whose object of knowledge is a polynomial, two *local praxeologies* can be noticed in Croatian textbooks: algebraic-functional, in the notation $MO_{P_{AF}}$, and functional, in the notation MO_{P_F}. Moving away from the functional approach is noticeable when considering the most common types of problems in a section about algebraic expressions ("simplify algebraic expression", "reduce algebraic fraction") and presented techniques for solving them, which are reduced to manipulation of symbols based on polynomial algebra for P_A. Considering the organization of almost all observed textbooks, praxeologies whose types of tasks are solving linear equations also belong to $MO_{P_{AF}}$ because the lesson on a linear equation precedes the lesson on a linear function. The same is true for praxeologies whose objects of knowledge are quadratic functions in textbooks for the second grade of high school. On the other hand, MO_{P_F} consists of praxeologies about linear function in the first grade, quadratic function in the second grade, and polynomials as one of the elementary functions in the fourth grade. Attempts to unite $MO_{P_{AF}}$ and MO_{P_F} are visible in some textbooks, such as in the presented textbook from 2005. For the same purpose, in some other textbooks, a lesson on the value of algebraic expression has been included, but none of this is enough to be able to say that $MO_{P_{AF}}$ and MO_{P_F} form a regional praxeology. Finally, the reduction of the polynomial-related logos, especially in first-grade textbooks, separated the praxeologies that were associated with it.

Undergraduate study program, syllabi, and course materials

In terms of content, the first year of U has hardly changed in the last twenty years, and since the introduction of the Bologna System in 2005, it has not changed structurally. The first year, as in most undergraduate study programs in mathematics in the world, is dominated by two *modules* (Bosch et al., 2021): algebraic and analytical. The module of analysis includes the first-semester course Introduction to Mathematical Analysis (IMA), and the second-semester course Mathematical Analysis I (MAI). The algebra module includes the first-semester course Linear Algebra I (LAI) and the second-semester course Linear Algebra II (LAII).

Introduction to Mathematics (IM) course is a first-semester course aimed at bridging the gap between high-school mathematics and university mathematics. In IM, among the elementary functions, polynomials are treated as real functions of one real variable. The course covers the knowledge of polynomials prescribed by the curriculum for mathematical gymnasium, only with a more rigorous approach. The characterization of the equality of polynomials and the characterization of the null polynomial, in high school usually introduced only as rules without explanation, are proved. The supplementary literature mentions that the null polynomial theorem need not hold over finite fields, and provides counterexample: „the polynomial $f(x) = x^3 + 2x$ is a null polynomial over $\mathbb{Z}_3$, and not all its coefficients are equal to zero". Also, after the theorem on the equality of polynomials, it was noted that a polynomial can be identified[4] with a series of its coefficients, so "the variable x can be understood as $x = (0,1,0,0,\dots)$". In courses belonging to the analysis module a polynomial is defined as a real function of one real variable. Students in the IMA learn about continuity, and in the MAI about differentiation (Taylor's theorem) and integrability. Thus, through the IM, IMA, and MAI MO_{P_F} praxeologies are formalized and supplemented, and students are equipped with new praxeologies whose object of knowledge is P_F.

In the algebra module, the approach to the notion of a polynomial can be twofold: formal and functional. LAI is a first-semester course in which, among other things, basic algebraic structures are introduced: group, ring, field, and vector space. Although students are introduced to some substructures (normal subgroup, quotient subgroup) and homomorphisms, the tasks are reduced to checking the structure, or homomorphism between structures. Therefore, it can be concluded that the development of structuralist praxeologies (Hausberger, 2018) begins in this course. Polynomials over a field of real numbers with standard operations are treated as an example of a commutative unitary ring and real vector space. Unlike other courses of the first semester in LAI, a polynomial is introduced formally, as the finite formal sum of powers, i.e., polynomial in variable x is an expression of the form (1), where $n \in \mathbb{N}_0$ and $a_k \in \mathbb{R}$, for each $k \in \{0,1,\dots,n\}$. For some students, this will be a re-encounter with the definition because, in some high school textbooks, a polynomial is defined in the same way; while in others (such as observed one from 2005), some kind of hybrid definition is introduced between this formal and functional approach because $p(x) \in \mathbb{R}$. This redefines or upgrades $MO_{P_{AF}}$ praxeologies. Polynomials over an arbitrary field are also mentioned in LAI materials (and finite fields are introduced in the section dedicated to fields), but the tasks are focused exclusively on polynomials over a field of real numbers. Polynomials with coefficients from an arbitrary ring, and the functions induced by polynomials, are part of the fourth-semester course Algebraic Structures. LAII is a course dedicated to linear operators on the finite-dimensional vector (unitary) spaces, so polynomials in this course have an important role because of the characteristic and minimal polynomials of a linear operator, but also because vector spaces of polynomials to a certain degree (including null polynomial) are one of the most common examples for the domain or codomain of linear operators. Although the theory is given in full generality (over an arbitrary field), the tasks are limited to vector spaces over fields of real or complex numbers. As LAI, LAII is marked by general logos blocks and specific praxis blocks. Both approaches to a polynomial can be found

[4] Identification in this course for a student can only mean establishing a 1-1 mapping.

in the prescribed literature (example of literature in English: Friedberg et al., 2018, p. 10) for LAII, even though in some of it the explicit definition of a polynomial is missing (e.g., Hefferon, 2020). The authors seem to refer to this notion as already known (*illusion of transparency*), as evidenced by the frequent use of the phrase "usual operation" (e.g., Hefferon, 2020, p. 88-89) which refers to operations with polynomials. This may mean that the students sometimes have to conclude from the context which approach it is. Let us give one such example from the course materials (similar examples can be found in, e.g., Nicholson, 2013, p. 336, 355), in which it was necessary to examine whether the mapping $f: \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ given by the rule

$$f\big(p(x)\big) = 2p(-x) \qquad (2)$$

is linear operator, where it was previously written that $\mathbb{R}_2[x]$ denotes the set of all polynomials with real coefficients in the variable x of degree less than or equal to 2. If $p(x)$ is a formal polynomial (an expression of the form ax^2+bx+c, where $a, b, c \in \mathbb{R}$; like in, e.g., Nicholson, 2013, p. 546), then students may wonder what $p(-x)$ is. Namely, in this case, x is a formal variable, it is not a real number, and in none of the prescribed textbooks the composition between formal polynomials was introduced. On the other hand, the IM, IMA, and MAI courses insist on the difference between a function and the value of a function at a point, so in the case that p is a real function of a real variable, the ostensive (2) can also be confusing for students. In that case, some of the ostensives $\big(f(p)\big)(x) = 2p(-x)$ or $f(p)(x) = 2p(-x)$ (as can be found in some prescribed textbooks) are more appropriate. Praxeologies, whose objects of knowledge are operators of derivation or integration, are important factors in connecting the two dominant modules. In the algebra module, these operators are introduced formally (without the use of limits) on the vector space of polynomials.

TAUGHT AND LEARNED KNOWLEDGE

The interviewed high school teachers had almost the same experiences and thoughts on the issue of textbooks and equipping students with praxeologies whose object of knowledge is P. Teachers believe that textbooks are important in the teaching process, but because they consider them deficient, inaccurate, and poorly structured, they do not rely heavily on textbooks, especially in terms of explanation and lessons order. They mentioned that 30 years ago there was a school subject in mathematics gymnasiums that was dedicated only to polynomials and that this notion was gradually marginalized, so according to newer textbooks, students learn almost only about second-degree polynomials. The experiences and thoughts of the interviewed teachers are in line with the results of a questionnaire conducted at the very beginning of the first semester among 45 students. When asked to define the notion of a polynomial, about 44% of students did not provide an answer, but they often mentioned binomials and trinomials. When they were asked to provide an example of a polynomial, these students mostly answered by giving an example of a quadratic equation. Around 13% of students explicitly wrote that a polynomial is an equation, while almost 16% of students wrote that a polynomial is an expression, but they wrote it as an algebraic equation. Only one of these students knew the general form of the algebraic equation, while the others cited the general form of the quadratic equation. Around 22% of students defined a polynomial as an expression of a certain form. Only two students of that 22% wrote a general form of a polynomial in one variable, and the rest wrote a general form of a second-degree polynomial. Only 2 of 45 students defined a polynomial as a real function of one real variable, and these students knew the rule of mapping for polynomials of arbitrary degrees. Immediately after the first two questions, each student was individually asked to explain the meaning of the symbols in the definition and examples he/she had written. Almost all students used the ostensive x for the variable in their definitions and examples. When asked what x is, all but four students who knew the general form of a polynomial (whether they defined it as an algebraic expression or defined it as a function) answered that x is unknown. In interviews[5] with university teachers, we learned that teachers who teach IM are well aware that students do not distinguish between polynomials and algebraic equations. They pointed out that first-semester students generally do not see the need for characterization of equality of polynomials and characterization of null polynomial. However, the situation changed after the first semester. At the beginning of the second semester, 2 of the 30 students wrote that polynomial is an expression, but they wrote it as an algebraic equation. 60% of students defined a polynomial as a real function of a real variable. About 23% wrote the definition of a polynomial from LAI, but when they were asked what x is in that definition, all students answered that x is a variable with $x \in \mathbb{R}$. 3 out of 30 students did not answer the question at all.

Teachers who teach LAI said that the exams in which students should check whether a mapping, whose domain or codomain is a set of polynomials, is a homomorphism of groups or rings are arguably the worst solved in exams. One of the teachers said: "Students simply do not know what to do with these tasks, and they manage to pass the exam thanks to other tasks." Mentioned praxeologies appear and complement at the very beginning of the LAII course through tasks in which it must be checked whether the given mapping is a linear operator. For example, if a mapping is given with (2), students are generally not sure of the linear combination of what the mapping has to keep, so they often write $f\big(p(\alpha x + \beta y)\big)$, where $\alpha, \beta \in \mathbb{R}$. Also, when students have to determine a kernel of a linear operator, whose domain is a vector space of polynomials to a certain degree, they often solve algebraic equations. It is also interesting how students sometimes unknowingly generalize the results proved for polynomials of one variable (polynomial equality theorem and null polynomial theorem). One such example is the problem from the third questionnaire in which three linear functionals $F, G, H: \mathbb{R}^3 \longrightarrow \mathbb{R}$ were given by the rules $F(x, y, z) = 2x + z$, $G(x, y, z) = 2y$ and $H(x, y, z) = x + y + z$, and it was necessary to check whether

[5] The first part of the interview with the university teachers consisted of a general question asking them, in case they noticed, to list the difficulties that students have related to the notion of a polynomial, which spans several generations within the first-year courses they teach. For the second part of the interview, questions were asked regarding the course they teach and the answers they had provided in the first part of the interview.

$\{F, G, H\}$ is the basis for the dual vector space of $\mathbb{R}^3$. Students either didn't know how to solve this task or set out to test linear independence the way they did assume that $\alpha, \beta, \gamma \in \mathbb{R}$ are such that it holds

$$\alpha F(x,y,z) + \beta G(x,y,z) + \gamma H(x,y,z) = 0$$

from which they got $x(2\alpha + \gamma) + y(2\beta + \gamma) + z(\alpha + \gamma) = 0$ and then the system

$$\begin{cases} 2\alpha + \gamma = 0 \\ 2\beta + \gamma = 0 \\ \alpha + \gamma = 0 \end{cases}$$

but they did not know how to correctly argue how they came to the system. Usually, they only described the technique they used, and the extremely common conclusion was: "This is how it is done." Given that students have difficulty with the previous example, which is one of the simplest types of tasks when it comes to dual vector spaces, recognized in the literature as a notion that carries obstacles (Vleeschouwer, 2010), it can be assumed what happens when the environment is dual vector space of polynomials to some degree.

CONCLUSION

While Croatian high school textbooks have similar praxis blocks, they differ in logos blocks about the notion of a polynomial P, especially in first-grade textbooks. Two local praxeologies are observed: $MO_{P_{AF}}$, which leans towards an algebraic approach to a polynomial, and in some textbooks may contain some parts of a functional logos; and MO_{P_F} based on a functional polynomial approach. It is certainly difficult for a high school teacher to unite $MO_{P_{AF}}$ and MO_{P_F} in some way, so depending on how much the teacher relies on the given textbook, students can enroll in the undergraduate study U equipped with praxeologies with very different logos blocks. However, it seems that $R_{HS}(hs, P)$ is marked by the dominant type of tasks in high school whose object of knowledge is P - by solving the quadratic equation. Given the traditional organization of mathematical knowledge, followed by the organization of study U, the theory of P consists of the theory of a polynomial introduced as a function P_F and the theory of a polynomial introduced as a formal algebraic object P_A. Praxeologies whose logos are those theories do not unite until the course of Algebraic Structure in the fourth semester. From the obtained results we can conclude that $R_U(s_1, P)$ is characterized by the definition of P as a function, and techniques and technologies characteristical of P_A, which fail to be justified by functional logos. In the first year of U, the functional logos is probably still underdeveloped, as a result of the transition from high school, which is an institution dominated by the praxis blocks (Winsløw et al., 2014). In addition to the presented didactic obstacles, the results indicate that the formal approach to P could carry some epistemological obstacles, and the role of *non-ostensives* (polynomial, polynomial function, variable, formal variable, $p(x)$) should not be neglected. All the above are assumptions that still need to be explored and considered in further transitions through undergraduate and graduate study of mathematics. The departments of mathematics in Croatia must not ignore the fact that the first generation of high school students will graduate in the school year 2021/2022 according to new curricula, which can bring new phenomena in the secondary-tertiary transition.

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Difficulties and strategies of first-year students in formal reasoning using definitions - The borderline case of constant sequences

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Mastering formalism is key to learning university mathematics. It is particularly characterised by the consequent use of definitions. Specific difficulties may arise when dealing with borderline cases that satisfy a given definition although they do not look prototypical (e.g. a constant sequence). To work out the learning potential of such examples and difficulties for first-year university students, we conducted an exploratory study with 21 students investigating their coping with the borderline case of a constant sequence. We identified two major difficulties (symbolic representation and characterisation as a process) and four main strategies (change of representation, recall of known borderline cases, use of definition, use of previous content). Finally, we discuss how borderline cases can be usefully integrated into mathematics teaching.

Keywords: Teachers' and students' practices at university level; Transition to, across and from university mathematics; definitions; formal thinking; borderline cases.

DEFINITIONS IN THE TRANSITION TO FORMAL MATHEMATICS

At the beginning of mathematics studies, mathematical formalism becomes very important. While argumentation based on examples, visualisations or intuitive reasoning is common in school mathematics, university mathematics mostly relies on proofs based on axioms and definitions. In school, definitions serve to describe already known concepts. In contrast, objects are only created through a definition in higher mathematics (Edwards & Ward, 2008). The secondary-tertiary transition thus marks a transition "from describing to defining, from convincing to proving in a logical manner based on those definitions" (Tall, 2002, p. 20). Tall continues:

> This transition requires a cognitive reconstruction which is seen during the university students' initial struggle with formal abstractions as they tackle the first year of university.

Students often do not use definitions in their reasoning (Edwards & Ward, 2008). Instead, they use justifications such as the sole use of examples or argumentation based on the intuition, even if tasks cannot be solved without definitions (Alcock & Simpson, 2002; Holguin, 2016). This is partly because students are often unaware of the role definitions play in argumentation (Edwards & Ward, 2008). Thus, many of students' reasoning strategies are no longer valid in higher education (Alcock & Simpson, 2002). Students may then struggle with the new ways of working, feel excluded from the new discourse, and significantly lose motivation (Liebendörfer, 2018, chapters 9.3 & 10.4).

One approach to understanding the difficulties with formal reasoning is to distinguish between the notions of "concept image" and "concept definition". According to Tall and Vinner (1981), the concept image is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152), while the concept definition is "a form of words used to specify that concept" (p. 152). The university discourse requires the concept definition to be used in argumentations, especially in cases where concept image and concept definition seem to contradict. However, students often only refer to their concept image when arguing, even if it does not fit the definition (Edwards & Ward, 2008).

Previous research has mainly focussed on helping students to enrich their concept images and strengthen the connections to the concept definition (e.g. Alcock and Simpson, 2017). However, it lacks insights into how students learn to generally argue formally using the concept definition. To better understand students' difficulties and strategies, we aim at analysing how students deal with concepts when their concept image does not fit the concept definition. This leads to the following research questions:

RQ1: What difficulties do students have in working with the constant sequence?

RQ2: What strategies do students use to cope with these difficulties?

THEORETICAL BACKGROUND: STUDENTS' DIFFICULTIES WITH FORMAL DEFINITIONS

According to Tao (2007, p. 1), the transition from school mathematics to university mathematics includes a transition from the "pre-rigorous stage" of mathematical education to the "rigorous stage". In the "pre-rigorous" stage, which covers years at school, learning is informal and intuitive with a focus on examples and calculations. The "rigorous" stage, which is in Germany situated at the beginning of university studies, contains strongly theory-based mathematics that demands precise and formal ways of working. The "post-rigorous" stage is only attained in later years of study. When students are familiar with the rigorous mathematical ways of working, intuitive reasoning can again be increasingly used to support or guide the formal argumentations. Consequently, the initial studies is the stage in which the most rigorous mathematics is required in order to avoid common mistakes and eliminate misunderstandings right from the beginning. Rigorous work is intended to destroy false intuition and strengthen good intuition (Tao, 2007). Formal mathematics does not exclude intuition and visualisation for generating ideas, but must always consist of proofs in the end. Thus, in the "rigorous stage", students should learn what mathematically valid argumentation consists of, that is formal deductive reasoning based on definitions and axioms.

Tall (2008) adds a theoretical view on this transition given by "three worlds of mathematics" used to describe the development of mathematical thinking. The “conceptual-embodied world” is based on the perception of objects and their properties in the real world. Mathematical thoughts arise from visual objects, patterns and experiments. Then, the “proceptual-symbolic world” arises from the embodied world by the use of symbols that develop through actions and represent “thinkable concepts”, such as the concept of a number that arises from counting. The “axiomatic-formal world” refers to formal concepts based on set-theoretic definitions and logical reasoning using proofs based on axioms and definitions. While in school there is a transition from embodiment to symbolism, the transition to higher education marks the transition to formalism. At this point, students have to learn that argumentations that were still valid in symbolism are no longer so in formalism. Consequently, one of the learning goals at the beginning of the university is mastering between visualisation and the formal world of mathematics (Nardi, 2014).

The very nature of formal argumentation based on definitions can be especially demonstrated by so-called “borderline cases” (also called “pathological” or “strange examples”; Mason, 2002). These cases are examples of a given definition that violate some very typical properties that are central to the existing concept image. In borderline cases, the first intuition may no longer be certain or it may contradict the definition. Tall and Vinner (1981), for example, asked students whether the real function $f(x) = \frac{1}{x}, x \neq 0$ was continuous. Students often gave wrong, intuitive reasons, such as "the graph is not in one piece" or "the function is not defined at the origin" (p. 167) to argue that it was not continuous. A formal argumentation based on the definition of continuity would give the correct answer.

Similarly, borderline cases of sequences have been investigated. The concept image of some students includes that a convergent sequence is always either monotonically increasing or monotonically decreasing. Thus, an alternating convergent sequence is often not recognised as such. Moreover, the common idea that a sequence must not assume the limit leads to not accepting a constant or finally constant sequence as convergent (Vinner, 1991). Some misconceptions also relate to the acceptance of a sequence as such. The concept image of many students only includes sequences that are defined by a single term with a variable, typically “n”. As a result, sequences are not recognised as such if they are defined differently for even and odd indices or if they are constant (Roh, 2005; Tall & Vinner, 1981). Such sequences thus represent borderline cases, especially with regard to the definition of a sequence.

METHODS AND STUDY DESIGN

To explore students’ difficulties (RQ1) and strategies (RQ2), we conducted a study with 21 students of a preliminary course for mathematics and computer science students and for preservice mathematics teachers for higher secondary schools. For this group, the intended study programme will build strongly on formal mathematics, which is why mastering formal reasoning is an important learning goal for them. Of the participants, 12 were female and 9 male, and they were between 18 and 23 years old. All participants of the preliminary course were invited and participation was possible on a voluntary basis either alone or in pairs. This resulted in 13 interviews.

The interviews took place via Zoom and were audio- and videographed. The screen of a participant was shared and captured in order to be able to follow the editing processes. Beforehand, an interview guideline was prepared and the three interviewers were introduced to the guideline, the mathematical content, and the aim of the study.

At the time of the study, in the third week of the preliminary course, basics of logic as well as sets and functions had already been worked on, as well as calculating with absolute values, inequalities and sums. After that, the students worked on a digital learning environment on the topic of sequences, in which the use of definitions was to be promoted. It was completed by all preliminary course participants instead of taking part in the synchronous online lecture and the tutorials. The students were to work independently on the learning environment, in which sequences were first introduced and defined. In doing so, they were to watch existing videos and explanations as well as solve some tasks provided. A sequence was defined in the learning environment as follows:

> A sequence (of real numbers) is a function $f: \mathbb{N} \longrightarrow \mathbb{R}, n \mapsto f(n) \coloneqq a_n$.
>
> The values are called the elements of the sequence. The whole sequence is usually notated in a shorter form: $(a_n)_{n\in\mathbb{N}}$.

After examining some examples of sequences and exploring possible representations of sequences (piecewise defined, recursive, etc.), the question was asked in the learning environment whether $(a_n)_{n\in\mathbb{N}} = (12)_{n\in\mathbb{N}}$ correctly defines a sequence. Only after working on this task were they introduced to the definition of a constant sequence.

Once the students had completed these tasks, we conducted semi-structured guideline interviews were we reviewed the tasks and asked subject-specific questions about this section of the learning environment. This paper refers to the review of the task presented above. Students were asked how they had worked on the task. The correct answer was already known to the students at the time of the interview, so that it was a reflection on their own working process. If students did not explain their answers on their own, they were asked to clarify their difficulties in more detail.

The interviews were transcribed and then analysed using a structuring qualitative content analysis (Kuckartz, 2016). According to our research questions, the students' statements were first deductively coded for difficulties and strategies. The statements in these categories were then inductively categorised for emerging themes. The resulting codes were allowed to overlap, and in fact, several categories were recognised simultaneously in some statements.

RESULTS: STUDENTS' DIFFICULTIES AND STRATEGIES

We report the analysis of the part of the interview relating to the above task, which lasted no longer than three minutes each. All participants had worked the example in advance and most of them had reported difficulties or at least irritations when working the selected task. Inductive coding yielded two major problems and four main strategies, see figure 1.

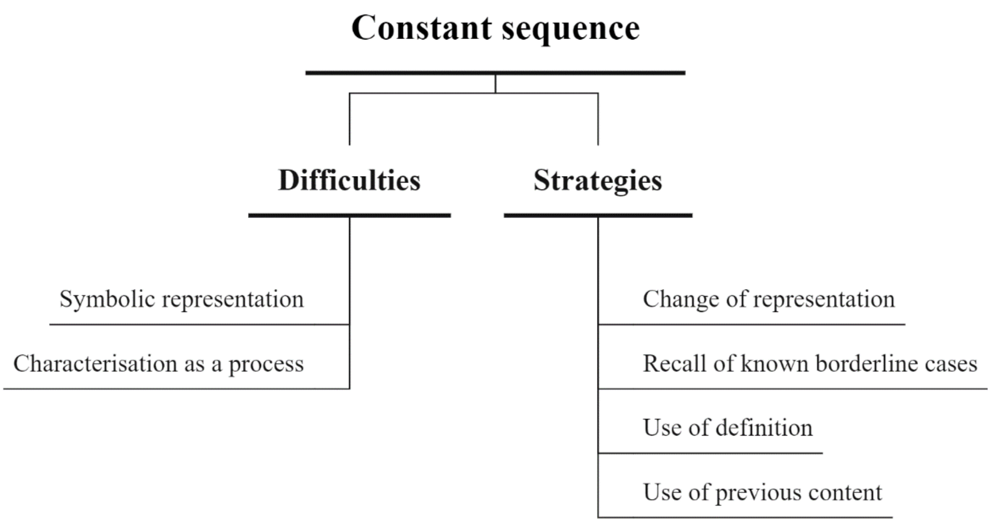

Figure 1: Category system identified in the qualitative analysis.

To answer the two research questions, we describe the subcategories and illustrate them with student utterances. The excerpts from the transcripts were translated from German, with some filler words omitted for better readability. We indicate the participants' numbers in brackets.

Difficulties in working with the constant sequence (RQ1)

The two main difficulties with the constant sequence were identified in terms of symbolic representation and in terms of the characterisation of a sequence as a process.

The first difficulty concerning the symbolic representation refers to students' uncertainty caused by the unfamiliar symbolic representation of the defining term. Whereas in the definition of a sequence and in all the examples presented before, the defining term included an "n", this was now no longer given, so that students were irritated:

> Before, we only had examples where *n* was somehow contained in the second part of the term. And that's why we somehow assumed that *n* always had to be integrated in it. (1)

Since the representation differed from the previous ones, the assumption was made that it was not a sequence: "In my head it was still, there has to be a variable, otherwise it can't fit." (13)

The second main difficulty describes the characterisation of a sequence as a process. In this case, students, similar to the understanding for functions, assumed a covariation of the sequence elements, so that they must differ from the previous one. This also leads to the sequence not being recognised as such: "And it is often the case that a function, depending on what value for *x* you insert, a different *y* comes out, and here it was more or less fixed." (6) As the values did not differ in the given example, there was also a contrast to the previous examples and the concepts built up through them: "Because I just thought, the next value must be different from the one before or/ [...] I just had that in my head, that there was a change. And has to happen." (4)

Strategies in working with the constant sequence (RQ2)

We identified four main strategies students used to address these challenges: the change of representation, the recall of known borderline cases, the use of definition, and the use of previous content.

The first strategy was the change of representation. In this case, a representation of the sequence was used, which differed from the defining term, to answer the question. We identified two representations that were used by the students. The first representation was the graph of a sequence: "I first thought about what such a graph could look like." (15) The second was the tabular representation to decide whether there is a sequence at hand: “So I first made a table for the different *n*'s and then also for the corresponding values.” (20)

The second strategy in this context was to refer to similar borderline cases in the same sense, thus violating the same properties. Their properties were then transferred to the current example to answer the question. First of all, the constant sequence can be compared to the constant function from which the students knew that this still represents a function:

> So I just imagined a function that is simply like/ Well, we know *f* of *x* equals twelve, then twelve is simply assigned to every value of *x*. And that's how I imagined it for the sequence. (5)

Another borderline case that was familiar to the students was the sums in the sigma-notation without an indicating variable, for example $\sum_{k=1}^{5} 10$. The fact, that the notation without this variable is still valid, was transferred to the example of the constant sequence:

> I think we've already had similar cases with the sums, where you have *k* equal to one to five and then the sum sign somehow just shows ten. [...] Yes, but I also transferred that to it, actually. (1)

The third strategy was the formalistic use of the definition of a sequence. This relates to the argumentation that the example represents a function, for example by stating that every natural number is assigned to twelve:

> And then I thought to myself, okay, then I looked again briefly at the definition, and the definition was simply that a value is really assigned for each *n*, and in principle, that is because the value 12 is assigned for each value. And then I said, okay, then of course it's correct. (14)

The fourth category of strategies consists of the use of previous content to find the answer. This includes the content of the learning environment such as previous examples or explanations as well as content of the preliminary course or other sources. For example, some students compared the given task with previous examples and tasks in order to transfer them to the current example: “I first had a look at what this expression in brackets means and I looked at this illustration up here again, that is, in the previous task.” (21) This strategy cannot be defined as narrowly as the three strategies before it. It can also directly merge into one of the strategies named before it. However, it did not always do so, so we consider it to be a strategy in its own right.

We identified some overlaps in the use of strategies as, for example, the comparison to functions and the graphical representation: "I always make the comparison to functions. I thought about it like just a straight line, it's also a function." (14)

In many cases, the definition was only applied after preliminary considerations had already been made, for example the graphical representation from above:

> „I first thought about what such a graph could look like, and then I looked at the definition, that an *n* is always assigned to a real number, that is, that it is always a function, and then it fit.“ (15)

The comparison with other borderline cases was also partly validated by definition. The student comparing the sequence to a constant function continues:

> „And that's how I imagined it for the sequence. Yes, then the condition was that each *n* from the natural numbers should be assigned a concrete value. I then thought that the sequence is correctly defined.“ (5)

DISCUSSION

In an exploratory study, we confronted 21 students of a preliminary mathematics course with a constant sequence to answer the questions of what difficulties and strategies they show in reasoning whether this was actually a sequence. The analysis of the 13 interviews (mostly in pairs) resulted in two main difficulties (symbolic representation and characterisation as a process) and four strategies (change of representation, recall of known borderline cases, use of definition, use of previous content) to face them.

Limitations

The small sample yielded only a limited number of difficulties and strategies. As the excerpt presented was part of a larger interview, the focus of the interview was not only on this task, which is why no detailed follow-up questions had been prepared beforehand. It should also be noted that the results depend strongly on our choice of the example. For instance, the constant sequence is an example that targets notation and graphical representation, so there are corresponding difficulties here as well. So one should also look at examples, which are borderline cases of the definition in a different way.

Theoretical implications

Whereas the first difficulty is already known from the literature and was reproduced in this study, the second difficulty based on the dynamic notion of sequences seems new. Both difficulties relate to a contrast and apparent contradiction to students’ previously constructed concept images. This creates uncertainty and thus the assumption that it is not a sequence, resulting in tensions between concept image and concept definition.

We propose not to conceptualise such tensions based on individual examples, but to categorise them in terms that are more general. In this particular case, we have already found two prototypical categories: The tensions due to familiarity with certain symbols and the tensions caused by process-based conceptions. In both respects, the object dealt with does not look like what is familiar. These categories fit into Tall's (2008) proceptual-symbolic world, as this world refers especially to symbols and processes. Future research could classify the main tensions between concept image and concept definition of objects from first-year mathematics to delineate a learning trajectory for a way into mastering formalism.

Concerning the strategies, we do not discuss the fourth strategy because it is not specific to our question. The other strategies can be classified according to Tall and Vinner (1981). The first two strategies refer to the concept image that either had been built up about sequences themselves (several representations) or was present about other concepts (other borderline cases). These strategies help students to think about what the solution can be. They also serve to "revisit your intuitions on the subject" (Tao, 2007, p. 2) to get a personal conviction of the solution. The third strategy is based on concept definition and is the one that is finally formally accepted.

It is worth noting that many students first used one of the two strategies corresponding to the concept image and then moved on to the definition. This suggests that the change of representation and the search for analogies to known borderline cases may be helpful in finding a solution. Through these two strategies, students gain some conviction in what the solution will be. We may thus think of a three-step approach to answering questions that heavily rely on mathematical formalism. As students will most likely have some intuition from the beginning, the first step is to experience and become aware of the difficulties in form of tensions. Such an awareness seems to have to be developed in the transition to formal mathematics in general, for instance also when evaluating theorems. Once students start critical reflecting on the formal definition, their own intuition needs to be doubted and questioned in the second step. The third step is then the transition to formal reasoning in order to finally answer the question. The first two identified strategies (change of representation, recall of known borderline cases) relate to reasoning by analogy and are allowed and desired in the second step, but no longer in the third step, where only the third, formal strategy is allowed. However, the first two strategies help in the second step to distinguish between good and bad intuition in the sense of Tao (2007).

Since we have already identified strategies to address such problems without concrete instructions to the students, it seems that “handling the formalism” can be seen as a competence of its own to resolve tensions between concept image and concept definition. This is a general competence that is not linked to concrete content. Thus, the learning goal of handling the mathematical formalism can be described more precisely in an overarching way and independent of content. That this learning goal is also relevant for teachers is shown by school-relevant borderline cases such as the question of whether $0,\overline{9}$ equals 1. This question, which often leads to the intuitive answer that $0,\overline{9}$ less than 1 (Tall & Vinner, 1991), can also be dealt with using the three-step approach described above, for example by changing the representation and using the number line in the second step.

Practical implications

Following the supposition that "handling the formalism" can be conceived as a general competence, university teachers have to decide whether they want to include fostering this competence as a learning goal in their lectures. This will depend on the teaching context. For example, this competence is probably less important in mathematics for engineering than in pure mathematics. Mason (2002) writes: "There is considerable controversy between lecturers as to whether it is advisable to show students strange examples" (p. 25). Based on our findings concerning the three-step approach, which suggest that exactly such examples can be supportive in moving to formal reasoning, we suggest that the use of such examples is helpful in fostering this competence. They can help to establish learning strategies and to learn the distinction between good and bad intuition. This means that such examples should not be avoided, but should be used explicitly to correct false intuitions. It seems particularly helpful to use borderline cases when the content is still simple, as the strategies described can then be made comprehensible without many difficulties on the content. Especially the last strategy, the use of definitions, is new at the university and has to be taught to the students before they can apply it themselves. We therefore suggest creating borderline cases as early as possible, linking them directly to definitions, and explicitly making analogies to other examples in order to provide the students with this strategy.

The difficulties with mathematical formalism are a part of the difficulties in the transition from school to university. Many students struggle because they are required to prove theorems based on definition. Our research refers only to a first step, namely the coordination of concept image and concept definition. However, this step seems necessary before students can realize how important definitions are and that they should be used in proofs (Alcock & Simpson, 2002). In particular, students are sometimes helpless because their known ways of argumentation based on analogy or visual reasoning are no longer accepted. Yet, we could see that strategies based on analogies and images may be valid and helpful to clarify students' own intuition. Only when they need to move to the formal argumentation, these strategies no longer help. It might thus help them to reflect on the roles of concept image and concept definitions in dealing with formalism.

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Didactic transposition of concavity of functions: From scholarly knowledge to mathematical knowledge to be taught in school

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*In an institution **I**, a praxeology $\wp$ is generally a modification of a praxeology $\wp^*$ coming from a collective of institutions **I***, where the modification is conceptualised by the phenomenon of institutional transposition. This paper presents a praxeological analysis of the concept of concavity of functions as expressed in a mathematics textbook for Norwegian upper secondary school. The analysis shows how the institutional (here, didactic) transposition has "moved" the mathematics presented in upper secondary school away from the mathematics taught at the university and how this transposition has resulted in a poor* logos *block of the mathematics to be taught.*

Keywords: Transition to, across and from university mathematics; concavity; didactic transposition; praxeology; teaching and learning of analysis and calculus.

INTRODUCTION

The theory of didactic transposition (Chevallard, 1991) was introduced in 1985 by Yves Chevallard. The didactic transposition process refers to the transformations an object of knowledge undergoes from the moment it is produced by scholars, to the time it is selected and designed by noospherians to be taught, until it is actually taught (and studied) in a given educational institution (Chevallard & Bosch, 2014). When doing didactic transposition analyses, the empirical unit is enlarged to encompass data from outside of the mathematics classroom. This reflects the insight that to study teaching and learning of mathematics in the classroom, it is not enough to study what students and teachers are thinking and doing: the mathematics taught becomes itself an object of study. The researcher studies transformations between the following instances: the scholarly mathematical knowledge as it is produced by mathematicians; the mathematical knowledge to be taught as officially formulated in curriculums and as presented in textbooks; the mathematical knowledge as it is actually taught by teachers in classrooms; and the mathematical knowledge as it is actually learned by students (Bosch & Gascón, 2006). The didactic transposition process taking place between the mentioned instances is illustrated in Figure 1.

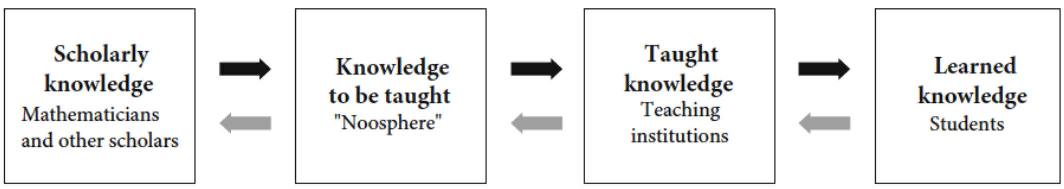

Figure 1. Didactic Transposition Processes (adapted from Chevallard & Bosch, 2014, p. 171)

In the research presented here, we have studied the didactic transposition of concavity of functions from scholarly knowledge to mathematical knowledge to be taught in secondary school. We have analysed transformations between the mathematical organisation of concavity of functions (in particular, its *logos* block) as expressed in a university textbook on calculus (Lindstrøm, 2016) and the mathematical organisation of the same theme as expressed in a mathematics textbook for Grade 12 (Kalvø et al., 2021). As asserted by Winsløw (2022), the calculus presented in mathematics courses at the university is indeed the result of a didactic transposition of the calculus of the 18th century. So, the mathematics presented in the university textbook analysed here is itself a body of transposed knowledge. An analysis of the transformations that this knowledge has undergone from the scholarly mathematical knowledge is however beyond the scope of this paper.

The transformations of concavity of functions that have taken place between the university textbook and the school textbook have been studied through a praxeological analysis. Generally, praxeological analyses, together with analyses of didactic transposition processes that specific knowledge objects have undergone, help us understand which mathematics is taught in school, and why it has become so. Our study centres on the following research question: *What are the transformations that the notion of concavity of functions has undergone during the didactic transposition process from the knowledge taught at the university to the knowledge to be taught in Norwegian upper secondary school?*

THEORETICAL TOOLS

The study reported here has been conducted in the framework of the anthropological theory of the didactic (ATD; Chevallard, 2019). A praxeology of a body of knowledge is in the ATD a *model* of this knowledge. This model is a unit composed of four components: T, τ, θ and Θ (sometimes referred to as "the four t-s"), where T is a *type* of tasks, τ is a technique (or a set of techniques) to solve the tasks, θ is a *technology*, that is, a discourse describing and explaining the techniques, and Θ is a theory, that is, a discourse justifying θ. T and τ belong to the *praxis* block of the praxeology, whereas θ and Θ belong to the *logos* block. A praxeology $\wp$ is written: $\wp = [T / \tau / \theta / \Theta]$.

A praxeology $\wp$ is usually the product of the activity of an institution or a collective of institutions $\boldsymbol{I}$. It is often the case that this "product" is the result of an *institutional transposition* of a praxeology $\wp^*$ living in a collective of institutions $\boldsymbol{I}^*$ to a praxeology $\wp$ that has to live within $\boldsymbol{I}$ and thus has to satisfy a set of conditions and constraints specific to $\boldsymbol{I}$ (Chevallard, 2020). This is the case when $\boldsymbol{I}$ is a collective of "didactic" institutions, that is, institutions declaring to teach some bodies of knowledge, such as secondary schools for example. This is referred to as *didactic transposition* of $\boldsymbol{I}^*$ into $\boldsymbol{I}$. Often, in this case, it is observed that $\wp$ is a "simplification" of $\wp^*$ through various processes. For example, it may be that a certain type of tasks T in $\boldsymbol{I}^*$ becomes useless in $\boldsymbol{I}$. It may be that a particular technique is inefficient, or that it leads the average user to make many mistakes. Moreover, in the process of transposition, it is likely that $\wp^*$

has been greatly simplified and thus distorted so the technology does not really justify the proposed technique. Finally, the theoretical elements are often implicit, repressed, or taken for granted. Therefore, for those who want to analyse a praxeology living in a given institution, the theoretical component is hard to bring to light. This is shown in the analysis section below.

METHODICAL APPROACH

The methodical approach is essentially that of *didactic transposition analysis* (Chevallard, 1991). The didactic transposition analysis of the concerned body of knowledge $\mathscr{k}$ presented here involves a comparison of praxeological analyses of two different "copies" (i.e., "transposed" versions) of $\mathscr{k}$ as they appear in two different institutions. The data are the mentioned textbooks (in Norwegian)[1]: The first is *Kalkulus*, an introductory textbook on calculus for the university, published in 2016. It is written by Tom Lindstrøm, professor of mathematics at the university of Oslo. The second is *Mønster* [Patterns]: *Mathematics R1*, a Grade 12 mathematics textbook for a theoretical programme at upper secondary school, preparing for university studies in science, technology, engineering, and mathematics. It is part of a textbook series for the national curriculum since 2020, written by Tove Kalvø, Jens C. L. Opdahl, Knut Skrindo, and Øystein J. Weider, all serving as teachers in mathematics at (different) upper secondary schools. The reasons for the choice of these books are: *Kalkulus* is an introductory textbook used in the first calculus course taken by students enrolled in teacher education programmes for Grade 8–13 at several Norwegian universities; *Mønster* is part of a brand-new textbook series for the theoretical programme; it is not a revised version of an old series as are two other textbook series for the same programme (i.e., Borgan et al., 2021; Oldervoll et al., 2021).

ANALYSIS OF A DIDACTIC TRANSPOSITION PROCESS

We present here an analysis of *concavity of functions* as treated in the textbook *Mønster* (Kalvø et al., 2021, pp. 208–224) and compare it with the treatment of the same topic in the university textbook *Kalkulus* (Lindstrøm, 2016, pp. 313–321)—which we regard as closer to scholarly knowledge. The aim is to bring to light the didactic changes this knowledge object, as presented in *Mønster*, has been subjected to.

The *Logos* Block of Concavity of Functions in *Kalkulus*

In Chapter 6.4 of *Kalkulus*, with heading "Discussion of Curves", there is a section entitled "Convex and Concave Functions" (pp. 283–288).[2] The author starts with a geometrical definition of the concepts of *convex function* and *concave function*:

> **6.4.5 Definition** The function f is called *convex* on the interval I if every time we select two points $a, b \in I$, then no point on the line segment between $(a, f(a))$ and $(b, f(b))$ will be

[1] Quotations from these textbooks have been translated into English by the first author.

[2] For functions, being convex and concave is synonymous with being "concave up" and "concave down", respectively (as used by e.g. Adams & Essex, 2018).

below the graph of $y = f(x)$ (see Figure 2).[3] We say that f is *concave* on I if every time we select two points $a, b \in I$, then no points on the line segment between $(a, f(a))$ and $(b, f(b))$ will be above the function graph (see Figure 3). (Lindstrøm, 2016, p. 314)

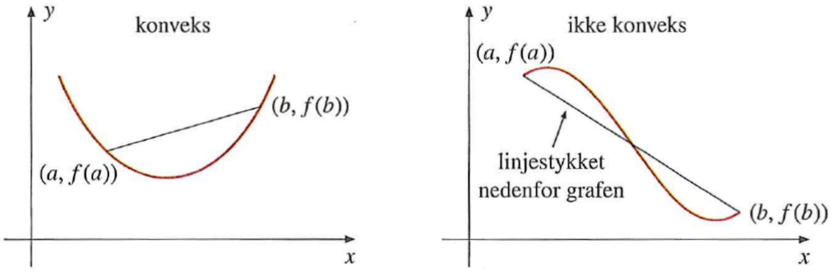

Figure 2. Convexity of a Function (taken from Lindstrøm, 2016, p. 314)

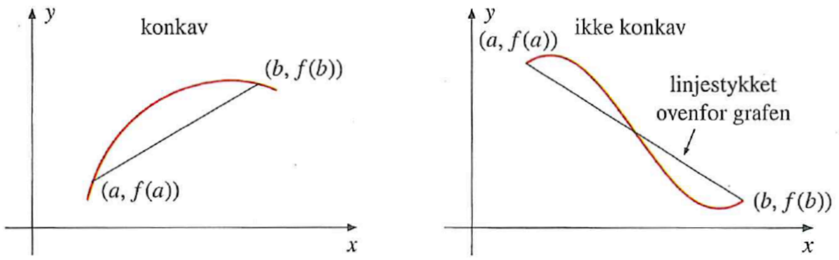

Figure 3. Concavity of a Function (taken from Lindstrøm, 2016, p. 314)

The author continues to build up elements in the *logos* block, which allows him to deduce a connection between concavity and the second derivative of a function twice differentiable. To be able to use the mean value theorem in the proof of the theorem that establishes the sought relationship, the following lemma using difference quotients is presented (p. 315):

6.4.6 Lemma A function is convex on an interval I if and only if the following applies. For all points $a, b, c \in I$ such that $a < c < b$, we have

$$\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(c)}{b-c}. \qquad (1)$$

Correspondingly, f is concave on I if and only if

$$\frac{f(c)-f(a)}{c-a} \geq \frac{f(b)-f(c)}{b-c}. \qquad (2)$$

The author writes that the two statements can be proved the same way and presents a proof for the convexity part of the lemma:

Proof: Assume first that f is convex. Then the point $(c, f(c))$ cannot be above the segment connecting $(a, f(a))$ and $(b, f(b))$, and we must have the situation shown in Figure 4a.

When we compare the slopes k, k_1, and k_2 of the three segments in the figure, we see that $k_1 \leq k \leq k_2$, which means that

[3] Figures taken from *Kalkulus* used in this paper are given titles by the authors (figures are untitled in the source). Moreover, we have renumbered them to have continuous numbering of figures.

$\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(c)}{b-c}$.

If we omit the middle part, we get (1). (Lindstrøm, 2016, p. 315)

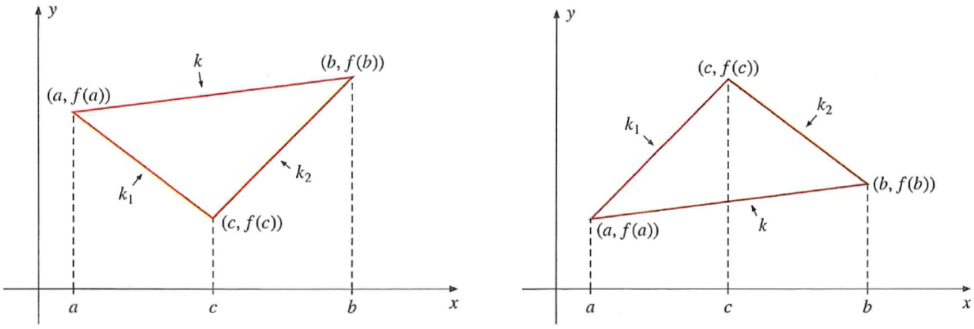

Figure 4. Convexity (left image) and Non-convexity (right image) (taken from Lindstrøm, 2016, pp. 315–316)

The next step of the author is assuming that *f* is *not* convex and showing, consequently, that (1) does not hold:

> Since *f* is not convex, we can find points a, b, $c \in I$ so that $a < c < b$ and (c, f(c) is above the segment connecting (a, f(a)) and (b, f(b)); this means that we have the situation shown in Figure 4b.
>
> We now see that the ratio between the slopes is $k_1 > k > k_2$ – in other words
>
> $\frac{f(c)-f(a)}{c-a} > \frac{f(b)-f(a)}{b-a} > \frac{f(b)-f(c)}{b-c}$.
>
> If we omit the middle part, we get the inverse inequality of (1). ■ (Lindstrøm, 2016, pp. 315–316)

Then everything is ready to state the theorem that is the central theoretical element in the author's mathematical organisation of the *logos* part of concavity of functions:

> **6.4.7 Theorem** Assume that f is continuous on an interval I and that $f''(x) \geq 0$ for all inner points $x \in I$. Then f is convex on I. If instead $f''(x) \leq 0$ for all inner points of I, then f is concave on I. (Lindstrøm, 2016, p. 316)

The proof addresses the convexity part of the theorem, using the above lemma:

> **Proof**: Choose three points a, b, $c \in I$ so that $a < c < b$. According to Lemma 6.4.6, it suffices to prove that $\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(c)}{b-c}$. By the mean value theorem, there exist two numbers $c_1 \in (a, c)$ and $c_2 \in (c, b)$ so that $\frac{f(c)-f(a)}{c-a} = f'(c_1)$ and $\frac{f(b)-f(c)}{b-c} = f'(c_2)$. Since $f''(x) \geq 0$, f' is increasing and, consequently, $f'(c_2) \geq f'(c_1)$ [because $c_1 < c < c_2$]. Hence $\frac{f(c)-f(a)}{c-a} = f'(c_1) \leq f'(c_2) = \frac{f(b)-f(c)}{b-c}$. ■ (p. 316)

After this, two examples are given that discuss convexity/concavity. The second example introduces the notion of *inflection point* with this formulation: "a is an inflection point for *f* if *f* is continuous at a and there exists an $\varepsilon > 0$ so that *f* is convex on one of the intervals $(a - \varepsilon, a)$, $(a, a + \varepsilon)$ and concave on the other" (p. 318). An inflection point is a point where a function changes from being concave to being convex or vice versa. This is succeeded by 18 tasks that address appearances of curves more broadly.

The above is a brief account of the "scientific" treatment of the concept of concavity in *Kalkulus*, which describes the constituent parts of the *logos* block of concavity: 1) a *definition* of convex/concave function; 2) the *mean value theorem* (proved in a previous section) used in the proof of a lemma to be used in the proof of a central theorem with respect to the concept at stake; 3) the mentioned *lemma* (with proof); 4) the central *theorem* (with proof) declaring a connection between the sign of the second derivative and the concavity/convexity of a function; 5) a *definition* of inflection point.

The Mathematical Organisation of Concavity of Functions in *Mønster*

Here, we analyse the treatment of concavity of functions in *Mønster* (Kalvø et al., 2021) One remarkable point is that the section devoted to concavity issues is entitled "The Second Derivative". The question of the concavity of functions is thus presented as an application of the notion of second derivative. The words concave and concavity do not appear in the textbook: they are replaced by the expressions "hollow side" (*hul side*)[4]—the side that faces either down or up—and "curvature" (*krumning*), respectively. We will see that this is a "symptom" of the treatment of concavity by the given textbook. These notions appear in the following passage:

> We compare this with the graph of f and see the following:
> • When f'' is negative, f' is decreasing and the graph of f turns its *hollow side* down.
> • When f'' is positive, f' is increasing and the graph of f turns its *hollow side* up.
> A function with a graph turning its hollow side up or down is not linear. We say that the graph curves, and we mark the *curvature* of the graph with an arc below the sign line (see Figure 5). (Kalvø et al., 2021, pp. 211–212)

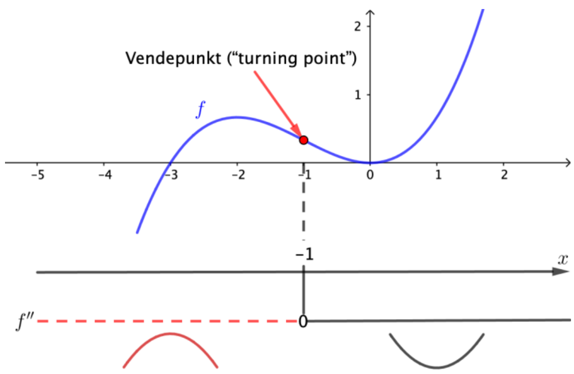

Figure 5. Sign Line for the Second Derivative (adapted from Kalvø et al., 2021, p. 211)

In this way, words that are traditional in mathematics (concave/convex, concavity/convexity) but which, a priori, mean nothing to the students, are replaced by expressions (hollow side facing down/up, curvature) that make sense in everyday

[4] Throughout the paper, italicized words in parentheses refer to Norwegian words used in *Mønster*.

language, which in this case have a metaphorical value, and which are used here as *definitions*. The same is true for *inflection point*, where the authors use the notion "turning point" (*vendepunkt*).

> The point where the graph goes from facing the hollow side up to [facing] the hollow side down (or vice versa), is called the *turning point*. At the turning point, the sign of the derivative changes. (p. 211)

The *technology* of this technique (using a sign line for the second derivative) is in fact reduced to a minimum. The authors of the textbook have adopted a "naturalistic" approach to functions. They do so by considering a specimen function regarded as generic, in this case the function defined by $f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2$ for all $x \in \mathbb{R}$. The graphs of f and $f'(x) = \frac{1}{2}x^2 + x$ are shown in Figure 6 (adapted from Kalvø et al., 2021, p. 211). Looking at the graph of f, we "see" that f first increases, reaches a maximum at a point that appears to be –2, then decreases and reaches a minimum at $x = 0$, before increasing again. Let us then try to determine "visually" the intervals in which f is either concave or convex.

The function f is first concave, up to a value x_0 somewhere between –2 and 0; then it becomes convex after x_0. How can we determine x_0?

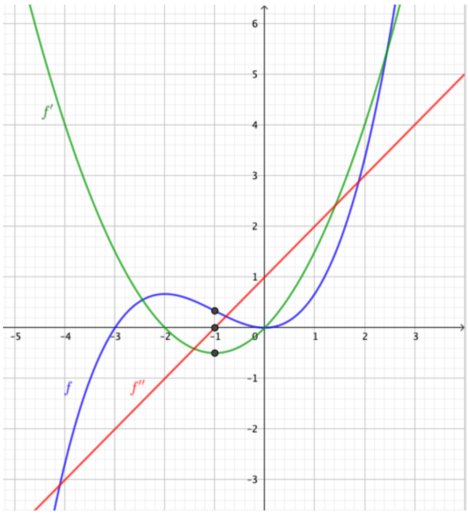

Figure 6. The Graphs of *f*, *f*′, and *f*″

To do this, we need to look not at the values of the derivative, but at how the derivative varies—that is, how the slope of the tangent to the graph of f' changes.

Instead of examining the graphs of f and f', it is technically more concise to simply examine the graph of f'' (see Figure 6). Here, $f''(x) = x + 1$. The second derivative f'' is therefore represented by a straight line with slope 1. It is strictly negative when $x < -1$, zero for $x = x_0 = -1$ and strictly positive when $x > -1$. The reader can examine two animated GIFs, where the first GIF (first link) highlights the *values* (positive, negative) of f' while the second GIF (second link) highlights the fact that f' is *increasing* or *decreasing*.

> https://commons.wikimedia.org/w/index.php?title=File:Tangent_function_animation.gif&oldid=507127692
>
> https://upload.wikimedia.org/wikipedia/commons/7/78/Animated_illustration_of_inflection_point.gif.

If we look at the treatment of concavity in *Mønster* as a certain ***praxeology***, we can analyse it as explained in the following paragraphs.

The *type of tasks T* studied is formulated more allusively than explicitly. A task *t* of type *T* consists in determining the curvature of the graph of a given function f and finding its possible turning point(s). This involves determining intervals of $\mathbb{R}$ on which f is either "concave down" or "concave up" by examining a "sign line" as shown in Figure 5.

The notion of concavity is *hinted at*, rather than properly defined. This is made possible, among other things, by a linguistic "manipulation" which is one of the keys to the didactic transposition carried out by the authors: the words *concavity* and *inflection point* do not appear. "Concavity" is replaced by the expression "hollow side down/up" associated with the expression "curvature"; "inflection point" is replaced by "turning point". While the words "concavity" and "inflection point" are relatively opaque words in ordinary language, and therefore require comments, if not a precise definition, the expressions by which they are replaced belong to everyday language and are known to all, which authorizes the authors not to say more about them.

The *technique* τ to perform a task $t \in T$ consists in calculating the second derivative f'' (differentiating a function: type of tasks T_1) and studying its sign (determining the sign of a function: type of tasks T_2). In essence, both T_1 and T_2 are assumed to have been studied beforehand and to be now largely routinised. The only new feature is that, given the function f, the derivatives f' and f'' must be calculated successively.

The *technology* θ of the technique τ is reduced to next to nothing. One would expect that when f is concave down, the authors would point out to their readers that *the slope of the tangent decreases*. Instead, they invite them to observe, on *the graph of f'*, that f' is decreasing. Even more so, they do not care to mention the equivalence of various properties such as

- the slope of the tangent decreases;
- the curve is below its tangents;
- for any point a on an interval I on which f is defined, the function $r_a\colon x \mapsto r_a(x) = \frac{f(x)-f(a)}{x-a}$ decreases. (In Figure 7, we have for example: $r_a(x_1) > f'(a) > r_a(x_2)$.)

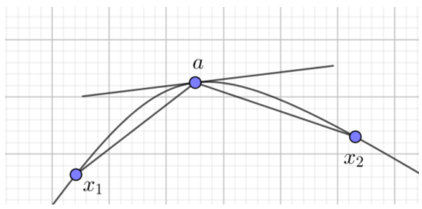

Figure 7. Chords and Tangent for a Function *f*

The third of these properties corresponds to Lemma 6.4.6 in *Kalkulus* (explained in the previous section).

Finally, about the praxeology of concavity in *Mønster*, we have uncovered that there is no *theory* Θ justifying the technology θ. This can be explained by two factors: first, θ is almost non-existent; second, the authors make no real attempt to justify what little exists of θ. We would like to state here what the theory (according to the ATD) would be: a system of statements (definitions, axioms, lemmas, theorems, corollaries…) from which we can derive a justification of θ. Let us suppose, for example, that we want to justify the fact that, when the derivative f' decreases, “the curve is below its tangents”. We have (see Figure 8): $f(a) + \varepsilon f'(a) - f(a + \varepsilon) = \varepsilon f'(a) - [f(a + \varepsilon) - f(a)]$. According to the mean value theorem, there exists $\gamma \in (0, 1)$ such that $f(a + \varepsilon) - f(a) = \varepsilon f'(a + \gamma\varepsilon)$. We thus have: $f(a) + \varepsilon f'(a) - f(a + \varepsilon) = \varepsilon f'(a) - \varepsilon f'(a + \gamma\varepsilon) = \varepsilon[f'(a) - f'(a + \gamma\varepsilon)]$. Since f' decreases, $f'(a) > f'(a + \gamma\varepsilon)$ and therefore $f(a) + \varepsilon f'(a) - f(a + \varepsilon) = \varepsilon[f'(a) - f'(a + \gamma\varepsilon)] > 0$. ■

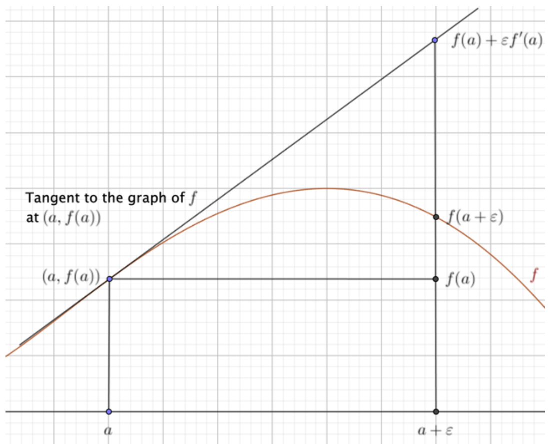

Figure 8. Decreasing Derivative

In that case, we could look for the mathematical “principles” that justify the mean value theorem and the tools used to establish it (e.g., Rolle’s theorem).

DISCUSSION

The presentation of concavity of functions in the secondary school textbook is but a “technical notice” expressed in a casual way, with as little mathematical “*logos*” as possible, most likely to make it accessible to a wider range of students. This contrasts with the presentation of the same theme in the university textbook, where we found a *logos* block consisting of definitions and proved results (theorems, lemma). In the school textbook, the notion of concavity has been substituted by an *application* of the notion of second derivative and, consequently, there is an exclusion of questions where concavity could have come into play. There are two other mathematics textbook series for the theoretical programme in upper secondary school in Norway: one is written by Borgan et al. (2021), the other by Oldervoll et al. (2021). They have a very similar treatment of concavity of functions, using exactly the same notions as the textbook analysed here.

How can we summarise the effect of didactic transposition on the notion of concavity of a function as it manifests itself here? The main fact is that, while in the university presentation, the *graphical* notion of concavity is *mathematised*, in secondary school textbooks it remains non-mathematised: concavity is *to be seen* on the graph of the function. At best, authors simply translate this *visual* property by saying that the slope of the tangent to the curve decreases or increases. This visually established property is then translated mathematically by the sign of the second derivative. The crucial gain is obvious: the subtle work required to mathematise the graphical notion of concavity is avoided, so that its presentation is accessible to a wider audience.

Another gain stems from an “iron law” of curriculum crafting: a new item benefits from appearing as an “application” of an established item—here the notion of second derivative. Is there a loss? Yes, there is. Whereas, at university, under appropriate regularity conditions, one can *prove* that, if a function is concave down, its second derivative is negative, and *conversely*, at the secondary level, for lack of a *mathematical* definition of concavity, students will miss this particular opportunity for a simple, founding experience in their mathematics education: tackling a theorem, and then its reciprocal. Didactic transposition thus surreptitiously makes its mark, and sometimes takes its toll, on students' and teachers' *praxis* and *logos* by distorting and, often, damaging the mathematical equipment which is available to them.

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How to choose relevant mathematical content to address Klein's second transition? The case of the inner product

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Keywords: inner product, Klein's second transition, secondary education, hypotheses, challenges.

INTRODUCTION AND RESEARCH QUESTION

Research dedicated to issues in relation with the transition between secondary and university education have been developed for long (Gueudet 2008). More recently, specific work has been carried out on what is referred to in the literature as “Klein's double discontinuity” (Winsløw & Grønbæk, 2013), and more specifically with “Klein’s second transition” (Gueudet et al., 2016), that is the transition between academic mathematics and secondary mathematics for (future) teachers.

The report of TWG3 during INDRUM2020 highlighted Klein’s second transition as a promising avenue of research in many regards. Among other things, the issue to find relevant topics with strong epistemological foundation that could allow teacher students to deeply understand crucial links between university mathematics and secondary mathematics in a professional perspective was particularly emphasized (Biehler & Durand-Guerrier, 2020). The research presented aims at contributing to the following related main issue: how to choose relevant advanced mathematical content to address or study Klein’s second transition and to create and design capstone courses?

MAIN ASPECTS

The purpose of the proposed poster is to present a methodological and inductive preliminary work in order to highlight possible main challenges, tools or levers or hypotheses with respect to the above main issue. To do this, we focus on the notion of inner product in the Euclidean plane, as it is taught in secondary education in France. In this regard, the underlying theoretical framework is minimal and based upon didactic transposition between scholarly knowledge and knowledge to teach, and between knowledge to teach and taught knowledge.

By considering the case of the inner product, we first illustrate in which regard the relevance of a notion for future teachers with respect to Klein’s second transition could be measured. For this purpose, several examples are developed in the poster that highlight different aspects – either related to university curriculum, secondary curriculum or to the relations between them (both directions) – that can help teacher students to mobilize a “higher standpoint on mathematics”:

- the choice of a definition, and the articulation between the definition and proof processes, these aspects being particularly rich in the case of inner product;

- the use of specific tools that shed light on the resolution of a problem: the example of the distance between a point and a line will be developed;
- the choice and use of an appropriate representation register, and the underlying flexibility to mediate between them;
- the importance to search for and solve inconsistencies and “vicious circles” in the curricula.

HYPOTHESES AND CHALLENGES

The above considerations give us the opportunity to formulate hypotheses with respect to teachers or future teachers that are consistent with respect to Klein's second transition. One of those is the fact that academic knowledge must allow teachers to have tools to control and analyse the choices related to didactic transposition on the one hand, and "everyday teaching" on the other hand. Also, in our view, academic knowledge is knowledge that is useful for the profession of mathematics teacher. It is the responsibility of mathematics education researchers to find relevant content for Klein’s second transition, to create links and finally to design situations or capstones courses with respect to these dimensions.

In our view, this preliminary study highlights a new issue that need to be addressed in future didactic research: the “coherence” of taught knowledge in classrooms from the point of view of didactic transposition. In view of the various aspects presented above, it seems to us that this is a relevant criterion for designing capstone courses. This study could also be a first step to constitute a reference for researchers to model the possible mathematical activities for the considered notions within several theoretical frameworks.

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Fostering Enculturation in the Transition to University Mathematics through Processes of Informal Learning – Poster Proposal

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Keywords: Transition to, across and from university mathematics; Teachers' and students' practices at university level; Digital and other resources in university mathematics education; Informal learning.

MATHEMATICAL ENCULTURATION, TRANSFORMATIVE EDUCATION & INDIVIDUAL LEARNING PROCESSES

The presented project is researching ways of understanding and fostering the process of enculturation in the transition towards university mathematics, especially focusing on the individual and self-regulated learning processes within and outside curricular and didactic framing. To that end, the project understands enculturation as a process of transformative learning (Mezirow, 1997) and education (Koller, 2018), therefore, as the transition of a learning person from one mathematical culture into another (Perrenet & Taconis, 2009) accompanied by crises, cultural breaks (Pepin, 2014), and shifts within the mathematical world view and beliefs of the individual (Schoenfeld, 2016).

Studies have shown that such adjustments of personal beliefs and views happen throughout the involvement with mathematics, but especially occur in the transition from school to university mathematics and relate to the high dropout rate of mathematics students in the first year of university (Egger & Hummel, 2020; Hochmuth et al., 2021). Within that transition students often struggle and, in some cases, fail to overcome the cultural break they encounter when starting to acquire university mathematics (ibd.). Nevertheless, enculturation is seen as a necessary part of the socialisation into mathematics as a science, since it concerns basic concepts such as formalism and axiomatics, logic, rigor, and proof (Hochmuth et al., 2021) and, therefore, must be promoted (Thomas et al., 2015).

As enculturation is seen as an individual process following an emergent and performative path, the project is focusing on self-organized learning and coping mechanisms applied by the students, as they encounter breaks within their comprehensions and tools. For this it is integrating the concept of informal learning into the didactic discourse as a description of the self-organized and intentional transformative search for new and functional solutions for a problem (Arnold, 2016).

RESEARCH QUESTION, STUDY & POSTER DESIGN

Based on the outlined concepts the project is asking how informal actions of learning within and outside of university courses trigger and accompany individual processes of mathematical enculturation in the transition towards university mathematics and how it can be described as transformative education. The corresponding study will use a mixed-method design: A quantitative survey with students of the first semester will concern the students' beliefs about mathematics before (pre-test) and after (post-test) the first semester. It will try to pinpoint events and topics of crises leading to changes in those beliefs. Within the semester a group of students will be accompanied through a string of qualitative interviews and a digital questionnaire application, both concerning the occurrence and the individual dealing with the mentioned learning crises, as well as the usage of informal learning in these contexts.

The poster will outline the theoretical framework of the project and give an overview of the mixed-method study design and its questionnaires. Also, first results of the pre-test will be displayed. A mix of texts, schematics, and diagrams will be used to visualize the project.

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Coherence of discourses of asymptote in the university transition

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INTRODUCTION

The notion of asymptote appears across different educational levels and mathematical domains. It developed from geometry, through calculus, into the notion of asymptotic behaviour in numerical and asymptotic analysis (Katalenić et al., 2021). Hence discourses of asymptotes in transition between educational levels should be coherent, that is, aligned and increasing in complexity.

Previous research showed that issues with discourses of asymptote emerge in particular educational settings (Berger & Bowie, 2012; Dahl, 2017; Katalenić et al., 2022; Mpofu & Pournara, 2018), some of them persisting with different participants − secondary and university students and practising teachers. This study investigated discourses of asymptote in a broader mathematical context and in the transition from upper secondary to university education.

FRAMEWORK AND METHODOLOGY

We used the Anthropological theory of the didactic (ATD) (Chevallard, 1999) and Commognition theory (Sfard, 2008) to examine discourses of asymptote in different settings covering; (1) upper secondary mathematics textbooks, (2) university students' answers to questionnaires, and (3) academic mathematicians' answers in interviews on discourses of asymptotes. Commognition theory enabled us to scrutinize the observed discourses and the ATD to compare them across institutional settings.

RESULTS AND DISCUSSION

Several discourses of asymptote emerged in our study of (1) and (2). They were content specific, separated from practice and incoherent across different settings. Mathematicians from (3) held different ideas about endorsing emerged discourses (Table 1). For example, the discourse of asymptote as a tangent line to the curve at the infinite point emerged in an analysed textbook and as an informal narrative in students' answers. However, it was endorsed only by an expert in geometry.

Emerged discourses of asymptote	**Endorsement**
Asymptote of a curve is its tangent line at the infinite point.	Rejected/Endorsed
Asymptote of a curve is a line that the curve approaches but does not intersect or touch.	Conditionally endorsed/Endorsed
A line is an asymptote of a curve if the distance from the curve to this line tends to zero as points on the curve move away from the origin of the coordinate plane.	Endorsed

Table 1: Discourses of asymptote emerged from secondary textbooks and questionnaires with university students and their endorsement from interviews with mathematicians

CONCLUSION

The study of asymptotes should be appropriate and representable in mathematical and educational context. Based on the results of our study, we propose the development of discourses of asymptote in the transition from the upper secondary and across the university education. The discourses that we suggest are coherent across settings, increasing in complexity and aligned with the notion of asymptotic behaviour in applied mathematics.

POSTER DESIGN

The poster will be focused on different aspects of discourse of asymptote in mathematics, results from literature review, methodology and results of our study, and suggestions for attaining coherence of discourses of asymptote. When appropriate, we will use diagrams, figures and tables to present the course of the study.

Keywords: teaching and learning asymptote, transition from secondary and across university mathematics, Anthropological theory of the didactics, Commognition theory.

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CASPER: A framework to assess the difficulty of exercises in terms of semiotic registers

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Keywords: Teachers' and students' practices at university level; Transition to, across and from university mathematics.

INTRODUCTION

In this poster we present the first step of a design and research project whose aim is to support university mathematics teachers in their task design activity for first-year students. We focused on semiotic representation registers (Duval, 2006) intervening in mathematical tasks. We developed a framework to assess and adapt the difficulty of mathematical exercises, in terms of registers used and conversions of registers needed, implicitly or explicitly. We called this framework CASPER: *Categorize Activities Systematically with imPlicit / Explicit semiotic Registers*.

THEORETICAL FRAMEWORK

Duval (2006) introduces the notion of semiotic representation registers as semiotic systems allowing transformations of representations. He distinguishes between two activities: treatments (within the same register) and conversions between two different registers. Conversions between registers are both a source of difficulties and an essential lever for conceptualization processes. Indeed, students shall separate objects from their representations, that is, they must acknowledge the existence of "objects" behind semiotic representations. This is only possible when they are facing multiple representations of the same object. Moreover, the conversion process can be more difficult than a simple encoding. It is the case when representations are not *congruent*, that is when the different signs in both registers cannot be mapped (1) bijectively and (2) univocally, or (3) if the mapping changes the reading order between source and target representation.

Even at the university level, confronting students to these issues is needed and requires a careful design of tasks. University students are expected to develop a flexibility in terms of semiotic registers, and this is one of the difficulties at the secondary-tertiary transition (Gueudet & Vandebrouck, 2022). How can we design a framework to situate and adapt the flexibility required by an exercise with respect to semiotic registers?

THE CASPER FRAMEWORK

Defining implicitness with respect to semiotic registers in activities

A register is *explicitly displayed* when it appears in the exercise's text. It is *explicitly mentioned* when the register is suggested without being displayed. It is *explicit* if either of these two situations holds. It is *implicit* if it is in the expected answer without being explicit in the exercise's text. When there are several registers, the procedure(s) for solving the exercise will involve a conversion of registers. This change is *implicit* when no indication is given to the student about when, how and why they performs the conversion. Otherwise, it is *explicit*.

CASPER : Categorize Activity Systematically with imPlicit / Explicit semiotic Registers

Our framework analyses activities according to 4 categories:

1. Explicit single register (a.k.a. treatment).
2. Multiple explicit registers with explicit conversions
3. Multiple explicit registers with implicit conversions.
4. Multiple registers where at least one is implicit.

In our poster, we present examples with the associated CASPER categories. We also provide examples of how it can be used to adapt a given activity. This could be interesting for teacher to address the secondary-tertiary transition issue.

Empirical evaluation

We conducted an experiment with 28 first-year undergraduate students in a science and technology degree. They were randomly assigned into 4 groups (similar in terms mathematical proficiency, according to their semester 1 grades). Starting from 4 mathematical activities on different themes, we created variations of the exercises according to the 4 CASPER categories for a total of 16 exercises. Participants of each group individually performed the same 4 exercises (covering the 4 activities and the 4 CASPER levels), yet overall, the different groups covered the 16 exercises.

The goal of the experiment was to validate the CASPER framework. We expect the categories to be difficulty levels. Thus, we measure the average success ratio for the different categories. Given the small data situation, we did not reach statistically significant results, but these preliminary results are promising.

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TWG2: Teaching and learning of analysis and calculus

TWG2: Teaching and learning of analysis and calculus

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INTRODUCTION TO THE THEMATIC WORKING GROUP

The thematic working group TWG2 on "Teaching and learning of analysis and calculus" continues the working groups on "Calculus and analysis" from the two previous INDRUM conferences (see Trigueros et al., 2021; Vandebrouck et al., 2021). 24 authors and coauthors contributed nine papers and two posters to this working group at INDRUM2022. The research in this working group deepened our understanding of these classic topics in university mathematics education. The contributions were diversified in terms of the selection of specific concepts but also included more general aspects beyond the teaching and learning of a single concept. In addition, new aspects of more advanced topics from university-level calculus and analysis were investigated.

The thematically coherent nature of TWG2 allowed the contributors to discuss a diversified range of topics related to the teaching and learning of analysis and calculus at large. Accordingly, the research presented in this TWG focused on both the teaching and learning of specific mathematical topics or on more general questions dealing with the teaching and learning of courses related to calculus and analysis. Approximately half of the contributions focused on epistemological analyses and individuals' learning of central mathematical topics such as functions of one and several variables and their derivatives as well as integrals in calculus, real analysis, and complex analysis. About the other half of the contributions addressed more general questions related to courses in calculus, real analysis, and beyond. These contributions included lecturers' practices such as defining and enabling students' participation in proofs, the use of interactive tasks in instructional videos, the adaptation of paper and pencil activities to a dynamic geometry environment, and longitudinal effects of the Covid-19 pandemic on students' performances in examinations. Research in this thematic working group also included aspects of theoretical interest related to different frameworks such as the use of schemas in APOS theory, commognition for the study of lecturing and proof, and the relation between basic mental models (BMM) and "personal meanings."

Two sessions for the presentation of each of the paper and poster contributions were organized according to the main topics covered in the contributions. To initiate fruitful discussions for each paper, the presentation sessions were followed by thematically focused discussion sessions. Each group of authors and co-authors prepared a "reaction" to another paper, in which they summarized the central aspects related to the research questions, theoretical and methodological frameworks used and raised discussion with prompts to the authors. More open discussion sessions were held to identify larger strands of research questions and the need for further research in university mathematics education on the teaching and learning of analysis and calculus.

<table>
<tr><th></th><th>Epistemological inquiry</th><th>Cognitive inquiry</th><th>Instructional inquiry</th></tr>
<tr><td rowspan="3">Theme 1:
Integrals in calculus, real and complex analysis</td><td>Hanke: Vertical coherence in the teaching of integrals? An example from complex analysis</td><td></td><td></td></tr>
<tr><td colspan="2">Kouropatov, Noah-Sella, Dreyfus, & Elias: An epistemological gap between analysis and calculus: the case of Nathan</td><td></td></tr>
<tr><td colspan="2">Dreyfus, Elias, Kouropatov, Noah-Sella, & Thompson: Personal meaning vs mental models for integral</td><td></td></tr>
<tr><td rowspan="3">Theme 2:
Functions one and several variables</td><td colspan="2">Bouguerra & Ghedamsi: An investigation of potential changes in students' images of derivative at the entrance of the university</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">Lankeit & Biehler: Different interpretations of the total differential and how they can be reconstructed in textbooks for multivariable real analysis</td><td>Trigueros, Martínez-Planell, & Borji: Development of the differential calculus schema for two-variable functions</td></tr>
<tr><td colspan="2">Trigueros, Orozco-Santiago, & Martínez-Planell: Learning two variable functions using 3D dynamic geometry</td></tr>
<tr><td rowspan="4">Theme 3:
Lecturer and student attainment</td><td rowspan="3"></td><td rowspan="3"></td><td>Umgelter & Geisler: Analysing the quality of advanced mathematics lectures regarding the presentation of definitions – the case of real analysis lectures</td></tr>
<tr><td>Griffiths & Palau: Returning to the classroom after taking online classes during the Covid-19 pandemic: A longitudinal study of student attainment</td></tr>
<tr><td>Karavi, Lipper, & Mali: Investigation of metarules in lecturing for enabling students' participation</td></tr>
<tr><td></td><td colspan="2">Krämer & Liebendörfer: The effect of interactive tasks in instructional videos on students' procedural flexibility</td></tr>
</table>

Table 1: Central themes and inquiries in TWG2

The papers dealing mainly with instructional inquiry include the study by Umgelter and Geisler on lecturers' presentation of definitions. The authors propose and discuss a protocol with four categories to describe the presentation of definitions: *motivation of concept, description of definition, giving examples and counterexamples,* and *mental* *or visual forms of representation*. Then, they use this protocol to examine the mathematical exposition by two lecturers. In a poster, Karavi, Lipper, and Mali report on an investigation of lecturers' discursive actions that may assist students' participation in the university mathematical discourse. As an example, they discuss lecturers' discursive actions when presenting a proof (*giving an idea on how to start the proof, sharing key ideas of the proof,* and *bringing means for emergence of the proof*) as reflecting the governing metarule *while proving, an idea of how to start is needed*. Also dealing with instructional inquiry, Griffiths and Palau examine differences between online and face-to-face Calculus 3 courses. They compare the performances of two groups of students who took a face-to-face Calculus 2 course prior to the pandemic, one of which went on to take an online course and the other remained face-to-face. Griffiths and Palau compare the student groups' grade point averages of their midterm and final grades. Then, the authors also compare the students' performance in a subsequent face-to-face differential equations course. Their results show that the face-to-face students performed better. Finally, Krämer and Liebendörfer compare the performance of students who were presented with instructional videos without interruptions to those who had interactive tasks interspersed throughout the videos. They use the context of polynomial differentiation with the product and chain rules to examine students' flexibility in choosing one or the other technique. The comparison showed a weak advantage of the use of interactive tasks. This paper can also be considered to have a component of cognitive inquiry.

In one of two papers mainly dealing with epistemological inquiry, Hanke observes that the creation of vertical coherence between different mathematics courses can be problematic. He shows that the mean value interpretation for integrals of real-valued functions of one real variable does not generalize to the case of complex path integrals in an immediate way. He further supports his observations with an interview with a complex analysis lecturer. Lankeit and Biehler propose a "model of meanings" for the total differential of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. The model considers the contexts of *analytic-algebraic, geometric, approximation,* and *real-world interpretations*. The authors examine different definitions of the total differential and how the notion and related content could be developed. They also apply the model to analyze the meanings presented in three German textbooks on the subject.

Two papers may be considered as dealing with both epistemological and cognitive inquiries. In one of them, Kouropatov, Noah-Sella, Dreyfus, and Elias argue that there is a substantial difference between real analysis and calculus. According to the authors, calculus requires thinking within an extra-mathematical context, while analysis has an intra-mathematical origin, which leads to didactical challenges. They support this claim with an interview with a student. In the other paper, Dreyfus, Elias, Kouropatov, Noah-Sella, and Thompson argue that "basic mental models" (BMM) and "personal meanings" are different constructs and use the context of integration to contrast the differences. Using task-based interviews, they reconstruct the BMMs and personal meanings of three students related to integrals. The authors argue that personal meaning-making is a cognitive activity as opposed to the epistemological activity of structuring mathematical content and that the focus on personal meaning adds to the normative layer captured in BMMs on integrals. Furthermore, in their poster, Bouguerra and Ghedamsi combine a praxeological analysis of the presentation of the derivative in Tunisian textbooks with students' concept images at the entrance of university.

Finally, two papers were classified as mainly dealing with cognitive inquiry. In one of them, Trigueros, Martínez-Planell, and Borji use APOS theory to examine the construction of the differential calculus schema for two-variable functions. They explore the notions of schema components and the types of relations between schema components to give empirical evidence of this complex construction. In another paper, Trigueros, Orozco-Santiago, and Martínez-Planell adapt activities originally developed for a paper and pencil environment to teach basic ideas of two-variable functions in a GeoGebra environment. The authors use student interviews and students' written productions during the semester to show the potential of this approach. However, they also observe that student performance was not as good as that in the paper and pencil environment and discuss possible reasons. This last paper can also be considered to have a component of instructional inquiry.

FUTURE DIRECTIONS

Several directions for future research were identified during our group discussions. The distinction between calculus and analysis proposed by Kouropatov, Noah-Sella, Dreyfus, and Elias gave rise to some discussion as it seems to be more visible in some countries than in others. For example, in Germany, calculus roughly corresponds to parts of school-level analysis or to application-oriented and less proof-based courses for non-mathematicians. As another example, the paper by Lankeit and Biehler about the total differential of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ could be considered to deal with analysis, while the one by Trigueros, Martínez-Planell, and Borji on the differential calculus of functions from $\mathbb{R}^2$ to $\mathbb{R}^1$ might be considered to deal with calculus. Would the explicit discussion of this distinction help to better interpret and apply the results of these related papers? There is room here for further research.

There was also much discussion on BMMs and their relation to more cognitively oriented research, like in the paper by Dreyfus, Elias, Kouropatov, Noah-Sella, and Thompson. BMMs might be understood in different ways by different researchers, which might hinder researchers in embedding their own work in the body of literature. Other papers, like those of Hanke, and Lankeit and Biehler also gave rise to the discussion on BMMs. More research on the relation between BMMs and other concepts in mathematics education and epistemological analyses appears to be warranted.

The epistemological analysis of some papers invited reflection on its possible use as a foundation for further cognitive research. For example, from the point of view of the epistemological analysis by Lankeit and Biehler, we may ask whether the study of functions from $\mathbb{R}^2$ to $\mathbb{R}^1$ as in the paper by Trigueros, Martínez-Planell, and Borji helps students generalize from few to higher dimensions.

Some of the topics studied were new, mainly because they are more advanced and have rarely appeared in the research literature before. This invites further research on these topics such as complex analysis (Hanke) and the general multivariable function (Lankeit and Biehler).

Finally, the role of teachers and resources in the teaching and learning of calculus and analysis also invites further research, for example how to interweave face to face and online instruction? The contributions by Griffiths and Palau, Krämer and Liebendörfer, and Karavi, Lipper, and Mali are examples in this regard. There also remains the need to study how to interweave paper and pencil and technology; for example, Trigueros, Orozco Santiago, and Martínez-Planell show that this is not a straightforward endeavor.

Overall, this group presented a wide variety of novel and interesting issues that resulted in several productive discussions.

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Personal Meanings versus Basic Mental Models for the Integral

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While there are parallels between what has been called, in the literature, Basic Mental Models (BMMs, Grundvorstellungen) and what has been called Personal Meanings, there are fundamental differences between them. In this paper, we work out some of these differences, using the notion of integral as example. Roughly summarized, our findings are that BMMs, including individual ones, are epistemological whereas Personal Meanings are cognitive. Here epistemological refers to a content analysis, often from a didactic point of view, and hence is informative, for example, for curriculum developers; cognitive refers to individual students' personal conceptions, and hence is of interest, among others, to teachers.

Keywords: Teaching and learning of analysis and calculus, Grundvorstellungen (Basic Mental Models), Personal Meanings, Integral.

INTRODUCTION

Greefrath et al. (2021) have introduced Basic Mental Models (in German: Grundvorstellungen; henceforth BMMs) for the definite integral. They claim that students use their individual BMMs when they solve a problem. On the other hand, Thompson (2013) has introduced the notion of students' personal meanings and has shown many students and teachers lack such meanings for some central basic notions of mathematics. The question arises, what do students use when they solve a problem or answer a question, in our case about integrals; specifically, how is it possible that students who lack meanings use individual BMMs? We conclude that there are fundamental differences between individual BMMs and personal meanings. The aim of this paper (and its research question) is to clarify the differences and similarities between individual BMMs and personal meanings.

THEORETICAL BACKGROUND

Basic Mental Models (BMMs, Grundvorstellungen)

A fundamental idea produced and used by the German tradition of didactics of mathematics is the notion of Grundvorstellung, often translated into English as Basic Mental Model or BMM. According to Greefrath et al. (2021),

> Normative BMMs are interpretations of a mathematical concept that learners should generally and ideally develop. These BMMs are identified by didactic analyses of the mathematical concept... They can be used as educational guidelines and to specify learning

objectives for mathematics lessons. This can provide orientation to teachers when designing and organizing their lessons. (p. 650)

That is, BMMs are theoretical constructs based on a content analysis, and as such they can be useful in the determination of learning objectives and the design of instructional materials. For the notion of definite integral, Greefrath et al. (2021) have identified four BMMs: area, reconstruction, average and accumulation. For example, the accumulation BMM "considers the definite integral of a function as the limit of a sum with a large number of small terms" (p. 650), which are products, and hence this BMM emphasizes the process of integration. In this paper, we relate to only three of these four BMMs because the average BMM did not play a role in any of our data.

In addition to normative BMMs, Greefrath et al. (2021) also consider individual BMMs, which "are the specific manifestations of normative BMMs in a person" (p. 654). They developed an instrument for assessing students' individual BMMs for integral; when using this instrument, students are presented with an argument that uses one of the four BMMs and are asked whether the argument reflects the student's own line of thought. The instrument has a high validity for the students' choice of one of the four BMMs, as shown by expert evaluation.

Students' personal meanings

Taking a Piagetian approach, Thompson (2016) focuses on what comes to a person's mind upon encountering a situation; meaning is what the person imbues to the situation. A person's meaning in a situation is what comes to the person's mind immediately, together with what is ready to come to mind next. Thus, meaning also has an implicative nature. The meaning of an understanding is the space of implications that the current understanding mobilizes—actions, operations, or schemes that the person's current understanding suggests.

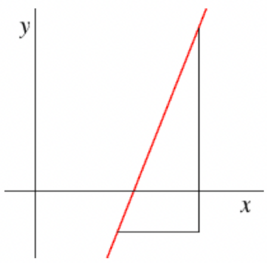

Figure 1: The slope item

To make this more concrete, we present the following item about slope developed by Thompson (2016). Subjects are shown the graph in Figure 1 and told that it represents the relationship between two co-varying quantities P and Q, whose measures x and y are related by $y = mx + b$. Students are first asked to give an approximate numerical value of m given that the x – axis and the y – axis use the same scale. In a second question, they are asked for the numerical value of m if the scale of the y-axis is changed so that the distance between 0 and 1 is two times as large as the original one.

Thompson (2016) presents results from high school teachers to the slope item. We have informally presented the same item to mathematics education MA students. The results were similar. Most respondents give values between 2 and 3 to the first question. Most of these respondents doubled or halved their value in response to the second question.

The value of m in the relationship between the co-varying quantities P and Q does not depend on the scales of the axes in the graph used to represent this relationship. Hence, respondents whose meaning of slope in the above situation is the ratio between the relative changes in P and in Q, will not change the value of m from the first to the second question. Respondents who change the value of m from the first to the second question, hold a meaning for slope as a property of the triangle representing the relationship rather than a meaning of slope as a property of the relationship itself.

INDIVIDUAL BMMs ARE NOT PERSONAL MEANINGS

Our motivation for writing this paper is a question addressed to us repeatedly when presenting initial results from a project investigating Israeli high school students' personal meanings for derivative and integral (e.g., Elias et al., 2022; Noah-Sella et al., 2022). We were asked why we did not use individual BMMs as theoretical framework for our research. This question appears to be well taken since individual BMMs are, at least in the case of the definite integral, well elaborated, an instrument for assessing them exists, and this instrument has been used with a large sample of first year mathematics students in German universities: Why should we not use the existing instrument, whose reliability and validity have been shown, and administer it to a suitable sample of Israeli high school students?

The brief version of our answer to this question is that BMMs do well what they are designed to do namely to reveal students' knowledge about mathematical concepts, and about integrals specifically. Such knowledge may be assumed to have been constructed (or memorized) during an epistemic journey led by a teacher. However, BMMs have not been designed to assess students' personal meanings. In the remainder of this paper, we will present three examples illustrating differences between BMMs and personal meanings. We will conclude the paper by giving a more elaborate answer to the question why we do not use BMMs as theoretical framework.

EXAMPLES

The examples for students' personal meanings presented below have been selected from among more than 50 task-based interviews held for the purpose of designing a set of questionnaires to assess high school students' meanings for derivative and integral. Using Thompson's definition of personal meaning, we decided on the following five criteria as being potentially indicative of the interviewee's personal meaning.

Distinctive language. Utterances may reveal personal meaning if they refer to mathematical notions in terms distinctive to the interviewee. By this we mean that the terms have not been used by the interviewer or in the task, and were not prompted in any way, but are the interviewee's own terms.

Repetition. Utterances may allude to personal meaning if they contain terms that repeat themselves, whether within a certain task, or across different tasks and contexts. Repetition signifies that the concepts associated with these terms are readily available and relevant to the interviewee, and therefore are a salient part of their thinking.

Reasoning. Utterances that are intended to explain or justify mathematical notions or claims are indicative of the interviewee's thought process, and therefore may point to personal meaning.

Unexpectedness. If an utterance by the interviewee is unexpected for the interviewer, then it was not solicited by the interviewer or the task and may be an expression of the interviewee's personal meaning.

Statement of opinion. Utterances explicitly qualified by the interviewee as their own belief, opinion or interpretation will be regarded as indicative of personal meaning. This includes, for example, utterances containing phrases such as "to me", "in my opinion", "the way I understand it".

Nathan

Nathan is an experienced high school teacher. He had completed several analysis courses at university and is very skilled in the subject. He was interviewed with the aim of clarifying his personal meaning for integral (see Noah-Sella et al., 2022, for details). The three tasks used in the interview with Nathan dealt with the length of a segment of the graph of a function, with the mass of a wire, given its mass density, and with the amount of money in a bank, given the cash flow.

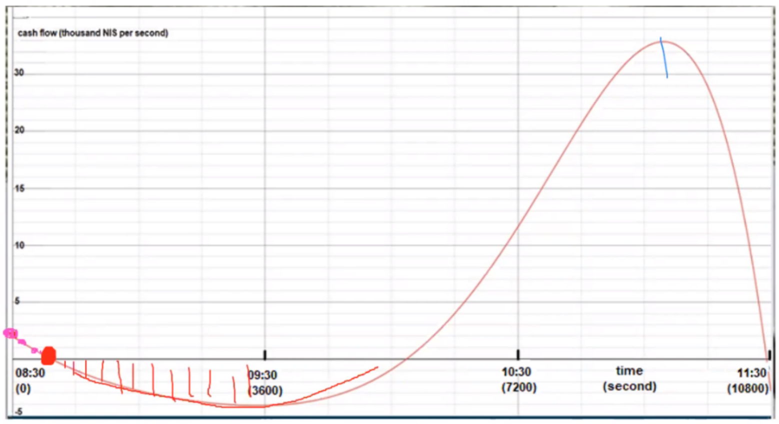

Figure 2: Nathan's markings on the cash flow graph

The cash flow task was presented in a graphical setting, leading Nathan to draw vertical (red) lines from the horizontal time-axis down to the graph representing the cash flow (Figure 2, between about time 8:50 and 9:30), and to explain

Nathan: My logic is that when Δx approaches zero, or is even equal to zero, the size of the – I don't want to say rectangle, it's a line - it has no width. It's just a

> line, and since its width is zero, when we add up all these lines, we will get the area under the curve.

Nathan sums little bits, thus using the accumulation BMM. He "adds up" all the lines to obtain the area and interprets the area as amount of cash, thus using the area BMM. And when asked how he would add up the lengths in practice, he said "adding up the y-coordinates of all the points, the infinite number of points". To Nathan, that was what the integral was doing, and for the computation he would find an antiderivative and use the second part of the fundamental theorem, that is the reconstruction BMM. Thus, Nathan's personal meaning for integral encompassed the three BMMs.

Nathan's personal meaning for integral was consistent across tasks. When asked about the curve length, Nathan wanted to "take all the points and add them up", which he elaborated as "take two points, calculate the distance between them, and make Δx approach zero. Then I'll get the length of a single segment… If I integrate this, I will get the length of this segment". Nathan first lets Δx approach zero, and then "integrates" or sums the values. We mention in passing that Nathan's explanations are completely in line with what Oehrtman (2019) has called the collapse metaphor.

What exactly Nathan imagines adding became most clear in the mass from density task; in this task, Nathan claimed that the given mass density function did not serve him to find the mass of the wire; rather that he needed a function giving the mass at each point, and then he could integrate these point masses to find the mass of the wire.

We conclude that Nathan's personal meaning for integration is that the integral is a technique that sums the values of the integrand at each point of the interval of integration. From Nathan's explanation in the curve length task, it appears that he may have developed this meaning in an effort to tackle the role of limit in the definition of the integral. He uses this meaning explicitly in combination with the accumulation BMM in all three tasks; he combines it with the area BMM in the cash flow task; and he combines it with the reconstruction BMM in both, the cash flow and the curve length task. He is familiar with the three BMMs and uses them freely in his explanations. But what he interprets the integral to be, to do and to mean is to add values of the integrand. This interpretation is independent of the BMMs and reaches across all three of them.

Oren

Oren is a 12^{th} grade student who is taking advanced track mathematics. The task on which Oren's interview was based relates to $\int_0^x f(t)dt$, where f is a step function with two positive values (Figure 3).

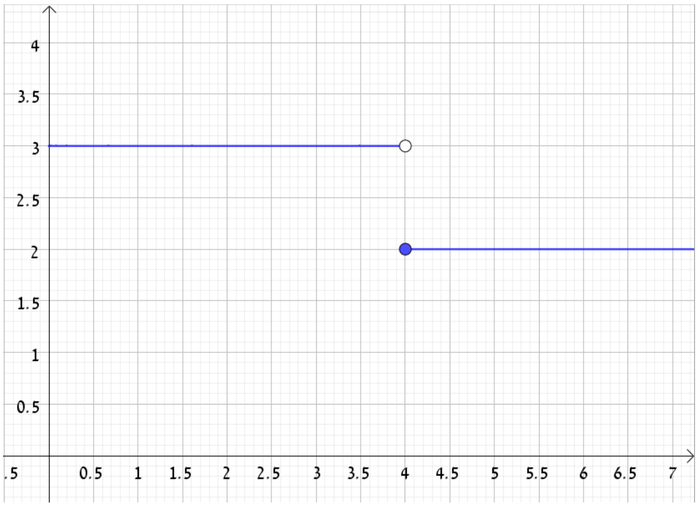

Figure 3: the graph presented to Oren

Oren related to $\int_0^x f(t)dt$ as an "accumulating integral" and distinguished it from a "regular integral". When asked what accumulates, Oren explained that "what accumulates are values" and "I like to look at it as with some image, for example accumulation of money in the bank. If we accumulate positive values, the amount of money grows." The discontinuity of f did not disturb Oren – he explained that the integral simply did not accumulate a value there, whereas at all other x's it does. So, one might conclude that Oren's individual BMM for integral was accumulation. However, this was not at all the case. When presented with a hypothetical student K who used antiderivatives in the same situation, Oren explained that "the integral is the opposite of derivative, and that's what K did when he took the antiderivative. If it were a regular integral, I would agree with him, but since it is an accumulating integral, he makes a mistake". Similarly, when presented with a hypothetical student A who used area in the given situation, Oren explained that "the integral is the area caught between the graph – the ceiling – and the x-axis - the floor. What's between them, that's really our integral". But for Oren, an accumulating integral is not represented by area: "When I think about an accumulating integral, I don't think about area; I think about values that I accumulate, y-values."

Finally, when presented with a student C who added a constant of integration, Oren was "in a dilemma how to relate to the constant of integration. In a regular integral, we add a constant of integration. I am not sure whether we do that in an accumulating integral; my intuition says no. In an accumulating integral, you don't speak about a primitive function, to which you could add an arbitrary constant that would then disappear when you differentiate. I don't agree with C. He would be correct if we spoke about a regular integral, but we don't."

Like Nathan, Oren flexibly relates to the three BMMs, rather than preferring one of them. However, there is a disconnect for him. He links the expression $\int_0^x f(t)dt$ to the "accumulating integral" or the accumulation BMM, which is separate from and behaves differently from a ″regular integral″ that can be thought about using the area BMM or the reconstruction BMM.

In contrast to Nathan who linked the three individual BMMs by a common way of thinking about them, Oren separates the accumulation BMM as being a completely different entity with different properties from the "regular" area and reconstruction BMMs. Oren's personal meaning consists of two distinct notions of integral, "regular" and "accumulating". His meaning for integral does not reside in an individual BMM but rather in how he views, connects, or separates the three BMMs. The "regular" integral is evaluated by means of an antiderivative (reconstruction) and returns the value of the area of a fixed static region enclosed by the graph of the function and the x-axis. The "accumulating" integral, on the other hand, has nothing to do with area or reconstruction; it sums values.

Nadia

Nadia was interviewed on an item parallel to Oren's, about $g(x) = \int_0^x f(t)dt$, where f is a step function with two positive values (Figure 3). Her item was slightly different, however: The integral was described as representing the area under the graph of f. Nadia was presented with the thinking of 5 hypothetical student. With the thinking of two of them she did not at all identify; we discuss the other three.

Hypothetical student V said that to find the integral, one needs to find the antiderivative in each of the two subdomains. Nadia explained V's reasoning as follows: V splits the domain into two subdomains, notices that f is the derivative of g, and hence does, in each subdomain, the operation inverse to differentiation, which is finding the antiderivative. Hypothetical student T said that area accumulates at a certain rate up the point of discontinuity and then continues to accumulate at a different rate. Nadia's reaction was that she was familiar with both ways, the one of V and the one of T, but that she slightly prefers V's way. Hypothetical student P looked at the integral as describing area accumulation like T but claims that the accumulation starts again from 0 at the point of discontinuity. Nadia points out the similarity between P and T, but not the difference. She identifies with both, the thinking of T and of P, but not as closely as with the thinking of V.

Our interpretation, so far, is that Nadia appears to be at ease with and able to explain the area BMM, the reconstruction BMM and the accumulation BMM, with a slightly higher affinity for the reconstruction BMM than for the other two.

Next, Nadia was presented with four potential graphs for g (Figure 4). All graphs are positive and have the correct slopes; the only difference between them is at the point of discontinuity of f. Nadia was asked, which of the graphs represent the function $g(x)$. Nadia kept vacillating between the three discontinuous graphs for about 10 minutes, giving no clear reasons for her choices, and struggling: "it's a bit difficult to explain" and "it feels strange". At one stage, she explained that "it can't start from a lower point as in 1 and 3" because "if you do an integral, if you separate the two parts…, you like, have to add one to the other”, and later “because we do the addition… there is a jump between them because it can’t be 100% continuous”. Indeed, she preferred, at different times, Graphs 1, 3 and 4, but never Graph 2.

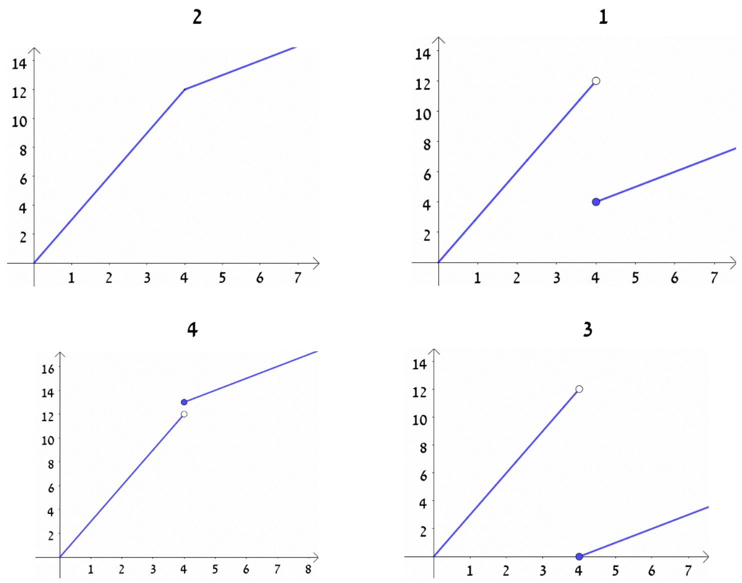

Figure 4: The graphs proposed to Nadia for $g(x)$

As the conversation went on, she ventured “it depends how I look at this... you can take it as a question about definite integral, and you can take it as question about area; and each one has a very different meaning”; “I say they [the graphs] can all be correct; it’s just a question of how I try to look at them”; ”Graph 3, for example, can be if I look at the first segment, and then I like restart the area”; “I choose Graph 1 because I don’t completely restart from zero”; and she concluded with “[if the horizontal axis represents x and the vertical one $g(x)$] then I would choose Graph 4”.

Nadia’s only definite argument was against Graph 2; her exclusion of Graph 2, as well as her acceptance of Graphs 1 and 3, contradict accumulative thinking. We conclude that her earlier support for the way of thinking of students T and P may have been only declarative. The BMMs of area, and reconstruction play an explicit and recurring role in Nadia’s thinking about the graphs; however, she expresses a clear disconnection between definite integral and area, claiming that they lead to different graphs (209); these different graphs are incompatible with each other, as well as with a meaning of area for the definite integral – the growth of area can’t be discontinuous.

Nadia’s meaning for integral contains elements of all three BMMs, but the elements from each BMM lead her to ways of thinking that are incompatible with the other BMMs. For example, Nadia’s seemingly strong grounding in the reconstruction BMM was too weak to play a role for her when choosing a graph. Nadia’s meaning for integral at the time of the interview cannot be associated with a specific BMM, nor with a coherent fusion of two or all three BMMs.

CONCLUSION

BMMs and meanings are not orthogonal, as the examples show. In Oren's meaning for integral, all three BMMs play a role, but this meaning decomposes into two notions for integral, something that might not have become evident, had we only looked for BMMs. BMMs may also blend rather incoherently in a student's mind as for Nadia. On the other hand, a student's meaning for integral may be consistent across all three BMMs, as for Nathan; Nathan has a strong but mathematically erroneous meaning for integral that is quite independent of the BMMs. The meaning of none of the three students can be clearly associated with one BMM; quite the contrary: for each of the students, all three BMMs play some role, and these roles differ greatly from student to student.

The test developed by Greefrath et al. (2021) lets students choose between 4 options. This is useful for the purpose for which the test has been designed. But as pointed out in the previous paragraph, the test is not likely to reveal students' meanings. Students' meanings are not neat enough to be categorized into 4 slots that have been defined by theoretical content analysis. In addition, the BMM a student uses may strongly depend on the context of the situation presented to the student. For Oren, the area BMM played a minor role. For Nadia, who was presented with what may appear to an expert to be the same problem (except that the area interpretation was mentioned in the description of the integral), the area BMM played a central role.

As epistemological constructs, BMMs are useful for comparing students' meanings to desired meanings for integral, for assessing how close students' meanings are (or are not) to the desired meanings offered by the BMMs. We note that Greefrath et al. (2021) themselves explicitly distinguish between BMMs and concept images; this seems a reasonable conclusion. We conclude that the role of BMMs in assessing students' meanings that differ from the desired meanings is limited.

The researcher who intends to investigate meanings should certainly be aware of BMMs and ask whether and how these meanings are related to BMMs. But when we ask about a student's meaning, we do not, or not mainly ask in terms of which models students think with. We ask how the student thinks about and with these models. In the BMM instrument, Nathan might have expressed the same high appreciation for the area and for the accumulation answers; and we would not have learned how he thinks about the integral, namely as a sum of values (or lengths) rather than a sum of products or areas (or small rectangles).

The BMM instrument is doing well what it was designed to do, but it does not assess students' meanings. BMMs have an epistemological role in the design of instruction. Designers will want to introduce several BMMs sequentially and coherently. Going beyond epistemology, Thompson (2013) has pointed out the importance of meanings for mathematics teaching and learning to become productive. The investigation of meanings requires suitable tools, and the development of these tools requires intensive interviewing (Thompson, 2016). It is a complex long-term effort.

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Returning to the classroom after taking online classes during the Covid-19 pandemic: A longitudinal study of student attainment

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This mixed mode longitudinal study compares the level of attainment between a face-to-face and an online class of Calculus 3 during the Spring 2021 semester, considers the attainment level of the students in the prior Calculus 2 course, and then goes on to compare the results of the same students when they all returned to campus to take the subsequent Differential Equations course. The results show that although the students were of a similar level of ability when entering the Calculus 3 course, the face-to-face students did much better once the semester began, with a statistically significant difference in the scores on both the first midterm exam and at the end of the course. This difference was maintained in Differential Equations. We discuss the causes of the disparity using the quantitative and qualitative data collected, along with the implications for how to best allocate resources in a way that blends the positive aspects of different teaching modalities.

Keywords: Teaching modalities, online learning, attainment, calculus, Covid-19.

INTRODUCTION

The Covid-19 pandemic led to a substantial increase in the amount of attention given to teaching modalities, with much of it concentrated on the rapid and forced transition to online learning and its effect on students and instructors. In this paper, we look at aspects of online education in the context of the pandemic and use data from students taking the university calculus sequence in the United States to see whether weaker students had a greater propensity to take online alternatives to regular face-to-face classes, how the online students performed compared with face-to-face students when taking the same course, and how students who have taken classes online during the pandemic adjusted when returning to the classroom.

LITERATURE REVIEW

Looking across all disciplines, a small number of the initial studies discussing the changes to teaching and learning modalities caused by the Covid-19 pandemic reported a benefit to the enhanced use of digital technologies. Almusharaaf and Khahro (2020) reported that of the 283 undergraduate students surveyed in Saudi Arabia, most were highly satisfied with the (online) course modality, the knowledge and skills that they gained, and the level of engagement, while Gopal et al. (2021) found that a clear majority of the 544 business and hotel management students that they questioned in India considered online teaching to be valuable even though their first experience of it came during the pandemic. An often-cited caveat is that students must have the necessary internet connection and related support needed to make the transition and this is especially relevant in developing countries. A study of 115 students in Indonesia by Bahasoan et al. (2020) found that almost all of them experienced difficulties related to network connectivity and data quotas.

By contrast, a large number of studies have brought to light the disadvantages caused by the sudden shift to distance learning. A study of 270 undergraduates conducted by Aguilera-Hermida (2020) found that the students had a strong preference for face-to-face instruction, and that those with this preference struggled to adapt to the change in modality. Students reported their biggest challenge to be concentrating while being at home due to the distractions of family members, having a tendency to view their home as a space for relaxation rather than work. Even among mature students studying professional disciplines (Bączek at al., 2021; Sarwar et al. 2020), technological issues and the lack of interaction reduced the perceived effectiveness of online classes.

Academic struggles during the pandemic were found to be exacerbated by issues related to student mental health. A study by Pragholapati (2020) emphasized that the academic performance of students is affected by issues that go beyond the modality of instruction and found that at least a quarter of the student participants suffered from Covid-19 induced anxiety. Research carried out by Mendoza et al. (2021) confirmed the existence of severe anxiety in a significant part of the sample of mathematics students sampled during the pandemic.

Given the documented struggles that students endured during the transition to online classes, it is not surprising that many wanted to return to campus as soon as possible. In a study of 604 students in Romania, Gherhes et al. (2021) found that most of the respondents wanted to return to face-to-face learning after the pandemic. This is in line with the study of students in Florida by Griffiths (2020) that found a significant proportion of undergraduate students still believe that on-campus instruction offers the highest level of education, with a difficulty in maintaining focus and the lack of interaction with the instructor and other students cited as the primary drawbacks to taking classes online.

Quantitative studies of student attainment using different modalities are more difficult to conduct, especially during the Covid-19 pandemic, as there was often no choice given to students. Such studies are challenged by the need to maintain the same pedagogical practices and must also consider whether any discovered disparities were present at the beginning of the course. As a result, the first articles involving attainment relied mostly on student perceptions. A study by Hashemi (2021) of 1231 students in Afghanistan found that academic performance during the pandemic was reduced, while 88% of the undergraduate biology students surveyed in Florida by Hacisalihoglu (2020) indicated that their education had been negatively impacted by the enforced switch to online learning.

In this study, we consider many of the issues described above, along with how students in a Calculus 3 course taught using two different modalities fared once they all returned to campus to take the subsequent Differential Equations course. We look at the results from both semesters, as well as those from the previous semester when the students took Calculus 2, to ensure that any disparities between the two groups were not present at the outset. As a result, we consider the following three research questions:

1. As measured by their grade in Calculus 2, was there a difference in ability between those who registered for the online class of Calculus 3 and those who registered for the face-to-face class?

2. Was there a difference between the scores of the two groups of Calculus 3 students at the beginning of the course, and did that change by the end of the semester?

3. When all the students returned to the campus to take Differential Equations, was there a difference in performance between those who had taken Calculus 3 online and those who had taken it face-to-face?

THEORETICAL FRAMEWORK

While it is the body of recent literature described earlier that frames this study in the context of the pandemic, a more general framework comes from the community of inquiry model, which was applied to online learning by Garrison et al. (1999). It emphasizes how the overall educational experience is affected by a combination of the teaching presence, the social presence, and the cognitive presence (see Figure 1).

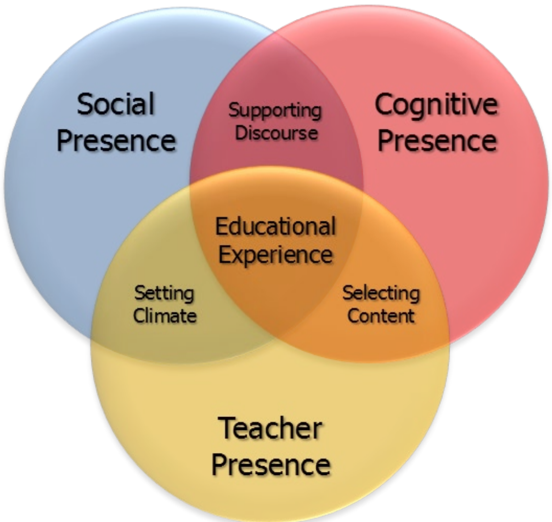

Figure 1: Elements of the educational experience

Teacher presence refers to the design of the curriculum, the creation of course materials, and the nature of the assessments, which will typically be adapted according to the instructional modality. Social presence refers to how students are able to project their own identity into the learning experience through interaction with their instructor and their peers. This is perhaps the most difficult aspect to replicate when moving from face-to-face instruction to online learning, especially if the instruction is asynchronous. Cognitive presence refers to how students develop understanding through reflection and communication, and is again dependent on the learning environment, something that is often beyond the control of the instructor when students are outside the classroom.

METHOD

The population initially consisted of 49 students taking an online class of Calculus 3 during the Spring 2021 semester, and another 49 students taking the course face-to-face with the same instructor. Aside from the instructional modality, the students in the two classes were taught in the same manner, with lectures, midterm exams, and the final exam being almost identical to one another. The first research question caused us to analyse the grades obtained by the students in the two classes when they took and passed the prerequisite course, Calculus 2. After filtering out students who took the course either in high school or at a two-year college (which led to unequal sample sizes – 36 in the online class and 35 in the face-to-face class), we converted their grades to GPA points according to the usual four-point scale (A = 4, B = 3, C = 2, D = 1, F = 0) and ran a Welch's t-test to see if there was a difference in the mean values.

The second research question required us to analyse the difference in performance between students in the online class of Calculus 3 and those in the face-to-face class by considering the scores from the first midterm exam and the end of semester scores. A Mann-Whitney U test was performed in both cases. The third research question caused us to analyse the performance of the online students when returning to campus to take the subsequent Differential Equations course and compare it with those whose face-to-face modality did not change. By converting their grades to GPA points according to the usual four-point scale we were able to run a Welch's t-test to analyse the mean values. At the conclusion of the semester, students were sent a brief survey questionnaire to gather qualitative data regarding their perception of online courses.

RESULTS

Was there a difference in attainment between the two groups before entering the class?

Since the two groups were non-overlapping and the sample sizes different, a Welch's t-test was applied to determine if there was a difference in the course grades (as measured by GPA) between the scores of the 36 online students and the 35 face-to-face students in the prerequisite Calculus 2 class, which everyone took face-to-face. No statistically significant difference was found ($t = -0.26$, $df = 66$, $p = 0.80$), although the online students in Calculus 3 had a slightly lower GPA in Calculus 2 (2.96) than the face-to-face students (3.00).

Was there a difference in attainment between the two groups during the course?

Given the nonparametric nature of the data, and the fact that the two groups were independent, a Mann-Whitney U test was applied to calculate whether there was a difference between the scores of the 49 face-to-face students (median = 70) and the 49 online students (median = 60) on the first midterm exam. A statistically significant difference was found ($U = 868$, $z = 2.077$, $p = 0.038$), with the face-to-face students performing better.

Similarly, when the overall semester scores were tabulated, the Mann-Whitney U test was again applied to calculate whether there was a difference between the scores of the face-to-face students (median = 81.05) and the online students (median = 75.6). Again, a statistically significant difference was found ($U = 461.5$, $z = 2.095$, $p = 0.037$), resulting in an average GPA of 2.83 in the face-to-face class and 2.36 in the online class.

Was there a difference in attainment between the two groups when taking the next class?

Since the two groups were non-overlapping and the sample sizes different, a Welch's t-test was applied to determine if there was a difference in the course grades (as measured by GPA) between the scores of the 22 students who took Differential Equations after taking the online class of Calculus 3 and the 28 students who took Differential Equations after taking the face-to-face class of Calculus 3. The average difference was almost half a grade (0.43 GPA points, compared with a difference of only 0.1 GPA points in their Calculus 2 grades), with the mean GPA in the former class being 2.27 and the mean GPA in the latter class being 2.70 ($t = -1.16$, $df = 40$, $p = 0.13$). These results are summarised in Figure 2.

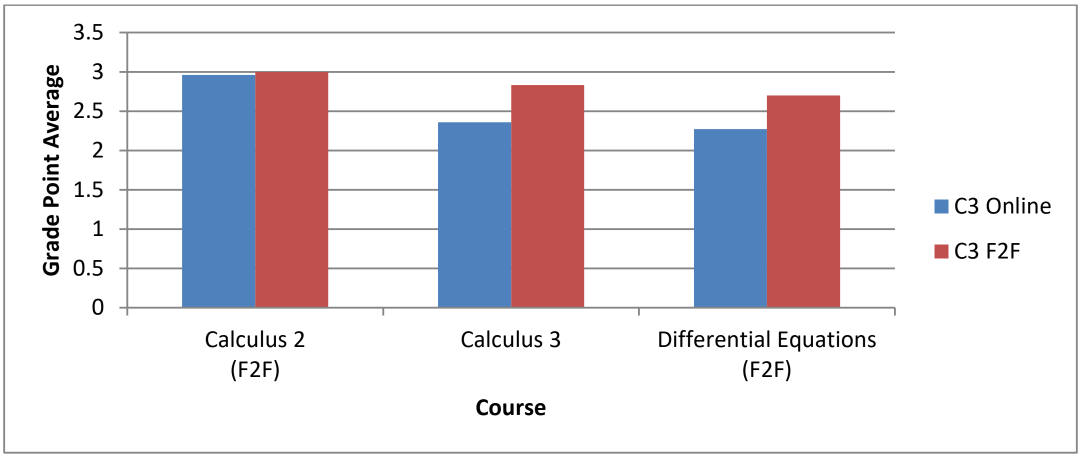

Figure 2: A comparison of attainment by modality among the different courses

Qualitative data

Having recently taken classes both face-to-face and online, students were asked for their preferred modality, with the most representative feedback included below. Even though some students believed online classes to be easier, almost all preferred the face-to-face format, believing that they learned more and were better able to focus, with many citing how they value the increased interaction with the instructor and their peers.

> I firmly believe that face-to-face classes are better for interaction and learning since they are more personable. Online classes have their purpose, but face-to-face delivery makes the material stick more easily. In addition, being able to connect and learn from the professors and with other students makes dry content more tolerable and usually helps to secure foundational knowledge. Another advantage to face-to-face classes is

> being able to ask questions and receive immediate answers which leads to quicker comprehension and subject clarity. Overall, I would only choose online classes if I knew the class was very easy, if I knew the class material was not important to my goals, or if I needed more time for other classes.

Students were asked whether they believe that learning mathematics is easier in a face-to-face class rather than one taught online. There was unanimous agreement among the participants that this was the case.

> I believe that online math classes make the subject matter more difficult to comprehend for most students. They are much less engaging. I am okay with self-learning but reading the textbook constantly is not my strength. This especially matters in a class where I need to fully comprehend the material for application and future classes.

Finally, students were asked if they believe the pandemic will have a long-term effect on the number of classes taught using either online or mixed mode modalities. Again, there was unanimous agreement that this will happen.

> Absolutely. I think many courses will choose to offer hybrid classes so that they include the benefits of both modalities, including flexibility and interactivity. I also think that many general education courses will continue to stay online as many students require this flexibility due to time conflicts, other priorities, or perhaps a disability. I think it is good to have the choice so students can find a balance which works for them.

DISCUSSION

The first research question led to an analysis of whether the students in the online class were weaker to begin with versus their counterparts in the face-to-face class, as measured by their performance in the prerequisite course. It was found that the difference between the two cohorts was negligible. This contrasts with the paper by Protopsaltis and Baum (2019) which found that weaker students are more likely to take online college classes but end up doing worse in them. However, in this instance, the unique circumstances surrounding the pandemic caused enrolment in the various modalities to go beyond educational strategies, with many students choosing online classes due to health concerns and the scarcity of face-to-face options.

Despite there being little variation in the calibre of the two groups of students upon entering the Calculus 3 class, the results show a statistically significant difference between the scores of those taking the class face-to-face and those taking it online, with the former group doing better from the outset. There was a statistically significant difference between the two groups on the first midterm exam, and this disparity was maintained throughout the semester. The results are in line with several studies (Alpert et al., 2016; Dynarski, 2017) that have questioned whether online classes are as effective as the equivalent classes taught in-person. Protopsaltis and Baum (2019) described how there is a strong body of evidence that has emphasized the critical role of frequent and meaningful interaction between students and instructors, which is often lacking in online courses. The difficulties that students appear to have in online classes appear to be magnified in theoretical disciplines such as mathematics (Griffiths, 2020; Smith et al., 2008; Xu & Jaggars, 2014).

That being said, one needs to consider the circumstances caused by the pandemic when putting the results in context, given the unique challenges faced by students and instructors. One important factor is the effect that the Covid-19 pandemic had on the mental health of students, which almost certainly contributed to the relatively poor performance of those taking online classes. Several studies have found that students across the globe struggled with aspects of mental health during the pandemic (Essadek & Rabeyron, 2020; Wang et al., 2020), leading to a detrimental effect on learning outcomes. In addition, faculty were forced to adapt to new teaching modalities at short notice and without much training. Although every effort was made in this instance to ensure a level playing field in terms of the quality of instruction and the integrity of the testing process, it is difficult to ensure that the level of interaction is maintained when teaching an online class. Jones and Sharma (2020) caution that teachers who are good in a physical classroom will not suddenly transform into great online instructors.

There have been few longitudinal studies assessing the longer-term impact of taking classes online, but the results here, showing an average difference of 0.43 GPA points between the students in Differential Equations who had taken a face-to-face class of Calculus 3 and those who had taken an online class is very much in line with the comprehensive study including this topic conducted by Bettinger et al. (2017) who found a difference of 0.42 GPA points for courses in the same subject area as the treated course. This indicates that some initial remediation might be needed for students returning to the campus having taken classes online.

The qualitative data collected by this study indicates that students believe that the pandemic will have a lasting effect on higher education, with a wide range of teaching modalities continuing to be offered. However, there was general agreement that the increased level of engagement with faculty and other students offered by face-to-face courses aids comprehension, and that this is particularly true with regard to theoretical disciplines such as mathematics. These findings are in line with other studies which indicate that learning mathematics is different from learning other subjects (Mullen et al., 2021) and that, more broadly, students prefer face-to-face instruction when theoretical concepts must be deeply understood (Paechter & Maier, 2010).

CONCLUSION

While some contend that face-to-face instruction will remain the dominant method of educational delivery (Gilbert et al., 2021), there is little doubt that online courses will continue to be offered in greater numbers compared with the situation before the Covid-19 pandemic. However, the results of this study are in line with several others in showing a significant difference in the level of attainment between students taking classes online and those taking the same class face-to-face. The central issue therefore is how to best implement the positive aspects of online instruction in a way that retains the elements of engagement that enable students to do better in face-to-face classes.

This appears to be magnified in mathematics and other scientific disciplines. Significant pedagogical development is needed to overcome the reduced interaction that students experience when taking online classes and educators should be particularly aware of this when designing courses where a significant amount of theoretical material must be understood. Trenholm et al. (2016) indicated the urgency of moving toward student-centred pedagogies when designing online mathematics courses, and this issue will become increasingly acute post-pandemic when more and more such courses are expected to be offered.

This study also confirms that any disadvantage experienced by students in online courses can have a lasting effect when they return to the classroom. As a result, educators and administrators need to be judicious in how resources are allocated, especially in courses that begin a sequence such as Calculus 1. While it is tempting to take heavily populated lower-level courses and deliver them online, it has been repeatedly shown that weaker and less experienced students struggle more with the aspects of self-motivation and time management required to succeed in online classes (Bettinger et al., 2017; Kalman et al., 2020; Xu & Jaggars, 2014). Blended modalities that incorporate the positive aspects of both online and face-to-face courses perhaps offer a better alternative, with several studies making the case that technology adds to the learning experience when it supplements rather than replaces face-to-face interaction (Griffiths, 2015; Protopsaltis & Baum, 2019). An initial period of remediation may also be required for students transitioning within the same discipline from online courses to face-to-face instruction, especially when part of a sequence.

There are several avenues for further research that follow on from this study, including an exploration of whether similar results are obtained when broadening the research questions to larger populations and different subject areas within mathematics. Over time we would expect a convergence in outcomes among the different modalities as students and instructors adjust to the new normal.

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Vertical coherence in the teaching of integrals? An example from complex analysis

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Students encounter integrals in a wide range of mathematical domains, often in relation to the ideas of measuring and mean values. Using the integral as an example, this paper discusses one aspect of the teaching and learning of advanced mathematics, namely, the use of core ideas of mathematical concepts to establish vertical coherence between mathematical domains in teaching. It will be explained to what extent the ideas of measuring and mean values can be used for complex path integrals. Then, the epistemological considerations in this paper are enhanced with a case from an expert interview. This case exemplifies the application of the two aforementioned ideas to complex path integrals, but it also points to a potential overgeneralisation from real to complex analysis.

Keywords: Teaching and learning of specific topics in university mathematics, epistemological studies of mathematical topics, complex analysis, integrals, mean values.

INTRODUCTION

Mathematics educators at university are driven by the question what thriving teaching at the university level may look like. The transition to university mathematics, the double discontinuity in mathematics teacher training, or teaching mathematics in service courses or for the workplace have already been studied quite a lot (e.g., Biza et al., 2016; Gueudet et al., 2016). In addition, mathematics education research has emphasised that learners' previous conceptions of a mathematical concept influence their mathematical thinking when the mathematical concept reappears in other contexts, has been further abstracted or modified (e.g., Biza, 2021; Kontorovich, 2018; McGowen & Tall, 2010). Moreover, some authors suggest grounding mathematics teaching in core ideas (e.g., "basic ideas" and "aspects", vom Hofe & Blum, 2016; Greefrath et al., 2016; "fundamental ideas", Vohns, 2016). However, less research and didactic materials are available for more advanced mathematics (≈2nd year at university onwards). In this vein, Winsløw et al. (2021) summarise:

> Material that identifies fundamental or central ideas, provides insight into learning difficulties or obstacles for the students and that shows possible remedies. Such content-specific mathematics education knowledge (didactics) is available for teaching at school level, for instance to know about different ways to approach and organise the teaching of derivatives or integrals (cf. Greefrath, Oldenburg, Siller, Ulm, & Weigand, 2016). Similar expositions are inaccessible or unavailable when it comes to more advanced subjects (e.g., linear algebra) and their teaching at university level. (p. 74)

Particularly in the context of the sequencing and modularisation of mathematics study programmes, it is important to create at least some kind of coherence, for example in the sequence from calculus and analysis in one and several variables to metric space theory or complex analysis (cf. Hochmuth et al., 2021). Generally speaking, if mathematics educators intend to create coherence between different mathematics courses in undergraduate curricula, they need to identify intersections where lecturers can draw on students' prior knowledge.

In this paper, this issue is investigated using the example of integration. Accordingly, the paper is mainly of epistemological nature. The growing body of literature in mathematics education on complex analysis (e.g., Gluchoff, 1991; Hancock, 2018; Hanke, 2020; Oehrtman et al., 2019; Soto & Oehrtman, 2022) indicates a lack of endorsed, but also idiosyncratic, interpretations of complex path integrals. For example, Hancock's (2018), Oehrtman et al.'s (2019), Hanke's (2020), and Soto's and Oehrtman's (2022) case studies indicate that both students and experts struggle to interpret complex path integrals or to express what is accumulated here, and that they occasionally blend interpretations from real to complex integrals. Therefore, I address the question *"Which epistemological challenges arise when the ideas of mean value formation are transferred to complex path integrals?"* mostly from a subject-matter didactical point of view (Greefrath et al., 2016). Additionally, excerpts from an interview with a lecturer of complex analysis illustrate the transfer of the two aforementioned ideas to complex path integrals and a potential overgeneralisation from real to complex analysis. Finally, this epistemological study is more broadly embedded into challenges for the teaching and learning of advanced mathematics.

THE CROSS-CURRICULAR CONCEPT OF INTEGRAL AND MEAN VALUE

Different kinds of integrals appear at various places in undergraduate curricula. Thus, they may be considered as polysemous "cross-curricular concepts", that is, they are "reconsidered in different domains" and their "domanial shift and the substantial change are potential sources for students' difficulties and mistakes" (Kontorovich, 2018, p. 6). Therefore, it seems quite likely that learners may wonder how a newly introduced integral connects to previously encountered integrals—after all, all these notions are baptised "integral" and symbolised with the sign $\int$. Thus, it is important to know what the interpretations of some integrals look like in the context of the others and which constraints arise when these interpretations are transferred.

Aspects and basic ideas of real integrals

As mentioned in the quote by Winsløw et al. (2021), there are guidelines for the teaching of Riemann integrals (e.g., Greefrath et al., 2016). The pairing of *basic ideas* (or sometimes also translated with *basic mental models*; vom Hofe & Blum, 2016) and *aspects* of a mathematical concept (Greefrath et al., 2016) was developed in German subject matter didactics. An *aspect* of a mathematical concept is "a subdomain of the concept that can be used to characterize it on the basis of mathematical content" (Greefrath et al., p. 101). In this sense, an aspect of a mathematical concept is an idea that can lead to a definition of the mathematical concept. A *basic idea* of a mathematical concept is a "conceptual interpretation that gives it meaning" (Greefrath et al., 2016, p. 101). Basic ideas link the mathematical concept "back to a familiar knowledge or experiences, or back to (mentally) represented actions" (vom Hofe & Blum, 2016, p. 230). Greefrath et al. (2016, pp. 114–116) identified three aspects of the Riemann integral:

(i) The *product sum aspect* characterises it in terms of limits of Riemann sums and emphasises the idea of a generalised sum.
(ii) The *anti-derivative aspect* characterises it as the difference between a primitive function of the integrand at the upper and lower limits of integration.
(iii) The *measure aspect* asserts that Riemann integrals satisfy "fundamental properties of measure" when applied to measure areas, lengths, or volumes (p. 115).

Basic ideas for Riemann integrals are as follows (Greefrath et al., 2016, p. 116–121):

(i) The *basic idea of area* interprets Riemann integrals as the signed area enclosed by the graph of the integrand.
(ii) If one considers the integrand as the rate of change of a quantity, the *basic idea of (re)construction* asserts that this quantity can be reconstructed in terms of the integral.
(iii) The *basic idea of accumulation* identifies integrals as accumulations of quantities.
(iv) The *basic idea of average value* relates integrals to averages.

In the basic idea of accumulation, the integral is interpreted as a continuous version of sum "obtained by accumulating or aggregating multiple partial products" (Greefrath et al., 2016, p. 120). Similarly, in the basic idea of (re)construction, integrals reconstruct a quantity whose derivative is the given integrand. The basic idea of average is reasonable for two reasons. First, by the *mean value theorem*, there is a $\xi \in [a, b]$ such that $\int_a^b f(t)\,\mathrm{d}t = f(\xi)(b-a)$ if f is continuous. Second, the integral is like a continuous version of the arithmetic mean: If $[a, b]$ is partitioned into n equidistant subintervals $[t_{k-1}, t_k]$ and $\xi_k \in [t_{k-1}, t_k]$ $(k = 1, \dots, n)$, the arithmetic means $\frac{1}{b-a}\sum_{k=1}^{n}\frac{b-a}{n} f(\xi_k)$ converge to $\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$ as $n \to \infty$ if the integral exists. In addition, for $A \equiv \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$ we have $\int_a^b A\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t$.

The bottom line is this: All these aspects or basic images emphasise different facets of integrals, but it is always the *integrand* that is summed up, used for (re)constructions, or averaged. Having encountered one or multiple of these aspects or basic ideas, it is quite natural to wonder what these may look like for "new" integrals (e.g., double integrals, Lebesgue integrals, and of course, complex path integrals).

Complex path integrals

As in calculus and analysis of one or several variables, integrals appear in complex analysis along with the concept of derivative for complex functions. These *complex path integrals* $\int_\gamma f(z)\,\mathrm{d}z$ are *path* integrals since the domain of integration is a path $\gamma: [a, b] \to \mathbb{C}$ and the integrands are complex-valued functions f on the trace $\mathrm{tr}(\gamma)$ (for simplicity, we will assume that γ is continuously differentiable and f is continuous). One can define complex path integrals via *product sums* and accordingly connect them to the product sum aspect (e.g., Needham, 1997), or directly set

$$(*) \qquad \int_\gamma f(z)\,\mathrm{d}z := \int_a^b f\big(\gamma(t)\big)\cdot\gamma'(t)\,\mathrm{d}t$$

(e.g., Lang, 1999). This is very similar to the definition of *real* path integrals, where γ' is also multiplied to the integrand [1]. The difference, however, is in the choice of product (e.g., complex multiplication vs. scalar product in $\mathbb{C} \cong \mathbb{R}^2$).

Complex path integrals satisfy properties like "$* \mapsto \int_\gamma * \,\mathrm{d}z$ is $\mathbb{C}$-linear", "$* \mapsto \int_* f(z)\,\mathrm{d}z$ is additive", and the inequality "$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq L(\gamma) \max\limits_{z\in \mathrm{tr}(\gamma)} |f(z)|$", where $L(\gamma)$ is the length of γ (Lang, 1999). These are quite similar to properties of Riemann integrals, which are also linear with respect to the integrand, additive with respect to boundaries of integration, and satisfy $\left|\int_a^b g(t)\,\mathrm{d}t\right| \leq |b-a| \max\limits_{a\leq t\leq b} |g(t)|$. Alongside these properties, the interchangeability of complex path integrals with uniform limits of integrands, only a small number of properties are required to reach the first milestones in a first course on complex analysis, such as *Cauchy's integral theorem* or the *residue theorem* (e.g., Lang, 1999) [2]. However, even though complex path integrals share properties like those of Riemann integrals, they also behave quite strangely. Gluchoff (1991) summarises this succinctly:

> My experience is that students are mystified on first exposure to this concept, and working examples by the formula $\int_a^b f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t$ can be a baffling experience; what sense is a beginning student to make of the results
>
> $\int_{|z|=1} \mathrm{Re}\, z\,\mathrm{d}z = \pi i$ or $\int_1^i \frac{i}{z}\,\mathrm{d}z = \frac{-\pi}{2}$? (pp. 641–642; notation adopted)

For example, in the left integral, the integrand is purely real and the path of integration is distributed uniformly around the origin, yet the integral is purely imaginary.

An average interpretation of complex path integrals

We can now ask what is measured in the case of complex path integrals. If $h = u + iv$ is a complex-valued function on $\mathrm{tr}(\gamma)$, then $\operatorname*{av}\limits_{z\in\mathrm{tr}(\gamma)}[h(z)] := \frac{1}{L(\gamma)}\int_\gamma u\,\mathrm{d}\mathbf{s} + i\frac{1}{L(\gamma)}\int_\gamma v\,\mathrm{d}\mathbf{s}$ is the mean value of h along γ, directly analogous to the case of Riemann integrals [1]. However, this is not what the complex path integral measures. For simplicity, assume that γ is regular and has length $L(\gamma) \neq 0$, and T is the "unit tangent vector" $\gamma'/|\gamma'|$. Then, for each $n \in \mathbb{N}$ consider an equidistant partition $z_0, z_1, \ldots, z_n$ of $\mathrm{tr}(\gamma)$ such that $|z_k - z_{k-1}| = L(\gamma)/n$ and points ξ_k between z_{k-1} and z_k on $\mathrm{tr}(\gamma)$ ($k = 1, \ldots, n$), Gluchoff (1991, p. 642) shows that

$$(\star) \qquad \frac{1}{L(\gamma)}\int_\gamma f(z)\,\mathrm{d}z = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\xi_k)\frac{z_k - z_{k-1}}{|z_k - z_{k-1}|} = \operatorname*{av}_{z\in\mathrm{tr}(\gamma)}[f(z)T(z)].$$

In contrast to the basic idea of average value for Riemann integrals, the average here does *not* refer to the integrand f, but to $f \cdot T$! This is remarkable and odd at the same time because ($\star$) is derived analogously to the basic idea of average value in the real case (Gluchoff, 1991). Therefore, it cannot be said that complex path integrals (modulo the constant $L(\gamma)$) represent the average of f along γ.

Gluchoff's (1991) average interpretation also conflicts with other properties of integrals and mean values students encounter in courses on calculus and analysis:

- The *mean value theorem* (see above) does *not* directly generalise: For example, if γ is the path from along the line segment from $c \in \mathbb{C}$ to $d \in \mathbb{C}$, there need not exist a $\xi \in \operatorname{tr}(\gamma)$ such that $\int_\gamma f(z)\,\mathrm{d}z = f(\xi)(c-d)$ (e.g., $f(z) = e^{iz}$, $c = 0$, $d = 1$). Instead, there are $\xi_1, \xi_2 \in \operatorname{tr}(\gamma)$ such that $\int_\gamma f(z)\,\mathrm{d}z = (d-c)\left(\operatorname{Re}\left(f(\xi_1)\right) + i \operatorname{Im}\left(f(\xi_2)\right)\right)$ (Rodríguez et al., 2013, p. 109).
- If A is the average of f along $\operatorname{tr}(\gamma)$ (i.e., $= \underset{z\in\operatorname{tr}(\gamma)}{\operatorname{av}}[f(z)]$), it is *not* true that $\int_\gamma A\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z$. Let me repeat: This is because the "correct" average involves $f \cdot T$!

Gluchoff's (1991) average formula seems rather unknown (at least, I could barely find references to it in the literature). This suggests that average values are misleading or do not occur in complex analysis, doesn't it? No:

Cauchy's integral formula, average values, and another example

Recall that *Cauchy's integral formula* for a holomorphic (i.e., complex differentiable) function f on an open neighbourhood of the ball $B_r(\omega)$ of radius r around $\omega \in \mathbb{C}$ asserts that $f(\omega) = \frac{1}{2\pi i}\int_{\partial B_r(\omega)} \frac{f(z)}{z-\omega}\,\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} f\left(\omega + re^{it}\right)\mathrm{d}t$ (Lang, 1999, p. 145). Hence, $f(\omega)$ is the average of f on the boundary of the ball (Needham, 1997, p. 428). However, this average is *not* the complex path integral of f along $\partial B_r(\omega)$ itself modulo one of the constants $1/2\pi$, $1/2\pi i$, or $1/2\pi r$ ($2\pi r$ is the length of $\partial B_r(\omega)$). In fact, the complex path integral $\int_{\partial B_r(\omega)} f(z)\,\mathrm{d}z$ is 0 by *Cauchy's theorem* (Lang, 1999, p. 116)!

Another example may once again illustrate what we have seen so far. Take $f(z) \equiv 7$ and a path that traverses the boundary of the unit circle $\partial B_1(0)$ once anticlockwise. Then, the average of f along that path clearly is $\underset{z\in\partial B_1(0)}{\operatorname{av}}[f(z)] = 7$. Moreover:

- $\int_{\partial B_1(0)} f(z)\,\mathrm{d}z = 0$ by explicit computation or Cauchy's integral theorem,
- $7 = f(0) = \frac{1}{2\pi i}\int_{\partial B_1(0)} \frac{f(z)}{z-0}\,\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,\mathrm{d}t$ by Cauchy's integral formula for $\omega = 0$ and $r = 1$, but
- $\underset{z\in\partial B_1(0)}{\operatorname{av}}[f(z)T(z)] = \underset{z\in\partial B_1(0)}{\operatorname{av}}[7iz] = 7i \underset{z\in\partial B_1(0)}{\operatorname{av}}[z] = 0$ (since $T(z) = iz$ for $z \in \partial B_1(0)$).

THE CASE OF SEBASTIAN

In an interview study of mine, lecturers in complex analysis were asked how they interpret complex path integrals (cf. Hanke, 2020). Sebastian is a professor of mathematics, who has also taught complex analysis. His case is chosen for this paper to underline the previous observations on the interpretation of integration as measuring or averaging, but also to underline the subtleties when applying these interpretations to complex path integrals. Before asking for an interpretation of $\int_\gamma f(z)\,\mathrm{d}z$, the interviewer recalls the basic image of area for integrals in real analysis. Sebastian immediately interrupts and rejects it by saying "I find this bad". Instead, he explains his general view on integration (excerpts were translated from German and slightly polished for better readability):

Sebastian: […] I would always tell my pupils: Actually, one should think about mean values, in particular when one has Lebesgue integration in mind, and measures. It is about measuring. And, uhm, this geometric intuition [of area; EH.] can destroy this higher dimensional situation. […] And therefore I find it much better if one imagines: integration is mean value formation.

By referring to Lebesgue integration, Sebastian presents integration as a cross-curricular concept. The pedagogical decision that students "should think about mean values […] and measures" with respect to integration underscores the potential practise of including core ideas to the teaching of integrals. Doing so, Sebastian rephrases the basic idea of average value and the measure aspect for a broader context of integration. In particular, he does not only not take up the basic idea of area the interviewer mentioned, but additionally, he values this idea as unhelpful since it "destroy[s] this higher dimensional situation". At this point, Sebastian is likely referring to $\mathbb{C}$ as $\mathbb{R}^2$, which eventually implies that the graphs of complex functions are subsets of $\mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$.

So far, Sebastian's utterances are about integrals in general, not specifically complex path integrals. Accordingly, the interviewer asks what the mean value is taken of:

Interviewer: And which/ mean value of what?

Sebastian: Yes, of what's, uh, in the integrand, so to speak. […] Yes, uh, for me this is simply the mean value of the complex numbers, which I grab along this path. Therefore this is again a complex number because it does not have a geometrical area meaning, but mean value formation over the objects, which one quasi sees along the path. And, uhm, in my view this has nothing to do with area. […] This is actually the rotation that one measures on the plane. And this/ here we are again at what we discussed previously, that these, uh, complex numbers always have this character of an amplitwist.

Sebastian describes himself as an actor who "grab[s]" complex numbers along the path to find their mean value. Since this mean value is a complex number, he rejects the area interpretation again. However, Sebastian emphasises another geometric idea: When repeating his general view on integration as measuring "the objects, which one sees along the path", Sebastian links this process to "the rotation […] on the plane" and "complex numbers" to "amplitwists" (cf. Needham, 1997).

Sebastian: […] And on the other hand, I just have these values of the function f of z and, uh, f of z does now what we have seen previously, yes, this now maps some

> portion of what one has here with this [grid; EH.; see Fig. 1] […]. And, uhm, geometrically speaking, this involves such a stretching and a twist probably, yes, so where the grid points are somehow distorted or so. […] And I average this effect along this path so to speak. [… a few turns later:] Okay, for me the imagination is that I think of a small neighbourhood at each point and from that I see how the effect of f is in some abstract [plane; EH.; see the right part of Fig. 1] lying somewhere else. […] I have not thought about that before, uh. Simply/ this four dimensionality is a hurdle, at least for me, to imagine this better. In particular when it comes to those path integration issues. So the number f of z really is a linear amplitwist for me. And this effect is averaged along this path and this is what the integral means to me.

Here, Sebastian describes that the integrand f acts on grids in small neighbourhoods of the points on $\mathrm{tr}(\gamma)$ by mapping them to another plane (right part of Fig. 1). This "effect" is "averaged along this path" and the values $f(z)$ are identified with "linear amplitwist[s]". Additionally, Sebastian emphasises that "four dimensionality is a hurdle" for him, particularly at this point when imagining the "path integration issues".

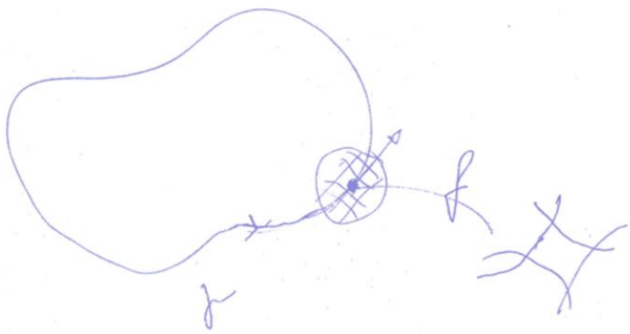

Figure 1: Sebastian's drawing of a path γ and the effect of a function f as amplitwist.

It seems that Sebastian omits the γ' or T in his interpretation and focuses mainly on $f(z)$. In particular, while drawing the arrow at the thickened point in Fig. 1 (possibly representing γ' or T), he explains that "the parametrisation of the path […] will be quasi cancelled or neutralised right by the definition of the path integral". Although the "parametrisation" is in fact not negligible (see, e.g., $(*)$ and $(\star)$), let me emphasise again that Sebastian is adopting the measuring and mean value interpretation to complex path integrals during the interview for the first time. Thus, even though his interpretation may not be fully compatible with those in the literature, it takes into account the geometrical operation induced by the function values on the trace of the path as "amplitwists" (see in particular Needham, 1997): Since *holomorphic* functions are linearly approximable, small changes in the input (Δz; "small grid" in Fig. 1) cause small changes in the output (Δf), which are approximately given as copies of Δz rotated and dilated by the derivative: $\Delta f \approx f'(z)\Delta z$ ("distorted grid" to the right of f in Fig. 1).

DISCUSSION

This paper investigated the teaching of a cross-curricular topic of integration using the ideas of measuring and mean value formation, especially in relation to complex path integrals. It has been shown that these ideas are applicable to complex path integrals, but they do not work the same way as for real integrals. Gluchoff (1991) showed that $\int_\gamma f(z)\,\mathrm{d}z/L(\gamma)$ is the average of $f \cdot T$ along γ rather than f. Additionally, Cauchy's integral formula relates a function value to the mean value of the function along the boundary of a circle. The fact that the expert Sebastian applied the ideas of measuring and mean value formation to complex path integrals underlines their relevance. However, this case also illustrated the potential overgeneralisation of the relationship between integrals and mean values for Riemann or Lebesgue integrals in the sense that Sebastian thought that complex path integrals yield the mean values of the integrands instead of the mean value of their products with the unit tangential vectors attached to the path of integration (modulo the constant $L(\gamma)$).

Now, since the transfer of some interpretations of integrals to complex analysis is subtle, it is important to ask what this means for the epistemology, teaching, and learning of integrals in advanced lectures, in particular complex analysis. For example, one may ask whether the ideas of measuring or averaging are *locally* important for *some* integrals or *globally* for the cross-curricular concept of integral per se, which then must be worked out locally for each integral again. Of course, one might simply dismiss these ideas for the case of complex path integral in teaching. Then, lecturers still face the question where discrepancies between different interpretations of integrals like measuring or means values occur and what their possible resolutions may look like. Additionally, appropriate interpretations of integrals in complex analysis remain missing. Besides, it seems that one can get along in complex analysis quite well without any interpretation of complex path integrals at all. For building theory in courses on complex analysis, it seems sufficient to state and use a few properties of complex path integrals, which are stored in propositions (e.g., that it is $\mathbb{C}$-linear, additive with respect to paths, and restricts to the Riemann integral for real functions on paths along real intervals, etc.). At least, this is what most texts on complex analysis I consulted do. Is this satisfactory? Probably not as Gluchoff (1991) illustrated.

One difference between the teaching of calculus / real analysis and complex analysis seems to me that in calculus / real analysis visual, physical, or other vivid interpretations are readily available and well known so that it is hard to *not* mention them in class. Therefore, the "real" concepts connect quite well to real life experiences and applications. Perhaps this abundance of interpretations of real integrals leads to the assumption that there should be similar (or new) interpretations for complex path integrals as well (cf. Hanke, 2020). Hence, it seems relevant, at least to me, to reflect on the use of interpretations of cross-curricular topics for their potential applicability in more advanced courses.

At present, we cannot yet foresee how the transfer or potential overgeneralisation of the mean value or measure interpretation may affect the way students interpret or work on tasks on complex path integrals. Rather, more research is needed on how individuals interpret complex path integrals or how they relate the various types of integrals they have encountered (cf. Kontorovich, 2018).

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NOTES

1. Recall that *real path integrals* for a scalar function h and a vector field $(u, v)^T$ are given by $\int_\gamma h \, \mathrm{d}\mathbf{s} := \int_a^b g(\gamma(t)) \cdot |\gamma'(t)| \, \mathrm{d}t$ and $\int_\gamma (u, v)^T \, \mathrm{d}\mathbf{T} := \int_a^b \left(u(\gamma(t)), v(\gamma(t))\right)^T * \gamma'(t) \, \mathrm{d}t$, where $*$ represents the scalar product.

2. I cannot “prove” this claim in this short paper but a careful look at a textbook on complex analysis of your choice will reveal that only a handful of properties of complex path integrals are required to prove milestones such as Cauchy’s theorem, Cauchy’s integral formula, and the residue theorem (e.g., Lang, 1999).

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An epistemological gap between Analysis and Calculus: the case of Nathan

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The terms Analysis and Calculus are widely used in mathematics. It seems that the professional community dealing with research on the teaching and learning of analysis and calculus is gradually realizing that Mathematical Analysis and Calculus are not one and are not the same subject, no matter how closely related they are. We agree that there is a substantial difference between them, leading to genuine didactical challenges. The study reported below provides empirical evidence supporting this claim.

Keywords: Teaching and learning of analysis and calculus, Integral, Fundamental Theorem of Calculus, Meanings

INTRODUCTION

Being mathematics educators, we teach mathematics. Some of our students are future mathematicians. Some of our students are future engineers. Some of our students don't know yet what they want to do after graduation. Considering this, we should ask ourselves: what are we teaching to whom?

Let us consider the known controversies between Newton and Leibniz (Hall, 1920; Meli, 1993; Garber, 2008). Newton's approach was to use mathematics as a tool to explain and further understand natural phenomena. Leibniz's approach to mathematics was intra-mathematical, math for math's sake. He sought to better understand and to investigate mathematical objects and their abstract structures.

In this paper, we will distinguish between Mathematical Analysis and Calculus, building on the work of Topic Study Group 12 "Research and development in the teaching and learning of Calculus" (ICME, 2004) and of the Working Group "Didactic contrasts between Calculus and Analysis" (PME-NA, 2021). The difference and possible tension between Mathematical Analysis and Calculus is under discussion in the professional community (additionally to the groups mentioned above see, for example, Katz & Tall, 2012; Moreno, 2014). As there is currently no solid theoretical framework for this issue, we will describe our own developed approach.

By Mathematical Analysis, we invoke the formulations of Cauchy, Weierstrass, etc. Mathematical Analysis therefore deals with functions, limits, variables. This is done in a logical-symbolic and formal way. On the other hand, Calculus deals with quantities that vary in magnitude, rate of change and accumulation. The quantities covary with each other and have dimensions and units. Calculus requires a variational way of thinking within a natural extra-mathematical context. In this paper we will focus on integration and accumulation, it is therefore important to note that "The central idea of

Calculus to quantify accumulation is not that of antiderivative; it is 'isolate in small intervals-multiply-add'" (PME-NA, 2021). The multiplication is a multiplication of the relevant average rate of change by the length of the small interval.

It seems that the professional community dealing with research on the teaching and learning of analysis and calculus, is gradually realizing that Mathematical Analysis and Calculus are not one and are not the same subject, no matter how closely related they are (see for example: Katz & Tall, 2012; Moreno, 2014; PME-NA, 2021; Rogers, 2005). Analysis is more pure mathematics. Calculus is more applied mathematics. We agree that there is a substantial difference between them, leading to genuine didactical challenges. The study reported below provides empirical evidence supporting this claim.

THE STUDY

Nathan is a valued and experienced teacher at Middle and High School (Advanced Level). Nathan has completed several Analysis courses at university and is very skilled in the subject.

Nathan volunteered to be interviewed in the framework of a research project aiming at identifying students' meanings for rate of change (RoC) and accumulation.

The semi-structured interview with Nathan included the following tasks:

1. Evaluating the accumulated amount of cash, given a graph of the cash flow function.
2. Calculating the length of a curve representing a smooth function.
3. Finding the mass of a thin wire, given its mass density function.

The interview was recorded and transcribed.

FINDINGS

First task – accumulated cash given the RoC graph

In the first task, Nathan was given a graph of cash flow at a certain bank as a function of time from 8:30 to 11:30. Nathan was able to give reasoned and appropriate estimates regarding the behavior of the accumulated cash in this scenario, by effectively linking the accumulated cash with the area between the graph and the x-axis. This included, for example, an estimation of when the bank had less cash than the initial sum (**Error! Reference source not found.**; inscriptions in blue and orange are Nathan's), which was given to be 5 million Shekels.

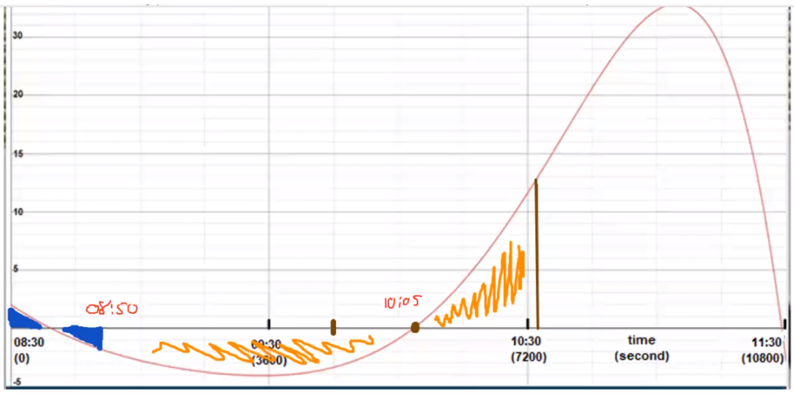

Figure 1: The area marked orange shows when the bank has less cash than initially

Even though Nathan identified the area as representing the amount of cash, and used this to correctly answer the questions he was asked, the collapse metaphor (Oehrtman, 2009) was prevalent throughout his explanations (translated from Hebrew) – "How much money exited the bank? The sum of the length of this segment and the length of this segment and the length of this segment, etc. etc., an integral on the graph of the function between 8:40 and 10:00" (**Error! Reference source not found.**).

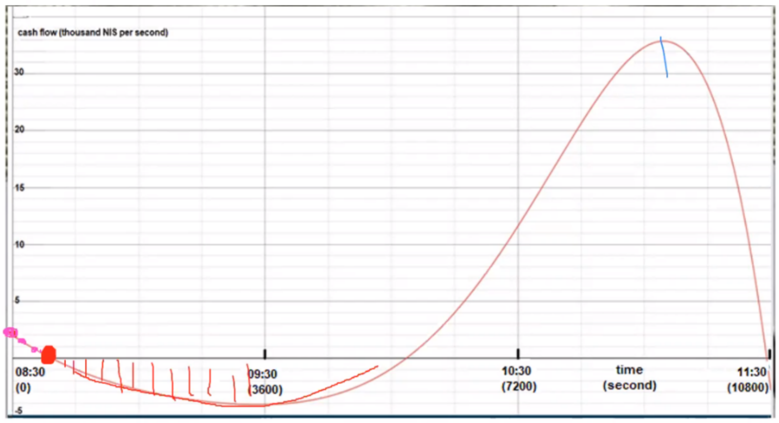

Figure 2: Nathan explains that the amount of money exiting the bank is the sum of the lengths of the red lines

Nathan's conception of integral is based on notions of limit. This is evident from both his use of the collapse metaphor, which is a way of reasoning about limits, and from the mathematical terms he invokes (for more details regarding the collapse metaphor manifestation in Nathan's case, see Noah-Sella et.al., 2022). Example of this can be seen in the following two exchanges:

Nathan: My logic is that when Δx approaches zero, or is even equal to zero, the size of the – I don't want to say rectangle, it's a line. It has no width. It's just a line, and since the width of this line is zero, when we add up all of these lines, we will get the area trapped under the curve.

Interviewer: Can you explain to me what adding up lines is?

Nathan: Adding up lines? Or attaching lines?

Interviewer: Whichever is more convenient for you.

Nathan: Adding up, actually adding up the y-coordinate of every, I mean adding up the lengths of all the lines, meaning adding up the y-coordinate of all the points, the infinite number of points.

Interviewer: You're adding up an infinite number of values, how are you doing that in practice?

Nathan: Using limits, and using an integral on a graph of a function.

…

Nathan: [The integral] sums the y-values when Δx approaches zero. I mean, that's the visualization I have in my head for an integral, that's how I perceive it, like rectangles with zero width – straight perpendiculars.

Interviewer: Is there such a thing as a rectangle whose width is zero?

Nathan: There is a straight line. A rectangle whose width approaches zero. It comes from Riemann and Darboux sums.

Two examples how to interpret points on the graph had been presented by the interviewer in the introduction of the item: "at 8:30 am (time $x = 0$ seconds), $y = 2$, that is 2000 Shekel per second entered the bank's accounts; at 9:30 (time $x = 3600$ seconds), $y =$ -4.1, that is 4100 Shekel per second left the bank's accounts". Although Nathan read these statements aloud accurately, when he recalled the first statement, he omitted the "per second" from the units of measurement.

Although Nathan answered all of the questions correctly, and provided detailed justifications involving the areas, when the interview concluded he expressed a lack of confidence that stems from not having an algebraic representation of the cash flow function – "I felt like I was missing something. The lack of ability to see the function, even if I don't know how to integrate it, even if I can't actually feel it, something here felt very amorphic, and made me somewhat insecure".

In response, Nathan was presented with the algebraic representation of the cash flow function. He exclaimed – "Great! This I can work with!" and proceeded to say "Now I feel like I have something to fall back on, if I'm completely at a loss. […] If I want to substantiate the answer I gave, and to make sure I answer correctly, I know I have the analytical tools, the actual analytical tools to deal with this thing".

Second task – evaluating the length of a curve

In this task Nathan was asked to evaluate the length of a curve given both the formula and graph of a function, and a closed line segment. This question has no extra-mathematical context and is well suited for a Mathematical Analysis approach.

Nathan described how he intends to find the length: "I want to take two points, calculate the distance between them, and make Δx approach zero. Then I'll get the length of a single segment, and that's going to be for any pair of points on the graph of the function. If I integrate this, I will get the length of this segment along the graph of the function".

At first, Nathan struggled with the development of a formula for the length of the curve. This prompted the interviewer to ask for an estimated length. While one might expect Nathan to divide the curve into segments of width h, and repeat the above calculation for some finite h, he instead gave an upper and lower bound using secants and tangents. In spite of his initial struggles, Nathan eventually managed to derive the formula for the general case (Figure 3). While it is possible that he had learned the formula at university, his behavior and responses during the interview suggest that he is at the very least reformulating it, and not reciting it from memory.

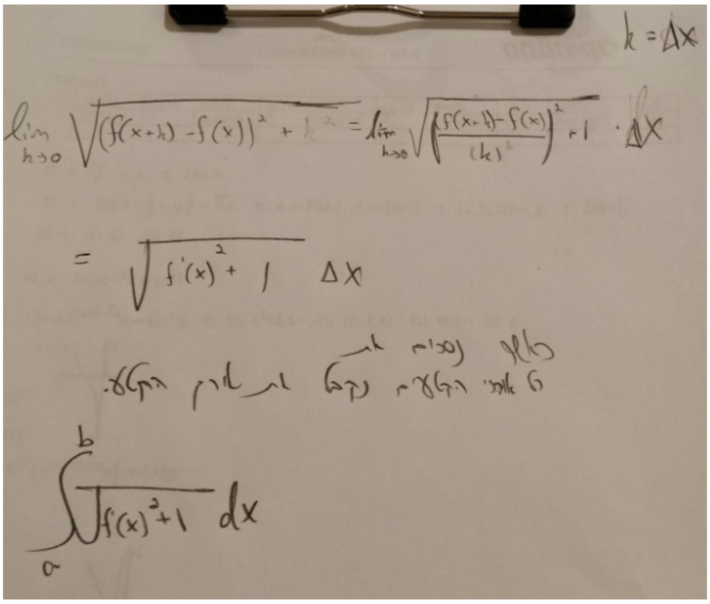

Figure 3: Nathan's solution to find the length of a curve (Hebrew text translation: When we add up all the lengths of the segments we will get the length of the segment)

It seems that Nathan dealt with the second task using a well-established approach from Analysis. He approximated the required length by summing the lengths of small chords. He calculated the length of each small chord using the Pythagorean theorem. Then he used a limit for the purpose of formalizing the idea that the length of the chord is getting progressively smaller (Figure 3). As a result, Nathan gets a formula ($\lim_{h\to 0}\sqrt{\left(f(x+h)-f(x)\right)^2+h^2}$) that structurally (!) doesn't fit with Nathan's knowledge regarding Riemann sums – an expression multiplied by Δx. This is where Nathan started to struggle with the task. Eventually, Nathan succeeded to manipulate the formula algebraically to get a known (to him) representation of the limit of Riemann sum ($\lim_{h\to 0}\sqrt{\left(f'(x)\right)^2+1}\ \Delta x$). Only after this process, he was ready "to replace" this representation by definite integral.

Third task – finding the mass given its density

In this question, Nathan was asked to calculate the mass of a straight wire, given its density (**Error! Reference source not found.**; for the original task in Hebrew, see Figure 5). At first, Nathan immediately replies with the correct integral, reasoning that "at point X, if the density is [the length of the segment] AX then the mass is x, and therefore $\int_0^8 x\,dx$ will give the mass of the wire". Though his result is correct, he does not distinguish between density and mass in his explanation. When the interviewer reminds him of the units of measurement, a conflict arises for Nathan which causes him to lose confidence: "I'm in a spiral with myself".

> Below is an 8 meters long wire AB.
>
> A____B
>
> This wire is made from cutting edge material. The density of the material is not constant.
>
> The density can be calculated in the following way: the density (measured in grams per meter) at any point X on the wire is numerically equal to the length of the segment AX.
>
> Find the mass of the wire (in grams).

Figure 4: Mass from density task

In response to the conflict, Nathan retracts his answer and decides to change course, attempting to find a function representing the mass at each point. This causes him within two minutes to exclaim "all of my confidence is lost", citing that "the density versus mass thing is really confusing me" and "I can't decide with myself what will be the mass at a specific point, I want to take a certain point, express the mass there, and this function will be the integral". He further explains this by saying that if the mass at every point is given by $g(x)$ then the total mass will be $\int_0^8 g(x)\,dx$ (**Error! Reference source not found.**).

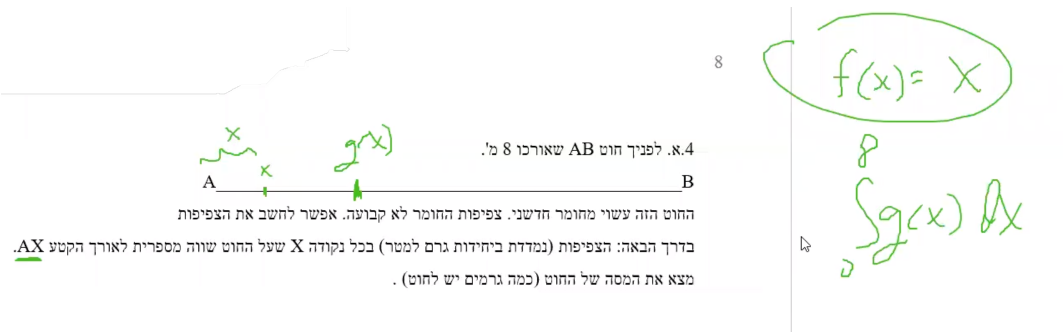

Figure 5: $f(x)$ denotes the density at each point. $g(x)$ denotes the mass at each point, and the integral is the mass of the entire wire

DISCUSSION

Nathan's conception of integral is greatly influenced by the collapse metaphor. In the second task, he invokes the collapse metaphor in the beginning – "I want to take all the points and add them up". When asked what adding up points means, he describes a procedure of calculating the distance between two points on the graph and taking the limit as Δx approaches zero. This illustrates his use of the collapse metaphor to make sense of limits. Thus, Nathan's thought process is rooted in limits, which is a characteristic of Analysis.

Nathan's use of the collapse metaphor seems to lead him to develop the misconception that integrating is summing the y-values of the integrand. Thus, to him the integrand does not represent the rate of change of the quantity accumulated, and dx is a notation and not a quantity, and therefore unitless. This might explain why Nathan changed the given RoC function's units of measurement to Shekel in the first task, and to grams in the third task. It appears that he unwittingly omitted "per second" and "per meter" respectively to amend these quantities to better fit his own meaning of integration and accumulation. Unsurprisingly, when this omission was pointed out to him in the last task, a cognitive conflict followed. This is consistent with Nathan having a background in Analysis, since variable quantities in Calculus have units, while in Mathematical Analysis variables and functions are dimensionless.

It is also important to contemplate why Nathan performed better in the first and second task than in the third. In the second task, one might suggest that the absence of extra-mathematical context means that units of measurement will not be a cause of conflict. However, the first task is set within an extra-mathematical context. We suggest that the graph given in the first task allowed Nathan to convert the problem to a geometric problem involving areas, thus circumventing the need to identify the given function as a rate of change function. This is important to note, since rate of change is a concept central to Calculus. In the third task, since there was no graph given, relating to quantities, rate of change and accumulation was necessary. Consequently, Nathan was unable to complete the task.

The connection between the collapse metaphor and Mathematical Analysis can also be seen in Nathan's use of language. When explaining the dimensional collapse, he isn't referring to the extra-mathematical context of the task, to rate of change nor to accumulation, but to function, limit, and Riemann and Darboux sums. This suggests that Nathan's meaning for integral was formed within Mathematical Analysis.

In addition, Mathematical Analysis deals mainly with functions, whose properties are analyzed mostly using algebraic and symbolic tools. Thus, Nathan's wish for the formula of a function in the first task, and his reaction upon receiving it support the notion that he is more at ease in Mathematical Analysis than in Calculus.

This claim is also supported by Nathan's performance in the second task. When asked to find the length of a curve, he uses formulas and limits. While the development of these formulas could be interpreted as based on the practice of "isolate in small intervals-multiply-add", which is a Calculus practice, when explaining his reasoning, it appears that Nathan operated with the intent of "isolate in small intervals-add", with the multiplication arising as a byproduct, due to his familiarity with the structure of Riemann sums. The multiplication is essential to 'Calculus thinking' since it embodies accumulating from a rate of change. Furthermore, when prompted for an estimate of the length he did not use the same principle. One might expect that Nathan would divide the curve into small segments, approximate their length using a chord, and sum the lengths of all the chords, thus using accumulation reasoning. Instead of dividing the curve into smaller segments, he examined the curve as a whole, looking for an upper and a lower bound for the entire length, using a tangent and a secant. This suggests that he views the length of the curve as a static value (Analysis), and not as an accumulated quantity (Calculus). Consequently, it seems that the Riemann sum technique is only available to Nathan in an algebraic or symbolic context, central to Mathematical Analysis, and not in a numerical context, central to Calculus. Finding the formula for the general case, rather than finding a solution for the given curve is in accordance with the preference for formal proof in Mathematical Analysis.

Considering all of the above, we argue that Nathan shows a high proficiency in Mathematical Analysis problems, and difficulties in Calculus problems, strengthening the assertion that these are epistemologically distinct.

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The Effect of Interactive Tasks in Instructional Videos on Students' Procedural Flexibility

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Especially in higher education, mathematical procedures must not only be mastered in isolation, but also selected and applied flexibly. Considering the growing importance of digital learning, we report on the results of a pilot study to investigate how interactive tasks in instructional videos may help to enhance students' flexible procedural knowledge in deriving polynomials. Economics students (N=43) were given a digital self-learning environment with videos and follow-up learning tasks. In a randomized controlled trial, they either watched videos that were interrupted by interactive tasks or videos without interaction. We found that the students with the interrupting interactive tasks performed significantly better in the post test and therefore seem to have gained greater procedural flexibility.

Keywords: Digital and other resources in university mathematics education, Teaching and learning of analysis and calculus, Derivations of polynomials, Procedural flexibility, Procedural knowledge

INTRODUCTION

In university mathematics, flexible procedural knowledge is considered a key skill (Maciejewski & Star, 2016). It is particularly important for first-year students who, according to Maciejewski and Star (2016), have "only rote procedural ability" (p. 299) at their disposal. Digital learning environments offer special potential for practicing mathematical procedures independently of time and location. They may include instructional videos that explain and illustrate procedures or convey strategies.

Videos have become increasingly popular in higher education, at the latest since the outbreak of the pandemic. However, whereas tasks naturally promote cognitive activation for students, instructional videos typically lack activities that help students to maintain their attention. Therefore, instructional videos often lead to only passive use which might limit the learning effect.

The promotion of flexible procedural knowledge has been investigated in several classroom settings and has been applied to solving equations (e.g. Rittle-Johnson et al., 2012; Rittle-Johnson & Star, 2009) and deriving polynomials (e.g. Maciejewski & Star, 2016). To our knowledge, it is unclear how these findings can be adapted to digital learning environments. In this paper, we focus particularly on how instructional videos can support students' cognitive activation.

We created instructional videos on the product and chain rule to derive polynomials and enriched them with interrupting interactive tasks using the H5P-tool. In a randomized controlled trial, we investigated their effect on students' performance. Our research question is as follows: How do the interactive tasks affect students' learning regarding the acquisition of procedural flexibility?

THEORETICAL BACKGROUND

Procedural flexibility

Star and Newton (2009) define procedural flexibility as the "knowledge of multiple solutions as well as the ability and tendency to selectively choose the most appropriate ones for a given problem and a particular problem-solving goal" (p. 5). Their term "appropriate" refers to the most efficient strategy which is determined by the fewest steps to execute, the effort involved and one's familiarity with the problem type. Later, Liu et al. (2018) put forward a finer classification of the concept by distinguishing potential and practical flexibility. By potential flexibility, they refer to the knowledge of multiple strategies for solving mathematical problems without demanding their correct application. In contrast, the term practical flexibility addresses the performance, thus the concrete use of those strategies.

According to Maciejewski and Star (2016), flexible procedural knowledge can be taught. In a study on calculating derivatives, they successfully conducted an experiment using side-by-side, multiple solution comparison assignments. They found that these assignments had a greater effect on the students' procedural flexibility than a standard set of exercises without any comparison. They also figured out that self-reflection enriches the development of procedural flexibility. It is rather unclear, however, to what extent these results can be transferred to digital learning settings and, in general, how to design online environments for self-directed learning to foster students' flexible procedural knowledge.

Rittle-Johnson et al. (2012) essentially propose two ways to improve both conceptual and procedural flexibility heavily drawing on learning with worked examples: comparison of worked examples and self-explanation. While comparing two correct solutions, students should also consider whether one procedure is more efficient that the other. In this context, the so-called tri-phase flexibility assessment is proposed by Xu et al. (2017). During the first phase, students must solve problems as quickly and accurately as possible. Then, they are asked to generate multiple strategies for each problem. In the third phase, they evaluate their procedures and select the most innovative and appropriate one. The assessment has been evaluated in a study on equation solving. Xu et al. (2017) found that potential flexibility was higher than practical flexibility and that most students were not able to use innovative strategies.

Cognitive activation in digital learning environments with instructional videos

Instructional videos offer specific benefits for self-directed learning. They especially enable dynamic presentations of contents (Cooper & Higgins, 2015) and offer multiple options for visualisation. In mathematics, the speaker can demonstrate how to correctly use mathematical language in oral and is able to add colloquial explanations to lower the threshold. Instructional videos consider the potential heterogeneity as learners can adjust the playback speed, rewind, stop, and watch the video repeatedly.

However, instructional videos lack any means for cognitive activation because they are typically passively consumed. Kulgemeyer (2022) indicates that instructional videos might support an illusion of understanding since animations make explanations appear easier to understand than they are. An illusion of understanding is therefore “the false belief that an explanation was easy, a topic has been thoroughly understood and requires no further instruction” (Kulgemeyer, 2022, p. 3).

Combining instructional videos with follow-up tasks might be a way to counteract the low level of cognitive activation. This idea is supported by Lehner (2018) who states that tasks are powerful means to manage cognitive activation both in analog and digital learning settings. According to Lehner (2018), especially tasks with multiple solutions are useful to enhance cognitive activation.

H5P (https://h5p.org) is a tool to create in-video tasks that can be integrated in moodle. It offers a variety of task formats such as single-choice and multiple-choice questions, cloze texts, and true/false questions. They can pop up automatically during the video and one can adjust that the video stops until the task is completed. In terms of possible didactic scenarios, those in-video tasks can be used to check prior knowledge, to underline central cognitive conflicts, to summarize or to review newly gained knowledge or to think ahead.

METHODS

Participants

We started the study with 51 economics students from a German university. Since eight students did not complete the session, we restrict our analysis to the 43 students who participated until the final tasks were answered. The participants attended a mathematics bridging course which has been offered as an optional support for the course “Mathematics I for Economics”. Due to the pandemic, the bridging course took place virtually via Zoom.

Procedure

The participants were randomly assigned to one of two breakout rooms. The digital learning environment, the pre- and the post-test and the questionnaire were uploaded to a moodle course in which the students were inscribed. The learning environment was intended for self-learning, but the first author and another researcher familiar with digital learning each supervised one breakout room. The students could contact them in case of technical or organizational issues. The procedure took 60 minutes.

Before the self-learning unit, the participants completed two introductory tasks in which they had to apply the product rule and the chain rule. After the treatment, the students were given four final tasks. For given functions, students were to decide which of two proposed strategies was more efficient.

Treatment

During the treatment, the participants independently worked on a digital learning environment. The learning environment consisted of two instructional videos, one on the product rule (03:48 min for the intervention group, 04:02 min for the control group) and one on the chain rule (06:08 min for the intervention group, 06:39 min for the control group), each followed by three follow-up tasks on the respective rule. We opted for screencasts and handwritten style when producing the videos. The tasks were implemented in the videos using the H5P-tool.

In each video, we formed the derivations of two to three polynomials by contrasting two solutions. The application of the power or chain rule was compared to first transforming the function term, e.g., by applying the power laws or the binomial formulas, and then deriving it without the respective rule (see *Fig. 1*). The number of calculation steps and the effort involved were considered the main criteria to select the most reasonable strategy. The cognitive demand of the strategies and what is considered a step still need to be elaborated.

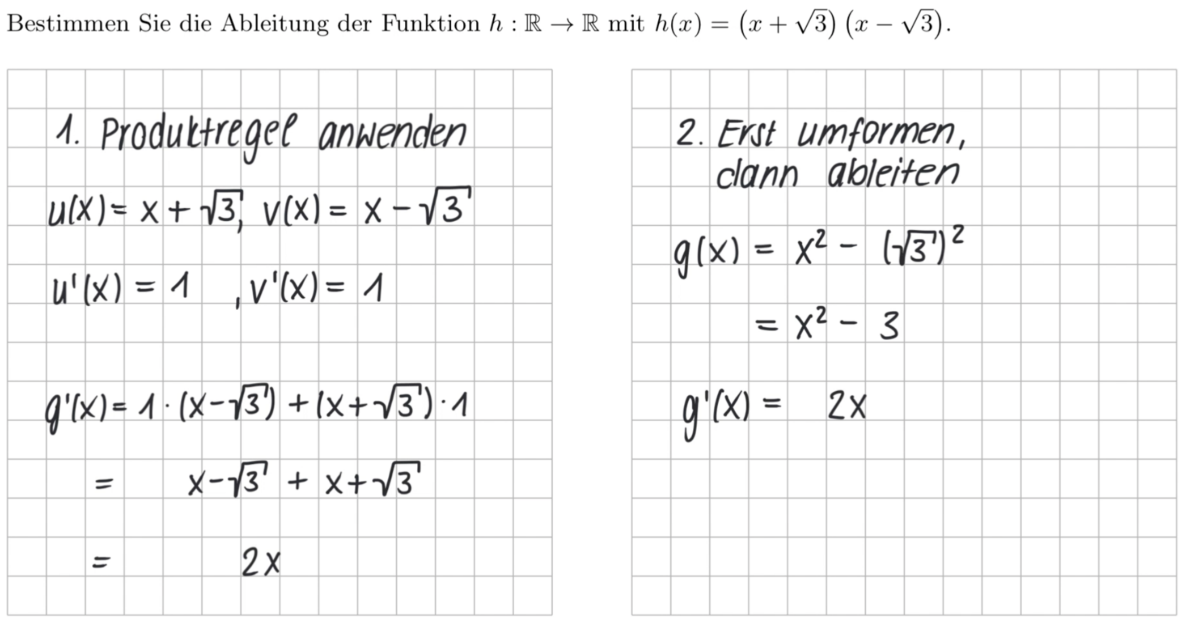

Fig. 1: Multiple solutions on the task "Calculate the derivative of the function $\boldsymbol{h}: \mathbb{R} \to \mathbb{R}, \boldsymbol{h}(\boldsymbol{x}) = \left(\boldsymbol{x}+\sqrt{\mathbf{3}}\right)\left(\boldsymbol{x}-\sqrt{\mathbf{3}}\right)$"

The two treatment conditions differed by the presence of interactive tasks in the videos. Each video comprised three interactive tasks, see *Table 1*. Whilst the intervention group was confronted with these tasks popping up automatically during the video, the control group was given videos without interaction. In their videos, however, the interactive single-choice questions (see *Table 1*) were asked verbally, and the summary provided by the cloze text was also given verbally. The other parts of the learning environment were kept equal.

Video 1: Product rule	Video 2: Chain rule
1. (single choice): „Transform the term $x^3 \cdot x^5$ using the power laws. What does the transformed term look like? "	1. (single choice): „We can form the derivation of $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = (x+1)^8$ with the chain rule. What are the functional equations of the outer function u and the inner function v here? "
2. (single choice): „What is the first derivative of $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 + \pi \cdot x - \sqrt{3} \cdot x - \pi \cdot \sqrt{3}$?"	2. (single choice): „Transform the term $(x^2)^3$ using the power laws. What does the transformed term look like? "
3. (cloze text): „For products of two monomials, for example $x^3 \cdot x^5$, it is reasonable to transform the term with the (power laws/binomial formulas/product rule) before forming the derivative. For products of two sums that are difficult to multiply out, it is reasonable to apply the (product rule /chain rule /power rule) to form the derivative. For the special case $(a+b)(a-b)$ meaning that the summands match in each case, we can transform the term with (third binomial formula /sum rule /product rule) before forming the derivative."	3. (cloze text): „For sums with two or more summands that are exponentiated with a (high/low) exponent, the (product rule/chain rule/power rule) is the best choice to form the derivative. For monomials, for example x^2, that are potentiated, it is more reasonable to transform the term with the (power laws/binomial formulas/chain rule) before forming the derivative."

Table 1: Overview of the in-video interactive tasks

After having watched the video on the product rule, the students completed the following tasks. The follow-up tasks after the video on the chain rule have been structured analogously.

Task 1: Form the derivative of the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 \cdot x^7$, once by applying the product rule and once again by first transforming the term and then forming the derivative.

Task 2: Decide which of the following strategies is the most reasonable to form the derivative of the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = 4x^3 \cdot 2x^6$ (without calculation).

a) Applying the product rule
b) Applying the power laws and then forming the derivative

Task 3: Decide which of the following strategies is the most reasonable to form the derivative of the function g: $\mathbb{R} \rightarrow \mathbb{R}, g(x) = (x-1)(x+1)$ (without calculation).

a) Applying the third binomial formula and then forming the derivative
b) Applying the product rule

We developed the components of the digital learning environment presented above mainly building upon research on procedural flexibility as well as on cognitive activation. The definition of procedural flexibility according to Star and Newton (2009) comprises two essential aspects: the knowledge of multiple solutions and the ability to select the most suitable one for a given task. We aimed to address both facets of the definition in the videos and follow-up tasks.

The videos on the product and chain rule primarily address students' knowledge of multiple solutions by presenting different procedures to derive the given polynomials. Considering the distinction between potential and practical flexibility put forward by Xu et al. (2017), the videos therefore aim to foster potential flexibility because students were not prompted to carry out any procedure in terms of practical flexibility.

The concrete performance was demanded in the follow-up tasks after the videos that were therefore intended to enhance students' practical flexibility. Especially the first task after each video served this purpose. Students had to find the derivative of a polynomial using two different procedures (applying the product or chain rule vs. transforming the term before forming the derivative). The second and third task aimed rather at the ability to select a suitable procedure to form the derivative of a given function. Thus, these tasks address the second facet of procedural flexibility mentioned above.

Considering cognitive activation, we opted for a combination of instructional videos and follow-up tasks to counteract the typically low level of cognitive activation of instructional videos. As quizzes are considered useful to activate learners (Lehner, 2018), we decided to enrich our videos with small interactive tasks not only to provide activating tasks outside the videos but also while watching them (only for the intervention group). Unlike the follow-up tasks after the videos, the interactive tasks (see *Table 1*) were not intended to directly foster procedural flexibility but to maintain students' attention. On a more functional level, the tasks served to activate students' prior knowledge (the first two tasks in *Table 1*) or to reflect the main aspects of the video (the third tasks in *Table 1*).

Instruments and data analysis

Before the intervention, students had to complete two introductory tasks, one on the product rule and one on the chain rule. The tasks were to find the derivation of a product of two polynomials ($f\colon \mathbb{R} \to \mathbb{R}, f(x) = x^2 \cdot (3x - 1)$) and of two chained polynomials ($g\colon \mathbb{R} \to \mathbb{R}, g(x) = (2x + 5)^3$). To get better insights into potential mistakes, we structured the tasks using cloze texts. First, the learners had to give the functional equations of the two polynomials, then they had to type in their derivatives before they finally applied the respective rule. The participants could achieve a maximum of eight points for each task.

For both introductory tasks, we calculated the mean value and standard deviation for every group. Additionally, we carried out a two-sided t-test to check whether the students were randomly assigned to the two groups.

After the intervention, we used the following tasks to measure students' knowledge gain. During the treatment, students already completed analogous tasks. In the treatment section, we already described that these tasks addressed the ability to select an appropriate solution for a given task, thus the second aspect of procedural flexibility. We put emphasis on this facet because it appears crucial to us that students can estimate the effort of a procedure before performing it. We developed all tasks (including the in-video and follow-up tasks) on our own.

Final task: Decide for each of the following functions which of the proposed strategies is the most reasonable to calculate its derivative (without performing the calculation).

1. $f_1: \mathbb{R} \to \mathbb{R}, f_1(x) = \frac{1}{3}x^2 \cdot 3x^4$
   a) Applying the product rule
   b) Applying the power laws and then forming the derivative *(correct answer)*

2. $f_2: \mathbb{R} \to \mathbb{R}, f_2(x) = (3x+2)(3x-2)$
   a) Applying the product rule
   b) Applying the third binomial formula and then forming the derivative *(correct answer)*

3. $f_3: \mathbb{R} \to \mathbb{R}, f_3(x) = (x^2-1)^5$
   a) Applying the chain rule *(correct answer)*
   b) First expanding the term and then forming the derivative

4. $f_4: \mathbb{R} \to \mathbb{R}, f_4(x) = (2x^3)^5$
   a) Applying the chain rule
   b) Applying the power laws and then forming the derivative *(correct answer)*

Each correct answer has been awarded one point, incorrect answers yielded zero points. We used SPSS statistics 28 for data analysis. For each task, we calculated the mean value and the standard deviation for both groups. To compare the intervention group and the control group and to see whether the former performed significantly better than the latter, we conducted two-sided t-tests for independent samples.

RESULTS

We first give some descriptive statistics. In *Table 2*, the means, and standard deviations (SD) for both the introductory tasks and the final tasks are listed. The intervention group (IG) got the in-video tasks during the treatment, the control group (CG) had no in-video tasks.

For both introductory tasks, there is only a slight difference between the means of the two groups that is not significant, *t(44)=1.386, p=.173 (product rule); t(43)=.316, p=.754 (chain rule)*. This indicates that the randomization worked as intended.

Task	Mean		SD	
Introductory tasks (min=0, max=8)	IG (n=27)	CG (n=19)	IG (n=27)	CG (n=18)
$f: \mathbb{R} \to \mathbb{R}, f(x) = x^2 \cdot (3x - 1)$	7.81	7.21	0.68	2.12
$g: \mathbb{R} \to \mathbb{R}, g(x) = (2x + 5)^3$	5.07	4.89	1.80	2.11
Final tasks (min=0, max=1)	IG (n=24)	CG (n=19)	IG (n=24)	CG (n=19)
1. $f_1: \mathbb{R} \to \mathbb{R}, f_1(x) = \frac{1}{3}x^2 \cdot 3x^4$	.63	.58	.50	.51
2. $f_2: \mathbb{R} \to \mathbb{R}, f_2(x) = (3x + 2)(3x - 2)$	.54	.58	.51	.51
3. $f_3: \mathbb{R} \to \mathbb{R}, f_3(x) = (x^2 - 1)^5$	.67	.58	.48	.51
4. $f_4: \mathbb{R} \to \mathbb{R}, f_4(x) = (2x^3)^5$	.75	.42	.44	.51

Table 2: Mean values and standard deviations for the introductory and final tasks[1]

In the data on the final tasks, we can see that, except for the second task, the students in the intervention group performed slightly better than the control group. We find the biggest difference between the means in the fourth task. The t-tests revealed that only the difference for this task was statistically significant, $t(41) = 2.27, p=.029$. The differences in the other tasks could also be explained by random, $|t(41)| < 0.579, p>.565$. Thus, the group with the interactive tasks performed significantly better in the last task. Cohen's d is .70 indicating a rather high effect size.

Due to the dichotomous coding, the scores can be interpreted as solutions frequencies. Instead of just under half of the control group, 75% of the participants in the intervention group were able to solve the task. However, the data overall indicate a small learning effect, especially in the control group. Solving approximately 50 % of tasks with two choices could simply be explained by chance. In the intervention group, those percentages are slightly higher, especially for the last task.

DISCUSSION

Coming back to our research question "How do the interactive tasks affect students' learning?", we can state that in-video tasks in a digital learning environment on the flexible use of derivation rules have affected our students' learning gain positively. We found that students completing those tasks performed significantly better in one task in the post-test.

[1] The group sizes differ in the intervention group because three participants completed the introductory and follow-up tasks after the videos but not the final task. Since all data was collected anonymously, we cannot retrace the students who did not finish all tasks.

However, the overall learning gain appears rather small to us. Possible reasons might include the format of the final tasks that only offer two options to respond and the short duration of the treatment (60 minutes). Since the students in the intervention group performed better in the post-test and especially significantly better in one of its tasks, we conclude that the additional cognitive activation via in-video tasks is fruitful for digital learning.

Limitations

Certain limitations of this study must be considered to correctly classify its results, including limitations concerning the sample and frame conditions, the measures, and the results.

The small sample size and the short duration of the treatment are created by the explorative design of the study. As some students did not finish the treatment, the sample size further reduced during the session. Despite the successful randomization, we did not get equally sized groups. Our study thus has low statistical power and we may have missed results that could be gained in larger samples.

The measures have great potential for improvement as they were not piloted. The first task used for the randomization check showed a ceiling effect (i.e., it was too easy). The final tasks do not correspond to the length of usually extensive performance tests. Therefore, we could not expect a precise measurement. Again, imperfect instruments likely blur results so might expect more clear effects in future studies.

Implications and future research

This study contributes to the promotion of procedural flexibility in digital learning settings we still know little about. Therefore, we adapted non-digital approaches that have been approved in classroom settings (see Maciejewski & Star, 2016; Xu et al., 2017). We think that we successfully developed a digital learning environment with interactive videos as most of the participants managed to complete the treatment and stated that they were satisfied with it.

We aimed at addressing both facets of procedural flexibility with our videos and follow-up tasks. The instructional parts, the videos, aimed rather at the knowledge of multiple solutions and the in-video tasks served to maintain the students' attention. The follow-up tasks addressed the concrete performance of those solutions and the ability to select the most appropriate one.

On a theoretical level, future research will be needed to investigate the participants' learning processes as they might show us how the learners really deal with the material and to what extent they reflect on it. On a methodical level, researching the use of a digital learning environment via Zoom turned out to be challenging. Research will be required on valid methods for digital settings. Finally, as practical implication, we recommend enriching instructional videos with in-video tasks that can be rather easily integrated with tools such as H5P.

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Different interpretations of the total derivative and how they can be reconstructed in textbooks for Multivariable Real Analysis

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The total derivative is an essential concept in first-/second-year university mathematics many students struggle with. However, its concrete teaching and learning gained little attention in mathematics education research. We present an analysis of the meaning of the concept "total derivative" for multivariable functions, providing a "model of meaning" using a framework including several relevant contexts of interpretation. Then, we comparatively analyze three different German "Analysis II"-textbooks dealing with differentiation in the multidimensional case with respect to our model concerning their use of contexts of interpretation. We find variability on the semantic level, especially pertaining to geometric interpretations of the total derivative.

Keywords: teaching and learning of analysis and calculus, epistemological studies of mathematical topics, textbook analysis.

INTRODUCTION AND THEORETICAL BACKGROUND

While the teaching and learning of derivatives in the one-dimensional case at the school level have been studied for some years, studies focusing on differentiation concepts in the multidimensional case are scarce. Martínez-Planell and Trigueros (2021) give an overview of studies regarding the teaching and learning of multivariable calculus. Most of these studies focus on functions $\mathbb{R}^2 \to \mathbb{R}$ and especially on a geometric approach to concepts of derivatives. Additionally, Weber (2012) addresses the interpretation of the directional derivative as a local rate of change in space. However, extensive subject-matter analyses of differentiability concepts in the multidimensional case are lacking but could provide suggestions for teaching in a way that might enrich students' concept images. We, therefore, want to address the total derivative comprehensively, analyzing its meaning in different contexts of interpretation which results in a "meaning model" for the total derivative, fusing and structuring parts of concept images shared by the mathematical community, and looking for these in several German textbooks.

Hußmann and Prediger (2016) formulate a framework for specifying and structuring mathematical topics to be learned. In particular, they differentiate between the formal and semantic level for a didactical analysis of a subject. The formal level includes the mathematical objects and phenomena in their formal presentation and logical structure, whereas the semantic level addresses the sense and meaning of a concept. The notion of "concept image" is often used to describe the meaning of a concept. Tall and Vinner (1981) define the concept image as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). This includes verbal associations as well as non-verbal ones. In this article, we do not analyze an individual's concept image but rather describe parts of a comprehensive concept image for the total derivative, which can be seen as an attempt for a structured fusion of the mathematical community's concept images (similar to Zandieh's (2000) framework for the $\mathbb{R}^1$-derivative), and analyze to which extent components of those are addressed in different textbooks. As Sierpinska et al. (2002) point out, the meaning of a concept is based on its relations with other concepts. We, therefore, also discuss relationships of the total derivative to related concepts like directional derivatives and partial derivatives.

Building on analyses of the meaning of the derivative in the one-dimensional case (Greefrath et al., 2016; Kendal & Stacey, 2003; Zandieh, 2000), we elaborated a framework for describing a comprehensive concept image for multivariable differentiation. It contains conceptually different definitions of the concepts (we call them "definition variants") and then provides several relevant contexts of interpretation. We identify the following relevant interpretive contexts: geometric, analytic-algebraic, approximation, and real-world models. The geometric context of interpretation is further subdivided into the abstract-geometric one, which deals with interpretations related to the graph of functions, and the real-geometric one, in which the abstract-geometric interpretations are transferred to the idea of hilly landscapes in the real world. The geometric context includes all interpretations in the Cartesian coordinate system, in relation to function graphs and the use of words such as "tangent" or - in the multi-dimensional context - tangent (hyper)planes, as well as all "classical" geometric terms such as straight lines and planes. In the analytic-algebraic context, inner-mathematical properties of the definition and the concept, motivations for the concept and relations to other concepts are dealt with that do not belong to other interpretive contexts. "Approximation" means that the derivative is used for a local approximation. Here, on the one hand, the derivative can be understood locally as a linear approximation for the difference of the values of the function at two (nearby) points. On the other hand, the derivative can be used in a local affine-linear approximation of the function. The interpretation in the "real-world model" includes the model-like application of the concept to real situations, for example in the context of a function $\mathbb{R} \to \mathbb{R}$ that represents the location dependent on time, the interpretation of the $\mathbb{R}^1$-derivative as the instantaneous velocity.

We will apply our conceptual analysis to various textbooks. However, we are aware that textbooks represent only offers for student learning. Studying students' usage of these textbooks is another topic, but we hypothesize that the meaning of derivatives we will reconstruct will constrain students' learning.

RESEARCH QUESTIONS

1. Which interpretations (from the one-dimensional context and beyond) in the various contexts are valid for the total derivative, and how do relationships of the total derivative to related concepts (e. g., partial derivatives, directional derivatives) enrich interpretations of the total derivative on the semantic level?

2. Which interpretations of the total derivative and its relations with other concepts can be found in various textbooks used in German Analysis II courses?

We thus carried out a subject-matter analysis of the concept of the total derivative, including its relations to related concepts using the framework described above. To do so, we discussed possible interpretations for the total derivative in each of the identified contexts extensively. We present our results as a model of meaning – providing a suggestion for a shared concept image of the mathematical community that can be discussed – in the following section. Our model of meaning is comparable to reference models in ATD (cf. for example Wijayanti & Winsløw, 2017), but we are not reconstructing praxeologies here. The model is then used for a textbook analysis of relevant books, for which we explain our method separately before showing results.

MODEL OF MEANING OF THE TOTAL DERIVATIVE

We can formulate various equivalent but conceptually different definitions of the total derivative (sometimes also called "derivative" or "(total) differential") on the formal level. There are two facets in which the definitions differ. On the one hand, the total derivative of a function $f: \mathbb{R}^n \to \mathbb{R}^m$ at a point x_0 can be either defined as a linear mapping $\mathbb{R}^n \to \mathbb{R}^m$ or as a matrix $\in \mathbb{R}^{m\times n}$. Matrices and linear mappings can be identified with each other since every linear mapping has a distinctive transformation matrix when $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with the canonical basis. The other facet is the formulation of the defining property for total differentiability of a function. A possible definition is the following:

> A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is called totally differentiable at $x_0 \in \mathbb{R}^n$ if a linear mapping $A_{x_0}: \mathbb{R}^n \to \mathbb{R}^m$ exists such that in a neighborhood of x_0 the following condition holds: $f(x_0 + h) = f(x_0) + A_{x_0}(h) + \varphi_{x_0}(h)$, with φ_{x_0} being a function that is defined in a neighborhood of $0 \in \mathbb{R}^n$ with values in $\mathbb{R}^m$ and $\lim\limits_{h\to 0} \frac{\varphi_{x_0}(h)}{||h||} = 0$. If it exists, this linear mapping A_{x_0} is unique and is called the total derivative of f at x_0, written as $Df(x_0)$.

Instead of introducing an "error function" φ_{x_0} and thus needing two equations, the condition could also be written out in the limit: $\lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)-A_{x_0}(h)}{||h||} = 0$.

Another possibility is a definition explicitly using partial derivatives. We assume that f is partially differentiable at x_0. We can determine the Jacobian matrix $J_f(x_0)$ containing the partial derivatives. The function f is called totally differentiable iff $\lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)-J_f(x_0)\cdot h}{||h||} = 0$. In this case, $J_f(x_0)$ is the total derivative represented as a matrix. The latter definition is a constructive one in the sense of Richenhagen (1985) while the former possibilities are relational-descriptive definitions in his sense, not telling how to compute the total derivative (which can be derived in a further step).

On the semantic level, we now take a look at the different contexts of interpretation, keeping relations with related concepts in mind.

Analytic-algebraic context: If a relational-descriptive definition is used, it must be shown that the total derivative of a given function at a point is unique. The question how to compute it remains open at first. While it can be done "by hand" by computing $f(x_0+h)-f(x_0)$ and finding the linear term, the connection to partial derivatives can also be used since it can be shown that total differentiability implies partial differentiability and the total derivative is given by the matrix whose components are the partial derivatives, which can be computed easily using techniques from the one-dimensional case. It is important to note that the matrix containing the partial derivatives is not always the total derivative because this matrix can be computed when all partial derivatives exist which, however, does not automatically imply total differentiability. There are two possibilities to check this: showing that $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)-J_f(x_0)\cdot h}{||h||}=0$ (necessary and sufficient condition) or that all partial derivatives are continuous (C^1-criterion, not necessary, but sufficient condition). These are applications of theorems that need to be proven on the formal level. The connection of the total derivative and directional derivatives makes it possible to interpret the total derivative at x_0 as the linear function that maps each vector $v\in\mathbb{R}^n$ to the local rate of change of f at x_0 in the direction of v. For $n=m=1$, the total derivative is the same as the usual $\mathbb{R}^1$-derivative. However, the definition of the total derivative does not look similar to the usual definition of the $\mathbb{R}^1$-derivative using the differential quotient. Like $\mathbb{R}^1$-differentiability and in contrast to other differentiability concepts in $\mathbb{R}^n$, total differentiability of f at x_0 implies continuity of f at x_0. If the total derivative of f at x_0 is seen as a linear mapping $h\mapsto Df(x_0)(h)$, it is clear that this function is always continuous (being a linear mapping between finite-dimensional vector spaces). A different question, however, is whether the function $x\mapsto Df(x)$ is also continuous. This is not automatically the case, thus introducing the need for the term "continuously differentiable" which is equivalent to all partial derivatives being continuous. These distinctions can cause confusion for learners.

Approximation context: In the first definition shown, the total derivative is introduced as a linear mapping that can be used to approximate the course of the function f near x_0. This idea is furthered by the equation in the definition, resulting in the approximation $f(x_0+h)\approx f(x_0)+A_{x_0}(h)$. The "roughly equal"-sign can be specified by the "error function" φ_{x_0}. In the other definitions, the approximation interpretation is also already contained, but more hidden than in the first case, since here, the definitions do not contain an equation of the form $f(x_0+h)=\cdots$. However, through the limit expression, it is also indicated here that the relative error made when writing $f(x_0+h)\approx f(x_0)+J_{f(x_0)}\cdot h$ is small.

Geometric context: Like the tangent in the one-dimensional case, the abstract geometric interpretation of the total derivative using a tangent plane does not follow directly from any of the definitions. Visualizing a tangent plane is only possible for $n=2, m=1$, but the wording is often used in other cases as well, generalizing the idea. The total derivative can be used to describe the tangent plane / tangent space as an n-dimensional affine subspace of $\mathbb{R}^{n+m}$. This can be understood as the graph of the total derivative (shifted so that the point $\left(x_0, f(x_0)\right)$ corresponds to the origin) if the total derivative is understood as a mapping. Using the definition variant "matrix of partial derivatives", the tangent plane / tangent space can be understood as the set $\{\left(h, f(x_0) + J_f(x_0) \cdot (h - x_0)\right) | h \in \mathbb{R}^n\}$, where the entries of the matrix correspond to the "slopes" of the tangent plane in the directions of the respective coordinate axes. If all directional derivatives exist, tangents to the graph exist in all directions. If the mapping that maps v to the directional derivative $D_v f(x_0)$ is linear, these tangents form a plane. That plane does not have to be a tangent plane to the graph of f at x_0. This geometric interpretation can help visualize that directional differentiability does not imply total differentiability, not even if additional linearity is given. The real-geometric interpretation is a transfer of this phenomenon to the three-dimensional visual space: If $f: \mathbb{R}^2 \to \mathbb{R}$ describes the height as a function of a point given by two coordinates, i.e., if its graph results in the relief map of a hilly area, then its total differentiability means that the corresponding hill appears locally like a plane (the tangent plane). Another geometric interpretation of the total derivative $Df(x_0)$ can be formulated using the composition of f with a curve $\gamma: I \to U \subseteq \mathbb{R}^n$ with $\gamma(0) = x_0$ and $\gamma'(0) = h \in \mathbb{R}^n$: The composed curve $\tilde{\gamma} = f \circ \gamma$ has then $\tilde{\gamma}(0) = f(x_0)$ and $\tilde{\gamma}'(0) = Df(x_0)(h)$. This can be imagined as follows: If x moves from x_0 with instantaneous velocity h, then $f(x)$ moves from $f(x_0)$ with instantaneous velocity $Df(x_0)(h)$. Using this curve γ, we introduced a notion of time to illustrate the meaning of the total derivative. The directional derivative can then be seen as the particular case of this, with γ being a specific curve that is a straight line ($\gamma(t) = x_0 + t \cdot v$).

Real-world model: In the multidimensional case, the "standard application" from the one-dimensional situation, instantaneous velocity, could only be used for one of the partial derivatives because there can only be one time dimension. There are many other applications for functions $\mathbb{R}^n \to \mathbb{R}^m$. Suppose f describes a real-world application, e. g. the temperature at each point of a room. In that case, the interpretation of a local rate of change can be used for the different components of the total derivative. The total derivative itself cannot be interpreted as "the local rate of change" in any way (only as the mapping that maps every direction to the local rate of change in this direction as described above). It can be used to approximate quantities in a small neighborhood, and the linear approximation itself can then be used for other applications, e. g. when the linearization of a vector field (for example, a force field) makes working with differential equations easier.

ANALYSIS OF TEXTBOOKS BASED ON THE MEANING MODEL OF THE TOTAL DERIVATIVE

Method for the textbook analysis

Textbooks are essential resources for students' learning. However, it is important to note that in mathematics lectures at German universities, there are usually no selected course books that are mandatory: The lectures typically stand on their own, and lecturers often recommend some textbooks for additional reading. In our analysis, we include three different "Analysis II" textbooks and call them "(F)" (Forster, 2008), "(G)" (Grieser, 2019) and "(H)" (Heuser, 1992). Two of them, (F) and (H), are "standard textbooks" that lecturers regularly recommend to their students for additional reading. (G) is not yet a published textbook but a lecturer's script (published online and published as a book soon). We decided to include (G) even though it is not yet published as a book due to its particularly rich presentation of the topic with a focus on fostering understanding that we felt would enrich our sample. The script can be used like a textbook already because it can be found online. We are interested in mathematical intentions of the textbooks (Pepin & Haggarty, 2001) and our analysis is "vertical" in the sense of Charalambous et al. (2010) since we looked at the various interpretations of the total derivative in the relevant chapters using our meaning model as a comparison and coding scheme. We used Bowen's (2009) approach to document analysis. At first, we selected the relevant chapters – those dealing with differentiation concepts in $\mathbb{R}^n$. We are interested in the introduction of the respective concepts and first theories about them. Therefore, we excluded the sections dealing with higher derivatives, Taylor's theorem, submanifolds, the implicit function theorem, etc. from our analysis. Then we divided the relevant chapters into small sections of meaning, e.g., definitions, theorems, examples, and accompanying text. We coded if and which interpretative context from our framework for the total derivative could be found for each section. Let the reader be reminded that in the analytic-algebraic context of interpretation, by our definition of this context in the framework, only comments on the semantic level (e. g., when it is explicitly stated that the relationship between total derivative and partial derivatives implies its uniqueness) are counted and not all properties that might be stated in theorems on the formal level. Since the analytic-algebraic interpretation contains many different possibilities, we work with several subcategories (see Table 1). In the geometric context, we differentiate between the notion of the tangent plane / tangent space and other geometric interpretations regarding the total derivative.

Results of the textbook analysis

In book **(F)**, we identified 14 relevant pages. In sum, we found very few comments on the semantic level. The introduction of the chapter about the total derivative comments on the meaning of the total derivative on the *analytic-algebraic* and the *approximation* context:

> "In this paragraph, we define the total differentiability of functions from open subsets of $\mathbb{R}^n$ into $\mathbb{R}^m$ as a certain approximability through linear mappings. In contrast to partial differentiability, one does not need to refer to the separate coordinates in the process; additionally, a totally differentiable function is automatically continuous." (Forster, 2008, p. 62, translated by the authors)

As can be seen from this excerpt, partial derivatives are defined before the total derivative is introduced. Total differentiability is then defined similarly to the first definition we stated above, but this is the only time the *approximation interpretation* is explicitly mentioned. In the *analytic-algebraic context*, it is explicitly remarked that the theorem stating that the transformation matrix of the total derivative contains the partial derivatives implies the uniqueness of the linear mapping and thus makes the definition of the total derivative possible. There are no *geometric interpretations* or interpretations in the *real-world model* for the total derivative (and not for partial derivatives, either), even though the chapter about partial derivatives starts with definitions and figures of level sets and contour lines.

In contrast, book **(G)** provides very rich interpretations. We found and analyzed 21 relevant pages. Concerning the *analytic-algebraic context of interpretation*: After the definition of the total derivative is stated, the question is asked whether an interpretation as a local rate of change as in the one-dimensional case is possible. In this book, partial derivatives and directional derivatives are not defined before the total derivative, and the desire to be able to compute the total derivative and interpret it as a local rate of change is presented as motivation for their respective definitions. The C^1-criterion is introduced as an essential criterion for checking for total differentiability. The *approximation context* is mentioned as well: It is explicitly stated that total differentiability means that the change of function values depends "almost linearly" on the change of x-values. The idea is then used for a heuristic argument for proving the chain rule. The *geometric* idea of the tangent plane is found throughout the chapter. The motivation for the total derivative is given by the idea of trying to generalize the idea of tangents for the $\mathbb{R}^1$-derivative. The illustrative idea of a tangent plane is introduced early, later the tangent plane is defined as the graph of the total derivative. This idea is also taken up later in argumentations about the total differentiability of example functions. It is mentioned that the Jacobian matrix contains the "slopes" of the tangent plane, and the idea of tangent plane and tangents of curves is used to make the connection of total derivative and partial derivatives plausible in an illustrative way. Another *geometric interpretation* of the total derivative, the idea of the transformation of a curve (described above), is also mentioned. Regarding the *interpretation in real-world models*, there is a section listing different real-world situations that could be modeled using functions $\mathbb{R}^n \to \mathbb{R}^m$ and the meaning of the total derivative in these contexts, e. g. the air pressure depending on the location (for which the meaning of the gradient – related to the total derivative – as the direction of strongest increase is thematized).

In book **(H)**, we found and analyzed 33 relevant pages. Here, partial derivatives are defined first, then the change behaviour of C^1-functions is addressed, and in the course of this, the *approximating* property of the total derivative is explained before the total derivative itself is defined. The motivation for looking at more than partial derivatives is the desire to analyze the change behavior of the function when change occurs in more than one direction. Like in book (F), the relationship between total derivative and partial derivatives is used to show the uniqueness of the total derivative on the *analytic-algebraic level*. Additionally, the C^1-criterion is introduced as the most important criterion for total differentiability. The *approximation interpretation* is given explicitly: The total derivative is introduced using the idea of approximating a function with “especially easy,” i.e., linear, functions, with reference to the theorem about approximation of C^1-functions and explicitly mentioning that change in different directions should be possible (in contrast to partial derivatives). The tangent plane is not introduced. A *geometric interpretation* is only given for other differentiability concepts: the partial derivative as the slope of the tangent of a curve, the gradient as the direction of greatest ascent, etc. *Real-world models* are considered throughout the sections regarding partial derivatives, especially in exercises at the end of chapters (e. g. concerning fluid mechanics or oscillating strings). The idea of a function modeling temperature on a thin panel is used for motivating that partial derivatives are not enough. The approximating idea is also applied to real-world models in tasks for C^1-functions, e. g. when propagation of uncertainty in a harmonic oscillator is addressed. These tasks are given before the total derivative is defined; after that, no explicit real-world applications are presented.

An overview of these results can be found in Table 1. An “x” indicates the interpretation was found in the book, “pd” in the abstract-geometric and real-world model contexts of interpretation means that the interpretation in this context for the total derivative was not in the book, but something similar for partial derivatives.

	Analytic-algebraic				**Approx.**	**Abstr.-geom.**		**Real world model**
	Computation using pd	Uniqueness using pd	Local rate of change	C^1-crit.		Tangent plane	other	
(F)		x			x			
(G)	x		x	x	x	x	x	x
(H)		x		x	x		pd	pd

Table 1: Overview of interpretations of the total derivative found in the three textbooks

DISCUSSION

We have worked out a variety of possible interpretations of the total derivative in the different contexts that we identified. We have also shown how the related concepts of partial derivatives and directional derivatives, as well as the relations with the $\mathbb{R}^1$-derivative and the concept of continuity, enrich the meaning of the total derivative. This is a plea for Sierpinska et al.’s (2002) notion of a theoretical system instead of considering only isolated concept images for each concept. A comparison of our meaning model with different textbooks shows that not all textbooks mention each of the possible interpretations. Not surprisingly, there are differences between the three books. The approximation context of interpretation is mentioned at least briefly in all three books and in two of the books in more detail. The geometric context is not always discussed. While (G) emphasizes geometric interpretations, (F) does not mention these at all, and (H) only includes geometric interpretations of other differentiability concepts. This lack of focus on geometric interpretations, not only in textbooks but also in many lectures, makes it difficult to apply findings from studies mentioned by

Martínez-Planell and Trigueros (2021) to German classes. In some but not all textbooks, real-world applications of the total derivative are given. Book (F) shows very few interpretations and mostly stayed on the formal level, not explicitly trying to further enrich the reader's concept image. Since (F) is a relatively thin book and only 14 pages regarding differentiability were identified, it was expected that this might show fewer contexts of interpretation than the other books. However, the number of pages addressing differentiability does not seem to be a reliable indicator for how rich the provided concept image is: At least in our case, we found fewer pages relevant but more different interpretations for the total derivative in (G) than in (H). Differing from Harel's (2021) findings when investigating six multivariable calculus books, none of the books in our study omitted the definition of the total derivative, and the idea of a tangent plane was not given in two of the books while it was important in all books in Harel's study. On the other hand, the idea of linear approximation was not as important in the books in Harel's investigation. These differences might illustrate a disparity between Calculus and German "Analysis" classes, usually more formal and abstract.

An analysis of three textbooks (which is a tiny sample) can only be the beginning of an analysis of the landscape of textbooks regarding differentiability concepts in the multidimensional case. It would be interesting to find prototypes of different textbook styles (first only regarding differentiability, but this could be extended to other subject areas, too). This would make recommendations for learners easier, knowing what style the different textbooks use. Important questions are what contexts are addressed and what is done to enrich readers' concept images. For lecturers, it might be a good idea to know which meaning of the total derivative in which contexts they want to convey and which textbooks go well with this. For example, if a lecturer focuses on tangent planes, working with a textbook that does not mention the abstract-geometric context of interpretation or the other way round might confuse learners. Further follow-up questions would be why lecturers address contexts of interpretation the way they do, how students work with the textbooks and what kinds of concept images evolve after working with these, and how these help or hinder them in their further studies.

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Development of the differential calculus schema for two-variable functions

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We use Action-Process-Object-Schema theory (APOS) to study the development of the differential calculus Schema for two-variable functions. This allows us to obtain information about students' constructions and also gives us information about the notion of Schema. We performed semi-structured interviews with a group of eleven students that had completed an introductory multivariable calculus course. We use data from two students to exemplify the types of constructions observed. To analyze the data, we use the Schema development triad and the notions of correspondence, transformation, and equivalence relations between Schema components. Our study contributes to a better understanding of these notions, how students relate differential calculus concepts, and how to support their learning and Schema development.

Keywords: Teaching and learning of specific topics in university mathematics, teaching and learning of analysis and calculus, APOS theory, schema, two-variable functions.

INTRODUCTION

The notion of Schema is an important component of APOS Theory. However, there is not much research using Schema to analyze students' constructions of different topics. The first research to further the study of the notion of Schema was by Clark et al. (1997), who introduced the notion of the Schema triad from Piaget and Garcia's work (1982) into APOS theory to study the development of the chain rule Schema for one-variable functions. More recently, Trigueros (2019) studied the development of the linear algebra Schema, underscoring for the first time the types of relations between Schema components. She operationalized the use of Schemas into the study of its constituent components and types of relations between them. Further research needs to explore the notion of Schema and its application to the study of the teaching and learning of mathematics. One of the goals of this study is to contribute to the discussion of Schemas in APOS theory.

This study is part of a second research cycle based on previous work by Martínez-Planell et al. (2015, 2017) and Trigueros et al. (2018). These studies explore students' constructions of plane, partial derivatives, tangent planes, directional derivatives, and the total differential, applying, for the most part, the APO part of the theory. The second goal of this study is to contribute to the second cycle of research on students' understanding of the differential calculus of two-variable functions.

THEORETICAL FRAMEWORK

The notion of Schema from APOS Theory is used as the theoretical framework in this study. Schema is a structure defined as the individual's collection of Actions, Processes, Objects, and other previously constructed Schema which are linked by different types of relations and that he/she brings to bear upon a problem situation related to a specific mathematical concept or topic (Arnon et al., 2014). Using the Schema point of view, the data analysis focuses on the possible changes in the components of the Schema and particularly on the relations that an individual shows to have constructed among the components of the Schema. A Schema can be considered coherent when the individual can discern between those problems that are within its scope and those that are not. A Schema may become an Object through thematization. This mechanism involves the possibility to apply new Actions or Processes to the Schema. An important characteristic of Schemas is that they are in continuous development. They are dynamic structures, but this development can be described through three recognizable stages, Intra-, Inter-, and Trans-, characterized by differences in the type of relations constructed among the Schema's components.

Any relation constructed between the components of a Schema may be classified as a correspondence, transformation, or equivalence relation. This classification of relations was introduced by Trigueros (2019) and is further detailed and explored here. *Correspondence relations* are those that result from the superficial comparison of structures in terms of similarities or differences. They may arise from the repeated observation of pairs of component structures that appear jointly in problem-solving situations, so the individual knows they are somehow or other related, but is not yet able to justify the relation. *Transformation relations* are developed when the individual discovers that some structures in the Schema can be grouped and related to each other in terms of other structures. These other structures play a role in the reasoning and justification of the connections in accordance with disciplinary practices. A relation between two components of such a grouping or that plays a role in explaining or justifying the interrelation between other component structures of a grouping, will be considered a transformation relation. Transformation relations can be distinguished when the relation is somehow justified. *Equivalence or conservation relations* involve the conservation of properties in which one structure is dependent upon the others. Equivalence relations can be distinguished when one structure is interchangeably used when solving a problem situation involving the other in accordance with disciplinary practices.

We will consider that a Schema is at the *Intra-stage* of development when its components are mainly isolated from one another and the type of relations that exist between components are for the most part correspondence relations. At an *Inter-stage* of Schema development, groupings of different components start to appear. At this stage of Schema development, transformation relations will be more prevalent so that the individual will be able to justify some of the relations by referring to others. At the

Trans-stage of Schema development, all the different components are related, conservation relations appear and there is evidence of coherence of the Schema.

Research questions for this study are: How can the study of the Differential Calculus Schema development inform us about students' learning? How does the study of students' construction of the Differential Calculus Schema for two-variable functions enable us to better understand the types of transformations between Schema components?

METHODOLOGY

The *component structures* of the Differential Calculus Schema we consider are slope, partial derivative, directional derivative, tangent plane, total differential, function, and gradient. These are the same components that were studied in the first research cycle, now explicitly including the pre-requisite components of functions and slope, and adding gradient vector, for which questions were added to the new interview instrument. While clearly there are other components that may be called upon in a problem situation involving the differential multivariable calculus, our interview instrument will only allow us to examine the chosen components.

For *correspondence relations,* we look for relations that only point to commonalities between Schema components or appear to result from memorized procedures. For example: given a graphical representation of a tangent plane to the graph of a function at a point, a student might be able to correctly do computations to approximate the change in the values of the function for a small change in input values but might not recognize it is only an approximation and not the exact value of the change, or might not be able to explain why it is an approximation. Such a student would be grouping function, tangent plane, and perhaps even total differential components in a correspondence relation.

For *transformation relations,* we look for some reasoning explaining or justifying the relation in accordance with disciplinary practices. In the example above, a student that is able to recognize that his computations will only give an approximation, or who argues that close to the base point the tangent plane and surface are very close to each other, will be giving evidence consistent with a transformation relation grouping the components of function and tangent plane.

For *equivalence relations,* we look for evidence of one structure being used interchangeably for another in a way that agrees with disciplinary practices, consistently when solving different problem situations. For example, a student who is asked to approximate the value of a function near a base point, given graphical information of the tangent plane might immediately recur to the total differential at the base point, using the Schema components of tangent plane and total differential interchangeably or a student might be able to understand the role partial and directional derivatives play in terms of local change of the function. The student would be giving evidence of an equivalence relation among Schema components.

Eleven students spanning the range from above-average (3), average (5), to below-average (3), as chosen by the professor, were interviewed twice (Interviews I, II) two weeks after completing an introductory multivariable calculus course at a mid-tier Iranian university. The two interviews were held on different days and each one lasted approximately 1 hr. Interviews were video and audio recorded, transcribed, individually analyzed by the researchers, and discussed among them until consensus was reached. Interviews were analyzed in terms of the relations established between the seven chosen Differential Calculus Schema components.

There were six problems in Interview I and eight in Interview II. Some of these problems were multipart. Students were familiar with some problem types (see below: I-3b, I-4, I-6a, I-7c) and unfamiliar with others (II-1, II-6). In this report we will be referring to the following problems, which we label to include the interview and problem number:

I-3b. Suppose the graph of $z = f(x, y)$ is as follows (Figure 1a). State the sign (positive, negative, zero) of $D_{\langle -2,1\rangle} f(4,0)$. Justify your answer.

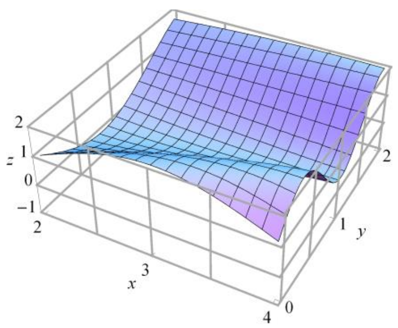
I-4. The following is a table of values of a differentiable function f of two variables (Figure 4). Approximate the value of $\frac{\partial f}{\partial y}(1,2)$ the best you can.

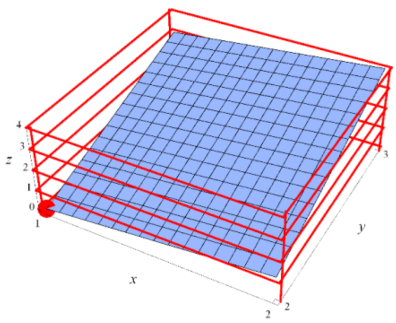
I-6a. The plane in the figure below (Figure 1b) is tangent to the graph of a differentiable function $z = f(x, y)$ at a point (1,2,0). What can you say about the change in the value of the function if x increases 0.02 units and y decreases 0.02 units?

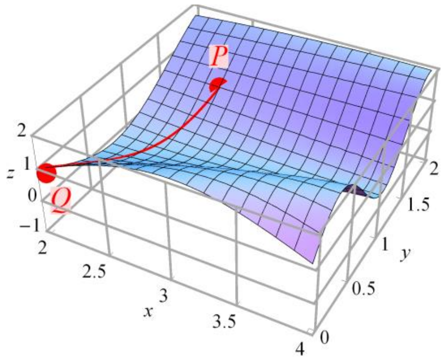
II-1. The graph of $z = f(x, y)$ is given in the figure below (Figure 1a). If $H(t) = \frac{\partial f}{\partial x}(t,0)$, draw how the graph of H may look for $2 \le t \le 4$.

II-6 Suppose point P moves towards point Q at a constant speed along the curve that joints them (Figure 1c). How may the graph of $D_{\langle 1,-1\rangle} f(P)$ as a function of time look?

II-7c. The following figure (Figure 1d) shows the contour diagram of the tangent plane of a function $z = f(x, y)$ at the point (0,3). Approximate the value of $f(0.04,2.97)$ the best you can.

a. 

b. 

c. 

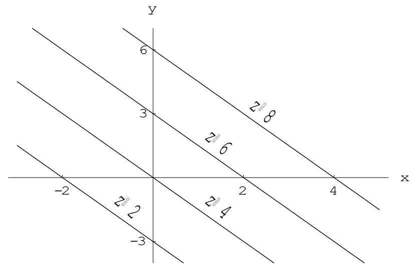
d. 

Figure 1: Figures for the interview problems

RESULTS

We show the results of two students, Hamid and Koorosh, relative to their Differential Calculus Schema development. With this, we show how we operationalized the study of Schemas by defining a collection of components and investigating the type of relations between components that have been established by students.

We found that Hamid interconnected all seven component structures with equivalence relations. So, we use him as an example of the Trans-DC (Differential Calculus) Schema development. We only show some excerpts from the interview to exemplify how he generally responded as it will not be possible to address all Schema components in this report.

In II-1, Hamid was given the graph of a function and was asked for the graph of $H(t)=\frac{\partial f}{\partial x}(t,0)$ (Figure 2). He showed to have an equivalence relation grouping the components of partial derivative, slope, and function.

> Hamid: I draw two axes, the vertical axis named $H(t)$ and the horizontal one named t. I need to compare the slope of tangent lines to graph f at the points $(t,0)$ in the x direction when t goes from 2 to 4.

> Interviewer: You should draw H.

> Hamid: Okay, the derivative with respect to x at the point (2,0) is the slope of this tangent line, the slope is positive so in the graph $H(t)$ we have a point here above the t axis and umm for t=2. If I continue in such a way for t=3 we have the point (3,0), based on the figure the tangent line is a horizontal line like this so the slope is 0. I now plot point (3,0) in the graph. From t=2 to t=3 the slopes of the tangent lines are decreasing. From t=3 to t=4 the values of slopes of the tangent lines are negative and they going to more negative and more negative, so we have a decreasing curve like this from t=3 to t=4.

In this problem, Hamid considered partial derivative and slope as interchangeable ("I need to compare the slope of tangent lines…). This suggests that the components of function, slope, and partial derivative are grouped in an equivalence relation where the properties of partial derivatives are conserved in the slopes of tangent lines.

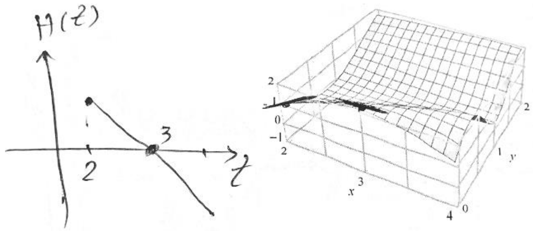


Figure 2: Hamid's work on problem II-1

In I-6a, Hamid was given the graph of a tangent plane and was asked to discuss change in value of a function. He grouped the components of tangent plane, function, total differential, partial derivative, and slope. Hamid related the components of tangent plane (given) and function (requested) through the total differential while treating total differential and tangent plane as interchangeable.

Hamid: I know $df = f_x dx + f_y dy$. Here we have $dx = 0.02$ and $dy = -0.02$. I have to find the values of f_x and f_y at the point $(1,2,0)$. Since it's a tangent line to the function f at the point $(1,2,0)$ so f_x is m_x and f_y is m_y. Based on the figure m_x is 1 over umm 2 minus 1 which is 1 so it will be 1, so m_x is 1, and m_y is 3 units to the up over 3 minus 2 which is 1 umm it will be 3 over 1 which is 3, so m_y is 3. The change in the value of the function is 0.02 times 1 plus -0.02 times 3 and umm the answer is -0.04.

Note that he justified the relation between tangent plane, partial derivative, and total differential when, while computing total differential, he says "since it's a tangent line [part of the tangent plane] to the function *f* at the point (1,2,0) so f_x is m_x and f_y is m_y." As we previously saw, in another problem (II-1) he evidenced an equivalence relation between slope and partial derivative, and in the next problem (II-7c) will show such a relation between tangent plane and function.

In problem II-7c Hamid was given a contour diagram of the tangent plane of a function $z = f(x, y)$ at the point (0,3). He was asked to approximate the value of *f*(0.04,2.97) as best as he could. In this problem, Hamid treated function and tangent plane as interchangeable in recognizing that approximate change is conserved. He gave evidence consistent with an equivalence relation grouping function and tangent plane.

Hamid: The point (0.04,2.97) is very close to the point (0,3), so to find the value of *f* at the point (0.04,2.97) I can find the value of *z* at the point (0.04,2.97) on the tangent plane.

Interviewer: Why?

Hamid: Because the graph of *f* and the graph of tangent plane are very similar to each other for the small neighbourhood of the point (0,3). I mean the tangent plane is an approximation for the function *f*.

Interviewer: Okay, approximate the value of *f*(0.04,2.97).

Hamid: At the point (0,3) the value of *z* or *f*(0,3) based on the contour diagram is 6. The equation of the tangent plane is $z-6=1(x-0)+\frac{2}{3}(y-3)$ so we have $z=6+x+\frac{2}{3}(y-3)$. If I put *x*=0.04 and *y*=2.97 then I have z equal to $z=6+0.04+\frac{2}{3}(2.97-3)$, and this is an approximation for *f*(0.04,2.97).

Hamid was not just doing a rote computation, he justified the relation when he said: "because the graph of *f* and the graph of tangent plane are very similar to each other for the small neighbourhood of the point (0,3) …"

While we only showed a few examples, Hamid related all components of the Differential Calculus Schema with equivalence relations. The interview instrument we designed does not allow us to inquire into the coherence of the Schema. Students knew they were being asked about the Differential Calculus Schema. They did not have to decide if a problem situation fell in the scope of the Schema. Nevertheless, we tentatively classify him as in the Trans-DC stage of Schema development.

Now we consider Koorosh as an example of a student at the Inter-DC stage of Schema development. At this stage transformation relations are starting to form, grouping different components of the Schema. There will still be some components related by correspondence relations and there may be some unrelated components. Again, we are only able to show a few examples of his work.

In problem I-3b, Koorosh gave evidence of a transformation relation grouping components of function, directional derivative, and slope. When given the graph of a function (Figure 3a) and asked for the sign of a directional derivative:

Koorosh: We are at this point umm if we move in the direction umm -2 units in the *x* direction and 1 unit in the *y* direction it's like we are moving in this direction (see Figure 3a for his direction). Looking at the figure we see the tangent line in this direction has a positive slope because the values of *z* increase so the sign of *D* which is the directional derivative is positive.

Note that he justified his answer. So, we classified this relation between function, directional derivative, and slope as a transformation relation. We did not classify it as an equivalence relation since, in problem II-6, when asked for the graph of $D_{\langle 1,-1\rangle}f(P)$ as a function of time as P moves towards point Q (Figure 3b), he did not evidence an equivalence between directional derivative and slope as may be seen in his answer; here, the directional derivative is initially negative but his graph of this derivative starts above the horizontal axis. There were however many instances where he showed the construction of correspondence relations as the next example shows.

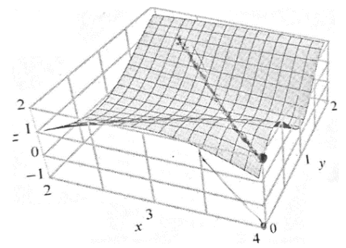
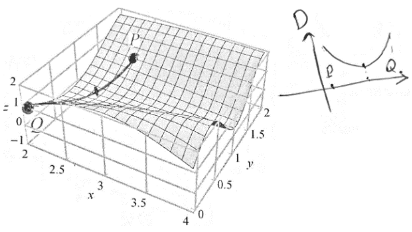
a.  b. 

Figure 3: Koorosh's work on problems I-3b and II-6

In problem I-4, Koorosh was given a table of values (see Figure 4) and was asked to approximate the value of $\dfrac{\partial f}{\partial y}(1,2)$ the best he could.

Koorosh: I need to compute the change of *z* over the change of *y* by holding *x* equal to 1. When *y* changes from 1 to 2 then *z* changes from 6 to 10, therefore f_y is $(10-6)/(2-1)$ and it's 4.

Interviewer: Is your answer the best approximation?

Koorosh: It's the derivative in the y direction and so my answer seems to be correct.

x / *y*	1	2	2.01
0	**5**	**7**	**7.04**
1	**6**	**10**	**10.06**
2	**8**	**14**	**14.10**

Figure 4: Table of values for problem I-4

Observe that Koorosh seems to be rigidly applying a procedure he could not justify. The difference between correspondence, transformation, and equivalence relation in this problem is best understood by comparing with the response of Hamid.

Hamid: It's the partial derivative with respect to *y*. I need to find it at the point (1,2). I need to fix the *x* equal to 1 so we are in the third row of the table where *x* is 1. For this row when *y* is 2 umm *f*(1,2) is 10. When *y* changes from 2 to 2.01 then the value of the function changes from 10 to 10.06 so the derivative at the point (1,2) and with respect to *y* is approximately equal to $(10.06-10)/(2.01-2)$ which is 6.

Here, Hamid aimed to relate the component of function (the given table) with that of partial derivative (the request of the problem). To do so, he related the components of partial derivative and slope as seen in the computation of slope as a quotient of "change". Here we consider average rate of change as another way to think about slope. The structures of slope and partial derivative are grouped and related to each other in terms of the structure of function (the table). Function plays a role in the reasoning of the connection when he recognizes the partial derivative "is approximately equal" to the slope he computed and uses the closest point to the base point in doing so. So, the relation between the slope, partial derivative, and function components in this problem satisfies the definition of transformation relation However, considering Hamid's overall performance, he consistently treated slope and partial derivative interchangeably in his other problem solving so we may classify this relation as an equivalence relation.

We may also compare Koorosh with Hamid in problem II-7c where he uses a contour diagram of the tangent plane to approximate a functional value:

Koorosh: The value of *f* at the point (0,3) is 6. When *x* increases 0.04 units then *z* changes as 0.04 times 1 which is 0.04 so *f*(*x*=0.04,3) will be 6.04. Now if *y* decreases as 3-2.97 which is 0.03 then *z* decreases umm 0.03 times 2/3. So the value of *f* at (*x*=0.04, *y*=2.97) will be 6.04-(0.03)2/3.

Observe that Koorosh could do computations relating tangent plane (the given contour diagram) with function (the requested value), but his argument lacks justification, suggesting a correspondence relation. Similarly, in problem I-6a, when given a graphical representation of the tangent plane and asked about the change in the value of the function if *x* increases 0.02 units and *y* decreases 0.02 units:

Koorosh: I first find f_y umm it's the change in *z* over the change in *y* which is umm using this line it will be (3-0)/(3-2) which is 3. Now I find f_x which is (4-3)/(2-1) umm which is 1.

Interviewer: You need to find the change in the function.

Koorosh: To compute it I use the differential formula which is $dz = dx + 3dy$. So it will be $dz = 0.02 + 3(-0.02)$ umm the answer is -0.04 and this means the function decreases 0.04 units.

Again, Koorosh grouped several Schema components: tangent plane (given graph), function (the requested approximation), total differential (what he chose to use to do the computations), partial derivative (to compute total differential), and slope (to compute partial derivative). He did this without any justification and without showing awareness in this or any other problem of the interview instruments of the relation between tangent plane and total differential, other than to do computations.

DISCUSSION AND CONCLUSION

The cases of Hamid and Koorosh exemplify how Schemas may be used to better understand students' construction in the differential calculus of two-variable functions. The types of responses these students showed were very different from each other. We found that Hamid showed evidence of a Trans-DC Schema development. This is assuming a Schema coherence we can't justify or inquire into with our instruments. Koorosh showed evidence consistent with an Inter-DC Schema development. Although it has not been stressed in this report, a closer examination of the relations established or not, can inform instruction and result in improvement of didactical activities.

This study enabled us to consider the definitions of the types of relations between Schema components and interpret them in a new context. In doing so, we gained a deeper understanding of the relations involved in the development of the Differential Calculus Schema. In future studies, we need to be more insistent in teasing out explanations and justifications from students in order to better distinguish the difference between the types of relations. Our study contributes to furthering the understanding of Schemas, their stages of development, and the correspondence, transformation, and equivalence relations among Schema components together with understanding how students' constructions may evolve in the course. Considering the work of all eleven students, we find there are many possible ways in which the DC-Schema stage of development appears, varying from weak to strong transition levels from Intra-DC to Inter-DC, and from Inter-DC to Trans-DC stages of Schema development. We also found that students showed a deeper understanding of the different Schema components in this second research cycle than in the first research cycle (Martínez-Planell et al., 2015). A constraint we have in the present study is that the interview instrument was not designed to allow for an investigation into Schema coherence. More research is necessary to study Differential Calculus Schema coherence.

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Learning two-variable functions using 3D dynamic geometry

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In this study, we redesigned successful paper-and-pencil activities and implemented them in an introductory multivariate calculus course supported by 3D dynamic geometry software. We used semi-structured interviews and students' written productions during the semester to analyze the use of technology in supporting students' learning. Considering the work of two students, we find that the software has transformative potential but that it did not promote students' learning as expected. We discuss possible reasons for this.

Keywords: Functions of two variables, APOS theory, 3D dynamic geometry, University mathematics.

INTRODUCTION

The modelling of natural phenomena generally does not depend on a single variable. Over the past decade, several researchers have studied students' understanding of two-variable functions (for a review see Martínez-Planell & Trigueros, 2021). Among these studies, there are some results exploring the use of physical manipulatives for the teaching of two-variable functions. For example, McGee et al. (2012) developed a set of tangible manipulatives and support materials for visualizing concepts related to points, vectors, surfaces, curves, and contours in 3D space and report their positive effect on student learning. Martínez-Planell and Trigueros (2019) used these manipulatives in their teaching and then researched student understanding, also reporting positive effects. Wangberg (2020) observed improved student understanding when using a different manipulative. Some articles consider digital technologies as a means to support visualization in multivariable calculus (e.g., Alves, 2012).

The use of tangible and virtual manipulators in teaching and learning situations is attracting growing interest due to the new possibilities offered by digital technologies (Soury-Lavergne, 2021). One of the possibilities offered by the digital is the passage from static supports (for example, paper and pencil), which allow us fixed figures, to dynamic supports, which would enable us to experiment with mathematical ideas in dynamic figures (Roschelle et al., 2017). Computer algebra systems (CAS), spreadsheets, and dynamic geometry environments (DGE) are the technologies most used in mathematics classrooms. According to Soury-Lavergne, "dynamic geometry is a generic term for a type of software that allows the construction on the screen of dynamic figures that can be deformed while retaining the geometric properties used at the time of their construction" (2020, p. 7). DGEs such as Cabri 3D or GeoGebra 3D can represent three-dimensional surfaces and thus can potentially support student learning of two-variable functions. However, if the student's activity is reduced only to typing algebraic expressions and seeing their representation on the screen, the dynamic aspect of dragging their free elements and observing what geometric properties of the figure are preserved is missed, thus limiting the technology's didactic potential. In this study we report on results obtained while using GeoGebra 3D together with didactic activities in the learning of two-variable functions.

Our research questions are: What mental constructions do college students show when using graphing activities of functions of two variables supported by 3D dynamic geometry? How do these constructions compare to those shown by students who worked with the same activities on paper and pencil?

THEORETICAL BACKGROUND

In APOS theory (Arnon et al., 2014), an Action transforms a previously constructed mathematical object and is perceived as external, i.e., it will be relatively isolated from the individual's other mathematical knowledge. Performing Actions does not allow per se the individual to justify them. Actions may correspond to mechanically executing a procedure by following explicitly available or memorized instructions. When an Action is repeated, and the individual reflects on it, he/she might interiorize it into a Process. A Process is perceived as internal, which means that the individual can think about the result of its application without following all the necessary steps and without recurring to external support. Different Processes can be coordinated into new Processes and can be reversed. These coordinations allow the individual to justify, imagine, and generate dynamic imagery of the Process. When an individual can think of a Process as a whole and can do or imagine doing Actions on it, the Process is encapsulated into an Object. The essential aspect of the Object structure is that Actions can be performed on it. We will not need to refer to Schemas in this article. APOS structures and mechanisms can be used to design a model of how students might construct a mathematical notion; such a model is called a genetic decomposition (GD). A GD is not unique, multiple GDs of the same concept can coexist, or different researchers can propose other models. What is important is that it is tested by data obtained from students. In general, a GD is used to design teaching activities that help students achieve the constructions conjectured in the GD. These activities are used in the classroom, then research is conducted with students, and, depending on the results obtained from the analysis of data, it may turn out that it needs to be revised and thus also the designed teaching activities. This opens the door to further research using the newly revised GD. One can continue doing research cycles until a GD is stable, that is, until the analysis of the data obtained reflects what the GD predicts about the construction of the mathematical notion at stake.

From its beginning, APOS theory considered technology as part of its teaching methodology by promoting programming as an instrument to encourage exploration, reflection, and concept building. Today, various tools offer new possibilities to promote student reflection. Drijvers (2015) distinguished three didactic functionalities for digital technology: (a) to do mathematics, (b) to practice skills, and (c) to develop conceptual understanding. To examine the different forms in which technology can be used in the classroom, Hughes (2005) developed three categories: technology as (a)

Replacement, (b) Amplification, and (c) Transformation. She defined technology as a replacement when "the technology serves as a different means to the same instructional end"; Technology as an amplifier when it "capitalizes on technology's ability to accomplish tasks more efficiently and effectively, yet the tasks remain the same"; and technology as transformation when it can change "students' learning routines, including content, cognitive processes, and problem-solving" (p. 281).

METHODOLOGY

Two groups of approximately 30 students each from a Mexican public university participated in this study in synchronous teaching, due to the Covid-19 pandemic, in the fall of 2021. Both groups worked with the activities based on the GD designed for the third cycle of the study by Martínez-Planell and Trigueros (2019). The activities were redesigned to enable the use of a 3D dynamic geometry environment (GeoGebra). The professor who taught this course is one of the authors of this article. Students first worked individually on each activity, then discussed it in teams of four students; during these discussions, the teacher visited the teams –these discussions were video recorded but have not been analyzed in this paper– and finally, a teacher-led whole group discussion was held. At the end of the semester, one student from each of the eleven teams was chosen to be interviewed. The researchers designed an instrument consisting of seven multitask questions to conduct semi-structured interviews with eleven students to test their understanding of the different components of GD. All the students were engineering students who had just completed this introductory multivariate calculus course. Each of the interviews lasted between 60 and 80 minutes and was video recorded. All the data were independently analysed by the researchers and conclusions were negotiated. The written work from the GeoGebra-based activities produced during the semester was obtained from all students and was used in the analysis.

The seven interview questions focused on functions of two variables. For the purpose of this report, only four of these questions are addressed. All of these questions were to be worked with paper and pencil, except where otherwise stated.

1. Draw in three-dimensional space the collection of points in space that satisfy the equation $y = 2$ and that are also on the graph of the surface $z = x^2 + x^3(y-2) + y^2$.

2. Let $f(x, y) = x^2$.

a. Represent in three-dimensional space the intersection of the plane $y = 1$ with the graph of f.

b. Draw the intersection of the plane $z = 1$ with the graph of f.

c. Draw the graph of f.

3. Let $f(x, y) = xsin(y)$.

a. What can you say about the intersection of the plane $x = 0$ with the graph of the function $f(x, y) = xsin(y)$? Represent the intersection in three-dimensional space.

b. Draw the graph of $f(x,y) = xsin(y)$.

c. You can use the GeoGebra scenario to graph [one was provided, as in Figure 2b].

4. State which figure corresponds to $f(x,y) = sin(x) + y$. Carefully justify your answer (see the figures below).

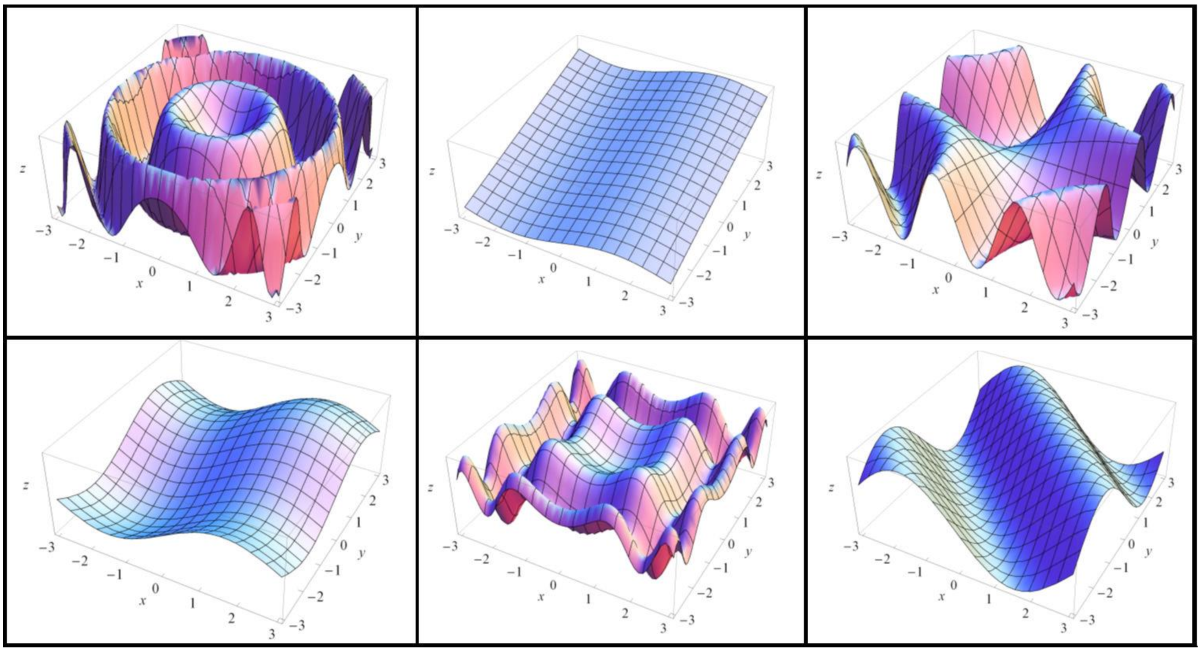
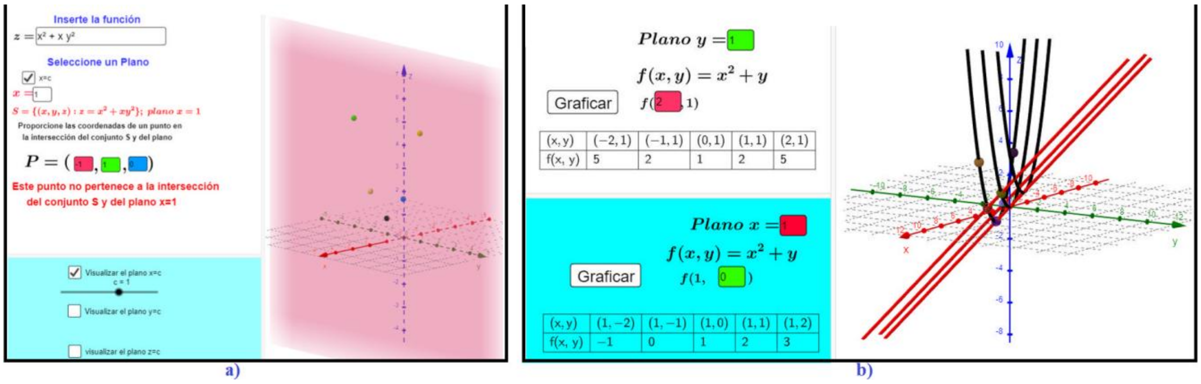


Figure 1: Surfaces for problem 4, the graph appears on the first row, second column.

Problems 1, 2a, 2b, and 3a directly deal with students' understanding of "fundamental planes" (planes of the form variable=constant) and the geometric meaning of substituting a number for a variable. Problems 2b, 2c, 3a allow to obtain information on students' understanding of "free variables" (situations represented by an equation with fewer variables than its geometric context). Problem 2c, 3b, 3c, and 4 may give information on students' graphing of two-variable functions. They may also give information on students' use of transversal sections, and thus, of fundamental planes.

Design of the Interactive Math Environment (IME)

We developed an IME based on the DGE, GeoGebra 3D. For the design, we considered the first five sets of activities used in the third research cycle of Martínez-Planell and Trigueros (2019). These had shown to be successful in helping students construct functions of two variables in a paper-and-pencil environment. IMEs consist of two or three Views (see Figures 2a and 2b). In the first activity, students are asked to make point-by-point constructions in which they intersect fundamental planes with surfaces, for example, students are given the set $S = \{(x,y,z): z = x^2 + xy^2\}$ and are asked to draw its intersection with the plane $x = 1$; IME helps the student observe their Actions on the screen, allowing for an automatic response that helps them identify whether a point they enter belongs to both the plane and the surface. Students are expected to interiorize those Actions into a Process where they can imagine the relation between the equation of a fundamental plane, its geometric representation, and placement in space.



a) b)

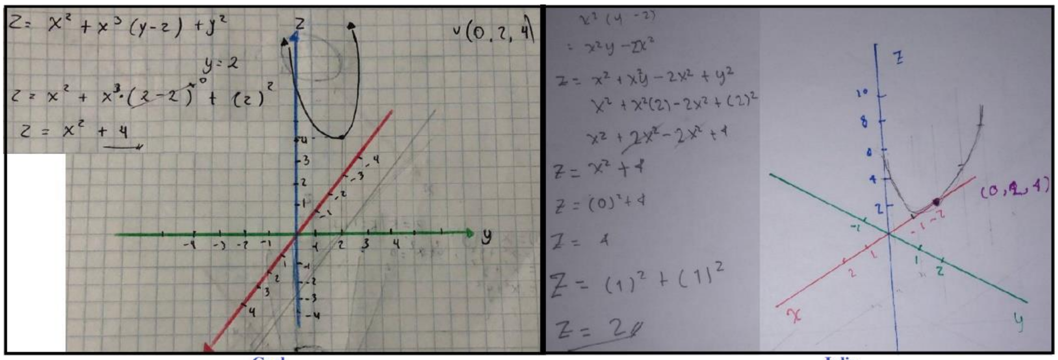
Figure 2: A scenario where students do a) Actions to construct a fundamental plane and b) Actions on fundamental planes.

Another activity guides the student to plot the graph of $f(x, y) = x^2 + y$ by drawing and placing a few curves using specific transversal sections, leading to reflection on the effect of giving different values to different variables. The student chooses the variable and value (e.g., $y = -1$, $y = 0$, $y = 1$, $x = 0$), GeoGebra plots the curve, and the student may verify by plotting points if desired. The dynamic imagery necessary to make sense of the situation was expected to help students construct a graphing Process. The need to do Actions on fundamental planes was expected to help then encapsulate fundamental planes into an Object (see Figure 2b).

RESULTS

The results obtained in this study were not as good as those of the third research cycle of Martínez-Planell et al. (2019) or the reproducibility study of Borji et al. (2022) in which the original GD-based activities were done with paper-and-pencil. In this report, we consider the results of two of the best students. This will enable us to discuss why activities with GeoGebra were successful in promoting some of their constructions but failed for other students. In doing that, we also give a more detailed account of specific student difficulties with free variables.

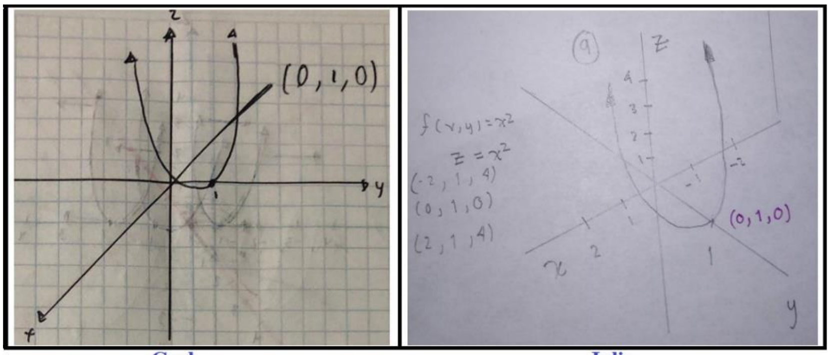
In question 1, both Julio and Gael showed the Process of relating graphical and algebraic representations of fundamental planes, as well as imagining their position in 3D space. The results suggest they had constructed the Process on the geometric meaning of substituting a number for a variable. This is consistent with a Process conception of fundamental plane. This construction seems to have been fomented by the use of GeoGebra during the in-class activities, as suggested by the fact that they graphed using the same colours for the axes as the GeoGebra activity (Figure 3).



Gael Julio

Figure 3. Graphical and algebraic representations of the fundamental planes.

In problem 2a, Julio and Gael could locate the intersection of $y = 1$ with the graph of $f(x, y) = x^2$ correctly in space (see Figure 4). Although this question involves the free variable y, all variables are explicit: students set $z = x^2$ and are told that $y = 1$. This gives further evidence of their understanding of fundamental planes, this time in the case of a "cylinder," meaning the graph of a two-variable equation that is to be interpreted in the 3D context.



Gael Julio

Figure 4. Intersection of the fundamental plane $\boldsymbol{y = 1}$ with the graph of $\boldsymbol{f(x, y) = x^2}$.

In problem 2c, both students used transversal sections to draw the graph of *f(x,y)=x*^{2} (see Figure 5). They drew parabolas resulting from giving positive and negative values to y, then connected these parabolas with straight lines (see Figure 5). Based on the interview and in-class work where they graphed cylinders and justified their reasoning, it seems that they could generate the needed dynamic imagery to join the curves. This is consistent with a Process conception of graphing two-variable functions. Moreover, as they did Actions on such planes in order to graph the cylinder, we considered they had constructed an Object conception of fundamental planes. Students' work on the GeoGebra scenario used during in-class activities to graph cylinders gives evidence of their use of technology as a Replacement since they did Actions of intersecting fundamental planes with surfaces when plotting point-to-point graphs. The evidence also shows its use as Amplification since it allowed them to automatically validate their calculations and display the points and/or curves in the 3D view (Hughes, 2005), enabling them to imagine the entire surface. However, this was not the case for most other students. Even though the instructions for the activities were careful to request justifications, most students did not construct a graphing Process. They tended to take the output of the computer in itself as a valid justification, and used technology as replacement, sidestepping the necessary reflection to interiorize Actions into Processes. In this case, we also considered that as lectures and group discussions were conducted virtually, because of the pandemic, it was difficult to develop a culture of discussion and justification in the classroom.

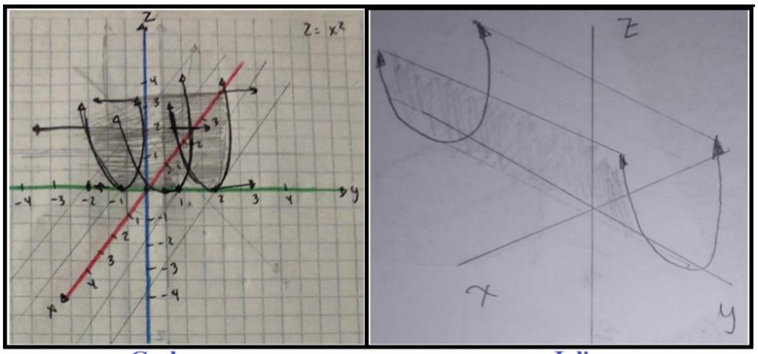


Figure 5. Graphs of $f(x, y) = x^2$ during the interview.

Responses to question 2b suggest that some situations involving free variables are treated differently by students. This difficulty had been observed before (Martínez-Planell & Trigueros, 2019). Both Julio and Gael wrote $z = x^2$ and substituted $z = 1$ to obtain $1 = x^2$. It can be observed that contrary to the situation of problem 2a, the variable y does not appear in their computations. So, both of them seemed to do the Action of setting $y = 0$, which led them to consider two points rather than two lines. The same behaviour regarding free variables was also observed in question 3a. Both Julio and Gael did the Action of substituting $x = 0$ in $z = xsin(y)$ to obtain $z = 0$. It seems that, from their perspective, y disappeared and so it again seems they set $y = 0$.

> Julio: It tells us that in the plane $x = 0$, we substitute in the function and we get 0, z is equal to 0, in three-dimensional space, ..., I guess it will be just a point [He drew the point (0,0,0)].

Students had worked on exactly the same problem in the in-class activities. Examining their Actions, we see that the GeoGebra scenario required them to enter different points satisfying both equations, and then the scenario would show their graph; then students connected the resulting y-axis points by performing Actions, apparently without reasoning algebraically why they could connect them. That is, the GeoGebra scenario did not induce students to reflect on the fact that $0 = 0sin(y)$ is true regardless of y. Students used technology in this activity as replacement (Hughes, 2005) as they could have performed the same point-by-point graphing Actions on paper-and-pencil. A similar response was observed in most other students. By considering their written response to the GeoGebra-based activities, it becomes clear that some students went beyond the scenarios provided and produced surface graphs using GeoGebra in a way that did not foster their reflection on fundamental planes and the geometric interpretation of the meaning of holding a variable fixed, thus partially explaining their lack of success.

In problem 3b, when asked to graph $z = xsin(y)$, it became apparent that all students had not constructed the prerequisite Processes of trigonometry to coordinate it with a Process of fundamental plane. Both Julio and Gael stated that they would use transversal sections as their strategy, both knew they were expecting a wave-like surface, but neither of them interpreted x as an amplitude nor took the sign of x into account:

Interviewer: how do you plan to draw the graph?

Gael: by transversal sections, ..., that would be the sine graph, which is basic trigonometry, which are like waves, ..., but I'm trying to remember how it behaves when it's multiplied by x? I know that if x weren't there, it would be represented as a galvanized sheet, but when you multiply x, it behaves differently.

Julio: What I would do is give values to x and y, ..., like a wave …

Most other students showed difficulty with trigonometry. This seems to be an institutional issue. It was not observed in the paper-and-pencil studies mentioned before. Julio and Gael succeeded in graphing the function using a GeoGebra scenario, to do Actions involved in completing a few transversal sections to obtain the surface (see Figure 6). The scenario seemed to also help Julio make sense of the intersection of $x = 0$ with $z = xsin(y)$:

Interviewer: Ok … in the first part when $x = 0$, do you remember the first part? … you told me that the answer was a point … Why do you think, you get a line, Julio?

Julio: Because … we are only giving values to y.

It seems that now Julio realizes that $x = 0$, $z = 0$, but that y can take any value.

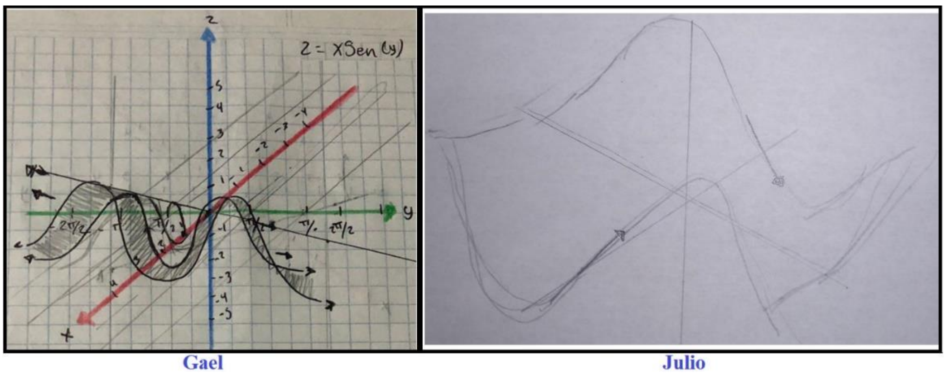


Figure 6. Graph of the function $f(x, y) = xsin(y)$ supported by the IME.

The above discussion underscores that even though students might have constructed a graphing Process, they need to coordinate the Process of fundamental plane with one-variable function graphing Processes, which may not have been constructed due to the pandemic or other institutional reasons. It also shows how technology's capacity to generate graphs can potentially support students understanding of free variables when used as transformation.

In question 4, when choosing the graph of $f(x,y) = sin(x) + y$, Julio chose the correct graph by doing an Action on fundamental plane. He mentioned that if $x = 0$, the function reduces to $z = y$, and therefore should result in a line with positive slope. There is only one such option. Gael gave signs of construction of dynamical imagery:

> Gael: … on the x-axis [meaning the x direction] it is going to be the graph of sine … then plus y, y would be a line on the y-axis [meaning the y direction] … when $y = 0$ I'll be left with only that, its wave in z … and it keeps on increasing.

Julio and Gael seem to imagine intersecting fundamental planes with surfaces as a technique for graphing surfaces, they seem able to do Actions on fundamental planes in order to form transversal sections, and show some evidence of generating dynamical imagery when graphing cylinders and functions. While this shows the potential of activities with GeoGebra, for most students it did not encourage the necessary reflection to interiorize their Actions into Processes.

DISCUSSION AND CONCLUSIONS

We observed that the activities worked in class helped Julio and Gael to construct the geometrical meaning of substituting a number for a variable in a 3D context, to recognize that the graphs of cylinders can be obtained by intersection with fundamental planes corresponding to the missing variable, and to use transversal sections as their chosen graphing strategy. The use of GeoGebra seems to have contributed to this understanding of fundamental planes and cylinders. Although this shows GeoGebra activities have the potential to be used as transformation, most students used the technology as replacement. We found that there are situations involving free variables, particularly situations when during a computation one of the variables "disappears," in which the GeoGebra activities did not foster the needed student reflection. Thus, some GeoGebra activities need to be redesigned to accomplish this goal.

Considering the entire student population, the results obtained in this study were not as good as those obtained in other studies based on the equivalent paper-and-pencil activities (Martínez-Planell et al., 2019; Borji et al., 2022), which rendered very positive results. We conjecture that one of the reasons for this difference was that classes for this experience were taught virtually due to COVID-19 pandemic. In this context, more attention to classroom culture and management issues are needed, particularly as it regards justification in a DGE. Also, there were other institutional factors, like students' knowledge of trigonometry, that affected outcomes.

When redesigning the activities, the purpose was that the IME would have a didactic functionality to develop conceptual understanding (Drijvers, 2015); however, the results show that this was not achieved in this teaching experiment. The shift from paper-and-pencil activities to 3D dynamic geometry technology is not a direct translation; activities have to be redesigned and institutional conditions have to be taken into account to allow both for the possibility to use the technology as transformation and for promoting students' reflection.

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Analysing the Quality of advanced Mathematics Lectures regarding the Presentation of Definitions – The Case of Real Analysis Lectures

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This paper reports from a study that investigates the quality of advanced mathematics lectures, in particular regarding the presentation of definitions. We compare the presentation of definitions given in two real analysis courses by two different lecturers using a structured observation protocol. The results show, that both lecturers present formal concept definitions with great importance, but they perform differently regarding the motivation of concepts, giving of mental or visual forms of representation as well as giving of examples and counterexamples. Therefore, students might have developed different concept images. Furthermore, one of the lecturers might put different value on the presentation of different kinds of definitions.

Keywords: advanced mathematics lectures, mathematics lecture quality, mathematics definitions, structured observation protocol.

INTRODUCTION

Advanced mathematics lectures are challenging for many students. Because of this, many students drop out from their mathematics study or change to another subject during their first year at the university (Geisler, 2020). Many researchers have questioned lectures as teaching format at universities (e.g., Fritze & Nordkvelle, 2003). Nevertheless, Fritze and Nordkvelle (2003, p. 328) say: “[T]he lecture survives, probably because it serves many functions not so well observed in the present research”. However, according to Viirman (2014, p. 512), “programs and resources designed to help university mathematics teachers develop and improve their teaching are not informed by data on the teaching practices actually used in tertiary mathematics teaching”. Indeed, we have found only a few studies concerning characteristics and the quality of mathematics lectures (e.g., Viirman, 2014; 2021; Rach et al., 2016). Since lectures remain a dominant teaching format in advanced mathematics, more research analysing the quality of mathematics lectures especially concerning lectures in the challenging study entry phase is necessary.

In this paper, we present results of an observation study concerning the presentation of definitions in mathematics lectures supported by a structured observation protocol. In the following, we describe our conceptualization of quality and a theoretical framework for quality of mathematics teaching as well as previous research regarding the presentation of definitions in advanced mathematics lectures.

THEORETICAL PERSPECTIVE

Quality criteria for mathematics lectures

According to Kiendl-Wendner (2016), university quality can be characterized according to the following features: forms including macro level (whole university or faculty) and micro level (single courses), and dimensions including process quality (clearly defined processes with standardized roles and procedures), result quality (achievement of the goals set) and structural quality (adequacy of resource allocation).

German and non-German undergraduate mathematics courses usually consist of lectures and tutorials (e.g., Pritchard, 2015). In our study we take a look at the micro-level-quality of mathematics lectures and investigate their process quality (Kiendl-Wendner, 2016).

Viirman (2021, p. 467) describes lectures as "a teaching mode involving one teacher and a large group of students with communication mainly directed from the teacher to the students […]". Additionally, Bergsten (2007, p. 48) completes lecture as a "time scheduled oral presentation on a pre-announced topic […]". Because empirical research concerning the quality of mathematics lectures is scarce, we have found only two frameworks that describe process quality criteria of mathematics courses. The first framework is theory based and comes from Rach et al. (2016) and includes two main categories: formal mathematics criteria that describe the presentation of definitions and theorems in lectures as well as presentation of solutions in tutorials, and general criteria (adapted from secondary school research (e.g., Clausen et al., 2003)) such as learner orientation, cognitive activation, instructional efficiency as well as clarity and structure. Presentation of solutions in tutorials has no relevance for us because we do not take a look on the quality of tutorials.

The second framework is practice based and comes from Bergsten (2007). It includes three main categories: mathematical exposition, general criteria and teacher immediacy. By teacher immediacy Bergsten summarizes personality and non-verbal behaviour of the lecturer such as facial expressions or gestures. These criteria have to be fulfilled for a high-quality mathematics lecture (Bergsten, 2007). Both frameworks include categories regarding formal mathematics and general criteria. The main difference is the category regarding teacher immediacy in lectures in the framework from Bergsten. Moreover, Bergsten names the main categories of a quality lecture but he does not describe them in detail. We have built a synthesis based on both frameworks to describe the quality of mathematics lectures. Our framework regarding quality of mathematics lectures consists of three main categories: mathematical exposition, general criteria and teacher immediacy. The first main category contains actions that are specific and particular relevant in mathematics lectures like presentation of definitions or presentation of theorems and proofs. The second main category summarizes general criteria that can also be used in non-mathematical lectures such as learner orientation, cognitive activation, instructional efficiency as well as clarity and structure. The third and last main category calls teacher immediacy and it contains lecturer´s personality and his or her non-verbal behaviour.

Presentation of definitions in mathematics lectures

Definitions take an important place in mathematics lectures (e.g., Halverscheid & Pustelnik, 2013) and are therefore also of major relevance for the quality of a mathematics lecture. In this paper we are going to present our findings concerning the presentation of definitions in mathematics lectures. Therefore, we use a theory of concept formation based on Tall and Vinner (1981). They define two important terms: concept definition and concept image. Concept definition describes "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). Concept image describes "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (ibid, p. 152). According to Tall and Vinner (1981), students´ concept image can be in conflict with the concept definition. A major challenge for lecturers is to help students building correct concept images to get a full understanding of a concept definition. Moreover, Capaldi (2020) distinguishes between formal/rigorous definitions that correspond to concept definition and informal/non-rigorous definitions that support students in building a concept image. Therefore, colloquial interpretations and visual characterizations (Capaldi, 2020) have to be used in mathematics lectures to illustrate concepts. Fukawa-Connelly and Newton (2014) emphasize the importance of examples in mathematics lectures for development of a correct concept image and use the term from Mason and Watson (2008) called example space. Nevertheless, the example space is just one of the puzzle pieces – such as formal statement of definition, used language and motivation – to build a correct concept image (Fukawa-Connelly & Newton, 2014). Capaldi assumes that lecturers´ activities concerning presentation of definitions in mathematics lectures influence the development of students´ concept images: students could develop different concept images in courses given by different lecturers (Capaldi, 2020).

Despite these arguments for the relevance of motivation for concepts, the use of examples and (visual) representations when presenting definitions, many scholars have stated that actually in many mathematics lectures only formal concept definitions are presented (e.g., Davis & Hersh, 1981). However, these perspectives were only rarely informed by empirical data. Empirical studies that rely on actual lecture observations draw a less clear picture: Fukawa-Connelly et al. (2017, p. 577) observed 11 advanced mathematics lecture courses and concluded that "Instructors present informal content, including examples, informal representations, […] during their advanced mathematics lectures, at least some of the time." Viirman´s (2014) qualitative study concerning quality of mathematics lectures is based on observations of seven different mathematics lecture courses by different lecturers each. He found out that there are construction routines lecturers use in their mathematics lectures like definition construction, example or counter-example construction and topic-specific constructions (Viirman, 2014). Moreover, he identified five different types of definition construction: stipulation (definition based on a reminder), exemplar (definition given according to the example), contrasting (definition given according to examples that differed in an important aspect), saming (definition based on crucial property of examples) and naming (definition based on naming of already constructed mathematical objects) (ibid.). A quantitative study regarding the quality of mathematics lectures from Rach et al. (2016) is based on observations and structured coding of a real analysis lecture course by one lecturer. The evaluation of data shows that the lecturer of this mathematics lecture course put value on a correct definition of concepts but he almost does not take time to motivate them (Rach et al., 2016). Moreover, his usage of examples and counterexamples as well as usage of mental or visual forms of representation is limited (ibid.).

THE PRESENT STUDY

Research question

The empirical research regarding the characteristics of advanced mathematics lectures is rare, particularly there is not enough empirical research "based on observations of actual lecturing" (Viirman, 2021, p. 467). Previous studies have collected data about mathematics lecturing either through observation of only one lecture or several lectures covering different topics without using of a structured instrument. Our goal is to use a structured observation protocol to enable comparisons of mathematics lectures. In this study we investigate how lecturers present definitions in their courses. In particular we want to answer the following research question:

In which way is it possible to identify differences in the presentation of definitions in advanced mathematics lectures given by two different lecturers and which differences can be found?

Methodology

First of all, we took a look at the study regulations in linear algebra and real analysis – as the courses students usually take during their first year – from several universities in Germany. It turned out that the themes in real analysis are quite similar across different universities. For this reason, we have decided to observe lectures in real analysis, in particular those sessions covering the topics sequences and series.

Next, we decided to observe the lectures with a standardized observation protocol to make our investigation systematic and enable comparisons between different real analysis courses. As afore mentioned, Rach et al. (2016) have already researched on the quality of mathematics lecturing at universities and developed a standardized observation protocol. This observation protocol is based on Tall and Vinner´s (1981) framework concerning definitions. We have merely adapted it slightly for our research.

The standardized observation protocol uses four categories to describe the presentation of definitions: (1) *Motivation of concept* (introduction of a concept), (2) *Description of definition* (presenting of a formal definition), (3) *Giving examples and counterexamples*, and (4) *Mental or visual forms of representation*. Each category can be evaluated in four grades: presented well, presented, presented poorly, and not presented. We assigned the grades the numbers 1, 2, 3, and 4 accordingly. There is an example (see Figure 1) how the category (4) *Mental or visual forms of representation* must be coded. We have coded all four categories for presentation of definitions.

The category is coded:

- When the concept is visualized by a graphic representation.
- When mental images are used.
- When misconceptions are mentioned that can be found for this concept.

Coding note: It is important to assess whether the representation is suitable for communicating the intended (desired) train of thought.

- *1 – presented well* Various visual representations or mental images are used. The connection to the formal definition of the concept must be drawn.
- *2 – presented* Visual and mental representations are used sensibly. The connection to the formal definition is not drawn.
- *3 – presented poorly* Visual representations are used, but they do not illustrate the concept to be explained or have mathematical weaknesses. Or mental representations are mentioned, but explained in a misleading manner or with mathematical weaknesses.
- *4 – not presented* No visual representations and no mental representations are addressed.

Figure 1. Observation protocol for the category *Mental or visual forms of representation* (Rach et al., 2016)

For this paper, we have observed two lecture courses in real analysis (we call them *Lecture A* and *Lecture B*) by two different lecturers from a large German public research university. These lecture courses are aimed at pure mathematics students and upper secondary pre-service teachers. We coded only lectures covering the topics sequences and series. Both lecture courses were previously video-recorded. For this reason, we have had a good opportunity to stop or to repeat the recordings and to make some notes. *Lecture B* started often with the repetition of the last definition from the previous lecture. We did not code repeated definitions because these repetitions mainly served as memory aid and were presented less elaborated than in the first occurrence. Moreover, to be sure that our measuring instrument is reliable, we asked another German researcher to code one lecture by each lecturer. In order to check the interrater reliability we calculated the Spearman correlation coefficient between both codings. The corelation of ρ=0.53, indicates a medium but sufficient correlation between our coding of both lectures and the coding of the second coder.

RESULTS

After coding definitions, we analysed which definitions were presented in both lectures. We noticed that there are some similar definitions in the *Lecture A* and *Lecture B* and built two groups: primary definitions (are presented in both lectures), and secondary definitions (are presented in only one lecture).

Lecturer A presented only primary definitions in his lecture course. *Lecturer B* presented primary definitions as well as secondary definitions. Table 1 shows arithmetic means (*M*) and standard derivations (*SD*) for above-named categories of definitions. The values in columns *Lecture A* and *Lecture B primary* include only primary definitions presented by *Lecturer A* and *Lecturer B* in their lecture courses, *Lecture B secondary* includes secondary definitions presented by *Lecturer B* in his lecture courses.

	Lecture A		*Lecture B primary*		*Lecture B secondary*	
	M	*SD*	*M*	*SD*	*M*	*SD*
(1) Motivation of concept	2.142	1.293	1.125	0.330	2.380	1.326
(2) Description of definition	1	0	1	0	1	0
(3) Giving examples and counterexamples	2.142	0.989	2.250	1.089	2.476	1.138
(4) Mental or visual forms of representation	2.857	1.124	1.875	1.268	2.619	1.463

Table 1. Comparison of the observed lectures (means and standard deviations of the coded categories)

First, we compare the characteristics of definitions in *Lecture A* and *Lecture B primary*. Arithmetic means suggest that the quality of *Motivation of concept* and using of *Mental or visual forms of representations* by *Lecturer B* was better than by *Lecturer A*. Both lecturers presented well the *Description of definitions*. This means, all formal definitions presented in lecture courses were presented mathematically correct. *Giving examples and counterexamples* was by both lecturers rather presented than poorly presented. Moreover, standard derivation shows that categories *(1)* and *(4)* are spread by *Lecturer A* on average between presented well and not presented, category *(3)* is spread between presented well and presented badly. It is different for *Lecturer B*: category *(1)* is spread between presented well and presented, categories *(3)* and *(4)* are spread on average between presented well and presented badly. We compared the results for presentation of *primary definitions* by *Lecturer A* and *Lecturer B* using the Wilcoxon test, revealing that the differences concerning *Motivation of concept* are only weakly significant ($p<0.1$) and those regarding *Mental or visual forms of representation* are significant ($p<0.05$); differences in the other categories are not significant. Below we present an example of encoding a primary definition of absolute convergence given by both lecturers in order to illustrate the aforementioned differences concerning the presentation of definitions between both lectures.

Before giving a formal definition of absolute convergence, *Lecturer A* tried to motivate it. He emphasized the importance of a new tool to decide whether a series is converging and he gave a review of previous definitions concerning convergence of sequences. However, *Lecturer A* did not mention what the criterion of absolute convergence of series is useful for and how it can be used in the future. Moreover, he did not show the connection between convergent sequences and convergent series. According to our observation protocol, we evaluated *Motivation of concept* as 2 – presented. The given formal definition of absolute convergence was mathematically correct and we evaluated *Description of definition* as 1 – presented well. Only one example (exponential series) was given, so *Giving examples and counterexamples* was evaluated as 3 – presented poorly. *Lecturer A* did not use any forms of informal representation so *Mental or visual forms of representation* was evaluated as 4 – not presented.

Lecturer B used another way to present a definition of absolute convergence than *Lecturer A*. First of all, he presented with regard to the alternating harmonic series a theorem concerning limit depending on the arrangement of the elements of a series. Furthermore, he explained how the alternating harmonic series can get different limits when rearranging the elements using a sketch. He mentioned, that these series could not fulfil many properties of finite sums such as associative property. Thereby, he motivated the importance of absolute convergent series. Next, *Lecturer B* gave the mathematically correct formal definition of absolute convergence as well as an example (geometric series) and a counterexample (alternating harmonic series). Moreover, he described the connection between convergence and absolute convergence: every absolute convergent series is also convergent, but not every convergent series is also absolute convergent. Therefore, we evaluated *Motivation of* *concept, Description of definition, Giving examples and counterexamples* and *Mental or visual forms of representation* as 1 – presented well.

Second, we take a look at secondary definitions presented by *Lecturer B*. Arithmetic means for categories *(1)*, *(3)* and *(4)* in *Lecturer B secondary* are between presented and presented badly and point to the mediocre quality of presenting definitions in this lecture. Compared to *Lecture B primary* we can see that the quality of secondary definitions based on categories *(1)* and *(4)* is lower. Standard derivation of these categories shows that they are spread on average between presented well and not presented. Category *(2)* is always presented well. We compared the results for presentation of *primary definitions* and *secondary definitions* by *Lecturer B* using the Mann-Whitney-U-Test. According to the results, only the difference regarding *Motivation of concept* is weakly significant (p<0.1) while the differences in other categories are not significant.

Comparing the results from *Lecture B primary* and *Lecture B secondary*, we can see that *Lecturer B* pays more attention to the categories *Motivation of concept* and *Mental or visual forms of representation* and *Giving examples and counterexamples* concerning primary definitions than secondary definitions. The arithmetic means for the category *Description of definition* in both cases is the same. Therefore, secondary definitions were correctly presented as well as primary definitions.

CONCLUSION AND DISCUSSION

Lecturer A and *Lecturer B* attach great importance to *Description of definitions* but they deal with *Motivation of concept* and *Mental or visual forms of representation* differently. Considering primary definitions, *Lecturer B* achieves better arithmetic means in these categories than *Lecturer B*. The category *Giving examples and counterexamples* is by both lecturers midrange. Moreover, there are differences in the presentation of primary and secondary definitions by *Lecturer B* regarding *Motivation of concept*, *Giving examples and counterexamples* as well as *Mental or visual forms of representation*. These categories achieve worse arithmetic means concerning secondary definitions. It seems that *Lecturer B* puts more value on presentation of primary definitions then on presentation of secondary definitions.

Students might have developed different concept images in the course of *Lecturer B* then the course of *Lecturer A* concerning primary definitions and maybe could visualize the concepts better. This assumption applies also to primary and secondary definitions presented by *Lecturer B*. Because of worser values for categories *(1)*, *(2)* and *(4)*, students might have developed different concept images concerning secondary definitions than primary definitions presented by *Lecturer B*. Summarizing, students taught by different lecturers probably develop different concept images even if they attend a lecture in the same university and have the same study regulations. A study that investigates the quality of presentation of definitions in lectures *and* the actual developed concept images by students could confirm these assumptions.

There are some limitations in our study. The sample of our study consists of only two different lecture courses. To be able to see more effects concerning quality of mathematics lectures, a larger sample is necessary. In addition, the correlation between our coding and coding of the second coder is only medium. Because of this, we will revise the observation protocol prior to further investigations. This concerns first of all the category *Giving of examples and counterexamples.* According to the observation protocol, examples as well as counterexamples must be presented to each given definition in the lecture in order to be coded as presented well. But we have the opinion, that not every definition must be supported by counterexamples. We will adapt the observation protocol accordingly. Nevertheless, the use of a structured observation protocol enabled a structured observation of mathematics lectures and a first comparison of their quality regarding the presentation of definitions. We could identify differences in the presentation of primary definitions by both lecturers as well as presentation of primary and secondary definitions by *Lecturer B*. The next step of our study is to extend our observation protocol to general criteria and teacher immediacy. Future studies should help to generate a more holistic picture of quality of mathematics lectures at German universities. The results of future studies should help us identifying possible connections between the quality of mathematics lectures and performance of students in their mathematics programs. Moreover, it would be relevant to analyse whether the quality of mathematics lectures influences students´ decision to drop out from their mathematics programs.

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An investigation of potential changes in students' images of derivative at the entrance of the university

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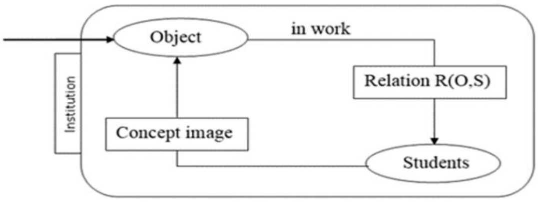
Keywords: concept of derivative, rate of change, networking, curriculum continuity.

RESEARCH PROBLEM AND THEORETICAL FOUNDATION

The concept of derivative holds an important place in calculus courses in both high school and the beginning of the university. Much research has been conducted to investigating the cognitive process involved in the learning of derivative. The results highlighted students' difficulties to deal with the relation between the rate of change, the slope of the tangent and the limit notion (for an overview see Bressoud et al., 2016). In Tunisia, the concept of derivative is introduced for the first time as an instantaneous speed one year before the end of high school (about 17 years old). Bouguerra (2019) considered that this introduction does not help Tunisian students understand the derivative at a point, because they have not learned that instantaneous speed is an instantaneous rate of change. She stated that the slope of a tangent is the main image students have of derivative and that its link to the instantaneous rate of change is not clear. In the other side, Ghedamsi et al. (2021) have shown that school derivative tasks focus mainly on computational and graphical activities while university tasks require more proof skills. In this research, we intend to identify any persistent difficulties for university students and investigate potential changes in their images. Our aim is not only to study university students' conceptions but also to examine how the concept itself is presented in university textbooks and curriculum and therefore try to find some connections between both. We deploy a networking (Bikner-Ahsbahs & Prediger, 2014) between the image frame (Tall &Vinner, 1981) and the praxeological model (Chevallard, 1999) to analyse institution choices and their link to students' understanding of derivative concept.

Chevallard (1992) stated that knowing an object means having a certain relation with it. So, praxeology model permits to describe what was taught and what could be learned and it may also provide general idea about students' mathematical activities and their relation to mathematical objects. However, the images frame highlights the existence of a dynamic cognitive entity, for a given mathematical concept, that differs from one student to another. In this sense, taught praxeologies could be considered as a potential element of students' concept image. Applying images frame helps knowing how students conceptualize the derivative and allows giving information on how students' images are constructed. A dialogue between praxeology model and image frame may provide tools to trace probable origins of students' difficulties by focusing on the way the concept is presented in textbooks and curriculum (figure). We argue that the two frames address complementary views on the teaching and



learning processes. Accordingly, our research question is: To what extent the way of presenting derivative in university textbooks and curriculum impact students' previous images of derivative?

Figure: Networking between the image frame and the tools of the ATD

METHODOLOGY

In the first step, we studied the first-year university official textbooks and curriculum of preparatory institutes for engineering and we identified the most important praxeologies related to the derivative. At a later stage, we conducted a questionnaire and an interview with 87 first-year university students from the same institutes.

FINAL REMARKS

Networking allows more explanation about students' understanding of derivative notion and its origin. Students'difficulties in using the rate of change to interpret definition of derivative at a point are the cornerstone issue. The results show that students' use of rate of change is limited to direct applications. Students' difficulties arise when moving from repetitive calculations to functional thinking, and they cannot get rid of the abstract aspect in which the formal definition is formulated. Eventually, in our case, the curriculum continuity enhances students' difficulties at the entrance to the university as well. Networking allows studying the research question and to conjecture that in Tunisia, students' difficulties about derivative are of several origins and that the way of presenting this notion in textbooks and curriculum is the most important one.

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Investigation of metarules in lecturing for enabling students' participation

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Keywords: Teachers' and students' practices at university level, teaching and learning of analysis and calculus, commognition, university lecture, metarule

Participation in the university mathematics community usually starts with the students attending mathematical lectures. One of the lecturers' main goal in teaching is to facilitate students' participation in this community (Sfard, 2008). The actions of the lecturer in the teaching that may assist students' participation are still underexplored in the university mathematics literature (Melhuish et al., 2022). Thus, in this study, we propose the commognitive framework to investigate the discursive actions that may assist students' participation in the university mathematical discourse as it gives a fine-grained analysis for a micro-level investigation.

For the investigation of the discursive actions, awareness of the metarules in the teaching is necessary. A mathematical discourse has its own sets of metarules, which are narratives that define patterns in the activity of the participants. These metarules result in routines, which are patterns of discursive actions. Consequently, our research question is "which are the discursive actions and the underlying metarules of the lecturer for supporting students' participation from the lecturer's perspective?".

METHODOLOGY

For this study, we investigated the teaching in online lectures of an introductory real analysis course. The lecturer in this case study is a mathematician with six years of teaching experience in this course. For the analysis, we coded the seven lectures using inductive and deductive thematic analysis with theoretical codes from Karavi et al. (2022). Through constant comparisons of the quotes under the same code, preliminary themes of discursive actions emerged. Then, while further investigating the discursive actions in relation to their appearance while proving, we identified implicit metarules. Operationalisation of the discursive actions and metarules occurred through constant comparisons.

RESULTS

We present briefly the results using excerpts from the episode of teaching of the characterization of compact sets: $K \subseteq \mathbb{R}\ compact \Leftrightarrow K\ closed\ and\ bounded$. In our interpretation, the lecturer supported students' participation in the proving processes through the performance of the following discursive actions: *making decisions on how to start the proving process* (e.g., "Let's assume that K is a compact set and then show the K is both closed and bounded. Well, let's do a proof by contradiction. Let's assume that a set K is not bounded"), *sharing the key idea of the proof* responding to the question "And now what?" that came after the statement of the theorem (e.g., "So, if you look very carefully what I'm doing in this proof, I'm almost using the same sort of proof that I use to show that the set of real numbers is not compact. Yeah, I am almost using the same trick here"), and *bringing the means for the emergence of the proof* (e.g., "Okay, so now my assumption is K is closed and bounded. And I'm going to reason from right to left. So, how do I show that K is compact? Well, the only thing I can do is to verify the definition"). These discursive actions are governed by the implicit metarule *while proving, an idea of how to start is needed*. The metarule is related to students' de-ritualization and an independent, product-oriented engagement with proving processes. The lecturer's discursive actions shifted the attention to the product and gave an idea to the students why specific actions took place.

DISCUSSION

Identifying the discursive actions and the metarules could give in future studies valuable insights into the ways of possible facilitation of the newcomers' participation in the mathematical community. Commognition can support micro-level investigation in the observational data from the lectures, aiming to explore lecturers' practices towards students' learning. As Pinto (2019) highlighted the differences among the examined lecturers were on a meta-level and related to their different views and experiences with teaching. Following Pinto (2019), the metarule affected the discursive actions that appeared in the lectures, shaping the lectures. In our case, the identification of the metarule facilitated the observation of de-ritualization instances behind the proving processes that may assist students' explorative participation. Awareness about the metarules and lecturers' views on them could provide understanding into the university mathematics lecturing than considering it as a monologue.

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TWG3: Teaching and learning of linear and abstract algebra, logic, reasoning and proof

TWG3: Teaching and learning of linear and abstract algebra, logic, reasoning and proof

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INTRODUCTION

In this group, there were 10 oral communications and 4 posters that were organised in two thematics: linear and abstract algebra, presented in the first parallel session; logic, reasoning, and proof presented in the second parallel session. The list of papers and posters is in an annexe of the document. The group comprised 19 participants from various countries. Each paper was allocated a 15-minute presentation, followed by 5-minute discussion with the audience. During the two first discussion sessions, a slot was devoted to the presentation of posters linked to the theme of the corresponding parallel session. For the discussion following each of both sessions, we split into three non-thematic subgroups around two papers or a paper and one or two posters. Previously to the conference, for each paper, two registered participants were invited to prepare a few slides to act as critical friends during the conference. These slides have been used during the discussion sessions in small groups. This was followed by a collective discussion.

During the third discussion session, we decided to split into two thematic subgroups: Linear and abstract algebra; Logic, reasoning, and proof, to discuss the papers and posters linked to the thematic and let emerge first ideas. Seven attended the subgroup on linear and abstract algebra, and 12 attended the subgroup on logic, reasoning, and proof. During the fourth session, we go on working in thematic subgroups to reflect collaboratively on the new ideas that have emerged during the conference. Each subgroup has prepared a report on its theme that has been presented in the closing session.

TRENDS AND PERSPECTIVES IN LINEAR AND ABSTRACT ALGEBRA

In this section, we provide a summary of presentations and discussions regarding linear and abstract algebra concluding with a brief synthesis of emerging topics/issues and questions, and further research directions.

The main topics in the papers and posters presented

There were four main topics in the papers and posters; Eigentheory (Piori and Lyse-Olsen & Fleischmann, Wawro & Thompson), vector spaces (Can, Aguilar & Trigueros), Gauss algorithm (My Hahn) and computational thinking (Turgut). Regarding eigentheory, we underline the following three main *themes*:

- Semiotic analysis of signs (including gestures) produced by students during a collective activity aimed at making sense of eigenvectors and eigenvalues.
- Students' conceptions of eigenvectors and eigenvalues
    - Representations and formal elements used in students' descriptions
    - Focus on task design for individual learning activities
- Student reasoning about eigenequations (or not) in quantum mechanics

Piroi explores eigentheory teaching and learning processes in her continuing PhD research and provided the preliminary findings. The focus of the presented paper was an investigation of students' *collective* meaning-making processes within the lens of the theory of objectification, as a sociocultural theory. The paper described an activity that was created especially to support these processes of objectification. University engineering freshmen worked collaboratively to rethink eigentheory principles and rules while working in small groups. The usage of various semiotic resources by students, as well as how they relate to one another and how they have evolved, are then discussed. Under the same topic, Lyse-Olsen & Fleischmann examine students' understanding of eigenvectors at an early stage of their linear algebra instruction. Students' various explanations of eigenvectors are examined in relation to the mathematical objects they choose to depict (algebraic, geometric, or abstract representations), as well as the *formalism* they employed. Students nevertheless demonstrated their ability to switch between many representations and descriptions and produce unique concept images, even when the modes of description that were presented to them appeared to impact their own choice of description.

Wawro & Thompson's poster focused on student reasoning in quantum mechanics regarding matrix equations as eigenequations. Wawro & Thompson particularly explored how students would be able to distinguish between eigenequations and (quantum mechanics) matrix equations, and how this connects to their justifications for eigentheory in both mathematical and quantum contexts.

Regarding the Gauss algorithm, My Hahn presented research about constructing online *cloze style* (fill-in-the-blank with drop-down menu) questions to help students improve their understanding of and ability to use mathematical language. Examples given from discussing solution processes for systems of linear equations were provided. The paper discussed the preliminary findings of the analysis as well as Steinbring's *epistemological triangle* as a potential analytical tool for (aforementioned) comprehension processes.

Regarding vector spaces, Can, Aguilar & Trigueros focused on a teaching strategy for the learning of the concept of vector space using non-standard binary operations with a diversity of sets to promote student reflection on the vector space axioms more generally. The design is based on the APOS theory using its ACE cycle as a didactic approach, and a group of engineering students solved the provided task. Using sets and binary operations that aren't typically covered in a first course in linear algebra encouraged students to think critically about the axioms that define vector spaces.

Regarding computational thinking (CT), Turgut presented an emerging framework for integrating CT into teaching and learning linear algebra. The framework refers to three teaching principles of linear algebra, theory of instrumental genesis and CT. The paper presents a vignette in terms of GeoGebra's specific tools, functions and commands to teach the system of linear equations within the lens of CT.

Discussions, emerging topics and questions

The variety of theoretical and analytical frameworks available to support and guide the various research goals and the flexible way of using these frameworks were one of the main points that were discussed in the group. Presenters referred to different theoretical/conceptual lenses, such as APOS theory, the three teaching principles of linear algebra, theory of objectification, multimodal paradigm and semiotic bundle, knowledge in pieces, symbolic forms, modes of representation/thinking and (levels of) formalism. The broad spectrum of lenses not only raised discussion about the theoretical/conceptual frameworks in themselves, but also brought us to elaborate on how they should be used, the possibility of selecting the topic as a point of departure, and the role of emerging frameworks about eigentheory in research.

The second point that the group discussed was the role of *task design* in our research. For example, where to start and which has a priority; (i) designing to overcome students' epistemological issues, (ii) designing innovative teaching-learning environments, or (iii) both in a synchronised way. The group discussed the function of guiding/orienting frameworks in task design too, like the role of Realistic Mathematics Education theory. The group also discussed the emerging role of CT (Wing, 2006) as a (possible) mediator context to create mathematical meanings of linear algebra topics (and also in other STEM fields), like exploring the system of linear equations and Gram-Schmidt diagonalisation etc.

A third point was about aspects and nature of *formalism* and *representations* in linear algebra. The role of representations, shifting between them (i.e. semiotic registers, in the sense of Duval) and how this could inform researchers to design tasks were discussed. The instructor's role was at the heart of discussion on some occasions, with a particular role to make a shared discussion at the end of each teaching episode.

Future directions for next INDRUM conferences

The following points have emerged after our thematic group discussions regarding linear and abstract algebra. The first point was about the role and purpose of theoretical/conceptual frameworks in our work. The group underlined that some frameworks provide orientation (for example in task design) and a better understanding of the observed phenomenon, but such frameworks come with some constraints. The second point was about the integration with computer science (e.g., CT, computer graphics etc.), and (possibility of) digital assessment of linear and abstract algebra in particular. The third point was the contribution of linear and abstract algebra to the professional development pre-service and in-service mathematics teachers.

TRENDS AND PERSPECTIVES IN LOGIC, REASONING AND PROOF

In this section, we first present the main issues that emerge from the papers presentations; then we present the main elements that emerged from the discussion, namely the new trends and the perspective.

The main issues in the papers and posters presented

The main issues in the papers presented in this thematic were: Multi proofs analysis as a means to foster conceptualisation (Viviane Durand-Guerrier); Comparison of various proof assistants at the interface between mathematics and computer sciences (Evmorfia Iro Bartzia, Antoine Meyer & Julien Narboux); the support of CAS (Kinga Szücs); the use of cloze test and multichoice questionnaire (My Hahn); students reasoning on visual words problems (Francesco Beccuti); the generic power of proofs in number theory (Véronique Battie); refutation beyond counter-examples (Alon Pinto and Jason Cooper); linguistic issues (Dimitri Lipper, Thomas Karavi & Angeliki Mali). During the discussion sessions, two main issues have emerged: mathematical and epistemological needs and overview of analysis criteria.

Mathematical and epistemological needs

There is a necessity to go beyond the illusion of transparency of mathematical knowledge (Artigue, 1991) for strengthening the a priori analysis in a didactic perspective. The question is “how to do this?”. Relying on the research experience of some of the participants, we list the following practices: write down our own proofs, individually and then sharing with our research team and beyond, including researchers and practitioners who specialise in the mathematical domain at stake; making a review of existing proofs in the literature as well in mathematics education as in domain-specific mathematics; analysing historical proofs; considering philosophical perspective; analysing experts’ practises; considering implicit and explicit norms among various communities. The discussion showed that the introduction of proof assistants and computer scientist methods raised new epistemological questions or calls for revisiting more classical ones. This appears as a challenge for research on proof and proving in mathematics education (Hanna, De Villiers & Reid, 2019).

Overview of proof analysis criteria

In the literature and in the participants' research experience and practice, there are various types of analysis criteria that are used in research on proof and proving, both for designing, and for a priori and a posteriori analysis, depending on the goal and the research questions. Nevertheless, we have tried to provide a non-exhaustive list as the first contribution to a collective state-of-the-art that the group considered worthwhile.

When the criteria were explicitly discussed in a paper presented in the TWG, we mention the name of the author(s) in brackets.

The first group of criteria refer to language, level of formalisation and semiotic register. A second group concerns norms and expectations, depending on the audience. A third group addresses the epistemological and pragmatic issue of the type of proof expected or provided, formal proof, operational proof, experimental proof (Dimitri Lipper, Thomas Karavi & Angeliki Mali) and Proof schemata: procedural versus conceptual, meaning versus ritual etc.

The fourth group of criteria concerns the proof structure analysis: organising and operative dimensions (Veronique Battie), data, hypothesis and introduction of objects (Viviane Durand-Guerrier), direct or indirect proof (Alon Pinto & Jason Cooper, visualisation (Francisco Beccuti). The fifth group of criteria refers to the precise analysis of a particular proof: steps in the proving process, logical validity, modes of inference, clarity, modularities, encapsulation, etc.

Perspectives for next INDRUM conferences

The first one is the relevance of revisiting epistemological questions in light of the increasing use of proof assistants in both research in mathematics and in mathematics education. The second one aims to deepen the logical dimension of analysis for the teaching and learning of proof and proving, which have already been shown to be relevant, but remain underrepresented in many research on proof and proving. The third one concerns the necessity of going on designing and implementing activities aiming at improving proof and proving skills at university, taking into consideration the specificity of the condition and constraints in university mathematics education. Another issue consists in exploring more systematically the possible contribution of proof and proving to address university mathematics students' difficulties, depending on the mathematical domain.

Finally, we consider that introducing specific work on proof and proving in University teacher training would be valuable, to initiate a change in the way proof is taught in general at university: moving from proof made in front of students, to students' proof elaboration and analysis.

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Generic power of number theory proofs

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To design generic proving multiple proof tasks, we use an epistemological tool in which the idea of generic power plays a crucial role. Thanks to a number theory case, we show how it works and then we open the discussion in a didactical way focusing on the Secondary-Tertiary transition.

Keywords: Teaching and learning of logic, reasoning and proof. Teaching and learning of number theory and discrete mathematics. Transition to, across and from university mathematics. Epistemological studies of mathematical topics.

INTRODUCTION

In our number theory didactical researches and University teaching practice, we develop activities we call *generic proving multiple proof tasks* (generic proving MPT) in reference both to (Leron & Zaslavsky, 2013) and (Dreyfus & al., 2012). We specifically work on tasks that contain an explicit requirement for proving a statement by using given proofs and where generic proofs play a crucial role. In this presentation, we study the design of such activities focusing on the main following research issue : how to compare several proofs of a mathematical result and how to choice some of them to design generic proving MPT ?

In the next section, we present our epistemological theoretical framework including the idea of generic power. Then, thanks to a number theory case study, we illustrate how it is helpful to analyze proofs in a generic comparative perspective. Before concluding, we discuss in a didactical way focusing on generic proving MPT at the Secondary-Tertiary transition (Gueudet & Thomas, 2020).

THEORETICAL FRAMEWORK

After summarizing Steiner's model of mathematical explanation, we present our epistemological tool to analyze number theory proofs and then introduce the idea of generic power by revisiting this model.

Steiner's account of explanation in mathematics

Steiner (1978) has presented a model of mathematical explanation that draws upon the distinction between explanatory and non-explanatory proofs. As Molinini (2012) sumps up :

> Steiner offers two criteria for considering a proof being explanatory :
>
> - Dependence on a characterizing property of an entity mentioned in the theorem (dependence criterion)

- Possibility to deform the proof by “substituting the characterizing property of a related entity” and getting “a related theorem” (generalizability criterion)

The *characterizing property* is defined by Steiner (1978) as it follows :

> My proposal is that an explanatory proof makes reference to a *characterizing property* of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property.

The generalizable criterion is called *generalizable proof* by Steiner (1978). This criterion has been introduced in Didactics of mathematics by D.Tall (1979) under the name *generic proof*.

Organising and operative dimensions and generic power

In our researches in didactic of number theory, we distinguish in proving (process) and proof (product of proving process) two complementary, and closely intertwined epistemological dimensions (Battie, 2009). The organizing dimension concerns the type of proof that is the “structural backbone” levels involved in proving or in a proof. For example, besides usual figures of mathematical reasoning, especially *reductio ad absurdum,* we identify induction (and other forms of exploitation of the well-ordering $\leq$ of the natural numbers), reduction to the study of finite number of cases, and factorial ring’s method. The operative dimension relates to those treatments operated on objects and developed for the implementation of choices involved in the organizing dimension. For instance, we identify forms of representation chosen for the objects, the use of theorems, algebraic transformations and all treatments related to the articulation between divisibility order (the ring Z) and standard order $\leq$ (the well-ordered set N) [1].

In a generic proving perspective, we make the hypothesis that closer we are in a proof to the characterizing property (Steiner, 1978), the more it will be no problematic to generalize from this proof. We propose to define the *generic power of a proof* in terms of accessibility (distance) to the characterizing property (Steiner, 1978). And an analyze in terms of organizing and operative dimensions permits to evaluate this accessibility. More precisely, by a process of proving admitted results embedded in operative dimension of the proof (process of “de-encapsulation” [2]) we are able to evaluate the complexity in terms of organizing and operative dimensions and to conclude: the less complex the resulting proof of this de-encapsulation is, the more we could have a high generic power in the initial proof.

We highlight that the generic power is introduced in a didactic perspective focused on proving and proof, and not in a philosophical perspective focused on mathematical explanation as proposed in Steiner’s model. Moreover, as discussed briefly in the next section, it seems to us that there is not evident link between generic power and explanatory one.

A NUMBER THEORY CASE STUDY

In this section, we propose five $\sqrt{2}$ irrationality proofs named proof 1 to proof 4 and proof 3bis and two others from Steiner (1978) respectively called by him *Pythagorean proof* and *proof invoking the Fundamental Theorem of Arithmetic (FTA proof)*. In a generic proving perspective, our analysis in terms of organizing and operative dimensions and generic power focus on how each proof can be used to prove the irrationality of $\sqrt{3}$ and to go towards the general result "Let *n* be a natural number, $\sqrt{n}$ is rational if and only if *n* is a square".

All proofs presented have the same main organizing dimension, a *reductio ad absurdum* reasoning, and the same first operative treatment, squaring to access to specific tools of number theory. From this point, proofs differ in terms of organizing and operative dimensions.

> **Proof 1**. By *reductio ad absurdum* suppose that $\sqrt{2}$ is rational, there are non-zero natural numbers a and b that $\sqrt{2} = \frac{a}{b}$. Without loss of generality we may assume that a and b have no factor in common. Now $2b^2 = a^2$ so a^2 is an even number. By a contrapositive reasoning we prove that a is an even number too: if not, we can write $a = 2k + 1$ (for some integer k) and then a^2 is also even because $a^2 = 4(k^2 + k) + 1$. So we have $a = 2a'$(for some integer a') and then $2b^2 = 4(a')^2$ becoming $b^2 = 2(a^{'})^2$. With the same reasoning, b is also an even number, contradicting our assumption. In conclusion $\sqrt{2}$ is not rational.

> **Pythagorean proof (Steiner, 1978).** Consider the Pythagorean proof that the square root of 2 is not rational: if $a^2 = 2b^2$, with $\frac{a}{b}$ reduced to lowest terms, then a^2 and thus *a* itself have to be even; thus a^2must be a multiple of 4, and b^2 - and thus b - multiples of 2. Since therefore $a^2 = 2b^2$ implies that both *a* and *b* must be even, contradicting our (allowable) stipulation that $\frac{a}{b}$ be reduced to lowest terms, it can be true, q.e.d. The key point here is the proposition that if a^2 is even so is *a*. This can be verified by squaring an arbitrary odd number $2q + 1$ showing that the result must be odd.

> **Proof 2**. By *reductio ad absurdum* suppose that $\sqrt{2}$ is rational, there are non-zero natural numbers a and b that $\sqrt{2} = \frac{a}{b}$. Now $2b^2 = a^2$ so a^2 is an even number. By a contrapositive reasoning we prove that a is an even number too: if not, we can write $a = 2k + 1$ (for some integer k) and then a^2 is also even because $a^2 = 4(k^2 + k) + 1$. So we have $a = 2a'$ (for some integer a') and then $2b^2 = 4(a')^2$ becoming $b^2 = 2(a^{'})^2$. With the same reasoning, b is also an even number and we can write $b = 2b'$ (for some integer b'). Therefore, from (a, b) we have (a', b') such as $\sqrt{2} = \frac{a'}{b'}$, $a^{'} < a$ and $b^{'} < b$. Then we have an infinite strictly decreasing sequence of natural numbers, contradiction. In conclusion $\sqrt{2}$ is not rational.

In operative dimension, these proofs have the same step proving that *a* and *b* are even numbers. The result "Let be a an integer, if a^2 is divisible by 2 then a is also divisible by 2" takes place and a contrapositive reasoning appears; this organizing dimension complexification would disappear if we admit the result. The operative dimension differs in how the object-fraction $\frac{a}{b}$ is specified : a and b have no factor in common (irreducible fraction) only in proof 1. This difference is connected to the main organizing dimension difference: in proof 2 the *reductio ad absurdum* is specified (Fermat's descent), in proof 1 and Pythagorean proof it is not. In terms of organizing and operative dimensions, proof 1 and Pythagorean proof are equivalent (and we note {proof 1, Pythagorean proof} hereinafter). To prove the irrationality of $\sqrt{3}$, we have to adapt the common operative step and the result becomes "Let be a an integer, if a^2 is divisible by 3 then a is also divisible by 3". The organizing dimension becomes more complex because a proof by separating cases appears in the contrapositive reasoning (we have two cases and not only one for the negation of "to be divisible by 3"). Finally, {proof 1, Pythagorean proof} and proof 2 are problematic to generalize as Steiner confirms:

> Consider the Pythagorean proof [proof 1] that the square root of 2 is not rational […] The key point here is the proposition that if a^2 is even so is a. […] Indeed for each prime p, one can separately verify that if p divides a^2 it must divide a also, though the proofs become more and more complex […].

It should be noted that with the result "for all p prime, if p divides the product ab of two integers then p divides a or b" (particular case of Gauss' theorem), the organizing dimension of these three proofs is not more complex (no contrapositive, no separating cases) and it is suitable to generalize.

> **Proof 3.** By *reductio ad absurdum* suppose that $\sqrt{2}$ is rational, there are non-zero natural numbers a and b that $\sqrt{2} = \frac{a}{b}$. Now $2b^2 = a^2$. Without loss of generality we may assume that a and b have no factor in common. Then a^2 and b^2 have no factor in common too (result admitted). And because $2 = \frac{a^2}{b^2}$ we have $b = 1$ so $a^2 = 2$. Contradiction because a is an integer. In conclusion $\sqrt{2}$ is not rational.

The main organizing dimension of both {proof 1, Pythagorean proof} and proof 3 is a *reductio ad absurdum* by specifying the object-fraction (irreducible fraction). However, the nature of the contradiction is not the same in these proofs and this difference is essential: proof 3 helps to find the necessary and sufficient condition "be a square" involved in the general result. Moreover, contrary to {proof 1, Pythagorean proof} and proof 2, with proof 3 no adaptation is needed to prove the irrationality of $\sqrt{3}$ and we can easily write a proof of the general result. We highlight that this ease to adapt proof 3 depends on the specificity of the operative dimension of proof 3. First, a crucial result is explicitly admitted : we have an encapsulation" in the operative dimension. Moreover, the unicity of the irreducible fraction is implicitly involved to deduce $b = 1$ from $n = \frac{a^2}{b^2}$; this unicity is equivalent from a logical point of view to Gauss' theorem.

> **Proof 3bis.** By *reductio ad absurdum* suppose that $\sqrt{2}$ is rational, there are non-zero natural numbers a and b that $\sqrt{2} = \frac{a}{b}$. Now $2b^2 = a^2$. Without loss of generality we may assume that a and b have no factor in common. Then a^2 and b^2 have no factor in common too (result admitted). And, from $2b^2 = a^2$, we deduce on the one hand a^2 divides $2b^2$ so 2 thanks to Gauss' theorem and finally $a^2 \leq 2$, and on the other hand $a^2 \geq 2$. So $a^2 = 2$. Contradiction because a is an integer. In conclusion $\sqrt{2}$ is not rational.

In terms of organizing and operative dimensions, proof 3 and proof 3bis have essential similarities: the main organizing dimension is a *reductio ad absurdum* by specifying the object-fraction, both proofs help to find the necessary and sufficient condition "be a square" involved in the general result, no adaptation is needed to prove the irrationality of $\sqrt{3}$ and we can easily write a proof of the general result from both, this ease depends on the specificity of their operative dimension (the same crucial result is admitted). The specificity of the operative dimension of proof 3bis is the joint utilisation of the (explicit) use of Gauss' theorem and the articulation between divisibility order and standard order ≤.

> **Proof 4.** By *reductio ad absurdum* suppose that $\sqrt{2}$ is rational, there are non-zero natural numbers a and b that $\sqrt{2} = \frac{a}{b}$. Now $2b^2 = a^2$. Let α be the exponent of 2 in the prime decomposition of a and respectively β for b. Then we have $1 + 2\beta = 2\alpha$. Contradiction (a non-zero odd number cannot be at the same time an even number). In conclusion $\sqrt{2}$ is not rational.

> **FTA proof (Steiner, 1978).** By using the Fundamental Theorem of Arithmetic - that each number has a unique prime power expansion - we can argue for the irrationality of the square root of two swiftly and decisively. For in the prime power expansion of a^2 the prime 2 will necessarily appear with an even exponent (double exponent it has in the expansion of *a*), while in $2b^2$ its exponent must needs be odd. So a^2 never equals $2b^2$ q.e.d.

In proof 4 and FTA proof, contrary to proof 2, the *reductio ad absurdum* is not specified. Indeed, contrary to proof 1, proofs 3 and 3bis, the object-fraction is not specified too. No result is admitted (except the FTA theorem) and at the same time there is no complexification of the organizing dimension. The crucial element of the operative dimension is the form of representation chosen for integers (prime decomposition). In terms of organizing and operative dimensions, proof 4 and FTA proof are equivalent (and we note {proof 4, FTA proof} hereinafter). As well as proof 3 and proof 3bis, {proof 4, FTA proof} is suitable to prove the irrationality of $\sqrt{3}$ and to give an idea of the general result and how to prove it as Steiner confirms :

> But by using the Fundamental Theorem of Arithmetic [proof 4] […] we can argue for the irrationality of the square root of two swiftly and decisively. […] Generally, the same proof shows that a^2 can never equals nb^2, unless *n* is a perfect square (so that all exponents in its prime power expansion will be even).

Nerveless, we remind that in both proofs 3 and 3bis, there is an encapsulation in the operative dimension (result admitted).

In echo to the previous section, the characterizing property involved in this case study is pointed out by Steiner (1978) when he refers to the crucial choice made in the operative dimension of {proof 4, FTA proof}:

> Our proof that $a^2 = 2b^2$, which uses the prime power expansions of *a* and *b* (and 2) [proof 4], conforms to our description, since the prime power expansion of a number is a *characterizing property*. It's easy to see what happens, moreover, when 2 becomes 4 or any other square; the prime power expansion of 4, unlike that of 2, contains 2 raised to an even power, allowing $a^2 = 2b^2$. In the same way we get a general theorem: the square root of n is either an integer or irrational.

The characterizing property is explicitly and directly used in the operative dimension of {proof 4, FTA proof} and identified in the form of representation chosen for integers *ie* the unique prime power expansion. That is why this proof is so suitable to prove both the irrationality of $\sqrt{3}$ and the general result. In proofs 3 and 3bis, we can identify this characterizing property (through the logical connection with Gauss' theorem) but it is not explicit, especially in proof 3. With proof 1 and proof 2, we lose access with it and the potential to generalize is very low. It all depends on how the result "Let a be an integer, if a^2 is divisible by 2 then a is also divisible by 2" is proven in the proof and it's not "because of linguistic considerations" as D.Tall (1979) concludes.

Finally, the analysis of this case study in terms of organizing and operative dimensions offers the result of the de-encapsulation process : following this increasingly order - {proof 1, Pythagorean proof} and proof 2 ex aequo, proof 3 and proof 3bis ex aequo, {proof 4, FTA proof} - we go through proofs with a growing generic power ; the less complex the resulting proof of this de-encapsulation process is, the more the generic power of the initial proof is high.

To conclude this section, we note that in the study of number theory proofs of $\sqrt{2}$ irrationality, especially about {proof 4, FTA proof} with the highest generic power, there is no explanatory power of the irrationality of $\sqrt{2}$. And, according to us, the reason is clearly exposed by Hardy and Wright (1945):

> […] many problems of irrationality which may be regarded as part of arithmetic. Theorems concerning rationals may be restated as theorems about integers; […] Thus (P) '$\sqrt{2}$ is irrational' means (Q) '$c^2 = 2b^2$ is insoluble in integers', and then appears as a properly arithmetical theorem. We may ask 'is $\sqrt{2}$ irrational?' Without trespassing beyond the proper bounds of arithmetic, and need not ask 'what is the meaning of $\sqrt{2}$?' We do not require any interpretation of the isolated symbol $\sqrt{2}$, since the meaning of (P) is defined as a whole and as being the same as that of (Q).

We suggest that the explanatory power (Steiner, 1978) of proofs invoking the fundamental theorem of Arithmetic is not related to the result '$\sqrt{2}$ is irrational' but to '$c^2 = 2b^2$ is insoluble in integers' one.

DIDACTICAL DISCUSSION

Proofs are normally presented in a step-by-step linear fashion, which is well suited for checking the proof's validity but we think that is not as good for communicating its mains ideas. In this way, specifically for number theory, an analysis in terms of organizing and operative dimensions permits to identify skills of proofs, especially in terms of generic power when appropriate. This identification is a crucial step to think about generic proving MPT at the Secondary-Tertiary transition, more specifically to elaborate tasks trying to help pupils and students to get more control on the two dimensions and their interactions.

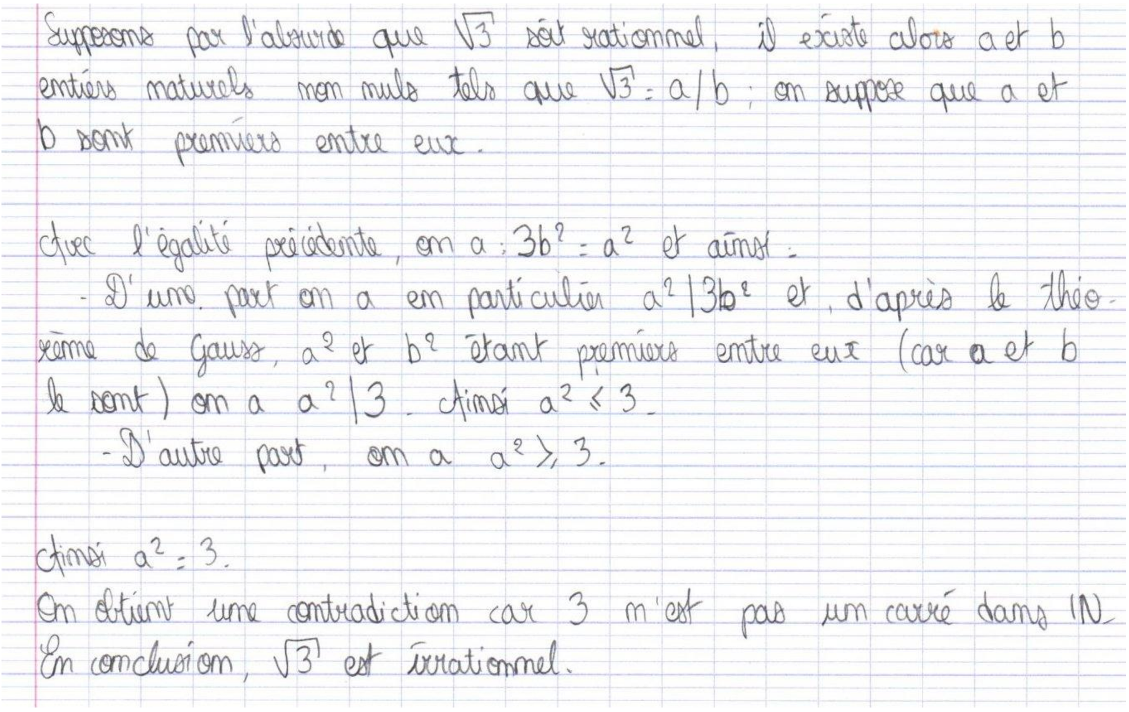
Providing proofs in a generic proving MPT may help pupils/students to go on in their mathematical research and motivate an active reading of proofs. However, beyond the elaboration of MPTs, the role of the teacher is challenging because there is some risk that pupils/students produce correct proofs without any deep understanding, especially any awareness of the generic power involved. This citation of Rowland (2002) about his researches with undergraduates is particularly relevant:

> I believe that the accounts given here of my work with undergraduates offer grounds for considerable optimism regarding the possibility of students "seeing" the generality we intend them to see in arguments based on particular cases. At the same time, it warns us against naïve complacency: we cannot be sure what they will see, and they may see considerably less than we might hope. (Rowland, 2002)

Back to the previous number theory case, we illustrate the risk mentioned above with a proof produced by a group of undergraduates ; this group has been asked to write down a $\sqrt{3}$ irrationality proof being inspired by proof 1, proof 2, proof 3bis and proof 4 they have been given. This group decided to chose proof 3bis ; the French version of proof 3bis is given below:

> **Proof 3bis (French version given to undergraduates groups).** Supposons par l'absurde que $\sqrt{2}$ soit rationnel, il existe a et b entiers naturels non nuls tels que $\sqrt{2} = \frac{a}{b}$; on suppose que a et b sont premiers entre eux.
>
> Avec l'égalité précédente on a $2b^2 = a^2$ et ainsi :
>
> - D'une part on a en particulier a^2 divise $2b^2$, et d'après le théorème de Gauss, a^2 et b^2 étant premiers entre eux (car a et b le sont) on a a^2 divise 2. Ainsi $a^2 \leq 2$.
> - D'autre part on a $a^2 \geq 2$.
>
> Ainsi $a^2 = 2$.
>
> On obtient une contradiction car 2 n'est pas un carré dans $\mathbb{N}$. En conclusion, $\sqrt{2}$ est irrationnel.

And the proof produced by the undergraduates group is the following :



Supposons par l'absurde que $\sqrt{3}$ soit rationnel, il existe alors a et b entiers naturels non nuls tels que $\sqrt{3} = a/b$; on suppose que a et b sont premiers entre eux.

Avec l'égalité précédente, on a : $3b^2 = a^2$ et ainsi :

- D'une part on a en particulier $a^2 | 3b^2$ et, d'après le théorème de Gauss, a^2 et b^2 étant premiers entre eux (car a et b le sont) on a $a^2 | 3$. Ainsi $a^2 \leq 3$.
- D'autre part, on a $a^2 \geq 3$.

Ainsi $a^2 = 3$.
On obtient une contradiction car 3 n'est pas un carré dans $\mathbb{N}$.
En conclusion, $\sqrt{3}$ est irrationnel.

$\sqrt{3}$ irrationality proof of undergraduates group.

It is a success. But it could be an illusion as mentioned previously in terms of an authentic understanding of proofs. Indeed, we can read that undergraduates proof is word for word the adapted proof 3bis. It is well-adapted by students for the $\sqrt{3}$ case study but, as detailed in the previous section, proof 3bis has a high generic power and adaptations to do are minor. The same phenomena has been observed with Grade 12 pupils (Battie, 2015).

Finally, to fully develop the didactic potential of generic proving MPT, the teacher may develop pedagogical and didactical methods to evaluate the authenticity of students understanding. For sure, written productions are insufficient and to stimulate discussions, maybe debates, in classroom could be a relevant way to complete written tasks proposed to pupils/students.

CONCLUSION

To design a generic proving MPT, we need to know how to compare proofs and to select a set of proofs with an authentic diversity in terms of generic power. Our epistemological tool presented in this paper appears as a solution for the number theory domain. “Infrequently used” (Hanna & al., 2012), generic proofs should have high didactic potential and, according to Rowland (2002), this potential is “virtually unrecognized and unexploited in the teaching of number theory” as we observe in our

University. This is all the more regrettable since pupils (Battie, 2015) and students (e.g. (Tall & al., 2012) with confirmation from our University teaching practice) seem to spontaneously prefer generic proofs.

NOTES

1. Among the numerous didactic researches on proof and proving (Hanna, 2020), we can put into perspective our epistemological point of view with the "structuring mathematical proofs" of Leron (1983). As we showed (Battie, 2007), an analogy is a priori possible but, as far as we know, Leron's view does not permit access that gives our analysis in terms of organizing and operative dimensions, namely the different nature of mathematical work according to whether a dimension or another and, so essential, interactions taking place between these two dimensions especially in proving. The same concern seems to appear when Selden (2012) see their formal-rhetorical and problem-centred parts of proofs as somewhat like our organizing and operative dimensions.

2. The term "encapsulation" is used due to the analogy with computer network: encapsulation is a method of designing modular communication protocols in which logically separate functions in the network are abstracted from their underlying structures by inclusion or information hiding within higher level objects. In the process of de-encapsulation, proving admitted results embedded in the operative dimension of a proof is like opening « black boxes » of this proof.

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Proof assistants for undergraduate mathematics and computer science education: elements of a priori analysis

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This paper presents an a-priori analysis of the use of five different interactive proof assistants for education based on the resolution of a typical undergraduate exercise on abstract functions. It proposes to analyse these tools according to three main categories of aspects: (1) language and interaction mode, (2) automation and user assistance, (3) proof structure and visualisation. We argue that this analysis may help formulate and clarify further research questions on the possible impact of such tools on the development of reasoning and proving skills.

Keywords: Teaching and learning of logic, reasoning and proof, Digital and other resources in university mathematics education, Transition to, across and from university mathematics, Novel approaches to teaching, Computer assisted theorem proving.

INTRODUCTION

Investigating the use of technology for the teaching and learning of proof and proving is an active topic in both communities of education research and computer-assisted theorem proving. The topic of proof has been garnering interest in the mathematics education research community for years. The 19th ICMI Study focused on six major themes relative to the teaching and learning of proof and proving. It led to the publication of a study volume (Hanna & de Villiers, 2012) with contributions from specialists of the field providing insight on these themes. The interactive theorem proving community has also shown interest in the use of proof assistants for teaching since at least 2007 and the workshop on Proof Assistants and Types in Education (Geuvers & Courtieu, 2007), followed since 2011 by the ThEDU workshop series.

Recently Hanna and de Villiers, together with Reid, coordinated another volume specifically focusing on the use of software tools for computer-assisted proof in education (Hanna et al., 2019), featuring contributions from researchers in both communities. In the introductory chapter they state that the book's goal is "to begin a dialogue between mathematics educators and researchers actively developing automatic theorem provers and related tools". The chapter concludes with the statement that "we know almost nothing of [proof assistants'] potential contribution to other roles of proof, such as explanation, communication, discovery, and systematization, or how they now may become more relevant as pedagogical motivation for the learning of proof in the classroom", implying that much research is still required in order to gain further insight on the convergence of both fields.

Proof assistants (henceforth PAs) are quite broadly used to teach logic, proof of computer programs and, increasingly, classical mathematical topics by teachers who are not researchers in the field of interactive theorem proving. In this paper we focus on the potential use of these tools for teaching proof and proving itself at the transition between high school and university. In this work we consider proof assistants as possible teaching tools and not as professional tools. The tool's underlying proof theory and the structure and size of its mathematical libraries are therefore not directly relevant. We will instead focus on the way each tool enables the development of skills related to proof and proving.

The questions which motivate this work can be phrased as follows:

- What are the possible effects of using PAs on students' learning of proof?
- What characteristics of PAs are likely to strengthen or hinder these effects?

In order to start addressing these questions we chose to analyse the resolution of a single typical exercise about functions using a selection of five different PAs (Coq, Isabelle, Edukera, dEAduction and Lurch, introduced briefly below). We solved this exercise using each PA in turn, with one experimenter building the proof interactively and two observers. Based on initial observations we designed an analysis grid to capture some of the tools' characteristics likely to have an impact on teaching and learning. We then revisited each resolution of the exercise and analysed it with respect to this grid. Our aim is to help distinguish aspects of each PA which may facilitate or hinder student's learning of the various skills involved in proof and proving (Selden, 2012) as a preliminary step to future research.

We first briefly introduce the concept of PA. We then present our case study before describing our analysis grid. We finish by raising additional questions regarding the possible impacts of each PA on the teaching and learning of proof and proving.

PROOF ASSISTANTS IN EDUCATION

The term *proof assistant*, or *interactive theorem prover*, refers to a software tool allowing a user to interactively construct a formal mathematical proof. Some systems are designed to work in a specific domain such as geometry, logic or the analysis of computer programs, while others are general-purpose. Additionally, proof assistants used in the classroom can be sorted roughly in two categories: some are built by the community of educators and others are designed by specialists of interactive theorem proving for research or other professional purposes.

The input languages of PAs are usually classified into two categories: *imperative* and *declarative* languages. In an imperative language the user orders changes to be performed on the *proof state* (the current set of declared variables and constants, assumed hypotheses, and goals) using a predefined set of orders (also called *tactics*). Each tactic consists in one or several deduction rules to be applied, or other manipulations of the proof state. Most tactics do not contain explicit mathematical statements. In a declarative language one provides assertions along with their justification, in a way similar to a natural-language proof. The statements are therefore written explicitly, using a syntax resembling mathematical language.

In simple cases the validity of each proof step is ensured by matching some of the available statements with the premise of a given deduction rule, substituting variables accordingly in the rule's conclusion, and verifying that each involved expression is well-typed. This may be complemented with other automation techniques, for instance to help searching for applicable rules, to perform automatic computations in restricted domains (arithmetic, algebra…), to assist in syntactic manipulations, etc.

In the terminology of Duval & Egret (1993), most PAs clearly distinguish the theoretical status (hypothesis, axiom, definition, theorem, conjecture) and operational status (premise, conclusion, external rule, goal) of each statement. This is done using visual hints, the syntax of the PA's language, or by separating statements between disjoint areas of the user interface. This may be an important feature in an educational context, since this distinction is known to be a source of difficulty for students.

We will revisit these characteristics below, when we detail the aspects which constitute our analysis grid, and illustrate them on our selection of proof assistants.

CASE STUDY: ANALYSING AN EXERCISE IN FIVE PROOF ASSISTANTS

The exercise we chose for this work is a typical elementary proof about sets, relations and functions commonly found in introductory courses about reasoning and proof, in both Mathematics and Computer Science curricula, and available or formalisable in all studied PA. The exercise text reads as follows, with minor variants:

1. Given $f: A \to B$ and $C \subseteq A$, show that $C \subseteq f^{-1}(f(C))$.
2. Given $f: A \to B$ and $C \subseteq A$, show that if f is injective then $f^{-1}(f(C)) \subseteq C$.

We chose this exercise because it involves few and fundamental mathematical concepts and little calculation. The required proofs are of a manageable size, yet not trivial for students. They involve the concepts of set, function, subset relation, direct and inverse image and injectivity. The definitions of these concepts require universal and existential quantifiers and implication, which students tackling the proof are required to be able to manipulate.

We now briefly describe the five PAs we chose to analyse in this work. Coq and Isabelle are professional systems which are also used for teaching. Lurch, Edukera and dEAduction were designed specifically for teaching.

Coq and *Isabelle* are free and open-source proof assistants. Coq was created in the 1980s in French academia (Coq Team, 2022). Isabelle was developed at University of Cambridge and Technische Universität München (Nipkow et al., 2002). Both have been used successfully to prove mathematical theorems such as the Feith-Thompson theorem, the four-colour theorem or the Kepler conjecture, and to prove the correctness of large-scale computer programs. They have also been experimented for several years as teaching tools in graduate or undergraduate curricula on various topics. A difference between the two lies in the kind of user interaction and language they offer. In this work, we only use Coq in imperative mode, and Isabelle in declarative mode using its Isar language.

Lurch is a free and open-source word processor built on OpenMath, that can check the steps of a mathematical proof (Carter & Monks, 2013). Lurch is designed for student use and was experimented for teaching in 2008 and 2013. To our knowledge, it is no longer maintained, but was kept under consideration due to its originality with respect to other PA. Its user interface is inspired by that of a word processor, proof checking being presented similarly to spell-checking: one can write text freely, then mark some mathematical expressions as meaningful and check their validity.

Edukera is a closed-source web-based graphical proof assistant loosely based on Coq (Rognier & Duhamel, 2016). It is no longer maintained but was kept in our study for the same reasons as Lurch. It was designed to help teach proof and proving as well as classical high school mathematics content including algebra and basic analysis. Its originality is to combine a point-and-click interface with a presentation of the whole proof mimicking human-written text.

DEAduction [2] is a recent free and open-source graphical interface to the LEAN proof assistant created by Frédéric Le Roux. It was specifically designed for teaching, and is under active development. It provides a purely point-and-click user interface.

By lack of space we cannot provide here a full account of the proofs of the exercise in each PA, but we will give additional details during the presentation. Interested readers may download proof files for this exercise in each studied PA online [1].

ASPECTS OF PROOF AND PROVING IN PROOF ASSISTANTS

In this section we describe the three main categories of aspects of PA we retained in our analysis, each including several criteria which are summarised in Table 1. Other factors of practical importance are left out of this study, such as type of license, availability, ease of installation, integration with learning management systems, etc.

Language and interaction mode	type of user input, imperative or declarative style, object naming, possibility of writing ill-formed statements
Automation and user assistance	mathematical libraries, rule selection and application, scope management, rule chaining and automated computation, type of feedback
Proof structure and proof state visualisation	global or local viewpoint on proof, status of statements, possibility to create new definitions and lemmas

Table 1: Categories of aspects of proof assistants and related analysis criteria

Language and interaction mode

The first category we consider relates to the nature of interactions between user and proof assistant. We focus in particular on the tools' linguistic, semiotic and visual characteristics. This includes the syntax and semantics of the input language, if any, the textual, graphical or mixed output language displayed by the PA, and more generally any kind of visual hints which carry proof-related meaning.

Type of user input. Interactions between the user and a PA generally include both mouse-based and text-based modalities, to varying degrees. In dEAduction and Edukera most interaction is mouse-based (through menus, buttons, drag-and-drop), textual input being only rarely required (for instance when introducing an existential witness). In Coq and Isabelle the user respectively types in tactics or proof text, both obeying a strict syntax. In Lurch the user experience is similar to that of "literate programming" where code is mixed with explanatory text. By default natural-language text is ignored and carries no semantics. "Meaningful expressions", whose syntax resembles that of standard mathematics, are then combined with one another to form deduction steps, which are then formally checked by the software.

Imperative or declarative style. Coq is an example of an imperative language. The user types in *tactics* which perform transformations of the current proof state. In Question 1 of the analysed exercise, to prove that $f(x) \in f(C)$ the user runs the command `unfold im`, which instructs the prover to unfold the definition of an element being in $f(C)$, yielding as new goal $\exists x_0 (x_0 \in C \wedge f(x) = f(x_0))$. In Isabelle the language is declarative: at every step the user has to declare what will be proved, i.e. she has to state how the goal will be transformed after she applies the next proof step. Assuming the hypothesis $x \in C$, referred to by label `Hx`, is available in the current scope, the user may type: `have "f x f ` C" using Hx by (rule imageI)`. This line attempts to prove $f(x) \in f(C)$ using the hypothesis $x \in C$ and the definition of the image of a set (`imageI`). Deduction steps in Lurch have a similar structure. In Edukera the user simply clicks the "def" button while the goal is selected and the definition of the image of a set is unfolded automatically. In dEAduction the user has to select the appropriate definition from a predefined list.

Object naming and referencing. In Coq and Isabelle the user can choose the names of hypotheses and objects when they are introduced. In Edukera, each line of the proof is automatically numbered, and is referenced whenever it is used as a premise in a deduction step. In dEAduction variables and hypotheses are automatically assigned fresh names (i.e. not bound in the current context). In Lurch one can choose custom labels for statements. Even though each tool offers different presentation choices and interaction styles, being able to refer to objects by name is essential to the structuration of a proof.

Automation and user assistance

Automation refers to all features facilitating the selection of a usable rule in a given context, the syntactic manipulation of statements (in particular regarding type checking, substitution and pattern matching), the chained application of rules, etc. Other features include the organised presentation of available rules and theorems, automatic scope management, and contextual hints or feedback. According to some PA designers' and teachers' testimonies, finding a good balance in the level of automation is a challenge, especially in an educational context where efficiently completing a proof may not be the main goal.

Mathematical libraries. Contrary to traditional proofs, most PAs provide libraries of definitions and theorems, and make their formal definitions easy to access. PAs may also provide additional assistance such as contextual search, automatic completion, online help, etc. Professional PAs like Coq and Isabelle provide thousands of proven mathematical facts. Edukera and dEAduction simply list predefined lemmas and definitions, sorted by topic, not all of which are available in every exercise.

Rule selection and application. One of the main actions when building a proof in a PA consists in performing a reasoning step by applying a theorem or a logical rule, or by substituting a symbol by its definition. Each PA provides a different level of assistance and automation for these tasks, mainly regarding the way a rule or statement is instantiated when it is used (i.e. its variables substituted by terms), or the way a given rule, theorem or definition is selected with respect to the current context. In Isabelle and Lurch, the user writes instantiated mathematical expressions, and explicitly invokes a rule by its name. The tool then checks that this instantiation is correct, and if so applies the rule. In Lurch, multiple rules may share the same name, in which case all matching rules are tried in order until one succeeds. In Coq, dEAduction and Edukera, commands to unfold a definition or apply a theorem are provided, either by invoking them by name or by selecting them from a list. In all three systems, pattern matching and substitution are performed automatically. For logical deduction rules, a varying degree of automation is offered. In some of the tools (Coq and Edukera in maths mode), generic commands are available to eliminate or introduce logical connectors and quantifiers. Only when ambiguity occurs is the user required to add input. In other tools, the user generally has to determine the outermost logical connective themself.

Scope management. According to teacher testimony and previous research on proof, keeping track of the scope of each variable or hypothesis is a source of difficulty for students, which sometimes leads to confusion between free and bound variables, or to circular arguments. In Coq and dEAduction, scope management is fully automatic and available variables and hypotheses are neatly gathered in corresponding areas of the interface. In Edukera, unproven statements are clearly distinguished from proved ones, scopes are visually materialised and can be selected when introducing new variables or hypotheses. Isabelle and Lurch also have syntactic or visual means to indicate scopes, but more work is left to the user to maintain them.

Rule chaining and automated computation. Some PAs offer possibilities for implicit or explicit "chaining" of rule applications. For instance, when applying a universally quantified theorem, Edukera offers to perform the introduction of the universal quantifier and introduction of implication in a single step. Coq also supports implicit chaining of rules: for example, a single invocation of the `apply` command to deduce $x=x'$ from the hypothesis $f(x)=f(x')$ using the injectivity of f successively unfolds the definition of injectivity, eliminates two universal quantifiers and one implication, and performs the associated pattern matching and substitution steps. Other tactics in Coq or Isabelle may perform further automatic transformations. Finally, in specific mathematical areas such as basic arithmetic or linear algebra, PAs may give access to fully automatic solvers, for instance when checking simple equations.

Type of feedback. Feedback varies from basic to very rich. In Coq and Isabelle little feedback is given, apart from error messages when a rule does not apply or when an expression is not well-typed. On the contrary, feedback in Lurch is very rich: there is a colour code to indicate the status of each statement (undischarged hypothesis, valid or invalid conclusion) and visual hints to highlight the scopes of hypotheses. Moreover, very complete feedback on rule application is provided, including a list of selected premises and an explicit substitution of variables.

Proof structure and proof state visualisation

This final category concerns the aspects of a PA related to how proofs are perceived and manipulated. There are two main design choices: in some PA, the whole proof text is visible at once, and users complete it by inserting new assertions. Work may be done progressively on several parts of the proof. In others, only the current goal and the current proof state is prominently displayed. Other aspects related to proof structure concern the users' possibility to decompose a long proof by writing down and separately proving intermediate definitions theorems which can then be reused.

Global vs local viewpoint on proof. In Coq or dEAduction, the user may visualise the sequence of invoked tactics and navigate through them to view the evolution of the proof state at each point. The proof as a whole is left implicit, it is never displayed entirely [3]. Moreover, the origin of each statement in the context (hypothesis of the theorem to be proven, previously proved fact, hypothesis in a proof by cases or by contradiction) is not displayed. In both tools, it is also natural to treat the goals in the order in which they are generated by the system. One may say the viewpoint on proof is local, with much information hidden. On the contrary, in Edukera or Lurch (or in a pen-and-paper proof), the proof state is implicit: it is composed of the list of open statements combined with the list of hypotheses which are assumed to hold in the scope of each open statement. Due to their declarative style and since proof texts in these two PAs rather closely imitates usual mathematical language, they offer a more global viewpoint on proofs without resorting to back-and-forth navigation through proof lines. Isar (Isabelle's language) combines both aspects by allowing both a complete, more or less human-readable proof text, and the ability to display the current proof state at each line of the proof.

Possibility to create new definitions and lemmas. DEAduction and Edukera do not allow the user to create new definitions or theorems, the user is on a "deductive island" imposed by the system. In Edukera, teachers can compose their own exercise sheets but they cannot create new exercises. Developing new theories is not possible for end users. Using Coq, Isabelle, or Lurch, the user is free to restructure her proof by introducing new lemmas or concepts.

Status of statements. As already stated, one may distinguish the *theoretical status* of a statement (axiom, lemma, hypothesis, conjecture, etc.) and its *operational status* (premise, conclusion, external statement) which may vary in the course of a proof: a statement may be the conclusion of a deduction step and the premise of another one. The status of statements is rather clear in all PAs (except Edukera where admitted lemmas/axioms, and proved lemmas are not distinguished). In DEAduction hypotheses of the exercises and other elements of the context are displayed in separate frames. Moreover, hypotheses used at least once as premises are greyed out. In Isabelle, local hypotheses introduced to prove universally quantified implications are syntactically distinguished. In Lurch, the validity of each step is displayed using a colour code. The operational status of statements is displayed using "bubbles".

POSSIBLE IMPACTS ON THE TEACHING AND LEARNING OF PROOF

As their name suggests, proof assistants relieve the user of some of the tasks usually associated with proving. While this may be desirable in a professional setting, it might become a hindrance when the goal is precisely to let students practice some of these tasks. Based on our analysis, we formulate a few hypotheses on the possible effects of the use of PAs in teaching regarding various possible teaching goals.

Possible effects on memorisation and formulation. When asking students to solve an exercise on functions, a possible prerequisite or desired learning outcome is that students intuitively understand relevant definitions (in our case those of set and function, set inclusion, direct and inverse image, and the notion of injectivity) and be able to state (and use) their formal definition. When using a PA where details of definitions and properties are always at hand, one may postulate that memorisation of formal statements is not required to "solve" the exercise. Rather, students may be required to read, understand and appropriately make use of them. However, it might be the case that being repeatedly presented with definitions and properties and putting them to use may actually help memorise them.

Possible effects on manipulation of formal statements. It has been remarked that performing substitution is one of the many difficulties of the proving activity. As we observed, the five PAs we studied differ in the way they automate the manipulation of formal statements. In three cases (dEAduction, Edukera and Coq), it may be possible to achieve a complete proof without actually having to write a single mathematical statement. Coq and Edukera automatically identify the outermost operator in a mathematical term. While this does not completely exempt the user from thinking about statements and anticipating which rules may be used next, it is not up to the user to actually figure out which substitution makes a statement match a given pattern, or how to apply it to another statement in order to use a rule. This is not the case in Isabelle and Lurch, where the user explicitly writes down mathematical terms, and the system simply checks if they are correct. In all cases however, *a posteriori* control and validation is possible, for example by replaying a step in imperative PAs. This may provide another way to practise skills related to formula manipulation, by reading and control rather than by writing. A related possible effect is that PAs may forbid certain incorrect manipulations, produce correct but unexpected outcomes or provide additional feedback (Lurch in particular provides rich and explicit feedback on substitutions). These retroactions are of course unavailable in a pen-and-paper proof.

Possible effects on the perception of proof structure. In our experience, users of imperative-style proof assistants such as dEAduction, Edukera and Coq may feel as though they are "pushing symbols around until it works", possibly not understanding why the proof went through. This may be strengthened by the fact that these tools automatically manage scopes and contexts, including the identification of each statement's operational status. Even though replaying previous steps is possible, these tools may act as "blinders", allowing one to entirely focus on the current proof state, possibly "forgetting" about other parts of the proof. This "tunnel" effect may even be strengthened by the fact that these tools automate several aspects of proving, which makes a trial-and-error exploration strategy more viable than in declarative-style PAs such as Lurch and Isabelle. Edukera stands out as a special case in that the whole text of the proof remains visible throughout, even though user input is mostly imperative and syntactic manipulations largely automated.

As we can see, different design choices in each PA entail different actions on the part of the user. Certain concepts (for instance that of substitution) intervene in all cases but quite differently, and may require different levels of proficiency from the user. One may argue that freeing students from certain tasks (writing syntactically correct statements, keeping track of variables and hypotheses' status and scope, memorising definitions and theorems, recalling what remains to be proven) may enable them to concentrate on the deeper ideas involved in a proof, and may contribute in overcoming these difficulties outside of the PA by mere "habituation". Conversely, one may object that acquiring these skills is indeed one of the intended goals of these activities, and that it is therefore essential to have students practise them and not rely on a tool's facilities. For a discussion of the possible effects of using a PA on the acquisition of proving skills, see for example (Thoma & Iannone, 2021).

Figuring out the actual effect of each PA on learning would of course require further research. It would be interesting to try and analyse student's proficiency with various proof-related competencies when using different types of PAs. Do PAs have an effect on known syntactic and semantic difficulties that students typically encounter when working on proof? Do they favour the development of higher-level competencies such as writing a full and correct proof on paper, or summarising the main arguments of a proof verbally? How much do these effects depend on students' backgrounds?

NOTES

1. See https://github.com/jnarboux/PA_a_priori_analysis for screenshots and source files in all PAs.

2. Deaduction website, including source code: https://github.com/dEAduction/dEAduction

3. Readers unfamiliar with PAs may consult the following site for examples of proof scripts along with corresponding proof states: https://plv.csail.mit.edu/blog/alectryon.html#alectryon.

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A tale of four cities: reflections of master's students in mathematics on a visual word problem

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A group of master's students in mathematics was asked to reflect on a visual word problem. No one of the students identified the correct solution and defended instead an incorrect one. The results are explained by the difficulty of the problem itself (reflecting into the students' overall difficulty in imagining its solution) as well as by the students' tendency to overgeneralize. Interestingly, some of the students reach a contradictory statement (which they do not dismiss or acknowledge as such) as a consequence of the effort to accommodate their own mathematical reasoning with what they perceive to be a normative characterization of the problem coming from the lecturer. I conclude by discussing psycho-pedagogical considerations on imagination and intuition with related issues of university curriculum reform.

Keywords: Teaching and learning of logic, reasoning and proof, Curricular and institutional issues concerning the teaching of mathematics at university level, Visualization, Imagination, Intuition.

INTRODUCTION

While in general research in tertiary education is extending beyond the level of undergraduate studies (Artigue, 2021, p. 14; cf. also Winsløw et al., 2018), there seems to be very little or no research at all on students of graduate programs or courses in pure and applied mathematics (Winsløw & Rasmussen, 2020, p. 883-884) and specifically on master's students in mathematics. This is possibly a consequence of the two-years master's programs in mathematics being a relatively new phenomenon in Europe. Thus, most research on postsecondary mathematics education appears to concentrate either on undergraduate programs/courses in mathematics and related disciplines or else on graduate programs specifically designed for teachers. The present study contributes to this under-researched field by investigating how master's students in mathematics reflect on an unusual visual problem.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

Research in visualization within mathematics education originated in the work of Alan Bishop and was later carried out by various authors: see Presmeg (2020) for a compendium. In this paper, I will follow the mainstream lineage of research developed by Abraham Arcavi (2003) and Norma Presmeg (2006) albeit explicitly stressing on some hopefully clarifying preliminary definitions inspired from the work of psychologist Efraim Fischbein (1987) as well as from the writings of mathematicians such as Felix Klein, David Hilbert and Henry Poincaré. These definitions will constitute the framework for carrying out the analysis of the case-study presented below. This framework can be understood as a systematization of the traditional understanding of cognitive steps happening during a working mathematician's process of proving.

Vision may be defined unambiguously as the faculty by which we directly see things which are there for us to see. On the other hand, *imagination* is the faculty by which we see what is not there to see (in mathematics this usually happens in connection with some properties one wants to prove or show). It may be divided into *passive imagination* (the act of representing to oneself something prompted to us from an outside source) and *active imagination* (the act of representing to oneself something not prompted from the outside). Furthermore, *intuition* is the faculty by which we generalize the properties that we see or imagine.[1]

Notice that the definition of intuition given above (essentially derived from Fischbein, 1987) is somewhat more specific than the usual meaning given to the term "intuition" (mostly found within philosophy of mathematics or mathematicians' introspective accounts) which is generally an umbrella term used by authors to characterize any informal way of grasping mathematical truths outside of formal reasoning. Indeed, for the great majority of authors "intuitive reasoning" is nothing but a synonym of "informal reasoning". Notice also that for simplicity and adherence to tradition, I take in this paper a clear a priori distinction between informal and formal reasoning, albeit agreeing with the philosophical stance taken by Giardino (2010) that the two forms of reasoning are really inextricably intertwined. Notice also that the literature has traditionally distinguished between internal and external acts of visualization. Presmeg (2006) assumes this distinction as unproblematic by adopting the Piagetian view that any act of external visualization depends on internal mental images. I do not want to delve into this issue here, but I would like to remark that the distinction must be made at the level of imagination, i.e., the distinction does not concern vision (always external) and intuition (always internal).

To get a concrete grasp of these definitions and to simultaneously give an example of how these can be applied to analyze mathematical processes, let us look at the usual proof of the following proposition: the opposite sides of a parallelogram are congruent to each other. Provided that we indeed know what a parallelogram is, we can draw it (as an act of passive imagination) by tracing two pairs of parallel lines as in Figure 1.a. At this point we can see the parallelogram $ABCD$ as a direct act of vision. Furthermore, in order to prove the proposition, we may (actively) imagine the segment AC (Figure 1.b) and consider the angles that this new segment forms with the lines. We are then able to conclude that angles DAC and ACB are congruent to each other (Figure 1.c) as well as angles CAB and ACD (Figure 1.d). Thus, triangles $\boldsymbol{DAC}$ and $\boldsymbol{ACB}$ are congruent (by known properties of congruence). Therefore, the opposite sides of the initial parallelogram are congruent to each other. Finally, it is by intuition that we realize that the property thus proved is not linked to the particular parallelogram considered, but holds in general for all parallelograms.

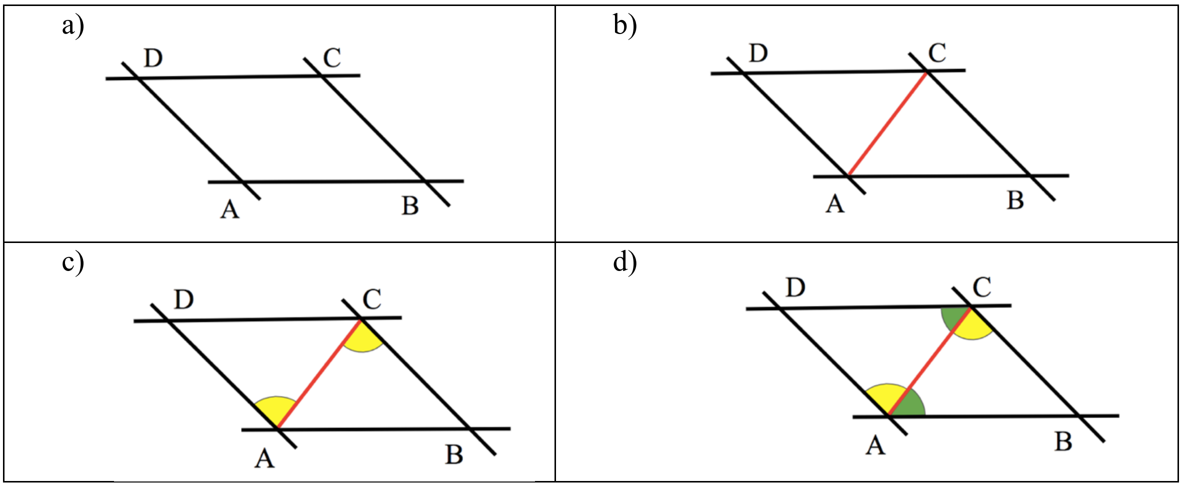

Figure 1: visual steps involved in the proof of the proposition

Now, in the passage above the crucial imaginative step (the proverbial "idea" one must have) is to consider the segment AC and reduce the proof of the proposition to the proof of the congruence of triangles DAC and ACB. In this example the segment or shape that one needs to imagine in order to complete the proof is almost evident to the trained eyes of a mathematician. However, this may not be the case for inexperienced pupils and, similarly, even experienced mathematicians might have trouble when solving a problem involving a difficult imaginative step which is not cued by the figure or diagram naturally representing the problem.

In this paper I will concentrate on a problem of this kind: the four cities problem, which I will describe below. My research question will thus be the following.

How do master's students in mathematics reason about the four cities problem?

Other than complementing the literature on university mathematics education, answering to this question will also contribute in general to the literature on students engaged in problem solving, which seems to have concentrated primarily on students of compulsory schools (cf. Verschaffel et al., 2020)

THE FOUR CITIES PROBLEM AND RESEARCH CONTEXT

The problem below was given to 28 students enrolled in a master's program in mathematics at the University of Turin, Italy. This is a competitive program focusing on pure and applied mathematics. The main requirement for entering the program is to have completed a three-year bachelor's in mathematics with good marks. Such bachelors, in the Italian university system (which does not offer a major-minor arrangement of credits but focuses instead almost entirely on mathematics) usually revolve around learning mathematical content knowledge in the form of theorems and proofs which were customarily tested by means of problem sets usually revolving on the application of these. The problem was given as part of a voluntary assignment within the students' first course in mathematics education. This is an elective course that students usually take in the first year of the master's program. During the course, other visual mathematical problems were presented, but this problem was the one which caused the most difficulties to the students.

Problem: Four cities are placed at the four corners of a square and an engineer wants to design a road which connects them. What path she has to choose in order to use the least amount of materials?

The problem is equivalent to the problem of finding the minimal path which connects the vertices of a square. The optimal solution to the problem is presented in red in Figure 2.d (modulo a 90-degree rotation), while Figure 2.a, 2.b and 2.c show paths which indeed connect the four cities but are not minimal.[2]

The following hint was given by the lecturer right after the statement of the problem: "the solution is not the path consisting of the square itself" in order to help the students exclude right away the path presented in Figure 2.a. However, this hint may have instead prompted some confusion since a portion of the students interpreted it "normatively", so to speak, as we will see below. After this, no other communication took place between the students and the lecturer.

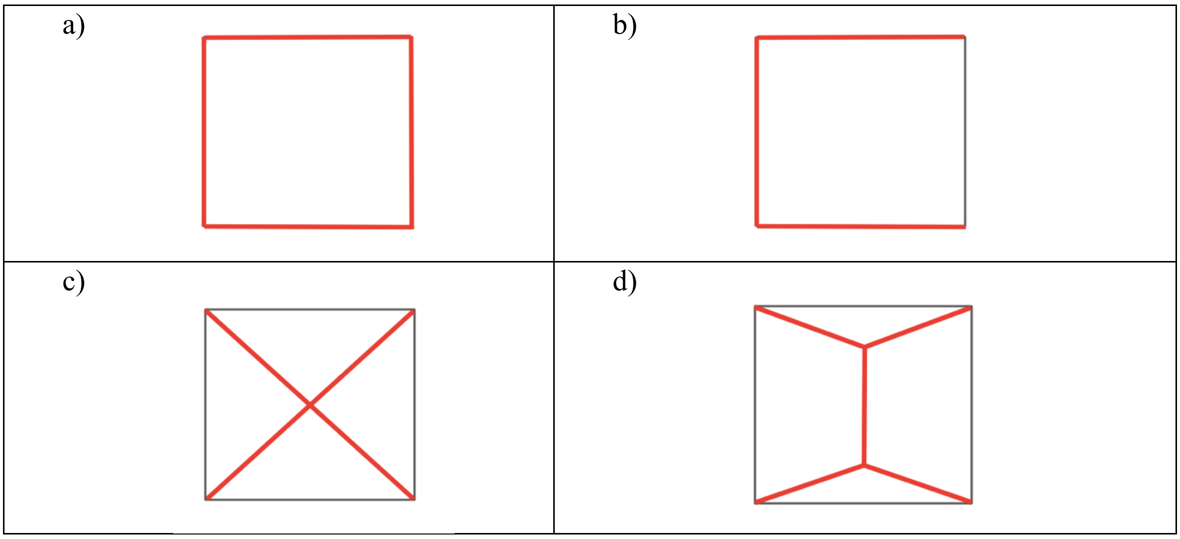

Figure 2: possible paths linking four vertices of a square

The participants were asked to form groups of three or four. In each group one student (the observer) had to write a report observing and elaborating on the way she and her colleagues reasoned about the problem. In the spirit of Arzarello et al. (2002), the students were invited to visualize the problem by means of a dynamic-geometry software (Geogebra) and given about one hour to solve the problem collaboratively.

METHOD

The data for this study consist of the written reports and the Geogebra files produced by the groups. A preliminary categorization of the groups was performed in terms of the final answer they gave to the problem.

Then the data were analysed by means of the framework described above. As seen in the case of the parallelogram theorem, a procedure of proof relying on visual representation can be segmented into steps linked to the given definitions of vision imagination and intuition. In that case, the analysis referred to an unproblematic and correct proof. However, nothing really prevents to apply it to any other correct or incorrect mathematical proof or proof attempt.

Thus, for each group of students, I analysed the respective texts in connection with the Geogebra files they produced in search for instances text and figures signalling acts of vision, imagination and intuition as connected to the logical structure of the argumentation they provided. In particular, the Geogebra files contained traces of both the students' acts of passive imagination (e.g., draw the initial square) as well as to their attempts to concretize products of their active imagination. These in turn were signalled by corresponding textual expressions describing attempts to add to their drawing new lines or figures. Finally instances of acts of intuition were similarly mostly signalled by textual data describing attempts at deduction and generalization.

In presenting the textual data, I translated the relevant passages from Italian as literally as possible.

RESULTS

Only one of the eight groups hinted at the correct solution. However, the students in this group admitted that one of them had already seen the problem before and hence they were excluded from the study. Among the remaining groups, four suggested that the solution was the one depicted in Figure 2.c: "the diagonals", while three groups suggested that the solution of the problem was the one depicted in Figure 2.a, "the square", and thus had to conceptually "accommodate" the aforementioned hint, as I will discuss below.

Analysis of two reports concluding that the solution is "the diagonals"

Let us now examine the reports of two representative groups (here called A and B) of the former portion of students. The remaining groups (C and D) had similar reports to Group B. Indeed, Group A drew a square together with its diagonals, and just wrote the following laconic sentence.

> The minimal path to unite the 4 cities is through the bisectors of the quadrilateral, given the fact that a straight segment is always the shortest way to unite 2 points.

Here the students enact a false deduction, or an over-generalization as described in (Fischbein, 1987): since the shortest way to unite two points is a straight line, then the shortest way to unite four points is simply two straight lines. This report does not furnish us with any clue as to how they arrived to consider the path consisting of the diagonals, or "the bisectors", as they say here.

On the other hand, a quotation from Group B's report may let us understand how these other students arrived at the same conclusion: the observer writes that his colleagues

[…] decided to represent the diagonals of the quadrilateral, since they thought that the best idea was that of starting from the properties offered by the quadrilateral […] they [then] asked themselves if there did not exist a path better than the one just deduced […] they then decided to construct a second quadrilateral and conjoin the vertices, not by the diagonals, but by segments located in a different way […] In conclusion both the girls agreed, in light of their reasoning and the tests performed, that the minimal path was the one represented by the diagonals of the quadrilateral.

Thus, the students in Group B chose the diagonals because they were "offered by the quadrilateral" itself. In other words, the imaginative step connected with the decision to conjoin opposite points in the parallelogram example above was, as they seem to mean, suggested or cued by the figure itself.

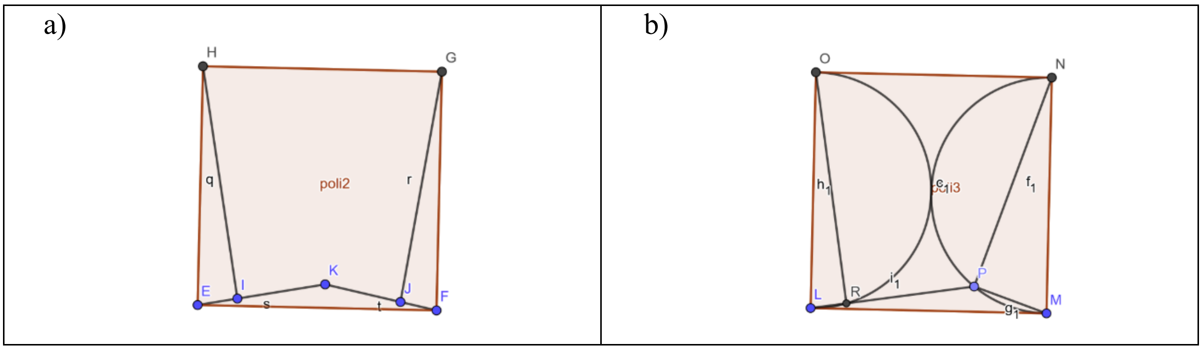

Figure 3: Geogebra protocols from Group B

Notice that these students also in the end overgeneralized as they considered two different configurations (their own drawings in the software are displayed in Figure 3.a and 3.b) and noticed that the path consisting of the diagonals was shorter than those, and then concluded that the former is shorter than any possible path. Interestingly, notice how the path displayed in Figure 3.a is not too distant from a correct solution.

Analysis of two reports concluding that the solution is (approximately) the square

What about the remaining three groups? As said before, the students in these groups stated that the solution the problem was the square itself but were puzzled by the hint which straightforwardly told them that this was not the case. Of course, the hint was specifically given in order to prompt students to think about other non-obvious solutions and help them in identifying the correct one. However, some of the students instead gave an argument for affirming that the solution was the square itself and then, since this had been excluded by the lecturer, proceeded to propose an "approximate solution" to the problem. Let us see how by analysing the reports of two representative groups (here called E and F) of this latter portion of students. The remaining Group G had a similar report as to Group F. Indeed, the observer of Group E wrote that her colleagues considered the diagonals first but then

After some reflection, they discussed on the fact that by considering the diagonals of the square, in order to visit all the cities, they necessarily needed to use one of the sides. Given the impossibility of doing this, they abandoned this idea.

This passage suggests that students in Group E were possibly also imposing to the problem the limitation that in order to visit all the cities a hypothetical traveller must not touch the same city two times (an interpretation which is at odds with the realistic setting within which the problem was presented). In any case, they were convinced that the solution to their interpretation of the problem had to be the square. Since this solution was ruled out by the hint, they then reasoned as follows.

> […] they thought of creating polylines [delle spezzate], not necessarily coinciding with the diagonals, which best approximated the perimeter so that their point of intersection lied on the square's axis. Initially they considered these just on two sides of the square while on the others they considered the diagonals. In order to understand if the minimal path was that formed by the polylines on two sides and the diagonals on the other two or rather was the path consisting of polylines on all four sides they decided […] to calculate which one was shorter […] Therefore […] their final conjecture was that of choosing, as minimal path, the one consisting of polylines which best approximate the square's perimeter.

What happened here? It appears that first the students did not imagine that other paths are possible, and as a consequence this led to a comparison whose result they thus generalized. Indeed, they constructed using Geogebra the two configurations displayed in Figure 4.a and 4.b below, then they calculated their respective perimeters (notice that for Figure 4.a this includes the dotted diagonals) and finally conjectured that the solution should be the latter.

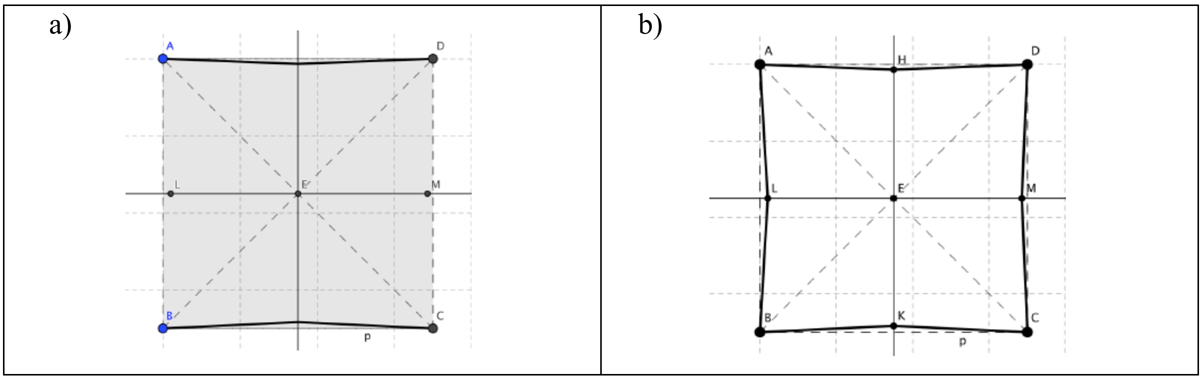

Figure 4: Geogebra protocols from Group E

Furthermore, what is perhaps most striking here is the struggle the students experienced in formulating the consequences of their conjecture. Since the latter points to the fact that the solution should be the square itself, but since also the lecturer had ruled out this possibility, these students are then forced to conclude an impossibility: the minimal path is the one which best approximates the square itself, despite the fact that no such unique path exists!

Similarly, Group F started by considering the diagonals of the square but then

> […] came the idea of creating a square in the centre [of the original square] whose side can vary between 0 and l [the side of the square] linking any of its vertex to one and only one city […]

At this point, they used Geogebra to represent this situation (as displayed in Figure 5 where point R can vary over side QN) and concluded that

> [...] point R must be as close as possible to point N for having the minimal path: this means that the two squares must have roughly the same side [...] The minimal path is given by the approximation of the square having as vertices the four cities.

Here again for these students "the" solution is "the minimal path" consisting of the best approximation to the square itself, despite this being a mathematical impossibility.

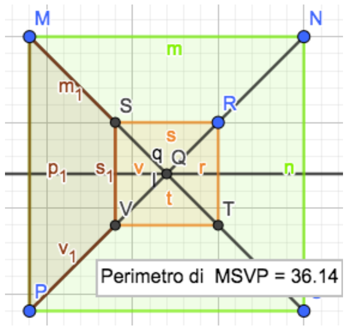

Figure 5: Geogebra protocol from Group F

CONCLUSION

In conclusion, these data may be explained by the difficulty of the problem itself (which in turns reflect into the students' difficulty to imagine the solution) as well as by the students' tendency to overgeneralize connected to false deduction. Furthermore, many of the students' imagination seems to have been crucially impeded by the fact that the square itself cued a false solution: in the words of (some of) the students, the square "offered" the diagonals as the solution for presumably all Groups A, B, C and D. A similar phenomenon is probably safe to assume having played a role also for Groups E, F and G in convincing them that the square itself (or its "approximation") was the solution. This fact is not surprising perhaps and possibly just tells us something about the difficulty of the problem chosen.

On the other hand, the proneness of master's students in mathematics to overgeneralize is more interesting given the fact that this phenomenon is present in a form or another in the reports of both the first and the second portions of students. This behaviour could be taken to be desirable and proper when conjecturing. However, the substantial easiness with which these students passed from the particular to the general could be regarded as problematic as pertaining to master's students of mathematics (i.e., students who supposedly are at the pinnacle of mathematical instruction). As a linguistic observation, notice the awkward usage of the term "deduction" in the quoted passage of Group B, where "deduced" is used as a synonym of "conjectured" or simply "thought of".

Moreover, with respect to Groups E, F and G, we have seen how the students prefer to conclude an impossibility rather than dismissing their own mathematical reasoning or rather than dismissing what they perceive to be a normative characterization of the problem coming from the lecturer. This is in itself an interesting phenomenon which could be interpreted as signalling a formal and "ritualistic" way of dealing with mathematical problems. Furthermore, if further replicated with different (but analogous) problems and by means of a larger group of students, similar results may point to the fact that for some university students in mathematics what they perceive to be a correct mathematical argument together with what they perceive to be a normative statement coming from the lecturer has stronger epistemic (or perhaps only cognitive) primacy over acknowledging the contradictory nature of their conclusions.

Finally, I do not mean to understand these preliminary conclusions in purely psychological terms as dependent solely on the students' internal faculties, them being completely separated from the context in which these students were immersed.[3] On the contrary, the students' difficulties I have outlined in this paper render possibly evident that the kind of mathematical training to which these students were exposed fails (at least in this case) as a training for problem solving involving a strong conjecturing component. More empirical data agreeing with these results may point to the fact that the type of mathematical rationality into which these students have been steeped in is very different from the kind of rationality which would ideally be that of a working mathematician. Indeed, a deeper and larger study into these phenomena would be required to reach more than tentative conclusions on this matter. Such study could perhaps suggest the need to develop the university mathematics curriculum in favour of a greater exposure of students to problems involving a stronger conjecturing component. It remains however an open question whether such exposure may in general succeed in the training of imagination and intuition and, in general, whether these faculties are susceptible to be trained at all. The latter in turn is a psycho-pedagogical matter over which further investigation would be needed.

NOTES

1. In accordance with the scholarship on visualization within mathematics education (cf. Arcavi, 2003), one can thus generally understand *visualization* in mathematics as all that concerns the faculties/properties/abilities above, i.e., the mode by which we bring mathematical objects at the attention of our senses, we manipulate them and we reflect on them, internally (i.e., in the mind) or externally via some material support, traditionally by hand-drawing on paper or, nowadays, by means of software-generated images.

2. An a priori analysis of the problem would be too long to give here. It is plausible to think that a path very similar to the one which is the solution of the problem must be reached by a unique act of imagination, which arguably seems to not be decomposable into simpler mental actions.

3. For instance, the behavior of Groups E, F and G just summarized may be perhaps explained by a difficulty in reasoning outside of the didactic contract the students assume to be in place (on this concept see Brousseau, Sarrazy and Novotná, 2020), connected to a difficulty in questioning the authority of the teacher.

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Mental constructions for the learning of the concept of vector space

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We present the results of the implementation of a teaching strategy for the learning of the concept of vector space. The strategy was implemented with a group of engineering students and its design is based on the ACE methodology from APOS theory. The results show that working with sets and binary operations different from those traditionally handled in a first course of linear algebra, promotes students' reflection on the validity of the axioms that define vector spaces and on the properties of the zero vector and the additive inverses.

Keywords: vector space, linear algebra learning, APOS theory

INTRODUCTION

Many of the particular obstacles that arise in the linear algebra teaching and learning processes are related to the nature of the elements that constitute this mathematical theory. Linear algebra is formed by a network of interconnected definitions, axioms, and abstract theorems. This results in frequent difficulties among students to succeed in higher level courses related to linear algebra. Thus, students end up confused and disoriented when trying to understand concepts related to this discipline, such as vector spaces, subspaces, linear transformations, among others (Dorier & Sierpinska, 2001).

This paper reports on the development and implementation of a teaching strategy to promote the understanding of the vector space concept among engineering students. Its design is informed by APOS theory. Due to the COVID-19 pandemic this teaching strategy was implemented online. The participating students had not received training in mathematical logic, nor did they have experience working with argumentative or demonstrative processes in their previous mathematics courses.

RESEARCH QUESTION AND OBJECTIVES

This research study is part of a wider project which aims to analyze the mental constructions evidenced by a group of students when working with the concept of vector space. Students had followed a linear algebra course where activities designed with APOS theory (Actions, Processes, Objects, Schemas) were used, and the APOS teaching methodology, involving Activities, Class discussions and Exercises (ACE teaching cycle), was followed. The study aims to address in general the following research question:

What mental constructions related to vector space do students manifest after finishing a course designed with APOS theory?

The teaching strategy used throughout the course included work on activities aimed at learning concepts that have been identified previously (Parraguez & Oktaç, 2010) as a requirement for the construction of the concept of vector space. This led us to consider another related research question:

What is the impact of introducing the study of the concepts of equality, set and binary operation, as background, in a teaching strategy aimed at supporting the construction of the concept of vector space?

The research Project conducted resulted in information regarding both questions. However, we decided to focus in this paper on results obtained from the data analysis corresponding to the second one. Space restrictions did not allow to include a detailed report on results obtained throughout the whole research experience.

LITERATURE REVIEW

In order to identify the mathematical concepts required for the learning of the concept of vector space, we conducted a literature review focusing on the main learning obstacles related to this concept (Can et al., 2021).

A first epistemological obstacle that students face comes from the level of formalism inherent to linear algebra (Dorier, 1998). In particular, when facing the concept of vector space, students may face many of the obstacles associated with the formalism and level of abstraction needed in its study. For instance, when working with the demonstrative and argumentative processes that are required to verify whether or not a given set is a vector space (Mutambara & Bansilal, 2018); or when working with the concept of zero vector and the additive inverse vectors of a space where elements are not necessarily n-tuples. These obstacles are also encountered when working with vector spaces where addition and scalar multiplication are defined differently from those traditionally defined in $\mathbb{R}^n$ (Kú et al., 2008; Parraguez & Oktaç, 2010).

Many students have difficulty recognizing the characteristics defining both the zero vector and the additive inverses. Some authors have suggested the use of unusual vector spaces as a way to address this situation, for example, vector spaces where the pre-established algorithms that traditionally work on typical vector spaces are not sufficient. In order to succeed, flexibility is necessary to identify those characteristics shared by the sets and the binary operations that constitute such vector spaces (Parraguez & Oktaç, 2012; Parraguez, 2013).

It is important to underline that, in spite of reports in the literature on the need to take into account the construction of the notion of set and binary operation before introducing vector space, it was difficult to find in the literature studies that take them into consideration as part of their research objective. We posit this may be due to researchers and teachers considering these concepts as something known by students, although this may not be the case for students taking a first course on Linear Algebra at the university, particularly if they are enrolled in fields other than mathematics.

The findings of this literature review put forward the need to consider the introduction of the concepts of set, binary operation, axiom, and function as requirements before dealing explicitly with the construction of the concept of vector space. Thus, in this paper we present a teaching strategy that considers all these elements.

The obstacles described above together with teaching suggestions given by some authors served us as the basis for designing a genetic decomposition addressing specifically the construction of the zero vector and additive inverses. They have also helped us to design a teaching strategy that introduces students to working with vector spaces whose elements are not necessarily n-tuples.

THEORETICAL FRAMEWORK

This research is based on APOS theory (Arnon et al., 2014). This theory intends to understand the constructions students need to learn a concept. Its main conceptual structures follow:

Actions are defined as the transformations applied to previously constructed Objects and that are somehow external to the subject that applies them. They are identified by the fact that the subject needs an external stimulus to apply them.

Processes are understood as the Actions interiorized by the subject, in such a way that the individual is capable of reflecting on such Actions without the need for external stimuli, and can describe them or even reverse the steps without the need to perform the steps operationally. Processes can be coordinated into other Processes and can also be reversed.

When an individual reflects on the operations applied to a particular Process, becomes aware of the Process as a whole and is able to perform new Actions on it, that is, can act on the Process itself, the individual has encapsulated the Process into an *Object*. Furthermore, if the subject is able to go back from the Object to the Process from where it comes, it could be said that the individual has de-encapsulated the Object into a Process.

A *Schema* is constructed as a coherent collection of mental structures (Actions, Processes, Objects and other Schemas) and the relationships between them. A Schema is described in terms of the mental structures that compose it and how they are related to each other. Schemas are constantly developing through a triad of stages: Intra- , Inter -, and Trans- . When a Schema has been constructed at the Trans- stage and can be considered as coherent, it can be thematized into an Object and new Actions can be applied to it.

METHODOLOGY

APOS theory research methodology begins with an analysis that leads to the development of a model for the epistemology of the concept to be studied. This model—which is called the *genetic decomposition* of the concept—is based on the description of the Actions, Processes, Objects and Schemas and the corresponding mechanisms needed in the construction of the concept or concepts of interest (Arnon et al., 2014). This model is not intended to be unique and needs to be validated through research. We describe the genetic decomposition below.

The genetic decomposition was used to develop a set of twenty tasks for the introduction of vector space. These tasks were organized in six activity sets to be used in the classroom. The design of the tasks included the construction of the concepts of equality, set, and binary operation—in addition to those related to the construction of the concept of vector space. As mentioned before, these elements were identified as a necessary background for learning vector spaces (e.g., Parraguez & Oktaç, 2010).

The teaching strategy followed the ACE cycle. The experience was conducted with a group of 20 engineering students enrolled in a first linear algebra course at a public university. It is important to underline that these students had not been introduced before neither to these concepts not to work with operations different to those defined traditionally for sets other than $\mathbb{R}^n$. These were the reasons to design tasks including different sets and operations. Also we considered that working with them would foster students' reflection on the need of the axioms defining vector spaces.

Students were organized in teams of four to work in the solution of the designed activity sets through six sessions. Due to the quarantine imposed by COVID-19, all the sessions took place on line.

The teaching strategy implemented in all sessions was the ACE Cycle: it consisted in students working collaboratively in small groups on the activities (A). Collaborative work was followed by whole class discussion with the teacher (C). These two steps were repeated several times during each session and homework exercises (E) were handed to students at the end of each class.

The first three sessions were devoted to work on the construction of the pre-requisite concepts, namely equality, sets and binary operations through tasks involving proving some vector space axioms related to them. The last three sessions consisted in defining of vector space followed by tasks involving different sets and binary operations where students had to prove if they were vector spaces. Students worked on these last tasks using paper and pencil and then programming computer codes to construct the vector space axioms.

All the work produced by the students during the sessions was kept and the video recording and work during interviews of one student in each group at the end of the semester was analyzed by the three researchers and results were negotiated. Analysis focused on describing and identifying important emerging ideas during small groups work, the evolution of students' contributions during whole group discussion and evidence of the construction and development of APOS structures related to the pre-requisite concepts and the vector space throughout the sessions and the interview for each student.

In this document we show results from three students that exemplify the role played by the use of tasks designed with the genetic decomposition and the construction of pre-requisite concepts on students' construction of the vector space concept.

Genetic decomposition of the vector space concept.

Genetic decompositions of the prerequisite concepts were designed taking into account results from the literature. The first three sets of activities implemented were devoted to students' construction of those prerequisites.

We proposed a new genetic decomposition for the vector space concept. It combines elements from the decompositions proposed by Parraguez and Oktaç (2010) and Arnon et al. (2014). This new genetic decomposition considers as prerequisites mental constructions of the concepts of equality, set, and binary operation. Next, this new genetic decomposition is introduced. It includes a mechanism to integrate the axiom schema in the construction of the vector space concept. The role of logical quantifiers of existence and universality to characterize the axioms that define a vector space is also considered.

The construction of the concept of vector space is based on the construction of the Schemas of the concepts of set, binary operation and equality: the student begins with the Action of calculating the result of applying a given binary operation to specific elements of the same set, and to the Cartesian products of sets. With these Actions, binary operations are conceived as functions with certain input arguments that in turn produce an output.

By working on the application of different binary operations on all the elements of different sets to analyze and describe their Actions, without having to carry out the operations explicitly, the interiorization of the set Process is promoted. This Process is defined in terms of a membership condition and the binary operation Process defined on a set that can be coordinated into a new one to originate a Process of the notion of sets with binary operations.

Continuing with the construction of the concept of vector space, the concept of axiom is constructed as follows: given a property of a binary operation involving the equality sign, the individual applies the Action of evaluating both members of equality on specific elements. This is done based on the definition of the binary operation to determine the validity of the equality stated by the property.

In order to continue with the construction of the vector space concept, the concept of axiom needs to be constructed as follows: Given a binary operation property involving the equal sign, the student does the Action of evaluating it in specific elements appearing in the two sides of the equality in terms of the definition of the binary operation concerned and then determines the validity of the given equality in terms of the specified property, When the individual reflects on the validity of a property of binary operations defined on a set, applied to all the elements of the set—involving the universal quantifier—, she interiorizes such Actions into a verification Process of an axiom with a universal quantifier. This Process, in turn, is coordinated with the Process set with binary operation to give rise to the verification Process of an axiom with universal quantifier for binary operations on sets.

This implies that the individual can analyze and discuss the existence of a specific element that satisfies an axiom, without the need to list all the elements of the set, and that she is able to determine the missing element in the expression that involves the axiom. This Process is coordinated with the sets with binary operations Process into the verification Process of an axiom with existential quantifier for binary operations on sets.

The two previous axiom- verification Processes are coordinated with the Processes corresponding to the axioms that define the vector space. This results in a set with binary operations Process (addition and scalar multiplication) that satisfy the axioms.

The resulting structure is a dynamic structure that depends on the set and the binary operations that are considered. This structure becomes static when the individual conceives the vector space as an object to which she can apply Actions such as finding a base, finding a spanning set, determining the linear dependence or independence of a set of elements interpreted as vectors, and applying linear transformations between vector spaces.

RESULTS AND DISCUSSION

As mentioned before, after presenting the definition of vector space students worked on testing the axioms and deciding if the given set was or not a vector space. We present now an analysis of transcripts form the interventions of three students when they addressed one of the tasks worked in session four posed to the students with the intention of observing the constructions that they evidenced when working with the concept of vector space.

Task.

Verify if the set $V = \left\{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in \mathbb{R}^{+}\right\}$ is a vector space with the operations defined below.

Vector addition: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & be \\ 0 & cf \end{pmatrix}$.

Scalar multiplication: $t \otimes \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^t & b^t \\ 0 & c^t \end{pmatrix}$.

When questioned about the algebraic closure of the defined addition, student 1 answered as follows:

Student 1: Yes, it does comply because... well... it does comply because in both matrices the elements that are taken, well, as it says there, as a rule, they belong to the positive real numbers and when doing the sum, well ... it will b the same. The sum of these matrices remains the same, so to speak, with the elements belonging to V.

Interviewer: Explain to me, what do you mean when you say that it stays the same?

Student 1: Yes... it could be, for example, ... a matrix of 1, 2, 0, 3... we do the sum of a matrix of 2, 4, 0, 6, and we would obtain a matrix of 2, 8, 0, 18.

A second student makes the following comments:

Student 2: It is like this... *ad*, *be*, 0 and *c* multiplied by *f*, but it was also replaced by numbers, like this with numbers, to verify that they belong to the positive real numbers as it is in the vector and they do belong… um, I wrote that... the matrix one is 1, 2, 0 and 3 and the other one is 4, 5, 0 and 6 and well... I replaced the numbers with the letters, and at the end I obtained 4, 8, 0 and 18.

Interviewer: Okay, and how does the calculation explain that the property is true?

Student 2: Because they belong to the set V, of positive real numbers.

To verify the closure property of vector addition, both students select specific vectors from the given sets and calculate the sum with those elements. This illustrates one of the first Actions most of the students perform when they have to verify that a set is a vector space. They select specific values to verify that the axioms are satisfied and, in some instances, they could conclude—based on particular cases—that such axioms are satisfied by all the elements in the set, without analyzing the validity of their conclusion on all the elements of the set. After analyzing the rest of student' interventions, we concluded that twelve students showed similar interventions corresponding to the construction of Actions.

When questioning a third student regarding the validity of the closure of the addition using the same task, he replies:

Student 3: Well they were like... It was a set like this... in the first row *a*, *b*, *c*, in the second row 0, *c*, and the other element would be, the first row *d*, *e* and the second row 0, *f*, so those two elements. That is, we apply the sum defined for the set... And, well, we got the matrix, so to speak, in the first line *ad*, *be*, 0 and *c* multiplied by *f*.

Interviewer: How did you get that?

Student 3: Because we applied the operation that was indicated for the set... from what we understood, it was like that. Because it says that *u*, the... the sum of vectors with *v* belongs to *V*. It means that if you take an element and add it with an element of *V* and... it means that these two elements are in *V* ... I don't know if you understand me... That is, the result is the one that has to be in *V*, right? because as I had already said before... The axiom tells us that *u* and *v* are elements that are in *V*... then obviously they comply with the... with the shape and structure of the set *V*. It means that, if when applying the operation to those two elements, yes... that axiom... for that axiom to hold true, the result of that sum must also belong to *V*... That's more or less what it means, right?

Student 3 argues about the validity of the closure axiom for the addition of vectors, using the definition of the set and the operation. He shows a generalization of the verification of the closure on different elements. This is evidence of a Process construction related to the verification of the axioms.

When questioned about the existence of additive inverse vectors, student 3 comments (see Figure 1):

Student 3: But if you notice… you are dividing *a* by *a*, well obviously if you have... let's say that *a* is equal to 3, 3 divided by 3 will give you 1. So if they realize, *a* over *a*, is practically 1 and, the others are 1 all those who are there...

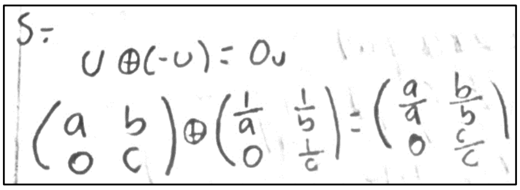

Figure1: Vector proposed by student 3 to verify the existence of additive inverses.

Interviewer: Well, there we have the image, so in the image we see a matrix *u* that is *a*, *b*, 0 and *c*, and it is being added to another matrix that is 1 over *a*, 1 over *b*, 0 and 1 over *c*... Can you explain that?

Student 3: Well, I took as a reference the one we already had in the fourth [axiom], since it is indicating that, if -*u* exists, it should lead us to the value of the additive neutral element. And well, since we had already stated that the additive neutral element... well, I put it as I had represented it before, then I used the 1 over *a*, since when doing the multiplication, I mean, when doing the addition operation, the multiplication would be done linearly and it would be like *a* over *a*, and it would give us as a result the additive neutral element that we had already obtained before.

Student 3 proposes the general structure of the additive inverse vector based on a generic vector *u* of the given set, and applies the binary operation defined on this set. In his intervention, student 3 describes the way in which the structure he proposes for the inverse vector can serve to prove the existence of the additive inverse elements for a specific value. This evidences a Process conception of verification of axioms with existential quantifier for binary operations defined on sets. After analyzing the rest of students' interventions, we found eight students showing evidence of a Process conception of vector space.

Finally, we found that only one student showed evidence of the construction of vector space as an Object.

In general, the mental structures that were introduced in the genetic decomposition of the vector space concept to address the verification of axioms with universal and existential quantifiers, were evidenced during the students' discussions that emerged when looking for specific elements that satisfied those axioms. This discussion involved reflection on binary operations and definitions of sets, which confirms the importance of these concepts for the construction of vector spaces as proposed by Parraguez and Oktaç (2010).

The coordination of the Process related to applying a binary operation to all the elements of a set, and the Processes that involve evaluating axioms with quantifier of existence and uniqueness, was evidenced in the students' discussion about the characteristics of the neutral element and the additive inverses. This coordination involved reflections on the particular characteristics of the elements that constitute a set and the way in which binary operations act on all these elements. This led the students to reflect on the uniqueness and universality of the neutral element and the additive inverses.

This situation paved the way for the verification of the particular characteristics of these vectors in a natural way within the construction of the concept of vector space, and is related to the Process of verification of an axiom with existential quantifier and the Process of verification of an axiom with universal quantifier for binary operations.

CONCLUSIONS

Working with vector spaces requires managing different sets of vectors, applying different types of binary operations and verifying that the axioms that define vector spaces are satisfied by the given elements of a vector space (Kú et al., 2008; Parraguez & Oktaç, 2012; Parraguez, 2013). The research reported in this paper has addressed these elements through a genetic decomposition proposed for the construction of the concept of vector space that includes the background that is considered necessary for learning this concept.

It was observed that working with different sets and analyzing their elements based on their definition, favors the ability to describe and propose the vectors that belong to a particular vector space. This allow students to recognize which elements are part of a set based on its definition, and propose specific elements that can be evaluated in binary operations.

The use of different binary operations turned out to be a strategy that favors the interiorization of the Actions carried out by the individual when evaluating the operation on particular elements. In particular, the work developed with different types of binary operations favors the ability to describe the way in which the operations act on all the elements of a given set. An observed phenomenon in this sense involves the use of literals to describe the way in which a given operation acts: when the student uses them to operate on a generic element of a set and describe the way in which an operation acts on all the elements of the set, she evidences the construction of a Process conception.

The didactic strategy based on collaborative work between students and with the teacher together with activities designed with the genetic decomposition proved to be effective in fostering students' reflection and understanding of the role of axions and their need in the definition of vector space.

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Analysing proofs of the Bolzano-Weierstrass theorem
A mean to improve proof skills and understanding of completeness

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In this communication, we will analyse several proofs of the Bolzano-Weiersrass theorem. We will first present the method by Bolzano in the memoire of the Intermediate Value Theorem that is known to have inspired the proof by Weierstrass, then we will analyse proofs available for both the set version proved by Weierstrass, and the sequence version. We will emphasize the diversity of modes of reasoning on the one hand, on objects and completeness characterisation on the second hand. We will finally suggest scenarios for undergraduates or for prospective teachers.

Keywords: Teaching and learning of logic, reasoning and proof, Epistemological studies of mathematical topics, Bolzano-Weierstrass Theorem, Proofs 'analysis, Teaching and learning of analysis and calculus.

INTRODUCTION

As accounted by Bergé (2010), undergraduate students having followed four courses on the set of real numbers might face still difficulties with task requiring a sound understanding of completeness. This motivates the search for activities able to contribute to this understanding. This communication falls within a wider research project aiming at identifying didactic means to improve the teaching and learning of the set of real numbers as a complete ordered set, considering the crucial role of epistemology in didactics of mathematics, and focusing on proof and proving. Our main didactic hypothesis is that logical analysis of proof fulfils three main functions: to control validity; to understand the strategy of the author of the proof; to contribute both to the development of proof and proving skills and to the appropriation of the mathematical content at stake. We have already discussed in other papers 1/ the potentiality of fixed-point theorems of increasing functions of a real subset in itself (Durand-Guerrier, 2016); 2/ the relevance of working on proofs of completeness in various settings of real numbers sets (Durand-Guerrier & Tanguay, 2018); 3/ The fecundity of approaching the real exponential function in the more general frame of real function satisfying the algebraic relation $f(x+y) = f(x)f(y)$ (Durand-Guerrier et al., 2019); 4/ the dialectical relationship between *truth* and *proof* in the case of the emergence of the Intermediate Value Theorem (IVT), while comparing Bolzano and Cauchy approaches (Durand-Guerrier, 2022). In this communication, we are extending our analysis to the *Bolzano-Weierstrass Theorem* (BWT) which plays an important role in Analysis. In his memoire on the IVT published in 1817, Bolzano used a method by *dissection* to prove (in modern term) the existence of a *least upper bound* for a bounded above set. Later, Weierstrass relied on the method described by Bolzano to prove the theorem that is named after the two mathematicians (Oudot, 2017).

Nowadays, we found in the international literature various proofs of this theorem. We will argue that analysing these proofs might contribute to deepen both proofs skills and conceptual understanding of the concept of ℝ-completeness, considering both the *version for sets* and the *version for sequences*:

1. BWT for sets: *Every infinite and bounded set of real numbers has at least an accumulation point.*
2. BWT for sequences: *Every bounded sequence has a convergent subsequence.*

We first present and analyse the Bolzano dissection method and four BWT proofs. Then, we propose some guidelines for designing a didactical engineering.

PROOFS OF THE BOLZANO-WEIERSTRASS THEOREM

Although the name of Bernard Bolzano is associated to the name of Karl Weierstrass, some authors claim that Bolzano neither prove, nor even enounce the theorem (e.g., Oudot, 2017). However, it is agreed that Bolzano used a *dissection method* in a lemma for the proof of the Intermediate Value Theorem (IVT), which inspired Weierstrass to prove that *any infinite bounded subset of the set of real numbers has an accumulation point*. (e.g., Oudot, 2017). We present and briefly analyse 1/ the dissection method by Bolzano; 2/ the main ideas of the proof by Weierstrass of the BWT for sets; 3/an alternative proof of the BWT for sets in a paper by Mamona-Downs; 3/ two classical proofs of the BWT for sequences in a textbook.

In Durand-Guerrier and Arsac (2009, p. 152), we provided evidence that analysing logically a mathematical proof requires both logical and mathematical competencies. We have identified relevant questions that we will use as a lens for analysing the proofs presented in this section: what are the *data and hypothesis*? *which objects are introduced along the proof, and with which aim? Which* are the *modes of reasoning*, the explicit and *implicit assumptions*? In the case of BWT, we will also focus on the *axioms of completeness,* and on the recourse to *potential infinity versus actual infinity*.[1]

The dissection method by Bolzano (1817)

In the paragraph 12 of the proof of the IVT, Bolzano enounce and prove the theorem[2]:

> Theorem. If a property M does not belong to all values of a variable x, but does belong to all values which are less than a certain u, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x has property M. (Russ, 1980, p. 174)

He initiated the proof by introducing a positive quantity $\boldsymbol{D}$ such that there is at least an element among those less than $\boldsymbol{u}+\boldsymbol{D}$ that does not hold the property $\boldsymbol{M}$ (such a quantity exists by hypothesis). Doing this, he implicitly introduces *a real interval* $[\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{D}[$ in which he will search the quantity $\boldsymbol{U}$. Then he considers sequentially the quantities $\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}}$, with $\boldsymbol{m}$ a null or positive integer. In other words, he introduces a *geometrical*

[1] This paper will not consider the logical issues involved in the use of quantifiers and connectives.

[2] For the theorem, its proof and the comments by Bolzano on this theorem, we use the English Translation by Russ (1980).

sequence with ratio $\frac{1}{2}$. It is sure, by the choice of $\boldsymbol{D}$, that for $\boldsymbol{m}=\boldsymbol{0}$ there is at least an element not satisfying the property $\boldsymbol{M}$ among the element less than $\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^0}}$. In case where whatever the value of $\boldsymbol{m}$, there exists always an element not satisfying the property M along the element less than u+$\frac{\boldsymbol{D}}{\boldsymbol{2^m}}$, then he concludes that $\boldsymbol{u}$ is the greatest of those quantities of which it can be asserted that all smaller $\boldsymbol{x}$ has property M. The proof of this assertion (not provided in this part of the memoire) relies on the fact that given a real number $\boldsymbol{v}>\boldsymbol{u}$, there exists an element between $\boldsymbol{u}$ and $\boldsymbol{v}$ that does not has the property; indeed there exists a positive integer $\boldsymbol{k}$ such that $\boldsymbol{u}<\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^k}}<\boldsymbol{v}$. Opposite, if for a certain rank *m*, for the first time all elements less than $\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}}$ holds the property $\boldsymbol{M}$, then he reinitiated the process with $\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}}$ playing the role of $\boldsymbol{u}$, and $\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^{m-1}}}$ playing the role of $\boldsymbol{u}+\boldsymbol{D}$. This is motivated by the fact that at this step, the quantity $\boldsymbol{U}$ should be searched in the interval $[\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}},\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^{m-1}}}[$. Bolzano notes that the difference between the two quantities is $\frac{\boldsymbol{D}}{\boldsymbol{2^m}}$.The process is then reiterated, considering the sequence[3]$\left(\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}}+\frac{\boldsymbol{D}}{\boldsymbol{2^{m+n}}}\right)$. Then either $\boldsymbol{U}=\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}}$ and the theorem is proved; or there is a value $\boldsymbol{r}$ such that $\boldsymbol{U}$ belongs to $[\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}}+\frac{\boldsymbol{D}}{\boldsymbol{2^{m+r}}},\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}}\frac{\boldsymbol{D}}{+\boldsymbol{2^{m+r-1}}}[$, and the process is iterated. There are then two possible issues: either there is a step where the value of $\boldsymbol{U}$ is reached, and the iterations stop; or the iterations do not stop and hence, it provides an increasing sequence in geometric progression of ratio $\frac{1}{2}$, with first term $\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^m}}$, each term being less than $\boldsymbol{u}+\frac{\boldsymbol{D}}{\boldsymbol{2^{m-1}}}$, with the difference between two consecutive terms decreasing to 0. This way corresponds to the modern method by dichotomy. Bolzano used then a previous lemma (§9) to conclude of the existence of a quantity $\boldsymbol{U}$ which is the greatest of those of which it can be asserted that all smaller $\boldsymbol{x}$ has property $\boldsymbol{M}$.

> *Lemma*. If, therefore, some given series has the property that each term is finite, but the change which it undergoes on every further continuation is smaller than any given quantity, provided only that the number of terms taken in the first place is large enough, then there is always one and only one constant quantity which comes as close to the value of this series as desired, if it is continued far enough. (Russ, 1980, p. 173).

In this proof, the data are *a property M*, *a variable* $\boldsymbol{x}$ and *a constant* $\boldsymbol{u}$; and the hypothesis: *M does not belong to all values of* $\boldsymbol{x}$*, but does belong to all values which are less than u.* Then there are several objects thar are introduced: a constant $\boldsymbol{D}$, a geometric sequence, and along the proof, some specific terms of the sequences identified by indexes of the power of 2; finally, a geometrical progression is introduced. To conclude to the existence, Bolzano refers to what we name nowadays the *Cauchy*

[3] In this proof, Bolzano does not use the term sequence, and the notion of interval; it remains implicit.

criteria of convergence for sequence, one of the possible axioms of completeness. It is noticeable that as Bolzano did not create the set of real numbers (Freudenthal, 1971, p. 387), the proof in the lemma 9 relies on an assumption that he could not prove (an axiom). In this proof, Bolzano recourse to potential infinity through iteration process.

The proof by Weierstrass of the BWT for sets

According to Oudot (2017), while in contemporaneous textbooks the BWT is most often given for sequences, Weierstrass proved the version for *infinite sets*. He first defined the notion of *accumulation point* for an infinite set.

Definition: an accumulation point of a given subset of the set of real numbers is a point such that each pointed neighbourhood has a non-empty intersection with the set.

He then proved by dichotomy, a method inspired by Bolzano, the theorem *for sets*: *Every infinite and bounded set of real numbers has at least an accumulation point.*

Here is a summary of the proof (translated from Oudot, 2017).

Proof - If a set A is bounded, then it is included in a closed interval $[m, M]$. As A is infinite, cutting the interval through its midpoint, we get two intervals so that at least one contains an infinite number of points of A. The process is then iterated on an interval with infinite numbers of points of A (there exists at least one), leading to the construction of a sequence of nested intervals with length less than $\frac{1}{2^m}$ at the step m. The conclusion follows because the set of real numbers being complete, it satisfies the property of *nested intervals with length tending to 0*: there exists a unique element belonging to each interval. This element is an accumulation point for A.

In this proof, the data is a set, with two hypothesises: the set is *bounded*; the set is *infinite*. Then an interval $[m, M]$ is introduced, and a process of dichotomy is performed. It is noticeable that the method is simpler than the method by Bolzano, thanks to the explicit introduction of *an interval*, that allows the recourse to *actual infinite*.

A proof of the BWT for sets in an educational paper

In a paper published in 2010, Mamona-Downs suggested that providing students opportunities to contrast the convergence behaviour of a sequence and the accumulation points of the underlying set of the sequence is worthwhile for undergraduates. In the paper, she provides a proof of BWT with recourse to the supremum property (p. 283).

> **"Theorem (Bolzano–Weierstrass)**: *Let S be a real set that is both infinite and bounded. Then S has an accumulation point*[4].
>
> As a preliminary step to the proof, the author explains that assuming the problem is solved, a candidate appears (the supremum of a well-chosen set, as shown in the proof below).

[4] The author indicates that there is no loss of generality f we suppose that S is a subset of a closed interval, [a, b] say.

Proof: Consider the set L: $\{x \in [a,b] : [a,x]$ contains none or a finite number of elements of S less than $x\}$. L is non-empty and is bounded by b. Thus sup (L) exists. Denote sup (L) by r.

Take any $\epsilon > 0$, and consider the interval $(r-\varepsilon, r+\epsilon)$. Now $[a, r-\epsilon]$ contains none or a finite number of elements of S, and $[a, r+\varepsilon\ ]$ contains an infinite number of elements of S. Then the interval $(\ r-\varepsilon, r+\epsilon)$ also contains an infinite number. This implies $(\ r-\varepsilon, r+\epsilon)$ contains an element of S[5], so r indeed is an accumulation point for S."

In this proof, the data is a set, with two hypothesises: the set is *bounded*; the set is *infinite*. There is a first introduction of an interval, relying on the hypothesis that the set is bounded, followed by the introduction of the set L, which is the left part of a cut, in Dedekind sense, of the interval $[a,b]$. This choice is motivated by the preliminary step because the supremum linked to this cut has been identified as a candidate to be an accumulation point. The axiom for completeness is *the existence of a supremum for any bounded subset of the set of real numbers.* Once done, the author names r the supremum and introduces a family of intervals centred in r. She uses implicitly that the complement of a finite subset in an infinite set is infinite, to assert that at each step, there is an interval with infinitely many elements of S; hence each interval of the family contains infinitely many elements of S; this proves that r is an accumulation point of S.

Two classical proofs of the BWT for sequences

In this section, we present two classical proofs of the BWT out of a textbook for undergraduates (Bartle & Sherbert, 2000). The two proofs of BWT are in the section 3.4 entitled "*Subsequences and the Bolzano-Weierstrass theorem*". The authors first enounce and prove a theorem guaranteeing the existence of monotone subsequence for any real sequence (p.78). We present the proof of this theorem because it is used in their first proof of BWT.

"3.4.7 Monotone Subsequence Theorem

If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.

Proof. For the purpose of this proof, we will say that the *mt*h term x_m is a "peak" if $x_m \geq x_n$ for all n such that $n \geq m$ (That is, x_m is never exceeded by any term that follows it in the sequence.) Note that, in a decreasing sequence, every term is a peak, while in an increasing sequence, no term is a peak.

We will consider two cases, depending on whether X has infinitely or finitely many peaks.

Case 1: X has infinitely many peaks. In this case, we list the peaks by increasing subscripts: $x_{m_1}, x_{m_2}, \dots, x_{m_k}, \dots$.. Since each term is a peak, we have

$x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k} \geq \cdots$.

Therefore, the subsequence (x_m) of peaks is a decreasing subsequence of X.

[5] Implicitly there is at least an element different from r.

Case 2: X has a finite number (possibly zero) of peaks. Let these peaks be listed by increasing subscripts $x_{m_1}, x_{m_2}, \dots, x_{m_r}$. Let $s_1 = m_r + 1$ the first index beyond the last peak.

Since x_{s_1} is not a peak, there exists $s_2 > s_1$such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we obtain an increasing subsequence (x_{s_k}) of X.

It is not difficult to see that a given sequence may have one subsequence that is increasing, and another subsequence that is decreasing." (op. cit. page 78).

In this proof, the data is a real sequence; there is no additional hypothesis. The notion of "peak" is introduced and depending on whether or not there are infinitely many peaks (actual infinity), one can construct either a decreasing subsequence or a finite sequence of peaks. In the former case, we have a proof of the theorem; in the latter case, an iterative process (potential infinity) provides an increasing sequence, and the theorem is proved. It is noticeable that this proof use only the order properties; the ℝ-completeness is not involved, so this theorem holds for example for rational sequences, while BWT does not. Both *actual* infinity and *potential* infinity are involved.

The authors move then to the BWT theorem for sequences for which they provide two proofs, the first one relying on the previous theorem.

"3.4.8 The Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers as a convergent subsequent.*

First proof - It follows from the Monotone Sequence Theorem that if $X = (x_n)$ is a bounded sequence, then it has a subsequence $X' = (x_{s_k})$ that is monotone. Since this subsequence is also bounded, it follows from the monotone convergence theorem 3.3.2 that the subsequence is convergent." (op. cit. p. 78-79).

Here the data is a sequence of real numbers; the hypothesis is that it is bounded. Thanks to theorem 3.4.7, the authors introduce a monotone subsequence of the given sequence. They use, without enouncing it, the assertion that *a subsequence of a bounded sequence is bounded*. The conclusion relies on theorem 3.2.2. (*A monotone sequence of real numbers is convergent if and only if it is bounded*). The completeness of the set of real numbers is required for the proof of the reverse implication. This theorem is among those that can be chosen as *axiom of completeness*. It was at the origin of the creation of irrational numbers by Dedekind, who proved it in his ordered complete system of real numbers (Dedekind, 1963). In this textbook, the axiom of completeness is the supremum one: *Every nonempty set of real numbers that has an upper bound has also a supremum (*i.e. a *least upper bound)* (2.3.6, p.37).

The second proof relies on arguments close to those of the proof by Weierstrass for the version *for sets* by introducing *the set of values of the sequence*.

"Second proof – Since the set of values $\{x_n : n \in N\}$ is bounded, this set is contained in an interval $I_1 = [a, b]$. We take $n_1 := 1$.

We now bisect I_1 into two equal subintervals I'_1 and I''_1, and divide the set of indices $\{n \in \mathbb{N} : n > 1\}$ into two parts:

$A_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I'_1\}$ $\qquad$ $B_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I''_1\}$

If A_1 is infinite, we take $I_2 = I'_1$ and let n_2 be the smallest natural number in A_1. If A_1 is a finite set, then B_1 must be infinite, and we take $I_2 = I''_1$ and let n_2 be the smallest natural number in B_1.

We now bisect I_2 into two equal subintervals I'_2 and I''_2, and divide the set of indices $\{n \in N : n > n_2\}$ into two parts:

$A_2 = \{n \in \mathbb{N} : n > n_2, x_n \in I'_2\}$ $\qquad$ $B_2 = \{n \in \mathbb{N} : n > n_2, x_n \in I''_2\}$

If A_2 is infinite, we take $I_3 = I'_2$ and let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 = I''_2$ and let n_3 be the smallest natural number in $B_{2.}$

We continue in this way to obtain a sequence of nested intervals $I_1 \supseteq I_1 \supseteq \cdots \supseteq I_k \supseteq \cdots$, and a subsequence (x_{n_k}) of X such that $x_{n_k} \in I_k$ for $k \in \mathbb{N}$. Since the length of I_k is equal to $\frac{b-a}{2^{k-1}}$, it follows from Theorem 2.5.3 that there is a (unique) common point $\xi \in I_k$ for all $k \in \mathbb{N}$. Moreover since x_{n_k} and ξ both belong to I_k, we have

$$|x_{n_k} - \xi| \leq \frac{b-a}{2^{k-1}}$$

whence it follows that the subsequence (x_{n_k}) of X converge to ξ." (op. cit. p.79)

In this proof, the data and the hypothesis are the same than for the first proof. The authors introduce the set of values of the sequence; as the sequence is bounded, the set is also; then, an interval is introduced to initiate the dichotomy[6], that provides both a sequence of nested intervals with length converging to 0, and a subsequence of the initial sequence. By the corresponding axiom of completeness, there exists a unique point in the intersection, that is proved to be the limit of the sequence. Comparing this proof with Weierstrass's one enlightens the links between the set version and the sequence version of BWT, and the links with axioms of $\mathbb{R}$-completeness.

The analysis of these five proofs confirms the potential of the BWT to work with undergraduates or prospective teachers on various characterisations of completeness, and on their links. In the second section, we provide some paths for designing a didactical engineering aiming at developing proof and proving skills together with contribution to a better understanding of the completeness of the set of real numbers.

GUIDELINES FOR DESIGNING A DIDACTICAL ENGINEERING

As Mamona-Downs suggested, we consider that introducing undergraduates or secondary mathematics prospective teachers to both versions of BWT and designing a didactical engineering[7] around the analysis of the various proofs would offer the

[6] The proof is done in case this set of values is infinite

[7] For didactical engineering in university mathematics education, see Gonzales-Martins et al. (2014)

opportunity to contrast the convergence behaviour of a sequence and the accumulation points of the underlying set of the sequence, and to contribute to the cognitive development of proof's skills (Dreyfus et al., 2012).

A possible scenario for undergraduates or prospective teachers

The scenario we now present has not yet been implemented but is based on our experience as a university teacher of undergraduates and as a teacher's trainer for prospective secondary mathematics teachers. The analysis of the five proofs in the first section is part of the a priori analysis; by lack of place, we will not go deeper in it in this section, but only indicates the guidelines of the proposal.

Asynchronous work – Reading the paper by Oudot (2017). This paper is online on the French website "CultureMaths" and provides historical and contextual elements.

First activity – In small groups, reading and analysing the excerpt of the proof by Bolzano presented above, including the lemma of §9. Possible questions are:

1. What are the data, the hypothesis, the objects introduced along the proof?
2. In which respect the method developed in the proof is related to the nowadays *proof by dichotomy*?
3. Considering your own knowledge, indicate in which conditions does the lemma of §9 used in the proof apply.
4. According to you, under which theoretical assumptions the proof is valid?

Second activity[8] – In small subgroups – Half of the subgroups works on the two proofs of the sequence version of the theorem provided by Bartle and Sherter (2000); the others work on two proofs of the set version – the one presented in Oudot (2017), and the one by Mammona-Downs (2010). The students are requested to write down an account of their works. Possible questions are:

1. What are the data, the hypothesis, the objects introduced along the proof?
2. Which are the theorems / axioms that are used explicitly or implicitly?
3. Which mode or reasonings do you identify in the two proofs.
4. According to you, under which theoretical assumptions the proof is valid?

Third activity – Collective work

1. Sharing the analysis by oral presentation with slides.

2. Discussing the following questions: where does the assumptions on completeness of the set of real numbers is called for in the different proofs? Are there some intermediate results that would hold in uncompleted ordered fields?

Possible further activities

1. Design and implementation of an algorithm for methods by dichotomy of the proof by Bolzano of the existence of the monotone subsequence and comparison with algorithmic methods in computer science (Meyer & Modeste, 2018).

[8] An alternative for prospective secondary mathematics teachers, is to ask them to provide proofs of the BWT.

2. Implementation of the proofs on automated theorem provers (Hanna & Yan, 2022) and combining pencil/paper proofs and formal proofs (Narboux et Durand-Guerrier, 2022).
3. Discussing the relationships between organising and operative dimensions in the various proofs (Battie, 2009).
4. Discussing infinity issues: actual *versus* potential infinity; other definition of the infinite set leading to alternative proofs (e.g. Eidolon & Oman, 2017).

CONCLUSION

In this paper, we have tried to highlight the opportunities offered by the Bolzano-Weierstrass Theorem, an important theorem in Analysis with a lot of applications, to promote both proof skills and understanding of $\mathbb{R}$-completeness. We have presented guidelines for a didactical engineering to be further implemented and refined.

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Modes of description and indicators of formalism in students' descriptions of eigenvectors

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Abstract. In the study presented in this paper, we investigate students' concepts of eigenvectors in an early stage of their education on linear algebra. The different descriptions used by the students for eigenvectors are analysed with respect to both their chosen representations of the mathematical objects (algebraic, geometric, or abstract), and the indicators of formalism used in these descriptions. We find that while the modes of description presented to them seem to influence their own choice of description, students still show their ability to switch between different representations and descriptions and provide individual concept images. However, some shortcomings concerning formalism and preciseness of their descriptions indicate that some mathematical properties and logical relations in the context of learning about eigentheory require particular attention in teaching and learning activities.

Keywords: Teaching and learning of linear and abstract algebra, teaching and learning of specific topics in university mathematics, eigentheory, concept image, modes of description.

INTRODUCTION

Linear algebra is of great use in many fields such as science and mathematics (Wawro et al., 2018). Over the last few decades, the problems in teaching and learning of linear algebra have received increasing attention by researchers in mathematics education (Dorier & Sierpinska, 2001). Eigentheory, the domain of mathematics concerning eigenvectors, eigenvalues and eigenspaces, is often described as a useful set of concepts across disciplines (Wawro et al., 2018). However, as students need to work with several key ideas simultaneously, eigentheory can be conceptually complex. In $\mathbb{R}^2$ and $\mathbb{R}^3$, eigenvectors can be understood geometrically as arrows that are scaled by the transformation or algebraically as the solutions to the eigenequation, but students may not be able to understand these interpretations from the start (Hillel, 2000; Wawro et al., 2019). Dorier and Sierpinska (2001) suggest that the many representations might contribute to the difficulties faced by students learning linear algebra. Wawro et al. (2018, p. 275) claim that research on the teaching and learning of eigentheory is "a fairly recent endeavour and is far from exhausted". In accordance with that, this study aims to contribute to the research on students' understanding of the concepts of eigenvectors and eigenvalues, by investigating which characterisation of eigenvectors and eigenvalues the participants chose in an early stage of their education on linear algebra. We work with the following overarching research question:

What characterises the students' conceptions of eigenvectors and eigenvalues?

THEORY

Concept image and concept definition

To describe our insight into the conceptions that the participants of our study had in the field of eigentheory, we make use of the terms *concept image* and *concept definition*, as introduced by Tall and Vinner (1981). The concept definition is a verbal definition that explains the concept in a precise and non-circular way (Vinner, 1983). According to Tall and Vinner (1981), it may be the result of rote learning of a formal concept definition, that is, a definition accepted by the mathematical community, often presented in lectures and textbooks. Alternatively, it can be the students' own reconstruction of it, that is, his or her personal concept definition (Tall & Vinner, 1981). For many people, there is also the concept image (Vinner, 1983). Tall and Vinner (1981) describe the concept image as consisting of all the cognitive structures associated with a concept. It can be non-verbal, but it might be translated into words. Thus, the concept image may consist of various representations as well as examples and non-examples associated with a concept. It is individual and dynamic, in contrast to a formal concept definition, which can be considered objective and constant (Tall & Vinner, 1981). One's concept image and concept definition may be more or less overlapping, contradictory or for some people, the concept image may be non-existent. According to Vinner (2002, p. 69), having a concept image is a necessary condition for understanding: «To understand, so we believe, means to have a concept image.». Given this, we argue that describing the students' concept image can, to some extent, provide information of their understanding of these concepts. Using this terminology, the overarching research question could be rephrased as follows:

> *What characterises the students' concept images and concept definitions of eigenvectors and eigenvalues?*

However, as an individual's concept image may be vast and multi-faceted, it is our perspective that it cannot be described in full detail in the scope of this study. Thus, we find it necessary to restrict our inquiry of students' concept images to specific aspects of them. In the following, we will explain our interpretation of Hillel's modes of description, the aspect of formalism, and how these ideas have helped in shaping two supporting research questions.

Modes of description

Hillel (2000) explains that a typical course in linear algebra applies several modes of description to objects and operations, as well as the transfers between them. These include the *abstract*, the *algebraic* and the *geometric* mode, and they can be applied to vector spaces of all dimensions. Within them, vectors and transformations have different terminology, notation, and representations associated with them. The abstract mode of description uses formal language and concepts from the general n-space like dimension, kernel and vector space. The algebraic mode concerns the concepts from the more specific theory of $\mathbb{R}^n$. Here, vectors are n-tuples and key topics include matrices, rank and solving linear systems. In the geometric mode, vectors can be considered as arrows, directed line segments or points, and transformations can be understood as corresponding to spatial actions, like rotations and translations. In this mode, key concepts like orthogonality can be visualised in 2- and 3-space but are used metaphorically in the general part of theory (Hillel, 2000).

The modes are different but not entirely disjoint. According to Hillel (2000), teachers make shifts within and between modes easily and frequently during lectures. However, several researchers have suggested that students struggle to work with these transfers (e.g. Lapp et al., 2010; Sierpinska, 2000; Stewart 2018). In particular, when working with $\mathbb{R}^n$, moving from the abstract to the algebraic mode can be a particularly confusing shift for students (Dorier & Sierpinska, 2001; Hillel, 2000). Hillel (2000) notes that the ability to understand how vectors and transformations can be represented differently within and between modes is key to understanding linear algebra. To further explore this aspect of students' concept images, identifying possible preferences and challenges they may have with these modes and transfers, we add a supporting research question:

> *1. What modes of description do the students use to explain the concepts of eigenvector and eigenvalue?*

The aspect of formalism

Another great challenge for students learning linear algebra is its formal character (Dorier, 2017). According to Dorier (2017), Robert and Robinet conducted research in France in the 1980s, showing that students felt overwhelmed by the many new definitions and theorems, and the students expressed concern with the use of formalism. Dorier et al. (2000) have researched students' difficulties with the generalised part of linear algebra, and they call this *the obstacle of formalism*. According to Dorier and Sierpinska (2001), students also have difficulties with understanding formal concepts in relation to their geometric interpretations. However, it is our perspective that the aspect of formalism needs further conceptualisation. In our study, we chose to define and identify particular elements of mathematical statements as "indicators of (lacking) formalism", as will be worked out in the next section. To further explore the aspect of formalism, we pose an additional supporting research question:

> *2. What indicators of lacking formalism can be found in the students' explanations of the concepts of eigenvector and eigenvalue?*

In this context, we would like to stress that we do not use the term "lacking" in any normative sense here, but only in the function of indicating the absence of something.

METHODOLOGY

Setting and participants

This study took place at the Norwegian University of Science and Technology in Trondheim with first- and second-year students in a basic linear algebra course. The students were majoring in mathematics and mathematics education. The teaching of this course included weekly lectures where the teacher presented key definitions, theorems and relevant examples using the blackboard and/or PowerPoint presentations. In addition, there were optional weekly exercise classes where the students could discuss tasks from the homework with each other and teaching assistants. To gain access to the exam, students had to complete and submit a minimum of eight out of twelve of these exercise sets and have them graded by a teaching assistant. Out of the 243 students who were enrolled in the course, 52 consented to participate in our study. We admit that our results may not be representative for the student body in the course, yet we argue that it is sufficient to say something about trends *within* the group of participants.

To understand the students' concept images and concept definitions of eigenvectors and eigenvalues, we designed four tasks as part of the students' weekly homework and collected the written works of the students. In addition to explaining the concepts of eigenvector and eigenvalue in their own terms, students were asked to determine whether and why statements about eigenvectors and eigenvalues are true or false, as well as use graphic representations to determine whether a given vector is an eigenvector corresponding to a matrix, and why/why not. These tasks were designed specifically to have students' work with multiple representations of eigenvectors, and consequently modes of description, and to test their abilities to move between them. The students were allowed to work on the exercises for one week and all aids were permitted. In this paper, we will only present our analyses of the first task and our focus is on part a): «*For parts a) and b), explain in your own words. You may also use drawings. a) What is an eigenvector? b) What is an eigenvalue?*». The purpose of this open phrasing was to elicit student thinking and learn about their concept images and concept definitions.

Method of analysis

The students' written works were collected through the digital learning platform Blackboard, that was used for the organisation of the whole course, and analysed qualitatively using a thematic coding approach in two rounds, each having a first and a second level. The coding in the first round was inspired by Wawro et al (2019). In the first level, descriptive codes were constructed inductively from single words or short phrases in the students' written answers. Codes such as «scalar multiple» or «transformation» were assigned to trace the modes of description in the students' answers. In the second level, codes were grouped into themes corresponding to Hillel's (2000) modes of description in an interpretative process.

For the coding in the second round, the students' explanations of the concept were compared to an "ideal" formal concept definition from the textbook used in the course, that is, *Elementary Linear Algebra* (2020, p. 291) by Anton Kaul:

> *If A is an $n \times n$ matrix, then a nonzero vector $\boldsymbol{x}$ in $\mathbb{R}^n$ is called an* ***eigenvector*** *of A (or of the matrix operator T_A) if $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$, that is, for some scalar λ. The scalar is called an* ***eigenvalue*** *of A (or of T_A), and $\boldsymbol{x}$ is said to be an* ***eigenvector corresponding to*** *λ.*

This definition contains all necessary specifications of the used symbols and precise relations between the occurring concepts and was therefore considered as fulfilling the highest relevant standard for formalism in the context of our study. We compared the answers of the students with this definition and identified which if these specifications were missing. These "lacks" were considered as "indicators of lacking formalism", and the categories obtained in this process are listed in the next section.

RESULTS AND ANALYSIS

Various modes of description

In this section, examples from students' works will be presented, together with their codes and how they were categorised as relating to the abstract, algebraic or geometric mode of description. For the purpose of this analysis, the tasks and the students' answers were translated from Norwegian to English. As the modes are not specific to eigentheory and the students gave only short explanations, it was necessary to make our own interpretation of this classification and restrict our analysis to single words or short phrases used by students. The codes, their explanation and prevalence obtained in the first round of coding are given in table 1. From a mathematician's point of view, many of these codes are interchangeable. However, we argue that this is not necessarily obvious to students and that realising some of these are interchangeable is related to having an advanced concept image. Some answers are complex, using both symbols and natural language, or connecting the concept of eigenvector to other concepts, while others are more condensed. Consequently, some answers were assigned multiple codes, while others were given only one or two.

Algebraic modes of description: Answers that describe eigenvectors by writing a symbolic definition similar to the one from the textbook, i.e. $A\boldsymbol{x} = \lambda\boldsymbol{x}$, were considered as using an algebraic mode of description. This was also the case for answers that rephrase this relation in natural language, i.e., a discursive definition of eigenvectors. From table 1, it is evident that most students described eigenvectors using the symbolic definition, a discursive definition or a combination of the two. For example, one student wrote: «*A vector $\boldsymbol{x}$ is an eigenvector if you can write $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where A is a matrix and λ is a scalar. More thoroughly explained, $\boldsymbol{x}$ is an eigenvector if a matrix multiplied by the vector returns the vector scaled by λ.*». In this example, the first sentence defines the concept of eigenvector as vectors fulfilling the eigenequation. In the second sentence, the student tries to elaborate by explaining the equation in natural language. As the answer gives both a symbolic and a discursive definition, it was coded accordingly and categorised as having an algebraic mode of description.

Abstract modes of description: Answers that relate eigenvectors to concepts from the more general part of theory were considered to have an abstract mode of description. Table 1 indicates that fewer answers were assigned these codes, as compared to the codes corresponding to the algebraic mode. Out of the abstract codes, «transformation» is the most recurrent within the data material, with 15 compared to 1–3 occurrences. The following answer was coded as «vector space», «image» and «transformation»: «*An eigenvector $v \neq 0$ is a vector in the vector space that doesn't change direction when it's imaged by a linear transformation. This means that if a square matrix is multiplied with this vector, the resulting vector will be a scalar multiple of the eigenvector.*». By describing eigenvectors in relation to multiple concepts from the more formal and general part of theory, the answer contains several elements corresponding to an abstract mode of description. As the student described an eigenvector as «not changing direction» (i.e., maintaining direction) and as a «scalar multiple», the answer also has elements from the geometric mode, and was additionally categorised accordingly.

Table 1: The table explains the 13 codes, how often they occur in the answers of the 52 students and their corresponding modes of description.

Mode of description	**Code**	**Explanation**	**Occurrence**
Algebraic	Discursive definition	Using natural language to explain the eigenequation, $A\boldsymbol{x} = \lambda\boldsymbol{x}$.	35
	Symbolic definition	Description with the eigenequation, $A\boldsymbol{x} = \lambda\boldsymbol{x}$.	31
	Linear system	Connects eigenvectors to the solution of a linear system.	1
	Linear	Describing $A\boldsymbol{x}$ and $\boldsymbol{x}$ as linear.	1
Abstract	Transformation	Description related to the concept of transformation, using the words «transforming», «transformation» etc.	15
	Span	Description related to the concept of span, using the words «spanning», «spans» etc.	3
	Image	Description related to the concept of image, using the words «image», «imaging» etc.	3
	Vector space	Description related to the concept of vector space, using the phrasing «an element of a vector space» or similar.	2
	Transformation definition	Description with the eigenequation in terms of a transformation, e.g. $T(\boldsymbol{x}) = \lambda\boldsymbol{x}$.	1
Geometric	Scalar multiple	Description using the words «scalar multiple», «scaling» etc.	21
	Maintains direction	Describing eigenvectors as vectors that do not change direction.	12
	Dynamic changes in size	Dynamic description, using words like «stretching», «shrinking» etc.	5
	Figure	Included a figure or sketch.	5

Geometric modes of description: Students who described eigenvectors by referring to some visual representation in $\mathbb{R}^2$or $\mathbb{R}^3$ were considered as using a geometric mode of description. This includes answers that described eigenvectors as maintaining direction or as being scaled under a transformation (or matrix multiplication), as well as answers where the student made some sketch showing the relation between the matrix, the eigenvector and the eigenvalue. An answer that was coded as both «maintains direction» and «dynamic changes in size» is the following: «*An eigenvector is a vector such that when multiplied by a matrix [it] won't change direction, but only length.*». In this example, the student correctly described how a matrix may change the length of an eigenvector and how its direction is preserved (however, the option of flipping the vector was not apparent in the students' answer).

Indicators of lacking formalism

The aspect of formalism was evaluated by comparing what was missing from the students' description of the concepts to the formal concept definition in their textbook. Table 2 gives an indication about which lacks were most prevalent in the dataset. The most common lacks among the students' answers appear to be not to mention the dimensions of the eigenvector $\boldsymbol{x}$ (omitted by 42 students), the matrix A (omitted by 28 students) or specify that $\boldsymbol{x}$ may not be equal to the zero vector (omitted by 36 students). From table 2, it is noticeable that the works of most of the students showed several lacks when compared to the formal definition. In the following example, the student correctly explained eigenvector and eigenvalue by referring to the symbolic definition, but the answer has lacks: «*An eigenvector is a vector $\boldsymbol{x}$ that can solve $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where A is a matrix and λ is called the eigenvalue.*». The student did not specify the dimensions of neither the vector («L/vector dimension») nor the matrix («L/matrix dimension»), did not rule out the eigenvector to be equal to the zero vector («L/nonzero eigenvector»), did not state that λ is a scalar («L/eigenvalue unknown») and did not explicitly state that the eigenvector and eigenvalue correspond to the specific matrix («L/eigenvalue vector» and «L/vector-matrix»). In our analysis, we found several examples where it remains unclear whether the student was aware of the connection between the matrix, the eigenvector and the eigenvalue. For instance, one student wrote «*An eigenvector $\boldsymbol{x}$ is a vector that is scaled when multiplied by a matrix.*». This could either indicate that the student thought of an eigenvector as corresponding to a specific matrix, or a misconception that an eigenvector is scaled by every matrix. In another case, the student did not mention the matrix at all: «*An eigenvector is a vector that can be scaled but does not change direction.*». Furthermore, three students gave answers where it is ambiguous if they were aware of all the ways a matrix may act upon its eigenvector(s). For instance, one student wrote: «*An eigenvector is a vector that is stretched either in a positive or negative direction [...].*». This could indicate the idea of an eigenvector as only being stretched (not shrunk etc.) when multiplied by the corresponding matrix.

Use of visual representations: In $\mathbb{R}^2$ and $\mathbb{R}^3$, eigenvectors and eigenvalues have visual representations. The task asked the students to explain eigenvectors and eigenvalues, but they were also encouraged to draw a sketch to illustrate the concepts. Out of the 52 students that participated in this study, only five drew a sketch supplementing their verbal description. Figures 1 and 2 give two examples of such sketches. In figure 1, the student drew a coordinate system and multiple arrows pointing in opposite directions to each other. In figure 2, the student drew the eigenvector $\boldsymbol{x}$ and the vector $A\boldsymbol{x}$ in a coordinate system with scales, as well as the equations $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A\boldsymbol{x} = -2\boldsymbol{x}$, thereby indicating the relationship between the matrix, the eigenvector and the eigenvalue $\lambda = -2$. Given this, the sketch in figure 2 is more detailed than the sketch in figure 1. However, as the sketch in figure 1 is not restricted to a particular eigenvector and eigenvalue, it could be interpreted as more general and dynamic. The student wrote that *«An eigenvector says about how much [sic] matrix stretches/shrinks things in a direction. The eigenvalue is how much [sic] eigenvector stretches/ shrinks.»*. This answer could indicate a developing concept image where the student is able to connect the concept of eigenvector to the geometric idea of scaling. The usage of very informal language here, for instance the usage of the term "things", leaves it unclear what object the student thought is being stretched.

Table 2: The table explains the seven codes for lacks and how often they occur in the answers of the 52 students.

Code	Explanation	Occurrence
L/vector dimension	The student does not specify the dimension of the eigenvector.	42
L/nonzero eigenvector	The student does not specify that the eigenvector cannot be equal to the zerovector.	36
L/matrix dimension	The student does not specify that the matrix needs to be square.	28
L/eigenvalue-vector	The student does not specify that the eigenvalue and eigenvector form a corresponding pair.	27
L/vector-matrix	The student does not connect the eigenvector to a specific matrix.	23
L/eigenvalue unknown	The student does not specify that the eigenvalue is a scalar.	7
L/act	The student does not describe the possible ways (i.e. stretching, shrinking, leaving unchanged, rotating by 180 degrees) in which the matrix may act upon its eigenvector(s).	3
L/matrix	The student does not mention matrix or linear transformation at all.	1

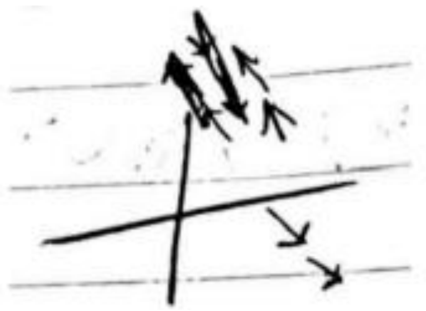

Figure 1: Student sketch apparently depicting the act of scaling.

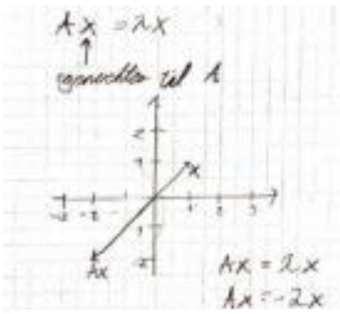

Figure 2: Student sketch showing the correspondence between the matrix, its eigenvector and eigenvalue.

DISCUSSION AND CONCLUSION

The purpose of this study was to gain more insight into students' descriptions of eigenvectors. Most of the students used an algebraic mode of description, which is also the mode the book and the lecture set their focus on. However, several students implemented multiple modes in their answers, indicating the development of their concept images. While the usage of only one mode of description in their answer cannot be considered as a proof of a concept image on a low level of development, we do think that an answer including several modes of descriptions and, even more significantly, some meaningful connections between these modes, can be considered as a strong sign of a further developed concept image. Only a handful of students included a sketch in their answer, despite there being an explicit suggestion to use drawings. This could be because the majority overlooked it, ignored it or perhaps because they did not know how to provide an appropriate sketch. Furthermore, few students connected the concept of eigenvector to the more abstract concepts of image, span or vector space. Of the students who did describe eigenvectors using concepts from the more formal part of theory, the majority used the concept of transformation. However, we wonder if students are aware of the nuances that distinguish a matrix from a transformation.

The works of the students presented a variety of lacks that may or may not result from flawed concept images. However, the results obtained in this study do not allow us to say for certain that these specific students had such misconceptions. It is also difficult to tell to which extent the rather open formulation of our task influenced the formalism of the answers given by the students. Concerning formalism, we got the impression that students are not used to focusing on this aspect in their weekly homework. If a higher level of formalism in the students' works is indeed desired by teachers, it may be constructive to target this shortage by emphasising why formalism is required in mathematical contexts. Furthermore, a discussion (either teacher-student or student-student) or task about «what if» could be productive. For example: «What would happen if we allowed the zero vector to be an eigenvector?».

Our analysis showed that several students gave a discursive rephrasing (as described in table 1) of the eigenequation (i.e. $A\boldsymbol{x} = \lambda\boldsymbol{x}$), omitting aspects of the formal concept definition such as the correspondence between the eigenvector, eigenvalue and its matrix. Other students defined eigenvectors without mention of the matrix or transformation at all. As eigenvectors are derived from their corresponding matrix (or transformation), these answers were interpreted as incomplete. However, whether such incomplete definitions were due to a lack of formalism, some losses caused by the translation from a (possibly non-verbal) concept image to a written description or actual misconceptions remains unclear. In future studies, we will work with similar research questions and address the challenges presented in this paper. We acknowledge that it can be problematic to characterise students' concept images from their written answers alone. We believe that by also analysing the students' answers to the other tasks we designed and conducting interviews with students, we can gain deeper insight into their concept images, and consequently, their procedural and conceptual understanding of eigenvectors and eigenvalues. Building on this, we aim to develop tasks that explicitly address changes between different representations and modes of description.

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Refutation feedback on student proofs beyond counter-examples

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Feedback on students' proofs is often intended to promote proof comprehension, yet formulating such feedback is a delicate task. In this study we investigate how refutation can be used for this purpose. We propose to extend Lakatos' notion of heuristic refutation to feedback that contains a refutation argument, possibly incomplete. Such feedback is heuristic in the sense that interpreting and utilizing it invites mathematical reasoning that can contribute to development of proof comprehension. Based on data from a Real-Analysis course, we show the importance of considering different kinds of refutation that are not based only on counter-examples, and demonstrate the nuances and subtleties of formulating feedback based on such refutation. Our findings suggest how professors can purposefully tailor feedback for particular didactic goals.

Keywords: Teachers' practices at the university level, novel approaches to teaching, professor feedback, heuristic refutation, real-analysis.

INTRODUCTION

Students' engagement with proof outside class is a central aspect of proof-oriented mathematics courses (Rasmussen et al., 2021). Students listen to their professors as they present proofs for the first time in lectures, and are expected to continue studying these proofs after class (Pinto & Karsenty, 2018; Weber, 2012). Weber (2012) observes that mathematics professors expect undergraduate students to do substantial work in order to understand a proof after it was presented in class, dedicating up to two hours to review proofs that are under ten lines long. In addition to reviewing the material that was presented in class, students are typically also expected to invest a substantial amount of time in writing their own proofs as part of their coursework, in order to promote further learning of proofs presented during the lectures (Rupnow et al., 2021). Feedback on students' proofs is widely recognized as having a key role in shaping and facilitating student learning between lectures (Moore, 2016; Pinto & Karsenty, 2018; Rasmussen et al., 2021). According to Moore (2016), professors' feedback on students' proofs is important for conveying norms and expectations, and for directing students' attention towards certain facets of the material, thus promoting proof comprehension.

While there has a been surge of empirical research of undergraduate mathematics teaching practices over the last decade, most attention has been given to what transpires inside classrooms, and research of professors' feedback on proofs that students submit as part of their coursework is fairly limited (Rupnow et al., 2021). Moore (2016) investigated professors' grading of students' proofs, and found that substantial variation in the scores assigned to similar proofs reflected to what extent flaws in the proofs indicated flawed comprehension. Moore's findings were corroborated and elaborated by Miller et al. (2018), who also highlighted the link between proof grading and students' apparent comprehension of the proof. Both Moore (2016) and Miller et al. (2018) concluded that when grading students' proofs, instructors were assessing not only the correctness of the proof or whether it adheres to the norms of proof writing, but also (and mainly) students' proof comprehension.

There is preliminary evidence that, similarly, feedback on proof aims to promote proof comprehension (Byrne et al., 2018; Pinto & Karsenty, 2018, 2020). Evidently, professors often opt to leave the required revision – or even the flaw – implicit in their feedback, and instead highlight certain locations in the proof, ask eliciting questions or request elaborations (Moore, 2016; Byrne et al., 2018; Pinto & Karsenty, 2020). Such feedback may be viewed as an invitation for the student to engage with the flawed proof not only to correct it, but also to promote proof comprehension. However, Byrne et al. (2018) examined students' interpretations of six types of feedback and found that when instructors' feedback did not explicitly provide the required revision, students typically failed both to identify the flaw in their proof and to understand how they are expected to revise it. Conversely, Byrne et al. (2018) observed that when feedback was explicit regarding the desired revision, students often followed the prescription in the feedback without being able to explain what was flawed in their original proof or how the revision addressed flaws.

In this paper we focus on a type of feedback not examined by Byrne et al. (2018), which was used extensively and specifically for promoting proof comprehension in a course we examined in previous studies (Pinto & Karsenty, 2018, 2020). The professor in this course, whom we call Mike, opted to provide students with incomplete refutations of their proofs, arguing that such feedback often enables students to identify and fix the flaw in their proof almost on their own, while also promoting comprehension of the proof. We recognize in student engagement with this kind of feedback an interesting and potentially useful extension of the notion of heuristic refutation (de Villiers, 2010; Komatsu & Jones, 2021; Lakatos, 1976; Pinto & Cooper, under review). Furthermore, we observed that Mike used various kinds of refutation arguments, suggesting that heuristic refutation can be extend beyond the case of counter-examples considered in the literature. In this paper we introduce the notion of *heuristic refutation feedback* (HRF) and explore the following question: What kinds of refutation, beyond counter-example, can be formulated as HRF, and how?

HEURISTIC REFUTATION FEEDBACK

The intricate connection between proof and refutation has a long and respected history, originating from the works of Lakatos (1976) that highlighted how an interplay between proving and refuting can generate new mathematical knowledge (Komatsu & Jones, 2021). Drawing on works of Lakatos (1976) and de-Villiers (2010), Komatsu and Jones (2021) use the notion *heuristic refutation* for a mathematical activity that goes back and forth between conjecturing, attempting to prove, discovering counter-examples, and revising the conjecture, its proof, or even the definition of mathematical objects at stake. Komatsu and Jones (2021) stress that the term heuristic comes to emphasize the revision activity that stimulates growth of knowledge following the refutation, not the refutation itself. Counter-examples play a key role in the heuristic refutation process by providing a trajectory for moving forward: the conjecture or the proof needs to be revised so as to neutralize the counter-example.

Both Komatsu and Jones and de-Villiers follow Lakatos in addressing a process of discovery in which a statement is not known to be true and may end up being revised if refuted. Lakatos (1976) called refutation of the statement *global* refutation, as opposed to *local* refutations, which challenge only one step in the proof or some aspect of the domain of validity of a statement. We wish to extend the notion of heuristic refutation to students' flawed proofs. Here, the global statement is known to be valid and will not be revised, yet we will claim that a particular kind of feedback on flawed proofs can invite students to engage in heuristic growth of knowledge when making sense of the feedback, reviewing their flawed proof with respect to the feedback, and attempting to revise their proof. To draw students into such heuristic activity, the feedback would need to unequivocally show that the proof is invalid, while leaving space for heuristic activity. Accordingly, we define *heuristic refutation feedback* (HRF) as any feedback on a flawed proof that contains a mathematical argument that indirectly implies that the proof is invalid. Here we extend the notion of global refutation to include cases where the refutation does not invalidate the statement, yet does reveal a structural failure in the proof, indicating that a local fix may not suffice, and that a different approach may be required. We emphasize that this definition extends Lakatos's and de Villiers' notion of *heuristic refutation,* which refers solely to engagement with counter-examples (local or global). While the literature recognizes different types of proof, which may have "diverse pedagogical properties and didactic functions in mathematics education" (Hanna & de-Villiers, 2008, p. 332), literature on refutation is generally restricted to counter-examples, and little is known about other kinds of refutation and how these may be utilized in teaching, particularly in the context of heuristic refutation.

Our definition of HRF extends the notion of heuristic refutation also in how the refutation argument may formulated. By definition, HRF contains a (possibly incomplete) refutation argument. The reconstruction of an incomplete argument can be a challenging task that entails inference and invention of implicit connections between the feedback and the proof. As such, it can be seen as a case of abductive reasoning, and illustrated with Toulmin's (1958) model of arguments, as discussed by Komatsu and Jones (2021). In this model, an argument includes, among other things, a claim (C), datum (D) that supports the claim, and a warrant (W) that describes how the datum supports the claim. Abductive reasoning, as discussed by Komatsu and Jones (2021), is a process that seeks to explain a surprising observation (claim) through inference of hypotheses (data) and recognition or invention of warrants. Komatsu and Jones (2021) distinguish between three types of abductive reasoning, according to whether students need to complete the missing datum or provide a missing warrant. While in the classroom activities Komatsu and Jones considered the teacher has a key role in orchestrating student reasoning, in the context of HRF, the only way to facilitate heuristic refutation is through careful formulation of the feedback. This includes not only selecting what to refute in the student's proof, and constructing an adequate refutation, but also deciding which elements of the refutation argument to present and how. Thus, we extend Komatsu and Jones's use of Toulmin model in two ways, by allowing each of the elements of the refutation argument, including the claim, to be stated implicitly, not only completely omitted. We exemplify and discuss these variations below.

METHODOLOGY

Data for this study were collected in a Real-Analysis course. The professor (Mike) is a mathematician who has been teaching for more than two decades and has taught this specific course several times. Prior research on Mike's goals and expectations with respect to this course (Pinto & Karsenty, 2018) revealed that he intended his feedback on students' proofs to be restricted as much as possible to refutations, stating that this kind of feedback affords opportunities for students to develop proof comprehension by looking for their own errors and for ways to correct them; develop the practice of testing their reasoning by trying to refute it; and develop a notion of validity of a proof that is absolute and independent of the professor's personal inclinations. Every week, Mike assigned a list of propositions to prove and examples to construct. Students submitted these proofs and examples electronically every few weeks. Mike did not grade students' submissions but provided written feedback. Seven students volunteered to participate in this study. The data corpus included a total of 57 submissions (5-12 submissions per student), and Mike's 2709 markings and comments.

Our first step in the analysis was to review all Mike's feedbacks to locate those that qualify as HRF and identify the refutation arguments therein. Often the feedback did not provide a fully argued refutation. In some cases, it was first necessary to identify what was being refuted, in particular in cases where the feedback refuted an implicit statement in the student's proof. When disagreements between the authors arose, they were discussed until agreement was achieved. Drawing on Toulmin's model of argumentation (1958) and on its application to heuristic refutation (Komtsu & Jones, 2021), we decomposed refutation arguments into three components (*Claim*, *Datum*, *Warrant*). As refutation argument typically coincided to some extent with arguments in the students' proof, we occasionally applied Toulmin's model to arguments within the students' proofs as well. Comparing the arguments in the feedback and in the proofs helped highlighting subtle and nuanced aspects of Mike's formulation of HRF. In this paper we focus on HRFs in which the datum was not a counter-example and show how Mike used different kinds of refutation while formulating HRF.

FINDINGS

At the beginning of the course, Mike defined the real numbers ($\mathbb{R}$) as an extension of the rationale numbers ($\mathbb{Q}$) that includes non-repeating decimals and asked that students will prove at home that this definition implies that every non-empty bounded subset of $\mathbb{R}$ has a least upper bound (Proposition 1). Mike then showed in class how Proposition

1 implies that ℝ is connected. One student, Alex, wrote a proof of Proposition 1 that relied implicitly on the connectedness of ℝ, and Mike provided the following HRF:

HRF1	Your argument is not only wrong, but it cannot be fixed, because you are not using any definition of real numbers, and hence whatever you write applies to ℚ [the rational numbers], for which the whole statement is false.

In this feedback, Alex not only learns that the proof is flawed, but is invited to verify that the theorem does not apply to rational numbers, then retrace the line of reasoning in the proof, only with rational numbers instead of real numbers, recognize where the connectedness of ℝ is implicitly used, and realize that the proof of connectedness of ℝ relies on the proposition she is trying to prove. While we have not included Alex's proof, we note that refuting it with a counter-example would not be straight-forward, since the proposition is correct, and the proof does not contain a false statement. Mike's feedback does not specify the critical flaw in Alex's proof or the required revision, but in inviting Alex to revisit the definition of real numbers, it provides her with a route for recognizing the flaw on her own. We refer to feedback that derives a false statement by adapting the proof or a part of it as *refutation by false implication.*

We stress that HRF need not rely on *global* refutations, as can be seen for example in the case of the Extreme Value Theorem (EVT), which posits that if *f* is a real-valued continuous function on a closed bounded interval *I* then *f* attains both a maximum and a minimum in *I*. In the lecture, Mike emphasized that EVT is not as obvious as it may seem, noting that an analogous statement for rational-valued functions would not be true, even if restricted to polynomials. In the home assignment, students were asked to prove that there exists a polynomial *f(x)* with rational coefficients that does not achieve an extremum when restricted to rational values in the interval *I=[0,1]* (Proposition 2). All the students' proofs of Proposition 2 roughly followed the same proof scheme, which can be described as follows:

- Step 1. Define a cubic polynomial *f* with rational coefficients.
- Step 2. Identify the critical points of *f* and ascertain that one or both are irrational.
- Step 3. Conclude that as a rational-valued function, *f* does not attain both a maximum and a minimum in *I*.

Notably, Step 1 and Step 2 do not necessarily imply Step 3, since *f* may attain its (global) maximum or minimum in *I* at the rational endpoints of *I*. Moreover, in general, polynomials may have both rational and irrational critical points. Thus, Step 3 should be warranted by showing that the cubic polynomial *f* achieves its maximum or minimum in *[0,1]* at the irrational critical points identified in Step 2. Most students left the warrant for step 3 implicit, as illustrated for example in Bailey's proof. Bailey defined $f(x) = \frac{7}{4}x^3 - 2x^2 + \frac{1}{6}x$, calculated the roots of *f'*, and stated:

> Utilizing the quadratic formula, we see that the roots of the derivative of this function are [...] irrational. It is clear that the maximum and minimum occur at these irrational roots,

and thus the polynomial does not achieve its maximum or minimum value for $x \in Q$ over the interval between 0 and 1.

Mike responded on Bailey's proof in the following way:

HRF2	*f(x)=x* does not have roots of the derivative (even among real numbers!) but it does achieve its maximum and minimum values on *[0,1]*.

Notably, the polynomial Bailey proposed achieves its extrema in *I* at irrational points, and therefore has no maximum or minimum when restricted to $I \cap \mathbb{Q}$. Moreover, the proof does not contain an explicit false statement. Nonetheless, the implicit warrant may be incorrect. Mike's feedback retraces Bailey's line of reasoning, drawing on the same data – the derivative of *f(x)* has no *irrational* roots – while replacing the polynomial *f* proposed by Bailey with *f(x)=x*, thus seemingly reaching a proof to an analogous statement, which is nonetheless false. Unlike HRF1, here the refutation is *local* as the proof could be readily fixed by adding an explicit (correct) warrant.

We note that formulating HRF to Bailey's proof entails attributing a flawed warrant to the unwarranted claim "*It is clear* that the maximum and minimum occur at these irrational roots". The warrant Mike's feedback attributes is: *For every polynomial f, if f'(x) has no rational roots then f(x) does not obtain a maximum or minimum in* $I \cap \mathbb{Q}$. We note that Mike could have chosen to attribute other warrants, for example: *For every closed bounded interval [a,b], if f'(x) has no roots in [a,b] then f(x) does not obtain a maximum in* $[a, b] \cap \mathbb{Q}$. In fact, this alternative path is reflected in Mike's HRF to Adrian's proof, which presented a line of argument very similar to Bailey's:

HRF3	These are critical points, but what makes you think that the maximum and minimum values of this f are achieved on [0,1] at these points? The points do not even depend on the interval! Do you mean that the maximum and minimum values on every interval [a,b] are the same? But this cannot be, because the polynomial is unbounded both above and below.

Attributing a false warrant to justify an unwarranted claim in a student's proof is not the only way to formulate HRF. In some cases, Mike's feedback altered data used explicitly but invalidly in the student's proof, as evident in his feedback to Dylan's proof of Proposition 2. Dylan, defined $f(x) = \frac{2}{3}x^3 + 2x^2 + x$ and stated:

To show [Proposition 2] we can demonstrate that the maximum value of the polynomial in this interval has no corresponding point in the specified domain. Since neither solution [of the equation *f'(x)=0*] is rational, we conclude that no least upper bound exists.

Mike responded to Dylan's proof in the following way:

HRF4	Note that both values of x [in which *f'(x)=0*] are outside the interval [0,1]. Thus, according to your logic, the range of your function does not have the least upper bound even over the real numbers. Contradiction?

Dylan's proof is different from Bailey and Adrian's proofs in that the polynomial it proposed is in fact a non-example, as the roots of *f'* reside outside the interval *I,* which implies that *f* is monotonic on *I* and thus obtains both its minimum and its maximum at the (rational) endpoints of *I.* Rather than refuting Dylan's proof by a counter-example, Mike's feedback drew on the misused data – the critical points – highlighting that their irrationality plays no role in the proof, and therefore an analogous line of reasoning could be applied to the same data only with *f* as a real-valued function, and reach the same conclusion in contradiction to EVT.

The cases considered so far admitted a wealth of refutation arguments. But, in some cases, formulating refutation feedback was not straight forward. For example, Charlie's proof defined $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x$, found a root *x* of *f'(x),* and stated:

> You can see that the first derivative equals 0 at x (x is between 0 and 1). The second derivative is positive at x, indicating that we've found a minimum in this interval, and x is not rational. However, the second derivative equals 0 at $x = 0$. In order to ensure that we have found a minimum for the interval *[0, 1]*, we still need to check that the value of *f(x)* is less than *f(0).*

Unliked the proofs of Alex, Bailey and Dylan, Charlie's proof provides an explicit warrant to why the irrationality of the critical point of *f* implies that *f,* as a rational function, does not obtain a minimum in the interval *I.* Charlie checks the sign of the second derivative of *f* at the irrational critical point $x \in I$, and rightly concludes that *x* is a local minimum of *f.* However, Charlie also notes that the second derivative of *f* is negative at every point of *I,* except that it vanishes at zero, and wrongly argues that in order to show that *x* is a minimum of *f* in *I* it is necessary and sufficient that *f(x)<f(0)*. Notably, whereas Charlie's line or reasoning is not valid, its conclusion for the particular *f* and *I* is true: *x* is indeed the unique minimum of *f* in *I,* and 0 is the unique maximum. Thus, showing that Charlie's reasoning is not <u>always</u> true entails the non-trivial task of constructing (or suggesting the existence of) an example in which all the data Charlie drew on can be used in the same way, leading to a false conclusion. Mike's feedback does just that:

HRF5	Apparently you see some connection between the sign of *f''(0)* and extremal values. Here is a counterexample: Consider f on the closed interval *[0,10]*. It has no local maxima, its 2nd derivative is positive on *(0,10]*, and *f(0)=0* is not a maximum, since, say f*(2) = 5/3 > 0*. Thus, according to your logic, the function does not achieve a maximum value on *[0,10].*

By expanding the interval from *[0,1]* to *[0,10]*, Mike's feedback demonstrates that the maximum of *f* is not necessarily achieved at a point where the second derivative is non-negative. The warrant Mike's feedback attributes to Charlie's proof and refutes is: *The minimum of f in I is achieved at points in which the sign of f'' is not negative*. Notably, this warrant is alluded to at the beginning of Mike's feedback to Charlie: "Apparently you see some connection between the sign of f''(0) and extremal values".

So far, we have examined HRF based on *global* and *local* refutations by false implication. Another type of refutation argument Mike used in his feedbacks is *refutation by contradiction.* This refutation argument, similarly to proof by contradiction, first assumed the student's proof or some statement therein is correct, only to reach a contradiction to the theorem that needs to be proved (and that is known to be true). For example, in one of the lectures, Mike presented the Peano curve as the limit of a sequence of curves $\phi_n: [0,1] \to [0,1]^2$. The curve ϕ_1 is defined as the curve that starts from the point (1,1), moves along the four edges of the unit square until it returns to (1,1), and then moves along the diagonal of the unit square to the point (0,0). The curve ϕ_{n+1} is defined reclusively by replacing every diagonal line in (the image of) ϕ_n with a curve that consisted of 8 parts, as illustrated in Figure 2. Points on the unit interval mapped to horizontal or vertical segments on the Peano curve were labeled by 's'.

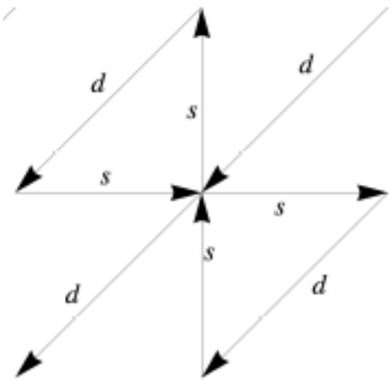

Figure 2: Construction of the Peano Curve

The students were asked to prove that the Peano curve is surjective. One student started the proof by claiming that "every point on the unit interval eventually falls into an interval labelled as *s*". Mike responded with the following feedback:

HRF6	All 's' points are mapped to points of the square with one coordinate rational; so, they don't cover the square. This contradicts the theorem.

Taking (implicitly) the student's claim as data, HRF6 posits that points of type 's' are mapped to plane points with one rational coordinate. Combining these two pieces of data together leads to the false conclusion that the Peano curve is not surjective.

We conclude by pointing out that refutation by counter-example can be seen as a case of refutation by false implication. To illustrate this, we return to Proposition 1. One student stated in the proof that "*every closed set [of the real numbers] can be written as the union of only finitely many closed intervals*". Mike addressed this statement:

HRF7	This is bluntly wrong: The Cantor set does not contain a single interval, but it is uncountable.

Here, the Cantor set is given as a counter-example to the false statement. At the same time, the argument can also be read as a false implication: applying the statement to the Cantor set implies that it is a finite union of closed intervals, which is absurd. We stress that the converse is not true, since, as demonstrated above, refutation by false implications can be used to refute proofs that do not admit false statements, and thus cannot be refuted by a counter-example.

DISCUSSION

This study, situated in the under-studied area of undergraduate mathematics teaching and learning outside class, aims to unpack nuances of a particular practice of undergraduate mathematics teaching – providing written feedback on students' flawed proofs. The work is guided by the premise that feedback on students' flawed proof can support development of proof comprehension rather than merely support the writing of correct proofs. We have proposed heuristic refutation feedback (HRF) as an extension of heuristic refutation (de Villiers, 2010; Lakatos, 1976) to conceptualize the activity of interpreting and utilizing refutation feedback on flawed proofs, and have demonstrated that formulating such feedback can be a delicate and thoughtful practice.

We have extended the notion of heuristic refutation (de Villiers, 2010, Komatsu & Jones, 2021) in two ways. First, we go beyond the notion of abductive reasoning (Komatsu & Jones, 2021), where a claim based on observation was taken as given and it is up to students to propose datum and/or a warrant, and consider the heuristic activity of completing a refutation argument that may contain only a claim, only datum, only a warrant, or any combination thereof. Second, we go beyond refutation by counter-example to consider refutation by false implication. In this we are extending the space of pedagogical applications of refutation in mathematics education.

Formulation of HRF was shown to involve several different pedagogical decisions that relate to the construction and selection of the refutation argument. Typically, students' flawed proofs can be refuted in more than one way. There is often more than one flaw in a flawed proof, and different flaws may indicate different issues of proof comprehension. Thus, formulating HRF may entail a decision about *what* to refute. Presumably, and this needs to be studied further, highlighting different flaws can provoke different engagement of students with their proofs, their flaws, their revisions and their comprehension thereof. Formulating HRF entails also decisions about *how* to refute. We have delineated two kinds of refutation arguments – *refutation by false implication* and *refutation by contradiction*, in addition to the familiar *refutation by counter-example*. We have demonstrated that in some cases, more than one kind of refutation argument is applicable. The literature suggests that different kinds of proof have different pedagogical advantages and afford different opportunities for learning (Hanna & de Villiers, 2008), and further research is need to investigate whether and in what sense this is also is also true for different kinds of refutation.

We have demonstrated how in some cases proofs may be flawed even if they do not contain an explicit invalid statement. Such proofs cannot be refuted directly by means of a counter-example, yet they may be refuted by identifying (or attributing) a flawed warrant and invalidating it. We have demonstrated how different warrants may be attributed to the same flawed proof, and lead to different HRF. We note that formulation of HRF entails also decisions about the extent to which different elements of the refutation argument (claim, datum, warrant) are explicated in the feedback (explicitly, implicitly or omitted). This aspect of the formulation of HRF is discussed in detail in a separate publication (Pinto & Cooper, under review). Thus far, the potential affordances of HRF for proof comprehension have only been substantiated theoretically (Pinto & Cooper, under review). Empirical research on how students engage with this kind of feedback, and how this engagement can contribute to the development of proof comprehension remains for future research.

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Objectification processes in engineering freshmen while jointly learning eigentheory

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In this paper we present the first results of an ongoing PhD study which investigates eigentheory teaching and learning processes. Drawing on a sociocultural theory, namely the theory of objectification, we study students' collective meaning-making processes. A specifically designed activity, aimed at supporting these objectification processes, is described. University engineering freshmen, working in small groups, are prompted to jointly reconceptualize eigentheory notions and rules and to solve some problems. Then a few excerpts of one small group's work are presented and analysed with a focus on students' use of different semiotic resources, their mutual relationship and evolution.

Keywords: teaching and learning of specific topics in university mathematics, teaching and learning of linear and abstract algebra, eigentheory, objectification, embodiment.

INTRODUCTION

Linear algebra is widely recognised to be a major obstacle for university freshmen. A growing body of literature has investigated the sources of these difficulties and the way students comprehend linear algebra concepts. Nevertheless, only a small number of studies has focused on eigentheory teaching and learning processes, despite its importance in different applications in STEM subjects. This paper describes the first results of an ongoing PhD project, concerning the didactics of this specific topic.

As described by Stewart & Thomas (2006), when eigenvector and eigenvalue concepts are introduced to students, the focus is turned too soon to the manipulation of algebraic representations. In a standard instructional sequence, the formula to compute eigenvalues, i.e. $det(A - \lambda I)x = 0$, follows their formal definition almost without delay. Immediately after, the algorithm to compute the eigenvectors associated to each eigenvalue is given. We agree that in this way students are provided with a trusty procedure and do not feel the need to elaborate further these concepts' definitions. As a result, "the strong visual, or embodied metaphorical, image of eigenvectors is obscured by the strength of this formal and symbolic thrust" (p.185). Most of the few studies concerning this topic, agree on the fact that consequently students prefer to rely on the standard algebraic procedure rather than draw on conceptual understanding to solve exercises and problems (Bouhjar et al., 2018; Salgado & Trigueros, 2015). Nevertheless, some of these researches bring evidence on how students' understanding of eigentheory could be enhanced by the use of dynamic-geometry software (Gol Tabaghi & Sinclair, 2013), inquiry-oriented instruction (Bouhjar et al., 2018; Wawro et al., 2019) or modelling activities (Salgado & Trigueros, 2015). However, the comprehension of how students develop and coordinate the interpretations needed for a deep conceptual understanding of eigentheory is not so clear and deserves further investigation (Bouhjar et al., 2018).

This research tries to fill this gap, analyzing how students collectively reinterpret an introductory standard frontal lecture on eigentheory, in order to construct a robust meaning for the presented concepts. We build on a sociocultural theory on mathematics teaching and learning, namely the Theory of Objectification (Radford, 2021). Hence, we are particularly interested in collective forms of knowledge production, with a focus on their multimodal features (Arzarello, 2006; Radford, 2014).

THEORETICAL FRAMEWORK

Radford (2010, 2021), defines the process of objectification as "the process through which cultural knowledge (Objekt) is progressively transformed into an object of consciousness" (Radford, 2021, p. 99). Students must engage in suitable activities in order to be able to transform cultural knowledge into *knowing* (p. 49). Through this activity, the student has the chance to encounter and attend mathematics as a cultural-historical system of thinking. This encounter does not happen all of a sudden but must be considered as a process; a process which is highly determined by the student's effort to attend the object of knowledge. The word *Activity* in the theory of objectification "refers to a dynamic system where individuals interact collectively in a strong social sense" (p. 29). To distinguish this specific formulation from activity as merely meaning "doing something", the notion of *joint labour* has been introduced (Radford, 2021). In joint labour, the acts of teaching and learning are not distinguished from each other any longer. In particular, students do not passively receive the knowledge in an "alienated" form of learning, but actively take part, through collective work, to the production of cultural social knowledge. Joint labour not only includes language as a mean for collective activity, rather encompasses the agency of body, matter, movement, rhythm, passion and sensations. Indeed, in order to become objects of consciousness, concepts must be actualised through material, sensuous activities (Radford, 2014). During the objectification process, students and teachers resort to multiple semiotic resources: written symbols, uttered and written words, diagrams, gestures, etc. These, together with object and tools, are intentionally used in social meaning-making activities in order to carry out actions aimed at fulfilling the goal of such activity: in Radford's theory (2001) they are called *semiotic means of objectification*. Since we are interested in analysing how these different signs jointly contribute to the process of knowledge objectification, they must be looked at in an integrated and systemic way, with attention to relationships and dynamics between them (Radford & Sabena, 2013).For this reason, methodologically speaking, it is important to analyse *semiotic nodes*, namely those segments of students' activities, in which different kind of semiotic resources intertwine and play a key role. In this investigation we emphasise the importance of the genesis of new signs, their evolution and the evolution of their mutual relationships in the process of objectification. Hence, we adopt the notion of *semiotic bundle*

(Arzarello, 2006), in order to perform an analysis of students' sign production and their evolution in time. It allows to have a more precise view on the way objectification is occurring. A semiotic bundle has been defined as:

> a system of signs […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello et al. 2009, p.100).

In this study, specifically, we will use two theoretical constructs originating in the field of gestures study, namely those of *growth point* and *catchment* (McNeill, 2005), to show how the evolution of the relationship between gestures and other semiotic resources can provide information about students' cognitive processes. A growth point is a cognitive mechanism that integrates linguistic and imagistic components (McNeill, 2005) and in a discourse is identified as "the starting point for the emergence of noteworthy information prior to its full articulation" (Arzarello et al., 2015, p. 22). The information condensed in a growth point could be progressively unpacked through a catchment, defined as an observable sequence of recurring gestural imagery (McNeill, 2015). Arzarello and colleagues (2015) have shown how catchments are produced by students in meaning-making processes of a new mathematical concept (Arzarello et al., 2015).

RESEARCH AIM AND METHODOLOGY

The investigation here presented has been conducted in an Italian public university in the fall term of 2021. The aim of the study was to analyse if/how students can objectify the concepts of eigenvector and eigenvalue, while engaged on joint labour in a specifically designed activity. Data were collected in three different linear algebra and geometry courses offered to first-year engineering students; in the Italian curriculum this is the unique linear algebra course offered to students in their first year of engineering studies, and covers standard vector-space theory (approximately: vector spaces, matrix algebra, linear systems, eigentheory, euclidean spaces). In total 64 students attended the activity and they worked divided in small groups of three, or in a few cases four students each. Sheets of paper used were collected for all the groups, while eight of them were video-taped during the whole activity. This last kind of data was necessary to collect, considering the theoretical framework that we have outlined. Indeed, from a methodological point of view, "the identification of the semiotic nodes and the semiotic means of objectification mobilised by the teacher and the students provides a kind of window to the investigation of objectification processes" (Radford, 2021, p. 106). We made sure that the recordings would capture not only the whole discussions, but also gestures and gazes produced by the students.

Activity design

As previously emphasised, activity is a key component of the objectification process. Even more, it is a key component of the investigation of this process, meaning that the design of an appropriate activity not only can support the process of meaning-making, but can also provide the observer with important information about how this process occurs and develops (Radford and Sabena, 2015). Another key component of the theory of objectification is classroom interaction, and this is why we shaped our activity as a small-groups work.

Because of institutional constraints - among others, the deeply-rooted habit in Italian engineering firs year courses of performing traditional blackboard frontal lectures and the extremely high number of attending students (around 200) per course - we had to accommodate the planning of our activity to the standard schedule of the linear algebra courses involved in the research and were not able to plan the activity as a first introduction to the topic. Consequently, we decided to perform a pilot study after the teachers would have conducted their frontal lecture of introduction to eigentheory. Because of this, we designed the first part of the activity as a collective review of the lecture to be performed during a two-hours tutoring class, which occurred a few days after the teacher's introductory lectures on the topic. The activity was guided and attended by the course tutor and/or the researcher author of this paper. We prepared guidelines that could direct the small groups in the meaning-making process. These guidelines comprised very open questions such as "How would you explain the concept of eigenvector to someone who has never heard of that before?". Students were not specifically asked to answer the question in a written or oral form, but could freely benefit from trying to answer to these questions in order to jointly making sense of eigenvalue and eigenvector concepts. They were free to use any tool and encouraged to use other resources that they had encountered, besides the book or notes taken during the lessons. In fact, the teachers of all the three courses had shown or suggested to use a GeoGebra applet to explore eigenvectors in two-dimensional space and to watch some videos about this topic retrieved from the web.

For the second part of the activity, we prepared a set of five problems. In this paper we focus on student's engagement in the first part of the activity, while students' solution strategies to the problems are left for future works. For this reason we will not further elaborate here on the design of the problems.

Research questions

Considering the outlined theoretical framework, we can phrase our research questions as follows:

1. Can our designed activity trigger and support first year university students' objectification process of eigenvector and eigenvalue concepts, and if so, how?
2. What information can the analysis of the evolution in time of the semiotic means of objectification mobilized by students give about these objectification processes?

For space reasons, we will limit to the description and analysis of one small-group's work, which we consider as illustrative of a trajectory for the objectification process towards eigenvectors and eigenvalues: we refer to it as Group 1. We will present three particularly significant extracts from their first part of activity and describe key semiotic nodes in their objectification process.

Tackling obstacles with the definition of eigenvalue and the formula $Ax = \lambda x$

The three students start from the guiding question "What are eigenvalues and eigenvectors and how would you explain these concepts to someone who is following a linear algebra course but still has not encountered this topic?"

They decide to write the answers on a sheet of paper and one student, that we will call A, takes on the task of writing. They glance at their lecture notes and start focusing on the term "eigenvalue". At the beginning, they seem to focus on writing a correct definition of the term, without really trying to make sense of the concept or to look for specific and possibly clear examples.

A: So I would say, starting from eigenvalues, that eigenvalues are values that can represent a linear transformation with a number.
B: Yes
A: Via a value …
B: Yes, at the end, if you think about it, if I'm not wrong, it is like multiplying the matrix of the associated function …

Student B, immediately starts focusing on procedures to find eigenvalues and A stops him and goes back to trying to find a definition. They keep looking for a reasonable definition until B's intervention leads them to facing another conflict:

B: because λ can be a 2x1 matrix
A: [thinks about it some seconds] No, λ is just a number
B: eh!
A: λI is the matrix
B: yes, ok, but you can think about λ also as a matrix, can't you?

The two students discuss about this conflict, each persuaded by his own idea. After a while, B understands that he is not able to make A understand his point with verbal language only. He starts writing formulas on his tablet. This is a first significant semiotic node to be analyzed in the group's activity. He insists on the fact that when finding the image of a vector, $f(v)$, a matrix that he calls M must be multiplied by that vector. He links then this idea to the formula used by the teacher and the textbook to define eigenvalues, namely $f(v) = \lambda v$. He correctly deduces the equality $Mv = \lambda v$, but interprets it as if λ must be a matrix as well, for the equality to stand. Stewart and Thomas (2006) have indeed described how the use of this formula can be a source of difficulty for students:

> One serious problem with Ax = λx for students is that the two sides of the equation are quite different processes, but they have to be encapsulated to give the same mathematical

> object. In the first case the left hand side is the process of multiplying (on the left) a vector by a matrix; the right hand side is the process of multiplying a vector by a scalar. Yet in each case the final object is a vector that has to be interpreted as the product of the eigenvalue and its eigenvector. (p. 186)

B's explanation of why he thinks that λ could be a matrix, shows how he has encountered this misconception. A seems to understand the reason of B's error, and tries to solve the conflict by rewriting the equality as $Mv = \lambda Iv$, so to make clear that λ is a scalar, while λI is a matrix. He keeps using this formulation from that moment on. We cannot say from the analysis of the rest of this segment of activity if B has understood his error; surely, as argued in Stewart and Thomas' work, reasons behind and ways to avoid this misconception need to be better studied.

As we have shown in this subsection, students struggle in finding a suitable verbal definition of Eigenvalue. In our opinion, their difficulty might be due to the fact that, ontologically speaking, it is challenging to think of an eigenvalue before even considering the existence of a linear transformation and of eigenvectors. In the following subsection, we will see how the comprehension of what an eigenvalue is can be supported by a geometric context. In fact, in it, we can define a linear transformation, and what happens to different vectors under its effect becomes more tangible.

Picturing a geometric example and gesturing as a meaning-making tool

An important shift in the advance of the activity, occurs when B suggests to use an example. In particular he suggests to consider an example offered by the teacher during the lecture. He refers to the teacher using a GeoGebra applet to explore and show the students a possible representation of eigenvectors in the two-dimensional space. Student C, who had not particularly got involved in the first part of the discussion, suddenly appears interested. He tries to recall the way eigenvectors could be identified in the applet, by gesturing with his two index fingers: first he moves them towards each other and then overlaps them (Fig. 1). These gestures allow a shift in the focus of the discussion: it moves from trying to define eigenvalues, to attempt to understand what eigenvectors are. After different efforts to verbally describing the situation, finally A states:

A: […] It is possible to find an eigenvalue associated with an eigenvector when the image of the linear application coincides…[B and C look baffled]
B: How to say it? Can we say "overlapping"?
A: When the eigenvector and its image overlap.

The three of them seem happy with this definition, but C, again with the help of gestures to make himself understood, shows that the words "coincides" and "overlapping" are not satisfactory because

C: With this definition it means that the vectors reach the same point (Fig. 1)

A then refines his definition with:

A: When they have the same direction.

C: Same direction and same sense.

At this point B steps in and, he too gesturing (Fig. 2a and 2b), shows that actually the eigenvector and its image can have opposite senses. The so refined definition satisfies the whole group.

Fig. 1

Fig. 2a

Fig. 2b

Objectifying eigenspaces

One last episode deserves being mentioned. Later in the discussion, the doubt about the number of eigenvalues that can exist for a same direction, triggers the need to bring eigenspaces into play. Talking about eigenvectors laying on the same line, B asks:

B: There are different values for λ, aren't there?

A: No

C: Why not?

A: No, because if a linear application let's say multiplies an eigenvector times 3, if you multiply the eigenvector times 3, its image is time 3, then times 9 with respect to the first one.

Providing this answer, A performs a gesture (Fig. 3) that is the first one of a series of repeated and very similar gestures that will have a key role in the development of the discussion. Apparently he starts gesturing – he almost hadn't done that yet during the activity – in order to align with his group mates' discourses. Obviously this is just our interpretation. In order to convince B and C that all vectors lining on an eigenvector's direction are associated to the same eigenvalue, he starts with this embodied idea of stretching different vectors in the span of (1,1) by the same factor 3:

A: if you, the vector (1,1)..(3,3), the vector (3,3) goes into (9,9). On the other way round if you take (-1,-1) (Fig. 4a) it goes into (-3,-3) (Fig. 4b)

The interesting part of this excerpt is the way the semiotic bundle evolves: from a gesture used to convey an embodied conceptualization of this property, A progressively moves to the use of written diagrams and then to symbolic formulas. Firstly, he converts the idea shown with his gestures into a diagram and then from this he shows to B and C how this idea can be formalized with symbols (Fig. 5) and to provide an almost correct proof of the fact that each vector laying on the same direction of an eigenvector is an eigenvector as well, associated to the same eigenvalue.

Fig. 3

Fig. 4a

Fig. 4b

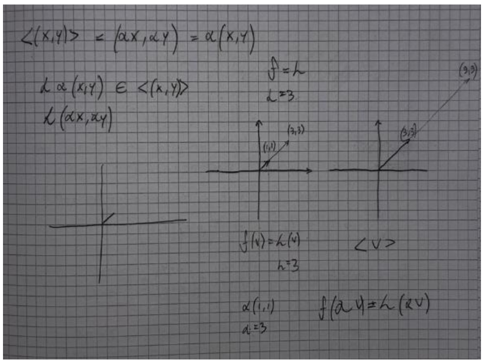

Fig. 5

We can identify A's recurring gesture as a catchment. He keeps replicating it, or a slightly modified versions of it, throughout his whole process of development of thought: from the example grounded in embodied reality, to the more formal formulation. The repeated gesture appears as the element of cohesion between these different levels of conceptualization, and that allows the other members of the group to follow and comprehend this development.

Moreover, the catchment generates from that first gesture (Fig. 3), accompanied by language. The idea guiding the described process seems to arise from this language-gesture integration that we have indeed identified as a growth point.

CONCLUSION

From the presented results we can outline some, however partial, conclusions. We can assert that the designed activity was suitable to make students engage in an objectification process. Firstly, students' management of time is a relevant indicator. As already stated, the whole activity lasted two hours. We had not recommended a partition of the whole available time, but were expecting students not to engage in the first part for longer than 25 minutes and that they would have hurried to start solving the problems. Unexpectedly, all the small groups engaged for at least 40 minutes in the first part of the task, before moving to the second one. We interpret this fact as an indication of the fact that students felt the need to really grasp the meaning of the concepts at stake. As we could notice from the recordings, students never settled for just repeating the definitions seen during the lecture. Rather, they tried with conviction to build strong meanings for those concepts and to pinpoint connections with other linear algebra concepts. As well, they tried to ensure that all the members of their group grasped the same meaning. Secondly, the analysis of students' means of objectification and their evolution and mutual relationships actually allowed us to study their collective meaning-making process. Thanks to the use of the semiotic bundle as an analytical tool, we could detect semiotic nodes in which the emerging and evolving relationships between signs help accomplish the objectification process (Radford & Sabena , 2015). It is particularly interesting to notice how students preferably appealed to different semiotic registers. Student A from the beginning privileged the use of oral or written verbal discourse, and this, despite his evident confidence on the topics, represented an obstacle for the objectification of eigenvalues. B apparently was more confident with symbolic manipulation and resorted different times to this kind of representations in order to connect to A's discourse. The role of C was relevant, even if from the beginning he seems to be the least confident on the subject. In the first part of the activity, he struggles in following the conversation and easily gets distracted. When they switch to a geometric example, he is able to actively engage in the dialogue using gestures, with which he is able to convey the intended meanings. In this case, it is clear how gestures, as also highlighted by previous researches (e.g., Arzarello et al., 2015), are not only a means for communication, but can be productive resources that help constitute thought. They are indeed key actors in the objectification process. Even more, it is the combination of these different semiotic resources in the bundle and conflicts arising between them, that allowed objectification to occur. "In fact, the activity through which knowledge is actualized is an activity of conflicting significations" (Radford & Sabena, 2015, p. 164). The intertwining of means of objectification activated by different students was possible only thanks to their joint labour. One last remarkable aspect is the fact that the observed group, despite required to deal with eigenvalues and eigenvectors, autonomously felt the need to deeply investigate the concept of eigenspace, in order to really understand them. This is a quite informative result, considering also the fact that research concerning eigenspaces teaching and learning is really limited (cf. Wawro et al., 2019), and will be more deeply described in future works.

To conclude, it is important to remark the fact that in this activity the course's teacher was almost absent. The issue of considering teachers' lectures and students' reflections in two separated moments poses a relevant question which requires further research also because of still scarce consideration in the literature. How can the teacher's role be integrated with a students' joint activity as that described? In future stages of our research, we are planning to move the focus to this aspect, whose investigation might provide further insights and perspectives to the same process of objectification.

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Toward a framework for integrating computational thinking into teaching and learning linear algebra

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This paper introduces an emerging framework for integrating computational thinking into the teaching and learning of linear algebra. To achieve this, we refer to the notions of three teaching principles of linear algebra, theory of instrumental genesis and computational thinking. Through the emerging framework, we present a vignette involving a set of activities using GeoGebra's specific commands, tools and functions. We approach the case of systems *of linear equations and limit ourselves to a linear algebra course whose students do not have a strong background in a programming language* (*much like the one for lower secondary mathematics teacher education programs in different countries*). *We propose several further steps to ameliorate the emerging framework.*

Keywords: Teaching and learning of linear and abstract algebra, Digital and other resources in university mathematics education, Computational thinking, Three teaching principles, Instrumental genesis.

INTRODUCTION

The term *computation* is considered one of the basic skills in school curricula; therefore, it could sound familiar to every mathematics teacher and mathematics educator (Li et al., 2020). When we combine computation with *thinking*, it immediately leads to a certain meaning: the practices of computer science, such as coding and programming. However, the notion of computational thinking is a way of thinking that encompasses a number of interrelated thinking skills (e.g., algorithmic thinking and problem-solving) that are beyond computing/programming practices (Lockwood, DeJarnette & Thomas, 2019; Wing, 2006). As a result, a growing body of research has recently attempted to define and characterize computational thinking in mathematics and science education (i.e., Kallia et al., 2021; Weintrop et al., 2016).

In addition to the growing interest in different levels of education (that involve various unplugged and plugged activities), a recent call for higher education has been raised by Lockwood and Mørken (2021). Lockwood and Mørken (2021) invite researchers to focus on machine-based computing activities in undergraduate mathematics education, especially those associated with practices of creating algorithms and running them through digital tools. However, this might require a certain level of maturity in programming languages (Buteau et al., 2020). Lockwood and Mørken's (2021) call has motivated us to focus on an undergraduate linear algebra course. Our question is: *How can computational thinking be integrated into teaching and learning linear algebra?*

We focus on linear algebra because it includes various mathematical notions (e.g., row reduction, echelon forms, linear independence, and rank), and different representations (e.g., equations, vectors, matrices and so on) that are interrelated with algorithmic thinking and problem-solving. Students are often challenged when they encounter such unusual (and new) ideas/steps/representations after high school, and "for a majority of the students, linear algebra is no more than a catalogue of very abstract notions" (Dorier et al., 2000, p. 85). We believe that the integration of the computational thinking perspective would be beneficial for linking different representations specific to linear algebra. Consequently, the current paper introduces an emerging framework for integrating computational thinking practices into linear algebra by grounding the framework in three perspectives as described in the next section.

CONCEPTUAL FRAMEWORK

Three teaching principles of linear algebra.

Learning linear algebra requires a certain level of coordination among different contexts, so designing the teaching setting for this has a core role in arranging the shifts and balance between the (new) notions and representations. Harel (2000) proposes three teaching principles that can be used for designing a teaching setting: concreteness, necessity and generalizability. The concreteness principle considers students' cognitive backgrounds and readiness for learning the proposed concept(s); this is strongly connected to student difficulties. The students need to be equipped with the proposed notions/concepts, and they need to have "… mental procedures that they can take these objects as inputs" (Harel, 2000, p. 180).

The necessity principle is about finding problematic situations that invite students into doing mathematics, and this should correspond to students' intellectual needs. Harel (2000) suggests that considering a *need for computation*, which means providing contexts that ask students to compute objects and explore mathematical properties, is the most effective way to invite students to start a mathematical discussion. This could enable students to find their way by elaborating a number of core ideas from their own work. The generalizability principle is strongly connected to the previous two principles because it enables students to arrive at a generalization in the end. The classroom activities, argumentation and teachers' orchestration of student learning should provide an environment where students move from their (own) work to generalization and the formation of ideas.

Harel (2000) highlights the use of *digital tools* for student exploration and geometry as a pedagogical context to enter a problematic set of situations. Following the three principles above, this context with digital tools should include a particular emphasis on the notion of the "need for computation" and development of ideas and generalization of the mathematical concepts. However, this is based on the manner of "tool use". The tool use shapes student thinking, and this process shapes tool use synchronously (Drijvers, 2019). Here, therefore, we point out the importance of estimation of student thinking (with tool use) to design the teaching-learning context. This brings us to the idea of "hypothetical [utilization] schemes" (Drijvers et al., 2010, p. 113) regarding tool use, which mainly comes from theory of instrumental genesis.

Theory of Instrumental genesis.

Theory of Instrumental Genesis (TIG) is based on the distinction between artifacts and instruments (Artigue, 2002). Here, an artifact can be any material, both physical and/or digital, but it is called a tool when used by the user for a particular aim. When the user develops one or more utilization schemes while using the artifact, we speak of instruments. Here, instruments involve the utilization schemes the user develops over time, in addition to the artifact(s). As a result, this can be simplified into the following formula: "Instrument = Artifact + Scheme" (Drijvers, 2019, p. 15). This process of scheme development is called instrumental genesis (Artigue, 2002). The process of instrumental genesis involves the development of both conceptual and technical elements. However, it is a subtle, continuous, and complex process. Techniques, which are the manner of tool use that lead to accomplishing a task (Artigue, 2002), are observable and explicit. The techniques give us clues regarding those utilization schemes that are invisible. Conceptual elements, on the one hand, convey the techniques that the user develops (over time); on the other hand, they are shaped by the artifact's affordances and constraints (Drijvers, 2019).

In the current paper, we focus on the *hypothetical* [utilization] *schemes* (Drijvers et al., 2010), (under the umbrella of TIG) that capture the synergy between techniques regarding the artifact and conceptual elements and their development. We hypothesize that the estimation of student thinking with tool use could help us design classroom activities with a particular lens that links the three teaching principles.

Computational thinking.

Computational thinking is an "umbrella term" (Kallia et al., 2021, p. 180) that involves a number of overarching and sophisticated skillsets, such as algorithmic thinking, decomposition, modelling, and abstraction. Wing's (2006) seminal description of computational thinking which "… involves solving problems, designing systems, and understanding human behaviour, by drawing on the concepts fundamental to computer science" (p. 33) opened the door to a growing body of research on computational thinking. Weintrop et al. (2016) define a four-category taxonomy regarding computational thinking in mathematics and science education (p. 135): "data practices, modelling and simulation practices, computational problem-solving practices, and systems thinking practices." The commonalities between Wing's (2006) and Weintrop et al.'s (2016) approaches concerning mathematics show a particular link to problem-solving, which means breaking a problem down into subproblems.

Recently, a particular characterization of computational thinking in mathematics education has been proposed by Kallia et al. (2021). This characterization has three main aspects (Kallia et al., pp. 179–180):

- *Problem-solving* (like understanding the problem, developing a solution strategy, performing the strategy),

- *Cognitive processes* (like abstraction, decomposition, pattern recognition, algorithmic thinking, modelling, logical and analytical thinking, generalization and evaluation of solution and strategies),
- *Transposition* (like phrasing the solution of a mathematical problem in such a way that it can be transferred/outsourced to another person or a machine).

The characterization above does not necessarily imply considering all aspects in a setting. For example, following a particular didactical aim, the topic and appropriation of the tools (both physical and digital) would not be practical if one tries to combine all the aspects described above. We concur with Kallia et al. (2021), who note:

> "… maybe some aspects of computational thinking are more critical than others and learning opportunities that consider computational thinking should provide opportunities for students to practice as many aspects as possible." (Kallia et al., 2021, pp. 179–180)

Therefore, based on the context, the teacher or educational designer can focus on specific aspects that invite students to perform (mathematical) explorations. Another fact is that the characterization above does not propose a particular set of tools, even though some of the aspects are directly related to computer science. If we go back to the context of linear algebra, there seem to be many topics related to computational thinking, for example, linear systems (particularly row reduction and echelon forms), matrix transformations and applications to computer graphics, the Gram-Schmidt process and so on.

The emerging framework for task design.

In this subsection, we relate the three teaching principles to those hypothetical utilization schemes with aspects of computational thinking. The first item that we need to address is that the backgrounds of the three teaching principles and TIG seem to be similar regarding students' mental development. The three teaching principles come from a Piagetian perspective (Harel, 2000), while TIG has foundations in both Piagetian (i.e., schemes) and Vygotskian (i.e., tool use) perspectives (Drijvers, 2019). Our particular aim is *not* networking these lenses and checking their grand theories, but rather considering and combining them into design tasks with a computational thinking lens. Regarding the shared theoretical background, there exists a link between tool use and the notion of the need for computation. Before beginning to explain this link, let us address the function of the concreteness principle in the emerging framework. The concreteness principle is carefully specific to the choice of the context. In other words, this principle is something that we can think of as a point of departure to think/decide about the setting. We claim that the following (interrelated) questions would be beneficial in setting the scene:

1) Which topic is going to be considered?
2) What is the (tentative) didactical aim/goal?
3) What do students know, what do they not know (*perhaps this is the most important one*), what would be *concrete* to the students' eyes, and why?
4) How to build on their existing knowledge/phenomenological experiences?

These questions bring us to the necessity principle, indeed to the notion of the “need for computation.” To invite students into a rich context that is open to exploration, argumentation, conjecturing, and testing conjectures, we underline the role of tools (Harel, 2000) as mediators:

5) Which tool(s) would be beneficial and why?
6) How would these tools function to achieve the didactical goal?
7) What kind of experience do the students have with the thought tools?
8) Which conceptual elements would emerge when students use the tools?

These four questions imply an estimation of the “manner of tool use” (Artigue, 2002) to design the teaching-learning environment, which brings us to the notion of hypothetical utilization schemes (Drijvers et al., 2010) and TIG. The main aim of these questions is to elaborate on the (hypothetical) techniques and associated conceptual elements that could help us picture/discuss the potentiality of the tools for the didactical aim. The responses to the questions (5 to 8) could be research-informed. A literature search for potential tools and manner of student use would be helpful here as well. The third point concerns embedding the aspects of computational thinking into the eight questions above. The designer could focus on certain aspect(s) in the sense of Kallia et al. (2021) by considering the following (general) question:

9) How would the context and tool enable students to engage with a problem-solving activity, cognitive processes, and transposition?

The generalization principle plays a central role here, and it is linked to the cognitive processes (aspect) of computational thinking. The designer could focus on the function of the tools at stake, along with how this would create a mathematical sense, meaning-making or would help students arrive at a conclusion. Hence, the designer could then finalize their didactical aim/goal. To conclude, we note that the combination and synergy among the nine questions above constitute the emerging framework for task design. Figure 1 summarizes the nine questions, which shows the components as the axes of a Cartesian view.

The designer can refer to Figure 1 by considering ordered triples (from three teaching principles to aspects of computational thinking) while brainstorming. For example, point *A* represents the role of concreteness, the tool and problem-solving and how these three (the selection of the context/aim, student pre-knowledge and deciding on the tool and problem-solving activity) would be aligned. As another example, *B* represents the triple of necessity, conceptual elements, and transposition. Focusing on *B* would help the designer think about and discuss how the need for computation would interlace with the targeted conceptual elements after instrumented activity. Through this way, it can be discussed how the solution of a mathematical problem can be transferred to another person/machine. It may not be necessary to discuss all the possible (27) triples here, however, we believe that Figure 1 would guide the designer as they consider and decide on the function of each component considered here.

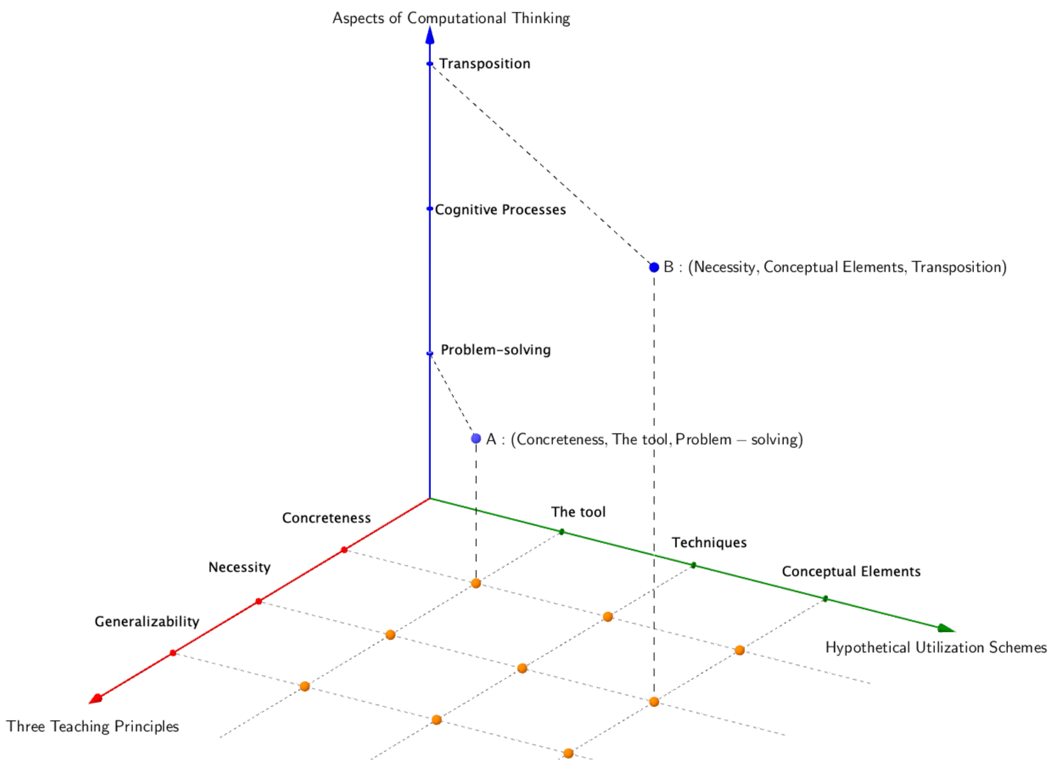

Figure 1: A Cartesian view of the emerging framework

A VIGNETTE

We present an exemplary vignette by following the concreteness principle (particularly point *A* in Figure 1). We have decided on the topic of systems of linear equations (SLE) for two main reasons. The first is the topic (in itself), specifically because SLE is one of the core topics in elementary linear algebra, which has many applications in different fields (Anton & Rorres, 2014). The second reason is SLE's dynamic geometry software availability. We have recently shown how a dynamic geometry environment creates coordination between algebraic and 3D geometric views regarding SLE (Turgut & Drijvers, 2021). The second question in the above subsection and notion for the need for computation help us consider the parameters in the task. As we have experienced (Turgut & Drijvers, 2021), the use of parameters in SLE creates a rich context to explore the geometry of lines and planes in $\mathbb{R}^3$. Therefore, we formulate a tentative aim: making sense of the role of parameters in SLE. We believe that the topic is more relevant after the matrix algebra topic and after the students have learned to solve SLEs on paper-and-pencil activities. Therefore, we plan to build on matrix algebra and consider that the target group does not know the role of the parameters in SLE yet.

The paragraph above can be summarized as addressing those questions from 1 to 4. Now, we focus on questions 5 to 8. In our exemplary case, we refer to GeoGebra based on three criteria. The first is GeoGebra.org's classroom function, where the teacher can design a set of activities/tasks and share the interface of the activity by providing a code. The second is the commands and some tools of GeoGebra that have been recently considered part of the computational thinking activity (van Borkulo et al., 2021). The software is now popular in many countries, so we consider that the target group is familiar with the basic tools/functions. The third is that we focus on the case where the target group does not have a programming language background.

Now, we briefly summarize hypothetical techniques and utilization schemes (based on Turgut & Drijvers, 2021) as follows. While exploring SLE solutions, students could refer to synchronic algebra and geometry windows that provides dynamic variations. The “Solve” command could be used to solve the given equations, and the software would provide different solutions where students could explore different values of the parameters. For example, in certain cases, there is a solution or no solution. Students could type and form matrices of given SLE through a spreadsheet window and attach sliders to matrices. The students later could also refer to the “Reduced RowEchelon Form” command to compute the echelon form of the matrices. This could enable students to see completely zero rows etc. in the matrix and its meaning in the SLE solution, helping them create a link between the role parameters of in row echelon forms. The latter command could also provide a meaning for infinite solution, single solution, or no solution. To plot lines and planes, the students could use the “Input” line and the “Intersect” command, which could provide a geometric feature of the role of parameters (like the intersection of planes and its meaning in the SLE solution).

Regarding computational thinking, we focus on algorithmic thinking and generalization with a problem-solving activity for solving a set of SLEs. In light of this, we re-design a set of activities borrowed from literature (Anton & Rorres, 2014; Turgut & Drijvers, 2021), which are divided into three episodes. The first starts with a figure to overview and link some key notions about SLE and the associated commands of GeoGebra. The first step of the activity (Episode 1) is presented in Figure 2.

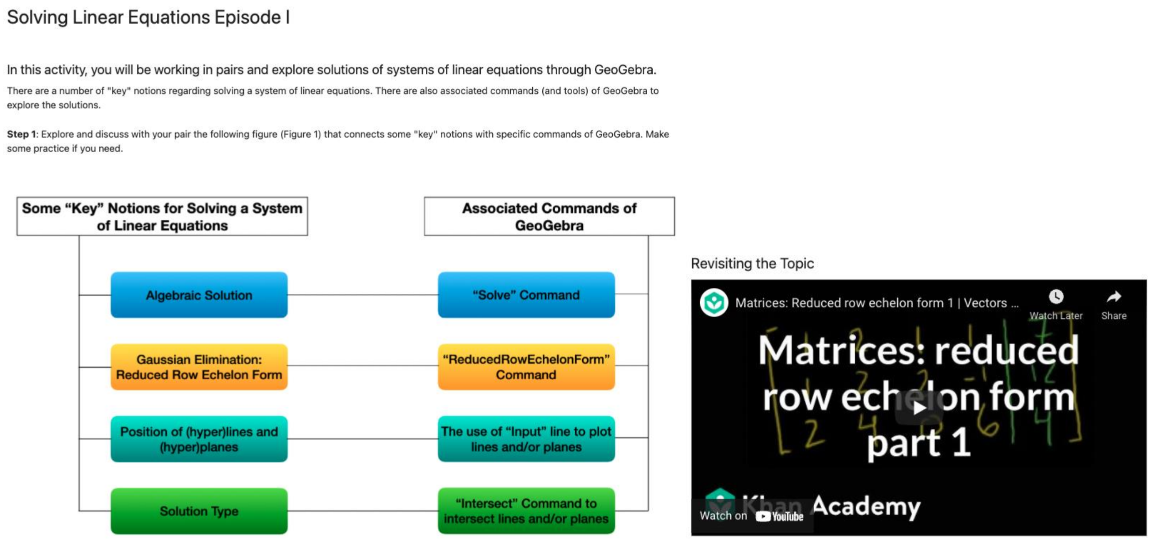

Figure 2: The First Step of Episode I

As a first step (Step 1 in Figure 2), the students discuss some key notions and associated commands of GeoGebra. They can also recall their knowledge by watching some topic-related Khan academy videos (e.g., reduced row echelon form, as seen on the right-hand part of Figure 2). Step 2 asks the students to sketch a tentative algorithm on an embedded (blank) GeoGebra applet to approach solving SLE (by keeping in mind the key notions and associated commands of GeoGebra, as in Figure 2).

Figure 3 summarizes Steps 3 and 4. The third step asks the students to solve a system of linear equations, including a single parameter k: $x + y + z = 4$, $z = 2$ and $(k^2 - 4) \cdot z = k - 2$. We provide a blank GeoGebra applet that includes an algebra window, and 3D graphics window with a spreadsheet window by considering the hypothetical utilization schemes explained on the previous page. For example, the use of the slider k could be referred to solve the proposed SLE.

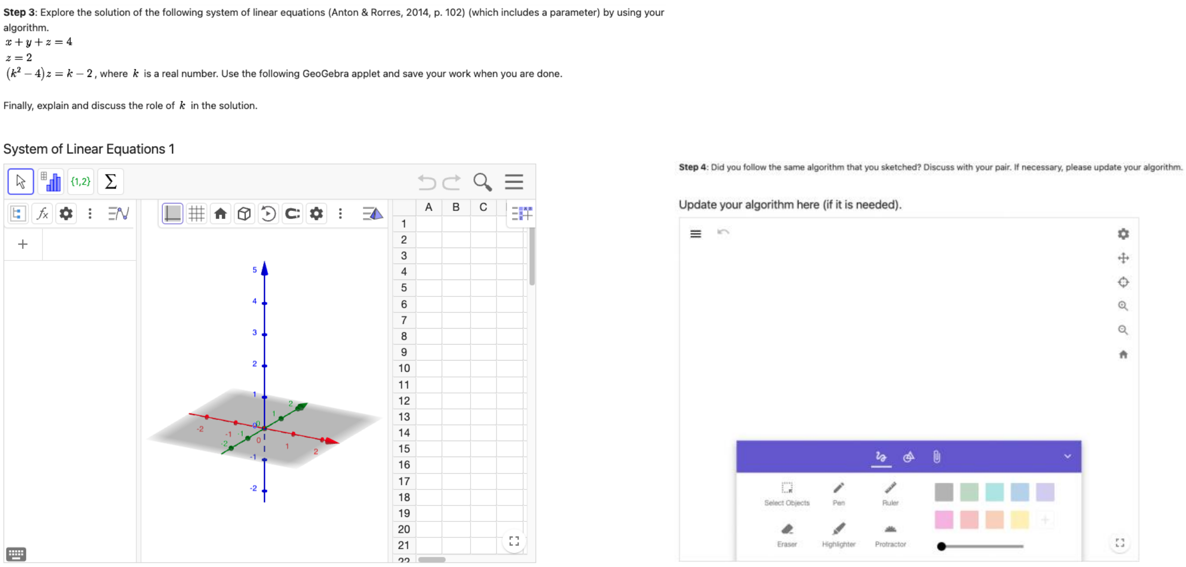

Figure 3: Steps 3 and 4 in Episode 1

In this part, the use of a slider as k is an estimated technique that could help to explore the dynamic effects of the parameter in the given SLE. As a fourth step, the students must discuss the initial algorithm after they have solved the given SLE with a single parameter. The final step of Episode 1 asks to discuss the role of k in the given system (the left-hand-side of Figure 4).

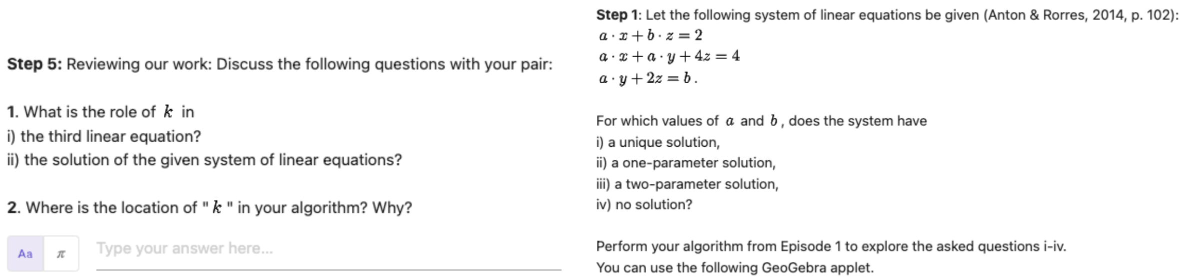

Figure 4: Step 5 of Episode 1 and Step 1 of Episode 2

The second episode starts with another system (the right-hand-side of Figure 4) that has two specific parameters: a and b. The user is asked to use the (updated) algorithm while solving the given system. As in Episode 1, as a next step, the students are asked to review the algorithm after solving SLE, and then discuss questions to overview the role of parameters in SLE. However, the parameter(s) in the given two SLE appear both in coefficients and known parts. Therefore, in the final episode, a specific system is proposed: $x + y + z = a$, $2x + 2z = b$, and $3y + 3z = c$, which has three parameters. In the final SLE, all three parameters are defined in the known part of the system. This episode also follows reviewing algorithms and making a generalization regarding the role of parameters in SLE by overviewing all episodes and (revisiting) all versions of algorithms.

CONCLUSIONS

In the current paper, we have introduced an emerging framework for integrating computational thinking into teaching and learning linear algebra. We present an exemplary case where the aim is to promote the knowledge and role of parameters in SLE and associated solutions. We note that the emerging framework needs further elaborations (e.g., through design-based research) to discuss its functioning in teaching learning settings. For example, the presented set of activities could be merged and field-tested with some come from matrix transformations and applications to computer graphics, and Gram-Schmidt process. As a limitation, the exemplary case is GeoGebra centric. Another context, such as R, Python or Trinket, could be focused on when designing machine-based computing activities (Lockwood & Mørken, 2021). These could be further steps to ameliorate the presented emerging framework.

Acknowledgement

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Die Konstruktion epistemologischer Dreiecke zur Analyse von Begriffsdeutungen bei mathematischen Lückentexten

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Abstract: Both in school mathematics and in university mathematics, cloze texts are used rather rarely to build up and develop mathematical language skills. It is therefore not surprising that the use of mathematical cloze texts in mathematics didactics has hardly been researched so far. In order to investigate the specific comprehension processes that play a role in the processing of mathematical cloze texts, an empirical study was conducted in which the verbalized thoughts of the participants were recorded in think-aloud protocols. This paper presents Steinbring's epistemological triangle as a possible analytical tool for these comprehension processes as well as the first results of the analysis.

Keywords: Teaching and learning of specific topics in university mathematics, Digital and other resources in university mathematics education, mathematical clozes, Processes of understanding , epistemological triangle.

EINLEITUNG

Das Erlernen der mathematischen Sprache stellt für viele Studierende eine große Herausforderung dar. Dies ist nicht verwunderlich, wenn man bedenkt, wie weit die Fachsprache der wissenschaftlichen Mathematik von der Alltagssprache entfernt ist: Sie ist nicht nur deutlich formaler und exakter, sondern auch zeitlos und unpersönlich. Geringe Formulierungs- und Notationsunterschiede bewirken zum Teil große semantische Unterschiede, weshalb präzises Lesen und Schreiben notwendig ist (Liebendörfer, 2018). Die Beherrschung der mathematischen Sprache ist für das Mathematikstudium zentral und Voraussetzung für ein tieferes Verständnis. Doch trotz ihrer Bedeutung findet eine explizite Thematisierung der Fachsprache häufig nur zu Beginn des Studiums statt, wenn ausgewählte Regeln der Logik gelehrt werden. „Danach bleibt die Vermittlung der Fachsprache weitestgehend implizit“ (Liebendörfer, 2018, S. 28). Dies trifft auch auf die zahlreichen Fachbegriffe zu, die in den Vorlesungen und Seminaren eingeführt und oft nebenher erlernt werden sollen, da bisher entsprechende Aufgaben, mit denen Fachbergriffe erlernt oder trainiert werden können, eher selten in der hochschulmathematischen Lehre verwendet werden. Eines der wenigen Beispiele dafür stellt die Mathematikvorlesung für Pharmazie- und Biologiestudierende der Philipps-Universität Marburg dar (Strauer et al., 2019). In dieser Veranstaltung werden seit dem WiSe 2017/18 Lückentextaufgaben mit Freitext-Lücken eingesetzt, um die präzise Verwendung von Fachbegriffen und die eigenständige Formulierung von Begriffserläuterungen zu üben. Als mögliches Mittel für das Erlernen und das Einüben der mathematischen Sprache wurden Lückentexte aber bisher kaum beforscht, sodass es nicht nur an Gestaltungskriterien und Empfehlungen für den Einsatz in der Lehre fehlt, sondern auch kaum Informationen darüber vorliegen, wie Studierende mit Lückentexten arbeiten (sollen), wie sie Lückentexte verstehen und welche Fähigkeiten sowie Fertigkeiten mithilfe von mathematischen Lückentexten überhaupt gefördert werden können. In diesem Artikel soll es daher um eine grundlegende Fragestellung gehen:

Wie gestalten sich die Begriffsdeutungen, die bei der Bearbeitung von mathematischen Lückentextaufgaben von Studierenden entwickelt werden, und wie hängen diese von den vorgegebenen Antwortoptionen ab?

Um diese Frage beantworten zu können, wurde in einer Studie im Frühjahr 2020 der Lückentext „Der Gauß-Algorithmus“ (siehe Abb. 3) als Diagnoseaufgabe eingesetzt. Bei den Teilnehmenden der Studie handelte es sich um zehn Studierende der Leibniz Universität Hannover aus unterschiedlichen Studiengängen[1], die im Wintersemester 2019/20 die Grundlagenveranstaltung „Lineare Algebra I“ besucht und im Rahmen dieser Veranstaltung Online-Tests als Teil der Studienleistung bearbeitet hatten. Den Aufgabentyp „Lückentextaufgabe“ kannten die Teilnehmenden vor der Studie daher schon. Für die Untersuchung wurden die Studierenden in der vorlesungsfreien Zeit zu freiwilligen Einzelinterviews mit der Autorin eingeladen, bei denen sie eine Lückentextaufgabe unter Laborbedingungen[2] bearbeiten sollten. Damit die kognitiven Prozesse der Studierenden erfasst werden konnten, wurden die Studierenden während der Einzelbearbeitung dazu angehalten, die Laut-Denken-Methode (Deffner, 1984; Bise, 2008) anzuwenden. Das bedeutet, dass die Probandinnen und Probanden während der Bearbeitung alle Gedanken, die aufkommen, laut äußern sollten. Die Audioaufnahmen wurden im Anschluss transkribiert und hinsichtlich der eingangs gestellten Fragen analysiert.

Da die Analysen noch nicht abgeschlossen sind, sollen in diesem Beitrag die Auswertungsmethodik und erste Analyseergebnisse diskutiert werden. Hierzu wird zunächst die theoretische Grundlage, das epistemologische Dreieck nach Steinbring, vorgestellt sowie die Analysemethodik erläutert. Abschließend werden die ersten Ergebnisse entlang eines Analysebeispiels illustriert.

DAS EPISTEMOLOGISCHE DREIECK

Um mathematisches Wissen erfassen, repräsentieren, kommunizieren und kodieren zu können, werden in der Mathematik bestimmte *Zeichen- bzw. Symbolsysteme* verwendet, wobei solche Zeichen/Symbole für sich allein keine Bedeutung besitzen (Steinbring, 2006). Die Bedeutung eines mathematischen Zeichens muss vom dem oder der Lernenden aktiv konstruiert werden (Steinbring, 2000; Steinbring, 2005).

Steinbring geht davon aus, dass sich die Bedeutung eines mathematischen Begriffs aus den konstruierten Wechselbeziehungen zwischen *Zeichen/Symbolen* und

[1] Studiengänge: Physik, Mathematik und Mathematik auf Lehramt (Gymnasium)

[2] Die Bearbeitungen fanden im Büro der Autorin statt. Die Studierenden hatten keine zeitlichen Begrenzungen für die Bearbeitung der Aufgabe. Die Lückentextaufgabe wurde von den Studierenden über einen von der Autorin bereitgestellten Laptop bearbeitet. Als Hilfsmittel wurden nur Papier und Stift zugelassen. Beides wurde ebenfalls gestellt.

konkreten oder abstrakten *Gegenständen/Referenzkontexten* ergibt. Allgemein können *Mathematische Begriffe* somit als „symbolisierte, operative Beziehungen“ (Steinbring, 2000, S. 34) verstanden werden, welche zwischen ihren kodierten Formen und „den sozial intendierten Deutungen“ (Steinbring, 2000, S. 34) bestehen. Mathematische Begriffe drücken demnach keine Dinge aus, sondern Beziehungen. In mathematischen Lehr-Lernkontexten wird mathematische Bedeutung also beispielsweise dadurch generiert, dass mögliche Bedeutungen von einem relativ oder teilweise bekannten Referenzkontext auf ein unbekanntes Zeichensystem, welches für den Lernenden noch keine Bedeutung besitzt, übertragen werden (Steinbring, 2005).

> Der Zusammenhang zwischen den Zeichen zur Kodierung des Wissens und den Referenzkontexten zur Etablierung der Bedeutung des Wissens lässt sich im epistemologischen Dreieck darstellen. (Steinbring, 2000, S. 34)

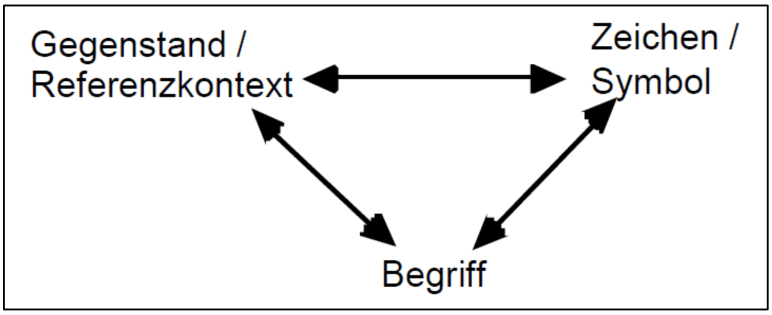

Abb. 1: Das epistemologische Dreieck (Steinbring, 2000, S. 34)

Der *Gegenstand/Referenzkontext* stellt bei Steinbring kein unabhängiges Element dar, das vorab eindeutig vorgegeben wird (Steinbring, 2005). Stattdessen wird der Gegenstand/Referenzkontext im Prozess der Wissensentwicklung „mehr und mehr in einen strukturellen Zusammenhang umgedeutet“ (Maier & Steinbring, 1998, S. 309). Darüber hinaus sind die Beziehungen zwischen den Eckpunkten des epistemologischen Dreiecks nicht explizit definiert. Vielmehr handelt es sich um ein ausbalanciertes System, dessen Eckpunkte sich wechselseitig stützen (Maier & Steinbring, 1998; Steinbring, 2000; Steinbring, 2005). Hierbei kann „eine Änderung in jeder der Ecken direkt Reaktionen in den anderen und den jeweiligen Beziehungen“ (Rieß, 2018, S. 91) auslösen, daher handelt es sich um ein Gebilde, das sich ständig verändert. Übertragen auf mathematische Lehr-Lernkontexte kann man sich dieses Spannungsverhältnis folgendermaßen vorstellen: Die Bedeutung mathematischer Zeichen/Symbole, die in einem solchen kulturellen Kontext verwendet werden, können aus dem bereits konstruierten Wissen und den bekannten Gegenständen/Referenzkontexten generiert werden. Es besteht allerdings die Möglichkeit, dass die Bedeutung sich mit der Beschäftigung neuer Gegenstände/Referenzkontexte ändert. Beispielsweise werden Variablen wie x im Mathematikunterricht mithilfe von Aufgaben wie $2 + x = 5$ als eine Unbekannte eingeführt. Das Symbol steht also für ein Objekt, das noch unbekannt ist, aber (unter bestimmten Konditionen) bestimmt werden kann. Beschäftigt man sich aber auch mit Aufgaben zu funktionalen Zusammenhängen wie z.B. $f(x) = x + 2$, so erhält das Symbol in diesem Kontext eine neue Bedeutung: Es steht für eine Veränderliche. Die Umdeutungen und Neuinterpretationen der Zeichensysteme und Referenzkontexte bewirken eine sukzessive Weiterentwicklung des mathematischen Wissens, sodass die „Deutung mathematischer Zeichen nicht zu endgültigen, eindeutigen Definitionen" (Akinwunmi, 2012, S. 51) führt. Die Vorstellung von einer komplexen Wechselbeziehung zwischen Zeichen/Symbol und Gegenstand/Referenzkontext bildet einen Kontrast zu anderen statischeren Zeichenmodellen (Ogden & Richards, 1923; Frege, 2019). Mithilfe des epistemologischen Dreiecks kann die mathematische Bedeutungsentwicklung mit einem stärkeren Fokus auf den Prozess untersucht werden.

Zu beachten ist allerdings, dass sich die Ecke *Begriff* in der Regel nicht explizit in den Äußerungen von Studierenden finden lässt. Betrachtet man das Beispiel in Abb. 2, so lässt sich schnell erkennen, dass das Symbol *3* für die drei schwarzen Kugeln und die drei Äpfel stehen kann, weil sie die Mächtigkeit der dargestellten Mengen ausdrückt. Diese formulierte Beziehung/Relation „drückt die Mächtigkeit aus" allein kann aber nicht die Komplexität des *Zahlbegriffs* abbilden. Um die Komplexität mathematischer Begriffe erfassen zu können, ist eine Akkumulation der aufgestellten Relationen zwischen Gegenstand/Referenzkontext und Symbol notwendig, die auf verschiedene Aspekte des Begriffs abzielen.

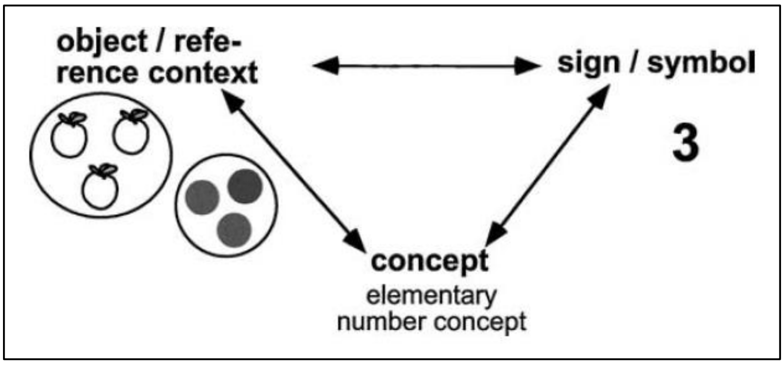

Abb. 2: Das (Steinbring, 2006, S. 141)

ANALYSEVERFAHREN

Mithilfe des epistemologischen Dreiecks können die Bedeutungen für die zentralen mathematischen Begriffe und Sätze, die von den Probandinnen und Probanden konstruiert wurden, identifiziert werden. Angelehnt an das Verfahren von Steinbring (Steinbring, 1993; Maier & Steinbring, 1998) wurden dafür zunächst die die beiden Ecken Zeichen/Symbol und Gegenstand/Referenzkontext sowie die Relation/Beziehung für jeden Fall bestimmt.

Für den vorliegenden Transkriptionsausschnitt wurden dazu die Äußerungen der interviewten Person in vier unterschiedliche Kategorien eingeordnet:

1. **Zeichen/Symbol**: Elemente, Begriffe, Phrasen oder Sätze, die den Probandinnen und Probanden unbekannt oder mit einer gewissen Art der Unsicherheit behaftet sind, d. h. dass sie den Teilnehmenden unbekannter sind als die Elemente der zweiten Kategorie.
2. **Gegenstand/Referenzkontext**: Elemente, Begriffe, Phrasen oder Sätze, die von den Probandinnen und Probanden herangetragen werden, um die Elemente aus der ersten Kategorie zu bestimmen. Diese Elemente sind den Teilnehmenden (relativ) bekannt.
3. **Relation/Beziehung**: Da die Ebene des Begriffs nicht direkt in den Äußerungen identifiziert werden kann, werden stattdessen Äußerungen kodiert, in denen ein Bezug oder eine Verbindung hergestellt wurde zwischen Elementen aus der ersten und zweiten Kategorie.
4. **Nicht zuordenbare Äußerungen**: Äußerungen, die sich in keine der oberen drei Kategorien zuordnen lassen.

Im Anschluss daran wurden die Wechselwirkungen zwischen dem Gegenstand/Referenzkontext und dem Zeichen/Symbol untersucht. Daneben wurden die aufgestellten Dreiecke auch miteinander verglichen, um den Prozess einer möglichen Bedeutungsverschiebung oder Änderung abbilden zu können.

BESIPIELANALYSE

Lückentextaufgabe und Transkriptausschnitte

Bei der ausgewählten Diagnoseaufgabe handelt es sich um eine Online-Aufgabe zum Thema Gauß-Algorithmus, die für den Einsatz in der Grundlagenveranstaltung „Lineare Algebra I" (WiSe 2019/20) entwickelt worden ist. Die Aufgabe stellt eine Mischform aus Lückentext und Auswahlaufgabe dar. Bei Lückentextaufgaben muss die Testperson einen unvollständigen Text ergänzen (Eichler, 1977). Welche Art von Element fehlt, kann je nach Aufgabe variieren. Denkbar wären beispielsweise Symbole, Wörter, Wortbestandteile, Wortgruppen oder gar ganze Sätze. Auswahlaufgaben, sind derart gestaltet, dass die Testperson aus einer vorgegebenen Auswahl an Antwortoptionen eine richtige oder mehrere richtige Antworten wählen kann. Existiert nur eine richtige Antwort, handelt es sich um eine *Single-Choice-Aufgabe*, im anderen Fall wird die Aufgabe als *Multiple-Choice-Aufgabe* bezeichnet. Bei der Aufgabe zum Gauß-Algorithmus gibt es sowohl Auswahlen mit Single-Choice als auch Auswahlen mit Multiple-Choice. Die Antwortoptionen, die für jede Lücke vorgegeben werden, sind nicht willkürlich gewählt worden. Die Erstellerinnen und Ersteller haben versucht, nach Möglichkeit nur falsche Antwortoptionen zu formulieren, die tatsächlich in der Mathematik existieren[3] und mindestens eine der folgenden Eigenschaften[4] erfüllen:

- Sie müssen zum dargestellten mathematischen Kontext passen.
- Sie müssen graphisch/formulierungstechnisch ähnlich zur richtigen Antwortoption sein. (Eine Verwechselungsgefahr sollte bestehen.)
- Sie müssen eine Fehlvorstellung enthalten.

Mit diesen Kriterien für die Antwortoptionen sollten die Lückentexte den Studierenden die Möglichkeit bieten, ihr Begriffsverständnis zu vertiefen, indem sie die zur Auswahl stehenden Ober-, Unter-, Gegen- und Nachbarbegriffe aus einem Themenkomplex miteinander vergleichen und gegeneinander abgrenzen können.

Der Lückentext „Der Gauß-Algorithmus“ besitzt vier inhaltliche Abschnitte. Der erste Abschnitt stellt eine allgemeine Einführung der Begriffe Gauß-Verfahren und LGS dar. Im zweiten Abschnitt werden die elementaren Zeilenumformungen sowie der grobe Aufbau des Verfahrens thematisiert. Der dritte Abschnitt beschreibt im Detail den Aufbau der Zeilenstufenform und im letzten Abschnitt wird die normierte Zeilenstufenform behandelt. Insgesamt besitzt die Lückentextaufgabe 19 zu ergänzende Lücken. Für die Beispielanalyse werden wir uns im Folgenden lediglich auf den ersten Abschnitt konzentrieren.

Abb. 3: Ausschnitt zur Lückentextaufgabe „Der Gauß-Algorithmus“

Die beiden folgenden Transkriptausschnitte stammen von einer Versuchsperson und beziehen sich auf den Aufgabenausschnitt in Abb. 3, genauer auf die erste auszufüllende Lücke. Im Transkript folgen sie – wie man an den angegebenen Zeiten sehen kann – kurz nacheinander.

[3] Nicht zugelassen wurden beispielsweise falsche Antwortoptionen, bei denen einfach nur ein Rechtschreibfehler enthalten ist, wie z.B. *Zailenstufenform* statt *Zeilenstufenform* oder Begriffe, die nicht existieren, wie z.B. *begrenzte Zeilenstufenform*.

[4] Da es für die Konzeption mathematischer Online-Lückentextaufgaben bisher in der Forschung keine konkreten Empfehlungen oder Kriterien gab, orientieren sich diese hier formulierten Kriterien an den Arbeiten von Vollrath und Roth (2012) zum Erarbeiten und Verstehen von mathematischen Begriffen. Da die Lückentextaufgaben zudem seit 2019 in den Grundlagenveranstaltungen Lineare Algebra und Analysis eingesetzt wurden, konnten die Lückentexte mithilfe von Ergebnissen aus regelmäßig stattfindenden Studierendenbefragungen zu den Aufgaben immer wieder überarbeitet und verbessert werden.

Erster Transkriptausschnitt:

ProbandIn: (seufzt) -ähm- Naja. Also, (.) ich weiß halt nur, was eine Zeilenstufenform ist und ich weiß ja auch, (.) also, dieses Muster innen und -ähh- und dass das beim Gauß-Algorithmus eben rauskommt. Dann bereinigte Normalform habe ich noch nie gehört und Spaltenstufenform (.) ist halt eben nicht das, was rauskommt. Also also/ Außerdem hat hat man diesen/ dieses Wort Zeilenstufenform, hat man auch schon irgendwie oft gehört. Also/#00:22:28#

Zweiter Transkriptausschnitt:

ProbandIn: Naja. (seufzt) Ich weiß nicht. Also, (..) egal, der Gauß-Algorithmus/ Ich weiß halt, was man damit macht. Man bringt den halt auf diese Zeilenstufenform und ich weiß auch gar nicht jetzt, wo ich darüber nachgedacht habe, was eigentlich eine Spaltenstufenform ist. Also, wahrscheinlich das gleiche irgendwie andersrum symmetrisch, aber/ (.) Ja. (lacht) #00:24:00#

Zuordnungen und Analyseergebnisse

Die folgende Tabelle enthält die Zuordnungen zu den aufgestellten Kategorien für die beiden Transkriptausschnitte[5]:

Nr.	Zeichen/ Symbol	Relation/Beziehung	Gegenstand/Referenzkont.
1	„eine Zeilenstufenform“	„innen“	„dieses Muster“
2	„eine Zeilenstufenform	„das beim [Gauß-Algorthmus] eben rauskommt“	„Gauß-Algorithmus“
3	„bereinigte Normalform“	„habe ich noch nie“	„gehört“
4	„Spaltenstufenform“	„ist halt eben nicht das, was rauskommt“	„Gauß-Algorithmus“
5	„der Gauß-Algorithmus“	„Ich weiß halt, was man damit macht. Man bringt den halt auf“	„diese Zeilenstufenform“
6	„eine Spaltenstufenform“	„Also, wahrscheinlich das gleiche irgendwie andersrum symmetrisch“	„diese Zeilenstufenform“

Tab. 1: Zuordnungen zu den Kategorien

[5] Die restlichen Elemente aus den Ausschnitten wurden der Kategorie „nicht zuordenbare Äußerungen“ zugeordnet und werden dementsprechend nicht in diesem Beitrag thematisiert.

Betrachtet man die Zuordnungen, so lassen sich verschiedene Aspekte erkennen:

1. Die gewählten Referenzkontexte können Elemente sein, die bereits in der Aufgabe enthalten sind wie *Gauß-Algorithmus* oder *Zeilenstufenform*. Gleichzeitig werden aber auch Referenzkontexte gewählt, die nicht oder nicht wortwörtlich im Lückentext enthalten sind, wie *gehört* oder *Muster*. Mit *gehört* könnten beispielsweise die akustischen Zeichen gemeint sein, die im Rahmen von Vorlesungen, Übungen, etc. kommuniziert wurden, d. h. mündliche Erwähnungen, Beschreibungen und Erläuterungen. Mit dem *Muster*, das sich *innen* befindet, sind möglicherweise die Art und Anordnung der Koeffizienten bzw. die Matrixeinträge gemeint, die eine Dreiecksform oder Treppenform bilden.
2. Die Relation zwischen den Zeichen und Referenzkontexten wird häufig vage und recht unpräzise formuliert. Deutlich wird dies zum Beispiel bei der formulierten Relation zwischen dem Zeichen *Spaltenstufenform* und dem Referenzkontext *Gauß-Algorithmus*. Hier beschreibt die Versuchsperson die Spaltenstufenform lediglich als das, was *eben nicht* beim Gaußalgorithmus *rauskommt*. Diese Relation ist insofern noch vage, als dass sie auf alle möglichen Referenzkontexte zutrifft. Eine Spaltenstufenform erhält man beispielsweise auch nicht, wenn man eine quadratische Ergänzung durchführt. Gleichzeitig entsteht die Vagheit auch durch den gewählten Referenzkontext, denn durch die Anwendung des Gauß-Algorithmus erhält man beispielsweise auch keine *bereinigte Normalform*, da dieser Begriff aus der Prädikatenlogik stammt.
3. Es wird durchaus mehr als ein Referenzkontext herangezogen, um ein Zeichen zu deuten. Das Zeichen *Zeilenstufenform* wird nicht nur mit *Muster*, sondern auch mit *Gauß-Algorithmus* in Beziehung gesetzt. Das Gleiche gilt für das Zeichen *Spaltenstufenform,* welches ebenfalls mit *Gauß-Algorithmus* ins Verhältnis gesetzt wird, aber auch mit *Zeilenstufenform*. Bei der *Spaltenstufenform* lässt sich durch die zweite konstruierte Relation sogar eine Art Entwicklung hin zur Präzisierung erkennen, denn die Spaltenstufenform ist nicht mehr nur das, was nicht bei der Anwendung des Gauß-Algorithmus entsteht. Der formulierten Relation kann man entnehmen, dass sie für die Probandin/den Probanden vergleichbar mit der Zeilenstufenform ist, nur *irgendwie andersrum symmetrisch*. *Andersrum symmetrisch* könnte wiederum eine unpräzise Beschreibung der Transponierten darstellen, da eine transponierte Matrix durch die Spiegelung der Ausgangsmatrix an ihrer Hauptdiagonalen entsteht.
4. Selbst bei einer einzelnen Lücke bleiben die Zuordnungen zu den Ecken des epistemologischen Dreiecks nicht stabil. Zu Beginn ist *Zeilenstufenform* ein Zeichen, das gedeutet wird. Im Verlauf der Bearbeitung wird *Zeilenstufenform* aber auch als Referenzkontext genutzt, um andere Zeichen zu deuten. Auffällig sind hier auch die Zuordnungen in Zeile zwei und fünf der Tabelle 1: *Gauß-Algorithmus* ist hier zunächst der Referenzkontext, mit dessen Hilfe

Zeilenstufenform gedeutet wird. Später werden die Rollen dieses Paares vertauscht. Die formulierten Relationen ähneln sich semantisch. Die zweite Relation ist sogar unpräziser, da sie suggeriert, dass der Gauß-Algorithmus in Zeilenstufenform gebracht wird, dabei müsste man eigentlich von der Ausgangsmatrix sprechen.

5. Wie die Zuordnungen in Zeile zwei und vier der Tabelle 1 zeigen, können gleiche Referenzkontexte und syntaktisch ähnlich formulierte Relationen herangezogen werden, um Unterschiede oder Kontraste zwischen den Begriffen zu verdeutlichen. Zeilenstufenform und Spaltenstufenform werden dadurch als Gegensatzpaar markiert.

FAZIT UND AUSBLICK

Die eingangs gestellte erste Frage *Wie gestalten sich die Begriffsdeutungen, die bei der Bearbeitung von mathematischen Lückentextaufgaben von Studierenden entwickelt werden und wie hängen diese von den vorgegebenen Antwortoptionen ab?* kann bisher noch nicht im vollen Umfang beantwortet werden. Die ersten Analyseergebnisse deuten aber darauf hin, dass durch die Vorgabe spezifischer Auswahloptionen bestimmte Prozesse der Bedeutungsgenerierung angestoßen werden.

Werden Begriffe zur Auswahl gestellt, die aus einem mathematischen Kontext stammen und in der Begriffshierarchie gleichwertig zueinander sind wie z. B. *Spaltenstufenform* und *Zeilenstufenform*, dann kann dies dazu führen, dass der Versuch unternommen wird, diese Begriffe voneinander abzugrenzen, indem jeweils eine Relation zu einem übergeordneten Begriff (in diesem Fall *Gauß-Algorithmus*) formuliert wird. Gewissermaßen wird hier eine gemeinsame Vergleichsbasis herangezogen. Daneben konnte aber auch beobachtet werden, dass Begriffe, die als Lösung für eine Textlücke zur Auswahl standen, direkt miteinander in Beziehung gesetzt wurden.

Die Deutungen, die entstehen, entsprechen keiner präzisen mathematischen Definition, sondern bleiben häufig vage. In einigen Fällen verbleiben sie (wie bei der Deutung für *bereinigte Normalform*) auf einer intuitiven Ebene. Im Verlauf der Bearbeitung können die Deutungen für ein Zeichen allerdings präziser werden, wie man am Beispiel *Spaltenstufenform* erkennen kann. Die formulierten Relationen sollten also nicht als etwas Isoliertes betrachtet werden, denn Zeichen/Symbole und ihre Gegenstände/Referenzkontexte bauen immer auf Wissen auf, das bereits im Vorfeld konstruiert worden ist, also anderen epistemologischen Dreiecken. Sie können sich gegenseitig ergänzen, wobei „die Ecken des epistemologischen Dreiecks als Summe der bisher mit dem Konzept […] verbundenen positionsgleichen Elemente zu verstehen" (Rieß, 2018, S. 93) sind.

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The characteristics of proof teaching in a first-year university mathematics course

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Keywords: teachers' and students' practices at the university level, teaching and learning of analysis, commognition, lectures.

INTRODUCTION

Mathematical proofs have a central role in developing, establishing and communicating mathematical knowledge. Given the importance of mathematical proofs, introductory courses in mathematics bachelor's degrees are dedicated to introducing students to the proving process. The goal of this study is to characterise proof teaching in university mathematics lectures by using the framework of commognition (Sfard, 2008). The basic tenet of this framework is that thinking can be conceptualised as communication with oneself. This poster focuses directly on proof teaching in an introductory mathematical lecture. The research question of this study is: what are the characteristics of proof teaching in a first-year university mathematics course?

METHODOLOGY

Six online lectures (out of 17) by an exemplary mid-career lecturer teaching a first-year university mathematics course on real analysis were analysed. Inductive thematic analysis was performed, using themes (characteristics of proof teaching) from Karavi et al. (2022). Thus, the following characteristics were used, flexibility (i.e., performing a proof in more than one way), bondedness (i.e., making connections between the different steps of the proof), applicability (i.e., discussing the application of a proving process in other situations), agentivity (i.e., making decisions for the proving processes, evaluating and showcasing how one can explore them), objectification (i.e., increasing the level of abstraction of a mathematical object) and substantiability (i.e., establishing the criteria to judge and reflecting on the essence and key ideas of the outcome of the proving process).

RESULTS

All quotes in this section were taken from the proof of the characterisation of compact sets (K in $\mathbb{R}$ is compact if and only if K is closed and bounded). We identified a structure for the teaching of proofs in the first-year university mathematics course under study that we discuss briefly in this section. When introducing a proof the lecturer first stated what exactly needed to be proved.

> The first statement is K as a subset of real numbers is compact, and the theorem asserts that this is equivalent to saying that K is both closed and bounded.

He showed how this theorem relates to previous proofs or results (bondedness)

> So, this connects back to two questions earlier on the chat, are all compact sets closed?

He explored what definitions and tools could be used and what proving approach will be taken (agentivity).

> Well, let's do a proof by contradiction. Let's assume that a set K is not bounded, so I assume K is compact, and I assume K is not bounded. And then I want to force a contradiction.

During the proof itself, the lecturer related the (steps of the) proof to other steps, proofs and results (bondedness).

> Okay, this is only half of the proof. If K is compact, then it is bounded. But I still need to show that K is closed as well. Okay, so that will be the next slide.

He applied previously known theorems and definitions (applicability).

> So here we go. Since x is a limit point of K, we know that there has to be a sequence x_n in K, such that x equals the limit of this sequence x_n.

He showed that (steps of the) proof can be performed in multiple ways (flexibility), and finally repeated the statement of what exactly was proved.

> So, here's one characterization of compact sets, a set is compact if and only if it is closed and bounded.

When closing the proof, the lecturer repeated the main ideas of the proof and explained them in other ways (substantiability).

> So, if I assume that K is compact, then the assumption that K is not bounded gives me a contradiction. And therefore, the compactness of K implies boundedness of K.

In this poster, we explored the proof teaching through the identification of characteristics that originated from the literature. Each characteristic appeared to have a specific function given its place in the proving process. These identified characteristics can be a starting point for future researchers who aim to further investigate proof teaching.

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Digitale Brücke zwischen Schule und Hochschule? Ausgewählte Vorschläge für die Unterstützung von Beweisprozessen durch CAS in der Hochschulmathematik

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Keywords: Teaching and learning of logic, reasoning and proof, digital and other resources in university mathematics education, CAS-assisted proofs.

Das Ziel des Mathematikstudiums ist Studierende in eine Wissenschaft einzuführen (Weber & Lindmeier, 2020), in der Beweise eine zentrale Rolle spielen (Hanna & Barbeau, 2008). Inwieweit allerdings CAS in den Beweisprozess einbezogen werden kann, ist noch nicht ausreichend erforscht (Jankvist & Misfeldt, 2019).

THEORETISCHER HINTERGRUND UND FORSCHUNGSFRAGE

In der Mathematikdidaktik wird in Anlehnung an Wittmann und Müller (1988) zwischen experimentellen, operativen und formalen Beweisen unterschieden, dieses Modell ist nach wie vor aktuell (Brunner, 2014). Experimentelle Beweise verifizieren einen vermuteten Zusammenhang anhand einiger konkreter Beispiele, zeigen aber nicht dessen Allgemeingültigkeit. Operative und formale Beweise sind demgegenüber deduktiv und unterscheiden sich voneinander nur in der Ebene der Darstellung: Während bei formalen Beweisen an durch Symbole dargestellten Objekten symbolische Operationen durchgeführt werden, stützen sich operative Beweise auf konkrete Operationen, die an konkreten Darstellungen ausgeführt werden.

Jankvist und Misfeldt (2019) identifizieren in Schulbüchern drei Typen des Einsatzes von CAS bei Beweisen. Beim ersten und beim zweiten Typ wird die Führung eines Beweises vollkommen oder teilweise dem CAS überlassen, beim dritten Typ werden bereits bewiesene Sätze an konkreten Beispielen überprüft. Diese Typen des CAS-Einsatzes lassen sich durchaus mit den Beweistypen nach Wittmann und Müller (1988) vereinbaren: Der erste und der zweite Typ stellen formale Beweise dar, deren Schritte allerdings ausgeblendet werden, sind also verkürzte formale Beweise. Der dritte Typ entspricht einem experimentellen Beweis. Das Modell von Wittmann und Müller (1988) eignet sich also, bereits erbrachte CAS-unterstützte Beweisprodukte zu klassifizieren, auch wenn hierbei das Modell nicht vollständig ausgeschöpft wird. Es stellt sich somit die Frage, inwieweit das Modell bei der Entwicklung von CAS-unterstützten Beweisen, d.h. beim Beweisprozess erfolgreich eingesetzt werden kann.

ERGEBNISSE

Der Einsatz von CAS ermöglicht vor allem Manipulationen an algebraischen Ausdrücken (Zehavi & Mann, 2009), demzufolge kann CAS bei der Vermittlung solcher Beweise verwendet werden, die überwiegend algebraische Umformungen erfordern. Solche Beweise findet man in der Hochschulmathematik vor allem in der linearen Algebra und der Zahlentheorie. In einem ersten Schritt wurden daher Beispiele aus diesen beiden Disziplinen mit Hilfe des CAS für alle drei Beweistypen entwickelt, wobei formale Beweise ohne Ausblendung der Beweisschritte erstellt worden sind. Dies wird an einem konkreten Beispiel (Abbildung 1) verdeutlicht.

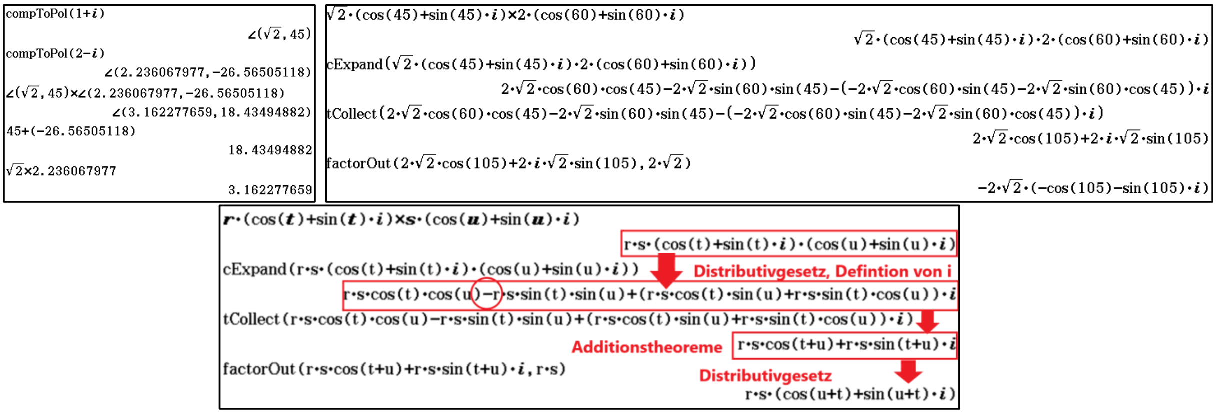

Abbildung 1: Multiplikation zweier komplexer Zahlen in Polarform (oben links: experimenteller Beweis; oben rechts: operativer Beweis; unten: formaler Beweis)

Weitere Beispiele werden im Beitrag präsentiert. Die Unterstützung des Beweisprozesses von Studierenden durch die Thematisierung solcher CAS-unterstützten Beweise wird im Wintersemester 2022/23 empirisch untersucht.

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Recognizing matrix equations as eigenequations or not: Examples of student reasoning in quantum mechanics

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Keywords: Teaching and learning of linear and abstract algebra, teaching and learning of mathematics in other fields, eigentheory, quantum mechanics.

Reasoning about mathematics is central in many of the scientific disciplines. Students often utilize mathematical concepts and procedures, mathematize physical constructs, and interpret mathematical entities in terms of physics (Uhden et al., 2012). For example, quantum mechanics problems often involve reasoning about linear algebra content such as matrix-vector operations, change of basis, eigentheory, projection, orthonormality, and inner products (e.g., Schermerhorn et al., 2019; Serbin & Wawro, 2022). Our broad research project investigates students' understanding, symbolization, and interpretation of eigentheory in quantum mechanics (US NSF #1452889). This poster will focus on the following research questions: in what ways do students recognize if quantum mechanical matrix equations are eigenequations, and how does this relate to their reasoning for eigentheory in mathematics and quantum contexts?

The data consist of video, transcript, and written work from individual, semi-structured interviews (Bernard, 1988) with ten volunteers from a senior-level quantum mechanics course at a medium-sized public research university in the United States. One interview question probed students' reasoning about three equations E1-E3 (Figure 1). E2 is a quantum mechanics eigenequation for a spin-½ system [1], and E3 is an equation in which the operation "flips" the spin state; E3 is *not* an eigenequation.

"I have a few equations prepared. For each one, I want you to explain what the equation means to you."

[E1] $A\vec{x} = \lambda\vec{x}$, where A is a 2x2 matrix, $\vec{x}$ is a 2x1 vector, and λ is a scalar

[E2] $\hat{S}_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$

"You mentioned both related to eigentheory. Please compare and contrast how you personally conceptualize eigentheory in the two situations."

[E3] $\hat{S}_z|+\rangle_x = \frac{\hbar}{2}|-\rangle_x$

Figure 1. The interview question used to gather the data analysed in this poster.

In this work, we adopt a theoretical stance consistent with the Knowledge in Pieces framework (diSessa, 1993). This assumes that students' intuitively held knowledge pieces are productive in some context and that knowledge change involves evolutionary refinement and reorganization of ideas. We conducted our analysis by iteratively examining the data for nuance in student imagery and noting relevant discursive cues (Gee, 2005) that we then organized into themes.

The results presented in the poster will focus mostly on student reasoning about E3. One aspect will delineate results related to student reasoning about if E3 was a valid equation. Of the four students who this engaged in this way, two of them used written calculations to eventually convince themselves of the equation's validity, and two of them believed it to be an untrue equation. The second aspect will delineate results related to student reasoning about if E3 was an eigenequation, with eight of them eventually determining that E3 was not an eigenequation. All eight in some way discussed the two kets in the equation not matching, either by reasoning about co-existing distinct vectors (a static view of the equation) or reasoning about not getting same vector back (a dynamic view of the equation); these are synergistic with results about E1 and E2. For example, one student stated, "So not so much an eigenvalue equation because we don't have the same vector on either side." In their examination of E3's structure, they seemed to leverage a static view of the equation as they looked for the same vector on both sides of the equal sign. This is consistent with Sherin's work on symbolic forms (2001), which interprets students' understanding of equations in terms of pairing symbol templates with conceptual justifications for the structure of the equation. The remaining two students displayed reasoning that indicated they knew E3 contained some aspects that related to eigentheory but were not sure if it was an eigenequation. For example, one student explained E2 in terms of measuring spin but voiced uncertainty about interpreting E3 with respect to measurement. The poster will include a broad synthesis of results across all three equations, highlighting instances of synergistic and potentially incompatible interpretations of the three equations, will offer pedagogical implications related to linear algebra, and will discuss avenues for future research such as the use of symbolic forms in mathematics education research.

NOTES

1. Spin is a measure of a particle's intrinsic angular momentum. Possible spin states are represented by normalized kets $|\psi\rangle$ which behave mathematically like vectors. The eigenstates $|+\rangle_x$ + and $|-\rangle_x$ for the spin-½ operator $\hat{S}_x$ correspond to the two possible spin measurements of $\pm\frac{\hbar}{2}$ along the x-axis, encapsulated in the eigenequations $\hat{S}_x|\pm\rangle_x = \pm\frac{\hbar}{2}|\pm\rangle_x$.

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TWG4: Teaching and learning of mathematics for engineers, and other disciplines

TWG4: Teaching and learning of mathematics for engineers, and other disciplines

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THE CONTRIBUTIONS IN TWG4

These proceedings evidence the rich content of the presentations in TWG4. Several "other disciplines" were considered by the authors: engineering, physics, chemistry, life sciences, with different foci that we briefly evoke here.

Some studies concern interventions, from their design to their implementation and the evaluation of their impact for students. Pollani and Branchetti designed and implemented a course for future mathematics teachers about mathematics-physics interdisciplinarity, where the students investigate the characteristic features of the two disciplines and their boundaries (Akkerman & Bakker, 2011). Cabrera, Vivier, Montoya and Vandebrouck, referring to the Mathematical Working Spaces theory (Kuzniak et al., 2022), explore student learning in an experimental course where trigonometric polynomials and Fourier series are used for modelling sounds. Rizzo introduces an active teaching approach, where Natural Science students collectively work on modelling tasks. Hernández-Méndez, Cuevas-Vallejo and Orozco-Santiago also implement a modelling-based course about differential equations and the harmonic motion. Rønning investigates the impact, in terms of students' perceived relevance, of a contextual learning approach to mathematics teaching, where students use mathematics to solve engineering problems. In these studies, concerning modelling-based courses, different kinds of technological tools play a significant role, providing in particular access to specific representations (e.g. Audacity representing the sounds as sinusoidal curves).

Several intervention studies in TWG4 concern Study and Research Paths (SRPs, Bosch et al., 2020) and refer to the Anthropological Theory of the Didactics (ATD, Chevallard, 2015). Lombard presents an epistemological analysis grounding the design of an SRP at the interface between mathematics and quantum mechanics. Markulin, Jessen and Florensa focus on the cross-disciplinary collaboration needed for managing an SRP in statistics for business administration (including the active participation of the client). Freixanet, Alsina and Bosch also implemented and SRP about statistics. In their study, the questions were proposed by the students (first-year future engineers) themselves, about the proposed topic of water as an indispensable resource.

While the researchers were involved in the design of the courses in the above-mentioned studies, some institutions also propose innovative courses. Salinas-Hernández, Kiliç, Kock and Pepin consider a challenge-based course for future engineers. Referring to the instrumental approach (Trouche, 2004), they investigate the use of resources by students in such courses, and how they perceive their learning both professional and in mathematics. The issue of "authenticity" is also important in the study by Hilger, Schmitz, and Ostsieker, which concerns the views of engineering students about application examples (provided in 'usual' courses), and how these views are linked to students' beliefs (Rooch et al., 2014) about mathematics.

'Ordinary' mathematics courses for non-specialists are also studied by other authors. Rogovchenko and Rogovchenko analyse Calculus and differential equation courses in terms of potential conflicts for student learning, in particular in terms of concept image (Tall & Vinner, 1981). Burr studies the teaching and learning of Numerical Analysis. Gueudet, Doukhan and Quéré focus on teachers' practices in mathematics courses for non-specialists, using ATD (Chevallard, 2015) and the concept of didactical praxeologies to identify specificities of these teaching practices.

Cuenca, Barquero and Florensa also refer to ATD, and analyse a reform of the engineering mathematics curricula in Ecuador. They evidence the stability of the contents of the courses, in spite of changes in their titles.

We note that, whatever their focus is, all these studies took into account the specific features of teaching mathematics to non-specialists. In what follow we briefly summarize the thematic discussions in TWG4.

INTERVENTIONS

One of the main discussions of the group, accordingly to the number of proposals including them addressed interventions in different programs and courses. The first aspect regarded the need to establish *collaborating groups* in order to reflect and to design the interventions. Secondly, the group addressed the need to systematize *the dissemination* of mathematics education research results in order to ensure an impact on the actual teaching practice.

The need for collaborations between researchers in mathematics education, mathematics teachers and teachers of the "domain" where the mathematics courses are taught (engineering, business administration, etc.) was considered as a necessity emerging from the analysis of the different contributions to the TWG4. Specifically, one of the proposals was to consider the members of the mathematic education research community as the brokers between the teachers of the domain and mathematics teachers to facilitate the design and implementation of new teaching proposals considering the need of each domain. This idea of networks of mathematic researchers and teachers was seen as a way to study and modify the conditions affecting the research-based proposals avoiding the fragility of the implementations leaded by a single person doing at the same time the role of researcher-teacher.

However, the development of these collaborations needs to consider several factors explicitly. For example, it is important to define the inputs considered and the outputs expected of these collaborating groups. Another aspect to be considered is the need to fix the institutional settings to ensure the ecological viability of interventions: as long as this activity is not considered as a part of the institutional activity, and that might involve different departments, their viability might be very fragile. The discussion also addressed the importance to reflect on the level for implementing interventions: lesson, course, or even full program.

Regarding the dissemination of the research results, the discussion considered two main proposals. Firstly, the participants considered that there is a need to develop resources around the teaching of mathematics to non-mathematicians addressed to mathematics teachers. One of the points that were considered is the need that these resources should be developed by mathematicians, mathematics educators and non-mathematicians working together. A second aspect that was considered to facilitate the dissemination of research results is the need to develop professional development proposals further from pedagogical courses. One of the specificities of undergraduate mathematic teachers is often the need to hold a doctorate in mathematics and some pedagogical course while the courses on didactics are often seen as complementary.

MODELLING

A second aspect that was addressed during the discussions of TWG4 was the consideration of modelling as one of the proposals that should be incorporated to the mathematics courses for non-mathematicians. However, the discussion revolved around the need to consider "authentic modelling tasks" coming from the workplace of engineers, biologists, etc. and avoiding giving a ready-to-use model to students. In other words, activities such as developing the model, validating and proving its accuracy that often do not exist in mathematics courses were considered by the participants as crucial as other activities such as using the model to obtain results which are very common in school settings.

A second aspect, related to the theories used to analyse modelling processes concerns the need to overcome the classical dichotomy between "non-mathematical" and "mathematical" contents proposed by diverse models. The discussion also considered important to enrich the different theories emerging from mathematics education with the theory of models in philosophy or the analysis of modelling in other disciplines such as engineering.

EPISTEMOLOGY

Finally, the need to develop appropriate tools to explicitly describe knowledge was considered as a priority by the participants. There was a clear consensus on the need to question of what is conceived as mathematics when addressing problematic phenomena in the teaching and learning of mathematics. In other words, the "classical knowledge labels" (such as derivatives, differential calculus, or statistics) are not a precise enough way to describe knowledge: the activity developed under these labels can be significantly different depending on the teaching proposal.

The role of proof also was addressed during the TWG4 discussions. Often, the mathematical activity of non-specialists does not explicitly include proofs: however, the models used are founded and considered valid because they had already been validated. A clear open question is how to deal with this “hidden” role of proof in the mathematics taught to non-mathematicians.

FUTURE DIRECTIONS

The future directions discussed in TWG4 were linked with the different issues discussed above. They concern both research and development, which are closely linked in these issues. Developing groups associating mathematics education researchers, mathematicians, specialists of other disciplines is a need for the design of productive interventions. Nevertheless, the conditions allowing the existence of such groups in various institutional contexts are complex and deserve a specific study. Similarly, understanding the conditions for a productive design of resources for university teachers – mathematics teachers teaching to non-specialists, or teachers of disciplines using mathematics – grounded in research results, and contributing to the dissemination of these results is a complex issue requiring further research. Would it be possible, for the “UME for non-specialists” research community, to write a ‘white paper’ presenting recommendations for stakeholders and policy-makers? Would it be possible for this young community to identify a list of “solid findings” that could support recommendations for practice, and/or professional development programs? The successful interventions designed are often very local; further research could work in the direction of interinstitutional projects, to evaluate the generalizability of these interventions.

Other research directions concern the epistemological basis of the studies, and the need to deepen our understanding of the links between mathematics and other disciplines. Including historical sources in our studies, which enlighten the historical genesis of the contents at stake, their raison d’être, appears as a promising mean for investigating this direction.

Finally, while ATD (Chevallard, 2015) seems to be increasingly used in TWG4, other frameworks were also present in the group (e.g. boundary crossing, the instrumental approach). The question of the theoretical frameworks relevant to identify and study research questions specific to mathematics for students following physics, chemistry, biology, engineering, economics or other subjects remains open.

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Modelling periodic phenomena through trigonometric polynomials using digital tools

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The teaching of trigonometrical functions usually is carried out from relationships with measurement of arcs and the unit circle. Within the properties of trigonometric functions, periodicity is one of the most important. We present the first results of a sequence of tasks whose objective was to model sounds using trigonometric polynomials and Fourier's theory, through experimentation in 1st university year with Audacity and Geogebra software. The circulations and activation of planes in the sense of the Mathematical Working Space of the students were analysed while the students' transits for the different stages of the modeling cycle. Among the main results, we recognize that students build the concept of trigonometric function from time-amplitude variables, avoiding the concept of angle or radian.

Keywords: digital technologies, teaching analysis, periodical functions, trigonometric polynomial, modelling,

INTRODUCTION

The study of periodic phenomena has been developed for centuries cultures such as the Babylonian and the Egyptian used the periodic to make predictions (Montiel, 2005). One of the first steps that transitioned trigonometric functions away from geometry was the recognition of their periodicity. In 1670, in *Wallis's mechanica* as sine curve with two complete periods may be found. The conversion and full admittance of trigonometric quantities to the family of functions was accomplished by Euler (Van Brummelen, 2021).

The first periodic functions taught in secondary education are the trigonometric functions sine and cosine, this is due to the importance they have within mathematics and science (Fourier series, electricity, waves, sound, harmonic oscillator, diffusion of heat, etc.). The teaching of these functions in secondary education may vary depending on the country or the educational institution, however, they are often taught based on the teaching of trigonometry.

In relation to the trigonometric function, research has recognized that for the student there is no distinction between ratios and functions, or at least that there is a mixture of concepts to solve problems related to functions (Tanguay, 2010; Winsløw, 2016; Loeng, 2019).

Sound is one of the periodic phenomena that is modeled by trigonometric functions. From a physical point of view, sound is a mechanical wave produced by a vibrating body that propagates in an elastic medium. Any sound, such as a musical note, can be described in physical terms by specifying the frequency, amplitude or air pressure, and timbre or waveform.

In the other hand, the integration of technologies in education has had a sustained boom during the last decades, however, the results have not been as expected (OECD, 2015). Due to its importance, over time the interest in carrying out studies on the uses of technologies to improve calculus concepts at the university level has increased. In addition, one main difference between secondary school and university work is the use of artifacts, mainly digital tools, to make calculations and develop multiple representations of mathematical objects. Artifacts and tools are widely used at secondary school whereas practices at university level tends to banish technologies from students' activities.

Based on the above, we have set ourselves the objective: study the mathematical work of first-year university students in a task of modelling periodic phenomena related to sound, which integrates technological tools. This task also allows students to conceptualize trigonometric functions from variables related to time and amplitude, rather than angles.

FRAMEWORK

Based on the proposed objective and the design of the task, we used the mathematical modeling cycle (Borromeo-Ferri, 2010) and the Mathematical Working Space [MWS] (Kuzniak, Montoya-Delgadillo & Richards, 2022) articulately.

The MWS allows describing and characterizing the mathematical work of an individual from a specific task. With this approach it is possible to describe how epistemological and cognitive elements are articulated in the solution of a task, showing the circulations between the activated MWS components (Kuzniak et al., 2022).

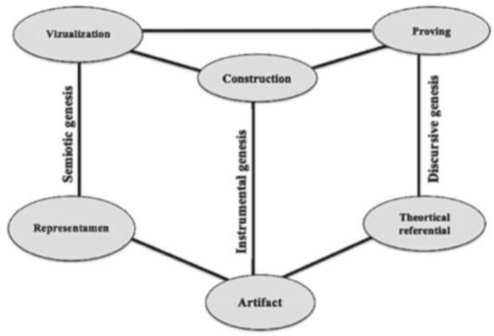

Fig 1a. Mathematical Working Spaces diagram (Kuzniak et al., 2022)

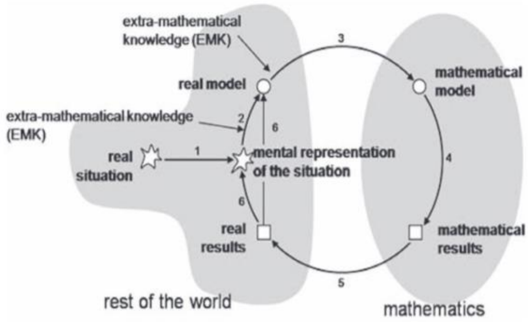

Fig 1b. Modeling cycle (Borromeo-Ferri, 2010)

The MWS consists of the articulation of two planes: Epistemological and Cognitive, which interact with each other, through a genesis process (Kuzniak et al., 2016; Kuzniak et al., 2022). It is worth mentioning that there is also an articulation between the different genesis, generating three vertical planes: [Sem-Ins], [Ins-Dis], [Sem-Dis].

Mathematical model is defined as a set of symbols and mathematical relations that represents, in some way (graphic, numerical or algebraic), the phenomenon in question. We incorporate certain concepts related to mathematical modeling from the cognitive point of view, for this, we will use the Blum-Borromeo modeling cycle (Borromeo-Ferri, 2010). The starting point of the modeling process is a real world situation, which the students read and understands, then idealized (2), simplified or structured to obtain a real model. The real model is mathematized (3), i.e., translated into mathematical language to obtain a mathematical model of the initial situation. A mathematical treatment (4) leads to mathematical results, interpreted as "real results" (5) which will be validated or not in relation to the model situation (6). Then, if the results obtained seem to be coherent, they will be consistent, they will be presented as predictions about the real situation (7).

CONTEXT AND METOD

The methodology that guides this research is Didactic Engineering (Artigue, 1995). To respond to the proposed research objective, the methodology used is qualitative (Bikner-Ashbahs, Knipping & Presmeg, 2015), where the Case Study (Yin, 2009) was used to know and understand the context of the students. The implementation is made up of 5 second semester undergraduate students in mathematical (2021-2022, period), who are taking the subject of *Mini-Project de Mathématique* who formed 3 work groups. Each group makes up a case. The first group (G1) was made up of one student, the second and third groups were made up of two students each (G2, G3). The proposal was developed during 4 classes of 2 hours each. The first task was developed during classes 1 - 2, the second task was developed in class 3, and the third task in the last session. As part of the course, students were given a document with a brief definition of Fourier Series and its application, from which the definition of trigonometric polynomial is presented. This last concept should be used by the students in this proposal.

The design of the proposal had the intention of generating a modeling process, which would encourage the incorporation of technological tools such as Geogebra or Audacity by students. Audacity is free software that allows you to record and edit audio. Audacity's interface provides a graphical representation of the sound using the variables time and amplitude. Both software's allow you to export the data in Excel files.

The participant observation technique was used by the researcher, since he was the group's teacher during the class sessions. The data has been extracted from the written productions of the students, from the geogebra and Excel files constructed by the students. Mathematical work in the sense of the MWS will be analyzed, as the students go through the different phases of the modeling cycle. We have considered the methodology carried out by Reyes-Avendaño (2020), in which analyzes the activation of the genesis, plans [Sem-Ins], [Ins-Dis], [Sem -Dis] and the circulations between them, in each one of the phases of the modeling cycle, indicating with arrows within the MWS diagram the activation chronology at each phase.

PRESENTATION OF THE TASKS

The objective of the situation is for students to model periodic phenomena through trigonometric polynomials presented as a Project that included three tasks. The first two tasks given to the students and the objective of these tasks will be described below, in the next section the results of these tasks will be discussed. The third task only its objective will be presented.

First task the objective was to recognize that pure sounds are those that are modeled by trigonometric functions. To accomplish this objective, two subtasks were designed. The subtask 1.1 given was: *Characterize the sounds present in the folder.* Students are presented with a folder with 10 different sounds, and the software Audacity to play these sounds. Students are expected to build the real model (RM) of the situation using their extra mathematical knowledge (EMK) in relation to music or physics. The classification is expected to be made in terms of the nature of the sound, the shape of the wave through what is observed in the Audacity interface or from the vibrating body that produces it. The transition to Mathematical Model (MM) is guided by subtask 1.2: *calculate the function associated with each of these sounds*. To do this, students export data from Audacity to Excel spreadsheet and using Geogebra's Two Variable Regression Analysis tool obtain a graphical representation of the sound. Mathematical Results (MR) obtained are given in terms of the analytical expression of each sound, thanks to the adjustment made by Geogebra. In the transition to the real results (RR) it is expected that students generate a second classification, since pure sounds can be adjusted by a sinusoidal function and complex sounds cannot.

In terms of MWS, Audacity is an artifact that allows students to activate semiotic genesis to construct RM, because they visualize the behavior of the sound graph in the software interface. This artifact allows students to obtain approximations of the periodicity of the wave to calculate the frequency. In the construction of RM and the transition to the mathematical world, it is the instrumental genesis that directs the activation processes of the semiotic genesis through the visualization of the wave and its representation on the Cartesian plane. It is the semiotic genesis that leads the testing processes when adjusting process is developed in Geogebra. Finally, the proof processes in the transition to RR can be based on the activation of the [Ins-Dis] and then [Sem-Dis] plane since the new classification is made in relation to the adjust made it and the analytical expression provided by Geogebra and what is visualized in it.

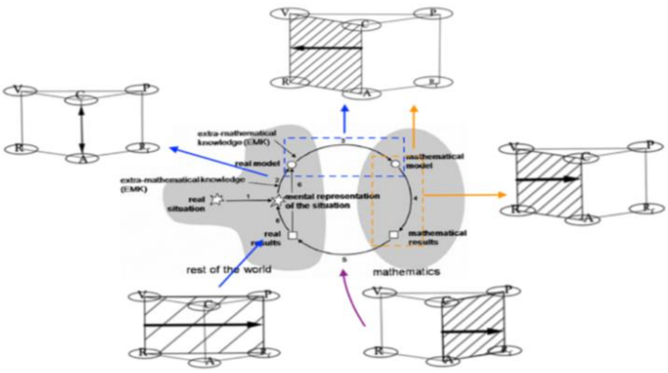

Fig 2. Genesis and plans activated during the modeling cycle of first task.

Second task. The objective was to determinate the trigonometric polynomial associated with the superposition of two pure sounds. To accomplish this objective, were designed two subtasks. Subtask 2.1 given to students is: *Using the Audacity mix tool, mix the sounds LA 440hz and DO 264hz present in the given folder and determine the function that models this sound.* The RM is obtained from the sound mixed in Audacity, which provides a graphic representation in its interface that allows it to be classified as a complex sound, following the classification made it in the first task. In the transition to MM, students are expected to relate the superposition of sounds to the sum of the functions that model each sound separately, obtaining that the function is $S(x) = \sin(2\pi \cdot 440\, x) + \sin(2\pi \cdot 264\, x)$, which can be written as $S(x) = \sin(2\pi \cdot 3 \cdot 88\, x) + \sin(2\pi \cdot 5 \cdot 88\, x)$ which allows transit from MM to MR, concluding that the function S is obtained as a superposition of two sound with a frequency of 88hz, whose period is approximately $T = \frac{1}{88} = 0{,}01136$. On the way to the RR, students can export the mixed sound data and graph it in geogebra, to graphically contrast it with the function S obtained earlier.

Subsequently, subtask 2.2 given to students was: *Determine the first coefficients of trigonometric polynomial from the data set provided by Audacity.* This subtask will allow the students to resignify the previously described model, since it will be obtained from Fourier Theory. In terms of modeling cycle, the resignification process is oriented from RR to MM. For the construction of the new MM, students must determine the period of the graphed wave, obtaining the interval $[t_0, t_0 + T]$, whose values they will use from Excel cells. After that, students must calculate the Fourier coefficients a_n and b_n as: $a_0 = \frac{1}{T}\int_0^T f(t)dt$, $a_n = \frac{2}{T}\int_0^T f(t) \cdot \cos(n\omega t)\, dt$ and $b_n = \frac{2}{T}\int_0^T f(t) \cdot \sin(n\omega t)\, dt$, with $\omega = \frac{2\pi}{T}$.

However, since the coefficients will be calculated from a data set, in the transition to MR, students must use the Riemann sum to approximate the value of the integral, therefore, they must calculate the coefficients as: $a_n = \sum_{t_0}^{t_0+T} f(t_n) \cdot \cos(\frac{2\pi}{T} t_n)$ and $b_n = \sum_{t_0}^{t_0+T} f(t_n) \cdot \sin(\frac{2\pi}{T} t_n)$. As MR students are expected to determine that $b_3 = 1$ and $b_5 = 1$. In the validation process towards the RR, students can use Geogebra's

PlaySound tool and analyze the difference between the sound created in Audacity and the one delivered by Geogebra, or graphically determine the error between the calculated trigonometric polynomial and the set of data delivered by Audacity.

Figure 3 shows the mathematical work developed during subtasks 2.1 and 2.2. Audacity is an artifact that allows groups to recognize the wave graph to construct the RM as a complex sound related with a superposition of sounds. Semiotic genesis is activated from the software interface in the transition to MM. Students can relate the wave modeled by Audacity with the function obtained through the visualization in Geogebra. They can use the graphical view as a pragmatic proof to determine that this function is the one that models the mix of sounds. In relation to subtask 2.2, when calculating the trigonometric polynomial with the data provided by Audacity, Excel spreadsheet is a first available resource. The organization of mathematical work in the spreadsheet is based on the students' theoretical referential of the concept of integral and Riemann sum. Students must organize a set of interrelated operations in the software to develop the calculation of the coefficients. It observes semiotic components associated with reasoning in relation to definitions and mathematical properties that are articulated, thanks to the intentionality given to the use of this digital artifacts, therefore, it is from the discursive genesis that the plane [Sem-Ins] is activated.

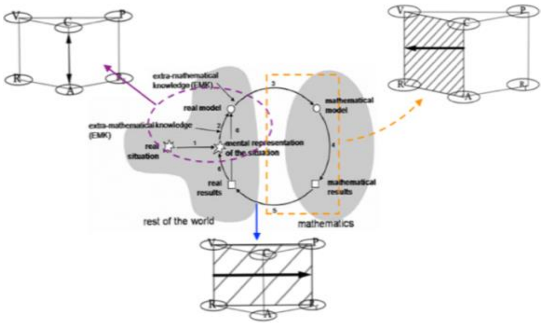

Fig 3a. Mathematical work developed in subtask 2.1

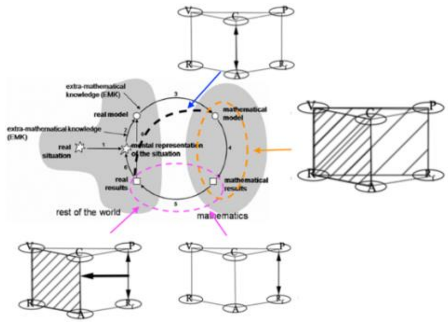

Fig 3b. Mathematical work developed in subtask 2.2

Third task. The objective of the third task was to model a complex sound through a trigonometric polynomial. The students were given only one task: *Determine the function that models this sound*. The students received an audio file with the sound of a flute playing the note LA. Students are expected to carry out a modeling process like the one developed in the second task.

DATA ANALYSIS AND RESULTS

The results obtained in the implementation of the elaborated proposal are presented below.

First Task

In relation to subtasks 1.1 and 1.2, the three groups achieved the proposed objective: to characterize pure and complex sounds and calculate the function associated to each sound. Students made a transit through the different phases of the modeling cycle when developing both tasks. They play the sounds in Audacity and that allows him to create a classification between pure and complex sounds due to graph provides them (Fig 5a) and their EMK. Particularly, G1 associates the pure sounds to sinusoidal functions whose algebraic expression is: $f(t) = Asin(tx + \varphi)$. Likewise, it describes complex sounds as those that have a fundamental frequency and other frequencies called harmonics (Fig 5b).

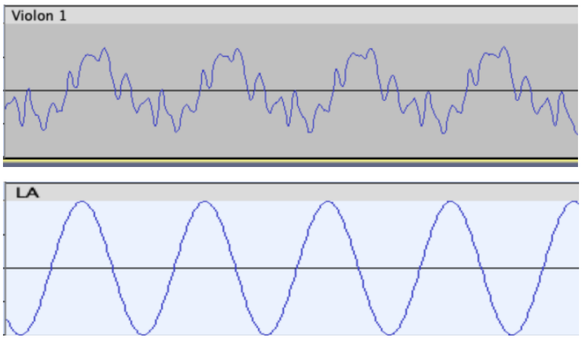

Fig 5a. view of violin and LA sounds in Audacity

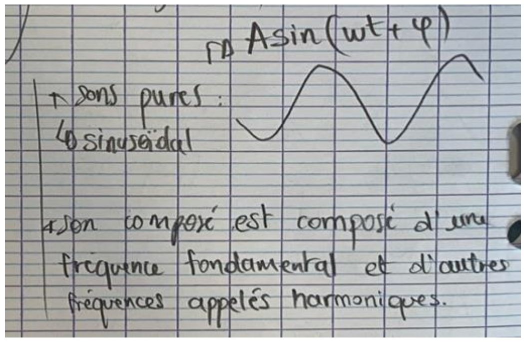

Fig 5b. G1 analysis, evidencing knowledge of music

Students used the Audacity data export tool and thanks to the Geogebra's tool *Two Variable Regression Analysis* get the graphical approximation. Fig 6 shows the function obtained (red one) for the adjustment of the LA 440hz sound. This value is sound closely approximates the theoretical result, which is $f(x) = \sin(2764.5992x)$.

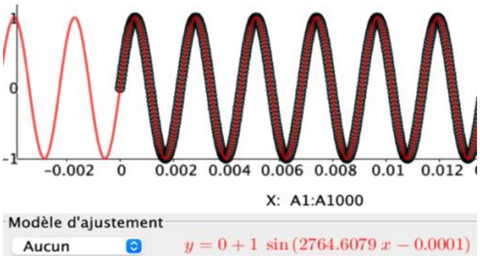

Fig 6. Adjustment obtained LA-tone 440hz made by G2

In relation to *a priori* analysis carried out, Audacity is an artifact that allowed students to activate semiotic genesis thanks to its interface. This artifact allowed the students to obtain approximations of the periodicity of the wave to calculate the frequency. In the construction of the RM and the transition to the mathematical world, it is the instrumental genesis that directs the visualization processes. In turn, in the adjustment processes, it is from the semiotic genesis that the mathematical work is oriented. Finally, in the transition to RR, the students classify that the pure sounds are those that are approximated by trigonometric functions as expected.

Second task

Subtask 2.1 was to determine the function associated with two-tone superposition created in Audacity. To perform this subtask, Audacity "mix" tool is used. The three groups were able to perform this task and determine the function associated with the superposition of sounds. In relation to the a priori analysis, students export the data from Audacity to Excel and then graph it in Geogebra. G1 and G3 after graphing the data set, immediately input a function that is the sum of $LA(x)$ and $DO(x)$. G2 uses the Geogebra adjustment (Fig 7a), without obtaining results, later it uses the strategy of the sum of the known functions (Fig 7b). Three groups determine that the function $S(x) = \sin(880\pi x) + \sin(528\pi x)$, is the one that approximates the behavior of the data.

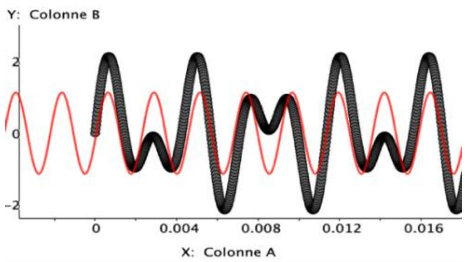

Fig 7a. Adjustment developed by G2

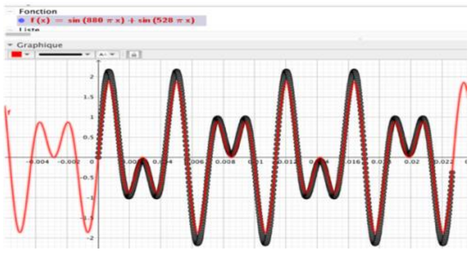

Fig 7b. Graph of the mix obtained by G3

In the validation process, groups mobilize the concept of approximation error through the graph of the function. In it they recognize that the distance between the values of the data set and those of the function is good except in the relative maximums and minimums of the function.

In relation to subtask 2.2, students were asked if they could determine the trigonometric polynomial associated with the first coefficients of Fourier Series using the data's provided by Audacity. In Fig 8 is presented how G1 construct their MM, previously be entered in the spreadsheet to calculate the coefficients using Riemann Sum knowledge, identifying the interval of integration as T = [0, 0.0113] and the fundamental frequency $\omega = \frac{2\pi}{0{,}0113}$ of the periodic wave.

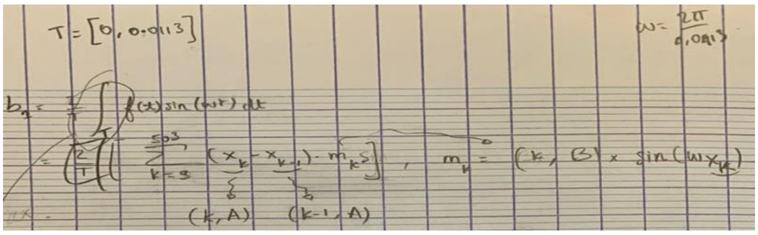

Fig 8. Expression to calculate the coefficients of trigonometric polynomial in Excel, G1

The spreadsheet developed by G3 is shown in Fig 9, unlike the above, builds the MM directly on the spreadsheet of Excel using Riemann Sum knowledge. In the developed programming, it is observed that G3 determines that the period is $T = 0.011$.

F3 =SENO(((2*PI())/0,011)*A3)

	A	B	C	D	E	F	G	H
1	Tiempo	Amplitud		intervalles		sinus	sinus*amplitude	aire
2	0	0						
3	0,00002	0,11503		0,00002		0,01142372	0,001314071	2,62814E-08
4	0,00005	0,22972		0,00003		0,02855605	0,006559896	1,96797E-07
5	0,00007	0,34373		0,00002		0,03997325	0,013740006	2,748E-07

Fig 9. Excel spreadsheet built by G3

Columns A and B are the ones that Audacity delivers when exporting the data. G3 creates in column D the distance between two consecutive cells of time variable, e.g., $D_3 = A_3 - A_2$, which represents the base of the rectangles in the Riemann method. In column F, G3 calculates the sine value for a given time, and then multiplies this sine value by its amplitude (column G). Finally, in column H, determine the area of each rectangle by multiplying the values in the columns D and G. The value of the sum, Excel delivers it by selecting all the values of column H, finally the value of the sum is multiplied by $\frac{2}{T}$.

As expected in the a priori analysis, the students mobilized knowledge from their theoretical referential related to the definite integral. The validation process towards the RR occurs in terms of the graphical behavior of the trigonometric polynomial calculated in relation to the set of values delivered by Audacity, however, they do not develop a testing process from an extra-mathematical point of view.

CONCLUSION

In terms of the **research objective**, we can conclude that students build the concept of trigonometric function as one that models pure sounds. Likewise, the trigonometric functions are defined in relation to the time and amplitude variables, proposing an approximation to these functions different that using the concepts of angles and radians. From this conceptualization of trigonometric functions, integrating knowledge of physics and music, students make sense of the phenomenon in question by relating elements of trigonometric functions such as amplitude and periodicity to the sounds reproduced. In relation to complex sounds, these encourage the student to mobilize the knowledge related to Fourier Theory to be able to model them mathematically.

In terms of digital artifact, audacity made a link between the two disciplines, music and mathematics. This allowed understanding that pure sounds were associated with trigonometric functions from the software interface. Audacity allows it to export data in Excel to be graphed in Geogebra, thus providing graphical, numerical and acoustic representations of sounds. Geogebra also allows to develop pragmatic testing processes based on the graphic representation obtained and the adjustment function delivered.

The design of a modeling proposal allows the connection of different mathematical fields through the integration of technologies. The proposal allows connecting knowledge of physics, music and mathematics, often disconnected, to conceptualize trigonometric functions as those that model pure sounds. This conceptualization promotes the understanding and modeling of periodic phenomena, such as sound, through the calculation of trigonometric polynomials associated with Fourier's Theory. We conclude that the design of mathematical modeling tasks integrating technological tools seems to be a good way to transition from high school to university.

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Mathematics curriculum reforms in undergraduate engineering: An institutional analysis

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This paper analyses the curricular reforms that have been carried out in the engineering mathematics curricula in the context of the Universidad Técnica Particular de Loja (Ecuador). Starting from diverse documentation related to a specific curricular reform, we conduct an analysis using the levels of didactic codeterminacy. More concretely, we analyse the criteria used to develop the reform, the real and effective changes that are reflected in the new programmes as well as the level of questioning of the knowledge to be taught.

Keywords: Teaching and learning of mathematics for engineers, Curricular and institutional issues concerning the teaching of mathematics at university level.

INTRODUCTION

Research on mathematical education in engineering education is a flourishing field, especially within the framework of the anthropological theory of the didactic (ATD) (Chevallard, 1992). The previous CERME (see, for instance, González-Martín et al., 2019) and the INDRUM conferences have also gathered and discussed research papers on engineers' mathematics education. Moreover, addressing both, problems related to the teaching and learning of mathematics for undergraduate engineers and the advances in creating suitable conditions for innovative proposals in mathematics for engineers (González-Martín et al., 2021). The analysis of how mathematics teaching programmes have been defined and implemented are important aspects to progress on analysis of the ecological conditions, in the terminology of the ATD, under which mathematics is taught for undergraduate students. With this respect, Bosch et al. (2021) focus on analysing some didactic phenomena related to the definition of university mathematics programmes, when comparing how they have been established in mathematics university degrees in some European countries. The research presented in Kaspary (2020) presents some advances in the analysis of the interaction between the different actors and institutions involved in programmes' definition.

Another area of research is that of discontinuities, already announced by Felix Klein (2016). The *first discontinuity* refers to the students' difficulties encountered in the transition from secondary school to university. Several studies have addressed this phenomenon, including those by Gueudet (2008) and Fonseca (2004) about mathematics and didactic discontinuities in the transition from secondary school to university. The *second discontinuity*, announced by Klein, focus on the discontinuity that mathematics students experience when moving from university to become secondary school teachers, when they are faced with the transposition of scholarly knowledge into knowledge that is to be taught.

In the case of mathematics education at engineering degrees, the recent work by Florensa et al. (2022) focuses on the characterisation of the *triple discontinuity* in mathematics education for engineers. Specifically, the first one is equivalent to Klein's first discontinuity (transition from secondary school to university), which according to Fonseca (2004) since in the tenth conjecture he states that:

> School mathematics has a strong pre-algebraic character in secondary school and undergoes an abrupt algebraization at the beginning of university education, given that in secondary school, equations and formulas are used as calculation algorithms, while at university they are used as algebraic models. (Fonseca, 2004, pp. 147, our translation).

This conjecture is evidenced in the study carried out by Cuenca and Granda (2020) at the Universidad Técnica Particular de Loja (UTPL) in which they found that students face difficulties when solving problems in contextualized situations, being accustomed to solving them in an algorithmic and mechanic way.

The second discontinuity occurs between mathematical and engineering courses. In other words, this is a discontinuity that is internal to the school of engineering and is associated with the difference between the mathematics courses compared to the engineering courses. The third discontinuity refers to the passage from engineering school to professional practice.

In this context, our research focuses on the curricula reform as it has occurred in the Universidad Técnica Particular de Loja (UTPL), selecting the mathematics training of undergraduate engineers. Specifically, we address the following research questions:

- RQ1. How is the need for a curricula reform justified? In what terms do the intervening "noospherian" institutions describe and justify this need?
- RQ2. What are the criteria this reform is based on?
- RQ3. Which are the real and effective changes that crystalise in the resulting mathematics programmes?

CONTEXT AND CURRICULAR REFORM

The university institution we focus on is the Universidad Técnica Particular de Loja UTPL, placed in the city of Loja (Ecuador). This is an institution that offers ten university degrees in engineering: agricultural, food, civil, geological, computer, telecommunications, chemical, industrial, logistics, and transport. These ten engineering degrees have a 9-semester structure, which is developed in four and a half years and with an academic load of 15 credits (not ECTS) per semester (where 1 credit is equivalent to 26,67 hours of teaching). As it is usual in most of engineering programmes (Romo, 2009), during the first four semesters all degrees have common subjects of mathematics.

In the context of the UTPL, the curricular reform was initiated from the request of the Higher Education Council to the higher education institutions to work on the reform of university degree programmes under the third transitory provision of the Academic Regulations issued by the council (CES, 2015). Thus, the process of restructuring the mathematics courses was initiated considering some results obtained by Cuesta et al. (2016), stating that university students had difficulty understanding and addressing problems, which are often non-adapted to the university degree they have chosen. This is aligned with González-Martín and Hernandes-Gomes (2020) and González-Martín et al. (2021) who state that mathematics content is generally taught separately from professional courses, which implies that there is a disconnection between mathematics and its application to engineering in contexts. Cuesta et al. (2016) also mention that one of the possible causes that pronounce these problems is in the curriculum itself. In the UTPL curriculum, until 2016, mathematics subjects included a wide range of content that grouped together two or more domains (e.g. “Mathematics” course covered contents on the domains of Geometry, Linear Algebra, and Statistics). Furthermore, there is not a compulsory proposed sequencing of mathematics subjects. Sometimes students, joining one advanced course (e.g. Calculus), have not passed courses with more basic content (e.g. Basic Mathematics).

Implementing of the curricular reform

The team in charge of working on the reform consisted of two teachers: the first with a university background in pure mathematics and the other with a background in mathematics education. This team oversaw the project of restructuring the mathematics courses. They were also responsible of writing the final report of the reform project (Cuesta et al., 2015). This team of specialists was supported by 4 lecturers from the Physicochemical and Mathematics departments, who carried out an analysis of the mathematics courses taught in all the university's degree before the reform in 2016. This project resulted in a final report on the redesign of Mathematics and Physics subjects, which was submitted to the UTPL authorities.

Degree	**Courses**					
	Pre-reform	**Credits**	**Year**	***Post reform***	**Credits**	**Year**
Geology and mining engineering	Calculus	6	1	Geometry fundamentals	3	1
				Linear algebra	3	1
				Univariate mathematical analysis	3	2
	Statistics	4	2	Multivariate mathematical analysis	3	3
				Differential equations and numerical methods	3	4
Civil Engineering	Mathematics	7	1	Geometry fundamentals	3	1
				Linear algebra	3	1
	Calculus	7	2	Univariate mathematical analysis	3	1
				Multivariate mathematical analysis	3	2
	Advanced calculus	6	3	Differential equations	3	3
				Numerical methods	3	4
Computer and	Mathematical fundamentals	4	1	Mathematical fundamentals	3	1
	Calculus	6	2	Linear algebra	3	2
	Statistics	4	3	Univariate mathematical analysis	3	2

Information Systems engineering	Quantitative methods	4	4	Differential equations and numerical methods	3	3

Table 1: Mathematics courses and workload pre and post-reform

Table 1 presents a summary of the different mathematics courses proposed in the pre- and post-reform curriculum, part of the three main engineering degrees in the UTPL. These curricular and programmes reforms are the result of, on the one hand, the specialist team who proposed the distribution of the courses and the prerequisites. On the other hand, the local teams who worked on the reform of the course programmes, selecting, and reorganising the contents.

DATA SELECTION AND METHODOLOGY

To analyse the reform process, the material to be analysed was selected by the authors of this paper. Firstly, three of the ten engineering degrees taught at the UTPL were selected: Geology and Mining Engineering, Civil Engineering, and Computer and Information Systems Engineering. The criteria for selecting them was that they are the engineering degrees with a higher load of mathematics courses in the post-reform. Then, the pre- and post-reform programmes of the mathematical subjects are used as empirical data for the analysis. Specifically, there have been analysed 3 pre-reform courses (Mathematics, Calculus, and Advanced Calculus) and 6 post-reform subjects (Univariate Mathematical Analysis, Multivariate Mathematical Analysis, Linear Algebra, Fundamentals of Geometry, Differential Equations, and Numerical Methods). We have decided not to include, in this first round of analysis, the courses on Statistics.

Secondly, the reports produced by the teams and commissions responsible for the reforms have been also important documents for the analysis. Finally, eight interviews have been carried out with the teachers in charge of these courses (five of them with experience in the pre-reformed courses) to complement the information about how the programmes are effectively implemented in the university classrooms. Due to space limitations, we do not include this last empirical data in this paper.

Levels of didactic codeterminacy as methodological tool for the analysis

The analysis of the empirical material has been done through the theoretical and methodological tool of the levels of didactic codeterminacy (Chevallard, 2002). Our aim is, in part, to analyse the conditions, also the constraints, set up by the curriculum reforms developed at the UTPL. To develop this analysis, we use the levels of didactic codeterminacy (see Figure 1, adapted from Barquero et al., 2014) as a common framework to illustrate and distinguish between the different conditions and constraints affecting the teaching and learning processes of mathematics courses for the training of undergraduate engineers. These levels go from the most generic level, civilisations, to the most particular one, the specific questions considered in a particular course. The lower levels refer to the way a discipline, mathematics in our case, is organised (in domains, sectors, themes, and questions) in a given teaching and learning process. The upper levels refer to the more general constraints coming from the way our civilisations and societies, through schools (universities, in particular) and their particular pedagogical conditions, organise the teaching and learning of the disciplines.

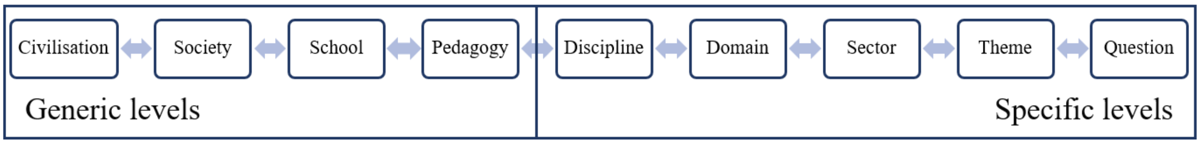

Figure 1: Scale of levels of didactic codeterminacy.

The second notion mobilised in the analysis is that of the "noosphere". The noosphere is understood as the set of institutions and agents belonging to these institutions that decides and delimits the mathematics to be taught in school institutions, in particular, at university. The "noosphere" goes beyond the community of teachers, and includes, for instance, agents that legislate on curricular changes, teachers' associations, university quality agencies, among others. Thus, considering the levels of didactic codeterminacy and the notion of the "noosphere", an analysis of the documents cited has been carried out and is presented in the following section.

RESULTS

Origin and need for the reform: proposals from the commissions

The reform arises in front of several needs that can be detected in the official documents considered and in the corresponding interviews with members of this commission in charge of the curriculum reform. Firstly, one of the stated aims is to close the gap between the Ecuadorian secondary education and the first years at university. Secondly, other aims are to homogenise the subjects, disaggregate them and ask about prerequisites before selecting the courses to be taken. The report drawn up by the commission of experts highlights that:

> […] The current curriculum at UTPL has a serious deficiency since there is no sequence of components. Under these conditions, students enrol in one course without having studied the necessary content taught in other course (Cuesta et al., 2015, p. 7, our translation).

Thirdly, and resulting from an interview with one of the two members of the commission, it is explained that the commission work consisted of analysing the content of the subjects: Mathematics, Basic Mathematics, Mathematics for the biological sciences, Pre-calculus, Calculus, Calculus I, Calculus for the biological sciences, Mathematics for the biological sciences II. It was then found that more than 80% of the content was similar in the different engineering specialties, leading them to work on unifying the content into common courses to then look for the specificities according to the engineering degree.

Last but not least, the other member of the commission highlights the importance of showing the usefulness of mathematics to thinking and solving problems in the engineering context. In one of the reports delivered, it is underlined that:

> […] After restructuring the subjects, it is expected to result in a course that: (i) enables the student to acquire the language of mathematics and the ability to use it to express and understand

> mathematics and the science under study, and (ii) that enables the acquisition of the fundamental concepts in mathematics in a meaningful way. This is precisely the most important challenge of mathematics teaching, whose current formalistic approach leads to the fact that the knowledge acquired in class is not useful for students.

At the *society-school level*, the reports underline the need of characterising better and smoothing the abrupt change between the mathematics taught at the Ecuadorian secondary education and that of the first mathematics university courses.

In relation to how the teaching of the different disciplines is organised, that is, placing at the *university-pedagogical level*, several aspects are emphasised. The first is related to the fact that, as the pre-reform subjects are arranged, there is a great deal of intersection in their contents. This leads to the proposal to unify them insofar as they are common and to integrate a specificity by promoting the use of contexts and situations specific to the studies being undertaken. As described above, it is proposed that in addition to the mathematics subjects providing a “mathematical language and the ability to grasp and use it in mathematics and other disciplines”, the ability to use this knowledge in the specific contexts of application in the professional field of engineers must be assessed.

At the *disciplinary-pedagogical level*, another agreement is when proposing a better sequencing of the subjects and to propose subject itineraries, which cannot be started without the prior approval of those taken. Consequently, the mathematics courses are proposed with less teaching load (from 6-7 credits in the pre-reform to 4 credits in the post-reform).

Effective changes in the resulting programmes

As already mentioned, we base our analysis of the mathematics courses’ programmes in the pre-reform (3 courses) and in the post-reform (6 subjects). The main tool here used are the specific or lower levels of didactic codeterminacy, which are used to analyse how a particular discipline is organised into domain, sector, theme, topics, and type of tasks.Figure 2 shows the results using the levels of didactic codeterminacy for the course of “Mathematics” (pre-reform course). Figure 2 and Table 2 show how the contents were redistributed, changed or reorganised in the post-reform courses proposed. Due to the space limitation, we are not able to include the analysis developed for the rest of the courses, but the results are available at https://bit.ly/INDRUM2022_RM.

Concerning Figure 2 (and the ones the reader can find in the repository), it is important to explain that different colours are used to indicate how the content of a pre-reform course is distributed in different subjects (indicated between them with different colours) in the post-reform. In the case of the analysis of the “Mathematics” course’s programme (pre-reform *vs* post-reform), the summary is illustrated in Figure 2. Based on the contents structure of the “Mathematics” course, the orange boxes indicate the contents that were assigned to the domain “Linear Algebra” that are transferred to the course of Linear Algebra in the post-reform, the dark green the ones assigned to the domain “Analytical Geometry” that are transferred in the post-reform as a course of Geometry fundamentals and, in light green colour, the new learning outcomes declared for each of the courses (“Linear Algebra” or “Geometry fundamentals”), and associated to specific themes.

Pro-reform course	Post-reform course	Comments about the changes in the post-reform programmes
Mathematics	Linear algebra	One new sector, ten topics, 35 types of tasks, and one type of task from the geometry domain is maintained.
	Geometry fundamentals	Two new domains, seven sectors, twelve themes and 45 types of tasks are added.
Calculus	Univariate analysis	14 new types of tasks.
	Multivariate analysis	One new domain, two sectors, seven topics, and 37 new task types.
Advanced Calculus	Differential equations	Two new sectors, three new themes and one new task types.
	Numerical methods	Not derived from any (pre-reform) course

Table 2: Changes identified in the courses in the post-reform

In the initial analysis of the course of “Mathematics”, two domains are identified “Linear algebra” and “Analytical geometry”. They are, in fact, delimited as two courses in the post-reform. Regardless of this change, the distribution of the sectors, topics and types of tasks remain almost the same plus some additions. Most of these “additions” are located at the specific levels of the type of tasks associated with each topic. In other words, the new additions do not introduce big changes to the rigid structure of the two post-reform courses.

A similar situation occurs with the calculus and advanced calculus courses. The changes observed are oriented to redistribute the contents in the new courses in the post-reform, as well as to add new types of tasks. Additionally, the inclusion of the numerical methods course, which does not derive from any course from pre-reform, includes a new domain and three new sectors "errors, roots, curve fitting", that helps to distribute better the contents.

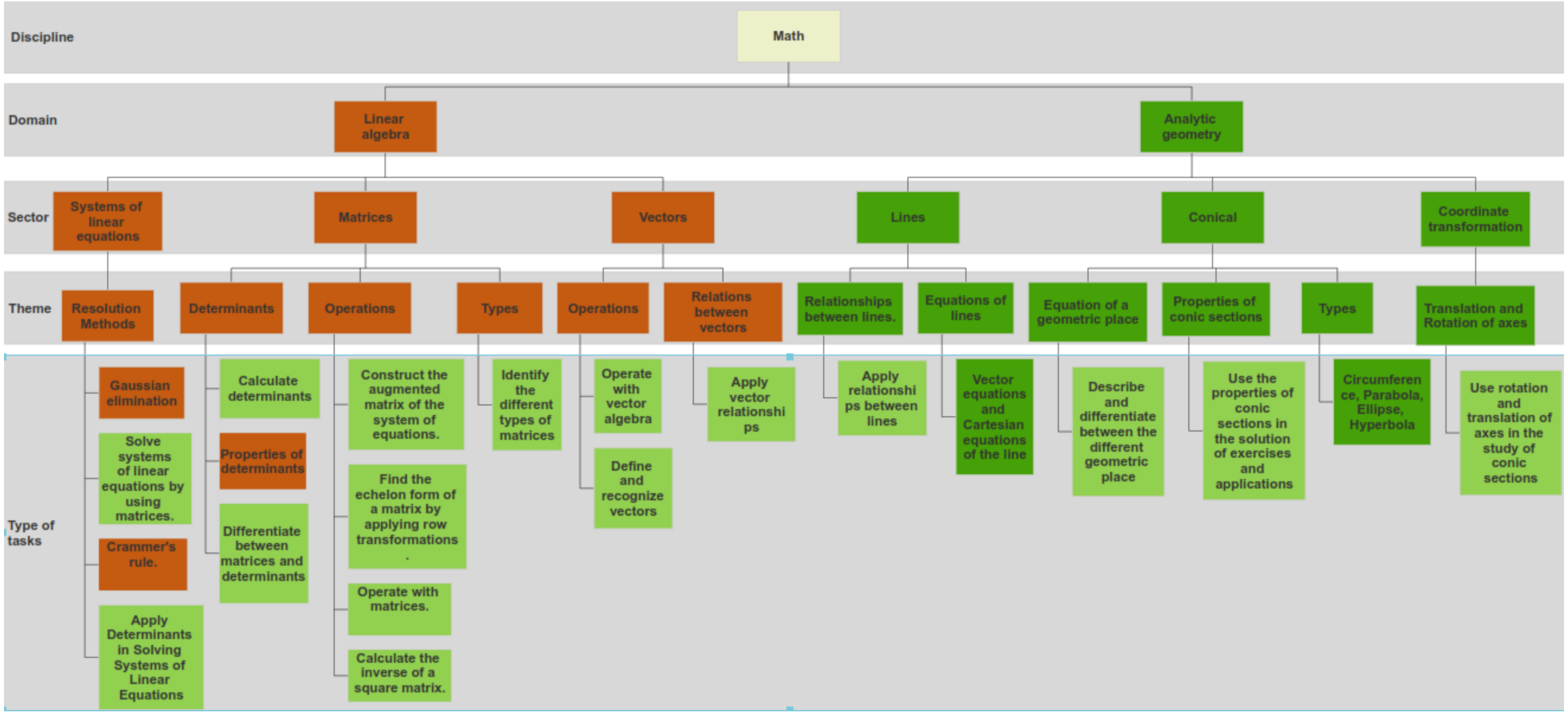

Figure 2: Analysis of the subject of Mathematics (7 ECTS, pre-reform) in relation to the levels of didactic codeterminacy

CONCLUSIONS AND OPEN ISSUES

This work can be considered as a first step of a broader analysis that is being conducted, and which will be enriched by the results of interviews with different teachers involved in the mathematical courses at UTPL. Nonetheless, the data analysed show that the noospherian institutions, in our case the Higher Education Agency and the commission in charge of the reform, play a crucial role as the initiator of the reform process. In addition, this agency establishes the main criteria to be followed, focused on softening the transition from upper secondary mathematics education to undergraduate programmes for engineers. Additionally, the analysis of the pre- and post-reform curricula and programmes reveal that the main changes have been introduced at the *pedagogical level*. That is, the changes have consisted of proposing a redistribution of contents into the different courses (disciplines-domains), a shortening of the courses (by a reduction of the number of credits) and the organisation of a pre-requisite system between courses. Most of the changes, besides this reorganisation of the already existing content, is the introduction of new content, most of the time at the very specific levels of the "type of tasks" or some new "topics". Another relevant change is the introduction of a new subject related to "Numerical Methods". In this case, the definition of the contents introduces more novelties than in the rest of the cases.

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INCORPORATING A STUDY AND RESEARCH PATH INTO A STATISTICS COURSE FOR ENGINEERING STUDENTS

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This paper describes a teaching proposal in engineering education based on the introduction of a study and research path (SRP) in the subject of Statistics of a Bachelor's Degree in ICT Systems Engineering. The conditions in which this SRP was designed and the first steps of its implementation are explained. Since an SRP starts with a generating question, special attention was paid to how the students worked on its formulation and the corresponding construction of the derived questions. With respect to the statistical knowledge mobilised, the problem of finding appropriate data in a format that can be applied appears as a key step to determine the study suitability of the generating question.

Keywords: study and research paths, statistics, anthropological theory of the didactic.

INTRODUCTION

Research on the implementation of *study and research paths* (SRPs) in university education is one of the developing areas of research in the Anthropological Theory of the Didactic (Barquero et al., 2021; Chevallard, 2015). SRPs appear as a specific type of enquiry-based teaching proposal that aims to make the prevailing pedagogical paradigm evolve from a knowledge-based study method towards a question-centered form of study. Their specificities compared to other forms of enquiry-based teaching formats are analysed in (Markulin et al., 2021). Various modalities of integrating SRPs into current university teaching have been explored (Bosch et al., 2020), together with the specific instructional devices implemented and the institutional constraints found. These constraints are mostly related to the prevailing pedagogical paradigm and the critical changes in the traditional didactic contract required by the new instructional proposal.

We present an experience related to the implementation of an SRP in a first-year course of Statistics for engineering students. Two previous studies present similar teaching proposals: three editions of an SRP in a Statistics course for second-year Business Administration students (Markulin et al., 2022), and an SRP in a Statistics course for Chemistry Engineering students (Quéré, 2022). Even if all the proposals correspond to the same theoretical model materialised in an SRP, numerous differences in the educational institutions, type of studies, course syllabi, students, group size, duration, form of implementation, etc. are observed. What they have in common is that the SRPs start from the consideration of an open question proposed by the teachers or by an external organisation – for instance a client or a firm – about the professional environment of the studies: a marketing study in the first case, quality control of pharmaceutical products in the second.

This paper presents a research study in process that is also based on the implementation of an SRP in a university Statistics course including different characteristics that will be explained later. The productivity of approaching a real question is taken from (Markulin et al., 2022) and (Quéré, 2022). In this respect, SRPs (as other project-based or enquiry-oriented instructional proposals) lead students to encounter many aspects of data analysis that are usually absent from traditional classroom statistics practice, like data gathering, cleaning, and sorting. The current research relies on some findings of previous studies about SRPs for engineering students like the use of questions-answers maps as a tool to describe, share and manage the steps and components of the study process followed (Bartolomé et al., 2019; Florensa et al., 2018), and the importance of the situation and external contract in which the initial question is formulated (Barquero et al., 2021).

The novelty of this SRP is the delimitation of the open question that generates the SRP. Instead of presenting the students with a situation where an open question is raised (by the teacher or by an external organisation), they are provided with a topic and are asked to formulate questions they find interesting related to it. Our research question is the following: *is it feasible to start an SRP with an open topic and let the students decide upon the generating questions by themselves? What are the consequences of this choice?* Given the fact that the project takes place in a statistics course, it goes without saying that the questions raised will be addressed using data available or produced.

This paper describes the experience of the first step of the SRP in the collective construction of the generating question. We first present the analytical tools used, then the educational context of the experience. After that, we describe the activities implemented in class and the results obtained. The last section puts forward some learnings that can help better understand the potential of SRPs and the different forms they can take, together with the limitations of the options chosen.

THEORETICAL TOOLS FOR THE ANALYSIS

We will use the Herbartian schema proposed by Chevallard (2011) to identify some key elements of an SRP. Its reduced form is $S(X; Y; Q) \hookrightarrow A^{\heartsuit}$ and indicates a didactic system S where a group of students X, with the help of a group of teachers Y, addresses a question Q to provide their own answer $A^{\heartsuit}$. In this case, no generating question was provided, but a topic T that would lead to some questions Q_i requiring their corresponding answers $A_i^{\heartsuit}$, being the new schema $S(X; Y; T) \rightarrow < S(X; Y; Q_n) \hookrightarrow A_n^{\heartsuit} >$. The developed form of the schema $[S(X; Y; Q) \rightharpoonup M] \hookrightarrow A^{\heartsuit}$ includes a *milieu* M with all the resources used by $S(X, Y, Q)$ during the enquiry: questions Q_i derived from Q, external answers or works $A_j^{\lozenge}$ elaborated by others that seem useful to address Q, empirical data D_k and other pieces of knowledge, virtual and material objects O_m:

$$[S(X;\, Y;\, Q) \rightarrow \{Q_i,\, A_j^{\lozenge},\, D_k,\, O_m\}] \hookrightarrow A^{\heartsuit}.$$

In the case of this SRP, the schema could be described as follows:

$$[S(X;\, Y;\, T) \rightarrow < S(X;\, Y;\, Q_n) \rightarrow \{Q_{ni},\, A_{nj}^{\lozenge},\, D_{nk},\, O_{nm}\}] \hookrightarrow A_n^{\heartsuit} >.$$

Q_{ni} being the questions derived from Q_n, $A_{nj}^{\lozenge}$ the answers or works that seem useful to address Q_n, together with empirical data D_{nk} and other pieces of knowledge, virtual and material objects O_{nm}. The implementation of SRPs produces important changes in the responsibilities assigned to the teacher and the students in the management of the different steps of the enquiry. These changes can be approached in terms of the evolution of the traditional *didactic contract* (Brousseau, 1997), and the tranfer of responsibilities from the teacher to the students.

DESCRIPTION OF THE INSTITUTIONAL SETTING

The *Universitat Politècnica de Catalunya* (UPC), like other European universities, has been going through a methodological transformation in education for some years, especially since the adoption of a competence-based curriculum within the European Higher Education Area. Educational innovation and student-centred methodologies have been developed in different areas, in particular since the implementation of the Research and Innovation in Learning Methodologies (RIMA) project by the UPC Barcelona Tech Institute of Education Sciences, and the foundation of the Journal of Technology and Science Education in 2011. Another significant boost was in 2017 with the creation of the Barcelona Science and Engineering Education (BCN-SEER), renamed to EduSTEAM, a research group in education mainly composed of senior researchers who, in addition to working in their areas of knowledge, had also been working on engineering education issues for some years (https://bcn-seer.upc.edu/en). After that, in 2020 the university was granted permission to offer a PhD programme in Education in Engineering, Science and Technology.

The SRP we are considering is part of the subject of Statistics offered in the second semester of the Bachelor's degree in ICT Systems Engineering in the Manresa School of Engineering – EPSEM, a campus of the UPC located one hour away from Barcelona. Although it is the first explicit implementation of an SRP at the UPC, it is some kind of evolution of previous experiences carried out by one of the authors in previous years (Alsina, 2022b). We can hence expect to find good conditions for its implementation.

Important institutional constraints also exist. UPC has engineering schools in Barcelona and in several nearby towns. However, they have a common Mathematics Department for most of them. The Manresa School of Engineering offers different Bachelor’s degrees: The degrees in Industrial Electronics and Automatic Control Engineering, in Mechanical Engineering, and in Chemical Engineering share all the first-year subjects, and the students are distributed into three groups that work in a coordinated way for each subject. The bachelor’s degree in ICT Systems Engineering has a different curriculum, but Statistics is offered in the second semester of the first year too, which means it shares the syllabus, the assessment criteria and a final exam with the rest of the degree programmes.

At the syllabus level, the subject is organised into five main blocks, each one with a list of exercises and problems to be solved: descriptive statistics, probability and random variables, probability distributions, fundamentals of statistical inference and quality control, and main components. Assessment of the subject is based on two individual written exams (70%) and three individual activities to be done with the software used in class (30%). If necessary, the students will have to take a final exam, which is common for all the degrees.

Before this experience, other project-based activities had been carried out in the subject of Statistics by one of the authors, who is acting as an observer and researcher in this experience. Due to the organisation of the subject, those activities were implemented in the degree of ICT Systems Engineering, taking advantage of the software skills of the students. Following the same criteria, the SRP is also being developed in the same degree by the first author of the paper (who will be referred to as the teacher) with the approval of the Mathematics Department at EPSEM. The implementation coincided with the change of software used in the subject, from Minitab to R and R-Commander.

To decide on the assessment of the subject incorporating the SRP, we encountered the constraint that the final test (in case of not passing the continuous assessment) had to be the same for all the degrees, and we were asked to keep individual activities. Finally, the assessment was agreed upon as follows:

- SRP (20%): final report (10%) and a poster with including an oral presentation (10%).
- Two individual written tests (60%)
- Two individual activities to be done with the software used in class (20%)

The SRP topic chosen was “water as an essential natural resource”, related to Education for Sustainability and the 2030 Agenda and its Sustainable Development goals (SDGs). This topic is linked to the UPC-led science dissemination project AquaeSTEAM, which seeks to propose scientific and technological questions, in which water is the common denominator. It is considered a challenge for students to explore, develop or create solutions, and find answers. On the website, educators and collaborators are provided with approaches, derived questions, resources, data and tools to work in the classroom (https://aquaesteam.upc.edu/).

To develop the SRP, the students were asked to work in teams of three or four members. The main operating tool used is the UPC virtual campus Atenea (a type of Moodle platform), in which the rest of the tools used in the project (padlet, Google Drive, Forms and Docs) are available. In this course, the lessons are structured into two weekly sessions of two hours each: one session with the entire group of students, and the other session with half the group, which is taught twice. The SRP is mainly carried out during the small group sessions, except when a pooling session is needed.

THE PROCESS OF FORMULATING A GENERATING QUESTION

In the very first session, for the students to gain insight into the whole subject, several activities were presented by the teacher. As an introductory activity, every student received a dice, and had to decide whether it was a trick dice or not. This activity follows a structure that allows the students to understand the subject, and have an overall perspective of what they are going to learn (Alsina, 2022a). It helps students to adopt a participatory role in the subject and work collaboratively, from the very first session, on an experimental activity directly related to statistics. It also allows them to gain self-confidence and use suitable vocabulary.

The SRP was then presented to the students: its objectives, the investigation process, the assessment criteria, and the schedule, as well as how it was going to be carried out: in teams of 3-4 students, and Atenea as the main tool used. See Figure 1 below for an image of the presentation of the course.

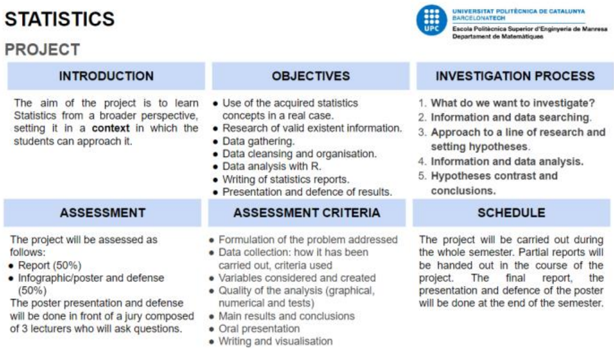

Figure 1. Project presentation (our translation)

Step 1: formulating questions related to water

Inspired by water and its relationship with sustainability, climate change, or energy, the students were asked to propose and discuss some questions they were interested in. It allowed knowing the interest of the students, to make their priorities emerge, and get them used to the activity of questioning the reality and formulating questions. The students formed teams and started doing some research, gathering information and data related to water. They were asked to share their work with the rest of the group through a padlet (https://padlet.com/mariajosepfreixanet/hhlc43z62pqd0xo3) (see Figure 2). They started thinking of what caught their attention, what they wanted to know. This had to be formulated as a question, possibly the generating question of their project.

In the next session, with all the students, each team presented its research and its project proposal. Some of the project proposals were the following:

- What effects does society have on water?
- Has Covid-19 contributed to an increased water consumption of the population?
- What is the variation of water in the rivers of Catalonia every year?

- How to maintain a clean swimming pool and avoid fungi?
- Has water consumption increased in the last years?

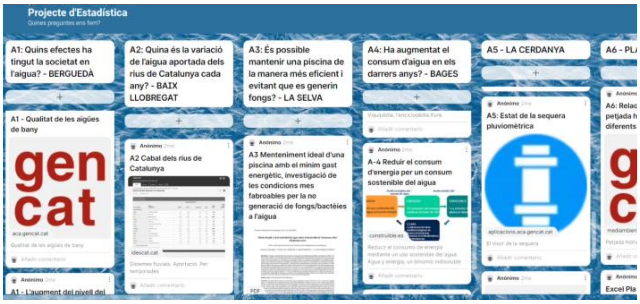

Figure 2. Padlet with the gathered information, data and project proposals

Step 2: linking questions with data

As a second step in the delimitation of the generating question, the teacher proposed the following task:

- *Write a short description of the aim of the study, the variables that are going to be analysed and why. What are these variables like?*
- *Include the link to a questions-and-answers map, which will be completed throughout the project.*
- *Enclose the data files you are going to use in your project.*

The teacher wrote a feedback report for each team and identified some difficulties most of the teams had when formulating the project proposals. In general, the questions were interesting, but some weaknesses appeared in the data provided:

a) Data format: wrong or difficult to analyse (five teams of 12 members)
b) Data content: not related or not answering the question (nine teams of 12 members)

With respect to (a), our interpretation is that the students lacked experience in the support software, and could not foresee the kind of data structure that was appropriate for the their project. As to (b), there may be two possible causes: difficulties in finding data, or in identifying statistical variables and their meaning.

A few examples of questions-answers maps (Q-A map) constructed by the students are included in Figure 3. It has to be noted that no Q-A map was provided by the teacher a priori, since the generating questions had to be given by the students. Nevertheless, the teacher explained what it was, the purpose of it and gave some examples. The teams only had to deal with the first level of the derived questions, without answers. Questions about the waste of water in Catalonia, or how the lockdown affected water consumption in Barcelona were topics of interest, but it is not easy to find data that can give accurate information about those issues.

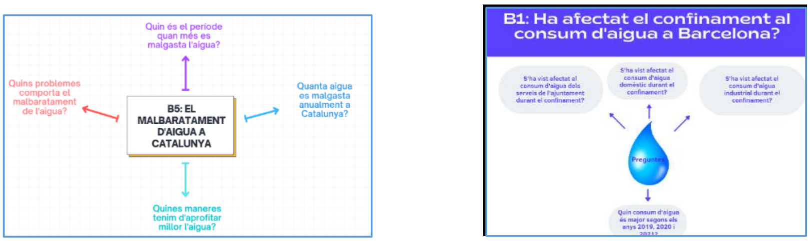

Figure 3. Two examples of questions-and-answer maps

Step 3: analysing the project proposals

Instead of writing a feedback comment to each team, the teacher decided to explain some descriptive analysis concepts as well as the first steps of how to use R for the students to be aware of the difficulties related to their data, and to have a critical perspective. It was also to help them identify what type of data they could work with. This took three more sessions. The assessment of this first task was postponed.

Univariate and bivariate descriptive statistics concepts, as well as written examples and exercises in R were explained in these three sessions. A questionnaire about the students' previous contact with statistics was used to easily gather data and introduce the basic tools for its description and analysis (Figure 4).

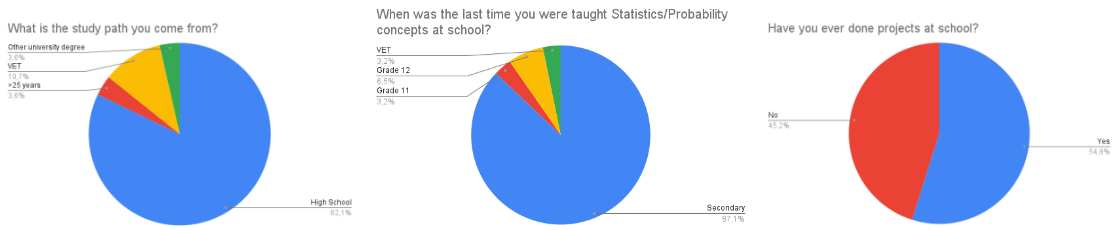

Figure 4. Results of the questionnaire on previous contact with Statistics

Once the descriptive statistics concepts and tools were introduced, each team of students had to analyse their own project proposal and the proposal of two more teams. A guide of questions was given to homogenise the analysis and to let the students think about the difficulties encountered by themselves. The questions to the students were asked are the following:

a) Data format: Can the provided data easily be analysed? YES/NO,
   Identified problem: The data are difficult to analyse, or have the wrong format
   - Can a descriptive study be implemented?
   - Can the data easily be uploaded to R?
   - Is it possible to find relationships between two variables? Which ones?
   - Is it possible to compare these variables with other data?

b) Data content: Can the provided data answer the main question? YES/NO
   Identified problem: The data do not answer the question

The students had to post this analysis as an answer to the project in the forum. Here are some of the results of the analysis of this activity, categorised by problems previously encountered by the teacher:

a) Data format: the data are difficult to analyse, or have the wrong format

Can the data easily be uploaded to R?

- A5 by A3: “No Excel document is provided, so it would be annoying to gather all the information manually.”
- B3 by B3: “No, there is a lack of tables of values.”
- B3 by B1: “No, the data are not presented in tables.”

Can a descriptive study be implemented?

- B3 by B2: “No, there is an important lack of data.”
- B4 by B3: “Information should previously be filtered. No tables of data are provided.”

b) Data content: the data do not answer the question

- B3 by B2: “In the water footprint webpage, you have access to the total national consumption and the national consumption per capita. You can also find information on natality, general health, or income per capita and compare it to water consumption (https://waterfootprint.org/en/resources/interactive-tools/national-water-footprint-explorer/)”
- A5 by A4: “the provided data indicate the water volume nowadays, a year ago, 5 and 10 years ago. It will not be possible to focus on the past 3 years”.
- A4 by A4: “we should search for new data about the water volume in reservoirs for more time periods. We could also consider a new variable about population increase, so we can analyse if there has been an increase of water consumption per inhabitant”.
- A3 by A1: “some more variables should be included: the presence of fungi, the quantity of fungi, the water volume, or the dimensions of the swimming pool (it may have a direct impact on how easily fungi grow) and if the swimming pool is private or public (public swimming pools tend to be bigger, water is more agitated and more people swim in them)”.

At the end of this task, the students were asked to fill in a questionnaire to know their opinion of the analysis, and their perception of their learning process. When asked about their learning, the students mentioned the limitations of their initial proposals (“Unnecessary data were provided.”, “The main question was too general.”, “The data provided corresponded to many different years.”), new aspects about data management and analysis (“Different approaches and how the others have used the data”, “To carry out a good descriptive analysis and to select proper data”, “To check if the revision and our data were correct”), and some improvement and ideas to move on with their project (“Possible mistakes we hadn’t taken into account previously”, “To find relationships between variables and inspiration for our project”, “Aspects the other groups and our group can improve”). In total, seven sessions were used to implement steps 1, 2 and 3: two sessions for step 1, one session for step 2 and four sessions for step 3.

DISCUSSION AND CONCLUSION

The experience here presented focuses on the first step of the enquiry described by the Herbartian schema: $[S(X; Y; T) \rightarrow < S(X; Y; Q_n) \curvearrowright \{Q_{ni}, A_{nj}^{\lozenge}, D_{nk}, O_{nm}\}] \hookrightarrow A_n^{\heartsuit} >$. We observed how the topic of water (T) proposed to the students led to the formulation of different generating questions Q_n and to the first steps of their study based on the identification of derived questions and the search of available answers $A_{nj}^{\lozenge}$ and data D_{nk}. The task proposed by the teacher in step 2 (linking questions with data) aimed at exploring the elements that could easily be integrated into the milieu during this phase of the enquiry. The teacher approached the students' difficulties by introducing new statistical tools and notions (O_{nm}) related to the concept of variable and the requirements of the statistical software used (R Commander). This work led the enquiry community to reject some of the proposed generating questions and to agree upon a few that appear to be suitable for the study – under the given conditions.

As far as the management of the SRP is concerned, the experience breaks some of the implicit clauses of the didactic contract, while maintaining others. Let us start with the first research question about the problem of choosing a productive generating question, which is an important issue related to SRPs (Markulin et al., 2021). By letting the students choose the SRP generating question Q_n instead of directly proposing it, a new responsibility is passed on to the students and, in a way, assumed by the entire study community composed of the students and the teacher. What is also shared with the students is the process of analysing the productivity of the questions raised and the viability of their study (their "studiability") concerning an important topic: the kind and quality of available data. This is an aspect of statistics that remains even more in the shadows than data cleaning and management, at least in educational contexts, and it certainly deserves more attention. It can be concluded that this way of starting the SRP has given the students the opportunity to learn the data suitable to be analysed statistically and the characteristics those data should have. However, this strategy took up more sessions than expected and represented an effort for the teacher to redirect the SRP. It is interesting to see how the use of questions-answers maps appears as a good strategy in this context.

Several aspects of the study process remained under the sole responsibility of the teacher. This is more than likely due to the prevalence of the traditional didactic contract. These aspects include the organisation of the tasks, the introduction of new concepts and tools, the pooling of results, and the planning of the sessions.

An important consequence shown in this paper is how much the movement of responsibilities from the teacher to the students results in the motivating and learning of a new type of statistical knowledge that would rarely appear otherwise. The conditions of "studiability" of the initial questions generating an SRP appear as a critical issue for the implementation of enquiry-based teaching proposals. Their approach does not seem to be only a question of the SRP design – therefore under the responsibility of the teacher – but of the SRP management – by both the teacher and the students.

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Teaching mathematics to non-specialists: a praxeological approach

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The study presented here concerns teaching practices at university for non-specialist students. Referring to the Anthropological Theory of the Didactic, our aim is to investigate the personal didactical praxeologies of university teachers (at the School, Discipline and Content levels) and to observe what can be considered as specific for non-specialists in these practices. We interviewed three experienced teachers with different profiles and collected their teaching resources. Analysing this data, we identified several didactical praxeologies specific for non-specialists at the Discipline level. At the Content level we give the example of a specific didactical praxeology and claim that many more exist, due to the mathematics-didactic codetermination.

Keywords: Anthropological approach to didactic, Teachers' and students' practices at university level, Teaching and learning of mathematics in other fields, Teaching and learning of mathematics for engineers

INTRODUCTION AND BACKGROUND

In their survey of research in University Mathematics Education, Biza et al. (2016) mention 'mathematics teaching at tertiary level' and 'the role of mathematics in other disciplines' as two emergent themes. Indeed, research about each of these two themes has developed significantly in the last few years. However, studies combining these two themes are scarce. González-Martín et al. (2021) note in their synthesis of research concerning teaching mathematics to non-specialists that many authors evidence ruptures between the mathematics taught in mathematics courses and in courses of other disciplines. Nevertheless, Pepin et al. (2021), focusing on mathematics in engineering education, observe that while several studies address the issues of instructors' expectations and their views about the mathematics that should be taught to future engineers, only a few authors investigate the ordinary practices of teachers in their mathematics courses for future engineers. Interviews with teachers having different backgrounds (studies in mathematics, in engineering, in physics, professional experience as engineer) evidence that they declare having different practices, regarding e.g. the links between mathematics and other disciplines, or the level of rigour expected. These differences can be a consequence of their different backgrounds (Hernandes-Gomes & González-Martín, 2016; Sabra, 2019).

González-Martín (2021) uses the Anthropological Theory of the Didactic (ATD, Chevallard, 1999) for analysing textbooks and teaching practices in two different engineering courses using the concept of integrals. Analysing interviews with two teachers, he observes that their practices seem to be strongly influenced by the textbooks they use; and that the mathematical content (integrals) finally plays a limited role in their courses. Following this work of González-Martín, we use ATD and the concept of didactical praxeologies to investigate teaching practices in the context of courses for first year non-specialists students. Our aim is to deepen our understanding of these practices, and elucidate the issues identified by the teachers and the strategies they develop to address these issues. Our work belongs to the DEMIPS [1] network in France (Theme 5: teachers’ practices at tertiary level).

THEORETICAL FRAMEWORK

The theoretical framework guiding our study is the Anthropological Theory of the Didactic (ATD, Chevallard, 1999). Chevallard considers that the knowledge taught is shaped by the institutions. In our study, secondary school and university are institutions; a mathematics course or a chemistry course for first year students are also institutions. How the knowledge is shaped is described in ATD by the concept of praxeology. A praxeology comprises four elements: a type of tasks T, a technique τ to perform this type of task, a technology θ which is a discourse explaining the technique, and a theory Θ grounding the technology. In mathematical praxeologies, the type of task concerns mathematics, e.g. T_{nsv}: "Compute the norm of the sum of two vectors". This type of task can be present both in mathematics and in chemistry courses, and will be associated with different praxeologies in each course. In didactical praxeologies, the type of task concerns the teaching of mathematical praxeologies in a given institution: "Teach the mathematical praxeology associated with T_{nsv}". A didactical type of tasks is associated with didactical techniques and technologies; the didactical theory usually remains implicit. The didactical and mathematical praxeologies mutually influence each other (Bosch & Gascón, 2001). The conditions and constraints underpinning any teaching or learning process (e.g. questions from the teacher to the students) can be located and analysed at different levels, classified in a scale extending from the more general to the more precise point of view. Florensa et al. (2018) separate this scale in an "Upper scale" (Humanity <> Civilisation <> Society <> School <> Pedagogy<>) and a "Lower scale" (<> Discipline <> Domain <> Sector <> Theme <> Question).

While most studies referring to didactical praxeologies focus on six predetermined ‘moments’ (see e.g., González-Martín, 2021), in our study we consider the personal praxeologies developed by teachers (Bosch & Gascón, 2001). We try to identify didactical types of tasks T, techniques τ and technologies θ. These didactical praxeologies of the teacher are empirical (Bosch & Gascón, 2001), developed by the teachers along their work in different institutions. The levels presented above also concern didactical praxeologies, and identifying to which level a didactical praxeology belongs can enlighten teachers’ practices (Florensa et al. 2018). Nevertheless as acknowledged by these authors this identification is complex; for this reason we have chosen here a simplified version of the codetermination scale: School <> Discipline (here mathematics) <> Content (e.g. vectors). What we call “School” includes the whole “Upper scale”. For example, "Ensure that students do personal work" is a didactical type of task at the School level; "Teach students how to present the solution of a mathematics exercise" is at the Discipline level, while "Teach the mathematical praxeology associated with T_{nsv}" is at the Content level. Drawing on these theoretical elements, the research questions we study here are:

What didactical praxeologies are developed by teachers teaching mathematics to non-specialists students? What is specific for non-specialists students in these praxeologies?

METHODS

The DEMIPS theme 5 group designed interview guidelines in order to investigate university teachers’ practices (in mathematics, physics or chemistry). During the semi-structured interview, after general questions about their teaching experience and the courses they deliver, the teachers were asked to focus on a particular course. They were informed ahead of the interview, and were asked to bring with them the material they used for this course. Concerning this course, they were firstly asked to present it and the resources offered by the institution to the students (e.g. digital platform). Then they were asked about their views on the students’ needs and potential difficulties, about their own practices (including the resources they design, how they design them, their collective work with colleagues) for the tutorials and for the students’ assessment. Interviews were conducted during the academic year 2021-2022 (9 interviews when we write this paper). The material brought for the interview (e.g. exercises sheets for students, exam texts) was collected, and the interviews were transcribed.

For the study presented here, we selected 3 of the 9 teachers. We chose teachers who focused in the interview on courses for first-year non-specialist students; and experienced teachers, who might have a rich repertoire of didactical praxeologies. The profiles of the three teachers chosen are presented in table 1 below.

Teacher	TC	TM	TP
PhD in	Chemistry	Mathematics	Theoretical Physics
Personal information	Female, 28 years teaching exp.	Male, 28 years teaching exp.	Male, 16 years teaching exp.
Course	Chemistry for biology students, including "maths reminders"	Mathematics for future chemistry engineers	Mathematics remedial course for students who gave up in another course

Table 1. Profiles of the three teachers interviewed and courses.

For analysing the data collected, we started by searching in the interviews the type of tasks mentioned by each teacher. The three authors of this paper confronted their initial analyses, and the level (S, D, C) of the corresponding task. While the difference between the School and the Content level is clear, sometimes the Discipline level and the other two can overlap. Then we searched the interviews for the didactical techniques used by the teachers, and possible explanations/justifications of these techniques (interpreted as technologies). We confronted the teachers' declarations with the material collected, and also analysed this material (in particular the solutions of the exercises) to identify mathematical and didactical praxeologies at the Content level.

RESULTS

Firstly we present our results concerning the School and the Discipline levels; then we give one example at the Content level. Analysing our data, we observed ten types of didactical tasks that were shared by at least two of our three interviewees; five for each of these two levels (Table 2).

School level	*Ensure that students complete personal work (T_{cpw})*	Ensure that students work during the tutorial (T_{wt})	Ensure that students work autonomously during the tutorial (T_{awt})	Ensure that students take responsibility for their learning (T_{rl})	Assess students (T_a)
Discipline level (in maths)	*Foster students' interest and engagement (T_{ie})*	*Teach basic maths tools (T_{bmt})*	Teach reasoning, justification and proof (T_{rjp})	Foster students' ability to tackle a new problem (T_{tnp})	*Restore students' self-confidence (T_{sc}) (not for TM)*

Table 2. - Didactical types of tasks shared by the teachers at the School and Discipline level. In italics: types of tasks potentially specific for non-specialists.

For the sake of brevity, we develop in what follows examples of praxeologies potentially specific for non-specialist students.

Example of a didactical praxeology at the School level

The didactical type of task "Ensure that students complete personal work" (T_{cpw}) was present in the three interviews but the techniques were different for the three teachers.

TC aims to make her students work *after* the classroom session by distributing a booklet containing many exercises for one chapter (collectively designed with other chemistry colleagues). Solutions are provided online *via* the institutional web environment and an online quiz - which mark counts for the global assessment - has to be filled by all the students. She also sends a reminder 2 days before the quiz is due. Before moving on to a new chapter, she spends 30 minutes during the tutorial to check that the work on the exercises has been done. Moreover in the final assessment all the skills are evaluated. She explains that since the booklet has many exercises, and the assessment covers all topics, at least some students will need to do the exercises as homework since they do not have enough time during the tutorial. Another technique used by TC for performing T_{cpw} is the organisation of group work during the tutorial. She explains indeed that students pursue the group work after the tutorial and that helps them to achieve their personal work.

For TM, the techniques for T_{cpw} aim more to foster the students' preparation for the tutorials. He asks the students to partly read the course in advance in the handout he has edited for them. If he finds on the Internet valuable videos about the contents to be taught, he posts them *via* the institutional web environment. Before the tutorial sessions, he emails every week or every two weeks the list of exercises to prepare. At the end of a chapter, he provides some solutions of the exercises (hand notes or software computations) *via* the institutional web environment. The techniques used are often justified by TM by the will to save time during the class. The videos also allow him to provide a visualisation of some mathematical phenomena.

TP declares in his interview that he thinks that the students do not work out of the tutorial sessions. Nevertheless the students have exercises sheets; the work during the tutorial concerns only a part of these sheets; a complete correction is provided and can support students' personal work. We consider this as a technique for this type of task, and note at the same time that TP does not trust this technique.

Examples of didactical praxeologies at the Discipline level

To foster students' interest and engagement (T_{ie}), TP chooses contextualised exercises (referring to physics, but also to day life contexts). TP justifies this choice by explaining that it is likely to foster students' engagement, but also that students are used to contextualised exercises at secondary school. Another technique used by TP for the didactical task T_{ie} is to explain to students that they will need these mathematical tools. Some mathematical exercises are inserted in TC's booklet for chemistry. In her interview she declares that she observed during the first semester that starting with these exercises was a mistake. For the second semester she plans to begin the lesson with chemistry exercises that motivate the mathematical exercises that will follow: this is a new technique that she will use for T_{ie}. TM does not make the connection with other disciplines and declares "I would think that we are not really here to foster interest… We are here to make them learn things." Nevertheless, he declares that he uses videos that he appreciates to foster students' engagement; we consider this as a technique for T_{ie}.

To restore students' self-confidence (T_{sc}), the technique used by TC is based on the exercise booklet. It contains many exercises, classified by difficulty level. Students can start with a more difficult exercise but go back to easier exercises if they fail. She justifies the use of this booklet by explaining that students are not confident in their abilities, depending on the options they took in high school (this concerns both mathematics and chemistry, the students are biology majors). TP aims to restore the self-confidence of his students by filling in their gaps: he attempts to restore their confidence by making them feel comfortable with the basic tools of calculation. He also wants the students to regain confidence in their reasoning, and urges them to check each step of this reasoning to be sure of its correctness. MP did not mention in his interview the need to restore its students' self-confidence.

Concerning the type of tasks "Teach basic maths tools" (T_{bmt}), in the booklet proposed by TC, some "boxes" are entitled "Mathematics reminder". They concern for example 2x2 linear systems, vectors or unit conversions: concepts taught at secondary school. Nevertheless the properties presented and the perspective on the concept is sometimes unfamiliar for the students, we discuss this with an example at the Content level. These "reminders" are sometimes followed by mathematics exercises; they were written by another chemistry teacher. This teacher had a long experience of teaching chemistry to selected students preparing to enter engineering schools, and TC trusts his experience concerning the students' needs in mathematics. TM considers that teaching basic mathematics tools is necessary but does not teach them himself since his engineering school dedicates one week at the beginning of the school year to an autonomous work of students on these basic mathematical tools for students who have just obtained their Baccalaureate (end of secondary school national exam in France). The technique used by TP is to make the students practise many exercises to develop their procedural fluency with basic contents: developing, factoring or solving first-degree equations. TP says in his interview: "the most important is to practice, practice, practice to develop their fluency [...]. It is important that they have the solutions, so that they can try, try, repeat exercises and check if their solution is correct."

Didactical and mathematical praxeologies at the Content level: norm of vectors

In this section we focus on the didactical praxeology used by TC for teaching the mathematical praxeology associated with "compute the norm of the sum of two vectors" (T_{nsv}). This praxeology is needed in chemistry within the theme entitled "polarity of molecules", for computing a "dipole moment". The students' booklet on this theme starts with a "mathematics reminder" about vectors. It includes in particular the general formula for the norm of a sum of two vectors, then it introduces the property presented in Figure 1.

The norm of a sum of two vectors that have the same norm but different directions can be written as:

$$\|\vec{u}+\vec{v}\| = 2\|\vec{u}\|cos\left(\frac{\theta}{2}\right)$$

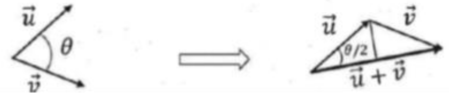
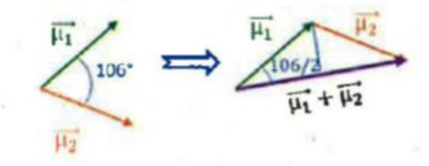


Figure 1. Extract of the "mathematics reminder" about vectors

This property provides a technique for T_{nsv}, when the two vectors have the same norm. We note that this technique is not taught in mathematics courses, at university or at secondary school. The figure (see figure 1) next to it can be considered as a technology;

but there is no associated text. Further in the booklet, exercise 3 asks for a justification of the formula, in the case of two given vectors (forming a 106° angle). The justification expected is presented in the corrected booklet (figure 2).



$\| \overrightarrow{\mu_1} + \overrightarrow{\mu_2} \|$ is two times the projection of $\overrightarrow{\mu_1}$ on $\overrightarrow{\mu_1} + \overrightarrow{\mu_2}$

With $\cos(\frac{\theta}{2}) = \frac{adjacent}{hypotenuse} = \frac{\frac{\| \overrightarrow{\mu_1} + \overrightarrow{\mu_2} \|}{2}}{\| \overrightarrow{\mu_1} \|}$.

Thus $\| \overrightarrow{\mu_1} + \overrightarrow{\mu_2} \| = 2 \| \overrightarrow{\mu_1} \| \cos(\frac{\theta}{2})$.

Figure 2. Justification expected for the formula of the norm of the sum

This justification implicitly uses several mathematical properties of the triangle formed by $\overrightarrow{\mu_1}, \overrightarrow{\mu_2}$ and $\overrightarrow{\mu_1} + \overrightarrow{\mu_2}$. The triangle is isosceles. Its perpendicular bisector is also its height (justification for the projection). Its vertex angle is 180°-106°, thus the basis angles are (180°-(180°-106°))/2=106°/2. The triangle formed by $\overrightarrow{\mu_1}$, $(\overrightarrow{\mu_1} + \overrightarrow{\mu_2})/2$ and the height is right-angled (justification for the cosine formula). For this mathematical task in the chemistry course, the praxeology is not the praxeology that would be expected in the mathematics course. TC confirmed in the interview that the students have not been able to solve this exercise.

We observe in this example several issues associated with the didactical praxeology for teaching T_{nsv} (which belongs to the praxeology at the discipline level T_{bmt}). Firstly, the "reminder" can in fact correspond to new knowledge. Here the property can be proven with secondary school knowledge, but it requires a complicated proof. Moreover students are not familiar with vector projections at secondary school. Second, for what TC identifies as a mathematical exercise, the kind of justification expected is very different from what would be expected in a mathematics course.

DISCUSSION

Are the didactical types of task (table 2) and the associated praxeologies specific for the target public of non-specialist students? In this section we discuss our results in order to answer this question. Our aim was not to compare the three teachers; nevertheless we also present some hypotheses about the differences between the didactical praxeologies they developed for the same types of tasks.

At the School level, we observed five types of tasks shared by the three teachers for their (non-specialists) first year students. During the first year at university, whatever the subject taught, the teachers have to ensure that their students complete personal work (T_{cpw}) out-of-class; that they work -autonomously- during the tutorial (T_{wt} or T_{awt}); that they take responsibility for their learning (T_{rl}). The teachers also need to assess all the students (T_a). We hypothesised that the praxeology associated with (T_{cpw}) could be specific for non-specialists students, since these students probably dedicate only a limited amount of time to mathematics. TM asks to prepare exercises before the tutorial. TC proposes a very elaborated booklet with many exercises whose solutions are available on the course's platform. She also fosters group work. TP proposes long lists of technical exercises with their solutions. This technique does not directly concern the amount of personal work, but its nature: TP wants that the students practice to develop their procedural fluency. It is both a technique for T_{cpw} (Institution) and T_{bmt} (Discipline). Thus we do not claim that we identified at the School level didactical praxeologies specific for non-specialists.

At the Discipline level, we also observed five didactical types of tasks shared by at least two teachers. In our analyses of the interviews, we did not find for the praxeologies associated with "Teach reasoning, justification and proof" and "Foster students' ability to tackle a new problem" elements that could be specific for non-specialists (the analyses are not presented here, due to space limitations). We contend that for the three other types of tasks, the praxeologies are specific.

The type of tasks "Foster students' interest and engagement" (T_{ie}) is specific because many non-specialist students are not motivated by mathematics. Two of the teachers (TC and TP) used as a technique the proposition of mathematics exercises in the context of another discipline, and we consider this technique as specific. Interestingly, TM made no links with other disciplines and chose to propose videos that he appreciated - he also declared that raising students' interest was not his role. This can be a consequence of his mathematical background (it is less natural for him to make links with other disciplines), but also of the type of engineering school and the role of mathematics in it. Indeed his students prepare for a competition; they are obliged to learn mathematics to succeed. The types of tasks "Teach basic maths tools" (T_{bmt}) and "Restore students' self-confidence" (T_{sc}) are specific as some of these students only have a limited mathematics background. Some of them have difficulties in mathematics (in particular TP's students who follow a remediation course); according to TP and TC, most of their students consider themselves as low-achievers in mathematics. As TC says: "they have prejudices about their level in maths […] they did not take the maths specialty in grade 12, so they feel suck at maths and they don't like maths". TM does not mention the T_{sc} type of task. This is linked with his teaching context: his students were high-achievers at secondary school and are self-confident.

At the Content level, we observed that the mathematical praxeology for "Compute the norm of the sum of two vectors" (T_{nsv}) was different from what would be expected in a mathematics course. Thus the didactical praxeology for the types of tasks "Teach T_{nsv}" (belonging to "Teach basic mathematics tools") is specific for non-specialists. The didactical technique used by TC (and her chemistry colleagues) is "present a brief summary of the mathematical properties needed and propose a few exercises"; the technology seems to include "the mathematical concepts and properties are not new for the students". Nevertheless this raises an issue, since some of the properties are in fact new, and some of the concepts like projections of vectors are not familiar for the students. Sometimes mathematics teachers at the beginning of university are not aware of what students precisely learned at secondary school; it is even more difficult for a chemistry teacher.

CONCLUSION

We observed in our analyses ten didactical types of tasks at the School or the Discipline level, shared by at least two of the three teachers we interviewed. Three of the didactical praxeologies at the Discipline level were specific for non-specialists students. At the Content level, we only presented one example of didactical praxeology, which was also specific. Drawing on previous works discussing mathematical praxeologies in courses for non-specialists (e.g., González-Martín, 2021), we hypothesise that most didactical praxeologies at this level are specific; indeed the mathematical and didactical praxeologies are co-determined (Bosch & Gascón, 2001). In order to validate this hypothesis, but also to improve our understanding of the didactic stakes in the teaching of mathematics to non-specialists, we plan to pursue our analyses with regard to the links between mathematical praxeologies and didactical praxeologies, to understand better the epistemological dimension of didactical praxeologies for non-specialists.

The theoretical approach we have chosen in terms of didactical praxeologies at three different levels allowed us to analyse the practices described by the teachers and the teaching resources they designed. This first step was needed to examine the specificity of their practices. We contend that this praxeological approach of the teachers' practices can contribute to our understanding of teaching at university level, for non-specialists or for other students. We plan to continue our study with more interviews with teachers intervening in diverse courses for non-specialists, and to observe their courses to confront these observations with the teachers' declarations.

The existence of specific didactical praxeologies for non-specialist students (developed by experienced teachers, in this study), suggest that novice teachers could benefit from a specific training. Research in mathematics education could contribute to formulating propositions for such a training.

NOTES

1. Didactic and Epistemology of Mathematics, interactions with Computer Science and Physics at Tertiary level, https://demips.math.cnrs.fr/recherches/pratiques-des-enseignants/

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A possible relationship between students' mathematical views and their evaluation of application examples

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Mathematics lectures play an important role in most engineering university courses. However, students often miss the reference to applications. Therefore, different application examples were integrated into a math lecture for first year engineering students. In our research we analyse a possible relationship between the students' evaluation of these examples and their mathematical views by using correlations and regression analysis. Our data of 143 students suggest that the stronger the students' applicationist view is the better the application examples are rated. If the student in addition views math as a toolbox, then the examples are evaluated more motivating.

Keywords: Teaching and learning of mathematics for engineers, Teaching and learning of specific topics in university mathematics, Mathematical Applications, Mathematical views.

INTRODUCTION

Mathematics have an important value for engineering studies at universities (Alpers, 2013). However, mathematics and the engineering courses are not necessarily interlinked (Tang & Williams, 2019), and the students miss the relationship between these disciplines, especially in the first year of their study (Harris et al., 2015). This missing link leads to low motivation and interest (Rooch et al., 2014) and a high drop-out-rate (Heublein, 2014). One possibility to concatenate mathematics with applications is to integrate good applications into the course.

What constitutes good applications from a students' perspective might depend on different factors. One factor lies in the properties of the application task, such as authenticity and length (Wolf & Biehler, 2014). Other factors might be the teaching person who has a great influence on the development of learners (Hattie et al., 2009), the presenting person or the fit to the study program (Schmitz & Ostsieker, 2020). The students' mathematical views might also influence what students see as good applications, since mathematical views are central to the learning of mathematics (Goldin et al., 2009).

We focus on examples integrated into the lecture. In order to better understand when students appreciate an application example, we investigate more deeply: In what extent is there a relationship between students' mathematical view and their evaluation of application examples? More knowledge about which factors are related to how well students find application examples allows for a more goal-oriented development of such examples. This might lead to more interest and motivation on the part of the students.

THEORETICAL BACKGROUND

Applications in math courses

Many different projects and concepts have been invented in order to place mathematical modelling into the engineering education. These projects and concepts range from short, less realistic tasks on only one sub-competency of modelling to authentic modelling projects which take place over a longer period of time (Greefrath et al., 2013, p. 23).

Given the institutional constraints, in a highly frequented math course at the beginning of the engineering studies smaller application examples and tasks can be implemented and enrich the mathematical content.

An application should treat extra-mathematical problems which are as real as possible so that a connection between mathematics and applications is created (Niss et al., 2007). The most important feature of application tasks and examples is thus the authenticity of the real problem (Greefrath et al., 2013). For math lecturers (who are usually mathematicians) the construction of such applications is quite challenging. Wolf (2017) developed a concept for the design of "good" application-oriented tasks which fit into a usual first-year math course for engineering students, and a test instrument to measure the criteria of the concept that can be evaluated by students.

In a lecture, rather examples are integrated. They differ from tasks in that learners do not have to solve the problems themselves. Assuming that the criteria for good tasks also apply to good examples, we expect that examples can also be constructed and evaluated according to the scales in Wolf (2017) or their equivalents regarding examples. We focus our research on "applicability" and "authenticity" as they are independent of the aspect of problem solving and can be directly transferred from tasks to examples. Additionally, we transfer the importance of "motivation" (also described in Wolf, 2017) to application examples and consider the property "motivational" as an indicator of a good example.

Mathematical views

The students' beliefs can be described by the construct of mathematical views. Grigutsch et al. (1998) developed a system of four dimensions ("process", "application", "schema", "formalism") to characterise a person's beliefs regarding mathematics. This belief system has an additional structure: Correlation analyses in Grigutsch et al. (1998) show that the process and the application aspect correlate positively with each other, and the schema and the formalism aspect correlate positively with each other, but the process aspect correlates negatively with the schema and the formalism aspect. Therefore, the authors distinguish a static view of mathematics (schema and formalism) versus a dynamic view of mathematics (process and application).

This belief system was transferred to engineering students by Rooch et al. (2014). The belief aspects they consider relevant in engineering studies are "process", "application", and "toolbox" (without the formalism aspect). The process aspect describes mathematics as a creative field of discovery, ideas and trial, the application aspect describes mathematics as useful for everyday tasks and problems, and the toolbox aspect, which corresponds to the schema aspect in Grigutsch et al. (1998), contains the view that mathematics consists of tools like algorithms and formulas. We take our inspiration from Rooch et al. (2014) as they have successfully used the aspects to study students' estimation of real-world examples.

Impact of mathematical views

Beliefs have an important impact on learning processes:

> Individual attitudes towards mathematics … are an essential influencing factor for mathematical … learning processes. They describe … the context in which pupils see and do mathematics. They influence how students approach mathematical tasks and problems and how they learn mathematics. (Grigutsch & Törner, 1998, p. 3)

The dynamic view of mathematics is often related to more motivation and interest (Köller, 2001) and better performance in mathematical tasks (Tossavainen et al., 2021). Mischo and Maaß (2012) found out that utility beliefs of mathematics (and by lower values, beliefs that mathematics means applying rigid schemas) can predict the modelling competence in general.

There is not much research on the impact of mathematical views to the evaluation of modelling resp. application tasks or examples. Maaß (2005) reveals that college students who have a static view of mathematics strictly disapprove modelling examples while the other students are partly positive or very positive about. We examine a possible relationship between students' mathematical views and their evaluation of application examples.

RESEARCH QUESTIONS

Four application examples were integrated in a math course for first-year engineering students. We investigate to what extend there is a relationship between students' mathematical views (in the aspects "process", "application", "toolbox", see Rooch et al., 2014) and their evaluation of application examples (regarding the criteria "authenticity", "applicability" and "motivational", see Wolf, 2017).

First, for each of the four examples separately, we look at the relationship between an aspect of the mathematical view and a criterion of the evaluation of the example:

(A) To what extend does a view aspect correlate with an evaluation criterion?

Then we investigate how the mathematical view as a whole influences the evaluation of one example:

(B) What predictors for the evaluation of the examples can be found in the mathematical view?

METHODOLOGY

Design

This quantitative study involved 143 engineering students (76 mechanical engineering, 28 energy and building services engineering, 37 renewable energy, 2 other) at a university of applied sciences in Germany, who participated in a first-year math lecture in 2019.

After explaining how to solve ordinary differential equations of first order, four application examples out of the engineering disciplines, named “Example C”, “Example W”, “Example A”, and “Example R”, were presented in the lecture.

Instruments

After the presentation of all four application examples the students answered a paper-based survey questionnaire.

The instrument to measure the students’ mathematical views consists of the three scales “process”, “application”, and “toolbox” taken from Rooch et al. (2014), to be rated on a five-point Likert scale (1 totally disagree – 5 totally agree). The Cronbach’s alpha values (0.49 for process, 0.66 for application, and 0.58 for toolbox) are similar to those in Rooch et al. (2014), one a bit worse, the others a bit better.

To measure the evaluation of the application examples, suitable items from the three scales “motivational”, “applicability” and “authenticity” by Wolf (2017) were chosen and modified by replacing the word “task” by “example”. Respondents were given options using a six-point Likert scale ranging from 1 (totally disagree) to 6 (totally agree) to measure their agreement on the questionnaire statements.

From Wolf’s scale “motivational” we took the five of the six items which fit to our setting. A sample item is “The reference to an application has aroused my interest.” The reliability coefficients (Cronbach’s alpha) had the acceptable values of 0.83 (Example C), 0.83 (Example W), 0.85 (Example A), and 0.85 (Example R).

All three items from Wolf are used for the scale “applicability”, one being “This example has improved my ability to work on tasks with an engineering context.” The reliabilities are also acceptable, with values of Cronbach’s alpha of 0.805 (C), 0.777 (W), 0.826 (A), 0.830 (R).

The criterion “authenticity” is checked with only one item (the second item from Wolf was skipped, since the reliability of the resulting scale was quite poor). The item is “The reference to application is authentic: a real engineering problem is solved with the help of mathematics.”

The data were analysed in an explorative manner using the IBM SPSS Statistics software. The relationship between students' views and their evaluation of an example was calculated utilising the Pearson product-moment correlation technique. The multiple regression analysis (method forward) was performed to find out the best predictors of students’ views in their evaluation of an example.

RESULTS

Preliminary analysis: mathematical views

The results of the descriptive statistics indicate students' views are based on mean scores ranging from 3.60 to 3.74 (Table 1). This indicates that the students' views of mathematics include in quite equal measure all three aspects process, application and toolbox. The values of the standard deviations (Table 1) reveal that the group of students is quite heterogeneous concerning their mathematical view.

	mean value	standard deviation
process	3.74	0.53
application	3.64	0.69
toolbox	3.60	0.58

Table 1: Mean value and standard deviation of all view aspects.

A correlation analysis shows that the dynamic aspects "process" and "application" correlate positively (see Table 2), but do not exhibit multicollinearity (which is important for the regression analysis later). No significant correlation exists between the static aspect ("toolbox") and the dynamic aspects.

	process	application	toolbox
process	1	0.507**	-0.043
application	0.507**	1	-0.062
toolbox	-0.043	-0.062	1

Table 2: Pearson's correlation coefficient for the view aspects. **The correlation is significant at the 0.01 level (1-sided and 2-sided).

The correlation patterns for the mathematical views reproduce the findings from the literature (Grigutsch et al., 1998).

Preliminary analysis: evaluation of the application examples

The descriptive analysis for the evaluation criteria reveals that all examples were evaluated as being very authentic, and seem to be rather motivating and applicable (Figure 1). Example C stands out due to a very good evaluation in all criteria, better than all other examples.

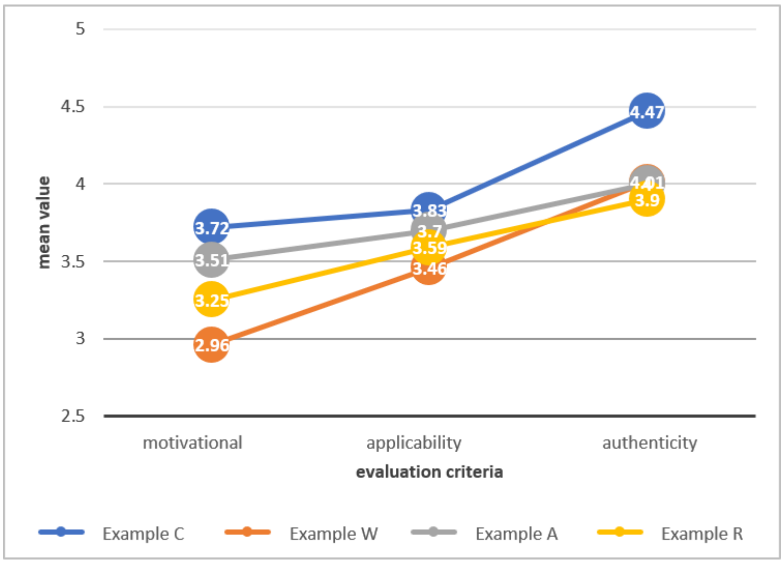

Figure 1: Mean values of the evaluation criteria for all application examples.

Relations between the mathematical view and the evaluation of application examples (A)

To determine whether there is any significant relationship between the mathematical view and the evaluation of an application example the pairwise correlations between particular view aspects and evaluation criteria for each of the four examples were tested using the Pearson's correlation analysis.

The result is presented in Figure 2. A line between a view aspect and an evaluation criterion is drawn if there is a significant correlation. The emerging significant correlations are all positive (from 0.17 to 0.43).

The illustration shows that the criteria “authenticity” and “applicability” are positively correlated with the application aspect in all examples. Furthermore, the criterion “authenticity” and “motivation” are positively correlated with the process aspect and toolbox aspect, respectively, in all examples except C.

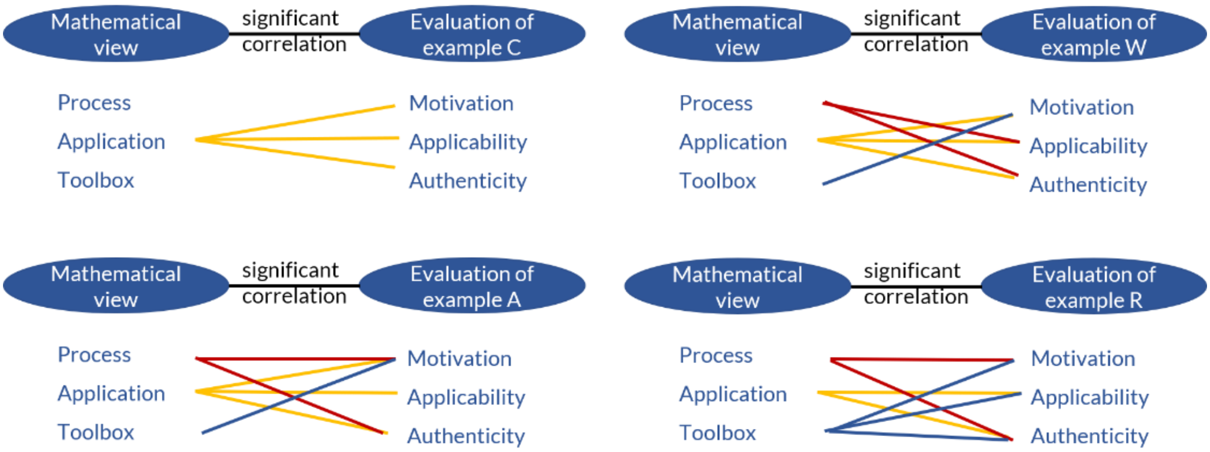

Figure 2: Significant correlations (p<0.01, 1- and 2-sided) between view aspects and evaluation criteria in each application example (all positive).

Mathematical view as predictor of evaluation criteria (B)

A multiple regression analysis was performed to identify the best predictors in the mathematical view that influence students' evaluation of the application examples. The view aspects serve as independent variables, and the dependent variable in the analysis is a particular evaluation criterion of one example.

The result of the analysis is similarly presented as before (see Figure 3). Each arrow leads from a significant predictor in the mathematical view to an evaluation criterion. The R-square value indicates that 5.1%-18.2% of the variance in one evaluation criterion is explained by the combination of view aspects. The quality of the model is weak to medium.

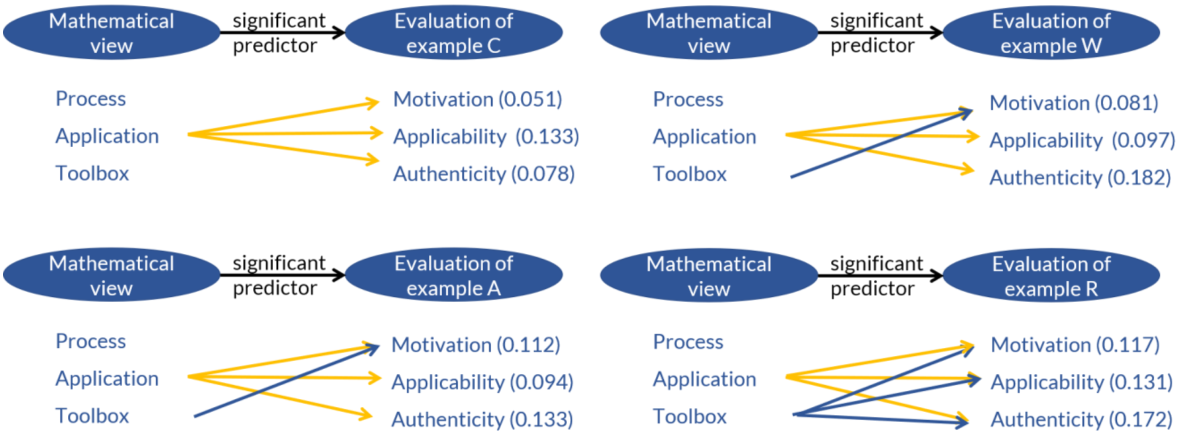

Figure 3: Predictors in the mathematical view for each evaluation criteria (value for R-square) in each example.

In every example, the aspect "application" is a significant predictor of all evaluation criteria, whereas the aspect "process" does not appear as a predictor. In example C, also the aspect "toolbox" does not appear as predictor. In contrast, in example R the toolbox aspect is a predictor for all evaluation criteria. In all examples except C the toolbox aspect is a predictor of the motivation criterion.

DISCUSSION AND CONCLUSION

In the present study, engineering students' mathematical view and their evaluation of application examples are measured and a possible relationship between them is investigated. In the following we summarise the results of the research questions (A) and (B), discuss limitations of the study and give an outlook on further research possibilities.

The correlation analysis (A) shows that students with a strong application aspect in their mathematical view tend to give good evaluations in almost all criteria in almost all application examples. In some examples, the aspects "toolbox" and "process" additionally correlate with some evaluation criteria. This result is only partly in accordance with the results by Maaß (2005) in the sample of college students since in her study the static aspect leads to disapproving application examples. One reason could be that we do not investigate the role of the formalism aspect which is part of the static view in Maaß (2005), another one that the participants of our study were engineering students.

The regression analysis (B) reveals that the aspect “application” predominantly occurs as a predictor of all evaluation criteria in all examples (as in the correlation analysis). This means, the more the application aspect is represented in the mathematical view, the better the evaluation of the example is, which similar to the results by Maaß (2005) in the sample of college students. The aspect “process” has no effect on any evaluation criterion in any example, in contrast to the positive correlations found in the correlation analysis. This occurrence could emerge from the fact that the bivariate correlations can be influenced by third variables.

In summary, the study shows that students with a strong application aspect in their mathematical view rate application examples better. This result seems to coincide with the fact that the dynamic aspect resp. the utility belief of mathematics leads to higher motivation and interest resp. better modelling competence (Köller, 2001 resp. Mischo & Maaß, 2012).

A limitation of the study concerns the generalization of the results, as the data came from only one university and from one lecture and one sample. Also, the model obtained by the multiple regression analysis is weak to medium. Other statistical methods should be tested.

Concerning the test instrument for the mathematical view, the value of Cronbach’s alpha for the process-scale is better in Rooch et al. (2014) than in our sample, and all view aspects are more reliable in Grigutsch et al. (1998). A reason for this could be that not all items from Grigutsch et al. (1998) were used in Rooch et al. (2014) and in our study. Improving the scales and eventually adapting them to engineering students should be investigated. Remark also that that participants are self-reporting about their mathematical views, so there are possible biases in the results.

To measure the evaluation of the application examples we used the scales by Wolf (2017) which are (to our knowledge) not evaluated in the sense that they have not been applied in a number of other studies. Here one could also try to improve the scales by changing items or adding other items.

The findings in our study reveal that the emerging correlations and predictors are quite diverse among the four examples. The application examples were of diverse nature, for example concerning the length of the presentation, the portion of modelling and computing, time consumption in preparation and the presenting person. Moreover, the examples showed applications from different study programs. Results in Schmitz and Ostsieker (2020) indicate that the evaluation of application examples differs depending whether the application context is related to the engineering degree program. This suggests that the correlation- and regression-pattern might depend on the application example. The role of the characteristics of the examples as well as other influencing factors could be investigated more intensely.

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Questioning the quantum world? A priori analysis of an SRP at the interface between mathematics and quantum mechanics

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This work is at the confluence of two groups of studies: on the one hand, research on didactic phenomena (and their consequences for teaching and learning) when mathematics is taught in an engineering or a physics class; on the other hand, research on conditions and constraints allowing to carry out the evolution from the prevalent didactic paradigm of visiting works to the novel paradigm of questioning the world, which are both brought out by the Anthropological Theory of the Didactic (ATD). In this dual context, we present in this paper the preliminary and a priori analyses of a Study and Research Path (SRP) set at the interface between mathematics and introductory quantum mechanics, in third year of bachelor's degree, at the University of Montpellier.

Keywords: Teachers' and students' practices at university level, Novel approaches to teaching, Teaching and learning of mathematics in other fields, Curricular and institutional issues concerning the teaching of mathematics at university level, anthropological theory of the didactic.

INTRODUCTION

In recent years, much work has been carried out on the teaching and learning of mathematics for engineers or on the role of physical modelling in mathematics curricula (see for instance Hausberger, Bosch & Chellougi, 2020, TWG2). However, the relative roles of mathematics and advanced physics are quite different: rather than seing mathematics merely as being used, both fields are then more on an equal footing (Lombard & Hausberger, to appear). In this work, we focus on the relationship between algebraic structures and theoretical physics, based on the example of Hilbert spaces in quantum mechanics. The study is carried out in the framework of the Anthropological Theory of the Didactic (ATD), which "aim (…) is the elucidation of human societies' relation to "the didactic," that is to say, to all the possible factors of learning. By adopting an anthropological point of view, it purports to embrace the didactic wherever it may show itself around us, paying special attention to the institutional constructions of knowledge and the conditions established to disseminate it." (Chevallard & Bosch, 2020a, p. 53).

Within the research program of the ATD falls the identification of "study paradigms", and the "contribution (…) to the paradigm of questioning the world" (Chevallard & Bosch, 2020a, p. 59; see also the next section). The study and development of Study and Research Paths (SRP) plays a key role in this endeavour (Bosch, Barquero, Florensa & Ruiz-Munzon, 2020). In this context, we have undertaken to set up an SRP at the interface between mathematics and introductory quantum mechanics at the

University of Montpellier. More specifically, this SRP is set in the context of quantum computing, and starts with the following generating question Q_0*: In what respect are quantum computers indeed quantum?* It is designed for students that are in third year of bachelor's degree (sixth semester). It consists of nine two- to three-hour sessions, distributed into three lab sessions and six classroom sessions. In this paper we present the results of the first two phases of the implementation of an SRP: the preliminary and a priori analyses.

Our work is thus in line with the past INDRUM conference "instructional proposals for university mathematics to move towards a change of paradigm, such as problem-posing activities, interdisciplinary projects or study and research paths" (Hausberger et al., 2020, p. 167). In this context, it aims at addressing the following issues (Hausberger et al., p. 167-168):

> How to find a "good" generating question for an SRP? Can the design and implementation of SRPs help us to rethink the contents of the course? […] How to look at university mathematics curricula from an interdisciplinary approach? How can the perspective of mathematical modelling contribute to it?

THEORETICAL FRAMEWORK AND METHODOLOGY

SRPs as part of ATD

In the paradigm of visiting works, also called paradigm of visiting monuments, works under study at school or university (theorems, formulae, methods) are "approached as (…) monument(s) stand(ing) on (their) own" (Chevallard 2015, p. 3). Notably, the raison d'être of these pieces of knowledge is never specified. The paradigm of questioning the world, as its name indicates, focuses rather on questions. In this paradigm, one sees pieces of knowledge as answers to be considered only when judged relevant to solve given problems, hence highlighting their raison d'être.

Study and Research Paths (Winsløw, Matheron & Mercier, 2013) constitute a contribution to the advent of this "oncoming counter paradigm" (Chevallard, 2015) in teaching institutions. "An SRP is an inquiry process which starts from an open-ended question and leads to a combination of investigation activities - to explore the question, and study processes - to obtain new information that will help in the research" (Bartolomé, Florensa, Bosch & Gascón, 2018).

Throughout an SRP, students are looking for answers to intermediary questions in lectures, in (text)books, on the internet, etc. All these "social system pretending to inform (…) some group of people about the natural or social world" are media in the sense of ATD, and they can become components of the inquiry "milieu", under the "dialectic of media and milieus" (Chevallard, 2006, p. 9). During the study of questions, the notion of systems and models (Barquero, Bosch & Gascón, 2019, p. 9) is also to play an important role, especially in the context of interface between disciplines we consider in this paper. The dialectics of media and milieu and systems and models should be important theoretical tools to monitor our experimentation.

Research questions

By means of this theoretical framework, we will now consider the following research issues: what are the conditions and constraints imposed on the implementation of an SRP at this level of studies, at the interface of mathematics and physics? What lies behind the choice of its generating question? How to avoid visiting works while maintaining research and teaching objectives?

Methodology

In order to prepare this module, we proceeded according to the Didactic Engineering methodology as it was applied to SRPs by Bartolomé et al. (2018): see fig. 1.

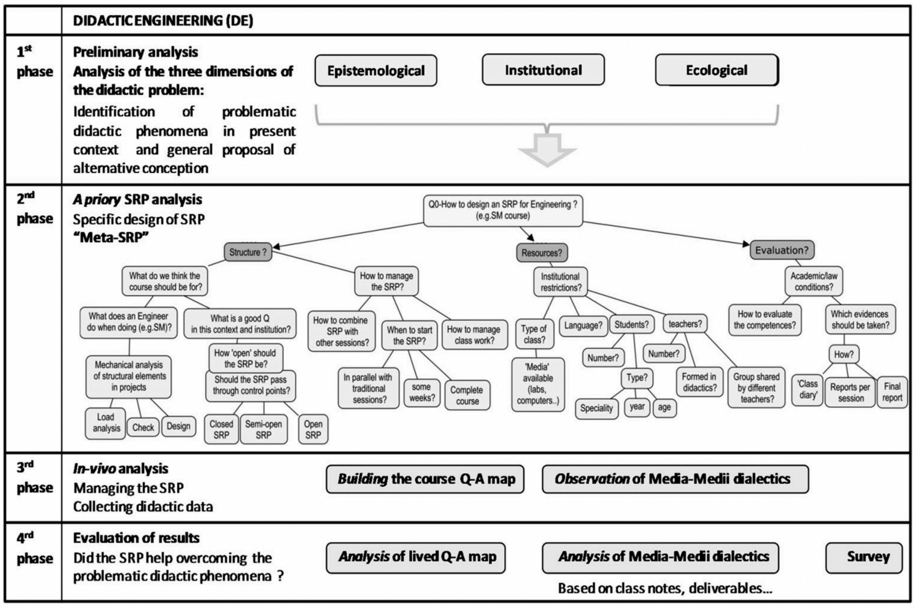

Fig. 1: The Didactic Engineering methodology applied to the implementation of an SRP (Bartolomé et al., 2018, p. 5)

First, we performed a preliminary analysis consisting in three parts:

- the study of epistemological aspects underlying the project, based on the study of primary and secondary sources in history of quantum physics and mathematics
- the institutional conditions of the SRP, based on interviews with professors at the University of Montpellier
- the ecological context of the SRP, based on the same interviews as well as analyses of relevant course material

Then, we carried out the a priori analysis which "includes the specific design of the SRP, including the selection of a generating question starting the study process, taking into consideration the conditions and constraints identified in the first phase." (Bartolomé et al., 2018, p. 6).

PRELIMINARY ANALYSIS

Epistemological aspects

From a study of historical epistemology (in the sense of « Histories of epistemic things » (Feest and Sturm, 2011, p. 288)) dedicated to the interplay between quantum theory and functional analysis during their respective early developments (1900-1930), we could extract several aspects that seem particularly relevant for the design of this SRP.

First, mathematics and physics exerted a mutual influence throughout the development of quantum mechanics. In particular, from 1925-1927, mathematicians could draw results from questionings arising from physics (Lacki, 2011). This culminated in von Neumann's introduction of Hilbert spaces as an abstract structure encompassing the variety of theories of quantum mechanics known at that time. So, a physical context may be fruitful to introduce higher level mathematics via the dialectics of questions and answers.

Then, models and formulations were abundant at the interface with successive attempts at unifications and simplifications, which the structuralist stance in mathematics finally helps to achieve (von Neumann, 1955, p. 28). We should thus expect the dialectics of objects and structures (Hausberger, 2017) will play a role.

Finally, interviews we performed with professors of physics and mathematical physics at the University of Montpellier lead us to consider such phenomena still occur in their day-to-day activity. More precisely, though there seems to exist a vivid practice at the interface between mathematics and physics when considering "scholarly knowledge", this does not seem to be the case anymore regarding "taught knowledge", showing a lack in the "didactic transposition" at the interface (Chevallard & Bosch, 2020b). This is one of the issues this SRP attempts to address.

Quantum computing appeared to be a subject that could put at play the aforementioned epistemological aspects. In addition, as the setting of an SRP should "be regarded – by the students, by their teachers (…) – as crucial to a better understanding and mastery of their lived world" (Chevallard, 2006, p. 7-8), quantum computing seemed to fit all the more. Lastly, specific institutional conditions in Montpellier would facilitate the implementation of an SRP on this topic.

Institutional conditions

Indeed, in 2019, the technology company IBM initiated a partnership with the University of Montpellier, in which several members of both the mathematics and physics departments are involved. This is why, in the first place, quantum computing emerged as a potentially workable setting for the upcoming SRP. However, it was yet to be found how to include such a project into the sequence of teaching units (TU) taught in this university. In Montpellier, physics and mathematics curricula are quite detached (see fig. 2). The bachelor (*Licence*) lasts three years (L1-L3), and it is divided into six semesters (S1-S6). Some TU taken by physics students are nevertheless taught by mathematicians: we filled them in blue. Lastly, our experimentation takes place during the sixth semester (S6), as a mixed TU (see below). It is shown in purple in the figure.

Several constraints weighted on the institutional implementation of the SRP, as it was to develop at the interface between mathematics and quantum mechanics. Firstly, physics students take their first quantum mechanics class at the fifth semester (S5, *Mécanique analytique et quantique*). Then, during the sixth semester, on the one hand, mathematics students have to take a "common knowledge" class (*Culture générale*), whereas, on the other hand, physicists take a TU devoted to doing an experimental research project supervised by a professor (*Projet tuteuré*). So, we could set up the SRP as a mixed teaching unit registered with both math and physics departments. It is the only such TU at this level of studies at the University of Montpellier. This way we could project nine two- to three-hour time-slots.

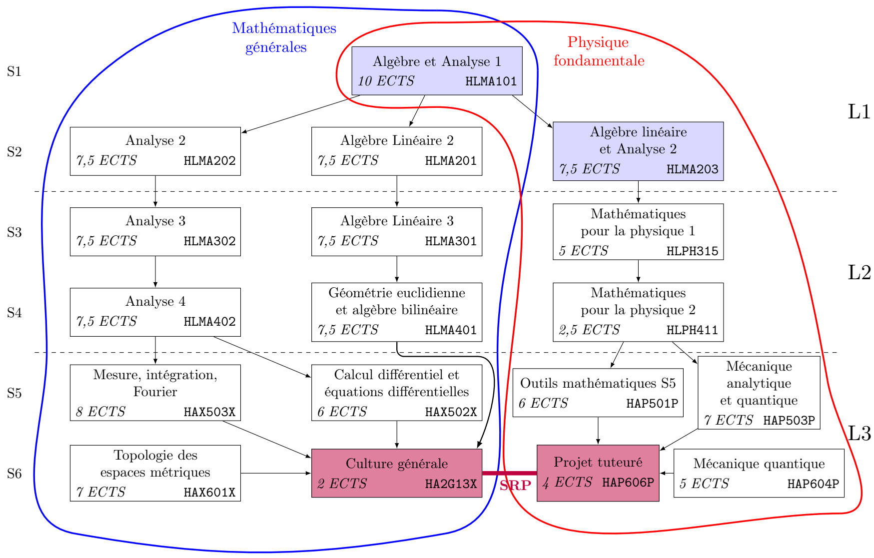

Fig. 2: Overview of some TU from the "General Mathematics" and "Fundamental Physics" programs that are taken by students participating in the experimentation

With this organisation came further conditions and constraints, mainly from the physics department. For instance, evaluation should include a peer-reviewing process among students taking this TU. This was actually a favorable condition. Indeed, defining the recipients of the answer to be given to the generating question is a crucial step. So, we extended the physics instructions to all students: to write a report and make a presentation their third-year colleagues could read and understand.

Ecological context

Physics students has already had a quantum mechanics course during the first semester, whereas students from mathematics never did (at least at the University of Montpellier). The content of this quantum mechanics course corresponded to parts of the first two chapters of *Quantum Mechanics, vol. 1* by Cohen-Tannoudji, Diu & Laloe (1991) which is often used in introductory quantum mechanics courses in France. We could analyse its content in a study which provides additional inputs into the preliminary analysis (Lombard et Hausberger, to appear).

For instance, a tension exists between the abstractness of the Hilbert space formalism and the necessity for students to develop operative skills in order to study actual physical systems or models. In particular, the raison d'être of some elements of the formalism is never specified (why an infinite number of dimensions? why Hilbert spaces and not Banach spaces or even pre-Hilbertian spaces, which are studied in second year by math students?). Of course, this is consistent with the fact this course provides an example of the paradigm of visiting works. By shifting towards the paradigm of questioning the world, this is another aspect this SRP wishes to address.

This tension particularly transpires when it comes to the passage from finite- to infinite-dimensional Hilbert spaces, as was actually acknowledged by a professor we interviewed. As a matter of fact, the course we studied began with the infinite-dimensional case, even though the students never encountered them, especially to solve eigenvalue problems as is customary in quantum mechanics. In this SRP we chose to go the other way, starting with what students may already be familiar with and going from finite- to infinite-dimensional models (for such a treatment in the common paradigm of study, see for instance Le Bellac (2013)).

A PRIORI ANALYSIS

Resources and assessment

In light of the previous considerations we undertook the a priori analysis of an SRP at the interface between mathematics and quantum mechanics, with quantum computing as its background. It would include both classroom sessions (six) and computing sessions (three), the latter being taught by an IBM representative already working with the University of Montpellier (denoted by P_{qc}). Besides, to enforce the conveyance of mathematics and physics content, we chose to let one professor from each field (P_φ and P_μ for physics and mathematics, respectively) teach once during the SRP. Finally, in order to comply with the constraints imposed on this TU by each department, we decided students would work in groups of three, by curriculum. So, the set of students may be denoted by $X = \{g^3_{\mu 1}, g^3_{\mu 2}, g^3_{\varphi 1}, g^3_{\varphi 2}\}$ (where g^3_μ and g^3_φ are math and physics trios, respectively) and the team of teachers by $Y = \{P_{qc}, P_\varphi, P_\mu\}$. So, we end up considering the didactic system $S(X,Y,Q_0)$ (Chevallard 2019, p. 72), where Q_0 is our generating question (see next section).

Before addressing the choice of the generating question, we can complete the answers to the following questions in our meta-SRP: "Q2) Human and physical resources" and "Q3) Student assessment" (Bartolomé et al., 2018, p. 6-7, and fig. 1 here), which otherwise almost fully ensue from our preliminary analyses. First, we planned on collecting question-answer maps per group in order to monitor the evolution of students' questioning, actually letting students draft them themselves.

Then, complying with the institutional constraints coming from physics, we opted for a final answer in the form of a written report and an oral presentation. This implied, in particular, to let a significant part of the investigation on the "other stage" (Chevallard, 1998, p. 17). This is a questionable choice, as our research would gain in monitoring as closely as possible students' questioning. On the other hand, it enforces the customisation of the final answer $A^{\heartsuit}$ produced by students, making it indeed close to their hearts (Chevallard, 2019, p. 100).

Generating question

We may now answer the "Q1) The SRP structure" block of the meta-SRP. The most critical point here is our objectives are always two-fold, as this SRP is at the same time a course and a research experiment (see also Markulin, Bosch, Florensa & Montañola, 2022, p. 3). For instance, regarding Hilbert spaces, we wish to put two phenomena at play: on the one hand, in relation to the paradigm shift we investigate, we wish to reinstall the raison d'être of their use in quantum mechanics; on the other hand, in the context of a PhD devoted to the didactics of algebraic structures, we wish to see them play a unifying and simplifying role. Actually, our epistemological analyses lead to the conclusion we could meet both these targets at once, provided the use of the structures comes as an answer to the generating question or to a question in the process.

As was already mentioned, the passage from the finite- to the infinite-dimensional settings is a crucial step in quantum mechanics, and on in which the Hilbert space structure plays an important role. The latter structure is indeed the good framework where the practice acquired in low dimension can the most easily be transposed to infinite dimension. Though it is common in introductory quantum mechanics to cover several situations where both finite and infinite-dimensional frameworks play a role, even though the peculiarities of the infinite-dimensional case are often hidden — and with it the necessity of a more general, hence abstract, framework (see Lombard & Hausberger, to appear). This is particularly so when it comes to quantisation (eigenvalue problems), as show for instance most treatments of the infinite quantum well, the simplest realistic model of a quantum computer. That is, quantisation in this case can only be accounted for on the base of truly quantum mathematics.

Actually, the context of quantum computing puts at play numerous models to describe the machine, from two-level systems to anharmonic oscillators. Each time, the question of quantisation is crucial, as quantum bits are quantised states spanning two levels (usually marked $|0\rangle$ and $|1\rangle$). So, working on models of quantum computers surely brings about many epistemological aspects our analyses have highlighted so far. Consequently, we put forward the following generating question *Q_0: In what respect are quantum computers indeed quantum?* As it stands, the questioning is however quite open, so we chose to add three questions, as is for instance done in Barquero (2009, p. 198):

Q_{0a}: What are quantum bits and how can you calculate with them?

Q_{0b}: What phenomena allow quantum computers to operate?
Q_{0c}: What characteristics of quantum systems are shared by quantum computers?

All that being said, a pitfall consists in wishing students go through certain questions for the sake of our research (for instance), thus leading us back to the "monumentalist" paradigm we wished to quit (on this point, see Chevallard, 2006, p. 8). Consequently, given this setting, we can only hope students will opt for a mathematically-leaning answer to our generating question.

Media and models

The choice of media gives us further latitude though, especially regarding how open the SRP will be. Indeed, in the case students actually enrich their milieu with them, media could channel the questioning towards given works, be they visited or not. So, media both increase the numbers of models of quantum computers under study and decrease the openness of the generating question, in order to balance its scope. During lab sessions, students will manipulate the software Qiskit. During class sessions, book excerpts or videos will enunciate facts about physical or mathematical models of quantum computers (such as the Block sphere, abstract two-level systems or quantum wells). In addition, the professors taking part in the experiment should play an important role in the media environment (see fig. 3). Lastly, we plan on producing tailor-made pieces of media, for instance to encourage the process of questioning about the links between the various models so introduced.

Date	Topic/model	Pedagogical and epistemic context
January 20th, 2022	First encounter and opening of the questioning	Video presentation of quantum mechanics and quantum computers
January 27th	First lab session	Taught by P_{qc}: first manipulations with Qiskit
February 3rd	Questions following the first lab session	
February 10th	Double quantum well	Intervention of P_{φ}: two-level systems in two-dimensional complex vector space
February 17th	Second lab session	Taught by P_{qc}
February 24th	Infinite well	Discrete infinite spectrum of eigenfunctions
March 3rd	Winter break	
March 10th	Third lab session	Taught by P_{qc}
March 17th	Finite Well	Intervention of P_{μ}; spectrum of eigenfunctions, both discrete and continuous
March 24th	Conclusion	Hilbert space as a common framework
April 15th	Peer-reviewing report	
April 18th	Written report	
April 21st	Oral exam	

Fig. 3: Practicable sequence of sessions and prefiguration of media and models at play throughout the SRP

CONCLUSION

In this article, we have given a concrete instance of the implementation of both the preliminary and a priori analyses of an SRP set up at the interface between mathematics and quantum mechanics. We have shown by way of example that careful studies of the institutional and epistemological contexts are in order. Lastly, we have described the rationale underlying our choice of the generating question, which is a crucial step in the design of an SRP.

The module we set up is “a subject totally organised as an SRP” (Bosch et al., 2020), which means in particular that no course adopting the paradigm of visiting works supports it. As a consequence, it seems necessary to find a balance between leaving the students’ questioning totally open (which amounts to leave aside tailored planning of learning as well as research goals) and closing it so much that the generating question Q_0 “becomes a mere decorative and opportunistic artefact” (Barquero, 2009, p. 93). So, we have arranged the conditions for a cohabitation of the two main paradigms. Indeed, as put Chevallard (2011, p. 40), the development of the paradigm of questioning the world “does not cancel the paradigm of visiting pieces of knowledge (*le paradigme de la visite des savoirs*), but rather places it otherwise, at both epistemological and didactical levels”. Organising this space needs careful preparation.

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Cross-disciplinary Characteristics of Study and Research Paths: Statistics for Business Administration

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Statistics is a discipline that allows implementing novel teaching proposals based on inquiry. Students' reasoning in the statistical context can be encouraged by implementing project-based learning in statistics courses. Statistical processes often involve other disciplines apart from statistics itself. We present an analysis of the staging of cross-disciplinary characteristics of an inquiry-based project in statistics for the degree in Business Administration. Three moments of cross-disciplinary collaboration are described and analysed from both the viewpoint of teachers and students. Broadening the perspective beyond the domain of statistics provides us with new insights regarding student engagement, challenges related to project organisation and management, and a venue for cross-disciplinary dialogue.

Keywords: cross-disciplinarity, statistics for business, study and research path, teaching and learning of mathematics in other fields, novel approaches to teaching.

INTRODUCTION

Traditionally, statistics has been taught in a lot of different university degrees and has recently gained more prominence in numerous professional areas. The importance of information technologies is growing (Nolan & Temple Lang, 2010), and so are the proposals for training students to gain computational and statistical competencies. As a consequence of the technological development in the past decades, a large number of statistics educators have embraced pedagogical novelties in their teaching. Project-based learning (PBL) (Batanero et al., 2013) in particular has been promoted as a design tool for nurturing reasoning in a statistical context (Wild & Pfannkuch, 1999). PBL is widely present in the literature on statistics teaching (Markulin et al., 2021a), and is commonly described from the perspective of the area of statistics and more general pedagogy.

The numerous implementations of project-based proposals present different characteristics depending on the way statistical knowledge and professional activity are conceived in each institution. Several designs of so-called *study and research paths* (SRPs), framed within the Anthropological Theory of the Didactic (ATD), have been implemented in statistics, in other disciplines, and in cross-disciplinary contexts at the university level (e.g., Barquero et al., 2021). The main aim of SRPs is to support the development of student knowledge in an area of study by posing meaningful and challenging generating questions to students (Jessen, 2014). According to Barquero et al. (2007), it is important for the generating question to be "of real interest to the students ("alive")" (*op. cit.*, p. 2052).

Considering the recent developments both in business and in statistics, a subsequent question is: What and how to teach Business Administration students? In other words, the organisational goal of PBL and SRPs needs to be adapted to the topic and complexity of the degree in which those tools are implemented to be in line with its specificities. In the following sections, we present the ATD as the framework to analyse the modelling aspects of the proposals. We describe the implementation of SRPs in a statistics course for a degree in Business Administration and explore some issues regarding cross-disciplinary collaboration when designing and implementing an SRP, and how it is perceived by the students.

THEORETICAL FRAMEWORK

Within the field of mathematical modelling, a dominant approach to its teaching and learning is modelling when addressing a real-world problem, transposing it to the realm of mathematics, and addressing the mathematical problem before moving it back to the real world validating the result found in the process (Niss & Blum, 2020). However, this approach might not capture important aspects of the real modelling activity, as it takes place in the workplace (Frejd & Bergsten, 2016). Some of these aspects were analysed in a study by Serrano et al. (2010). The authors argued that the SRP implemented on the sales forecast of a fashion brand shows the intertwinement of the extra-mathematical and mathematical (or statistics) domains when fitting real data and real-world problems in one-variable functions.

Initially, SRPs were proposed by Chevallard (2004) as a way to design the cross-disciplinary projects introduced into French secondary schools (Winsløw et al., 2013). An SRP is initiated by posing a generating question Q_0 to students, who consider it real (Barquero et al., 2007), and worth pursuing, though they cannot answer the question immediately (Jessen, 2017). To answer the question, they need to engage in the processes of study and research. The study process is characterised as the study of new knowledge, which the students decompose. This means they need to understand what it is made of, how it can be used, etc. During the research process, the students reconstruct the new knowledge (in combination with existing knowledge) into partial answers to the generating question (Jessen, 2017; Winsløw et al., 2013). Furthermore, we can see this as a process where students pose derived questions Q' to the generating question, which they construct answers to through study and research processes. Thus, the generating power of Q_0 can be considered as the degree to which it invites students to pose such derived questions. Winsløw et al. (2013) depict this dynamic through a question-answer map presented in Figure 1.

When working with SRPs, the intertwinement of the domains involved in a modelling activity is reflected in going back and forth between questions and answers, between study and research. Moreover, we see this interconnection in cross-disciplinary SRPs, when an answer in one knowledge domain leads to a question in others. This can be seen in the SRPs analysed by Jessen (2014) on mathematics and biology, where answers in biology turned into questions regarding differential equations in mathematics.

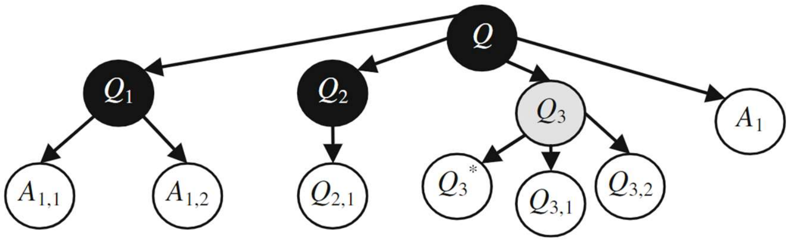

Figure 1: An example of a question-answer map (Winsløw et al., 2013, p. 271).

CONTEXT OF THE STUDY

We analyse the organisation of three SRP implementations in a statistics course in the second year of a bachelor's degree in Business Administration. Each of the implementations, during academic years 2019/20, 2020/21 and 2021/22 had its peculiarities concerning the topic of the project, the scope of the data to be gathered and analysed by the students, and the logistics of the implementation modality due to the COVID19 restrictions. However, the characteristics common to all the implementations were the collaboration with researchers in marketing, the existence of a "client" that was the facilitator of the project topic, and students organised in teams of 4-5 members acting as consultants for the client. More about the topics and the SRP implementations can be found in Markulin et al. (2021b, 2022a, 2022b).

RESEARCH QUESTION AND METHODOLOGY

Our work contributes to broadening the scope of SRPs in the domain of statistics in a Business Administration school by analysing constraints and conditions seen from the perspective of students on cross-disciplinary projects. The SRPs presented are implemented in a statistics course, although they could be implemented in other disciplines, encouraging different perspectives from which a certain project can be conceived. In this respect, we formulate our research question (RQ):

RQ: How can cross-disciplinary collaboration foster the management of a project in a statistics course in a Business Administration degree?

We will consider the issue from the macro-didactic perspective by analysing the collaboration of statistics lecturers and students, researchers in marketing, and external clients that are not usually involved in the education system. The approach we take is a qualitative analysis based on post-project interviews with the students. We comment on the experience of the project implementation, and finally, contrast it with the statistics students' reflections on the cross-disciplinary characteristics of the SRPs.

The post-project communication with the students is organised as semi-structured interviews, designed by a statistics teacher that also assumes a position as a researcher in didactics, together with five or six students. The students who participate in the interviews represent the students' working teams that participated moderately or actively in the project (according to the teacher's perspective and the students' final grades in the statistics course). The interviews are approximately 30-40 minutes long and structured according to the hypotheses on the SRP development put forward by the teacher-researcher in didactics. As shown below, the interviews are structured in such a way that broader problematic issues than simply the research question of the SRP addressed in this paper are analysed. The hypotheses (H) behind the interview script consider the:

H1. *Generating question and project aim.* The project comes from an external client who also presents the topic to the students. The generating question is a problem to be solved by student teams and is presented to an assessment jury at the end of the project to add realism to the project. An issue to discuss is the pertinence of the generating question, its driving force throughout the inquiry, and the extent to which the final answer presented by the students provides valuable information to the client.

H2. *Project survey and data collection.* The design of the project survey that students use for data gathering, the understanding of its components and its use to provide meaningful data to be analysed takes place in the interaction between statistics, marketing and the client's needs and possibilities. The interest in discussing its components is threefold: describing the students' involvement in the design part of the survey; the adoption of its raison d'être that fosters the arborescence of the derived questions to be statistically analysed; the insight into the challenges that students encounter when collecting data and the solutions they propose or implement to overcome the obstacles in such a "rudimentary" activity of collecting raw data and cleansing it before the statistical analysis. This is not usually part of a project students would expect in a statistics course but is essential for the profession of statistics.

H3. *Integration of the SRP in the statistics course.* The SRP is part of the statistics course, but the course is organised for the SRP to be its central activity. The classes preceding the project are organised in bi-weekly (14 days long) case studies or topic-related exercises that prepare the ground for the SRP to be implemented. The main issue of this section is the students' perspective on the integration of the SRP into the structure of the course. The connection between the bi-weekly studies and the project can be appreciated by recognising the importance of the provided software tools and knowledge for the project work, inquiry strategies developed before or during the project, cleansing the data, raising questions and synthesising results.

H4. *SRP organisation and management.* The SRP implementation during the statistics course, its development in student teams, the submission of the students' intermediate analyses reports, and the students' presentation of the final results as a poster or a slide show in front of the entire class are the organisational characteristics that are the most closely connected to pedagogical and school interventions.

A more detailed description of the interview design and the hypotheses supporting the script are described in Markulin et al. (2021b, 2022a).

Even though the different SRPs were implemented in the classroom under the guidance of the statistics teachers only, they were prepared thanks to the cross-disciplinary collaboration mentioned earlier. After analysing the students' interviews and identifying indications related to cross-disciplinary issues, especially concerning the first two hypotheses mentioned above, we present three moments in which collaboration is key:

1) agreement with the client,
2) design of the survey for data analysis,
3) understanding and exploiting the survey.

SRP EXPERIENCE AND DISCUSSION

The first moment: agreement with the client

Once a client willing to collaborate on the project is found, chronologically, the first step is to explore the potential of the problem's generating question. Figure 2 is an example of a statistics teacher's a priori test of a potential topic for the third SRP implementation. It can either be developed visually in a question-answer map (we tend to include only the questions in this kind of map, while the answers remain implied), or elaborated as a list of topics that can be derived from the initial generating question.

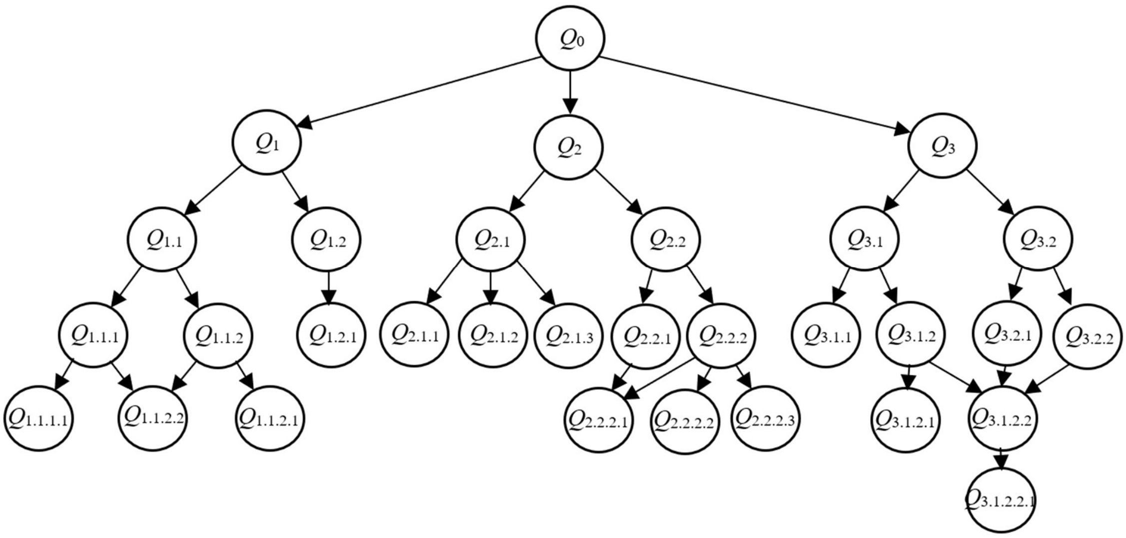

Figure 2: A priori question-answer map for the third SRP implementation

The generating question is Q_0: "What are the consumer habits and preferences of young people in Spain about sustainability, digitalisation, and leisure time?" The derived questions focus on sustainability issues (Q_1), experiences with digital services (Q_2), and leisure-time habits (Q_3). We will not go into detail about the map components in Figure 2, since it is not the focus of this paper, but the full text of the questions can be found at https://rb.gy/uzez4y. We will consider the moments based on collaboration. The first one is the final agreement with the client. It is worthy of note that, even at this early stage, the statistics teacher tends to leave the boundaries of the discipline to embrace the context of the potential data (something the students will eventually do as well once they start inquiring about the matter). The planning of the project development and its main focuses take place at the same time as the agreement with the client. Then, some of the SRPs trajectories proposed by the teacher are discarded, while new ones gain importance. This teacher-client communication is a pre-test of the communication that the client will later have with the different student teams. This interaction leads to improvements in the a priori analysis and adjustments are made to the teaching design, hence changing the focus of the data analysis at a later stage of the project. Moreover, this makes the situation more akin to genuine business practices.

The topic of the first SRP was vegetarian and vegan diets. Students found it interesting but were not very familiar with it. An illustrative example of this distance from the topic is the statement of one of the students:

> Student A: The topic is actually really good and interesting and the things we could see were really interesting, but it affects just a small group of people.

The second implementation concerned an SRP on a less restricted population, but it was locally oriented and about an initiative, the students did not know well – a cooperative supermarket looking for its optimal first location in Barcelona. The statement of one of the students reveals this fact:

> Student B: I think it was an interesting topic, different from what is normally dealt with in some projects by better-known brands or companies… I think it's a way that helped us to do a bit of market research, which, in the short term, will be something that we will have to do.

The third implementation was a collaboration with the marketing department of an international company about the attitude of young people in Spain towards sustainability. Thanks to the experiences from previous years, the interaction between the client and the students were given more importance in this implementation. Apart from the client presenting the project topic and ideas, an additional online meeting session with the client was scheduled. During that session, each student team met with the client. The students abandoned their role of "students" and adopted a more professional attitude. This was observed especially in those student teams that abandoned some of their initial problem proposals and redirected their inquiry adjusting it to the client's needs.

> Student C: At the meeting with the client, we presented our team's ideas on which to focus our analyses during the project. However, she told us to pursue certain ideas we presented, but also to discard some of them. We realised that not everything we proposed was interesting to her. We continued the analysis based on what the client suggested.

In the implementations where such interaction between students and the client during the project development was not possible to organise, the students sometimes tended to support what they considered to be the teacher's proposal at the expense of what was relevant to the client.

The second moment: design of the survey for data analysis

The second stage that requires collaboration outside the area of statistics is the design of a survey that will be used to gather the data for the project. It is a delicate moment since its production will directly affect the students' work and potentially create extreme actions, as we will show in the discussion on the impact of the collaboration of the third moment. Here, the statistics teacher teams up with researchers in marketing at the same business school. This collaboration is beneficial to both sides: the statistics teacher ensures the data to be gathered using the survey will suffice for the project's goal; the researchers in marketing get involved in a study in their area of interest and possibly enlarge their contribution to marketing research literature based on the ideas and results obtained from the Statistics project.

In the SRP implementations presented in this paper, most of the survey design responsibility was assumed by the researchers in marketing. They engaged in the survey design in teams. The teams differ slightly every year but mostly consist of one experienced researcher and two young researchers (PhD candidates at the business school where the Statistics course and SRPs took place). In the first SRP implementation, the survey was completely done by the marketing experts, since they were the initial "client". In the second implementation, the survey was designed in collaboration with researchers in marketing, the client, and the statistics teachers. In the third implementation, the survey was the final product of the collaboration between everyone involved in the project: researchers in marketing, the client, statistics teachers, and the students.

However, in the second implementation, when there was a chance to engage more with the client who posed the generating problem, the students hesitated to take part in the survey design. The following statement of one of the students shows that designing the survey might cause a feeling of uncertainty.

> Student D: If you had asked me to do it [design the survey] at the beginning or in the middle of the course, maybe I would have posed questions that at the end would be of little use for my analysis.

It seems the students lack knowledge, or confidence, to master the task of developing a survey to be statistically exploited. It is a challenge to statistically model a problem from the marketing domain. Collaboration with the researchers in marketing is, therefore, necessary and supportive for both students and statistics teachers.

The third moment: understanding and exploiting the survey

Students are the ultimate "miners" of the project survey. They are supposed to collect the data, clean them, and analyse them. Those actions are very common in contemporary business environments, in data science laboratories, and even in numerous professions remotely relying on statistics. However, most of the PBL literature focuses on competencies in data analysis and does not question the origin of the data and the cleansing process (Markulin, 2021a). In the SRPs here presented, the data origin appears to have stimulated even some lower achieving students to engage in the project, as student E states:

> Student E: …for the first time I think we have been able to work and contribute to a project that has not only been an ordinary task, but a real case…we realised that our work could provide something to the client.

According to the students' comments from the first two SRP implementations, the external origin of the survey caused certain detachment, even for high achieving students, such as Student F. Some radically stated that it would have been more useful if they could have used their own surveys (Student G), which might also be a sign of difficulties in recognising the relevance of the data coming from an external survey.

> Student F: I think the survey was super long. That's why I suppose we couldn't analyse everything.

> Student G: If we could start from the beginning and ask the questions that we want to answer from our data, it would have been clearer and more realistic.

Nonetheless, the process of cleansing and analysing the data eventually allowed the students to become familiar with the survey blocks and to start appreciating the design they were asked to follow. Student A describes this progressive process as follows:

> Student A: For example, when we got the answers and we started analysing the data, we saw that the survey was even clearer than we thought because there were correlations [between the answers to different questions in the survey].

It appears that the inclusion of external clients and survey facilitators represents both a productive condition and a challenging constraint for the students. In any case, this kind of organisation is the closest to a business environment that the teachers could have organised in a university setting under given conditions.

CONCLUSIONS

As shown in the previous section, the three phases involving the design and implementation of the SRP (agreement with the client, survey design and survey exploitation) lead to different levels of cross-disciplinary collaboration. Firstly, the existence of an external client generates two significant interactions, one between statistics teachers and the client in the first steps of the SRP, but also between the client and statistics students. As stressed earlier, the existence of a real client fostered the engagement of the teams. This phenomenon is already described in terms of *adidacticity* of the SRP by Barquero et al. (2021). It concerns observations on how the existence of an external client can favour the conditions under which the development of a final answer becomes the central activity during an SRP (Bosch et al., 2023). Consequently, there is a shift of the main purpose of the activity from the academic aspects such as course evaluation towards the production of reports for the clients.

Secondly, the interaction between marketing researchers and statistics teachers helps overcome the so-called *thematic confinement* (Barbé, Bosch, Espinoza & Gascón, 2005), that often exists in school institutions. However, this cross-disciplinary collaboration is still limited: marketing researchers that are also teachers in the same institution do not incorporate the survey design in their teaching activity. This reveals a clear restriction at the pedagogical level: the SRP is only implemented in one course. We think that an open issue of this research is related to the analysis of the restrictions hindering these cross-course collaborations to explore the ecological viability of cross-disciplinary SRPs.

Finally, the existence of different actors involved in the SRP design and implementation is an important challenge. The interactions between students, researchers, clients, and teachers need to be organised during the a priori and in vivo analysis of the SRP. This is often a new activity for lecturers that might find institutional tools to enable collaboration.

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AN EXPERIENCE OF EXPLORING THE BOUNDARY BETWEEN MATHEMATICS AND PHYSICS WITH PRESERVICE TEACHERS

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In this exploratory study, we investigate the potentiality of comparing mathematics and physics, from generic characterizations to interdisciplinary tasks and textbook analysis, as opportunities to foster secondary mathematics preservice teachers' awareness of the epistemic core of such disciplines. We designed and implemented a teaching sequence with Italian master students with a mathematical background, relying on a framework developed within a project about interdisciplinarity in preservice teacher education (IDENTITIES), and carried out three case studies analyzing data collected. We discuss the impact on students' conceptions and the development of learning processes at the boundary that our teaching sequence might have, as well as further reflections on how to make our activities more effective.

Keywords: Teaching and learning of mathematics in other fields, prospective teachers' education, rational behavior, learning potential, boundary crossing.

INTRODUCTION

The historical development of mathematics often reveals deep dialogues at the boundary with other scientific disciplines, particularly physics; stiffening the boundaries might lead to artificial and stereotyped views (see, for example, Boero et al., 2013; Branchetti et al., 2019), often accompanied by a characterization of the disciplines based, at least, on comparative definitions (Erduran & Dagher, 2014). Such an approach to the "disciplinarization" of knowledge not only hides the complexity of these dialogues but does not even mirror disciplinary authenticity, reachable by analyzing the scientific discourse, for example in articles or original texts (Branchetti et al., 2019). In previous works, it has been shown how a virtuous circle in preservice teacher education can be established: an interdisciplinary approach could help in understanding better the involved disciplines, while disciplinary knowledge could help in dealing with new problems not organized in a discipline yet (Satanassi et al., under review; Branchetti et al., 2019). Moreover, in Akkerman and Bakker (2011) it is stressed that while moving close to boundaries that separate/put in contact members of two different communities (in this case, disciplinary communities) there is a *learning potential* about the background of both communities, but whose fruitful activation depends on many contextual factors and is not trivial. Therefore, we address the following research problem: what processes might secondary mathematics preservice teachers enact when moving close to the boundary between mathematics and physics? Some activities to face and delve into this interdisciplinary exchange have been already explored (see Branchetti et al., 2019; Pollani et al., 2022). Among them, a teaching sequence about parabola, projectiles motion, and proof has been designed and implemented in different national and international contexts of secondary preservice teachers, to investigate whether and how learning potentials at the boundary might be exploited to make them develop an awareness of their view of mathematics and of the relationship between mathematics and physics. In this exploratory study, we qualitatively analyze the processes at the boundary between mathematics and physics that occurred in one of the national implementations, where the population consisted of preservice teachers with bachelor's in mathematics.

THEORETICAL FRAMEWORK

To explore our problem, we rely on a framework built on the notion of *boundary crossing mechanism* (Akkerman & Bakker, 2011), the *Family Resemblance Approach* (FRA) to the Nature of Science (Erduran & Dagher, 2014), and the *rational behavior* (Boero et al., 2013). Akkerman and Bakker highlight how in general the notion of *boundary* is ubiquitous and represents a dialogical phenomenon between communities, rather than a barrier. The authors then characterize four boundary crossing mechanisms: *identification*, occurring when a deep uncertainty of the line between disciplines leads first to question the core identity of intersections and then to renewed perspectives about disciplines; *coordination*, taking place if the cooperation between disciplines is required to keep the flow of work and the use of common tools; *reflection*, which happens through explaining and understanding the differences between disciplines, and thus enriching their identities; and *transformation*, which leads to a profound change, and even to new and in-between disciplines. We will use the terms *disciplinary* or *interdisciplinary learning potential* considering the increasing awareness respectively of disciplines, conveyed by identification and reflection, or of their interplay, conveyed by coordination and transformation. In designing our teaching sequence, to go beyond the stereotyped views of scientific disciplines, we referred to a characterization of disciplines developed within the FRA (Erduran & Dagher, 2014). According to the authors, the *epistemic core* of scientific disciplines is articulated in four networked categories, rather than disconnected fragments: *aims and values* (like objectivity, consistency, rationality, etc.), *practices* (like observation, argumentation, modeling, etc.), *methods* (like to generate reliable evidence and construct theories, laws, and models, etc.), and *knowledge* (like Euclidean geometry theory, atomic models, etc.). To identify the features of disciplinary and interdisciplinary discourses starting from the choices made in concrete examples, we also referred to the *rational behavior* (Boero et al., 2013; for textbooks and interdisciplinary contexts see Pollani et al., 2022), consisting of three interrelated dimensions: the *communicative* one for text presentation choices; the *epistemic* one for the choices related to identification and expounding of used facts; and the *teleological* one for pursued goals and strategies.

In this paper, the research question is: can any *learning potential* at the boundary of mathematics and physics as disciplines be actualized by our teaching sequence? In particular: what *boundary crossing mechanisms* might preservice teachers activate while characterizing mathematics and physics in different tasks? Whether and how do they become aware of their personal view about the *epistemic core* of the disciplines and able to question it? To address it, we analyze reports of 3 preservice teachers collected during the implementation of the teaching sequence carried with 25 master students in mathematics attending a mathematics education course, held at the University of Milan by LB.

In the following, we resume the teaching sequence of five two-hour lessons. In the first one, we brainstormed *What characterizes mathematics as a discipline? What are common aspects and the main differences with physics?* Then LB held the lecture "the FRA and the epistemic core of disciplines", followed by a questionnaire about the topic of the lesson, and the task *Deliver a written personal report on the characterization of mathematics as a discipline, also in comparison with physics, considering what emerged in the classroom, but also stressing your point of view*. We asked as homework: "Read this excerpt of a discussion between students about the task *Which curve is represented in the following images?* (see Fig. 1) *Gianni argues that the trajectory of the first image certainly represents a parabola, while we cannot say anything for sure about the others, but Francesca is not convinced: she says that we do not have enough information to establish that the first is a parabola, while on the others it is certain. Amina intervenes by saying that unknowing what context the images are placed in, we can never conclude.* Do you agree with one of the three? Which aspects of each position can be interesting, and which are questionable? How would you enter the debate and make it evolve to take a position?"

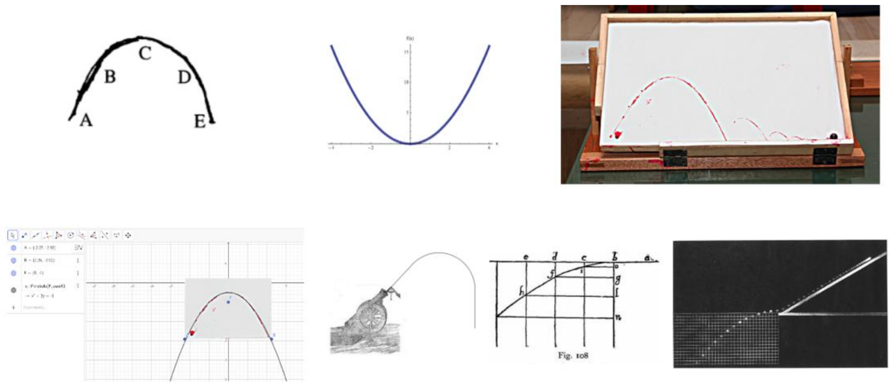

Figure 1: Images proposed to students to discuss curves and trajectories.

In the following 3 lessons, after an initial discussion about the tasks and the homework using FRA tools, three lectures were held: "The parabolic motion and the birth of physics as a discipline" (by Olivia Levrini, physicist), "Parabola in the history of mathematics and physics" (by LB), and "Habermas' dimensions of rationality" (by LP). In the fifth lesson, LP presented the analysis of an Italian physics textbook excerpt on the motion of projectiles using rationality, and then the task *Analyze in small groups with the lens of rationality the first part of the paragraph about horizontal initial speed. Deliver a text explaining in detail your* *group analysis*. Collective discussions were audio recorded, and the homework and two reports were collected. The first report led us to understand, at least partially, what were the students' points of view, so we could triangulate their answers in the second report, about textbook analysis. To help preservice teachers to catalyze and organize their conceptions, they were provided with the lenses of FRA and rationality, which we posit act as scaffolding factors. We hypothesized that the FRA could encourage them to characterize disciplines in terms of resemblances or differences, rather than definitions. Moreover, we conjecture that asking them to analyze a simulated debate and to detect in the first person the rationality in textbooks could bind and switch on their aims and values of disciplines, going beyond generic and stereotyped sentences, like "mathematics is the science of numbers" and "physics is the science of phenomena". In our analysis, we rely on the explicit information to search for boundary mechanisms, referring: to *identification* with differentiating phrases like "mathematics/physics is/is not"; to *reflection* with those like "from a physical point of view it is relevant", showing more awareness of a different relative disciplinary point of view; to *coordination* when opportunities for using mathematics to solve a physical problem are pointed out. The *transformation* is not considered relevant in this case. We identify how they are matched with aspects of the epistemic core and rational choices stressed as personally relevant. Referring to the FRA, we deepen our coding by searching if preservice teachers undertake a definitory or more blurred characterization of disciplines, if they show awareness of and in their processes, and if they refer to stereotyped views, meaning that they refer to generic and external terms or praxes.

DATA ANALYSIS

S1

FRA report	
The first words I would use straight away to characterize mathematics are: rigor, logic, abstraction, truth, and utility.	*Identification* of rather stereotyped *aims and values* of mathematics by listing single nouns.
Trying to describe a 'mathematical' process: we start from a real or invented problem, we try to identify the variables that influence the problem and isolate them, […] then we try to make the problem itself the most general and abstract as possible, […] finally we look for the limits revealed to us about the problem itself.	Non-definitory *identification* of stereotyped epistemic aspects of mathematics, again as a list; the repeated verb "to try" and the shift from "process" to "we" might show awareness.
Mathematics is the universal language to explain nature (quoting Galileo), it aims to respond to various practical problems, simplify the life of man, look for an order or a rule where there seems to be none, and explain what seems to be outside our control.	Definitory and stereotyped *identification* of mathematics, referring to a list of *aims* and *methods*.

Looking at the first words with which I characterized mathematics it is obvious that they also belong to other disciplines, such as physics.	The action of "looking at the first words" and recognizing resemblances with other disciplines shows an aware process of comparison of the epistemic cores.
Throughout history, mathematics and physics have interacted for a long time, […] geniuses have contributed to both, precisely because so close.	Historical generic *coordination* is pointed out, based on the "closeness" of the disciplines, without declining it epistemically.
Some of the main differences between mathematics and physics are that the former chose to be independent of reality, while the second has a continuous comparison and denial or verification with reality as appropriate.	*Reflection*, explaining some of the main differences in their willing choices, in one case with stereotyped *aims*, in the other with stereotyped *practices*.
However much the sciences need language mathematical and depend on it, mathematics has the same dependence on the sciences, since if it were not such a useful and versatile language, if it were not able to speak with and for the sciences, the mathematics would see its *raison d'être* disappear.	Instrumental stereotyped *coordination* and mutual need based on *aims* of utility and versatility of mathematical as a language, emphasized with "such a" utterance, and on its ontological status.
Homework	
The first image is a parabola (or perhaps a branch of a hyperbola?), and the second (probably) represents a parabola along the initial line ABC and then becomes a vertical line. The third has various parabolas, of different sizes and openings. The fourth again begins as a parabola and then assumes the course of what appears to be a horizontal line. The fifth consists of a branch of a parabola and then of an oblique line.	In her first statement a doubt is insinuated, that later becomes the possibility to state for a curve to be a parabola with a grade of uncertainty/probability, which seems to start to blur the initial "to be". All the statements are rather definitory and absolute, without much explanation.
I can't completely agree with any of the three, however how much Gianni and Francesca have positions that I partly share. Amina's position is the one with which the more I disagree, as context doesn't matter, it could certainly help us have more elements, but don't depend on it. Gianni is too rigid in excluding the parabola for the other images, it also takes a certain degree of adaptation/approximation. Francesca, on the other hand, approximates a little too much.	The first utterance reveals an aware attempt of *identification* of mathematics' *aims and values*. She justifies why she disagrees with definitory claims of the absolute truth of mathematics, and she considers truth independent from the context. She consciously identifies a new *value* ("a certain degree of adaptation"), but she does not state explicitly a standard for its acceptability.

Table 1: On the left are the original excerpts by S1, and on the right is our analysis.

In the questionnaire, she declared she had never reflected on this topic before. She was impressed by the choice to characterize and not define, and the composition of the *epistemic core*. What most impressed her is the inclusion of *aims and values*, which are terms used to describe a "real and alive person". She stressed that she appreciated that this framework does not only refer to concrete everyday actions but to less visible values, the essence of actions, which counts most. She complained that *practices* and *methods* are not easy to distinguish, and *knowledge* is too static, and does not consider that disciplines are knowledge in evolution (time-dynamicity), not an object that can be divided into fixed pieces (space dynamicity). In her textbook analysis, she referred to what the text does visually "put in evidence" (e.g., "it puts in evidence the paths", "[it could have shown] visually what it tells only by words"), as traces of *communicative rationality* and a *value* to be pursued. However, she focused more on "lack of rationality", up to questioning the text directly: for example, about the *teleological* dimension she criticized the statement "let us isolate *t* [the time variable]" with "why must *t* be isolated? […] it looks like a magic trick […] Why not write that you want to prove what is the motion path […] instead of leaving the reader unaware of reasonings behind the undertaken calculations?", where we can point out an *identification* mechanism and almost a defense of mathematics epistemology from being "a magic trick"; or, for example, about *epistemic* and *teleological* dimensions she pointed out how the "reasonable hypothesis" and the proof are not mentioned as such, implicitly referring to the hypothetical-deductive system, an aspect of epistemic core of mathematics.

S2

FRA report	
One of the first things that in my opinion characterizes mathematics, and which distinguishes it clearly from all other sciences, is the fact that it is not necessarily a pragmatic knowledge […] mathematics also makes sense to exist by itself, free from all its innumerable applications.	Conscious *identification* of a type of knowledge reached by mathematics, justified by its ontological status. The use of "not necessarily" avoids an overall definitory approach, unlike "clearly" could have made think.
Other characteristic that I would associate with mathematics is consistency, which is not exactness, but the fact that everything is consistent within a well-defined and defined axiomatic system.	Non-definitory and first-person conscious *identification* of and reflection on a *value* and its related knowledge structure.
Another characteristic that unites it to knowledge traditionally considered 'humanistic' is the fact that it is 'argumentative'.	Recognizing the resemblance of *practice* between mathematics and "humanistic knowledge".
The figure of the mathematician, for me, is that of a person who studies the mathematical world, which does not always coincide with the real one, although it may be its model;	Aware first-person *reflection* using rhetoric negation on the generic example of "the figure of the

interest in this mathematical world, however, would exist even without the link with reality, feeding on the sole desire to investigate the nature of abstract objects. […] mathematics is more akin to philosophy than to other purely scientific knowledge such as chemistry, biology, and others	mathematician", re-affirming the ontological status of mathematics and recognizing resemblances with philosophy.
Both mathematics and physics provide some of what Jürgen Habermas calls the dimensions of rational behavior.	Resemblance using the tool of rationality between mathematics and physics.
Homework	
The group agreed with Amina's statement, without having information on the context, on the reference system, it is not possible to have certain information. The statements of Gianni and Francesca are not motivated, to make the debate evolve it could be observed that in the first image there is no additional information concerning the graph represented, while in the other images there is information that accompanies the graph. Amina's position seemed to us the most reasonable, it underlines and highlights the importance of contextualizing each representation.	Conscious *identification* that the context and the information one can gain from it carries a certain degree of (un)certainty, and that contextualizing and motivating statements is necessary and help to increase their reasonability. The statements are formulated in general, without referring to a specific discipline.

Table 2: On the left are the original excerpts by S2, and on the right is our analysis.

In the questionnaire, she declared she had already reflected on the epistemology of mathematics, through personal readings and in a university course. She was impressed by the external rings concerning the institutional and social dimension of science: she considered important not only to be immersed in a discipline but to understand also how society sees scientific disciplines from the outside. In her physics textbook analysis, she started evaluating the coherence of choices with the authors' explicit aims (she says, "as authors promised"), and some generic visual criteria like "shortness, lightness, slenderness, clarity of images". All along the paragraph she stressed mainly the lack of rationality and explanations/motivations, with long and detailed arguments based on previous knowledge and considering possible student's point of view. She seems aware of the undertaken evaluating process (she often said, "[here] I point out/do not point out"), but not of her disciplinary point of view: indeed, she said, for example "It would have been more rational [first to have a general case and then particular cases because] they would have been reunited under a general theory", where she opposed a deductive approach as "more rational" to an inductive one. Furthermore, even if she also identified strengths, she did not motivate them in terms of the epistemic core of either physics or mathematics; for example, she said, "from this choice, it is easy for students to mistake of thinking that some physical laws are valid only in a specific case and not in others" and "without this concept to have been ever defined".

S3

FRA report	
Contrary to what is seen in advanced university courses in mathematics, in high school I always saw the latter as a subject completely different from any other, being then almost essentially practical. […] Mathematics is, on the other hand, exercises on exercises, a small theoretical introduction, and then again exercises on exercises. Personally, I think I did 20% of theory and 80% of exercises in the whole period of high school.	First a personal generic differentiation between mathematics and physics without traces of resemblances ("completely different"), based on a type of *knowledge*. Then a definitory *identification* based on stereotyped scholastic *practices* and his scholastic experience. Disciplines looked only as school subjects.
The similarity with a subject such as physics is almost evident: formulas and problems. However, physics on its part sees a more historical approach than mathematics. […] Unlike mathematics, however, the theoretical part of the subject appears much more present in physics (always in secondary school). The concept of proof was often associated, by any high school student who had not privately explored mathematics, with the physical realm. The structure of the verifications was perhaps the accomplice of this: many times, requests for proofs in physics appeared, seldom in mathematics.	Resemblance with physics rather stereotyped, based on scholastic *knowledge* and *practices*. Blurred attempts of reflection on differentiation based on epistemic aspects like *knowledge* and *practice*, but with disciplines looked only as school subjects.
I can summarize one last huge difference between mathematics and other subjects in the following sentence: the exception proves the rule, except in mathematics.	Hyperbolical and stereotyped differentiating *identification* of different *methods* between mathematics and other subjects.
Homework	
Reading the proposed discussion, we are more likely to agree with Amina, noting how important the context in which these images are placed is for us too, to feel confident in affirming whether or not they are parabolas.	Aware *identification* of the need of the context to reassure and "feel confident" in definitory classifying objects, referred to personal reasoning and not a specific discipline.
In class, we had thought of 'excluding' those that in our opinion were not parabolas (or not necessarily at least) by intervening on the drawings both manually and with GeoGebra […] For the fourth slide [I thought] to trace function profiles with GeoGebra that resemble that of the parabola.	Classification of curves is blurred ("not necessarily", "resemble"). Contextualizing can convince students about the "truth" behind the drawings: implicit *identification* of *values* and features of mathematical *knowledge*.

Table 3: On the left are the original excerpts by S3, and on the right is our analysis.

In the questionnaire, he declared he had never reflected on the topic before. He was impressed by the epistemic core of the disciplines, which considered something that students should learn from the very beginning because what appears simple and basic could be not so solid. He did not understand well the difference between characterizing and defining sciences, referring explicitly to the example of the mathematical definition of continuous functions, where defining is characterizing and vice versa. In his textbook analysis, he deemed each of the rational dimensions separately, starting with quantitative utterances about how much each rational dimension is in the excerpt. Through qualifying adjectives, he referred to generic and stereotyped *values* or *practices*, such as "explanations are simple and immediate", also with negative and comparative forms, like "choices do not seem unusual" and "to be more precise, it would have been more correct to write […] (domain, limitedness and compactness change)", even if the comparison (mathematical) term remains implicit. He also pointed out that "the setting is well-represented" and "even if the thesis is missed", which refers, again implicitly, to the mathematical practice of building a hypothetical deductive system.

DISCUSSION AND CONCLUSIONS

We observed three different cases during the same teaching sequence in the same context: S1 seemed to develop more personal reflections about mathematics and interdisciplinarity, S2 seemed to reinforce some views of mathematics but did not elaborate on her previous knowledge, while S3 seemed to refer only to school practices and not to deepen into epistemic issues. During our sequence, the FRA report fostered mainly an *identification* mechanism, as could have been expected from the questions, leading to characterize mathematics and physics as disciplines, but also as school subjects, as in the case of S3. This process did not always involve all the aspects of the epistemic core, was not always non-definitory and aware, and often led to explaining more differences, rather than resemblances. The homework, asking them to take a stand and being formulated in an interdisciplinary way, led to more aware observations, and made new epistemic aspects being mentioned, sometimes not so consistently with their declared view of the disciplines. Only in the case of S1 did our sequence seem to trigger a learning process at the boundary, blurring general and external sentences into more personal and contingent with the materials, supported both by the FRA and rationality. What we find particularly interesting is the fact that during the physics textbook analysis they seemed to use mathematical *values* to look at the presentation, up to questioning it openly, as in the case of S1. All of them seemed unconscious that they were referring implicitly to personal mathematical standards, which they assumed and defended as absolute and correct also analyzing a physics textbook. In conclusion, we provide preliminary answers to our question: from our analysis, it seems to emerge that some students grasped important epistemic aspects that we identified as *learning potentials*, but this did not work for all the students (see S3). The task of characterizing might foster itself mainly *identification*, and more rarely *reflection* and *coordination*. However, only in the case of S1, it seemed to occur a *learning process* at the boundary, triggered using both the FRA and rationality: indeed, S1 (critically) analyzed the textbook by questioning it openly, showing a more personal approach than the initial one, even if partially implicit and unaware. The overall weak awareness of the disciplinary point of view could be due to the homogeneity of the population's bachelors. To discuss this hypothesis, data from two other contexts, one national (Bologna, physics education course) and one international (summer school of the European project IDENTITIES, https://identitiesproject.eu), where the population was composed of secondary prospective teachers with a bachelor in mathematics *or physics*, are being analyzed, but we conjecture that this heterogeneous context could have led them to develop more "disciplinary awareness" and deeper reflections. These data might suggest a need for further reflections concerning a characterization of rationality in terms of disciplines. We will carry out further studies to check how these results are significant and generalizable, and how we can improve our teaching sequence to make it effective for most preservice teachers.

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Potential conflict factors in learning exact differential equations: An impact of institutional practices

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Many engineering students find university mathematics courses challenging and tend to adopt an instrumental approach to their studies. One of the difficulties pertains to students' inability to relate new material to their existing knowledge and skills. We employ anthropological theory of the didactic and the construct of the concept image for the analysis of two institutional praxeologies in Calculus and Differential Equations indicating potential conflict factors in students' understanding of the concept of an exact differential equation.

Keywords: teaching and learning of mathematics for engineers, curricular and institutional issues concerning the teaching of mathematics at university level, exact differential equations, anthropological theory of the didactic, concept image.

INTRODUCTION

Engineering plays a significant role in the modern society; the demand for engineers with better interdisciplinary and specialist skills is continuously increasing (Kent & Nossum, 2003). Future engineers need a wide spectrum of mathematical competencies and should comfortably use mathematics as a medium for communicating and developing ideas and concepts – "we need engineers who are at ease with it [mathematics] and who can take advantage of new ideas and use them appropriately even if they are expressed using advanced mathematics" (Blockley & Woodman 2002, p. 15). However, educational research acknowledges difficulties with students' conceptual understanding of mathematical disciplines which are often viewed as obstacles on the way to the engineering degree (Ditcher, 2001; Harris et al., 2015).

University courses on differential equations (DEs) are included in most engineering programmes, but related educational research is scarce, with "fewer than two dozen empirical studies published in top journals [in mathematics education]", which is quite surprising "given the centrality of differential equations (DEs) in the undergraduate curriculum, as well as the move away from a "cookbook" course to one that emphasizes modelling, qualitative, graphical and numerical methods of analysis" (Rasmussen & Wawro, 2017, p. 555). Exact differential equations (EDEs) is a classical topic; it is present in all traditional DEs courses and connects to many important concepts and methods in mathematics and physics including, for instance, integration of first-order linear DEs with variable coefficients, integrating factors, and first integrals. Rezvanifard et al. (2022) discussed difficulties with the learning of EDEs and a positive impact of a puzzle-based learning on students' conceptual understanding of this topic.

Recently, anthropological theory of the didactic (ATD) was employed by González-Martín and Hernandes-Gomes (2018, 2019) to analyse differences in mathematics and engineering courses. Hochmuth and Peters (2021) combined ATD with Weber's construct of ideal type to address variations in institutional praxeologies and individual student work in mathematics and engineering discourses. In this paper, we combine ATD (Chevallard, 2019) with the constructs of concept definition and concept image (Tall & Vinner, 1981) to explore potential conflict factors in the learning and teaching of EDEs. In contrast with the previous research on engineering education employing ATD, both institutional praxeologies in our case are within mathematics domain.

THEORETICAL FRAMEWORK / METHODOLOGY

The anthropological theory of the didactic

ATD furnishes an epistemological framework to describe mathematical knowledge as a human activity. A didactic system $S(X, Y, k)$ includes a class of students X, a team of teachers Y, and a piece of knowledge k from a body of knowledge K (a discipline D), in our case, mathematics. The theory of didactic transposition raises "the question of the precise nature of the piece of knowledge k which is the "didactic stake" – the thing to be taught and learned – in $S(X, Y, k)$" (Chevallard, 2019, p. 72). Importantly, this theory views knowledge as "a changing reality, which adapts to its institutional habitat where it occupies a more or less narrow niche" (Chevallard, 2007, p. 132).

ATD "hinges on an essential and founding notion: that of praxeology" posing that "all "knowledge" can be modelled in terms of praxeologies" (Chevallard et al., 2016, pp. 2615-6) used as building blocks for didactic systems. A praxeology consists of a task T, a technique τ (tau), a technology θ (theta), and a theory Θ (big theta). In ATD, the task T is performed using the technique τ. The technology θ is "a way of explaining and justifying or even of "designing" the aforesaid technique τ." The theory Θ "should explain, justify, or generate whatever part of technology θ may sound unobvious or missing" (Chevallard & Sensevy, 2014, p. 40). A praxeology is construed as the union of two "blocks," the *praxis part* $\Pi = [T/\tau]$ and the *logos part* $\Lambda=[\Xi/\theta]$. Notably, "it is the theoretical block that makes it possible to preserve the activity as a practice and communicate it to others, so that they, too, can participate in it" (Hardy, 2009, p. 344).

Didactic systems live in institutions understood as "any created reality of which people can be members" (Chevallard & Bosch, 2019, p. xxxi). For example, "a class, with its students and teachers, is an institution" (Chevallard & Sensevy, 2014, p. 2615). Institutional approaches significantly impact student inducting into mathematical practices (Hardy, 2009; Winsløw et al., 2014). Furthermore,

> It often happens that an object O lives permanently in an institution J and remains lengthily ignored by another institution I not unconnected with J, while being simultaneously unknown to the overwhelming majority of the persons subjected to I. … Consequently, for many teachers "the notion that inhabits "my" institution is exactly what this notion is, so that I can ignore all other institutions' definitions of it." (Chevallard, 2019, pp. 82-83)

To tackle this problem, it is important to compare the description of praxeologies for the same object adopted by different institutions.

> An institutional reference model of praxeologies involving a mathematical notion can be built to describe the practices and knowledge that an institution aims for students to develop – with interconnections between subjects, themes, sectors and domains related to the notion in question. (Winslow et al., 2014, p. 103)

Concept definition and concept image

One of the learning difficulties acknowledged by ATD stems from "a universal belief that any notion has a unique definition, independent of the institution that uses it" (Chevallard, 2019, p. 82). Introducing the constructs of concept definition and concept image, Tall and Vinner (1981) also recognised that this expectation is not justified.

> Compared with other fields of human endeavor, mathematics is usually regarded as a subject of great precision in which concepts can be defined accurately to provide a firm foundation for the mathematical theory. The psychological realities are somewhat different. Many concepts we meet in mathematics have been encountered in some form or other before they are formally defined and a complex cognitive structure exists in the mind of every individual, yielding variety of personal mental images when a concept is evoked. (p. 151).

The concept definition is usually regarded as a description of the mathematical notion accepted by the professional community, and this is what mathematics teachers strive to teach to students. However, "for each individual a concept definition generates its own concept image" (Tall & Vinner, 1981, pp.152-153) understood as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." The concept image "is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." Distinct parts of the concept image, termed evoked concept images, can be activated at particular times, including images that may appear conflicting – "if "conflicting" parts of the concept image are called at the same time then a sense of confusion, or conflict may appear." In this case, a potential conflict factor is defined as "a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition" (Tall & Vinner, 1981, pp.152-153). "The pre-eminence of the concept image is clear when it is time to act or to solve a concrete problem" (Gascón, 2003, p. 47); it often replaces the concept definition.

> In general, it is postulated that in informal learning of concepts (which is the most common), the concept image is utilised instead of the concept definition and also when the concept definition has been constructed (parting from the terms of the definitions, if these have already been introduced), this will tend to stay inactive in the mind of the person and may even be forgotten. (Gascón, 2003, p. 47)

The construct of concept image complements ATD in our analysis. ATD acknowledges possibilities for different institutional praxeologies built for the same mathematical concept and the concept image framework supports the evolution of concepts within institutions that creates potential conflict factors. Combining ATD with the concept image paradigm, we address “the challenge of empowering students with autonomy and insight into the raisons d’être and rationales of such [mathematical] work” (Winslow et al., 2014, p. 100). Focusing on praxeologies rooted in two mathematical institutions, a Calculus (C) class, and a Differential Equations (DE) class, we analyse two approaches to the concept of an EDE.

The research question addressed in this paper is: *What similarities and distinctions characterising C- and DE- praxeologies create potential conflict factors*?

TWO INSTITUTIONAL VIEWS OF EXACT DIFFERENTIAL EQUATIONS

Multiple mathematical organisations in the engineering curriculum

We explore mathematical organisations MO1 and MO2 in two courses, Mathematics 2, and Mathematics for Mechatronics. Mathematics 2 is taught to first-year engineering students and is based on the text by Adams and Essex (2018). The module includes, among other topics, functions of several variables, vector calculus, and line integrals of vector fields. Mathematics for Mechatronics uses the textbook by Boyce and DiPrima (2013) and is taught in the first semester of a master’s program. The course focuses on the methods for the solution and analysis of DEs, stability, and applications.

Vignette MO1 Conservative vector fields in Calculus (Adams & Essex, 2018, pp. 874-880). A vector field $\vec{F}(x, y)$ in two dimensions defined by

$$\vec{F}(x,y) = F_1(x,y)\vec{\imath} + F_2(x,y)\vec{\jmath} = \nabla\varphi(x,y) = \frac{\partial\varphi}{\partial x}\vec{\imath} + \frac{\partial\varphi}{\partial y}\vec{\jmath}$$

is called *conservative*, and the function $\varphi(x, y)$ is called a (scalar) potential of $\vec{F}$.

The equation $F_1(x,y)dx + F_2(x,y)dy = 0$ is called an *exact differential equation* if its left-hand side is the differential of a scalar function $\varphi(x, y)$.

A necessary condition for a conservative vector field. If $\vec{F}(x,y) = F_1(x,y)\vec{\imath} + F_2(x,y)\vec{\jmath}$ is a conservative vector field in a domain D of the xy-plane, then

$$\frac{\partial}{\partial y}F_1(x,y) = \frac{\partial}{\partial x}F_2(x,y) \quad \text{in } D.$$

If $\varphi(x, y)$ is a potential function for a conservative field $\vec{F}(x, y)$, the *level curves* $\varphi(x,y) = C$ of $\varphi(x, y)$ are called *equipotential curves* of $\vec{F}(x, y)$.

Example 1 (Adams and Essex, 2018, p. 877). Show that the vector field $\vec{F}(x,y) = x\vec{\imath} - y\vec{\jmath}$ is conservative, find a potential function and describe the equipotential curves.

Solution Since $\frac{\partial}{\partial y}F_1(x,y) = 0 = \frac{\partial}{\partial x}F_2(x,y)$ in R^2, $\vec{F}$ is conservative. For any potential function $\varphi(x, y)$, one should have $\frac{\partial\varphi}{\partial x} = F_1(x,y) = x$ and $\frac{\partial\varphi}{\partial y} = F_2(x,y) =$

$-y$. Integrating $\frac{\partial\varphi}{\partial x} = F_1(x,y) = x$ with respect to x, we obtain $\varphi(x,y) = \int x dx = \frac{1}{2}x^2 + C_1(y)$, where the "constant" of integration can depend on the variable y. Using $\frac{\partial\varphi}{\partial y} = F_2(x,y) = -y$, taking the derivative $\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 + C_1(y)\right) = C'_1(y)$ and equating it to $-y$, we deduce that $C_1(y) = -\frac{1}{2}y^2 + C_2$. Therefore, for any constant C_2, $\varphi(x,y) = \frac{x^2-y^2}{2} + C_2$ is a potential function for a vector field $\vec{F}(x,y)$. Equipotential curves defined by $x^2 - y^2 = C$ represent a family of rectangular hyperbolas.

Remark 1 A necessary condition is not formulated as a theorem and no proof is provided. However, restrictions on the topology of the domain are mentioned warning that a vector field may not be conservative in a domain that has holes (Adams & Essex, 2018, p. 879). The following example is provided as an illustration of this possibility.

Example 2 (Adams and Essex, 2018, p. 879-880). Verify that a vector field $\vec{F}(x,y)$ defined for $(x,y) \neq (0,0)$ by $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}\right)\vec{\imath} + \left(\frac{x}{x^2+y^2}\right)\vec{\jmath}$ is not conservative on the whole real plane including the origin.

Vignette MO2 EDE in Differential Equations (Boyce & DiPrima, 2013, pp. 95-100).

Given a DE $M(x,y) + N(x,y)y' = 0$, suppose that we can identify a function $\Psi(x,y)$, such that $\frac{\partial\Psi}{\partial x}(x,y) = M(x,y)$, $\frac{\partial\Psi}{\partial y}(x,y) = N(x,y)$, and such that $\Psi(x,y) = c$ defines $y = \varphi(x)$ implicitly as a differentiable function of x. Then

$$M(x,y) + N(x,y)y' = \frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\Psi[x,\varphi(x)],$$

and the DE assumes the form $\frac{d}{dx}\Psi[x,\varphi(x)] = 0$. In this case, the DE $M(x,y) + N(x,y)y' = 0$ is said to be an *exact differential equation*. Solutions are given implicitly by the equation $\Psi(x,y) = c$, where c is an arbitrary constant.

Example 3 (Boyce & DiPrima, 2013, pp. 95). Solve the DE $2x + y^2 + 2xyy' = 0$.

Solution One can guess that the function $\Psi(x,y) = x^2 + xy^2$ has the property that

$$\frac{\partial\Psi}{\partial x}(x,y) = 2x + y^2 = M(x,y), \quad \frac{\partial\Psi}{\partial y}(x,y) = 2xy = N(x,y),$$

and the given DE can be written as

$$\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y}\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{d}{dx}\Psi(x,y) = \frac{d}{dx}\Psi(x^2 + xy^2) = 0.$$

Thus, solutions to this equation are defined implicitly by $\Psi(x,y) = x^2 + xy^2 = c$.

Theorem 1 (Boyce & DiPrima, 2013, p. 96). Let the functions M, N, M_y and N_x where subscripts denote partial derivatives, be continuous in the rectangular region R: $\alpha < x < \beta,\ \gamma < y < \delta$. Then the equation $M(x,y) + N(x,y)y' = 0$ is an exact differential equation in R if and only if $M_y(x,y) = N_x(x,y)$ at each point of R. That is, there exists a function Ψ satisfying the equations $\Psi_x(x,y) = M(x,y), \Psi_y(x,y) = N(x,y)$, if and only if M and N satisfy the equation $M_y(x,y) = N_x(x,y)$.

Remark 2 The necessity part of the proof (Boyce & DiPrima, 2013, pp. 96-98) is constructive; it provides a method for finding the function $\Psi(x,y)$. A footnote to the theorem explains that the region may not necessarily be rectangular, but should be simply connected (that is, with no holes in its interior).

Example 4 (Boyce & DiPrima, 2013, pp. 98). Solve the DE $(y\cos x + 2xe^x) + (\sin x + x^2e^y - 1)y' = 0$.

Solution Observe that $M_y(x,y) = \cos x + 2xe^y = N_x(x,y)$, so the given equation is exact, and there should exist a function $\Psi(x,y)$ such that

$$\frac{\partial\Psi}{\partial x}(x,y) = y\cos x + 2xe^x = M(x,y), \qquad \frac{\partial\Psi}{\partial y}(x,y) = \sin x + x^2e^y - 1 = N(x,y).$$

Integrating the first equation, one has $\Psi(x,y) = y\sin x + x^2e^y + h(y)$. Differentiation of the latter equation yields $\sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1$, or $h'(y) = -1$, $h(y) = -y$. Then $\Psi(x,y) = y\sin x + x^2e^y - y$, and solutions are defined implicitly by the equation $y\sin x + x^2e^y - y = c$.

Praxeological analysis

Vignette MO1 *Goal:* Calculate the potential of a conservative vector field.

Raison d'être: demonstrate independence of a line integral of a conservative vector field on a path and use it for finding the potential and describing equipotential curves.

Steps in the concept definition for EDE in MO1.

1) Using the gradient of a scalar field, define a conservative vector field and its potential.
2) Using the differential of a scalar field, define an EDE in the 3D-space.
3) Provide necessary conditions for a conservative vector in 2D and 3D-spaces.
4) Define solutions of an EDE as equipotential curves.
5) Use the procedure of partial integration to find the potential.
6) Write the answer in the form of equipotential curves.

Vignette MO2 *Goal:* Develop an integration method for solving an EDE.

Raison d'être: use the total derivative for developing the integrating factor technique for solving first-order linear DEs and obtaining first integrals.

Steps in the concept definition for EDE in MO2.

1) Solve an EDE in Example 3 by rewriting its left-hand side as a total derivative.
2) Define solutions implicitly by an algebraic equation $\psi(x,y) = c$ and interpret the level curves of $\psi(x,y) =$ as integral curves of the given DE.

3) Consider the general case of EDEs and give the formal definition.
4) Formulate necessary and sufficient condition for a DE to be exact (Theorem 1).
5) Prove Theorem 1.
6) Use the constructive procedure in the necessity part of Theorem 1 to develop a solution guideline.
7) Solve an EDE in Example 4 using the procedure developed in step 6) obtaining solutions defined implicitly by an algebraic equation.

Both the object, an EDE, and the technique (solution method) are similar in MO1 and MO2, but the concept definitions differ. We argue that the concept images for EDE induced by two mathematical organisations are even more distinct. This signals possible conflict situations when students do not see important connections between mathematical notions and ideas. Note that the praxis parts in both mathematical organisations are well defined, $\Pi_1 = [T_1/\tau_1, \tau_2]$ and $\Pi_2 = [T_2/\tau_3, \tau_2]$. Partial integration technique τ_2 is exactly the same in both praxeologies, with minor variations in explanations. The technologies θ_1, θ_2 and θ_3 used in the two logos blocks for MO1 and MO2 justify the same mathematical procedure differently. This is due to the fact that MO1 only postulates a necessary condition for conservative vector fields without proving it and the solution method is introduced in Example 1, whereas Theorem 1 in MO2 furnishes both the exactness test and the justification for the solution method through a constructive proof. Finally, we observe that both logos blocks are incomplete and the theory Θ is missing in logos parts, $\Lambda_1 = [\emptyset/\theta_1, \theta_2]$ and $\Lambda_2 = [\emptyset/\theta_3]$. Therefore, the search for the theory takes us beyond these two mathematical organisations. Two theoretical results that can be used to fill in the gaps follow.

Theorem 3 (vector fields; Protter & Morrey, 2012, p. 478). Suppose that $\vec{v}$ is a continuously differentiable vector field with $\operatorname{curl} \vec{v} = \vec{0}$ in some rectangular parallelepiped D in space. Then there exists a continuously differentiable scalar field f in D such that $\nabla f = \vec{v}$. Any two such fields differ by a constant.

Theorem 4 (mixed partials; Young, 1908-09, pp. 163-164). Suppose $f(x,y)$ is defined in a neighborhood of a point (a,b). Suppose the partial derivatives f_x, f_y are defined in a neighborhood of (a,b) and are differentiable at (a,b). Then$(f_x)_y(a,b) = (f_y)_x(a,b)$, sometimes stated as $f_{xy}(a,b) = f_{yx}(a,b)$.

An elegant example due to Peano illustrates that Theorem 4 does not provide a sufficient condition (cf. Example 2).

Example 5 (Apostol, 1965, p. 358). Second-order mixed partial derivatives of the function $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases}$ are distinct, that is, $f_{yx}(0,0) = 1$, $f_{xy}(0,0) = -1$.

We summarise our praxeological analysis in the following table.

	Task, T	Technique, τ	Technology, θ	Theory, Θ
MO1	Find a potential of a conservative vector field T_1.	Test for conservative vector fields τ_1. Partial integration τ_2.	Necessary condition θ_1 for conservative fields without proof. Partial integration procedure θ_2 introduced in Example 1.	Theorem 3
MO2	Solve an exact DE T_2.	Exactness test τ_3. Partial integration τ_2.	The exactness test and partial integration procedure θ_3 in a constructive proof of Theorem 1.	Theorem 4

Table 1: Praxeologies associated with mathematical organisations MO1 and MO2

CONCLUSIONS

Both praxeologies serve their goals but praxeological analysis reveals two main reasons generating conflicting parts of the concept image. Firstly, the technique of partial integration τ_2 in both mathematical organisations is the same, but technologies θ_1 and θ_2 justify the same solution method differently. Secondly, since the theory θ is missing in both mathematical organisations, the technology lacks justification. It is known that incomplete logos part makes the preservation and communication of the practice difficult (Hardy, 2009). Our list of potential conflict factors in the definition of an EDE in MO1 and MO2 includes four contrasting items.

1) Defining an EDE, a DE-praxeology uses a derivative form under a default assumption that $y = y(x)$. A C-praxeology uses a more flexible differential form where any of two variables can be viewed as independent.

2) Solution of an EDE in a DE-praxeology is viewed as an implicitly defined function describing all solutions (integral curves). A C-praxeology defines them as equipotential surfaces or curves.

3) An EDE is considered in a rectangular domain in a DE-praxeology, but it is mentioned that the region has to be simply connected. A C-praxeology emphasises that the existence of a potential for a vector field depends both on the topology of the domain of the field and on the structure of the components of the field itself.

4) A DE-praxeology does not consider extensions to higher dimensions at all. A C-praxeology allows an easy extension of the notion an EDE and the formulation of an exactness test to a 3D case (thanks to differential form used for a DE).

These potential conflict factors induced by two institutional praxeologies may lead to significant variations in the construction and evolution of students' own concept images for an EDE. Gascón (2003) pointed out that students often use the concept image instead of the concept definition; the latter tends to stay inactive and may be even forgotten. It is likely that students' individual concept images of an EDE formed in a Calculus course will refer to one or more of the following: (i) conservative vector fields, potentials, and equipotential curves and surfaces, (ii) equal roles played by both variables and easy extension to higher dimensions, (iii) topological restrictions on the domain and components of a vector field. When students meet an EDE once again in an MS course on DEs, they may not recognise it because of quite significant differences in the two logos blocks $\Lambda_1 = [\emptyset/\theta_1, \theta_2]$ and $\Lambda_2 = [\emptyset/\theta_3]$.

The increasing demands for advanced mathematical thinking of engineering graduates require both the high-quality teaching and agreement between mathematics disciplines in the study curricula. This paper exposes hidden conflict factors in learning differential equations pointing toward the need for the harmonisation of mathematics courses in engineering programmes. We hope that our contribution will stimulate further interest of mathematicians and mathematics education researchers to this important issue.

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Mathematics students' self-reported learning experiences in a challenge-based course

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In this paper, we report on a study of 'students' learning experiences' in the context of challenge-based education in a higher education mathematics course. Using a case study approach, we investigated (1) how students perceive the role of the existing resources to help them solve their challenges during a one-week challenge-based course; and (2) how students experienced their learning in terms of mathematics and professional skills. Results point to (1) the crucial importance of human resources (e.g. problem owner) for such learning environments to link the mathematics to an authentic situation and develop the skills of an 'applied mathematician in the real world', and (2) a deeper understanding of appropriate methodological tools and their use for researching the concept of 'student learning experiences' in mathematics education.

Keywords: University engineering education, challenge-based education, innovative course, resources, students' learning experiences.

INTRODUCTION AND BACKGROUND

There are concerns in society, business, and industry that presently university engineering education does not sufficiently prepare students for the challenges of this century (e.g., societal problems, global warming, and sustainability), as indicated by the National Academy of Engineering (2018). In this context, an increasing number of universities are developing and implementing educational approaches that move from traditional teacher-centered teaching and learning processes to student-centered approaches (van Uum & Pepin, 2022). This shift, in turn, is related to forms of engineering education that become more relevant by contributing to the solution of societal problems through collaboration between industry and universities. One of these approaches is Challenge-based Education (CBE), which seeks to promote in students both the acquisition and production of disciplinary knowledge and the development of professional competencies (e.g., problem resolution, design capacity, ethical awareness, and multidisciplinary collaborative work). In this paper, we use the term CBE to include both learning and teaching processes. In CBE, students develop their knowledge and competences by collaboration on the solution of real-life problems derived from 'grand challenges' in society, often in a multidisciplinary setting. However, there are still several challenges in implementing CBE at the higher education level (Gallagher & Savage, 2020), in particular for fundamental disciplines such as mathematics and physics. In the case of mathematics, research is needed to understand what and how students can learn in line with a CBE approach and how they can be supported in their learning. Thus, we set out to understand students learning experiences in CBE, that is, to investigate the needs and benefits of this approach from the students’ perspective. For this, we carried out a case study in a Dutch university of technology with the aim to develop a deeper understanding of students' perceived learning experiences in an innovative master's course in mathematics: the modelling week. We are interested, in this paper, in (1) students' use and integration of resources when facing real-world problems and working with problem-owners from business and industry; (2) how students perceive their learning in such CBE environments. On the use of resources, we draw on the Instrumental Approach (Rabardel & Bourmaud, 2003; Trouche, 2004) to analyse students' learning through their interaction with different resources (Gueudet & Pepin, 2018; Pepin & Kock, 2019), particularly “when the curriculum changes from a teacher-centred to a student-centred one” (Pepin & Kock, 2021, p. 325). We ask the following research questions:

RQ1: How did students perceive the role of existing resources to help them solve their challenge during a one-week challenge-based mathematics course?

RQ2: How did students experience their learning in terms of mathematics and professional skills?

THEORETICAL FRAMEWORK

Challenge-based Education

In the transition to student-centred education, different approaches have been developed under the umbrella of inquiry-based education (Martin et al., 2007). One of these is CBE. Malmqvist et al. (2015) define learning experiences in CBE (by them termed challenge-based learning) as:

> A challenge-based learning experience is a learning experience where the learning takes places through the identification, analysis and design of a solution to a sociotechnical problem. The learning experience is typically multidisciplinary, takes place in an international context and aims to find a collaboratively developed solution, which is environmentally, socially and economically sustainable. (p. 87)

These authors add that problems in the context of CBE involve a greater challenge and complexity than those structured, for example, in problem-based learning. Challenges in CBE are linked to social challenges and often involve large open-ended problems (e.g., global warming), in which students have to define their distinctive challenge that they want to solve; this means that students experience greater uncertainty. Moreover, CBE challenges are inherently multidisciplinary. However, no agreed upon definition of CBE exists (Gallagher & Savage, 2020). Rather, a CBE learning environment may be compared to a range of criteria that it fulfills to a greater or lesser extent. The characteristics of a particular learning environment should then be considered holistically when determining whether it may be described as CBE (van den Beemt et al., 2022). For this reason also monodisciplinary courses or couses in a non-international context could, depending on their other characteristics, still be considered as CBE.

CBE is claimed to be motivating for students, because of the real-world character and the relevance of the challenges. Through CBE, it is said, students acquire and develop disciplinary knowledge, transversal competences while interacting and collaborating with multi-stakeholders (Gallagher & Savage, 2020; Membrillo-Hernández et al., 2019). Two relevant aspects of CBE are the definition of the problem, and the design and implementation of prototype solutions.

Additionally, we also consider it necessary to extend the definition of CBL-experience given by Malmqvist et al. (2015) by first pointing out what is meant by 'learning experience' and considering the different ways in which students are affected (e.g., depending on the communication with academic supervisors and other students) and what feelings they expressed (e.g., frustration, liking, interest in the activity) during the solution of a problem in the context of CBE. Thus, to consider the different facets of and agents in student learning experiences, we propose a first conceptualisation of 'mathematics students' learning experience' as the conjunction of two processes: A process of being affected and getting knowledge or skills (e.g., from tutors) from doing, seeing, or feeling things depending on learning goals; and a process of developing and applying knowledge or skills through the use of resources and different forms of collaboration.

The lens of resources

This study draws on the Instrumental Approach (Rabardel & Bourmaud, 2003; Trouche, 2004), to address the question of student learning experiences and development of competencies in CBE when they use different types of resources while solving problems. For engineering students in applied mathematics, we assume that the development of competencies is related to students' strategies when orchestrating and integrating different types of resources. This integration involves two processes: (1) instrumentation, where the affordances of resources influence student practice and knowledge; and (2) instrumentalization, where students adapt the resources to their own needs. In this way, this study draws on the categories of resources as outlined by Pepin and Kock (2021): curriculum resources (e.g., textbooks, teacher curricular guidelines, worksheets), social and cultural resources (e.g., conversations with tutors, peers, and friends), cognitive resources (e.g., concepts and techniques), and general resources (e.g., software, internet, and other digital resources).

THE STUDY

A case study approach (Cohen et al., 2007) has been used for this study, where the unit of analysis is the case of the modelling week in the mathematics department.

The context: University and modelling week

The study took place at a Dutch technical university that offers different educational programs in science, core engineering studies and social engineering studies. As part of its educational vision towards 2030, this university is in the process of shifting towards a student-centred education following the CBE approach as one of its main educational strategies. Through this strategy, the institution seeks for the future engineers to have a deep understanding of their discipline and to be able to work collaboratively in real world complex situations in multidisciplinary settings.

The study participants took part in a ‘modelling week’ for first year master’s students, which is part of a compulsory course in the Applied Mathematics Master's program. The ‘modelling week’ allowed students to work for a week on problems designed by stakeholders from outside the university (problem owners hereafter); these came from regional businesses and industry. The whole course consisted of three moments: (1) Kick-off, where information about the course was given by the problem owners and the creation of different teams (by the course leaders and university tutors) was carried out according to the areas of interest of each student; (2) Lego workshop, where students got to know each other and team dynamics were performed; and (3) Modelling week, where during one week (Monday to Friday) students worked in teams to find a feasible and effective solution to the problem, guided by university supervisors and problem owners. The modelling week ended with the presentation of the results of each team. Eight teams of 5-7 students each participated in the course and four of them agreed to participate in our research.

Data collection

Data were collected through non-participant observation of students working together and with their tutors (problem owner, mathematics tutor) during their meetings. These resulted in fieldnotes from these meetings and photos (e.g. of student writings). In addition, the following data collection strategies were used (Table 1):

Participants	**Instrument**
Students	Exit Cards, interviews, drawings, and surveys
Tutors	Interviews
Problem owners	Interviews

Table 1: Instruments for data collection from participants of modelling week

The exit cards were filled out by the students at three different data points (Monday, Wednesday, and Friday), and consisted of five questions to be answered by students: (Q1) Select the smiley that best describes your overall feeling about your work today and explain why you selected this smiley; (Q2) describe the most interesting things you learned today; (Q3) describe the activity you liked best today, and explain why; (Q4) describe the most important hurdle/difficulty you came across today for your progress, and explain why; (Q5) On the axes provided, sketch how you feel you have progressed towards your goal so far and use a few words to explain what you have drawn. The interviews were conducted at the end of the week, based on students' drawings of their resource system (Schematic Representation of Resource system-SRRS; Pepin et al., 2017). The SRRSs are a schematic representation of how students used and integrated different resources throughout the week. During the interview, each student was asked to explain his or her drawing.

Analysis strategy

To carry out the analysis of results, and in accordance with the objectives of this article, we focused on the analysis of the students' learning experience through exit cards, and students interviews in combination with SRRSs. The interviews (with problem owners and tutors), observation notes and surveys were backgrounded and not considered for the analysis reported on in this paper. For data analysis, we drew on Grounded Theory (Walker & Myrick, 2006) as it allowed us to organize and categorise the data collected into themes that in turn supported the descriptions and analyses, and fed the theoretical approaches used. Thus, we established two main categories of analysis with subcategories in each category: (1) Kinds of resources and support for: identifying the problem, guidance, and making choices. (2) Student perceptions of their learning experiences (including difficulties experienced): applying mathematics in the real world, and social skills. Through these categories we identified both the process of being affected and the process of developing and applying knowledge or skills; both processes in relation to the different resources mentioned by the students. To identify resources, we qualitatively analysed the sentences with words referring to resources (material, concepts, actors) to determine the beliefs, ideas, or motives guiding students' activity, as well as the contexts and situations in which they are used.

DISCUSSION OF RESULTS

Here we report results from a team of students (S1-7) involved in the problem entitled ‘stochastic durations in taxi route planning’ (referring to a non-deterministic process of transportation service). In making taxi transfers more efficient for elderly and disabled people, the students addressed the problem of "how do stochastic boarding times affect the quality of the realization of the planning compared to the constructed (deterministic) planning?"

For answering *RQ1*, we analyzed the interviews/SRRSs and exit cards (see Table 2). Due to space limitations, we present the SRRSs of two students: S2 and S7.

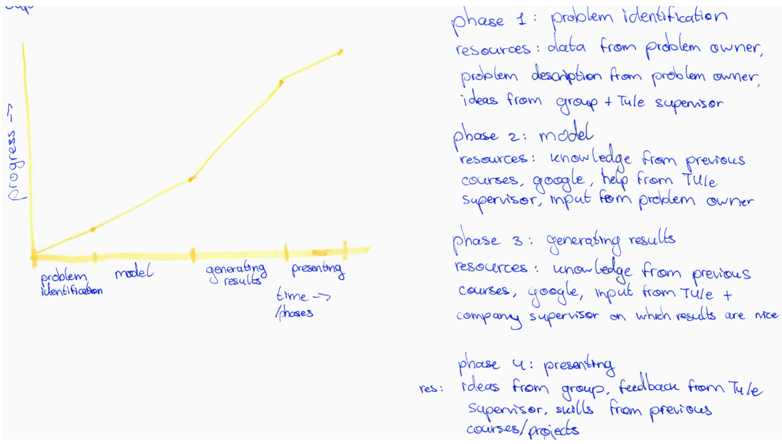

Fig 1: Schematic Representation of Resource system-SRRS from S2

S2 represents his/her experience in the modelling week divided into four phases. This diagram allows us to identify more precisely the resources that S2 used and incorporated in each phase. It can be seen how the problem owner appeared important for the identification of the problem and giving “inputs”; the supervisor was also relevant for giving “feedback”. Other resources available to the student were: the use of "knowledge from previous courses", "google", “ideas from group".

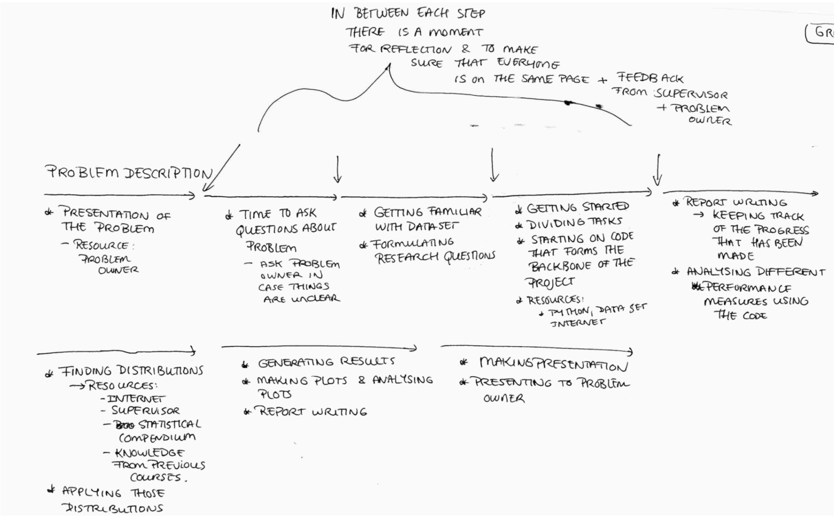

Fig 2: Schematic Representation of Resource system-SRRS from S7

S7 represents his/her experience in a continuous manner and specifies the different activities the group carried out. Two relevant resources that S7 perceived that helped them solve the problem were the problem owner and the supervisor through their feedback, which allowed the transition through the different stages. Other resources used were: “pyhton”, “data set”, “internet”.

The SRRSs are complemented with information from the exit cards in relation to the first category of analysis. The number in parentheses corresponds to the question number on the exit card and the day it was filled out (M-Monday, W-Wednesday, F-Friday).

	Identifying the problem	Guidance	Making choices
S1	We discussed the problem with our group in an organized manner (Q1.M)	The problem owner saying our results were valuable for their company and that they will use them (Q3.F)	brainstorming the problem and hearing everyone’s ideas (Q3.M)
S2		Discussing them with the problem owner and then changing our model according to the feedback (Q1.W)	Discussing with the problem owner and making sometimes difficult choices in modelling (Q2.W)

S3	I feel like we have got the problem clear, and know what we have to do (Q5.M)		
S5		Meeting with problem owner, a lot of insight (Q3.W)	
S6		Our meeting with the problem owner we got a lot of new insights, as well as meaningful feedback (Q3.W)	
S7	We were well organized and made concrete plans to tackle the project (Q1.M)		

Table 2: Exit card responses for "Kinds of resources and support" category

Table 2 shows that for four students (S1, S2, S5 and S6) in this group the role of the problem owner was essential as a guide. To identify the problem, group discussion was important (S1, S3 and S7). For decision making, the brainstorming (S1) amongst the group and discussions with the problem owner (S2) appeared crucial.

From the interviews/SRRSs and exit cards (Table 2), we can summarize the role of the available resources in the following way: (1) The importance of the problem owner: as guide through the process; answering questions related to the problem; assessing the solution and presentation. (2) The role of the tutor: giving tips with the mathematics/modelling, also in terms of resource availability and giving feedback on what they have done. (3) Importance of other social resources (e.g. group discussions) and general resources (e.g. google, python, internet).

For answering *RQ2*, we analyzed the exit cards (see Table 3) in relation to the second category of analysis.

	Applying mathematics in real world	**Social/professional skills**
S1	In terms of mathematics: what a simulated annealing algorithm is (Q2.M)	I did not really learn anything new. We just applied our knowledge. So you could say we gained more experience in applying theory to real life (Q2.W)
S2	How to present your results in a way that they are most interesting and understandable for the project owner (Q2.F)	making a plan of work together. Also including all team members, also the ones that are less active (Q4.M)
S3	I got a better idea of what mathematicians do in the real world and industry (Q2.M)	the skill of reading/understanding other people's code (Q2.W)
S4	To understand the meaning of the question and how to convert a question from reality problem into a math question (Q4.M)	how to extract the effective information from the questions (Q2.M)

S5	Starting coding. There were many places to start and deciding where is something difficult (Q4.M)	Not being to critical of your own work (Q4.F)
S6	The most difficult obstacle is to write the program (Q4.M)	to get the final results ready for our presentation (Q4.F)
S7	How we should interpret some part of the data (Q4.W)	new ways of comparing the results and new distributions and additions we could add to our code (Q2.W)

Table 3: Exit card responses for "Student perception of their learning" category

The learning that students perceived they acquired came from Q2 of the exit cards (see Table 3). Here we observed learning related to mathematics, for example, "a simulated annealing algorithm" (S1) or "coding" (S5); as well as professional skills, for example, "to get the final results ready for our presentation" (S6). The difficulties perceived by the students come from Q4 of the exit cards. Among the difficulties we note: "How we should interpret some part of the data" (S7), "to write the program" (S6) or "how to convert a question from reality problem into a math question" (S4). From the exit cards (Table 3), we can summarise the student perception of their learning experiences under two important points: (1) Some students pointed out aspects related to mathematics (e.g., a simulated annealing algorithm). However, it was striking that others said that they did not learn anything new in terms of mathematics. Vergnaud (2009) points out that the mathematical competencies that students acquire and develop are not restricted to linguistic and symbolic expressions (predicative form of knowledge) but are also required to consider actions in the physical and social world (operational form of knowledge). (2) About professional skills: they said that this was the main outcome, that they worked like 'real engineers' with authentic problems, and in a pressurised situation.

CONCLUSIONS

In terms of theory, we developed insights into students' learning experiences in relation to their own perceptions of what they learned and with which resources they learned. This, in turn, allowed us to approach a better understanding of the notion of "students' learning experiences" at the conjunction of two processes.

Coming back to our proposal to (re)conceptualize 'the mathematics students' learning experience': referring to RQ1, the results show the importance of considering a process in which students are affected by the PO and the tutor; at the same time, referring to RQ2, results show the second process in which students develop and apply both knowledge and skills to face their challenge. As we can observe, these two processes are in turn closely related to the use of resources, where two processes are involved: instrumentation and instrumentalization.

In terms of methodology, (1) we made a link between different methodological tools (SRRSs and exit cards) for grasping students' reflection on their development (operationalisation of the mathematics in an authentic situation), and the resources that they need for such development; (2) we found the usefulness of exit cards (as new tool) and SRRSs as "deepening tools" for receiving richer data; and SRRSs in combination with interviews as useful tools for examining integration of resources.

Finally, the results have implications for practice and are useful for course designers: e.g., choice of problem/project; support of students by tutor and problem owner; alignment of support by problem owner and tutor.

At the curricular level and in the context of the transition to a student-centered approach through the CBE approach, from the results we observe that this transition is not only moving away from traditional teaching, but it also entails a reflection on the new roles expected by tutors and the impact of their interaction with students. Thus, it is important to continue with research that accounts for: (1) how tutors and POs help, guide, and establish the balance between not leaving students completely alone and not solving problems for them; and (2) how this interaction is perceived by the students themselves through the use of resources.

ACKNOWLEDGEMENTS

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Between mathematics and computer science? A study of research and teaching in numerical analysis

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Keywords: Teachers' and students' practices at university level, teaching and learning of specific topics in university mathematics, numerical analysis, numerical methods, algorithms

RESEARCH TOPIC

Numerical analysis or numerical mathematics is a branch of applied mathematics that deals, among other things, with the mathematical analysis of algorithms or numerical methods that can be used to approximate certain quantities, such as solving partial differential equations (Funken & Urban, 2018). Since the goals and tasks of numerical analysis can vary widely among numerical analysis researchers depending on the research direction, the given characterization should not be taken as an exact definition, but rather as a possible description of numerical analysis.

Although numerical analysis content is part of many mathematics and teaching training courses and students take numerical analysis courses in various study programs, there is no research that addresses the description and analysis of the competencies that students should acquire in numerical analysis courses from the teacher's perspective (Burr, in press). Existing works mostly refer to how numerical analysis content can be implemented in school or how numerical methods can be taught in engineering courses (Bishop, 2013; Titz, 2018). To identify and describe the desired competencies of students when working on problems in numerical analysis courses and to develop appropriate teaching, learning and assessment methods from them, it seems interesting to investigate what actions and practices are observed among numerical analysis researchers and compare them with those of students. Based on Burr (in press), the aim of this poster is to present preliminary results from a qualitative research approach that reconstruct these actions and practices of numerical analysis researchers using expert interviews and qualitative content analysis.

RESEARCH QUESTIONS & METHODOLOGY

This poster aims to answer the following research questions:

1. What actions and practices can be identified among researchers in numerical analysis?
2. What similarities and differences can be identified between the actions and practices of numerical analysis researchers and the intended and subject-specific actions and practices of students in numerical analysis courses?

A total of 17 expert interviews were conducted to answer these two research questions. The results of these interviews are currently being evaluated using qualitative content analysis according to Glaeser and Laudel (2010). As mentioned at the outset, the understanding of the goals and tasks of numerical analysis can vary widely among numerical analysis researchers depending on their own research direction. For this reason, the research direction played an important role in the selection of the experts. For example, experts who deal with the numerical solution of partial differential equations in their research were interviewed. However, researchers were also interviewed, who work in a very practice-oriented manner and deal with topics from scientific computing and high-performance computing, among others.

PRELIMINARY RESULTS

Preliminary results of the expert interviews show that a wide range of specific actions and practices can be found in numerical analysis research, including both mathematical and computer science aspects. Among other things, numerical analysis researchers not only develop special algorithms adapted to the underlying problem, but also deal with the mathematical analysis of the respective algorithms. Implementing these algorithms on computers and validating and verifying the results of these algorithms are also part of the actions and practices of numerical analysis researchers. Comparing the actions and practices of numerical analysis researchers with the intended and subject-specific actions and practices of students in numerical analysis courses already shows many similarities, but also significant differences. Thus, students are also expected to implement algorithms on computers, but further development of algorithms seems to play a role only in numerical analysis research so far.

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Didactic proposal to re-signify the differential equation by modelling simple harmonic motion with the support of digital technology.

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Keywords: Second order differential equation, university mathematics, modelling, linearity, vibrations.

INTRODUCTION

According to Kwon (2020), differential equations (DE) are ubiquitous in applied mathematics and constitute an important component of the mathematics curricula of most universities. Rowland and Jovanoski (2004) mention in a study that in the teaching and learning of differential equations they suggest that, for students, the connections between a differential equation, its solution and what each of them can represent physically are not significant. In this research, we focused on simple harmonic motion, and more specifically on the analysis of the physical concepts involved, such as magnitude, time, velocity, acceleration, and energy, to model this phenomenon using a second-order differential equation and thus obtain a new meaning for this mathematical object.

THEORETICAL FRAME

Posner et al. (1982), characterize conceptual change as the modification of students' previous ideas so that they are replaced by those concepts accepted by the scientific community. The CUVIMA methodological model (Cuevas et al., 2017) is applied in our work for the organization of didactic activities that promote the understanding of simple harmonic motion and is composed of four frameworks, which are: 1) Reality framework in Physics, 2) Modelling framework in the device, 3) Conceptual analysis framework in Physics, and 4) Mathematical, conceptual analysis framework.

The objective of this research work is to contribute to find ways to promote the concept of differential equation in engineering students, through the modelling and interpretation of simple harmonic motion. How do engineering students re-signify the concept of differential equation through modelling simple harmonic motion?

METHODOLOGY

Phase 1.- Preparation and design

A pre-test was designed in order to collect previous ideas about simple harmonic motion, for this activity we used the Google Forms platform.

The context used to introduce mechanical vibrations consists of the movement of a spring in a bicycle that has a linear spring incorporated in its structure to dampen the movement. The main objective of this didactic activity is that the student is able to characterize the linearity property of the force applied to a spring that obeys Hooke's Law.

Phase 2.- Teaching experience

The research was carried out with two engineering students from a public university in Mexico. The activity was carried out in a single session which lasted 80 minutes.

RESULTS

In this paper, we present the results of a student with the pseudonym Andy, who was randomly selected from the sample. In the "Simple Harmonic Motion (SHM) Pre-test" activity, Andy answered all seven multiple-choice questions correctly. Andy's answer shows that he has knowledge about (SHM) and describes that one of its features is periodicity. Activity 1 consists of 12 multiple choice questions and two open questions, and the last exercise is to build a graph. Andy correctly answers all 11 multiple-choice questions. In particular, his answers to questions 1, 2, and 3 show that he manages to interact and interpret the simulated situation in the applet. Question 6 prompted Andy to explore the Activity 1 applet with more care and time. His answer shows that he can identify that a change in the value of the elastic constant, k, implies that the force applied to the spring will have a different magnitude, resulting in an increase or decrease in the length of the spring. Spring, that is to say, stretches or compresses with more or less difficulty.

DISCUSSION AND CONCLUSIONS

The activity was applied remotely, due to public health conditions and confinement. However, the results of the investigation are positive, since it shows us that Andy managed without difficulty to characterize the linearity property of the force applied to a spring that obeys Hooke's Law. This indicates that Andy has correct previous ideas about simple harmonic motion, and that activity 1 promoted in ratifying such ideas. In addition, the results obtained allow us to see that Andy can continue without objections with the rest of the activities to complete the investigation.

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We want news about Pingu: An explanatory action research case study in undergraduate service mathematics

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Keywords: Teachers' and students' practices at university level, Teachers' and students' practices at university level, Teaching and learning of analysis and calculus, Curricular and institutional issues concerning the teaching of mathematics.

MATH IS SCARY

Math is scary. For life science students, math is even scarier (Bishop & Eley, 2001). Students' failure in mathematics is usually attributed to their negative attitudes towards the subject (Goldin et al., 2016), or to the rupture of the self-regulation cycle of engagement, reflection, and anticipation (Schunk & Zimmerman, 1998). What is even worse, is that students "after taking Calculus showed a reduction in positive attitude about mathematics" (Rickard & Mills, 2018) and that "[b]etween the start and the end of the students' college calculus class, their confidence and enjoyment of mathematics dropped sharply, with confidence falling by half a standard deviation and enjoyment of mathematics by a third" (Bressoud, 2015).

I have been teaching for six years the Calculus class in the "Natural Sciences" programme, a BSc degree in Italian universities which, integrating life & earth sciences, focuses on the correlation between organisms, substrate, environment. Its students are often amongst the *weakest* STEM students with respect to mathematics competences that should have been acquired in high school (Rizzo, 2020).

What to do? Bressoud (2015) shows that *ambitious teaching*, i.e., active learning approaches, can be a solution.

WHAT IS GOING ON?

Pandemic at-distance teaching (Fall term 2020) brought the creation of short ad-hoc videos on all the *theoretical* contents in the syllabus. This made possible to devote most class time to active teaching, thanks to the capability of at-distance software to effectively implement rapid quizzes and the subdivision of students in small groups.

The Fall term 2021 was, in Milan, back in presence—but with the possibility for students to attend at-distance. The availability of the videos created for the previous term allowed a flipped approach, with most 2-hours slots devoted to a cycle of formative assessment (45') and to an explorative group activity (60') where students were called to apply the assigned concepts to the solution of a problem relevant to Natural scientists. E.g., using vector sums to compute the position of a dolphin given daily movements as detected with a GPS device; or using derivatives to study the weight of a nesting penguin and his chances of survival before the return of his companion.

[1] This work was supported by the "National Group for Algebraic and Geometric Structures, and their Applications"

THE THEORETICAL FRAMEWORKS

Classes were designed using the formative assessment cycle theory (Black & Wiliam, 2009) and Engeström's (1999) Activity Theory. Specific examples of the design will be shown in the poster.

THE RESEARCH QUESTION

Do we have an effective active learning approach? What does *effective* mean, in this context? It has been shown (Rizzo, 2020), that the chance of success—defined as passing the exam within the Academic Year—is closely related to the score in the national standardised entrance tests; for *effective* we will henceforth mean not dropping out, passing the exam, and later passing Statistics.

We do not yet have a quantitative answer to the first question, since students are still sitting exams. Although data will be definitive only in February 2023, partial results suggest a positive answer; moreover, we already have some quantitative results:

- Students have been observed talking about the problems outside of the classroom, on their way to the cafeteria.
- Students have reacted negatively to the news of the episodical *traditional* face-to-face lecture: "We want news about Pingu!" was notably a student's reaction.
- Attendance to lectures decreased to 40% rather than 25-30% at the end of term.

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Contextual learning of mathematics for engineers

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Keywords: teaching and learning of mathematics for engineers, novel approaches to teaching, contextual learning, mathematics and applications.

INTRODUCTION

I present some preliminary results from a project where mathematics is specially adapted to one particular engineering programme with the aim of making mathematics an active thinking tool when working with engineering problems, and improving the students' perceived relevance of mathematics for their study programme. The project is based on a contextual learning approach.

Several models for teaching mathematics to engineering students can be found, from mathematics as a general foundation subject to mathematics as an integrated part of the students' engineering specialisation. The first model may lead to students having difficulties to apply mathematics when needed in engineering courses (Carvalho & Oliveira, 2018), whereas the second model provides better opportunities for showing the relevance of mathematics for the engineering specialisation, but this model is expensive to implement.

MATHEMATICS AS A THINKING TOOL

At the Norwegian University of Science and Technology mathematics has traditionally been provided almost identically to all five-year Master of Technology (MT) programmes, without links to specific engineering fields. *Mathematics as a Thinking Tool* is a pilot project aiming at strengthening the connection between mathematics and the engineering fields, thereby hoping to increase the students' perceived relevance of mathematics, as well as making mathematics an active thinking tool in their work with engineering problems. There is evidence to show that many engineering programmes do not exploit the potential of mathematics in the early phases (e.g., González-Martín & Hernandes Gomes, 2017), and the project aims to change this situation by making mathematics and engineering courses mutually support each other. A basis for the project can be found in the Conceive, Design, Implement, Operate (CDIO) Initiative, which emphasises both *conceptual understanding* and *contextual learning* (Crawley et al., 2014). Examples of activities in the project are presented in Bolstad et al. (2022). I will present some answers to the question whether the students' perceived relevance of and motivation for mathematics differ for students within the project compared to those not in the project.

So far, the project has included two cohorts of the MT programme *Electronics Systems Design and Innovation* (Elsys) but from 2022 it will be expanded to two other programmes. A survey was distributed in the spring of 2022 to the first-year, second semester, Elsys students (n = 45) and in identical form to all the other first-year MT students (n = 494). The number of responses corresponds to a response rate of between 30 and 40%. Below are some of the questions asked in the survey, with answers (%) in brackets. Boldface numbers are for students from Elsys:

> How would you characterise your motivation for mathematics now compared to when you started your studies? (larger 22/**31**, about the same 44/**44**, smaller 34/**25**)
>
> I have seen why mathematics will be important for me later in my studies (completely agree 34/**82**, partly agree 38/**18**, partly disagree 21/**0**, completely disagree 7/**0**)
>
> In my work with other courses (i.e., not mathematics courses) I have seen the importance of learning mathematics (completely agree 37/**85**, partly agree 44/**13**, partly disagree 14/**0**, completely disagree 5/**0**)
>
> I don't think the mathematics I have learned in very relevant for my study programme. (completely agree 5/**2**, partly agree 25/**2**, partly disagree 44/**18**, completely disagree 26/**78**)
>
> So far, in my work with other courses (i.e., not mathematics courses), I have managed with the mathematics I learned at school (completely agree 28/**7**, partly agree 32/**11**, partly disagree 24/**49**, completely disagree 16/**33**)

The numbers indicate that the perceived relevance of mathematics is larger for students within the project than for the others. However, the motivation for learning mathematics seems to develop in a similar way in the two groups. The survey will be repeated in 2023 for all first-year students. Then it will be possible to compare the answers from the two new programmes with the results from 2022, when these programmes were not part of the project.

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TWG5: Teacher education in university

TWG5: Teacher education at the university

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TWO MEANINGS

In mathematics education research, the teaching profession occupies a place of pride, both as an important partner, as an object of study, and as a consumer of the ideas and results that researchers produce. Historically, the development and inclusion of mathematics education positions at universities are inseparable from the professional education of teachers. The ever-increasing importance and challenges of mathematics in schools have also led to considerable scholarly and political attention to conditions for improving the pre- and in-service education of mathematics teachers. While secondary school teacher education has been university based in most Western countries for centuries, this is increasingly the case even for primary schools. Indeed, mathematics education researchers frequently exercise their teaching duties in university-based teacher education, usually giving specialised courses on mathematics education for future teachers. Considering that such courses are a specific form of university mathematics education, we have the first meaning of the title of this TWG.

On the other hand, university mathematics education as a field of practice is usually construed as the teaching of post-secondary mathematics at the university level, especially within general areas like algebra and analysis. In general, undergraduate mathematics courses are often shared by a large variety of study programmes and, in many institutions, future teachers take such basic mathematics along with students aiming for other professions. It is thus difficult or impossible to draw exact boundaries between university mathematics education and mathematics teacher education, and several contributions to this TWG focus explicitly on the *connections* between general (or "pure") mathematics teaching at university, and the specific needs and courses of school mathematics teachers. This is the second meaning of the title of this TWG.

A third meaning arises from considering the profession of teaching mathematics at the university. The challenges of students with university mathematics have unquestionably led to an increasing interest in the practices and preparation of their teachers, and this (partly hypothetical) meaning of the title of our TWG is also represented by one contribution to the group.

MATHEMATICS EDUCATION COURSES AT THE UNIVERSITY

As suggested by Csapodi's poster on a recent reform of Hungarian mathematics teacher education, many countries and institutions distinguish sharply between courses on "mathematics" and courses on "teaching". When this is the case, the latter can be more or less specialised on “mathematics teaching”, and the specialised variants usually refer to mathematics education research as a scholarly base.

Barquero and Bosch present and discuss the idea of a *study and research path in teacher education* (SRP-TE), motivated by professional questions of a didactical nature, like “how to teach randomness and statistics in primary school”. In an experimental setting within primary school teacher education at the University of Barcelona, the students first carry out a mathematical activity, which has been designed by the authors with inspiration from the classical bottle situation (Brousseau, Brousseau, & Warfield, 2001). They then analyse their work from a mathematical and didactics perspective and proceed to design a similar path for primary school pupils (with an eventual implementation in some cases. The authors reflect on the similarities and differences between the SRPs-TE and the study and research paths (SRPs) proposed to non-teachers university students (in engineering, science or administration degrees). They consider the possibility to extend some instructional strategies and resources designed for teacher education to non-teacher students, in particular those related to the analysis of the mathematical activities carried out.

Hakamata, Otaki, Fukuda and Otani also experimented a study and research path based on the bottle situation, but now in secondary mathematics teacher education at Kochi University. The design and aim are different from the above study: to detect what mathematical praxeologies could be realised by the students, given that they had some university background in probability and statistics. The authors create a reference praxeological model of what they consider as relevant from this background to explore the bottle situation and analyse the students’ actual exploration in terms of this model. It turns out that students were able to mobilise some but not all of the anticipated elements of the model. The authors consider that students could have mobilised some of the missing elements with more explicit instruction from the teachers and that student errors in such an inquiry process may be important sources of learning.

LINKS BETWEEN UNIVERSITY AND SCHOOL MATHEMATICS

A majority of papers in this TWG consider, in one way or another, the links and gaps between university mathematics courses (or activities) that are not specific to future teachers and school mathematics or, more broadly, the activities of a school mathematics teacher. Most of the papers report on experiments to strengthen the links, while others seek to identify and explain the gaps.

Goor, Pinto and Karsenty present some aspects of the M-Cubed project in Israel, designed to explore interactions between mathematics researchers (called “mathematicians”) and experienced secondary school teachers of mathematics, as they meet to discuss video-recorded lessons from secondary school. The mathematicians focus on the characteristics of the mathematical problem while the teachers concentrate more on aspects related to the observed interaction in the lesson at stake. The authors consider that such interactions can be fruitful experiences of boundary-crossing but that facilitators or brokers are needed to overcome what might otherwise be discontinuities in the interaction.

Pustelnik also investigated the perspectives of mathematicians (in the above sense), but this time concerning their activity as lecturers in mathematics courses for future secondary-level teachers. From interviews, it appears that some lecturers occasionally try to show how the contents they teach are related to school mathematics, but none have recent or specialised knowledge about the latter. In the German university where the lecturers teach, they give similar but separate lectures to “mathematics students”, and “teacher-students” (different undergraduate programmes in Germany), but the lecturers believe the contents delivered in each case should be essentially the same.

Bübenbender-Kuklinski, Hochmuth and Liebendörfer examined, equally in the German context, the relevance for the teaching profession that mathematics teacher students attribute to mathematical domains which are mandatory in their undergraduate studies (such as calculus and linear algebra). They do so through a longitudinal quantitative study, which gives rise to interesting hypotheses, like students attributing less relevance to domains in which they feel insecure; however, supplementary qualitative data will be needed to confirm such causalities.

Hochmuth and Peters investigated, on their side, the qualities of the mathematical knowledge that German teacher students take away from their undergraduate studies. More specifically, in a graduate course specifically designed for these students, they found that the participants were largely unable to mobilise basic results from undergraduate analysis and linear algebra to carry out a qualitative analysis of ordinary differential equations, and to engage such analysis in modelling problems arising in other disciplines. The authors notice that both tasks are indeed highly relevant to the curriculum they prepare to teach.

The problems of connecting school mathematics and university mathematics are considered from an intervention perspective in three papers and one poster. Bauer and Müller-Hill propose a set of principles for designing tasks that could help teacher-students in Germany to deepen and professionalize the mathematical knowledge learnt in undergraduate mathematics courses of linear algebra and analysis, particularly in relation to proof. These tasks are worked with by students in modules that run in parallel with the courses, bringing promising results.

Huo and Winsløw also present a list of task design principles for students of a Danish university in a capstone mathematics course, which, unlike all university mathematics courses they have taken, is designed specifically with the needs of teachers in mind. The goal of the design is to produce tasks that can develop and assess students’ capacity to mobilise the mathematical knowledge studied at the university on secondary mathematics problems. Several examples of tasks are given and analysed according to the principles.

Viirmann and Jacobsen present a course design carried out at a Swedish university to cover mathematical and didactical content jointly and concurrently. The teachers (and authors) team teaches the course and personifies, through their research background in mathematics and mathematics education, the two strands of the course. The two teachers take turns to be the "main" teacher, and an example is given of how the other may inject comments and questions from his perspective is given, for instance, to highlight general perspectives from either university mathematics or didactics that apply to the topic at hand. The authors suggest that care should be taken in such a dialogue setting, not to lose the students.

The poster by Delori and Wessel presents an idea for connecting preservice teachers' knowledge of school algebra (like equation solving) and abstract algebra through design research, which at the time of writing was not yet carried out. The poster by Broley, Buteau and Müller outlines a sequence of modules, designed and tested at a Canadian university. One aim is to provide an experience of using coding to learn mathematics, which is particularly relevant to future teachers in times when new school curricula require pupils to get such an experience. Finally, the poster by Vinerean, Brandl and Liljekvist explain how preservice mathematics teacher's work with "Interactive Mathematical Maps" can help them gain a vision of mathematics as being constructed by humans and related to important problems.

EDUCATING UNIVERSITY MATHEMATICS TEACHERS

There were no contributions in TWG5 reporting on preservice education for university teachers of mathematics, but Gómez-Chacón, Hochmuth and Peters report on a workshop for a mixture of early career and more experienced university mathematics teachers from Spain, carried out in the context of the European PLATINUM project. The aim was to initiate participants to an inquiry-based approach to mathematics teaching and learning and to develop their skills to design mathematical tasks to be used in such an approach, based on collegial reflection, and taking into account concrete constraints in the participants' courses. The outcomes seem to confirm that task design could be an important and fruitful topic in in-service courses for university mathematics teachers, and (as to some extent demonstrated by previously outlined papers) a lever for implementing new results from university mathematics education research in practice.

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Resources to question mathematical knowledge: from teacher education to university mathematics

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Our research is centred on the design, implementation and analysis of inquiry-based teaching proposals based on study and research paths within the anthropological theory of the didactic. We briefly present a study and research path for teacher education (SRP-TE) designed and implemented in a mathematics education course for pre-service preschool and primary school teachers. After describing the main elements of the SRP-TE, we explain how it provides students with tools to question and analyse the mathematical knowledge involved in activities related to numeral systems, randomness, and modelling. We conclude with the possibility to extend the instructional strategies and resources designed for SRPs-TE to more general SRPs intended to teach mathematics outside the teacher's educational setting.

Keywords: Teachers' and students' practices at the university level, study and research paths, teacher education, resources, anthropological theory of the didactic.

STUDY AND RESEARCH PATHS AND TEACHER EDUCATION

From the approach of the anthropological theory of the didactic (ATD), our education systems today are situated closer to the so-called pedagogical "paradigm of visiting works" (Chevallard, 2015). This paradigm is mainly characterised by the introduction of a set of knowledge works teachers present to the students for them to study and know – or visit – its main characteristics and usages. The counter-paradigm of "questioning the world" helps analyse the limitations of the visiting works paradigm and study its potential to evolve. The paradigm of questioning the world focuses teaching and learning processes on the research and study of open questions. The visiting of works does not disappear, but is subjected to another purpose: the study of questions. That is, knowledge works are important insofar as they help develop answers to the questions addressed: their value is not intrinsic, but functional.

Several investigations within the ATD focus on studying the conditions that can facilitate the transition from the paradigm of visiting works to that of questioning the world, especially in the case of university education (Barquero et al., 2021). In this perspective, the proposal of *study and research paths* (SRPs) (Bosch, 2018) has been implemented in the past decades at different school levels. SRPs are instructional proposals specific to the paradigm of questioning the world that start with the approach of an open question not initially associated with a particular answer or piece of knowledge. Implementing SRPs requires creating new conditions in current educational systems, especially at the level of the didactic contract. Running them helps observe the institutional constraints that emerge during their implementation and hinder their development.

SRPs have also been proposed to support teacher education in what we call *study and research paths for teacher education* (SRP-TE). An SRP-TE also starts from a problematic question, in this case, related to the teaching profession. The initial proposal of (Ruiz-Olarría, 2015) starts by carrying out an SRP related to the teaching question addressed, and using the analysis of the SRP experienced as a tool for the design and eventual experimentation of an adapted SRP (Barquero et al., 2018). We consider SRPs-TE especially important to support collaborative work between researchers in didactics, educators, and teachers in the study of the conditions for the change of paradigm and the co-creation of instructional tools to design and implement new teaching proposals.

Despite the ill-defined position of teacher education in the context of university teaching and research, it seems appropriate to approach SRPs and SRPs-TE from a common perspective. On the one side, SRPs-TE can be considered as a specific kind of SRPs for degrees in teacher education, not different in essence from SRPs implemented in Engineering, Economics, Medicine or Administrative degrees. At the same time, the specificities of SRPs-TE and the fact that they include the design and analysis of an SRP can shed new light on the study of the institutional conditions needed by the paradigm of questioning the world.

In this paper, we start by presenting the specificities of the SRP-TE as a university training device. We then briefly introduce three cases of SRPs-TE that have been designed and implemented for pre-service primary school teachers at the University of Barcelona, and focus on the one concerning the teaching of inferential statistics in primary school. We are particularly interested in showing how, throughout the SRP-TE, educators transpose (from research to teacher education) some epistemological and didactic tools that help teachers manage didactic processes. We conclude with the possibility to extend the instructional strategies and resources designed for SRPs-TE to more general SRPs outside the teacher's educational setting.

AN SRP-TE ABOUT RANDOMNESS

Five modules structure the generic proposal of SRPs-TE as described in (Ruiz-Olarría, 2015; see also Barquero et al., 2018). However, each case requires its own adaptations according to the conditions of the instructional practice: sometimes not developing all the modules or splitting some of them. They are defined as follows:

> *Module* 0: Introduction of an initial question $Q_{0\text{-TE}}$, which is the starting point of the SRP-TE and which will progressively be addressed in all the modules. This initial generating question of TE is related to a teaching problem, and usually takes the form of *How to teach a content* ♥*?*

Module 1: Carrying out an SRP that appears as a possible answer to $Q_{0\text{-TE}}$. It is experienced with participants assuming the role of students, and the educator guiding its implementation.

Module 2: Analysing the SRP experienced from a didactic perspective.

Module 3: Adapting the design of the SRP and implementing it in a given school setting; observing and collecting data for further analysis.

Module 4: Analysing the SRP implementation and identifying its potential and limitations as a response to the initial question $Q_{0\text{-TE}}$.

Several SRPs-TE have been implemented for pre-school, primary, secondary and university teachers (Barquero et al., 2018). A common aspect of all these proposals is that they start by carrying out an SRP with pre-service teachers (module 1). In some cases, they are based on an SRP previously implemented with students; in other cases, the starting point is an SRP newly designed for the teacher education context.

In the case of the SRP-TE we are considering, the initial question $Q_{0\text{-TE}}$ is related to the teaching and learning of inferential statistics in primary school. This SRP-TE has been implemented since academic year 2014/15 at the University of Barcelona in the last compulsory course of Didactics of Mathematics (first semester, 6 ECTS) with groups of about 50 pre-service teachers. The participants are fourth year students of the degree of primary school teacher education. They work in groups of 3-5 members during the whole course. This is not the only SRP-TE implemented in this same course or in previous courses. Table 1 summarises the SRPs-TE that have been implemented under similar conditions, including the generating question $Q_{0\text{-TE}}$, the modules implemented, if the SRP-TE is based on a previously experimented SRP or not, and the approximate hours of work in the classroom.

Table 1: SRPs-TE implemented in the Didactics of Mathematics course

University year (over 4)	Generating question $Q_{0\text{-TE}}$	Implemented modules	Available SRP previously implemented	Duration
2	How to teach the rationale and usefulness of positional numeral systems?	0, 1, 2	No (Sierra, 2006)	30 hours
4	How to teach randomness and statistics in primary school?	0, 1, 2	What is hidden inside the bottle? (Brousseau et al., 2002; Granell & Barquero, 2019)	10 hours
4	How to teach mathematical modelling in primary school?	All	The cake box (Chappaz & Michon, 2003)	40 hours

The first SRP-TE is based on the work of Sierra (2006) that proposes to approach additive, additive-multiplicative and positional numeral systems from the analysis of their properties to not only represent numbers or quantities, but also order them and operate with them. The last SRP-TE takes a situation regarding building boxes with paper based on Chappaz and Michon's proposal as the initial question. The need to build boxes of different sizes to answer a baker's demand motivates the consideration of empirical, geometrical, numerical, and algebraic models (Wozniak et al., in press). Its analysis leads to considering specific notions to describe modelling processes, and to recognise the role of experimental work in mathematics.

The second SRP-TE is the one we are considering in more detail. Figure 1 summarises the content of its modules.

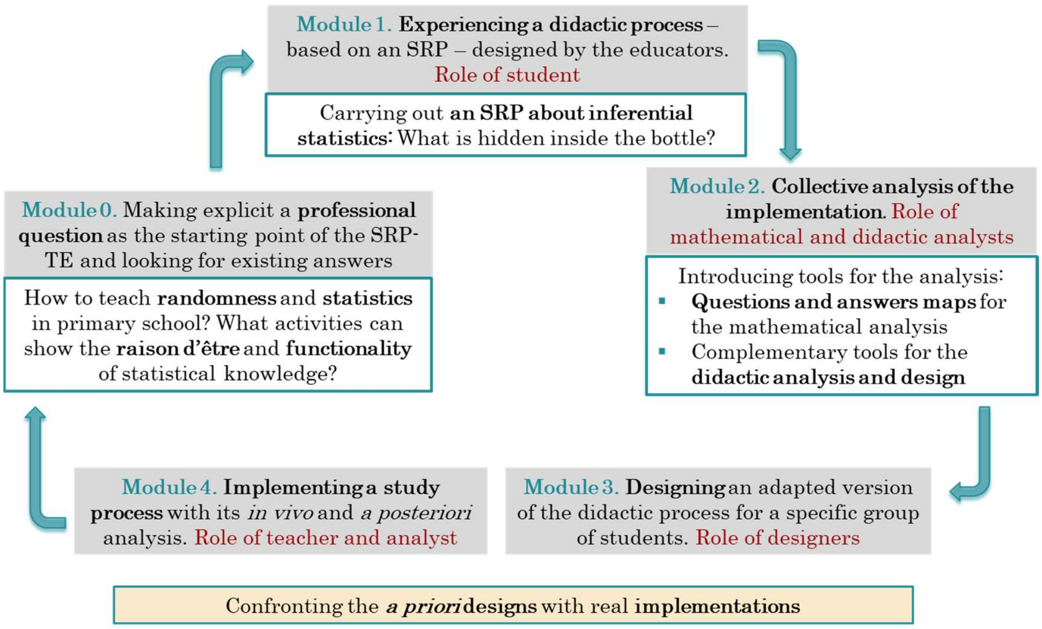

Figure 1: Modules of the SRP-TE about randomness and statistics.

In module 0, the following initial generating question $Q_{0\text{-TE}}$ is proposed: *How to teach randomness and statistics in primary school? What activities can show the rationale and functionality of statistical knowledge? How to design and analyse them?* The teachers are invited to search for available answers in the different accessible media (books, textbooks, curricula guidelines, etc.), which eventually include some instructional proposals coming from educational research about the teaching and learning of statistics in primary school. From this analysis, the teachers often conclude that most of the proposals are focused on introducing techniques for manipulating data for their graphical representation and the calculation of numerical statistics. There are not many activities that propose building an appropriate experimental milieu to deal with statistical variation.

In *module* 1, the teachers are asked to act as students, and experience a teaching project based on the proposal of an SRP entitled “What is hidden inside the bottle?”. This project comes from the activity designed by Brousseau et al. (2011) consisting of a long process of didactic engineering design, planned to be experienced during 26 sessions with grade 5 students. In its adaptation for this SRP-TE, we only implemented three sessions of two hours with the prospective teachers as an introduction to this project. The situation is presented by some coloured balls hidden inside an opaque bottle with a small hole in the lid. The experiment consists in predicting how many balls of each colour there are in the bottle and of what colour. We know the total number of balls inside the bottle, but we do not have access to them. The only way to collect data is by shaking the bottle, and registering the colour of the ball appearing in the hole of the lid.

The initial questions are “What is the number of balls of each colour inside the bottle? What hypotheses can we formulate about its content and colour distribution? How reliable are they?”. These same questions are presented in three different situations. The first one consists of having a bottle with 5 balls of two colours and no restriction with regard to data registration. In the second, each working team designs a bottle with 5 balls and 2 colours,. They exchange bottles with another team, and they ask the questions to be answered by the team receiving their bottle. In the third case, the bottles have 25 balls of 4 different colours. When implementing this activity, the student teachers are asked to report about their work and to explicitly describe the tasks and questions they deal with, the kind of strategies they use, and the answers obtained. At the beginning of each session, we share these reports in a common forum to have information about all the questions addressed and the answers provided.

Module 2 consists in the collective analysis of the implementation. The teachers are asked to change roles to become “mathematical analysts”. Some specific terms related to randomness and data analysis are agreed upon at this moment such as “event”, “sample”, “sampling universe”, “sample size”, “frequency”, “likely”, “unlikely”, “possible”, “impossible”, “hypothesis”, etc. The educator also introduces one of the main tools at this stage: the *question-answer maps* (Q-A maps) for the mathematical analysis of the process followed, consisting of a diagram of all the questions, answers and strategies appearing in the activity experienced and their connections (Figure 2).

Later, Q-A maps become a key tool for the analysis of the student teachers’ own experience of the SRP, as well as reference models to analyse the students’ work experienced in primary school. The strategy the educator adopts here is to share with students some classroom experiences with a similar SRP as the one developed in primary school to contrast their analysis with real students’ work. To do this, the educator counts on the case of “The COSMOS project: ordering Hama Beads for the school” (Granell & Barquero, 2019), which was implemented in grades 4, 5 and 6 at *Col·legi Sant Lluís*, a primary school in Barcelona. The teacher is a former student of the University of Barcelona who continues to collaborate with the group of researchers-educators. At this point, the pre-service teachers are introduced to the work the teacher prepared in collaboration with our research team regarding the design of the project. It includes the following materials: the *a priori* designs of the SRP and the lesson plans guiding the implementation, the students' answers to some activities, and video recordings of some parts of the sessions with grade 4 students.

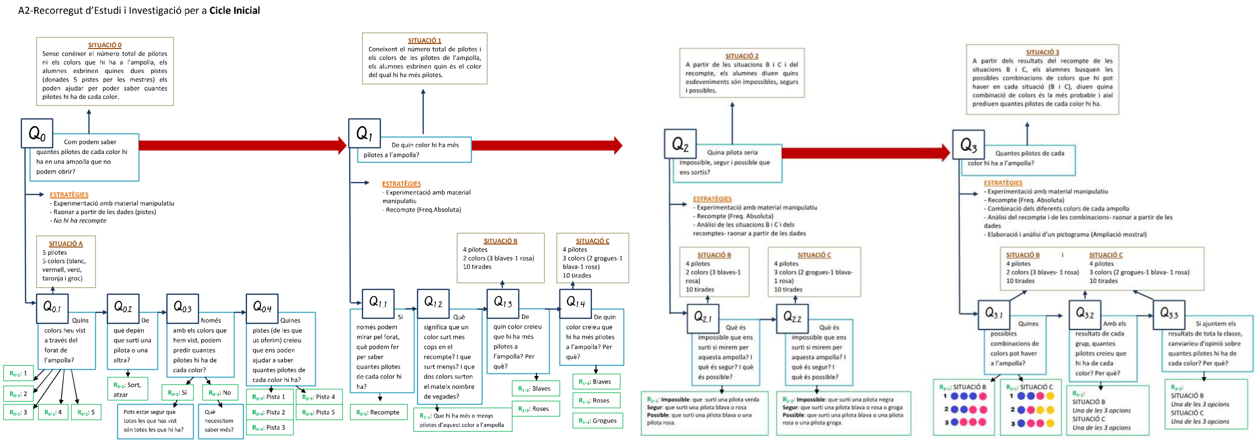

Figure 2: Example of a Q-A map proposed by a working team

SRPs-TE AND THE PARADIGM OF QUESTIONING THE WORLD

The general structure of an SRP-TE relies on several key features that are inherent to the paradigm of questioning the world. The first one is to start the SRP-TE with a professional teaching question $Q_{0\text{-TE}}$ the student teachers are asked to address together with the educators (who are, in this case, also researchers in didactics). The second one is that the final "product" is not supposed to be a previously known answer, as the questions addressed are open questions in mathematics education. Along with the modules of the SRP-TE, the aim is to initiate collaborative work between teachers and educators to search for already available answers for $Q_{0\text{-TE}}$ (in the form of teaching proposals), and to analyse them as potential teaching proposals. This collaboration leads to sharing, studying and building *mathematical knowledge for teaching* about the topic at stake concerning the initial and derived questions, which might include some "visits" of mathematics or didactics works.

Despite the specificities of the teacher education context, some SRP-TE features are also important for other SRPs (without TE). Q_0 might be an open question, of interest to the university context in which it should be posed and addressed. In the context of teacher education, as the profession behind is clear, searching initial generating questions legitimated by the profession does not seem difficult. On the contrary, what seems difficult to manage is their openness and the temporary and progressive character of the always partial and evolutive answers built. In other university contexts, the legitimacy and pertinence of possible Q_0 is a crucial aspect to be discussed by researchers and teachers, as well as with students. What seems easier to manage is the openness of the initial questions and the convergency or delimitation towards the answers produced. A strategy used on several occasions is to introduce an external demander *Z*, who is the person or group of people that comes up with the assignment (Barquero et al., 2021). *Z* can be also the recipient of the final answers, and the person or people responsible for evaluating and validating their pertinence. In the case of teacher education, the collaboration with an external validator *Z* could include agents from the teaching profession, or producers of teaching materials and resources.

Moreover, the collaboration between students (pre-service teachers) and teachers (educators) is productive as far as they jointly advance in establishing some *shared knowledge* about Q_0 and producing responses to Q_0 (or its derived questions). This is another important aspect of SRPs that collides with the traditional contract of the paradigm of visiting works. In an SRP, the teacher does not know the answer to Q_0 in advance. As Q_0 is an open and sometimes ill-formulated question, an answer does not always exist: it is defined and built during the enquiry process.

The other issue is how students and teachers can share, talk about, and refer to the knowledge they are constructing before getting the final answer to Q_0. In this regard, an important feature of SRPs-TE is that the *analysis of the SRP* is explicitly included as part of the enquiry. Therefore, educators introduce *epistemological* and *didactic tools* as far as they enable teachers to tackle the questions raised and elaborate answers to them. The didactic knowledge provided to pre-service teachers is not presented beforehand – which is the common strategy in the paradigm of visiting works – but introduced "on-demand" and motivated by its utility to address teaching questions. An important aspect in this regard is that educators are not supposed to teach future teachers "what and how to teach", but to help them in their search, construction, experimentation, analysis, assessment, etc. of different teaching proposals.

Another characteristic of an SRP-TE is that its module 1 – "experiencing an SRP" – aims to create a rich *empirical milieu* shared by teachers and educators to collectively identify and analyse some of the school conditions and constraints affecting the didactic designs and teacher actions. The description of the SRP that has been carried out generates the necessity of new tools (outside the traditional content description) for the mathematical analysis of this open activity. To address this necessity, educators introduce Q-A maps as an epistemological tool to analyse the dynamics of knowledge production. Q-A maps are very well accepted by teachers who use them not only as an epistemological tool for the *a posteriori* analysis of study processes but also for their *a priori* analysis—when they design the lesson plans to anticipate and evaluate possible paths to be followed by students. They also employ them in the *in vivo* analysis—when they use them as a tool to analyse real-time study processes for primary school students.

Finally, a specificity of SRPs-TE is that they include carrying out an SRP and its subsequent analysis. To distinguish between the two different paths, the strategy followed in the SRP-TE here presented consists of asking the participants to assume different roles during the instructional process. They start with the role of the "student" addressing an open question, then the role of the "analyst" of the study process that has just been followed, and finally the "teacher" role, including design and analysis responsibilities. This role-playing appears to be a successful strategy to facilitate the identification of different types of “mathematical” and “didactic” analyses, and to approach Q_0 from different angles, which can seem complementary but is necessary. These different perspectives on a single SRP, a more executive one and a more reflexive or analytical one, can also be extrapolated outside SRPs-TE. In an inquiry process of an initial question Q_0, the first aim is to produce an answer to Q_0. However, there is also a secondary more reflexive or “methodological” aim about the process followed, and the by-products that are worth preserving for further inquiries.

CONCLUSIONS: LESSONS FROM SRPs-TE TO NOURISH UNIVERSITY SRPs

As they are part of university education, teacher education degrees are faced with the same instructional problems as other degrees. Because researchers in didactics are usually involved in teacher education, they tend to consider these educational processes to be of a different nature. Being researchers, this was our spontaneous attitude in our research on SRPs, which were treated separately from SRPs-TE. At best, as mentioned before, some results of SRPs were introduced into SRPs-TE, like the use of Q-A-maps, and the exploitation of previously implemented SRPs within the SRP-TE process. What happens when we adopt the opposite perspective, and try to see which aspects of SRPs-TE can be passed on to SRPs? What can we learn from the teacher education experiences that could be useful—or at least worth trying—in other types of degrees?

The role-play strategy used in SRPs-TE can be taken as an example. It is motivated by the need to differentiate between carrying out a study process (the experimented SRP) and its description and analysis. This helps introduce an *in vivo* analysis during the SRP where the educator (acting as a teacher) comments on the strategies followed by the student teachers and by the teacher’s own strategy. In other words, SRPs-TE facilitate a type of analysis of the study process that reveals to be useful for managing an SRP: we can easily talk about the work that is carried out, which helps control it and better understand what we are doing and where we want to go. Therefore, the possibility of translating this *in vivo* analysis to university SRPs can be considered with the aim of improving their management and outcomes. At the end of the SRP, when the final question has been produced, a kind of complementary report about the process followed to produce the answer could be introduced. This is a very common type of professional process related to the need for accountability, assessment, and quality control that is rarely transposed in the classroom.

There is a more subtle issue related to the previous one. In SRPs-TE, the analysis of the SRP carried out is performed with didactic and knowledge resources that are sometimes elaborated for this purpose by research in mathematics education. A previously mentioned example are Q-A maps, but we can also think of less generic tools such as the description of data analysis or modelling processes, which require specific terms that are not always covered by the traditional mathematical terminology.

The institutionalisation of these tools that help describe and better understand (and control) the process followed is easier in the context of teacher education because the content at stake is not only “mathematics”, but also “didactics of mathematics”. It thus gives more flexibility to the educator to elaborate discourses about the mathematical activity “in process”, and also about the final answer produced. It is, however, more difficult to do so in another kind of degree when the teacher is supposed to teach “statistics” or “mathematical modelling” and not “a discourse about statistics” or “a discourse about modelling”. As if the difference between “mathematical” and “a discourse about mathematics” was so clear, or as if mathematics could be done without having a discourse about mathematics. In teacher education, it seems legitimate to teach Q-A maps and other epistemological resources elaborated by didacticians for the analysis of mathematical activities. It does not seem so legitimate in other kinds of degrees, although it is equally necessary.

Approaching university SRPs from the perspective of SRPs-TE helps us think about new strategies to develop. It also reveals new constraints linked to the nature of the knowledge to be taught and the aim of the instructional process: how the knowledge or activity at stake is conceived by the educational institution, what legitimates a given instructional process, etc. It also questions the boundaries between didactics and mathematics. This is a boundary that does not exist in teacher education and that may hinder rather than clarify what is done and what can be done in university education.

ADDITIONAL INFORMATION

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Activity Theory as a Base for Course Design in Pre-Service Teacher Education: Design Principles and Their Application in Two Examples

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Observations in practice show that pre-service teachers do not always experience and acquire mathematics subject knowledge and mathematics education knowledge in such a way that they are ready to effectively use this knowledge in their future careers when designing and implementing lessons. We develop an activity-theoretical framework that contributes to describing and explaining underlying discontinuity obstacles, and, as a developmental contribution, we use the framework to formulate and implement design principles for courses in pre-service teacher education aimed at counteracting discontinuity phenomena in the area of argumentation and proving.

Keywords: Teachers' and students' practices at university level; Transition to, across and from university mathematics; Teaching and learning of logic, reasoning and proof; Activity theory.

INTRODUCTION

Discontinuities between school mathematics and university mathematics have been recognized as a problem at least since Felix Klein's time (Klein, 1908). The first transition (secondary-tertiary) has been the focus of considerable attention in recent years (see Gueudet, 2008, for an analysis of phenomena and causes). A number of projects try to mitigate this "first discontinuity" by helping students to establish connections between the two "worlds", for instance by engaging them in interface tasks (Bauer & Partheil, 2009) that address the specific differences. The second critical transition, which occurs when pre-service teachers (PST) go back to school after graduation, has received less attention so far. The issue is whether students have acquired mathematical content knowledge and mathematics education knowledge in such a way that they can effectively use this knowledge in their future careers to design and implement lessons. Experience from practice, as well as from capstone courses as described by Winsløw & Grønbæk (2014), shows that one should not expect that this occurs automatically. In extreme cases, novice teachers may regard academic knowledge as unworkable in the "real world" of the classroom (Cavanagh and Prescott, 2007). They are then susceptible to the "familiarity pitfall" (Feiman-Nemser and Buchmann, 1985, p. 56), i.e., they might identify teaching with classroom practices that they experienced as pupils themselves – just as already Felix Klein observed.

The transition from university to school in particular means a transition from one mathematical practice to another, where PSTs experience discontinuous changes in various respects. Our hypothesis is that part of the problem of the second discontinuity is de facto based on a poorly mediated, possibly distorted experience of *differences between core activities* within different mathematical practices at school and at university. In the present paper we focus on the core mathematical activities of proving and argumentation. The following two desiderata motivate our work. (1) Provide a theoretical framework that would allow to grasp (that is, observe, describe, and explain) the practical impressions of discontinuity effects regarding the second transition as possibly interconnected general phenomena. (2) Find starting points for developing suitable formats for PST training courses in order to effectively address the respective issues of teacher education.

First, as a theoretical contribution, we develop an activity-theoretical framework that helps to analyze differences and commonalities between mathematical practices at school and university, and contributes to describing and explaining discontinuity obstacles. Second, as a developmental contribution, we use the framework to formulate design principles for courses in PST education and we will present two course designs where we have implemented these principles. Our concrete implementations show in particular that the three principles can be employed in a longitudinal setting (i.e., in a sequence of consecutive modules) as well as in a single module.

ACTIVITY THEORY AS A GENERAL CONCEPTUAL FRAMEWORK ON CORE MATHEMATICAL ACTIVITIES AND DISCONTINUITY PHENOMENA

In our approach, we build upon the work of Leontjew (1982) as developed further by Lompscher and Giest (see Bruder and Schmitt, 2016; Giest, 2008). Based on foundational work in the 1980s, a spectrum of activity-theoretical approaches emerged in the mathematics education literature, e.g. Engeström (2001), Jaworski et al. (2017), Cerulli et al. (2005). The approaches differ in particular in terms of the fields of activity they address. In our work, we consider argumentation and proving as a field of activity in school and university mathematics to study respective discontinuities between them.

From the point of view of activity theory, human activities always have a “dual character”: Namely, on the one hand as activities in the context of a communal, collaborative practice, and on the other hand as individual action.

Mathematical argumentation and proving as activities in collaborative practices

In our application and adaptation, we first look at mathematical argumentation and proof as activities in the context of collaborative practices. We distinguish three structural components: Motive, object and ways and means of action. The interplay of these three components in terms of activity theory can be described – in a very compact form – as follows. The motive drives activities that are directed towards an object. The object thus becomes the object *of* the activity. The concrete goals that can be pursued in an activity, as a realization of the motive, depend on available and suitable ways and means of action.

Along these three structural components, we can contrast two mathematical practices that are relevant for PSTs, university mathematical practice and school mathematical practice, and identify characteristic differences that are often experienced by students as dominant, for example:

- On the motive level, the deductive derivability of statements within globally ordered mathematical theories plays an important role in university mathematics, whereas school mathematics is more concerned with the verification of statements, which is conducted, if at all, with reference to locally ordered systems of statements.
- On the level of objects, we find a variety of explicit objects of proving in university mathematics, and in contrast often only implicit, hidden objects of argumentation and proving in school mathematical practice.
- On the level of modes and means of action, a difference between the two practices that is experienced as dominant can be identified with regard to the different roles of heuristic or generic arguments in the two practices.

In line with our hypothesis stated in the introduction, PSTs who experience such differences as dominant might not be able to integrate the meaning that argumentation and proving have as core activities in these different mathematical practices into their individual activity in a coherent way. Hence, they might also struggle to stage these activities properly in class, thus going through a “second discontinuity” in a problematic way.

Mathematical argumentation and proving as individual action

In order to capture this issue fully from the point of view of activity theory, we have, in addition to the collaborative community perspective, to consider the activities of mathematical argumentation and proving also with respect to individual action. Under this perspective, the triad of constitutive elements is similar to activity in collaborative practice, with a level of motives, a level of objects, and a level of ways and means of action.

We conceptualize the individual concretization of these three levels and their interplay in individual activity as follows (see Fig. 1): The objects that the individual has made their own and the available repertoire of ways and means of action form individual preconditions for action. These are at the same time preconditions for the individual to be able to consciously set concrete goals of action, and, accordingly, to be able to act on the object directed towards these goals. Such consciously set goals for action are influenced by superordinate motives for activity, which the individual, however, is not usually aware of at the stage of their concrete action. Rather, the goals mirror individual constructions of meaning in relation to the activity.

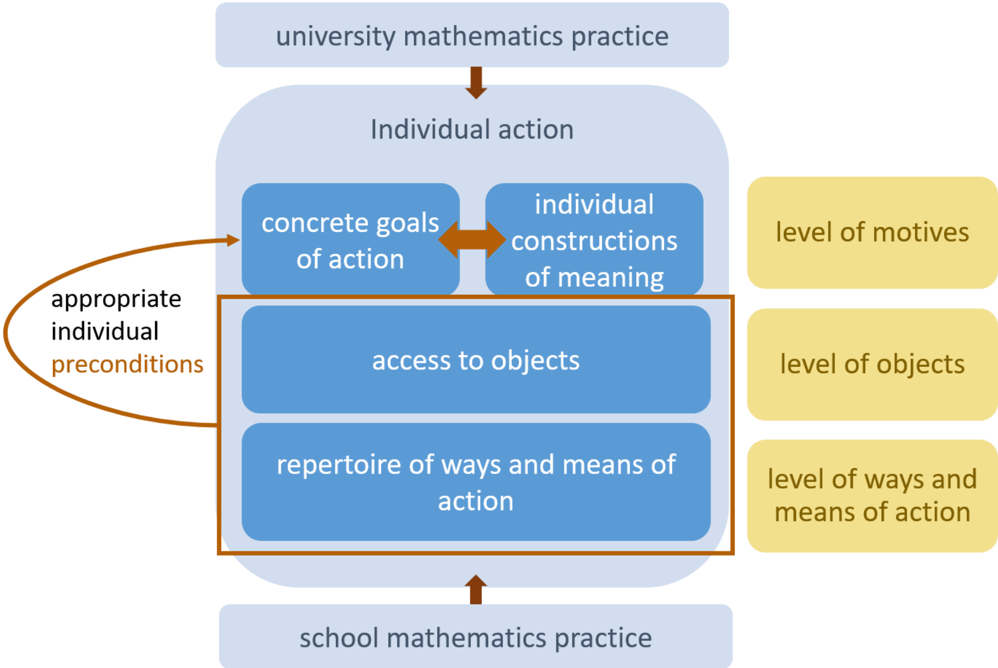

Fig. 1. Constitutive elements of individual action

First results of explorative empirical investigations of PSTs' lesson plannings and stagings, which we carried out on the basis of this activity theoretical framework (Bauer & Müller-Hill, 2022), indicate that there is indeed a need for action in PST educational practice: We exhibited phenomena and patterns in PSTs' stagings that indicate that they apparently lack effective background motives and corresponding goals and constructions of meaning, appropriate ways of action and access to suitable objects, in order to stage argumentation and proving activities in a meaningful way in the mathematics lessons they plan and conduct.

ACTIVITY THEORY AS A DESIGN BASE: DESIGN PRINCIPLES AND THEIR APPLICATION IN TWO COURSE DESIGNS

We understand the empirical observations mentioned above as a variant of double discontinuity, namely as an impact of transition obstacles that emerge between three areas of action and experience: the mathematics lessons experienced at school, university mathematics and didactics, and PSTs' own lesson stagings at school.

Accordingly, our main idea in using activity theory as a design base is that the presented activity-theoretical considerations on constitutive elements of core mathematical activities, together with empirical results such as those reported above, form a basis for the development of design principles for PST courses. The application of such principles is intended to help to turn discontinuity experiences productively, and hence to mitigate second transition issues such as, e.g., the observed phenomena described in Bauer & Müller-Hill (2022).

Design Principles

We suggest the following three design principles, each of which is formulated with reference to one of the constitutive elements of activity. A combined application of these principles is intended to counter the specific discontinuity issues that are revealed by the activity-theoretical explanations for the empirical phenomena mentioned in the previous section.

Principle 1: Engage PSTs in getting to know, applying, and assessing a variety of appropriate objects and ways/means of action regarding core mathematical activities, both in the PSTs' role as mathematically active individuals and as members of the collaborative mathematical practices of school and university.

Principle 2: Engage PSTs in explicating, reflecting and relating motives and goals of core mathematical activities, with respect to individual activity and with respect to activity within the collaborative mathematical practices of school and university.

Principle 3: Engage PSTs in explicating and reflecting content-related and didactical decisions within their own mathematical working processes as well as within their planning and implementation of mathematics lesson stagings, with the aim to support the development of appropriate and coherent individual constructions of meaning for core mathematical activities.

We will now show how these three design principles can be brought to fruition in concrete course designs.

ProPraxis – A Longitudinal Implementation of the Three Design Principles

The design principles are in this case implemented in a series of consecutive modules (see Fig. 2). In these modules, proving, learning to prove, and teaching proof is addressed both on the object level and on the meta level.

The subject-matter modules in the first three semesters (Linear Algebra and Analysis) serve as the starting point. They are related to Principle 1, as they engage students in getting to know a variety of appropriate objects as well as ways and means of action – both as mathematically active individuals (e.g. when solving homework problems) and as members of a collaborative practice of university mathematics (e.g. when they are exposed to the norms of the discipline through lectures and books).

The subsequent mathematics education module *ProfiWerk* (the title is a German shorthand for "*Professionalization Workshop"*) then implements in its first part the Principles 1 and 2, as it engages students in applying and assessing a variety of appropriate objects and ways/means of action, as well as in explicating and reflecting motives and goals of core mathematical activities. For instance, when the Toulmin model (Toulmin, 1958) is used to analyze proofs from university mathematics courses as well as proof products by pupils, students assess appropriate objects and establish connections between school and university practices. Or, when they practice and reflect on generic proofs, they connect proving as experienced in subject-matter modules with proving as an activity to be staged in the classroom. (See Bauer, Müller-Hill & Weber, 2021a, for details on Part 1 of ProfiWerk.)

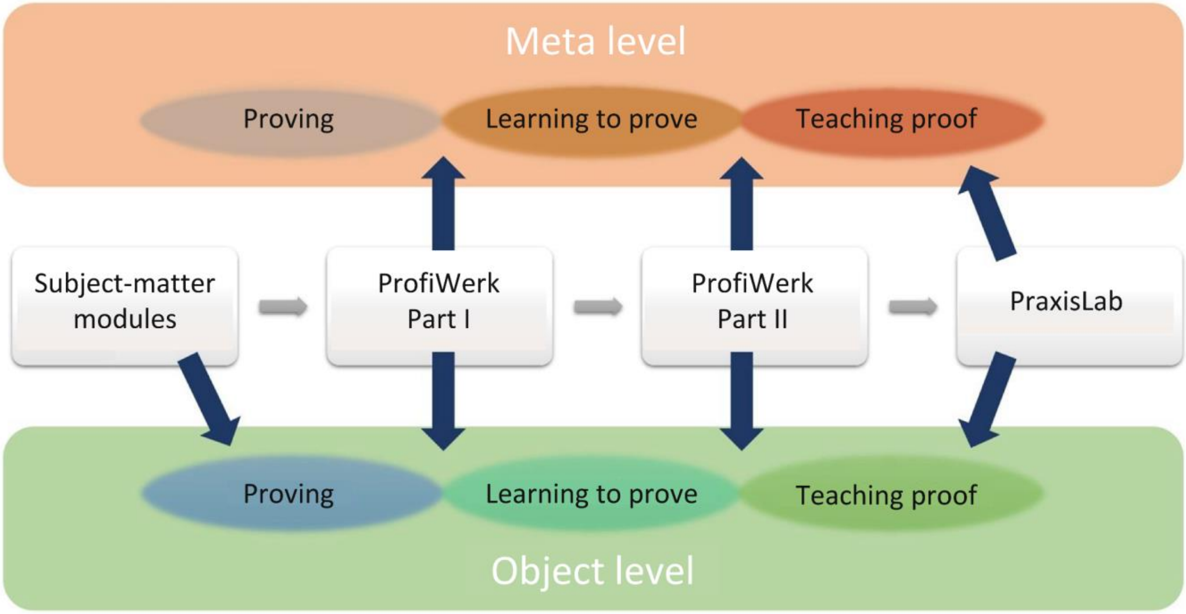

Fig. 2: Components and sequential structure of the longitudinal implementation

The second part of *ProfiWerk* focuses on problem solving as a core mathematical activity. Students solve problems and then analyze their problem solving process in terms of the heuristics strategies they used and in terms of the problem solving phases that occurred. They thus analyze and reflect on motives, objects, content-related decisions, goals, and ways and means of action in their working processes. So, in addition to Principles 1 and 2, Principle 3 is implemented here in a focused way. It is a decisive feature of the course that students work on problem tasks both at university level and at school level. Through explicit assignments they compare the respective problem solving processes, thus comparing and connecting the two practices. (See Bauer, Müller-Hill & Weber, 2021b, for details on Part 2 of ProfiWerk.)

The final module or the sequence, *PraxisLab*, consists of a field experience and an accompanying seminar. Here students get the opportunity to observe lessons and to try out their own implementations, which they plan based on motives, goals, suitable objects, and ways and means of action developed in ProfiWerk. As this entails explicating and reflecting content-related and didactical decisions within their own working, planning and staging processes, Principle 3 is fully implemented in this module.

ABEB – An Implementation of the Three Design Principles in a Single Module

The seminar design of ABEB (ABEB is a German shorthand for “Argumentieren, Begründen, Erklären und Beweisen im Mathematikunterricht”, in English: “Argumentation, reasoning, explanation and proof in mathematics lessons”) is one possible way to apply the design principles within a single PST seminar to initiate a productive turn of discontinuity issues.

The seminar agenda falls into four parts (see Fig. 3): The first and the last part are conducted mainly asynchronously (the closing session is the only exception), and include, as an application of Principle 2, individual reflexive writing tasks as core elements: The guided explication and reflection of motives, goals, and subjective core ideas regarding argumentation and proof in mathematics class as well as in university mathematics are central topics of the initial writing assignments. At the end of the seminar, the PST participants have to do an overall reflection that asks about changes in the views that they had written down at the beginning of the seminar.

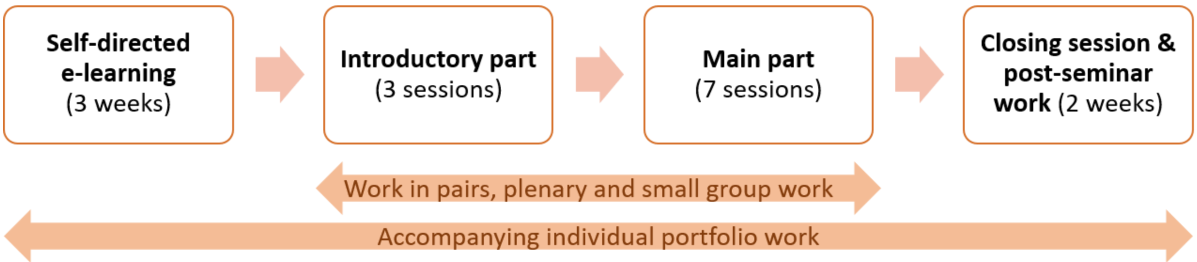

Fig. 3: Components and structure of the single-module implementation

These asynchronous parts also serve as the start and end points of accompanying individual portfolio work, which is one important form of learning during the whole seminar.

The middle part of the seminar agenda consists of two live parts, where work in pairs, plenary work and small group work are the crucial forms of learning. In the introductory part, participant PSTs get to know and practice techniques for proof comprehension and proof construction, in accordance with Principle 1. Participants are also engaged in dialogical reflection on explicated motives and goals for argumentation and proving, continuing the application of Principle 2 in the asynchronous parts. In the main part of the seminar, in each session one PST tandem applies and assesses the techniques learned in the introductory part on a given object, which could be a proof or a theorem on advanced school level. The tandem prepares an interactive presentation on their results for their peer students. This exemplifies Principles 1 and also 3, because the PSTs need to explicate and reflect content-related and didactical decisions to prepare their presentations.

A second component of each main part session consists of new input and collaborative activities on mathematical argumentation (e.g., the Toulmin model), mathematical reasoning (e.g., the interplay of abduction, deduction, and induction as in Müller-Hill, 2019, as well as the phases of argumentation and proving according to Boero, 1999), and mathematical explanation (e.g., invariance criteria for explanatory patterns, ibid.). This main part can be seen as a combined realization of all three design principles: PSTs are engaged in conducting and analyzing their own argumentation and proving processes, in analyzing pupils' work products and learning environments, and in developing approaches for possible implementation of argumentation and proof in lesson stagings.

RESULTS

The theoretical framework on core mathematical activities and the design principles, which we developed on its basis, provided strong guidance in finding course designs that address the empirically detected discontinuity issues regarding motives, objects, and way/means of action in their complexity already in early design stages. Evaluations of the ProPraxis module sequence, which has been running continuously since 2016, have been used as guidelines for adjustments and incremental improvements, but they did not point out the need for a fundamental reorientation. The ABEB seminar, first implemented in 2021, has entered a second iteration in 2022 with only slight modifications.

In our longitudinal concept (ProPraxis), the three design principles are implemented in a cumulative ascending manner: 1, 1+2, 1+2+(3), 1+2+3. In the ABEB seminar, we employ Principle 2 as a bracket in the sense of individual pre- and post-seminar engagement with motives, goals and personal core ideas on a meta level: **2**, 1+2, 1+2+3, **2**. This raises both the individual starting level of the participants and the collective starting level of discourse in the group. The intent is to support the subsequent application of all three principles in their quick succession of cumulatively ascending combinations.

In order to understand possible effects of the courses, we examined the reflections that the students wrote as part of their term papers. We briefly discuss two examples. Nadja (in the ProfiWerk seminar) states:

> Looking back now, I can say that my understanding developed over the course of the semester: Through various examples from everyday school mathematics, I was able to see that the concepts and explanations were often equivalent to or quite close to those of university mathematics, although at first glance one would not have thought so.

Apparently, Nadja has made new connections on the level of objects – she now relates argumentation in school and university to each other.

Timo states in his overall reflection of the ABEB seminar:

> In my initial reflection at the beginning of the seminar I stated that there seems to be a gap between university and school mathematics, which makes it only conditionally possible and useful to incorporate proofs into school lessons. [...] Especially with the help of the methods for understanding and constructing proofs, which were shown and practiced at the beginning of the seminar, are from my point of view well suited to include proving activities into math lessons in different ways at one point or another. [...] Motives like gaining knowledge and comprehensibility were not very important for me, because I was of the opinion that proofs would complicate the lessons in many places and hinder understanding. [...] Through the learned methods and procedures, through which proofs,

but also argumentation, justification and explanation phases can be integrated into the lessons, this point of view has changed.

His writing indicates that Timo has developed new constructions of meaning regarding proof under the impression of experiencing ways and means of action that were previously unknown to him. Furthermore, he has revised and extended his understanding of the range of motives for argumentation and proving in mathematics lessons.

CONCLUSION

We have presented an activity-theoretical conceptual framework on core mathematical activities and have shown how the design principles that we developed within this framework can be brought to fruition in concrete course designs in order to address and productively turn specific discontinuity issues. The interplay of the constitutive elements of the theory is mirrored in their implementation. As a consequence, elements that might otherwise have remained implicit were brought to the surface – they were given substantial weight and were specifically explicated and worked on in explicit student activities. Accordingly, we found evidence of new, coherent constructions of meaning in the students' reflections.

The fact that activity theory could be used so effectively here for the purposes of design convincingly underscores, in our view, the universal nature of the mechanisms it captures.

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Relevance attributions to contents in mathematics teacher education

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Many mathematics teaching students are highly dissatisfied with their studies and some criticize a lack of relevance. Hernandez-Martinez and Vos (2018) conceptualize relevance as a connection between subject matter, its usefulness, and the learner. To explore what would make mathematics teaching students ascribe more relevance to their studies, the first author examined links between university mathematics contents, students' perceptions of its usefulness and students' characteristics (Büdenbender-Kuklinski, 2021). In this paper, we focus on students' relevance attributions to explicitly asked-for contents and students' own ideas of relevant contents. Our results suggest that lacking relevance attributions might be linked to lacking criteria of relevance in university mathematics studies, to lacking recognition of connections between university mathematics and the later teaching profession and to students' insecurities. Based on our results we formulate hypotheses to guide future research into mechanisms behind mathematics teaching students' relevance attributions.

Keywords: relevance attributions, mathematics teacher education, insecurity, beginning of university studies, value construct.

DISSATISFACTION AND LACKING RELEVANCE

It is internationally common that mathematics teaching students at the beginning of their studies take courses in advanced mathematics taught by mathematicians (Even, 2011). There is evidence for students' high dissatisfaction with these courses, linked to their feelings of being unprepared for mathematics at university (Goulding et al., 2003). Often, future mathematics teachers do not see links between the mathematics they learn at university and the content they will teach at school (Zazkis & Leikin, 2010).

Mathematics teaching students' high dissatisfaction is also a problem in Germany as it may lead to dropping out of university (Geisler, 2020) and dropout rates are as high as 58% (Heublein et al., 2020). Student dropout frequently occurs at the beginning of university studies (Geisler, 2020) and mathematics teaching students particularly often consider dropping out of university (Blömeke, 2009). As some of the dissatisfied students criticize a lack of relevance (Scharlach, 1992), we explored students' relevance attributions to better understand the high dropout rates. Such knowledge could help in improving the communication about study programs and optimization possibilities between students and university officials. It might, for example, hint at relevance attributions' connections to contents' transferability to school teaching or school mathematics. Current support measures assume that presenting references between school and university mathematics support higher relevance attributions by mathematics teaching students (e.g., Eichler & Isaev, 2016).

EVIDENCE FOR LACKING RELEVANCE ATTRIBUTIONS

Several conceptualizations for relevance connect relevance to value constructs. For example, Vollstedt (2011) relates personal relevance to the concept of value and Neuhaus and Rach (2021) use the concepts of utility and relevance synonymously where utility is a value component according to Expectancy-Value Theory (Barron & Hulleman, 2015). Value describes a person-object relationship and while originally the Expectancy-Value theory focused on the value of achievement tasks (Wigfield, 1994), the object of value can be manifold, for example it could be mathematics contents ordered by topic area or complexity level. In this paper, we consider relevance as a value construct and explore what kinds of content beginning mathematics teaching students would ascribe such value to.

Earlier findings concerning the dissatisfaction of mathematics majors suggest that students criticize a lack of applicability of the study contents. Mathematics teaching students seem even more dissatisfied with their studies than mathematics majors (Brown & Macrae, 2005). In Germany, in the first semesters, teaching students and major students attend mathematics lectures together. These introductory lectures mainly treat topics of linear algebra and calculus. Applicability for first-semester mathematics teaching students would mean a connection to school mathematics and knowledge about didactical approaches concerning school mathematics. Hence, higher relevance attributions might be connected to a recognition of contents' applicability for school. As mathematics teaching students moreover criticize that the complexity in which contents are presented is too high (Göller, 2020), they furthermore might ascribe more relevance to contents that are rather basic than complex.

MODELLING POSSIBLY RELEVANT CONTENT

We model contents mathematics teaching students might find relevant based on a conceptualization introduced by leading German mathematical associations (DMV et al., 2008). It concerns study contents that should be mastered by mathematics teaching students by the end of their university studies. Of course, catalogues of contents are in themselves often controversial, but we chose this conceptualization as the authoring associations have an important standing in German educational policies. We focus on recommendations for the subject areas of arithmetics/ algebra, geometry, linear algebra, and calculus. The competencies in each subject area are subdivided into four different levels that differ "according to content expansion, conceptual elaboration and degree of abstraction and formalization" (own translation, DMV et al., 2008, p. 2). These complexity levels become more complex from Level 4 to Level 1 and each subsequent level assumes all competencies of previous levels. Level 4 includes the basic competencies of any teacher, regardless of the grade level they teach, and Level 1 includes competencies that a teacher teaching at the upper secondary level should still possess. Except for linear algebra, where competencies are described for Levels 3 through 1 only, competencies are described on all four levels for each subject area (DMV et al., 2008). Based on the lists for each subject area, we modelled potentially relevant content of the mathematics teaching degree program. This model of relevance content on the one hand comprises the dimension of subject areas and on the other hand the dimension of complexity levels. There are four different subject areas whose contents' relevance we researched. The contents were furthermore differentiated according to degree of abstraction and formalization. We illustrate the nature of the levels in the section "Methods" with example items. Based on this model, we explored how relevant students find contents that educational policies ascribe relevance to.

Of course, this model only covers a selection of university mathematics content that experts deem relevant for mathematics teaching students and there might be other contents students themselves find relevant. To find out which contents students name as relevant spontaneously, we asked students to name contents they themselves find relevant.

RESEARCH QUESTIONS

Given mathematics teaching students' dissatisfaction and criticism of lacking relevance, we constructed a model of possibly relevant study contents. In the following empirical part, we first explore to what extent these contents do indeed seem important to students. We explored the relevance ascribed to contents of different subject areas and of different levels of complexity as suggested by the model.

Research question 1: How important do mathematics teaching students consider contents of different subject areas to be?

Research question 2: How important do mathematics teaching students consider contents of different complexity to be?

In addition, we examined which topics are mentioned by students when asked to name relevant things in their studies.

Research question 3: What topics do teaching students themselves identify as relevant in their mathematics studies?

METHODS

Sample and research design

To answer the research questions, we conducted a longitudinal quantitative study with two paper-pencil surveys. In the first survey, 162 students participated, 78 of whom were female, and in the second questionnaire survey, again 162 students participated, 91 of whom were female. The two groups were overlapping, 109 participants took part in both surveys. Participation was voluntary and anonymous. The first survey took place in the second week of lectures of the winter semester 2018/ 19 in a course for first semester mathematics teaching students and the second survey in the penultimate week of lectures of the same semester in the same course.

Measurement instruments

For research questions 1 and 2, we developed a measurement instrument based on our model of relevance content. We created items that asked students to rate the importance of the content of various subject areas of different complexity on 6-point Likert scales.

These items' wordings cover all four complexity levels for all subject areas but linear algebra where items cover levels 3 through 1. They are very close to the formulations in the underlying conceptualization (DMV et al., 2008). Table 1 provides exemplary items for all four complexity levels for the subject area of arithmetics/ algebra. To answer the third research question, we applied an open-ended question: "Are there any other study contents that are particularly relevant to you? If so, which ones?"

	In my mathematics studies it is important to me that ...
Level 4	... I have a basic understanding of the aspect variety of natural numbers, fractions and rational numbers.
Level 3	... I can describe the limits of the rational numbers in the theoretical solution of the measurement problem.
Level 2	... I can explain the completeness of the real numbers using examples.
Level 1	... I master conceptual tools such as equivalence classes for the formal foundation of number ranges.

Table 1: Exemplary items for the relevance contents

Analysis

We conducted a mean value analysis and pairwise, two-sided t-tests with a significance level of 5% to answer the first two research questions. To answer the third research question, the answers to the open-ended question were evaluated using qualitative content analysis (Mayring, 2015). Based on the question "Which topics are mentioned in the open item?", we inductively formed categories.

RESULTS

Preliminary analysis

Survey		Arithmetics/ algebra	Geometry	Linear algebra	Calculus
T1	M (SD)	5.06 (0.74)	5.07 (0.77)	5.15 (0.76)	5.06 (0.73)
	N	102	68	100	105
T2	M (SD)	5.03 (0.75)	4.78 (0.88)	4.89 (0.90)	4.91 (0.93)
	N	123	97	153	141

Table 2: Relevance attributions to the subject areas

In a preliminary analysis of the relevance attributions to the subject areas and complexity levels, we found that there were high variations in the numbers of students that had evaluated the relevance of different subject areas and complexity levels. In the mean value analysis of the importance of the four subject areas (cf. Table 2) all mean values at both survey times were above the scale's mean and standard deviations were similar. Noticeably fewer students gave an assessment of the importance of geometry than of the other subject areas.

In the mean value analysis of the importance of the four complexity levels (cf. Table 3), all averaged relevance attributions were above the scale's theoretical mean, as well. Standard deviations of the variables increased with increasing complexities of the levels. Fewer students assessed the importance of more complex content.

Survey		Level 4	Level 3	Level 2	Level 1
T1	M (SD)	5.14 (0.63)	5.15 (0.62)	5.20 (0.67)	4.84 (0.98)
	N	132	89	83	69
T2	M (SD)	5.02 (0.78)	5.00 (0.75)	4.94 (0.81)	4.74 (0.89)
	N	126	109	109	103

Table 3: Relevance attributions to the complexity levels

Results concerning research question 1

We found little difference between the relevance attributions for the different subject areas (cf. Table 4).

	Arithmetics/ algebra		Geometry		Linear algebra		Calculus	
	T1	T2	T1	T2	T1	T2	T1	T2
Arithmetics/ algebra	n.s.							
Geometry	n.s.	<.001	n.s.					
Linear algebra	n.s.	n.s.	.041	n.s.	n.s.			
Calculus	n.s.	n.s.	.006	.043	n.s.	n.s.	n.s.	

Table 4: Mean differences between different subject areas at one survey point and same subject areas between survey points; p<.05

The only statistically significant mean differences were those between T1 relevance attributions to geometry and linear algebra, as well as geometry and calculus, and between T2 relevance attributions to geometry and calculus. While relevance attributions for all four subject areas were somewhat lower in the second survey, none of the mean differences was significant on a 5% level.

Results concerning research question 2

Less relevance was attributed to more complex content: Mean differences between relevance attributions to Level 1 content and content on all other levels at both survey points were statistically significant (cf. Table 5). From the first to second survey, relevance attributions concerning the complexity levels tended to decrease but only the mean difference between relevance attributions to Level 2 content at T1 and T2 was significant on a 5% level.

	Level 4		Level 3		Level 2		Level 1	
	T1	T2	T1	T2	T1	T2	T1	T2
Level 4	n.s.							
Level 3	n.s.	n.s.	n.s.					
Level 2	n.s.	n.s.	n.s.	n.s.	.034			
Level 1	<.001	<.001	<.001	<.001	<.001	<.001	n.s.	

Table 5: Mean differences between different complexity levels at one survey point and same complexity levels between survey points; p<.05

Results concerning research question 3

In responding to the open-ended question, students frequently stated that it seemed relevant to them to cover school mathematics content and mathematics didactics topics. Relevance was seen in addressing questions about the design of mathematics instruction and in addressing application references of mathematics. We found the categories "school mathematics/ mathematics relevant for school" (coded 5 times for T1, 5 times for T2), "didactics of mathematics" (coded 5 times for T1, 3 times for T2), "questions concerning the design of mathematics classes" (coded twice for T1, twice for T2), "applications of mathematics" (coded twice for T2), "mathematical topics" (coded twice for T2) and "history of mathematics" (coded once for T2). Most of these aspects have a strong connection to school and to school mathematics. Moreover, students made many more statements about what they found generally relevant in the teacher training program (e.g., didactics, psychology, internships) than about what was relevant in the mathematics program. For example, students would say "The study program should prepare us much more pedagogically". Many students mentioned they would ascribe more relevance if their studies showed them that the teaching profession was right for them ("Above all, I think that we, as teaching students, should have more didactics and also internships, so that you know exactly whether the teaching profession is right for you"). Criticism was also named frequently (for example, about studying together with mathematics major students – "Prospective teachers should not have to study with the mono-maths students!"). All categories are presented in the first author's dissertation project (Büdenbender-Kuklinski, 2021).

DISCUSSION

In this paper, we dealt with three research questions. In an exploration of how important students consider given subject areas, the mean value analysis indicated rather high relevance attributions to all four subject areas. Hence, students of mathematics teaching already seem to assess much of the content of their studies as relevant. There were less students who gave an assessment of the relevance of geometry. In a second mean value analysis, we explored how important students consider contents modelled on different complexity levels. Again, the students surveyed tended to attribute a high degree of relevance to the content of the various complexity levels though there seemed to be greater agreement about the relevance of less complex content which was seen as more relevant than more complex content. Our results also indicated that students ascribed less relevance at the end of their first semester than at the beginning. Analyzing an open-ended item to answer our third research question, we found that students named contents as relevant that were closely connected to school. We also found that some students connected a higher relevance to a personal feeling of security about their career choice. Finally, our analyses suggested that students rather criticized frame conditions of their studies than name relevant content.

Results hinting at rather high relevance attributions contrarily to earlier findings

That students found contents of various subject areas and complexity levels relevant matches findings by Bergau et al. (2013). They also found that teaching students wanted to acquire broad subject knowledge, which the students believed should go beyond the school subject matter. The often-described negative attitudes toward mathematics and the study of mathematics (e.g., Brown & Macrae, 2005; Scharlach, 1992) do not seem to be confirmed here. Maybe the negative attitude does not concern the study content itself but results from feelings at the transition between school and university like being overwhelmed or feeling insecure.

Lacking relevance attributions due to lacking criteria for what might be relevant

It would also be possible that the criticism otherwise expressed results from students not knowing what could be relevant for them as teaching students in mathematics studies. They could thus initially criticize a lack of relevance but could recognize a relevance if suggestions are made to them as to where it could lie. The lower relevance attributions to geometry might then also indicate that students feel uncertain about geometry's relevance. Geometry is not taught in the first semester so the uncertainty about its relevance might be connected to its missing appearance. If teaching students themselves are unsure at the beginning of their studies what could be relevant for them, the lack of treatment of geometry at the beginning of their studies could give them the feeling that it is not or hardly relevant at the university and they then adopt this assumed view as their own opinion. Our result of research question 2 that more complex content was seen as less relevant might also indicate that these were contents students did not recognize and thus could not assess as relevant.

Differentiation in relevance attributions

That students ascribed less relevance at the end of their first semester than at the beginning might indicate that they are more enthusiastic at the beginning and willing to ascribe more relevance but they lose motivation and enthusiasm throughout the semester after encountering various frustrations. As there were less students who gave an assessment of the relevance of geometry, students also seem to differentiate between different contents in their attributions of relevance, at least to a certain extent. Earlier research found that geometry was not seen as a central part of mathematics by teaching students in open impulses (Winter, 2001, 2003), which could be interpreted as assessing it as little relevant. However, the importance of geometry was recognized by students in the past when they were directly asked about it (Winter, 2001), whereas a relevance in the present work was attributed at least to a lesser extent than in the other subject areas, despite direct inquiry.

Relevance attributions' possible links to school references

That more complex content was seen as less relevant might also indicate that these contents seemed too far away from school mathematics for students and they thus found them irrelevant for their future job. This would fit with our finding concerning the open-ended question where students attributed a high degree of relevance to contents that closely connected to school. Hence, current support measures that present references between school and university mathematics to support higher relevance attributions by mathematics teaching students (e.g., Eichler & Isaev, 2016) seem to make a good approach.

Relevance attributions' possible links to feelings of insecurity

That students named contents with close links to school as relevant contents might also indicate that they have problems with university mathematics. Lower relevance attributions might then be connected to feelings of insecurity. Possibly mathematics teaching students attribute less relevance to things they feel more insecure about to protect their own self-esteem. This possibility would also fit with our finding that students ascribed less relevance to more complex contents that they might have problems with. If students felt less secure at more complex content, they might also have ascribed less relevance here to protect their self-esteem as it would be less destructive for their self-esteem to have problems with contents that are not relevant. The lower relevance attributions to geometry might also be connected to feelings of insecurity with geometry as students have not yet engaged with geometry at university.

Our finding that some students connected a higher relevance to a personal feeling of security about their career choice might hint at yet another kind of insecurity they might feel concerning their studies. Moreover, the finding that students rather criticized frame conditions of their studies than name relevant content in answer to our open question might suggest that their criticism of a lacking relevance is rather connected to a feeling of unease in their studies but not based on a clear idea of what is missing (cf. Wenzl et al., 2018).

IMPLICATIONS AND OUTLOOK

Our results serve as hypotheses to guide future research into mechanisms behind mathematics teaching students' relevance attributions. They suggest that students' relevance attributions might be supported by presenting links between university mathematics contents and the teaching profession, as has been done, but there might be more possibilities. Students could profit from reflections on relevance criteria at the beginning of their studies, for example embedded in bridging courses. If insecurities concerning mathematical content are connected to relevance attributions, it might help to provide students with success experiences in their studies. Of course, our findings only reflect initial responses. This served our purpose of exploring the suitability of our newly developed measurement instrument but must not be mistaken to be suitable to answer questions about student thinking and their final judgments behind relevance attributions. Interviews with students might give more insights into their actual thinking.

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Promoting Inquiry in Mathematics: Professional Development of University Lecturers

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Abstract. The paper reflects on the Inquiry Based Mathematics Education approach and the professional development of university mathematics lecturers. The main focus is on the design of tasks as a key aspect in lecturer training. The professionalisation activity we have established is rooted in a framework developed by the European project PLATINUM. We describe its basic ideas, the methodological approach and a didactical tool and focus on challenges and issues that arise in the design and implementation of mathematical tasks for an inquiry-oriented teaching. For empirical analyses, we use some data from a professional activity implemented at the Complutense University of Madrid in Spain and carried out jointly by a team from Madrid and Hanover.

Keywords: inquiry approach, professional development, university education

INTRODUCTION

The present study is located at the intersection of two issues: the methodological revitalisation of mathematics teaching towards inquiry-oriented approaches in the classroom, and the reflection about professional development of university teachers in a collaborative way. It is framed within the European project PLATINUM developed by researchers from seven European countries (Gómez-Chacón, et. al, 2021). One of the goals of the project was to develop and pilot a platform for the professional development of mathematics lecturers on a regular basis in the format of a “hands-on” workshop. The need for such a training platform reflects the current situation in university mathematics teaching, where the lecturer has to find a balance between preparing knowledge and reflecting on pedagogic methods, as well as bringing complementary areas of expertise together in subject-didactic considerations. On the one hand, mathematics lecturers often have limited or no access to information about contemporary pedagogical and didactic methods[1], which in turn might contribute to a lack of motivation to use them. On the other hand, lecturers, whose expert field is not mathematics education, cannot be expected to fully immerse themselves in mathematics education research[2] to become experts in contemporary mathematics pedagogy and didactics. We are aware of these boundaries and therefore focus on a developmental process that different tools can foster in terms of knowledge about different teaching methods and how to apply them. This can then also have a positive impact on student performance.

[1] While didactics is a discipline that is essentially concerned with the science of teaching and instruction for any given field of study, pedagogy is focused more specifically on the strategies, methods and various techniques associated with teaching and instruction.

[2] This is due to a number of institutional restrictions, e.g. time, interest, possibility, institutional position, and expectations.

PLATINUM and the workshop we conducted are grounded in, among other things, the idea of creating Communities of Inquiry (CoI) (Jaworski, 2020). Collaborating in CoIs effectively supports lecturers in IBME and fosters their professional development in teaching mathematics, such that teaching of mathematics on the university level supports the aim to achieve students' conceptual learning of mathematics. The theoretical model of IBME in higher education by the PLATINUM project (Gómez-Chacón, et. al., 2021) introduces three levels which all approach teaching and learning through principles of development and interaction.

The first level describes the inquiry in mathematics that is carried out by students and the lecturer in the classroom. Here student – lecturer interactions and student - material (e.g. tasks) interactions are essential. In the second level, lecturers reflect on the processes in the first level. Teaching material, e.g. learning tasks, are designed and adapted collaboratively based on the experiences in the classroom (i.e. first level). The lecturers discuss teaching and learning and give and receive feedback on the design of the learning tasks and their implementation. Together with lecturers, more experienced members and invited experts can promote and support professional development. In the third level, didacticians and educational researchers reflect together with the lecturers on the developmental process that takes place in the first and the second level, which also supports the developmental research. The boundaries between the first and second and second and third levels are crucial nodes in the development process. They are the communication and critical reflection and feedback nodes connecting lecturers and students and lecturers, researchers, and peers respectively. It is at this third level that the tools and reflections presented in this contribution are discussed, offering overviews of theoretical backgrounds for teaching. From this level, we carried out a previous collaborative work of deepening and discussing theoretical and practical aspects of IBME. We intend to support the reflective and evidence-informed teaching attitude of the workshop participants and promote different categories of reflections on the teaching and learning process.

Following this model and taking the institutional restrictions of university mathematics lecturers into account, we focus here on a small-scale approach of task (re)design. The question is how or in which directions tasks could be developed or modified, essentially what makes a task an IBME task. In the international workshops of the PLATINUM project, we dealt with this question in different ways and developed various instruments. For the present paper we choose one of the tools developed and used to characterise and further develop inquiry-based tasks previously at local level in Leibniz University of Hannover (see section LUH workshop in Gómez-Chacón, et.al., (2021)). Two central characteristics of this tool are, on the one hand, its simple and direct usability by lecturers and, on the other hand, it enables to the interpretation of the effects of the tool against an elaborated subject didactic background (here ATD was chosen).

In this contribution, we focus on our experience and data from the development of an International Workshop *Inquiry-based education in mathematics and professional* *development for university teaching* that took place at Complutense University of Madrid (UCM) in 2021, which focused on the implementation of models and materials in Inquiry based mathematics education (IBME) developed in the PLATINUM project. In this workshop we introduced, among other things, the tool to support lecturers and teacher students in developing and characterising IBME tasks (Gómez-Chacón, et.al., 2021). The research question that we try to answer in this contribution is: What leeways and scope for actions could we observe with this tool? What difficulties occurred?

The structure of this contribution is as follows: In the next section we describe the professionalisation workshop jointly conducted at the UCM. In what follows, we then focus on analysing data from this workshop and answering our research questions. To this end, we first address underlying theoretical and methodological aspects. We then present results of our analyses. Finally, we formulate some conclusions.

IBM PLATINUM INTERNATIONAL WORKSHOP

In the following we describe the institutional context and goals, the participants, and the pedagogy of the workshop.

Institutional context and goals

The concept of the PLATINUM workshop held in Madrid was developed with two boundary conditions in mind. First, we had to consider the methods and materials created locally by the three PLATINUM partners (Germany, Norway and Spain), that were developed specially to take the mathematical content into account. Second, the context of Professional development of mathematics teachers and lecturers at the institutional level at UCM. Here, the academic training of novice lecturers was a priority. IBME aspects worked out in the PLATINUM project had to be addressed within existing course structures for PhD students or assistant lecturers. From this point of view, there are principles of effectiveness in professional development that are promoted both in the content and in the training processes used. The following principles are focussed on the workshop: a) To have as a fundamental objective the participants' learning; b) To be based on the mathematical knowledge that the participants must teach and c) To be connected to the teaching practices of the participants, serving as a support for them. So, the objectives of the workshop were:

1) Practical initiation of university lecturers into an inquiry-based approach to teaching and learning of mathematics.
2) Development of methodological skills to (re)design inquiry-based tasks.
3) Knowledge of resources: examples of inquiry-based tasks and -projects in university mathematics teaching.

In the following, we will focus on 2), specifically taking the mathematical knowledge to be taught into account. The specific approach of the workshop was the focus on modifying already existing tasks to become more inquiry oriented. With this focus we also take into account institutional restrictions that lecturers have to face when they would have to develop new material from scratch.

Participants

Fifty participants from 8 different universities or mathematical research centres took part at the workshop. The profiles of the attendees were: new university lecturers of mathematics, mathematics research assistants, mathematical student-teachers of the master's degree in mathematics education[3]. In this contribution, we focus on this last group (21 participants from UCM).

Pedagogy of the workshop

The workshop was organised in the following steps:

Step 1. Task 1, given before the start of the workshop. The participants were given the following task: For the development of the workshop bring one or two math activities that you find interesting or that may be problematic to work with the Inquiry Based-Learning approach in university mathematics teaching. For those of you who are teaching, you can take it from the courses you teach. For those of you who are teacher students, you can take them from manuals and focus on the transition between high school and university or for the first academic year at the university.

Step 2. Information given in the workshop about characteristics of inquiry-based tasks. Working groups were formed and each group chose a task for the group work. In the workshop, presentations about IBME aspects of tasks were given and a didactical tool to help characterising IBME aspects in tasks was provided (cf. next section and Table 1). The task given to the groups was: Which aspects of IBME does your task already fulfil? Which aspects (and why) does/should a modified task fulfil? Prepare a presentation of your group work result.

Step 3. Sharing. Each group presented its modified task in the whole class group.

Step 4. Going back and reflecting. Each group returns to the design of the proposed task and analyses the following aspects in depth: 1) Formulate learning goals for the original task and for the modified tasks; 2) Formulate an expectation horizon for solving the modified task; 3) Fill in the table and explain your choices. What did you find useful for IBME teaching, why did you not use the table, or why would you change it.

THEORETICAL AND METHODOLOGICAL ASPECTS UNDERLYING THE DIDACTICAL TOOL AND THE ANALYSES

Tasks are an important part of learning environments. Lecturers should develop their design skills on IBME tasks and to modify existing tasks in the direction of promoting more IBME. To support professional development of lecturers in this

[3] There are two types of participants, students who are going to teach at secondary level, but also PhD students in mathematics who are doing it for qualification. In Spain there is a figure of university teacher, called "associate" and they are teachers who are going to teach at both levels, secondary and university

respect, one challenge is to reflect about what it means for a task to be an inquiry-based task in mathematics. For this we used a tool developed at the Leibniz University of Hannover to help identifying characteristics of inquiry-based tasks that focusses also on the mathematical content. The following dimensions are considered: a) Openness of the tasks (process open, open-ended, content-open), b) Enabling specific inquiry strategies (mathematical heuristics and developing solutions strategies), c) Enabling discourses on techniques, d) Enabling inner-mathematical knowledge linking, e) Enabling interdisciplinary knowledge linking[4]. The dimensions are compiled in form of a table to facilitate the use by lecturers (Table 1).

Content dimension / Task format	*Inquiry* - **strategies** are to be learned.	A **discourse on techniques** should be stimulated.	**Innermathematical knowledge linking** is to take place.	**Interdisciplinary knowledge linking** is to take place.
Closed				
Process-open				
Open-ended				
Content-open				

Table 1. Table of dimensions for inquiry-based tasks

An important aspect of using this tool is that it is not meant to classify tasks to be IBME tasks or not, so it is not meant for easy assessment. We see IBME-tasks as an open range of possible IBME orientations, were the dimensions above play major roles for (re-)designing tasks that allow to foster (eventually more) inquiry activities by students. The table and its use in the workshop represent a tool to help lecturers to collaboratively reflect about existing tasks, to redesign tasks with respect to the dimensions and to use it as a focus for discussion about tasks with colleagues.

The development of this tool incorporated basic concepts of the *Anthropological Theory of the Didactic (ATD)* (Chevallard, 1999). In the following we will shortly introduce the concept of praxeology from the ATD that will serve as an analysis tool later. In ATD a praxeology is a basic tool to model knowledge in so called "4T-models (T,τ,θ,Θ)". They consist of a practical block (i.e. the praxis, T and τ) and a theoretical block (i.e. the logos, θ and Θ). Bosch and Gascón (2014) concretise:

> A praxeology is thus an entity formed by four components, usually called the "four Ts": a type of tasks, a set of techniques, a technological discourse, and a theory. As activities and knowledge can be described considering different delimitations or granularities, a distinction is made between a "point praxeology" (containing a single type of task), a "local praxeology" (containing a set of types of tasks organized around a common

[4] Due to limitations of space we cannot go into details here. Please refer to (Gómez-Chacón, Hochmuth, et.al., 2021).

technological discourse) and a "regional praxeology" (which contains all point and local praxeologies sharing a common theory). (p. 69)

We will mainly use the concepts of point praxeology and local praxeology to analyse how the participants used the table and what kind of modifications they made to their tasks. The overall idea is to analyse the works of the participants and then draw conclusions for the impact and success. For the analysis of the group work and the interpretation of modifications in view of the table and its dimensions we reformulate them and their aspects in praxeological terms.

A next methodological step consists in embedding the tasks in the respective curricular context, outlining praxelogical landscapes and locating the initial tasks and the modified tasks in this map. Due to limitations of space, we will not go into the methodological details here, but we will illustrate the last aspect with an analysis vignette considering examples from the group results. We would like to notice that the participants did not know the ATD, nor was it our aim for them to learn about this theory. The restriction of the table to terms that seem technically evident, i.e. that can be used "superficially", makes sense in view of the participants and the goals of the workshop. We introduced the dimensions and gave short explanations in the workshop but didn't stress a precise definition of each dimension. They are formulated quite openly and such that there is room for interpretation by the participants of the workshops. This openness is deliberate because we did not want to focus on teaching the underlying ATD background of the dimensions but on promoting discussions among the participants. The meaning of the dimensions is allowed to be adapted to the context relevant for the participants. How the participants interpreted the meanings of the dimensions is also of interest.

RESULTS

In the following we present the results, regarding the dimensions of IMBE represented in the table and the usefulness of the table as a didactical reflection tool.

Inquiry-based tasks and developed dimensions

There were five groups, each (re)designing a task. Three groups focused on Solving Systems of Equations, one on the Derivative of Functions and one on the Concept of Vector Space. The transformation of the tasks essentially consisted of:

- a transition from specific more or less straight forward calculations in view of tasks which address several independent punctual praxeologies (focussing mostly on techniques), to extended tasks addressing local praxeologies, i.e. tasks mobilising also technology aspects of the content (describing, validating, questioning, exploring etc.) and fostering a common technological discourse.

- a problematisation from fixed procedural knowledge to a deeper understanding of definitions and concepts; the tasks were extended to focus on inner-mathematical knowledge linking that aims at connecting different mathematical concepts and overcoming the compartmentalisation of mathematical knowledge (Kondratieva & Winsløw, 2018).

Example 1: Derivative of a Function

The original task is a point praxeology with task T to “calculate the derivative of a function”, that addresses the techniques of applying rules for derivatives. Technological aspects are not addressed by this task. Group 4 modified this to a task with several subtasks addressing different approaches and changes of representation, e.g. analytical term to graph or table, graphical differentiation[5]. The subtasks become more and more open. Inquiry strategies are interpreted by this group as the use of mathematical heuristics (see Polya or Schoenfeld for other forms of inquiry strategies). Inner-mathematical knowledge linking is understood as changing of representations and accompanying activities (drawing the graph, calculating the derivative graphically). Each subtask, again, represents a point praxeology. But in the course of all subtasks, also technological aspects are addressed that support the overall forming of a local praxeology. Some examples are the request to compare different approaches used and to explain the properties of the studied function based on the results from previous subtasks. The modified task from Group 4 also shows potential for further development of IBME aspects, that are not explicitly realised in this step of the task development. We see both, the progress of Group 4 and the potentials of our ATD based analysis method as very satisfactory.

> Task: Study the derivative of the function f(x)=(x^3-7x+6)/(x+5) with different approaches and using different tools. a) Calculate the derivative of this function: f(x)=(x^3-7x+6)/(x+5); b) Using the definition of derivative, get the derivative again. Compare the result with a); c) Draw the graph of the function, observe the maxima and minima and calculate its derivative at those points; d) Generate a table with the value of the derivative at each point calculated graphically using slopes. Plot this data on a new graph.; e) What are the properties of the function (deduced from its derivative)? f) Try to factor the polynomials of the function and then try to derive taking into account this factorization.

Example 2: Solving Systems of Equations

This topic was worked on by three groups (Group 1, 2 and 3). It is interesting to see the variety of approaches to the same topic by the three groups.

Taking the (re)designed task from Group 2, the original task was a point praxeology with the task T to “find the solution of a system of equations using a specific method”, that address the technique of application of a concrete method. It was modified and opened up in several subtasks with:

- *respect to solution path*: it is proposed to find the solution of a system of equations previously explaining several methods and to let the students use the method they want to solve the problem

[5] We present the tasks developed by the Groups in the supplementary appendix.

- *respect to solution/result*: it is formulation of a system of equations which is compatible indefinite and propose each group to find a solution which in several cases would be different.
- *respect to knowledge used*: solving a system of equations geometrically and finding the relation with the real problem (giving meaning of the variables).
- *respect to knowledge applied*: geometrical interpretation: plane intersection (inner-mathematical knowledge linking). Science applications: balancing chemical reactions (interdisciplinary knowledge linking).

The result of the modification is the integration of various point praxeologies into local praxeologies. Interesting here is also the interpretation of the group, that openness of a task also means freedom of the students to choose.

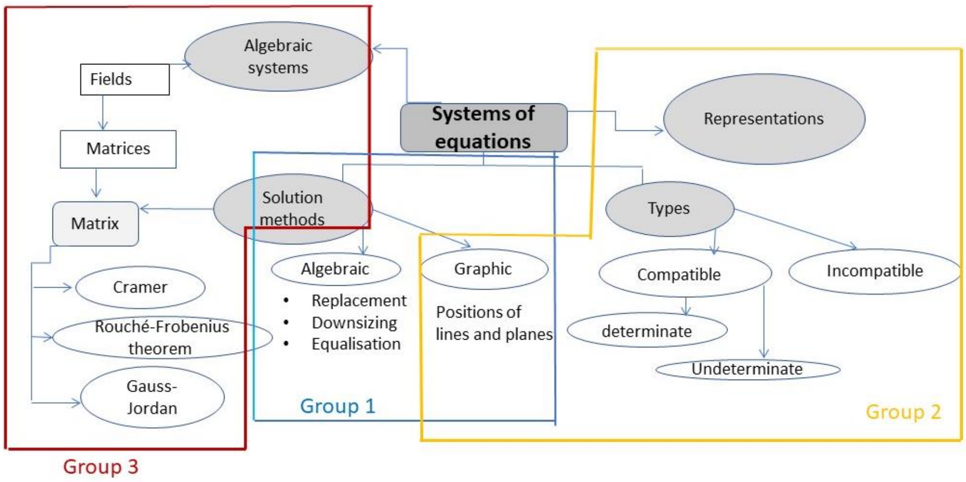

Figure 1. Concept map of the topic

Comparing all contributions of each of the three groups, we find that there is an evolution of the task and the related mathematical domain. The analysis of the conceptual structure of the topic gives rise to the concept map[6] shown in Figure 1, in which four fundamental concepts are organised: representations, types of systems, solution methods and algebraic systems. Some relationships are established between the concepts shown and the semantic structure of the problems. As an example of the relationships established between the concepts, we take the solution methods. These methods are divided into matrix, algebraic and graphical[7]. For the concepts associated with the graphical method, the tasks modified emphasise that concepts, algebraic representations of equations, types of systems of linear equations and the semantic structure of problems are the basis of the study when dealing with the graphical method. For some groups, the design of IBME tasks also entailed the possibility of questioning the curriculum and the predefined organisation of the content. The more or less strict organisation of subtasks seems at first to contradict the possibility of

[6] We also use a concept map instead of a praxeological map for space reasons, to provide a rough overview of the diversity of solutions among three groups. Even though they work on the same topic, they follow quite different paths.

[7] This is associated with the relative position of the lines and planes on the plane or on the space according to the dimensions of the system and the means in the determinate, indeterminate compatible systems or to the incompatible systems definition.

opening up tasks towards more IBME. But for those groups this was a method to keep up the compatibility of task and curriculum.

Table-tool valorisation

The workshop focused on work on inquiry dimensions of mathematical content. Thus, the investigation targeted mathematical knowledge, in particular with the development of technological aspects (from point- to local praxeologies). In view of a particular task, the table used helped to generate questions about its IBME characteristics and further development possibilities. Four of the five groups use the table, even the group that did not use it highlighted its benefits. In their own words: "*We did not use the table as such, but rather talked about the different contents that were in it, since in our case, the opening of activities only occurs in the last section… It is true that there can be several techniques, especially in the simplification of the matrix in section d, which allows us to affirm that in terms of techniques it is an open process*". The table was not simply applied by the students but provoked intensive group discussions. There were also difficulties that provoked reinterpretations of terms in it, that were interesting for the analysis. The researchers had ATD Theory as a background, so there are many aspects that are implicit and not transparent to the user. However, the analyses show that using the ATD conceptualisations that went into the development of the table also as an analysis tool for the students works, we see a lot more potential in the tasks and possible further developments than the students were able to produce. This is not a deficit of the students. The results are remarkably good for a short workshop with only a short introduction into the table, and without any introduction into the theoretical background. This also promotes the question how much of the theoretical background could be introduced to develop this kind of workshop and instruments further. This is an open balancing act between too much and too little theoretical background. And, so to speak, a basic problem for the further development of professional development workshops.

CONCLUSIONS

In our final section we bring all of the above together, addressing how the development of the workshop and the use of to support lecturer in developing and characterising IBME tasks opened up avenues of professional development. We have indicated the ways in which our work has related to PLATINUM's three-level model. The observations made can be interpreted against this background. The modification of the initial tasks has led to an increase in the complexity and depth. In the design, the participants tried to state the concepts according to the prescribed curriculum and to establish connections between different registers (geometric, algebraic) and regional praxeologies that are usually less explicit in the lessons. Regarding the three levels mentioned in the introduction the following aspects can be outlined: a) Students: Expansion of the ability to act, explicit addressing of technological/conceptual aspects, etc. b) Lecturers: Requirements for lecturers, pre-structuring etc., necessary or helpful, the exchange on how small-step and guiding necessary and helpful etc. c) Research: concept of training etc. but also needs for further research with regard to the modified task (beyond anecdotal evidence).

The tool table is one example of promoting connections between layers 3 and 2, i.e. between research and teaching - but not only the table alone. Also, its embedding in the overall workshop. However, these interactions must be balanced in the sense that we as researchers (layer 3) cannot "drag" lecturers into this layer (sometimes researchers are also lecturers, we also reflected on this double position in Ruge & Peters (2021). These results encourage to develop research-based instruments like the table and the workshop for professional development that promote collegial and collaborative reflection and discussion.

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Mathematicians and experienced teachers: crossing the boundary

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Academic mathematics (AM) has a central role in the preparation and professional development of secondary mathematics (SM) teachers, yet in recent years there is growing evidence that realizing the affordances of AM for SM teaching is not straightforward. This study is part of a long-term research program named M-Cubed, that seeks to investigate the processes involved in utilizing AM for informing SM teaching. In M-Cubed, mathematicians and experienced SM teachers watch videotaped SM lessons and discuss teaching alternatives. This study explores how, in this setting, mathematicians and teachers make sense of authentic teaching moves and decisions, how they invite each other to adopt new perspectives, and how these invitations are met. Findings provide insight into the social boundary between mathematicians and teachers, and into how opportunities for learning through crossing this boundary may be realized. We conclude by discussing possible implications for research and practice.

Keywords: knowledge for teaching, advanced mathematics, teacher education, cross-community encounters, boundary-crossing.

INTRODUCTION

Teachers' engagement with academic mathematics (AM) and interaction with mathematicians are key components in the secondary mathematics (SM) teacher preparation programs of mathematics teachers in many countries (Tatto et al., 2010). The literature suggests various potential benefits of teacher-mathematician encounters for SM teachers. For example, it has been suggested that such encounters may support the development of teachers' Horizon Content Knowledge (Ball & Bass, 2009; Wasserman, 2018); develop teachers' understandings *of* and *about* the discipline of mathematics (CBMS, 2012; Even, 2011); and inform instructional decision-making (Cooper & Pinto, 2017; Wasserman & McGuffey, 2021). Nevertheless, empirical studies have indicated that in practice, AM studies may only have a limited impact on SM teaching, thereby suggesting that it is far from trivial to translate knowledge of AM into knowledge for teaching SM (Biza et al., 2022; Zazkis & Leikin, 2010).

Relatively little is known about the processes involved in translating AM knowledge into knowledge for teaching SM, in part because such processes are often tacit and highly personal, in the sense that they are inspired by experiences of individual teachers in their particular AM studies (Zazkis, 2020). Research suggests that these translation processes build not only on deep understanding of AM but also on knowledge for teaching SM, as well as on knowledge about connecting AM and SM teaching (Dreher et al., 2018; Pinto & Cooper, 2022; Wasserman et al., 2018; Wasserman & McGuffey, 2021). In addition, Cooper and Pinto (2017) suggest that such a translation may manifest in back-and-forth moves between different perspectives on mathematics, its teaching and its learning, mutually enriching each other. All this suggests that realizing the potential affordances of AM for SM teaching is far from trivial, and that teacher education should look beyond development of teachers' AM knowledge in order to support and encourage teachers' in utilizing AM in practice, specifically in the context of teacher-mathematician interaction (Biza et al., 2022; Wasserman et al., 2018).

This study is part of a long-term research program, named M-Cubed, that investigates processes involved in utilizing AM for SM teaching (Pinto & Cooper, 2022). In M-Cubed (***M****athematicians*, ***M****athematics teachers*, ***M****athematics teaching*), small groups of mathematicians and experienced SM teachers view videotaped SM lessons and jointly inquire into mathematical issues and pedagogical dilemmas that they recognize therein. This research setting can be seen as a laboratory for generating and studying implications of ideas and perspectives from AM, for the consideration of authentic instructional situations. Within M-Cubed, the present case study investigates processes underlying realized and unrealized opportunities for utilization of the mathematicians' perspectives and knowledge for informing SM teaching. We conceptualize and study these processes in terms of *boundary crossing*, as we discuss in the next section.

THEORETICAL BACKGROUND

The design and conduct of M-Cubed are informed by the literature on boundary-crossing, which characterizes dialogical learning processes in cross-community interactions (Akkerman & Bakker, 2011). They define *boundaries* as a social and cultural discontinuity in action or in communication. Researchers have suggested that teacher-mathematician interactions may be framed as boundary encounters (Goos & Bannison, 2018; Pinto & Cooper, 2018, 2022). Pinto and Cooper (2018, 2022) have stressed that discontinuity in these interactions is manifested not merely in disagreements between the two parties but rather in tacit, incommensurable perspectives, which are difficult to recognize and make explicit, and that such discontinuity can hinder communication and collaboration between mathematician and teachers. Specifically, in terms of the *practical rationality* of teaching (Herbst & Chazan, 2020), mathematicians tend to have a strong obligation to the discipline, i.e., to mathematical precision and rigor, even when faced with pedagogical dilemmas, while teachers tend to have a strong obligation to students and their wellbeing and to the functioning of a class as a social entity. These different obligations can lead teachers and mathematicians to endorse conflicting courses of action. Pinto and Cooper (2018, 2022) have shown that such tensions can challenge both parties to engage in boundary crossing (Akkerman & Bakker, 2011), namely, to make explicit and possibly reconsider their positions, thus making public and visible their processes of exploring, elaborating and refining potential utilizations of AM for informing SM teaching.

The literature indicates various factors that may encourage and support boundary-crossing (Akkerman & Bakker, 2011). For example, productive communication across discursive boundaries can rely on *boundary objects* (Star, 2010), that is, objects that are, on the one hand, robust enough to maintain a common identity across communities working with them, yet, on the other hand, flexible enough for the parties to interact with them differently. In M-Cubed, the videotaped SM lesson episodes function as boundary objects. The episodes are selected by the researchers from the archive of the VIDEO-LM Project (Karsenty & Arcavi, 2017). Selection is based on various criteria that help in identifying potential opportunities for boundary crossing. The research question that guided this case study was: What are the opportunities for boundary crossing in teacher-mathematician interactions, and how may they be realized?

METHODOLOGY

Data for this study consist of 7.5 hours of videotaped discussions in three M-Cubed sessions (2.5 hours each) held in 2020, with the participation of five mathematicians (herein marked as M1-M5), five SM teachers (marked T1-T5), and two mathematics educators, the first and second authors of this paper. All of the participants expressed interest in, and openness to, the project's premise of learning from and with one another about the affordances of AM for teaching SM. All the mathematicians and teachers have more than a decade of teaching experience. Three of the five mathematicians have taught mathematics courses for teachers. All five teachers have experience as students in AM courses. The first session of the three documented for this case study was held in-person, and the other two were conducted remotely due to the COVID-19 pandemic. Within data reduction, we analyzed plenary discussions that took place before and after participants watched the videotaped SM lessons, as these discussions were particularly rich in cross-interactions between teachers and mathematicians. These discussions were fully transcribed and selectively translated to English by the authors.

The data analysis had three focal points: the emergence of a boundary in the M-Cubed discussions; opportunities for boundary crossing; and the realization of these opportunities. The first stage of analysis focused on identifying potential boundaries. We examined how different participants interpreted the goals of the joint inquires and how they approached them: the dilemmas they addressed, claims and observations they made, and justifications they provided. We then considered how different speaking turns relate to one another, not only in the sense of whether the participants agree or not, but also in the sense of being comparable, in terms of the questions that are being explored, and the implied grounds for endorsing or rejecting answers. The second stage of analysis focused on identifying opportunities for boundary-crossing. Here we looked for explicit or implied invitations from a participant to participants from the other community to make explicit and explain tacit aspects of their perspectives, or to adopt a new perspective. Finally, in the third stage of analysis, we explored whether and how invitations for boundary crossing were accepted, guided by different processes of dialogical learning identified in the literature, such as reflection and hybridization (Akkerman & Bakker, 2011; Pinto & Cooper, 2018, 2022).

FINDINGS

Our analysis highlighted various manifestations of boundary and boundary-crossing. In this paper we focus on one particular instance from the M-Cubed sessions that illustrates both realized and unrealized opportunities for boundary crossing. We first provide an overview of the videotaped lesson episode that the participants discussed, and of mathematical ideas that were explored or implied in this discussion. Then, we discuss the boundary that emerged between the teachers and the mathematicians, three opportunities for boundary crossing that were unrealized, and one opportunity that was.

A student's unexpected question on Apollonian circles

The M-Cubed session discussed herein revolved around a short exchange between a SM teacher, whom we will call Ms. L., and three SM students in a 12^{th}-grade advanced-track class. The exchange took place in an Analytic Geometry lesson, after the teacher presented and solved the following problem:

What is the locus of points whose distance from (4,0) is twice their distance from (1,0)?

The answer is an Apollonian circle of radius 2 centered at the origin (see Figure 1).

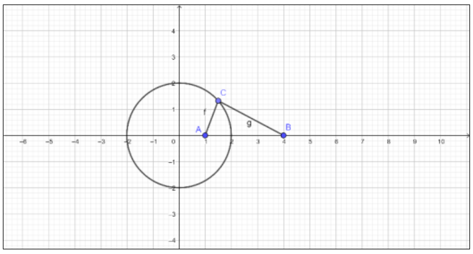

Figure 1: **The locus of points whose distance from (4,0) is twice their distance from (1,0)**

After Ms. L. concluded the solution, a student, whom we will call Ophir, asked a question, which triggered the following short exchange:

Ophir	Can I define circle in this manner?
Ms. L.	You're asking if it will always come out to be a circle?
Ophir	Of course, it will.
Ms. L.	You all heard his question? Can he define a circle according to this property?... If the points are not (4,0) and (1,0)?
Daniel	It depends on the relationship between the two points.
Ms. L.	What if they are general points? (k,0) and (p,0)?
Gefen	Why zero?
Ophir	I can take those two points and do this (gestures a translation), and the circle will simply move together with them.
Ms. L.	Ok. This will be part of what I will assign, as general proofs. Your questions are excellent, I just want to know if it will always be a circle, and if it will always be [centered at the origin]. I see you're starting to guess. And now a new question.

Before moving to the M-Cubed discussion, we offer a few observations. It is important to note that in Ophir's question – "Can I define a circle in this manner?" – the term *in this manner* is not clear. It appears that Ophir suggested generalizing the solution in some manner, but exactly how is open for interpretation. For example, Ophir may have meant that two points in the plane, other than (4,0) and (1,0), may be considered. Ms. L. seemed to have interpreted Ophir's question as referring to *any* two points on the x-axis, and Gefen and Ophir generalized even further to *any* two points on the plane with the same y value. It is also not clear what Ophir meant by *define*. Ophir may have referred, as Ms. L.'s response seems to suggest, to the existence of a sufficient condition ("it will always come out to be a circle"), but he may have implied a necessary condition, or a definition, i.e., a sufficient *and* necessary condition. Thus, there are many different ways to derive a general mathematical statement from Ophir's question. Here are a few options:

- For any two given points *A* and *B* on the plane, the locus of points whose distance from point *A* is twice their distance from point *B* is a circle.
- For any positive *k*, the locus of points whose distance from *(4,0)* is *k* times their distance from *(1,0)* is a circle.
- For any positive *k* and for any circle *C* with radius *k* on the plane, there exist points *A* and *B* on the plane such that *C* is the locus of the points whose distance from point *A* is *k* times their distance from point *B*.
- A circle is the locus of the points whose distance from some point *A* is *k* times their distance from some point *B.*

Notably, some statements are true, and some are not. In the discussion between Ms. L. and her students, various statements were implied, but none were made fully explicit.

Unrealized opportunities for boundary crossing

After watching this exchange between Ms. L and her students, the participants engaged in exploring alternative responses to Ophir's question. The conversation involved 57 talk turns among two mathematicians and four teachers. At first, even though all the discussants engaged with the same task, two distinct conversations took place. In one conversation, the mathematicians made mathematical observations while suggesting interpretations to Ophir's questions. In the second conversation, the teachers made observations about the classroom environment, while considering merits and risks of alternative teacher responses. These conversations were distinct in the sense that even though the mathematicians and teachers took turns and appeared to be responding to one another, they did not address questions and observations raised by discussants from the other community, as the following talk turns exemplify:

90	M1	By the way, the student at the end who realized that it is possible to obtain all circles in this manner, it would be very interesting to unpack his thinking, what he fixed and what he changes in his head.
101[1]	M2	At first, [Ms. L.] did not even understand what [Ophir] was asking. She said: "You ask if it will always turn out that way". That is not what he asked at all. He asked if it could be taken as a definition of a circle.

[1] Some turns were omitted to maintain coherency, as evident in the turn numbering.

104	T4	It's a dilemma I have in nearly every lesson, especially with high-ability students. There will always be one or two [students] who will drag me there. It comes with a price. My first inclination is to play along, but then I lose all the rest [of the students].
105	M1	And if Ophir would have been told "let's find the parameterization, just be careful of straight lines". Is it too dangerous?
107	T1	Why not put it to everyone? "This is what he asked, what do you think?"
111	T3	[…] They could benefit from it.

We consider these distinct conversations as an indication of a boundary. The teachers and the mathematicians appeared to be interpreting the goals of inquiry differently and to be utilizing distinct types of knowledge, consequently leading the two communities to develop insights separately instead of learning from and with one another.

Our analysis identified three unrealized opportunities for boundary crossing, where implicit invitations made by discussants to the whole group, to examine a specific issue from a particular perspective, were not picked up by members from the other community. For example, in turn 90, M1's comment "it would be very interesting to unpack his thinking" can be seen as an invitation for an exploration of different mathematical statements that could underlie Ophir's question, yet in the turns that followed, the teachers did not react to this invitation. In turn 105, M1 asks a question that is within the teachers' area of expertise ("Is it too dangerous?"). In doing so, M1 invites teachers to examine a concrete teaching reaction, albeit from a perspective focused on how the mathematical discourse could be developed in the classroom. Here again, M1's invitation is left unanswered. Conversely, in turn 107, T5 invites all participants to consider a possible teacher response ("Why not put it to everyone?") without specifying what mathematical idea could be addressed. This question initiated a discussion among the teachers about how different students may respond (turns 108-127), without any participation on the part of mathematicians.

A realized opportunity for boundary crossing

Although M1's first two invitations for an exploration of what mathematical statement could underlie Ophir's question were left unanswered, he continued raising this issue. In turn 128, M1 circles back to Ophir's intentions:

128	M1	It's not clear what [Ophir] meant. That is, I would really like to know what quantifiers he had in mind. Maybe in high school it's difficult to expect it […], but in university I would have said: okay, tell me "there exists such and such", phrase it as a clear logical statement. What do you mean?

Here, M1 interweaves the two different conversations. He places a focus on the mathematics but also elaborates a concrete teaching response and questions its appropriateness for SM, while addressing his own teaching approach as a university lecturer. Presumably, this interweaving may have been the trigger that led the teachers to respond to M1's third invitation, leading to a boundary-crossing event where M1 and the teachers built on each other's statements in a joint exploration. This exploration started with one teacher somewhat contesting M1’s comment (“Isn’t it clear what he meant? To us as teachers?”), which led M1 to further elaborate his perspective:

131	M1	Is it clear to you? I mean, is it that for any circle there exist two points such that the ratio [of distances] is one to two? Or that for any circle there exists [a ratio] such that for these two points [this is the ratio of distances]? What is [Ophir] quantifying on?
132	T1	The way I understood it is: can you define a circle in this manner.
134	M1	What is “in this manner”?
135	T1	With two points…
139	T3	In the same form.
140	T4	The distance from A is twice the distance from B.
141	M1	Perhaps not twice the distance? Perhaps [the ratio] is a parameter?
142	T3	Not necessarily twice.

We observe that in this short exchange, the teachers started considering various interpretations to Ophir’s question, addressing different ‘parametrizations’ and ‘quantifiers’ as M1 suggested, such as the relative position of the two points (turn 139) or the ratio between the distances (turns 140-142). Thus, the teachers crossed the boundary, no longer focusing only on pedagogical risks and merits, but actively investigating Ophir’s question from a mathematical perspective. Whereas this conversation was cut short, and the teachers and the mathematicians did not follow through and explicated the possible interpretations of Ophir’s question, nor have they derived pedagogical implications, there are still indications that this brief boundary crossing was meaningful for the teachers, for example by helping them to make explicit why they appreciated Ophir’s question. On several occasions, teachers remarked that Ophir’s question is ‘beautiful’ but without elaborating why. Following this exchange, one teacher remarked:

147	T5	Still, it is a beautiful question because [Ophir] knows one definition [for a circle] and the question was whether the same shape can be defined in another way.

Here, T5 observes that the beauty of the question is linked with the fact that a student, who already knows one definition for a circle, tries to establish another definition. The word "still" implies that even if the student may not have thought of all the mathematical possibilities that the discussants started to unpack, the question is nevertheless beautiful in exploring alternative, equivalent definitions.

DISCUSSION

Our aim in this study was to investigate *what* opportunities for boundary crossing may rise in teacher-mathematician interactions that revolve around concrete SM instructional situations, and *how* these opportunities are realized. The analysis we presented, of a joint exploration between mathematicians and experienced SM teachers regarding possible reactions to a student’s unexpected question, has revealed both realized and unrealized opportunities for boundary crossing.

The study highlights mathematicians' and teachers' different ways of making sense of instructional situations and inquiring into possible pedagogical dilemmas that these situations may introduce, for example in terms of what they notice and how they interpret what they notice. Our analysis demonstrates how these differences may create a boundary within teacher-mathematician interactions. When exploring possible alternative reactions to Ophir's question, at first the conversation divided. Even though the teachers and the mathematicians engaged in the same task, they interpreted the task differently, approached it from different perspectives, and responded to members from their own community. In particular, the teachers rarely engaged in mathematical discourse, whereas the mathematicians expressed less interest in translating their interpretations and observations into practical pedagogical insights. This dynamic does not encourage cross-interactions and thus limit opportunities for boundary-crossing.

However, our analysis also suggests how this dynamic may change. On several occasions, discussants made explicit invitations for the group to inquire into possible reactions to Ophir's question from a specific perspective. At first, these invitations were confined to one perspective – of the teachers' or of the mathematicians' – and were only accepted by members of the same community. But when an invitation by a mathematician interweaved both perspectives, it seemed to have 'broken' the boundary, leading teachers to adopt, even if only momentarily, the mathematician's perspective on the instructional situation, leaving aside the question of possible merits and risks of different reactions to Ophir's question, and exploring instead how the question can be interpreted mathematically in different ways.

Our study underlines the profound difference between interaction and boundary crossing. Previous studies have emphasized the importance of teachers' interaction with mathematicians and of teachers' exposure to the ideas and perspectives of AM (CBMS, 2012; Wasserman et al., 2018). From this perspective, the M-Cubed session we investigated may seem highly beneficial for teachers, since the mathematicians and teachers appeared to be engaged in cross-interactions and responded to one other in what may seem, at least from a surface examination, as a joint inquiry. However, careful analysis of this session reveals that teacher-mathematician interaction, even when it is focused on SM teaching and even when it includes exchange of ideas and perspectives, may not suffice for a meaningful engagement of the teachers with AM or with mathematicians' perspectives. Such interaction can be visualized as a movement – of the mathematicians and the teachers – on two parallel lines of inquiry, leading to different answers to different questions, and thus not providing a real possibility for the two sides to agree or disagree with one another, and learn from their cross-interaction.

Finally, we make three observations on the boundary-crossing event we presented. First, this event focused on different ways of 'mathematizing' a student's question so to unpack implicit mathematical ideas that could be addressed in a response to the question. However, this inquiry did not materialize into pedagogical insights and concrete responses, and more research is necessary to determine whether this particular inquiry represents a more general form of utilization of AM for SM teaching, and what could be the affordances of such utilization. A second observation is that this boundary-crossing event involved a shift of the teachers from their own inquiry to that of the mathematicians. In this study, such shifts were much more common than shifts of the mathematicians from their inquiries to those of the teachers. Further research may help to shed light into this observed phenomenon. A third observation relates to the key role of brokering in teacher-mathematician interaction (Akkerman & Bakker, 2011; Pinto & Cooper, 2018, 2022). As we have shown, opportunities for boundary crossing can be subtle and nuanced, thus could be easily missed. Further research of the realized and unrealized opportunities for boundary crossing in teacher-mathematician interactions can help facilitators of such interactions to identify and utilize boundaries that may otherwise remain implicit, and proactively encourage boundary-crossing.

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Statistical modelling in the Brousseaunian guessing game: A case of teacher education in Japan

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This paper argues that inquiry into what we call the Brousseaunian guessing game evokes statistical modelling where various concepts and methods in inferential statistics are integrated into relatively large knowledge organisation. To achieve this, an implementation of inquiry-based learning was conducted in a course of teacher training programme in a Japanese university, and we analysed the process of inquiry by using some notions offered by the anthropological theory of the didactic. In particular, the concept of modelling was formulated by the notions and thereby the description of the modelling process revealed that the probabilistic and statistical knowledge was developed meaningfully for the students in their autonomous inquiry.

Keywords: Teaching and learning of specific topics in university mathematics, novel approaches to teaching, inquiry, reference praxeological model, trophic relation.

INTRODUCTION

This paper reports a result of a *didactic engineering* conducted within the research program of the *anthropological theory of the didactic* (ATD, hereafter). In the didactic engineering, an inquiry-based statistics course was designed with the intention of examining the possibility of what we call the *Brousseaunian guessing game* as an inquired material. The guessing game, whose original setting was presented in Brousseau, Brousseau, & Warfield (2002), is a situation given by the following description:

> There is a box containing a large number of red and white marble balls with equal proportions, and we have three empty plastic bottles whose inside cannot be seen from outside (figure 1). Now, five balls are picked out and put into one of the bottles without anyone looking at the balls. We do this for the other bottles. Under these conditions, the followings are allowed for each of the bottles: 1) Drawing out a ball from the bottle to check the colour of the ball and put it back; 2) Repeating the first operation.

Figure 1. The bottle.

The original game was designed to teach probability and statistics for elementary school students. It was referred as one of a typical teaching environment to enable students to learn the piece of knowledge involved—namely, *fundamental situations*. In contrast, the present study aims to investigate what probabilistic and statistical knowledge can be developed by university students through inquiring into questions raised from the guessing game.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

To investigate the possibility of the guessing game, we need some models that enable us to describe the process of knowledge development in the inquiry. In this paper, the notion of *praxeology* offered by the ATD is used to do this. Praxeology, a term made of the *praxis* and *logos*, is an epistemological model of any human activity, which consists of the quadruple of a *type of tasks* T, *technique* τ, *technology* θ and *theory* Θ. A praxeology is denoted by $\wp$, that is, $\wp = [T / \tau / \theta / \Theta]$. The pair of a type of tasks and a technique is called the *praxis block* of a praxeology. On the other hand, the *logos block* of a praxeology consists of the pair of a technology and a theory. The praxis block and the logos block of a praxeology correspond to *know-how* and *knowledge* of a human activity, respectively. Therefore, a praxeology is an epistemological model for grasping both any human practice and knowledge.

Let us introduce here a theoretical construct that has not been used very often in the ATD for describing the growing process of praxeologies, that is the notion of *trophic relation* (cf. Chevallard, 2022). Any praxeological entity $\wp_0$—which is able to not only represent a praxeological organization $\wp$, but also stand for its part metonymically, i.e., T, τ, θ, or Θ—can get larger and more sophisticated through integrating other praxeological entities $\wp_1, \wp_2, ..., \wp_i$. Such relations of $\wp_0$ to $\wp_1, \wp_2, ..., \wp_i$ are called the trophic relations—$\wp_1, \wp_2, ..., \wp_i$ are the *foods* of $\wp_0$. The case of the trophic relation of $\wp_0$ to $\wp_1$ is denoted by: $\wp_0 \hookleftarrow \wp_1$, where the left arrow with hook "$\hookleftarrow$" means *eating*. Here, a *praxeological food* is anything that is taken in for evoking a larger praxeology from the original praxeology, and all such taking in is called *eating*. Foods can come from various sources, such as books, the internet, teachers, and other media, as well as from knowledge students already have. The sequence of the eating by $\wp_0$ of $\wp_1$, $\wp_2$, $\wp_3$ in succession can be denoted by: $[[\wp_0 \hookleftarrow \wp_1] \hookleftarrow \wp_2] \hookleftarrow \wp_3$. Let us emphasize here that this diagram should be read as the following: $\wp_0$ successively eats $\wp_1$, $\wp_2$, and then $\wp_3$. Such succession of trophic relations is actualised through the time flow with different *trophic moments* like the moment for "eating $\wp_2$". In this paper, we regard a series of trophic moments as a process of *modelling*. Namely, any *praxeological eater* $\wp_0$ functions as a *system* to be modelled, as well as the eating by $\wp_0$ of a praxeological food $\wp_1$ brings about a new *model* $\wp_0'$ of the system $\wp_0$: $\wp_0' = \wp_0 \oplus \wp_1 = [\wp_0 / \wp_1]$. This means that the model $\wp_0'$ of the system $\wp_0$ has the praxeological entities which belong to $\wp_0$ or $\wp_1$. Any whole process of such praxeological genesis starting from $\wp_0$ implies the *trophic span* of $\wp_0$: $\rho(\wp_0) \stackrel{\text{def}}{=} \{\wp \mid \wp_0 \hookleftarrow \wp\}$. Precisely speaking, the elements of a trophic span are dependent on reference instances $\hat{\imath}$ (for example, students), according to whom—denoted by $\hat{\imath} \vdash$—, a given praxeology eats other praxeologies: $\rho_{\hat{\imath}}(\wp_0) \stackrel{\text{def}}{=} \{\wp \mid \hat{\imath} \vdash \wp_0 \hookleftarrow \wp\}$.

In this way, any process where the mathematical praxeologies further develop is understood as *mathematical modelling* in the ATD (cf. Garcia et al., 2006). We note here that modelling is not limited to activity that mathematises some non-mathematical systems. Regardless of whether the system is in an extra- or intra-mathematical problem, the activity is identified as a modelling if the tasks are accomplished through constructing some models of the system. Let us add short, but crucial remarks about the expression "statistical modelling" in this paper. We do not use this expression in its usual sense within statistics education research, which seems to refer to some *methodological* flow of modelling process (e.g., the so-called PPDAC cycle). By contrast, within the ATD, the word "modelling" means the genetic process of praxeological organizations from a more *epistemological* point of view.

In light of the trophic relation, our spontaneous and ambiguous question about statistical modelling is refined and defined as the following: To what extent can the trophic span involved in the guessing game be developed in our setting, and how can the trophic moments progress?

A REFERENCE PRAXEOLOGICAL MODEL OF THE GUESSING GAME

In the ATD, a *reference epistemological model* of the concerned praxeologies—that is, a *reference praxeological model*—has to be built before designing and implementing some teaching experiments (cf. Bosch & Gascón, 2006). This is because we are trying to describe didactic phenomena by letting us emancipate ourselves from the dominant epistemology about the praxeologies. Otherwise, since such a dominant epistemology is taken for granted and conditions our view of understanding didactic realities in some way, we cannot capture what occurs, what the characteristic is, and especially what does not occur.

Our reference praxeological model $\wp_{ref}$ for the statistical modelling consists of two praxeologies—the experimental one $\wp_{exp}$ and inferential statistical one $\wp_{IS}$: $\wp_{ref} = \wp_{exp} \oplus \wp_{IS}$. The first one $\wp_{exp}$ starts with the type of tasks of *estimating the population proportion reliably*, which is denoted by T_0. This can be accomplished by different techniques, but at the beginning of the inquiry, it would be carried out by *a lot of experiments* (τ_{exp}). Although τ_{exp} can be justified by some technological-theoretical elements such as the *principle of convergence of relative frequencies* or the *law of large number*, this kind of justification is provided seldom unless being asked to explain the technique purposely. Accordingly, $\wp_{exp}$ is described as $\wp_{exp} = [T_0 / \tau_{exp}]$.

The second one $\wp_{IS}$ consists of two sub-praxeologies—the classical statistical one $\wp_{CS}$ and Bayesian statistical one $\wp_{BS}$: $\wp_{IS} = \wp_{CS} \oplus \wp_{BS}$. Both of them begin with the awareness of the fact that although the reliability would increase as the number of experiments increases, the experiments must be stopped at some point. This limitation brings about the question of "how do we reliably and economically judge the contents of the bottles?" Then, this would lead to the type of tasks T_1 of *determining the minimum sample size required for* T_0 and T_2 of *accomplishing* T_0 *with as small sample size as possible*. While the characteristics of T_1 is that the sample size is predetermined *before* starting experimentation, that is not the case with T_2. Accomplishing the type of tasks T_1 and T_2 evoke $\wp_{CS}$ and $\wp_{BS}$ respectively. In other words, in terms of trophic relation, T_1 and T_2 can eat the praxeological entities of $\wp_{CS}$ and $\wp_{BS}$ respectively.

We can assume various technical means such as *hypothesis testing*, *interval estimation*, and *maximum likelihood estimation* for T_1. Let us denote them by τ_{hyp}, τ_{int}, and τ_{lik} in order. For example, τ_{hyp} would be used for judging whether the balls in the bottle are all red or four red balls and one white ball are in the bottle. When judging that it is not (3, 2) but (4, 1), τ_{int} can be put in use—hereafter, the combination of the colour of the balls is denoted as an ordered pair of the number of the red and white ball(s). The technique τ_{lik} would be also used in this kind of subtype of tasks. These techniques can be supported by two technologies: one is the technology θ_{dis} that has various *probability distributions* such as *binomial distribution* and *normal distribution* as its elements; the other one is θ_{con} that consists of the definition and properties of *conditional probability*. The techniques τ_{hyp} and τ_{int} are justified by θ_{dis}, on the other hand, τ_{lik} is explained by θ_{con}. For these technologies, the *definitions of probability and random variable* and *basic properties of them* belong to the theory Θ_{pro} as the theoretical elements. Thus, $\wp_{CS}$ is described as $\wp_{CS} = [T_1 \,/\, \tau_{hyp}, \tau_{int}, \tau_{lik} \,/\, \theta_{dis}, \theta_{con} \,/\, \Theta_{pro}]$. As for $\wp_{BS}$, the technique τ_{Bay} of the *Bayesian update* would be used because some successive means are required for T_2. This technique is also supported by θ_{con} and therefore, Θ_{pro} functions as the theory of $\wp_{BS}$: $\wp_{BS} = [T_2 \,/\, \tau_{Bay} \,/\, \theta_{con} \,/\, \Theta_{pro}]$. Bayesian statistics is a praxeological organization whose key technique is the repetitive update of probability based on Bayes' theorem and newly gathered data. In contrast with classical statistics, that allows us to accomplish statistical inference without the samples of sufficiently large size. T_1 is more classical statistical, while T_2 is more Bayesian statistical.

Let us add here about the possible paths in the process of modelling in $\wp_{IS}$ in terms of the *dialectic of questions and answers*, which fundamentally promotes any inquiry (cf. Chevallard with Bosch, 2019). As mentioned in the above, τ_{hyp} of $\wp_{CS}$ would emerge as an answer when questioning "how do we judge it is (5, 0)?" This question comes from the situation where we pick out a red ball many times in a row from the beginning of the experimentation. In this case, τ_{hyp} consists of probability of repeated independent trials. Also, τ_{lik} of $\wp_{CS}$ can be available to the question "what population is the most likely to produce the obtained sample?" if the students have learned the conditional probability by then. On the other hand, τ_{int} of $\wp_{CS}$ needs to be told or instructed by some *media*, which is something with the intention of teaching. However, it is natural to refer such media when the students encounter the question "how do we estimate the population proportion?" and then they would be able to learn the technique. As for τ_{Bay}, we assume that the students will not be able to use this technique without any direct instruction by the teacher(s). This is because few of basic statistics textbooks commonly used in Japan cover the Bayesian inference. However, it is also assumed that once directed, τ_{Bay} is fully available because many concepts and techniques on the Bayesian update can be brought about into the *milieu* from various media such as statistics textbooks for general readers and YouTube, etc. In addition, when the questions about those techniques—"what is *p*-value?" and "where does '1.96' in the estimating interval come from?" for example—arise, the answers would be found or built in the logos block $[\theta_{dis}, \theta_{con} \,/\, \Theta_{pro}]$ of $\wp_{IS}$. In particular, inquiry into questions about τ_{hyp} and τ_{int} mostly leads to θ_{dis}, while inquiry into τ_{lik} and τ_{Bay} generally leads to θ_{con}. The figure 2 shows the reference model.

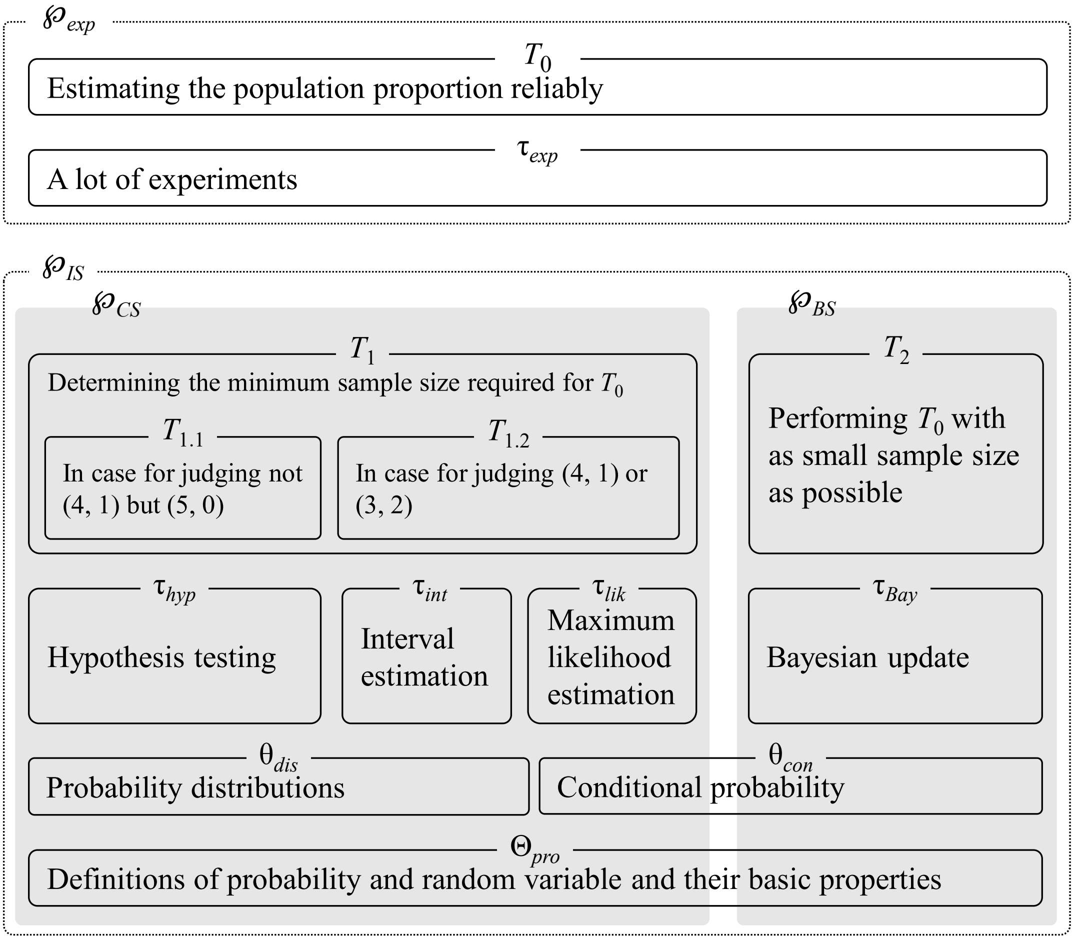

Figure 2. Our reference praxeological model of the guessing game.

CONTEXT OF THE COURSE

The implementation was conducted in the second term in 2020 for four students who were taking the training program for secondary mathematics teachers. Two of them were third, one was fourth and the other was graduated students. They had learned probability theory and statistics in the university by that time. However, they had not acquired the knowledge to apply the techniques and technological-theoretical elements mentioned in the reference model on their own. The course which was the part of the teacher training program consisted of 13 sessions of 90 minutes. All sessions were audio and video recorded. The teacher was the first author of the paper.

In the first session, the explanation of the aims and outline of the course was given. Firstly, the focus on the statistics and inquiry-based learning was introduced with reference to the current educational situation. More precisely, the aims were applying statistics in inquiry into some questions and, by doing so, developing probabilistic and statistical knowledge meaningfully for the students themselves. Then, the teacher explained the “guessing game of the contents of the bottles” to them.

For the setting of the guessing game, the following initial question was presented to the students: "How many red and white balls are in each of the bottles?" Before starting the inquiry, the teacher emphasised that any predetermined knowledge as the goal of the course did not exist and that whatever the students needed was available throughout the course, including the internet and the library. In other words, there were neither a sole goal nor a single path determined by the teacher. We had intended to update the *didactic contract* that the students spontaneously and implicitly signed, because it is assumed that they had been familiar with the conventional *didactic paradigm*—that of *visiting works*. If the emphasis was not done, the students would not inquire the questions but explore what is expected for them as the "inquiring activity". In short, we wanted to avoid possible negative effects from the old contract, while making the new contract of *questioning the world* explicit.

The outline of the course was as follows: the part of the first inquiry (4 sessions); midterm presentation (1 session); second inquiry (4 sessions); analysis of the entire inquiry (2 sessions); final presentation (1 session). In both the midterm and final presentation, the students made the presentations for three mathematicians and two mathematics education researchers in the university, who were not involved in the course. In designing the course, we had intended that each presentation would be an opportunity to promote the inquiry, or more precisely, the modelling. This is because the questions, comments, and critiques from them to the techniques and/or technological-theoretical aspects must lead the students to reconsider the praxeologies they developed. This evokes the praxeologies that takes the existing praxeologies as the systems. Therefore, the opportunity of making the presentations and getting the feedbacks on them was expected to promote the modelling activity.

In this paper, we describe and analyse the statistical modelling in terms of the trophic relation and trophic moments, focusing on the part of the first inquiry, midterm presentation, and second inquiry.

PROCESS OF THE REALISED STATISTICAL MODELLING

First inquiry (session 2 to 5)

In the session 2 and 3, in response to the initial question, the students tackled the task of *guessing how many red and white balls are in the bottles*, which corresponds to T_0 in the reference model. Some students took 70 times drawings to make their judgment, and other students said that at least 15 to 20 times are needed. The latter students also claimed that they thought it was enough to take 20 times to make a judgment. We can see here the trophic moment: $T_0 \hookleftarrow \tau_{exp}$. Namely, the praxeology grew into $\wp$ at this point: $\wp = [T_0 / \tau_{exp}]$. According to that, the question "how many times do we need to judge the contents reliably?" arose. For this question, the teacher taught that a value of 95% is commonly used as the level of the required reliability.

In this phase, the students encountered an interesting case: red balls appeared 18 times in a row from the beginning, and then a white ball appeared at the 19th drawing for the first time. Since the students had been thinking that the contents of the bottle were (5,

0) by the 18^{th} drawing, they were surprised at this result. Considering that such a case might be quite rare, they began to find the probability that the case happened under the hypothetical setting that the contents of the bottle were (4, 1). As a result, they found that the probability was less than 2% and concluded that it was very rare case. This is the spontaneous use of the idea of hypothesis testing. They successively tackled the subtype of tasks $T_{1.1}$ of *determining the minimum sample size to judge that the contents are not* (4, 1) *but* (5, 0). Namely, the situation or milieu evoked the trophic moment which brings about the more sophisticated praxeology $\wp'$: $\wp \hookleftarrow T_{1.1}$ and $\wp' = [T_0, T_{1.1} / \tau_{exp}]$. Technically speaking, $\wp'$ should be denoted as $\wp_{IS}'$, but "*IS*" is omitted for simplicity. Then, by applying τ_{hyp}, they built their own answer "if the red balls are put out 14 times in a row from the beginning, we can judge that the contents are (5, 0) with 95% reliability". We can see that the moment when $\wp \hookleftarrow \tau_{hyp}$ and the construction of $\wp''$ occurred here: $\wp'' = [T_0, T_{1.1} / \tau_{exp}, \tau_{hyp}]$. It must be noted that this is a possible abuse of the significance level. The students found $(4/5)^{14} < 0.05$ and wrongly concluded that the "probability" of not being (4, 1) was more than 95% according to the calculation of 100 – 5 (%). They implicitly used the following statement which is generally incorrect: $P(A|B) = P(B|A)$. We will mention about the abuse of the significance level in the last section.

In the subsequent sessions, the term of hypothesis testing was introduced by the teacher with reference to τ_{hyp}. The students then tried to apply this technique further to define a criterion for *judging whether the contents are* (4, 1) *or not*. They constructed $\wp'''$ as the product of the moment when $\wp \hookleftarrow T_{1.2}$: $\wp''' = [T_0, T_{1.1}, T_{1.2} / \tau_{exp}, \tau_{hyp}]$. However, they were unable to obtain the solution because of the increased number of variables in the combinatorial calculation. Regarding the introduction of the term, it had been expected not only to facilitate the application of the technique, but also to function as encountering with the logos block of the existing praxeologies. In fact, the students referred to some websites and studied the procedures of hypothesis testing. Nevertheless, the terms "null/alternative hypothesis" and "*p*-value" were not used to properly explain τ_{hyp} in preparing the midterm presentation. This would be because they were not familiar with the statistical concepts, such as random variables and probability distribution etc., in the descriptions.

Midterm presentation (session 6) and second inquiry (sessions 7 to 10)

In the midterm presentation, the students reported on the progress of their inquiry up to the point to the university teachers. Then, the teachers provided the feedbacks by raising the questions and making the comments. In the comments from them, the technological-theoretical elements that are directly related to $\wp'''$ appeared. One of the teachers said that the situation of the guessing game could be interpreted by using binomial distribution. Another teacher suggested that the technique the students used could be sophisticated by utilising the terms regarding the procedures of hypothesis testing and the concepts of the probability distribution. These are the elements of θ_{dis}. In addition, there was a comment that Bayes' theorem could be applied to the guessing game. This is an element of a different technology θ_{con} that is not directly related to the technique τ_{hyp} but connected with τ_{Bay}. The students did not understand all of these feedbacks on the spot. They were to learn them as needed in the subsequent inquiry.

In the phase of second inquiry, the students referred to some media in response to the feedbacks and attempted to learn the elements of θ_{dis}. They were not able to sufficiently understand them because the connection of the statistical concepts with the problematic situation of the guessing game could not be recognised. Namely, they could not find how to model the situation in terms of random variables. However, the very fact that the elements of θ_{dis} were needed to justify and explain τ_{hyp} was understood. In this sense, we can see here the moment when $\wp \hookleftarrow \theta_{dis}$; that is, the praxeology grew into the larger one: $\wp^{(4)} = [T_0, T_{1.1}, T_{1.2} / \tau_{exp}, \tau_{hyp} / \theta_{dis}]$. Moreover, while referring the media, one of them found a description about the interval estimation in a mathematics textbook. The student got to know how to use this technique to tackle $T_{1.2}$ with the help of the teacher. Let us note that the only a part of the theoretical aspects was explained to the student in the phase. For example, the teacher taught what the population and sample proportion were in this case. Then, the technique of interval estimation was shared within the group by the student, that is, the moment when $\wp \hookleftarrow \tau_{int}$ occurred and the more complexed praxeology was constructed: $\wp^{(5)} = [T_0, T_{1.1}, T_{1.2} / \tau_{exp}, \tau_{hyp}, \tau_{int} / \theta_{dis}]$. To be specific, a table was created in Excel with the number of drawing and the sample proportion as variables, and the 95% confidence interval of the population proportion was output. According to the result, the students found that 97 drawings were necessary and sufficient to uniquely estimate the contents of the bottles.

As for the response to the feedbacks on Bayes' theorem, the students also studied it by using some media—some video contents from YouTube in particular. Then, one of them learned the theorem based on conditional probability and understood how to find posterior probability from the assumed prior probability and obtained data. In fact, the student was able to calculate the posterior probability of being $(n - 5, n)$ for $n = 0, 1, \ldots, 5$, under the condition she actually got from the experimentation. However, when being asked whether the technique would be useful for the guessing game, she answered that it looked useless because "To me, the idea seems to be that there are bottles of all types.Then, one thinks 'this is the bottle with *that* type because this [posterior] probability was the highest'. This is not the situation of this problem, so I think this idea is hard to adopt". This explanation can be attributed to the gap between the tasks of the classical statistical type they were working on at that time and possible tasks of the Bayesian statistical type which is evoked by the technique τ_{Bay}. The other students agreed with the decision not to use either τ_{Bay}. According to the students X, $\wp$ did not eat such a praxeological entity in the inquiry: $X \nvDash \wp \hookleftarrow \tau_{Bay}$. Finally, the findings mentioned above became their final answer.

Summary

Let us here summarise the answer to the research question, which is concerned with praxeologies that can be constructed in inquiry into the guessing game. We have seen the praxeological organisation $\wp^{(5)}$ was finally developed from $\wp = [T_0 / \tau_{exp}]$. This indicates the trophic span $\rho(\wp) = \{T_{1.1}, T_{1.2}, \tau_{hyp}, \tau_{int}, \theta_{dis}\}$, as well as the range of the constructed knowledge $\{T_0, \tau_{exp}\} \cup \rho(\wp)$. Comparing $\wp^{(5)}$ with the reference praxeology $\wp_{ref}$, the entities integrated in θ_{dis} ($T_{1.1}$, $T_{1.2}$, τ_{hyp}, and τ_{int}) are included in $\rho(\wp)$, while those in θ_{con} (T_2, τ_{lik}, and τ_{Bay}) are not. In addition, $\wp^{(5)}$ did not have the theoretical discourse Θ_{pro}. The following descriptions make the comparison clear:

$$\wp_{ref} = [T_0, T_{1.1}, T_{1.2}, T_2 \,/\, \tau_{exp}, \tau_{hyp}, \tau_{int}, \tau_{lik}, \tau_{Bay} \,/\, \theta_{dis}, \theta_{con} \,/\, \Theta_{pro}],$$

$$\wp^{(5)} = [T_0, T_{1.1}, T_{1.2} \quad /\, \tau_{exp}, \tau_{hyp}, \tau_{int} \qquad /\, \theta_{dis} \qquad / \qquad].$$

Regarding the progression of the modelling, it has been already shown in the above section. In terms of trophic relation or trophic moments, it can be summarised as follows: $[[[[[T_0 \,/\, \tau_{exp}] \hookleftarrow T_{1.1}] \hookleftarrow \tau_{hyp}] \hookleftarrow T_{1.2}] \hookleftarrow \theta_{dis}] \hookleftarrow \tau_{int}$. Each moment when a praxeological entity eats another one leads the progression of the modelling. As we have seen, questioning how to accomplish the tasks, applying the techniques used in the previous cases, referring the media, and receiving the feedbacks through disseminating their own findings played the essential roles for the moments.

Among these actions, *studying new works* is important, though it may be paid little attention in inquiry-based pedagogy. In the ATD, a process of visiting some pieces of knowledge *associating with* their *raisons d'être* is called *study and research activity* (SRA). Any authentic inquiry has the action of studying as a part of or a form of it. The students studied the procedures of hypothesis testing, how to apply interval estimation, and what is Bayes' theorem, etc. In short, they constructed their statistical praxeologies through interaction with these works obtained from the media. Since the study was undertaken for the purpose of answering the questions, the praxeologies had their *raisons d'être*, and therefore, a statistical SRA developed in our case.

Furthermore, what could be called the *democratic relationship with knowledge* was observed (cf. Chevallard, 2006). The students selected knowledge to be used, that is, praxeologies to be eaten, by themselves. We can see it in the situation where they judged that τ_{Bay} could not be applicable for T_1 despite the teacher's instruction. In ordinary classes, students usually receive knowledge taught by the teachers and attempt to make use of it. The decision observed in the implementation is the result of renewing the didactic contract and indicates that autonomous inquiry has taken place.

FINAL REMARKS

In this last section, we want to point out two didactic precautions for emancipating teachers from traditional preoccupations in the reproduction of inquiry into the Brousseaunian guessing game. First, the following fact should be recognised in considering supervision of inquiry: direct instruction of knowledge can be effective in some cases. Teachers may expect students to learn everything possible on their own in supervising inquiry. This leads to the teachers avoiding directly teaching the concerned praxeological entities even if it is reasonable. In fact, the teacher of our implementation did not actively teach the technological-theoretical elements of the praxeologies. Here we see a didactic phenomenon that could be called *refraining from direct instruction*.

However, as mentioned in the above, the praxeologies developed to answer the questions have their *raisons d'être*. Thus, even if some works involved in the logos blocks are taught by the teacher, they would not be *monumentalised*. Accordingly, it is suggested that not to refrain from instructing more than necessary is important.

Another suggestion is concerned with a phenomenon that can be called *abuse of praxeological entities*. We have seen that the students naïvely replaced the (conditional) probability—which is, de facto, the *likelihood* in this context—with its inverse probability. Teachers tend to feel guilty when they overlook misunderstandings of their students. Indeed, mistakes are usually regarded as *antididactic* events under the paradigm of visiting works. By contrast, under the paradigm of questioning the world, any possible flaw involved in knowledge needs not always be avoided in advance; in fact, it can be welcomed in inquiry as *probably didactic* events. This is because the failure can be recognised later, and this facilitates the inquiry by deriving new questions on the mistakes. Since there are many distorted explanations of different works in media especially about statistics (at least in Japan), this phenomenon is likely to be common. We should not have to be afraid of error, but to consider how to take advantage of them for making moments of promoting inquiry.

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Student Teachers' Development of Introductory ODE Learning Units - Subject-specific and further Challenges

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What mathematical knowledge do student teachers for grammar schools have at the end of their studies and how do students succeed in linking their knowledge with subject-didactic considerations in the development of online learning units? This paper reports observations on both questions from the context of subject-specific and subject-didactic courses in German Master programs. The observations show that students generally have considerable difficulties in using standard knowledge from Analysis and Linear Algebra lectures. The linking of mathematical knowledge with subject-didactic considerations also poses considerable challenges. In view of the findings, we will argue that it might be helpful to broaden the view in research regarding transition issues and, in particular, to consider institutional-societal conditions.

Keywords: teacher education, ordinary differential equations, online learning units, subject-specific knowledge, institutional-societal conditions.

INTRODUCTION

After more than 100 years, Felix Klein's dictum of the double discontinuity has not lost its relevance. Both, Klein's critical diagnosis of the actual teaching of mathematics in schools and that, in order to improve the situation, it is the task of the university to train future teachers as well as possible, still seem to be true. Klein's goals have been taken up in various ways in the last decades, for example, concerning capstone courses near the end of university studies (Winsløw & Grønbæk, 2014): Students should be shown connections between university mathematics they have already learned in order to use them meaningfully as a resource for their professional lives. Klein's suggestion of specific bridging courses at the beginning of university studies is also being taken up in many places in Germany by a variety of measures (Hochmuth et al., 2022).

A premise of Klein's (1908) Elementary Mathematics form a Higher Standpoint course is that the students are well versed regarding university mathematics and were, in principle, able to represent mathematics as a science in its own right to grammar school students. Now, in this respect, it can be asked whether the actually learned knowledge from the basic first year courses of Analysis and Linear Algebra is in fact available to students in such a way that they can use it flexibly and adequately in order to acquire new mathematics that is closely related to school mathematical knowledge. This requirement addresses both techniques and rationales of content and concepts, such as representations of functions or of solutions of linear equations, derivatives as rates of change or tangent slopes. In addition to the availability of subject-specific knowledge, it is also a question of whether and how students, who have already attended substantial parts of their compulsory courses in subject didactics and pedagogy, can use this mathematical knowledge in preparing and reflecting online learning units. Surprisingly, the state of knowledge and the related subject-didactic ability to act with regard to academic mathematics strongly linked to school mathematics among student teachers, who have more or less completed their studies, has not been investigated in detail.

This paper reports and discusses observations regarding those questions. We consider Master courses, in which students are asked to prepare online learning units on selected topics about Ordinary Differential Equations (ODEs) and accompanying essays reflecting the units from a subject didactical point of view. Elementary aspects of ODEs are a suitable choice because they use concepts from both Analysis and Linear Algebra in a way that has strong links to subject-didactic reflections of school mathematics. In addition, there are many contexts of use, for example in physics or biology, with models that can be assessed as school-related (e.g. harmonic oscillator, pendulum, and predator-prey models). Last but not least, there is a wide variety of literature that presents ODEs at different levels and didactically diverse ways: deductive and concept orientated (e.g. Hirsch et al., 2012), application-oriented (e.g. Bryan, 2021) or even inquiry oriented (Rasmussen et al., 2018). The courses considered here represent an opportunity to exploit the potential of advanced mathematics in mathematics teacher education addressed by Hochmuth (2022), opening up a view of mathematics that most student teachers do not encounter in current courses. One important didactic goal of the Master courses was the exploration and learning of the subject-specific preparation of mathematical knowledge for online learning units that are oriented towards concepts of inquiry-orientated learning (Artigue & Blomhøj, 2013; Jaworski, Gómez-Chacón & Hochmuth, 2021). Of course, it is also an empirical question which potentials students actually realise under the current restrictive institutional-societal conditions of study programs and the contradictions induced by this in the relationship between learning opportunities and learning resistances. Our qualitative analysis of the students' developments provides some evidence with regard to those issues shedding light on the state of knowledge and subject-didactic abilities.

The contribution is structured as follows: The next section provides information on the teaching-learning context of the courses. In particular, mathematical and subject-didactic pre-knowledge, the literature provided and knowledge taught about ODEs, and the objectives of the learning units to be developed by the students are addressed. In the following section, in view of the above reflections, two research questions are formulated and briefly outlined regarding theoretical backgrounds and methodological issues. Observations regarding the research questions are then presented in the successive sections. In the concluding section, results are discussed and possible conclusions as well as further research issues are addressed.

TEACHING-LEARNING CONTEXT AND DATA

The observations reported are about Master courses for student teachers for grammar school at two German Universities. All students had successfully attended not only the basic courses on Analysis and Linear Algebra, but also courses about Numeric, Stochastic, Geometry and Algebra. Specific pre-knowledge about ODEs was only available for a few. Such knowledge then came, for example, from the teaching of the exponential function in Analysis or of normal forms for matrices A and observations regarding e^A in Linear Algebra, or from their eventual second study subject Physics or another science. Therefore, introductory teaching units about ODEs were provided, which were mainly based on application-oriented literature (Bryan, 2021) and on inquiry-oriented presentations (Rasmussen et al., 2018; Gómez-Chacón et al. 2021). Fundamental theorems about unique solvability, the continuous dependence of initial conditions and parameters, and the stability of hyperbolic equilibrium points were mentioned and illustrated, but not proved. Instead, emphasis was placed on phase diagrams (in 1D and 2D), detailed phenomena-oriented treatments of linear systems (esp. equilibrium solutions, asymptotic behaviour) as well as applications such as the harmonic oscillator, the string pendulum, and predator-prey systems. Following the IODE course (Rasmussen et al., 2018), the notion of rate-of-change equation and directional- and vector fields were dominant. Elementary tasks mobilising changes of representation (terms, solution curves, phase diagrams) and their respective use and interpretation in application contexts were focussed.

In the first half of the semester, the described contents were taught in lectures with integrated exercise units. From week to week, the tasks of the exercise units were to be worked on and, in addition, a few tasks introducing new contents should be explored. In the second half of the semester, eight groups of three to four students each were accompanied in the development of online learning units. The learning units to be developed should cover introductory and slightly advanced topics including linear systems (with emphasis on 2D), harmonic oscillator (modelling various phenomena including resonance, possibly double oscillator), predator-prey models and bifurcations (in 1D and in 2D exemplified by pitchfork and Hopf bifurcation). In addition to the literature already mentioned, specific parts from (Chow et al., 2012; Hirsch, Smale & Devaney, 2012) and diverse internet resources, such as a school project work on the justification of periodic solutions of the Lotka-Volterra model, were provided or referred to. In addition, possibilities for the use of digital tools like GeoGebra and Applets in Wikis were introduced in the first half of the course. Moreover, subject-didactic concepts for the preparation of mathematics for teaching units in general (Barzel et al., 2012; Hußmann & Prediger., 2016) and inquiry-orientated units in particular were recalled or introduced (Winter, 1989; Jessen, 2017) with specific foci on representations and types of linking mathematical knowledge.

The exam consisted of the preparation of an online learning unit and an accompanying essay. Based on subject-didactic reflections, the essay should describe, explain and justify the respective preparation of the mathematical content and the methodical-didactic design of the learning unit. The following observations mainly refer to the integrated exercise units, the development of the online learning units and the units themselves. The accompanying essays are considered insofar as design elements to be recognised in learning units are reflected with respect to their overall inquiry orientation.

RESEARCH QUESTIONS

Based on the data described in the preceding section, two research questions are considered in the successive sections. In addition to formulating the research questions, we also sketch the respective theoretical backgrounds. The data analyses have been guided by qualitative content analysis (Gläser & Laudel, 2009).

RQ 1: What students' deficits from Analysis and Linear Algebra can be observed in the exploration of knowledge about ODEs?

The deficits concern not only factual knowledge about mathematical objects, symbols and their definitions, but especially techniques and rationales and their interrelationship. In particular, it is about concepts, such as the concept of derivative or the flexible use of different representations of functions. Our view is oriented towards the 4T-model of the Anthropological Theory of the Didactics, although analyses in this respect are not be made explicit in this contribution, if only for reasons of space.

RQ 2: What kind of linking of mathematics to application contexts using ODEs can be observed in the student's elaborations?

One focus of the courses was on the mathematical description of basic phenomena that could be assigned e.g. to Physics or Biology. Dynamic and structural properties of the phenomena should have motivated the use of concepts from Analysis and Linear Algebra. Otherwise, phenomena also served to interpret and validate achieved mathematical results. The students were familiar with the basic structure of the modelling cycle and respective modelling tasks presented in school as well as from their subject-didactic courses. Our theoretical background is complemented by insights from studies on the use of mathematics in engineering and beyond (Hochmuth & Peters, 2021; 2022).

ON RQ 1: SUBJECT-SPECIFIC DEFICITS REGARDING ANALYSIS AND LINEAR ALGEBRA

Students were given the task of relating rate-of-change equations to representations of slope fields and to justify their assignments. Considerable hurdles were observed both in the interpretation of the equations and in their representation by slope fields. Regarding for example equations like $\frac{dy}{dt} = t - 1$ (Rasmussen et al., 2018, p. 1.5) students could hardly detach themselves from drawing the right-hand side function itself into a coordinate system and then integrating it directly. Students failed in sketching the slopes, that are, in this case, independent of y (!), or, finally, in sketching the solutions that depend on a constant. Also expressions like $\frac{dy}{dt} = y^2 - t^2$ (Rasmussen et al., 2018, p. 1.5) turned out to be rather difficult. Students got confused by the simultaneous occurrence of the variables t and y. Basically, in each point

(t, y) of the coordinate system a straight line with unit length and the right slope must be drawn. Against the background of the drawn slope field, it is then a matter of detaching oneself from point-by-point interpretation of both the symbolic and the iconic representation and of passing over to local or global conceptions of functions, i.e. to consider y as a function dependent on t, or to think about graphs of functions whose tangents correspond to the drawn straight line segments of the slope field.

The tasks require a flexible handling of punctual, local and global perspectives on functions and their respective iconic and symbolic representations. In this respect the teacher students seem to be on a similar level of knowledge as college students, targeted by the IODE material, although they already completed their mathematical study. Against the background of Klein's second discontinuity, subject-specific knowledge underlying didactic considerations in the sense of basic ideas of derivatives and functions hardly seemed to be available. From this observation, it is comprehensible why in the development of the online learning units but also in lesson plans for schools about e.g. derivatives, didactic-methodical considerations are seldom substantiated by subject matter, but instead pedagogical considerations dominate (Hochmuth & Peters, submitted).

While students were essentially able to deal with vector fields in the generic case of planar linear ODEs with constant coefficients, i.e. cases where the matrices possess linear independent rows and the origin is the unique equilibrium solution, and to use in such cases publicly available GeoGebra applets, the planar system $\frac{dx}{dt} = -3x - \frac{1}{2}y$, $\frac{dy}{dt} = 6x + y$ leads to severe hurdles in getting an overview of the solutions and in particular about the equilibrium solutions. No group was able to come up with a complete answer. This task also originated from the IODE course (Rasmussen et al., 2018, p. 10.13): The respective goal is to explore what happens if the matrix belonging to the right side has linear dependent rows, i.e., the kernel of the linear mapping is a one-dimensional subspace (the trivial case of a zero matrix is excluded from considerations), which geometrically represents a straight line through the origin consisting of equilibrium points. The other non-equilibrium solutions can then be represented by straight lines parallel to each other intersecting the straight line of equilibrium points and, since the corresponding eigenvalue is -2, converge to the intersection points. Of course, the task could systematically be solved by a more or less canonical approach. But such an approach was only sketched in the lecture and not trained. The intention of giving this task was to enable students to combine knowledge about 2D-matrices (or linear mappings) with geometric and analytical considerations. On the one hand, such ideas already play a role in school mathematics and, on the other hand, they form the subject-specific basis of related didactic considerations. Unlike college students, the student teachers have successfully passed courses that provide such knowledge in several contexts (multivariable Analysis, Linear Algebra, Analytical Geometry). However, it was hardly possible for the students to use such knowledge in this new context. Of course, one does not really need such taught knowledge to explore the given situation, instead might reflect about it in an elementary and direct way, but this was not possible either. The latter is the concern of the task in the IODE context: The insight into such phenomena should motivate to take a closer structural look and to discover structural reasons for such solution patterns. Conversely, the goal of basic university lectures like Analysis and Linear Algebra was to provide structural knowledge for ordering such phenomena systematically. But neither such a transfer, nor successful ad hoc explorations appear in the students' works.

A particularly rough case of misunderstanding related to Linear Algebra showed up in the following assertion presented in a submitted learning unit: The linear system $x' = Ax$ is solvable if $\mathrm{rg}(A) = \mathrm{rg}(A|x')$ holds. Obviously the later could formally be noticed, but actually makes no sense.

In the situation of considering elementary situations of bifurcations the following nonlinear system was considered (Hirsch et al., 2012, pp. 162):

$$x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 + y^2 x), \qquad y' = x + \frac{1}{2}y - \frac{1}{2}(y^3 + x^2 y).$$

With respect to this system, the linearisation at the origin (0,0) should first be determined and the corresponding local phase diagram be sketched. No group of students could find the linearisation directly from the given equations, which hints on a missing conceptual understanding of, e.g., the Taylor expansion in higher dimension. Instead, they tried (many calculation errors occurred) to calculate the Jacobi matrix formally. The eigenvalues are $1/2 \pm i$, which means that the Hartman-Grobman-Theorem is applicable and locally the phase portrait could be sketched, i.e., locally the solutions of this system spiral away from the origin. Then the students should transform the system to polar coordinates (r, θ), which leads to

$$r' = \frac{r(1 - r^2)}{2}, \quad \theta' = 1.$$

Neither during group work through the course nor as homework a correct solution was worked out by any student. For some of the students, hurdles start with the fact that they did not realise that the chain rule had to be applied. And if this was recognised, it could not correctly be executed. The chain rule for functions of several variables is a standard content of Analysis 2 and polar coordinates are typically treated at the latest in the context of the substitution rule for multidimensional integration. Interestingly, students with Physics as second subject also failed. Because of all these hurdles, the interesting global structure of the solutions of the nonlinear system, which would remain undiscovered by focusing only on the local considerations around the origin, could not be appreciated and not adequately presented in developed learning units. Likewise, it could not be appreciated that such insights were possible by (rather simple) qualitative considerations in the context of phase diagrams, in particular without explicitly calculating solutions of the (in the beginning) complicated-looking system of differential equations.

Summarising, the presented examples show considerable technical deficits, which made it impossible to acquire and represent rationales in the somewhat more advanced field of ODEs. Generally, the students could not rely on conceptual knowledge with respect to either multidimensional Analysis or Linear Algebra. Both in the course and its exercise units and in the elaboration of the learning units, these deficits obstructed an appreciation and appropriate use of illustrative representations such as vector fields and phase diagrams. This suggests that students moved away from inquiry oriented and rationales-focused presentations of ODEs in their own elaborations of learning units and returned to small-stepped and calculation-focused elaborations.

ON RQ 2: USING MATHEMATICS IN APPLICATION CONTEXTS AND MODELING

ODEs allow to place mathematics in the context of everyday but also in physical, biological, chemical or technical contexts. The literature used in the course presented extensive chapters on topics from these areas and demonstrated how the qualitative approach focused on in the course lead to interesting insights, often without complex calculations and usually without solving the equations explicitly, which in fact is often not possible in the case of ODEs. Several application contexts are also used in the IODE material to promote and motivate a basic understanding of concepts and mathematical relationships. Thereby, concepts such as rate-of-change equations and tools like phase diagrams, allow students to understand the mathematical concepts even without explicit knowledge from application fields. They potentially enable students to get insights in dynamic interrelationships and, above all, phenomena to be modelled on the basis of everyday ideas. However, it is also obvious that basic mathematical deficits as addressed in the previous section make such epistemic processes significantly more difficult, which can be seen in several rather problematic derivations of models in the developed learning units.

An interesting observation in this context is that insight into mathematical deficits was averted by the teacher students by locating the hurdles instead in insufficient knowledge of application fields, such as Physics. This might be related to characteristics of known modelling cycles (Blum & Leiss, 2005) and dominant ideas of “applicationism” (Barquero et al., 2011), which suggest that modelling essentially takes place in an extra-mathematical world: If modelling does not work out, then, of course, it is due to missing knowledge in the extra-mathematical world. In other words: Underlying ideas about the role and the use of mathematics in application contexts turn out to be a kind of ideological obstacle, here possibly for the purpose of psychical relief in view of the experience of failure. Of course, this is a hypothesis which has to be reviewed in further research.

Conversely, short-circuited argumentations transferring directly from modelling contexts to mathematics can be found: in learning units and accompanying essays we find a lack of noted needs for justification and proof of such argumentations. Thus, besides the phenomenon of a strong separation between context and mathematics, we also found phenomena of instant identifications, hence an implicitly assumed identity of reality and mathematical model. Related to this, there is also the phenomenon that mathematical results are sometimes directly applied to reality. The possibility that, in addition to explicit assumptions made, further and uncontrolled assumptions are included in models, and that every calculated and proven property thus also represents a validation possibility of the respective model as such, is only acknowledged in principle, but rarely considered in practice.

Summarising, any finely woven interweaving of various mathematical discourses related to slightly different ways of talking and doing mathematics as well as justifying validity (Hochmuth & Peters, 2021), which could and should also have subject-didactic relevance in school contexts, cannot be observed.

DISCUSSION AND OUTLOOK

There is only few research so far about the academic mathematics knowledge of teacher students at the end of their studies and its availability for developing subject didactically reflected learning units. This paper focuses Master's programmes in which teacher students have to mobilise knowledge from basic lectures in a way that is professionally relevant both to the subject and the subject-didactic. Thus, these courses are in the context of the second discontinuity addressed by Klein. The focus of the reported observations was on (non-) available knowledge from introductory courses about Analysis and Linear Algebra as well as their use in inner and extra-mathematical contexts. Regarding subject-specific knowledge, there are considerable deficits with respect to both techniques and rationales. It seems that a central premise of Klein's concerning his Elementary Mathematics from the Higher Standpoint, namely reliably available university knowledge, is hardly given. Problematic claims of a life-world orientation expressed by the students contribute to questionable results with regard to the use of mathematics even in the context of simple application situations. Moreover, the subject-specific deficits at least add to the fact that teaching materials that have been clearly prepared in the sense of an inquiry orientation and in which ideas are to be introduced and used in a concept-oriented way are transformed by the students into small-stepped, calculation-oriented learning units.

In recent years, the focus in university mathematics education research has been on the first discontinuity, i.e., the transition from school to university. Against a broad background of theoretical and empirical analyses, a wide variety of measures has been developed and established (Hochmuth et al., 2021; Hochmuth et al., 2022). However, if one looks at our results also from the point of view of observations in (Hochmuth & Peters, submitted), where we reflected on problematic aspects of the prevailed societal determined formation of learning processes, one is led to the following questions: Are the approaches and orientation of those measures adequately specified? The observed deficits do indeed show that students can mobilise very little university mathematics knowledge after three years of university mathematics studies that are successful in the sense of the examination requirements. How do the measures work in this respect, and, in particular, could they potentially contribute to teacher students never gaining access to university mathematics?

Similar study conditions at other universities at least suggest that the reported phenomena are not exceptional, although our insights do not allow statements regarding their frequency or representativeness. In order to systematically deepen our observations, more research with a substantial subject-specific reference and with a critical view on institutionalised teaching-learning relationships seems necessary. In view of the officially successful study efforts a crucial question concerns the following: How must university teaching be constituted in which “learning processes are possible in which, beyond [...] mechanisms of influence and control, real [...] experiences and insights can be gained” (Holzkamp, 1991)?[1]

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[1] Wie muss eine Hochschullehre beschaffen sein, in der „Lernprozesse möglich sind, in denen jenseits […] Beeinflussungs- und Kontrollmechanismen wirkliche […] folgenreiche Erfahrungen und Einsichten gewinnbar sind“ (Holzkamp, 1991).

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Mathematicians' Perspective on lectures in mathematics for secondary teacher candidates

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Mathematical content knowledge is a crucial part of teachers' professional competence and is necessary for developing pedagogical content knowledge. In most cases, mathematicians, who are not familiar with demands on teachers' content knowledge, teach the mathematics courses for mathematics teacher candidates. To learn about goals and teaching practices in courses for teacher candidates, we interviewed mathematics professors about their teaching and view on mathematics for teacher candidates, in particular about connections between mathematics at school and at the university. The interviewees mostly do not have specialized mathematic content knowledge for teaching in mind, but follow a trickle-down hypothesis regarding the mathematics needed in school.

Keywords: Teachers' and students' practices at university level, Transition to, across and from university mathematics, specialized content knowledge for teaching.

INTRODUCTION

Nearly 100 years ago, Felix Klein expressed the "double discontinuity" as a major problem in mathematics teacher education (Klein, 1924). The first discontinuity is concerned with the transition from school to university and the second discontinuity describes the return to school. When teacher candidates enter the university from school, they cannot link mathematics from school with the mathematics they learn at the university. After forgetting most of school mathematics, they are send back to school to teach just this mathematics. Wu (2011) formulated and criticized the second discontinuity as "Intellectual trickle-down hypothesis" where teaching advanced mathematics will lead to the needed content knowledge of teacher. Klein and Wu distinguish mathematics at school from mathematics at university. Deng (2007) describes differences between mathematics as academic discipline and as a school subject, which are important for both discontinuities. The second discontinuity is special for teacher candidates and raises the question: How and what kind of mathematics should teacher candidates learn at the university?

The COACTIV-study (Baumert & Kunter, 2013) showed that content knowledge plays an important role for developing pedagogical content knowledge. In addition, it is mostly build during studies at the university (Baumert et al., 2010). Therefore looking at the content knowledge taught at the university is important for developing teachers' professional competence.

In Germany, teacher candidates often do not visit specialized courses in mathematics, but attend the courses together with mathematics students (Gildehaus et al, 2021). Since mathematicians teach these courses, it can be expected that this courses are not orientated on teacher candidates. Even when they are separated mathematicians mostly teach the courses for teacher candidates. In this way, the mathematicians act as teacher educators (Leikin et al., 2018). That makes it important to learn about their goals and view on mathematics teacher candidates have to learn.

THEORETICAL BACKGROUND

Based on Shulmans (1986) model of professional competence of teachers COACTIV (Baumert & Kunter, 2013) formulates four aspects of teachers' professional competence: Beliefs, values and goals, motivational orientations, self-regulation, professional knowledge. The professional knowledge is further divided into five domains of knowledge: content knowledge, pedagogical content knowledge, pedagogical/ psychological knowledge, organizational knowledge and counselling knowledge.

Teachers' content knowledge is described as "teachers' understanding of the mathematical concepts underlying the content taught in middle school" (Baumert & Kunter, 2013, p. 34) and is explicitly distinguished from academic research knowledge. Therefore, teachers should possess a deep understanding of school mathematics.

We will now describe two models that describe teachers' subject matter knowledge in more details. The first approach extends content knowledge with a new construct: School related content knowledge. This construct explicitly deals with Klein's second discontinuity and therefore relates school mathematics to university mathematics. We will use school related content knowledge to investigate the connections seen by the lecturers between their lectures and the mathematics used in school.

The second approach describes four levels of mathematical knowledge teachers should reach. The normative model is specifically designed for mathematics teachers' education and does not distinguish between school mathematics and university mathematics, but describes different situations, in which mathematics is used. Using this literacy model, we will describe the level of mathematics the lectures want the mathematics teacher candidates to reach.

School related content knowledge

Dreher et al. (2018) introduced a new dimension of teacher professional competence: School related content knowledge. This dimension directly relates university mathematics to school mathematics. School related content knowledge consists of three facets: curricular knowledge, a top down direction from academic mathematics to school mathematics and a bottom-up view from school mathematics towards academic mathematics.

The curricular knowledge describes knowledge about reasons why topics are treated in school and how topics are connected over the curriculum. The top-down facet starts at the academic perspective and covers reducing and decompressing mathematical ideas for teaching. This view is also about the right definition, tasks and explanations adequate for students of a specific grade. The bottom-up perspective respectively starts at school mathematics. It is concerned with the underlying proofs for claims and assumptions implicitly made in school mathematics. Are these claims and assumptions justified and which concepts of academic mathematics lay behind them.

School related content knowledge was empirical separated from mathematical content knowledge and pedagogical content knowledge, even though it has high correlations with both constructs (Heinze et al., 2016). Using this construct as specialized teacher knowledge Hoth et al. (2020) could not support the trickle-down hypothesis, showing that school related content knowledge and university related knowledge develop independently from each other. Jeschke et al. (2021) also found effects for school related content knowledge on teacher action-related competences.

Literacy model

Based on Macken-Horarik (2012) Bauer and Hefendehl-Hebeker (2019) develop a Literacy-Model explicit for mathematics teacher education with four levels: Everyday Literacy, Applied Literacy, Theoretical Literacy, and Reflexive Literacy.

The first level describes Mathematics that is visible in everyday live for everyone. The second level focuses on Mathematics in relation to other sciences and presenting specialized tools and algorithms for solving concrete problems, without looking at the theory behind the tools. The third level deals with the mathematical theories. Here lays the knowledge written in mathematical papers and monographs. This level is addressed, when students learn about definitions, propositions and proofs. The fourth level deals with mathematics as a discipline, and it`s work methods, principles and regulations, that often stay implicit. The four levels are consecutive. Therefore, students can only reach one level if they already reached the previous level.

The authors see achieving the last level as the goal of teacher education. It is not only a nice to have overview of the other levels, but necessary for teachers as representatives of Mathematics. Even though the number of lectures in Mathematics is limited, the teachers should not stop at a lower level.

Previous Findings

Only few studies concerning the design of courses for teacher candidates exist. Ableitinger et al. (2020) interviewed seven teacher educators of one university after all mathematics courses were designed specifically for teacher candidates. The results show that educators do not think in categories like content knowledge or about teacher practices, but see mathematics for teacher candidates in comparison to mathematics for mathematics students. However, they had the discontinuity for teacher candidates in mind. In addition, personal beliefs were a major factor in the reasoning.

Yan et al. (2020) asked 24 lecturers what teacher candidates should know about university mathematics and let them design a hypothetical calculus course for teacher candidates. The main goals laid in the nature of mathematics as a discipline. The teacher candidates should experience mathematical investigations and see connections within and beyond mathematics. Yan et al. (2022) also describe the value in advanced mathematics that educators see: In addition to the increased epistemological awareness and connections between mathematical ideas, it lays in the problem-solving abilities. The lecturers also give some connections between advanced mathematics and school mathematics.

When looking for connections between advanced mathematics and school mathematics de los Ángeles et al. (2022) found, that educators had difficulties in finding connections between advanced mathematics and content of lower secondary classes. It was easier for the educators to find connections to upper secondary content. Their results suggest that the educators do not plan to teach these connections in their courses. The reasons for that were unawareness, missing time, teacher candidates should build the connections and building these is part of professional practice.

Overall, these results indicate that mathematicians do not have in mind a specialized content knowledge for teaching in their lectures. They mostly think of a university mathematics perspective, even though they recognize differences between school mathematics and mathematics at the university.

RESEACH QUESTIONS

This study wants to give answers for two research questions. Both are concerned with the design of mathematics courses for teacher candidates:

1) How do mathematicians design lectures for teacher candidates? Do they differ from lectures for math students?
2) Do mathematicians have specialized content knowledge in mind when designing courses for teacher candidates?

METHODOLOGY

Because all interviews were conducted with lecturers from one university, we will first describe the structure of studies at this university and afterwards the participant and the interview manual.

Mathematics courses for teacher candidates in Magdeburg

In Germany, teachers have to teach two subjects and therefore study two subjects. In addition, the study programs of mathematics teacher candidates are separated from the study program in Mathematics. Sometimes lectures in Mathematics are the same for both study programs, which is not the case in Magdeburg. Most mathematics courses are joint with students from other subjects or specific courses for teacher candidates.

The structure of the studies for future teachers in Magdeburg follows the typical structure in Germany. In the first semester students takes the courses in Calculus 1 and Linear Algebra 1. Both lectures are joint with students in Physics and mathematical engineers. In the second semester, the teacher candidates take the courses Calculus 2 with the same group of students and geometry, which is a special lecture for teacher candidates.

Further compulsory lectures are Numeric and Stochastics. Stochastics is a lecture specific for teacher candidates and numeric is a joint lecture with students in Mathematics and other subjects. There is also a lecture in History of Mathematics combined with a seminar with a selected topic as part of the bachelor studies. The last lecture in the bachelor studies and the lecture in the Master of Education are required elective courses.

Participants

Overall, eight interviews are part of this study. All of the participants are members of the Faculty of Mathematics at the Otto-von-Guericke Universität Magdeburg. One participant has a postdoc position; the other seven participants are professors. Two professors are women.

All of the participants have given at least ten courses in Mathematics. Six of the participants have at least given one course specifically for teacher candidates, five of them a course of the first year. The other two professors have only given elective modules. Only two participants have given more than ten courses for teacher candidates.

Interview Descriptions

The guided interviews consist of two parts. The first part addresses teaching of teacher candidates. The second part deals with the relationship between mathematics at school and the mathematics at the university.

The first part consists of four segment, dealing with the following aspects: Describe a course for teacher candidates you teach this semester or have taught before. Describe a dream course you would teach, if there were no limitations. Which courses should teacher candidates and mathematics students visit together? Comment on three given principles for a seminar.

The second part starts with questions about goals in school mathematics: What do you think are the goals in mathematics in school and what should be the goals? Afterwards the interviewees should comment on four statements. The first three statements concern the relationship between school mathematics and mathematics at the university, e.g. “One has to understand, that mathematics at the university has nothing to do with your job as a teacher”. The last comment deals with the fourth level of the literacy model.

All interviews were conducted online and the questions were displayed on the screen. The durations of the interviews are between 44:10 and 1:08:06.

All interviews were recorded and transcribed by the author, who afterwards analyzed it using qualitative content analysis. The author translated the quotes.

RESULTS

We will start with first research question regarding course design and specifically at the first semester lectures in Analysis in Linear Algebra.

Course design

The lecturers described the lectures for teacher candidates to be similar to the courses for students in mathematics.

> I6: But it isn't actually, let's say, regarding the volume of content and structure, far away from the other Analysis lecture.

The Linear Algebra course was just one semester instead of two semesters, so it was shorter:

> I2: I would say it was a condensed Linear Algebra 1 and 2, proof-oriented on the blackboard.

Overall, the lectures in the first semester are similar to the lectures of students in mathematics. The proof orientation mentioned in the last quote meets the third level in the literacy model. Some lecturers explicitly separate the proof oriented lectures from lectures for students in e.g. economics, which focus on using algorithms and therefore only reach the second level.

However, there are still some specific features of lectures for teacher candidates. The lecturers describe them as more concrete, having more examples and applications and less technically. For example, not all theorems are proven in full generality, showing a slight shift to level two in the literacy model:

> I8: That means, at technical complicated things I will simplify it.

Advanced lectures, that teacher candidates and mathematics student attend together, are mostly designed for mathematic students:

> I2: I honestly have to admit, I didn't take [teacher candidates] into consideration

> I6: Actually, there is no reason, to change anything for the teacher candidates

However, by choosing topics for students presentations or oral exams the lectures differentiate between teacher candidates and mathematics students. The chosen topics need less previous knowledge or are less theoretical and are less relevant for further lectures, which teacher candidates do not visit.

Interestingly the interviewees do not see large differences in the mathematics competencies of math students and teacher candidates at the beginning of university, e.g.:

> I2: Now, when I say it, I would say, differences are not that big

On the other hand, they describe motivation as different between both groups. This is also justified by the clear goal of being a teacher:

I8: and with some teacher candidates, it is […], that their ambition is not, to do it best as possible, but to somehow pass.

The lecturers also described a course for teacher candidates without limitations and constraints. They give different approaches and some are quite different from typical courses. However, it is striking, that only one of the courses is actually specific for teacher candidates. When they start describing the course they talk about teacher candidates, but when asked, if it is specific for teacher candidates, only one lecturers can confirm this. This seminar should explicitly connect university mathematics with school mathematics:

I5: selected topics from university mathematics with their references to school praxis and would try, but I cannot do this alone

In this case, the lecturer wants to visit schools with the students and talk about the content afterwards. The lecturer wants to show connections between school mathematics and mathematics at the university, which relates to school related content knowledge, but he can not to this alone, as the second half of the quote shows.

Overall, the lecturers for teacher candidates are oriented on the courses for mathematics students, but seem to have a slight lower level concerning the literacy model. However, the lecturers stress the third level. The fourth level is not mentioned for any student group. Only one lecturer mentions a link between school and university in a theoretical course design, while the other lecturers design courses not specific for teacher candidates.

Connections to school mathematics

The lecturers mention some direct connections between their lectures and mathematics in school:

I2: as solving systems of equations is something, which is not too far away from what is needed at school

I7: that we proof all the theorems, which are typically in school. Something like angle-theorems, Pythagorean Theorem, existence of perpendicular lines, unique parallel line, such things we discus there.

These quotes show direct connections identified by lecturers, where the content is the same in school and at the university. However, the lecturers mostly mention these connections in special cases, but not as a main goal of their courses. They are also not described as opportunities for building specialized knowledge for teachers.

Even though the lecturers mention connections to school mathematics, it seems more like a background knowledge then referring to some kind of school related content knowledge.

Lecturers rather want the teacher candidates to have an overview of mathematics. They should know more than just mathematics they teach in school:

I7: Just, to widen the view, and to put it in a bigger context.

I8: to not only learn an epsilon more than, what you do in school, but much more

Thereby the knowledge of university mathematics and experiences of university mathematics is helping in teaching better:

I7: That they somehow get the students to freely exchange and discuss, either with the teacher or with the classmates, to uncover fallacies and help each other.

I8: that it is important, to understand complicated things, to be able to explain simple things well.

The lasts quotes implicitly include the trickle-down-hypothesis. Students have to learn complicated mathematics to understand easier mathematics and are then able to explain it to students in school. Therefore, they need a good overview of mathematics.

Overall, the math lecturers see the teacher candidates as a specific group. Nevertheless, they think from a mathematics point of view. Therefore, teacher candidates need knowledge in mathematics and this knowledge helps teaching. They mostly do not have specialized content knowledge as described in the theory section in mind.

DISCUSSION

Overall, the mathematicians describe lectures for teacher candidates mostly as modified, especially shortened, lectures for mathematic students. Specifying lectures for teacher candidates means making it more concrete, showing applications, examples and using content, which is also part of school mathematics. In mentioning that the lectures for teacher candidates are proof oriented, mathematicians expect at least the third level of the literacy model. It is not astonishing that teachers should know that level of mathematics in the mathematicians view. However, they explicitly state the second level as not enough.

They also draw connections between university mathematics and school mathematics. A student, who has more complicated knowledge in mathematics, can explain mathematics better in school. In addition, university mathematics is a more complex and general case of school mathematics, so the teacher should know some of it, to be some steps ahead of the students. Therefore, in both perspectives the trickle-down hypothesis is implicitly mentioned.

The connections between school mathematics and university mathematics are also not concrete. The lecturers only describe shared contents and do not link the needed to knowledge to teach with the knowledge of their courses. Therefor they do not describe a top-down or bottom-up perspective regarding the shared topics. However, two lecturers want to foster theses connections, but cannot, because they are missing knowledge about mathematics in school.

Overall, the lecturers do not emphasise specialized content knowledge, but want teacher candidates to have an overall solid knowledge of mathematics. The lectures are mostly contrasted with lectures for mathematic students, what is in line with the previous findings.

These results describe interviews from one university and with a small group of lecturers. That means all of the lecturers are working at the same institution. Therefore, it is necessary to repeat such interviews with more lecturers from other universities to understand what is special and what can be generalized.

Similar we only discussed results directly to one course or some courses. Since a specialized mathematical knowledge is investigated, it would also be interesting, which view on school mathematics the mathematicians hold. What the teacher candidates will or should do in school will influence the teaching at the university. At the same time, despite some evidence, it is not clear, whether the described concepts of specialized content knowledge for teachers lead to actual learning effects of students.

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Teacher learning from team teaching an advanced course in mathematics and didactics for prospective upper secondary teachers

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We report on a course in advanced mathematics and didactics aimed at prospective upper secondary teachers, taught jointly by a mathematician and a mathematics educator using a team teaching format. The course is novel in that it covers mathematical and didactical content concurrently and jointly. The present paper focuses on the team teaching aspect, with the aim of investigating what opportunities for learning, in particular for the teachers, arise from such a teaching practice. Drawing on commognitive theory, we analyse a teaching episode that displays how the team teaching might contribute to creating opportunities for meta-level learning, but also how implicit assumptions made by the teachers might create challenges for some students.

Keywords: Teachers' and students' practices at university level, Novel approaches to teaching, Upper secondary mathematics teacher education, Team teaching, Meta-level learning.

INTRODUCTION

There is some consensus among mathematicians and teacher educators (e.g., Dreher et al., 2018; Leikin et al., 2018) that prospective upper secondary mathematics teachers need a solid knowledge of central topics of upper secondary mathematics, such as calculus and algebra, including some more advanced topics. However, research (e.g., Wasserman et al., 2018; Zazkis & Leikin, 2010) has shown that both prospective and practicing upper secondary mathematics teachers have difficulty seeing the relevance of more advanced mathematics courses to their teaching practice. In 2021, the authors of this paper received funding from Uppsala University for developing a course partly aimed at addressing this problem. The design of the course is novel in that it covers mathematical and didactical content concurrently and jointly, not as two separate parts of the course. To this end, the course is taught through *team teaching* (Friend et al., 2010), where two teachers are present in the classroom conducting the teaching together. In this paper we will focus on this aspect of the course, with the aim of examining what opportunities for learning that might arise from team teaching. Although opportunities for student learning will be discussed, we will concentrate on opportunities for teacher learning afforded by the team teaching. We will also discuss some challenges that arose in the teaching of the course.

CROSS-DISCIPLINARY COLLABORATIVE UNIVERSITY TEACHING

There are several models for collaborative teaching (Friend et al., 2010, p. 12), ranging from "one teacher, one assistant" models, via parallel and alternative teaching, through to team teaching, "in which both teachers lead large-group instruction by both lecturing, representing opposing views in a debate, illustrating two ways to solve a problem, and so on" (ibid., p. 12). The research literature on co-teaching of mathematics at the university level mostly consists of case studies in the context of elementary teacher education. For instance, Ford and Strawhecker (2011) developed a blended mathematics content/method course, co-taught by a mathematician and a mathematics educator in a format that included team teaching to a limited extent.

Studies involving co-teaching of university-level mathematics are rare. Lehmann and Gillman (1998) report on a semester of collaborative teaching. However, in that case both teachers were mathematicians. More directly relevant to the present study is the work of Zaslavsky and colleagues concerning a course in Mathematical Proof and Proving, taken primarily by prospective secondary mathematics teachers and co-taught by a mathematician and a mathematics educator. Sabouri et al. (2013) focus on the professional reflection of the teachers, building on one specific classroom situation to discuss "ways in which the collaboration between these two experts made them conscious of each other's considerations and of the importance of questioning their assumptions and negotiating them" (p. 312). Meanwhile, Cooper and Zaslavsky (2017) conducted an in-depth analysis of a teaching episode to investigate the two teachers' views on proof and proving as they come across in their teaching, and the affordances of these views on the students as future teachers. Perhaps unsurprisingly, the analysis revealed the mathematician to be more concerned with the content and presentation of the proof, while the mathematics educator paid more attention to the thinking of the prover, what Cooper and Zaslavsky label the "human element" of the proof. They also note that the teaching mostly took the "one teach, one observe" approach, and that a teaming approach might have created greater opportunities to discuss and reflect upon differences in views. In conducting their analyses, Cooper and Zaslavsky drew on a discursive approach to teaching and learning, the commognitive framework (Sfard, 2008). This framework also forms the theoretical backdrop of the present study.

UNIVERSITY MATHEMATICS TEACHING AS A DISCURSIVE PRACTICE

From a commognitive perspective, mathematics and mathematics teaching are seen as discursive practices, characterized by the *words* and *visual mediators* used, the *narratives* told about mathematical objects and their relations, and the repetitive patterns, *routines*, of the discourse (Sfard, 2008, p. 131-133). As a patterned activity, mathematical discourse is governed by two types of rules. *Object-level rules* are "narratives about regularities in the behaviour of objects of the discourse" (ibid., p. 201). Most of what we think of as mathematical rules or facts belong to this category, for instance, differentiation rules or the distributive law. *Meta-level rules,* on the other hand, govern the actions of the discursants, that is, they regulate "the activity of the discursants trying to produce and substantiate object-level narratives" (ibid., p. 201). An example could be rules for what counts as an acceptable mathematical proof. Meta-level rules (metarules for short) behave differently from object-level rules. They may *evolve* over time and vary between contexts. Moreover, where the rules governing the mathematical objects are explicitly articulated as mathematical narratives, metarules can be *tacit*. Indeed, Sfard distinguishes between *endorsed* and *enacted* metarules, where the former "are explicitly recognized as a person's own" (ibid., p. 204), while the latter are inferred by an observer of the discourse. Furthermore, metarules are typically *normative* and value-laden; *constraining* rather than deterministic; and *contingent* rather than necessary (ibid., p. 202). It should be pointed out, however, that not all metalevel rules satisfy all these characteristics. They should therefore not be taken as defining the notion of meta-level rule; the defining property of a meta-level rule is rather that it governs the actions of the discursants rather than the objects of the discourse. For more detail on this, see Viirman (2021, p. 469-470).

In commognitive terms, learning is defined as change in the learner's mathematical discourse. Such change can be of two types: *object-level learning,* which involves expanding one's discursive repertoire of objects and narratives about them; and *meta-level learning*, which involves changes in the meta-level rules of the discourse (Pinto, 2019, p. 4). Typically, meta-level learning involves discursive change that is not initiated by the learners themselves, and that evolves through repeated engagement with the new discourse. For instance, in the move from school to university, the meaning of, and rules governing, many familiar mathematical objects and narratives change in ways that are not always made explicit. A few studies have examined university mathematics teaching from the perspective of meta-level rules and learning. Viirman (2021) investigated how the discursive practices of seven university mathematics lecturers teaching first-semester courses served to model mathematical discourse, and Pinto (2019) compared the opportunities for object- and meta-level learning offered by two teaching assistants through their tutorial lessons in a Real Analysis course. As Pinto points out, "while instructors do not typically engage in open and explicit discussions about the meta-rules of their discourse in the course of their lectures, their recurrent actions, and their comments about their actions, provide glimpses into their mathematical discourse and its underlying meta-rules" (p. 4).

Using this terminology, we address the question of how and to what extent team teaching, with its element of teacher dialogue, can create opportunities for meta-level learning. Although we are interested also in opportunities for student learning, here our main focus is on the learning opportunities created for the teachers. For instance, we hypothesize that the dialog inherent to the team teaching might help make explicit enacted metarules both concerning mathematics and mathematics teaching. Before addressing the research question, however, some information on the course and the teaching, as well as on data collection and analysis, is needed.

METHODS

Setting – the course, the teachers, the teaching

The course is positioned late in the upper secondary teacher education program and runs with 1-2 two-hour sessions a week, a total of 30 sessions, throughout the spring semester, in parallel with the students' final thesis work. The students have taken three semesters of mathematics prior to enrolling in this course, including, for instance, linear algebra and calculus of one and several variables. For most students, these semesters are placed early in the program. Thus, their prerequisite knowledge is not always up-to-date.

For assessment purposes, the course consists of a mathematics part, 10 ECTS [1] credits, and a didactics part, 5 ECTS credits. The mathematical content of the course is taken from abstract algebra and real analysis, with a focus on topics relevant to upper secondary mathematics. For algebra, this means mainly ring and field theory, for instance Euclidean domains and field extensions, but only some rudiments of group theory. Applications include a treatment of the classical problems of geometric constructions (proving, for instance, that squaring the circle is impossible) and a discussion of the insolvability of the general fifth-degree polynomial equation. In analysis, we begin by detailing the construction of the real numbers through Cauchy sequences and then give a rigorous treatment of concepts familiar from upper secondary school calculus, such as limit, derivative and integral. Didactical topics covered include, for instance, the epistemological structure of calculus and well-known student difficulties with calculus topics, but also didactical perspectives on notions such as definitions, examples, proof, generalisation, representation and classification. The course is assessed through a variety of means, including two closed-book exams on mathematical content; an open-book exam on didactical content; and group and individual presentations on mathematical and didactical topics.

As already mentioned, the course is taught concurrently by the authors of this paper. The first author, ME, is a researcher in mathematics education, but also has a solid background in advanced mathematics (a masters degree in mathematics plus a number of additional courses at the doctoral level), whereas the second author, M, is a research mathematician, but also an award-winning teacher of university mathematics, professionally involved in didactical networks within the science faculty at Uppsala University. In the teaching of the course, the second author takes main responsibility for the mathematical aspects, and the first author for the didactical ones, but both are knowledgeable enough in the other's domain of expertise to be able to engage in meaningful dialogue. In planning and conducting the course, we wanted to establish connections between university and school mathematics teaching. Wasserman (2018, p. 6) describes a spectrum of such connections, on the level of content, disciplinary practice, classroom teaching and modelled instruction. We aim at achieving connections of all these types, for instance through explicitly pointing out content connections and ways in which the content can inform the students' future teaching practice, but also through encouraging student meta-reflection and through viewing our own teaching as a source of examples of practice, to be used for didactical reflection. We also try to achieve a high level of interaction with the students, by engaging them in conversation around the mathematics rather than just lecturing.

Data and analysis

So far, the course has been given once, with nine students enrolled. All sessions (except the first four, due to technical difficulties) were video-recorded. In addition, the first author has kept notes from the joint planning of the course, and we also conducted some informal interviews with the students. So far, we have conducted preliminary analyses of a small part of the material, focusing on teaching episodes where there was much interaction between the two teachers, or between the teachers and the students. In particular, we were looking for exchanges where the focus was on meta-level rather than object-level discourse, that is, where discussion revolved around, for instance, more general aspects of mathematical discourse, on implications for teaching practice, or on didactical aspects of the teaching currently taking place. From these exchanges we then selected instances which we deemed particularly representative or enlightening. We then analysed these further, with the aim of investigating how the dialogue between the teachers, or between the teachers and the students, influenced the opportunities for learning, both for students and teachers, arising from the teaching. The final decision to make opportunities for teacher learning the focus of this paper led us to select the particular episode presented below. All dialogue was originally in Swedish, and has been translated by the first author.

RESULTS

A first, general, observation is that at the very onset of the course we realised that we had unrealistic expectations of the mathematical maturity of the students. We knew that it was some time since they had studied mathematics, and in the information distributed to the students prior to the course we emphasised the need to review material from earlier courses in algebra and calculus in preparation for this course. In particular, we designed the first session around the algebra of integers and polynomials, and asked the students to review the statements and proofs of the fundamental theorem of arithmetic, and of Euclidean division of integers and polynomials. We intended to have the session revolve around a discussion of the structure of these proofs, and of abstract properties of integers and polynomials more generally, to prepare the students for the introduction of the concept of a ring in the second session. However, it soon became clear that most of the students had struggled with the proofs, and we instead had to spend much of the session explaining them in detail. When we reflected on the session afterwards, M first attributed the student difficulty to particularities of the proofs. However, ME, who had been less directly involved in the teaching during the session, and thus had greater opportunity to reflect on the discourse as it unfolded, observed that some of the difficulties seemed to originate in lack of experience with proving more generally. Further reflection and observation during the following session led us to agree on ME’s point of view.

Hence, we deemed it necessary to devote more teaching time to presenting and explaining the mathematical content. This placed more of the responsibility of the teaching on M, and made it more difficult for ME to interact, since there was less time available for didactical and mathematical reflection. Moreover, it was decided that ME should concentrate much of his teaching contributions on issues of proof and proving, and this is also the topic of the example episode that is presented and analysed below.

A teaching episode – proving an equivalence

This episode is taken from Session 6, and appeared in the context of proving a property of the valuation ν in a Euclidean domain, namely that $\nu(a) = \nu(1) \Leftrightarrow a$ is a unit (has a multiplicative inverse). The form of the statement prompted some meta-level reflection from M, prior to the presentation of the proof, concerning the two implication claims implicit in an equivalence:

M	Often, you cannot prove these two claims together, that is, show this [points at the equivalence written on the board] directly, rather you show one claim at a time, first this one [points from left to right in the equivalence] and then this one [points from right to left], or the other way around, depending upon what you feel like. I think that we'll go this way first [points from right to left], because it is easier.
ME	That's a bit interesting, because in many contexts, when you first encounter the idea of proving things, in number theory and things like that, then you often work with expressions that you reformulate in various ways, and then these equivalences will somehow hold all the way through.
M	Unless you do something like extracting roots or something like that.
ME	Yes, exactly, so if that is, like, your entry point to proving things, then you run the risk of forgetting that a claim like this typically needs to be split up and treated in two separate parts, because you're somehow used to being able to do everything at once.
M	A typical problem with the presentation of proofs like that often is that the student starts with what they are supposed to prove. You write down the equality you want and then you manipulate it until you get the equality you had to begin with or the other way around or whatever, and sometimes it's like, if you interpret it kindly, those are the calculations that you need to do, but the structure of the proof is unclear if you do it like that, OK? It's always better if you need to prove an equality (…) even if you figure out what the process is by fiddling with the equality yourself, at least when you present it start with one side and rewrite it until you get the other side 'cause then you'll have a clear direction of your argument. OK? Don't start with the claim, the equality, and start working on both sides at once, because it is not systematic.
ME	No, because we don't know that the equality holds to begin with.
M	For instance that, yes [laughs].
S	So, then, one direction is a is a unit and then that [referring to the claim $\nu(a) = \nu(1)$] holds, or if that holds then a is a unit?
M	Exactly, it's those two. So that, an equivalence [writes on the board: "An equivalence arrow is two implications"] an equivalence arrow is two

implications, OK? So, we have A is equivalent with B [$A \Leftrightarrow B$] means that A implies B [$A \Rightarrow B$] and that B implies A [$B \Rightarrow A$].

In this dialogue, the respective statements by M and ME prompted responses that elaborated on what was just said, adding further levels of reflection. From the point of view of an insider to the discourse, what was happening could be described as follows: M began by stating a general principle of proof, which prompted ME to reflect on potential didactical problems with the simple types of claims that typically are secondary school students' first contact with proof, which in turn caused M to highlight a typical student mistake with proof and to formulate an explicit metarule for this kind of proof. Prompted by a student comment about the particular proof under consideration, he also put the initial observation about equivalences and implications into symbolic form on the board.

However, given the students' difficulties during the first sessions and their lack of recent engagement with mathematics, it was likely too optimistic to assume this degree of insidership to mathematical discourse from many of them. A closer analysis of the dialogue reveals a number of places where M and ME made implicit assumptions on the students' familiarity with mathematical discourse, and in particular on their ability to move between object-level and meta level discourse.

The example given, in quite abstract and imprecise terms, by ME in his first statement assumed both that the students understood what kind of proof situations he was referring to (proof through algebraic manipulation) and that they were capable of reflecting on these at the meta-level, as examples of ways of mathematical reasoning. In his response, M aimed to complete the argument with details (not all algebraic manipulations result in equivalences), but again it was implicitly assumed that the students could follow this meta-level argument. For M, as an insider, it was clear what situation ME was referring to, but possibly it was less clear to the students.

In his next statement, ME reached his intended conclusion by connecting the two strands of the argument; if your main encounters with proof are through proving algebraic identities, often as a sequence of equivalences, then you can easily be led to think that this is how you typically prove equivalences. However, this again assumed that the students were able to follow this quite abstract meta-level reasoning, reflecting on what unites and separates two forms of mathematical argumentation, one of which had only been described too abstractly and with lack of precision. When M then connected this to a well-known type of student mistake, the fact that ME's argument concerned proofs of algebraic identities was clear to him as an insider, but in fact this is the first place where the term 'equality' (or 'identity' – the word 'likhet' in Swedish can mean both) was explicitly mentioned. The point was possibly further obscured by the fact that M was also unable to resist making a joke in the process (the manipulations resulting in getting back where you started). Still, M did formulate a useful explicit meta-rule for proving equalities, and at least one student (S) had apparently been able to keep track of the original narrative, prompting him to also explicitly state the connection between equivalences and implications.

The deepened analysis presented above indicates how ME and M made implicit assumptions on familiarity with mathematical discourse that possibly posed a challenge for the students. The analysis suggests at least two possible sources of problems. First, ME's comment was highly abstract, lacking a concrete example to specify the kind of situation he was talking about. Second, the language he was using was not sufficiently precise. For M, with access to a large and well-organised "library" of mathematical, as well as didactical, objects and examples, ME's statements had clear referents, despite their vague formulations. However, even if the students might very well be aware of the relevant examples, it is unlikely that such vague prompting would suffice for them to recall them in this context. Still, there are tensions here. If ME had taken the time to support his remark with a carefully presented example given with precise details, it would have ceased being just a remark, thus disrupting the ongoing mathematical argument. Moreover, ME's comment was prompted by an observation made in the moment and had to be made before work on the actual proof started, meaning that the time available for coming up with a supporting example was limited.

Lest we paint too bleak a picture here, despite these misgivings the dialogue format of the team teaching did support a deepened didactical reflection in the moment, in the process creating opportunities for meta-level learning concerning proofs and proving, although perhaps not for all students. In particular, here it led to an enacted metarule being made explicit for the students, in a way that would likely not have happened had only one of the teachers been doing the teaching.

DISCUSSION

The analysis presented above exemplifies how team teaching led both M and ME to realise the need for constant awareness of students' less developed mathematical discourse. Despite M and ME being didactically knowledgeable teachers, in their meta-level discussion they nevertheless lose track of this. Still, the episode also illustrates how team teaching can contribute to making metarules visible to students, thus creating opportunities for meta-level learning by providing insight into the "explicit discussions about the meta-rules of their discourse" (Pinto, 2019, p. 4) typically only glimpsed in teaching. But it also suggests ways in which team teaching can create opportunities for teachers' in-the-moment reflection, as one teacher is not solely responsible for managing the lesson. In the episode analysed, comments made by M prompted ME to reflect on the teaching taking place, a reflection that continued after the particular session. In this way, the team teaching also contributed to teacher learning. Only a small part of the data has been analysed so far and we expect further analysis to provide stronger support for the value of team teaching for meta-level learning.

Widening the perspective from this particular episode to the course as a whole, between the teaching sessions M and ME reflected on and discussed the teaching, providing analysis and critique. This process led to both teachers gaining new insight into their teaching practice, in a way that would have been more difficult, and possibly would not have happened, had either been the sole teacher of the course. This resonates with the observations made by Sabouri et al. (2013), where the collaboration between the mathematician and the mathematics educator “made them conscious of each other’s considerations and of the importance of questioning their assumptions and negotiating them”. Examples of learning that led to changes in teaching practice include the realisation of the crucial importance of timing when making comments, in order not to disrupt an ongoing argument, and of keeping comments short and precise. Moreover, ME has come to realise that the (over-)abstraction evident in the episode above is a recurring tendency in his teaching that he needs to work on, while M has noted how impatience sometimes causes him to not allow students enough time to reflect and formulate their arguments when conducting group discussions. Here it is also worth noting that the process of researching your own practice in itself creates opportunities for teacher learning, which are enhanced if you are two people doing the analysis.

There are also a few general observations to be made in relation to previous research. We note that the modelling of mathematical discourse that Viirman (2021) found in his analyses of university mathematics lecturing occurred also in this context. Moreover, the different roles discerned by Cooper and Zaslavsky (2017), where the mathematician took main responsibility for content, while the mathematics educator focused more on the “human element”, were less present in this course. In the episode discussed above, both M and ME made didactical reflections about proving. Indeed, it was a conscious choice when planning the course to try to avoid this strict separation of roles, showing by example that mathematical and didactical reflection go hand in hand.

We also want to make some remarks concerning difficulties we have encountered when trying to implement team teaching. Similarly to what Cooper and Zaslavsky report, much of the teaching took the “one teach, one observe” approach, rather than the more interactive format we had aimed for. This was due partly to us not being able to do the detailed planning of the sessions together, and partly to the need for more presentation of content described earlier, which sometimes caused ME to refrain from, for instance, initiating discussion of the lecturing of M as a model of instruction (Wasserman, 2018) since he knew that we were pressed for time. Moreover, in the light of the analysis above, there is a risk that the kind of reflection needed to make sense of meta-level comments made in the moment might contribute to student cognitive overload. In conclusion, however, so far we feel that the team teaching has a definite potential to contribute positively to student and teacher learning, particularly at the meta-level. Moreover, it is a very enjoyable form of teaching, something also highlighted by, for instance, Lehmann and Gillman (1998), and we recommend others to try it if they get the opportunity.

NOTES

1. European Credit Transfer and Accumulation System. One academic year corresponds to 60 ECTS credits.

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Task design for Klein's second discontinuity

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Since the 19th century, studies of mathematics at university have been a main component of the usual preparation for teaching at secondary level. Already around 1900, Klein pointed out that specific measures are needed to ensure that the university mathematical preparation becomes useful to the teacher, and he insisted that universities themselves must take responsibility for these measures. In this paper, we discuss this problem, as it presents itself in 2022, and we present and exemplify some principles of task design which are intended to support students' mobilisation of university mathematical knowledge in relation to specific mathematical challenges for high school teachers.

Keywords: 3, 15; Klein's second discontinuity, task design, ATD.

THE PROBLEM

The gap between university mathematics and secondary school mathematics has widened considerably in the 20th century. At the same time, the two institutions and their mathematical disciplines are not monolithic. At universities, mathematics has developed into several related, yet quite different disciplines of teaching and research, including the various domains of "pure mathematics" but also other disciplinary branchings referred to by labels such as applied mathematics, data science, computer science, statistics and even parts of engineering, finance and other profession oriented sciences. All of these are to some extent "references" for secondary school level mathematics: particularly at the upper secondary level (with students aged 15-16 onwards), various "streams" are present in most countries, which not only offer more or less mathematics, but also mathematics which can be more or less closely related to the university level forms of mathematics. This complicates both of the transition problems, described by Klein as the "double discontinuity" (Klein, 1908/2016; Winsløw & Grønbæk, 2014). One cannot simply talk of one "school mathematics" and of one "university mathematics".

Here we will consider only the second discontinuity, between university studies of mathematics (in some form) and teaching secondary level mathematics (in some form). It arises for university students as they prepare to become secondary level teachers. In most countries, this involves some mixture of university mathematics studies (some are in fact designed for teachers while some are not) where in general the second discontinuity must be considered. We do not consider generic educational components here, but only those parts that are directly aimed at adapting the future teachers' mathematical knowledge to presumed needs for teaching at the secondary level, with its variation and the rest of the students' university mathematical preparation in mind.

Even in European countries, the organization of teacher education programmes – and in particular, the part we focus on here – varies considerably. It can be considered as a bridging problem, where the continents to be bridged are two curricula: the mandatory programme of mathematics units studied at university (excluding the profession oriented part), and the secondary curricula in which the student will be teaching. In many countries (such as France and Denmark), there is a "consecutive" organization where the professional part comes last; in other countries, like Germany, a more "parallel" organization can be found.

We will now further delimit our problem, considering with Watson and Ohtani (2015, p. 3) that "the detail and content of tasks have a significant effect on learning; from a cultural perspective, tasks shape the learners' experience of the subject and their understanding of the nature of mathematical activity; from a practical perspective, tasks are the bedrock of classroom life". This leads us to the following general problem:

What are possible principles for designing tasks which, within a consecutive model, allow future teachers to adapt their mathematical background to the professional tasks of teaching mathematics at secondary level? In particular, how do these principles relate to mathematical tasks worked on by the future teachers at university, and by students in the schools where they prepares to teach?

We note that similar questions were studied by Bauer (2013) within the parallel model found in Germany. Naturally, the answers proposed here cannot exhaust the full range of relevant principles for task design related to courses in the consecutive model, but the focus on "adapting their mathematical background" will nevertheless allow us to propose a reasonably complete set of principles. Our discussion, at the end of the paper, will focus on the extent to which the proposed principles may be adapted or even extended to other similar, but different contexts. However, even before that, we need to furnish a more precise framework for the above problem, and then present our context, principles, and some examples of their use.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

We adopt, from the anthropological theory of the didactic (ATD), the notion of *institution*, which, are roughly speaking, social systems. This wide and unclear definition can be made more precise (see e.g., Chevallard, 2019, p. 92): human beings occupy, throughout their lives, various positions p in different institutions I, and for each of these positions, certain relationships, denoted $R_I(p,O)$ are required to certain objects O (O can be, for instance, knowledge objects, physical entities etc.). One can even define the elusive notion of institution as being a configuration of positions, each defined by a set of such relationships which occupants of the position is required to have in order to occupy p within I. For instance, to be a teacher t in a school institution S, it is required to hold certain relationships to a number of didactical tasks, to hold certain degrees etc. – these tasks, degrees and so on all being objects whose existence for S is confirmed by the relationships held by (some) positions in S.

Institutions may come in types such as schools and universities. Institutions may also, at least apparently "share" objects, for instance in the sense that they label certain objects in the same way. Yet, what is labelled, for instance, "real numbers" and "algebra" may not only differ from institution to institution, but even from position to position within these.

The second discontinuity has been modelled, within this framework, by Winsløw (2013) as pertaining to passages of the type

$$R_U(\sigma,\omega) \to R_S(t\ ,O)$$

where U is the university institution, σ is a student in U, ω is a (mathematical) knowledge object to which σ is required to hold the relationship $R_U(\sigma,\omega)$; and S is a school institution, t is a mathematics teacher in S, and O is an object to which t is required to hold the relationship $R_S(t,O)$. For the passage to be meaningful, it is naturally expected that $R_U(\sigma,\omega)$ is of some relevance to $R_S(t\ ,O)$, so that the latter could be supported by the former, presumably with some further development. If this development occurs, at least in part, already within the university institution, we can rewrite the above passage as

$$R_U(\sigma,\omega) \to R_U(\sigma,O) \cong R_S(t,O)$$

where $R_U(\sigma\ ,O) \cong R_S(t,O)$ indicates an approximate similarity of the relationship obtained by σ within U and the relationship to be held by t within S. If working with a task of type T can achieve the passage $R_U(\sigma,\omega) \to R_U(\sigma,O)$, at least in part, we write

$$R_U(\sigma,\omega) \xrightarrow{T} R_U(\sigma,O)$$

In this paper, we now consider the following research question:

Given a relationship $R_S(t,O)$ required to occupy t in S, how could some T be designed so that σ could develop $R_U(\sigma\ ,O)$, with $R_U(\sigma,O) \cong R_S(t,O)$, based on some $R_U(\sigma,\omega)$? In other words, what principles can be formulated for the design of T ?

Here, we present and explore four principles which have progressively been identified in the course of more than a decade of task design in the context described in the next section. The principles each focus on O at one of the praxeological levels (type of task, technique, technology and theory – for definitions of these ATD notions, see e.g., Chevallard, 2019, pp. 91-92).

P1. In case O is a *mathematical type of task taught in S*, T is simply a task of which is somewhat more demanding – but otherwise similar – to O, with the additional demands being satisfied by drawing on some $R_U(\sigma,\omega)$.

P2. With O as in P1, T requires σ to *pose* a task of type O, based on some $R_U(\sigma,\omega)$ which may also lead to a more theoretical or structured relation $R_U(\sigma,O)$.

P3. If O is *one or more mathematical techniques* (authentic or imaginary, correct or erroneous), which t should be able to foresee and assess, then T could ask σ to foresee or assess O while drawing on some ω;

P4. If O is *a segment of mathematical technology or theory*, which t should teach or otherwise know, then T could demand that σ establishes whether O is mathematically consistent with ω – for instance, can be proved based on ω.

Klein's discontinuities focus on the future teachers' relation to (school) *mathematical* objects. Klein points out that it is potentially useful for future teachers to establish such relations $R_U(\sigma,O)$ from the "higher standpoint" of university mathematics (an element of which we denote here by ω); but that doing so requires deliberate support measures within U, here conceived as engaging σ in work with carefully designed tasks T. The above principles then distinguish, but do not exhaust, important cases for the construction of T, as we will show through examples.

To prevent misconceptions, we also underline that the research question – and therefore the list – does not pretend to cover all task design that may be relevant to mathematics teacher education. Indeed, future teachers also need to develop didactical knowledge (both practical and theoretical) that cannot be directly supported by elements of "pure" mathematics as learned in standard university courses.

We will now outline the concrete context in which the four principles have emerged, and then present and analyse some examples of concrete T designed with them.

CONTEXT

The principles P1-P4 have progressively been made explicit in the theoretical terms given above, as they were developed and used within a concrete context by the first author (since about 2009). This context is a course, called "Mathematics for the teaching context" (UvMat), offered at the University of Copenhagen to students who do a minor in mathematics in view of becoming high school teachers.

We now outline what $R_U(\sigma,\omega)$, and in particular ω, could be in this context (for minor students σ). Before UvMat, σ has taken at least 1.25 years' credit of mathematics courses, covering: one- and multi-variable calculus, linear algebra (including axiomatic vector space theory), ordinary differential equations, abstract algebra (rings, fields and groups), differential geometry, discrete mathematics, statistics and probability, and analysis up to Fourier and metric space theory. The calculus part involves some level of computer algebra use.

We note that the mathematics courses drawn on are basic courses in the bachelor programme on pure mathematics. They focus primarily on theory development, and students are required to solve relatively theoretical tasks (except for the calculus part) involving deductive reasoning. Virtually no examples are studied of how the theory applies to solve practical problems outside of pure mathematics.

What, by contrast, could $R_S(t,O)$ – and in particular O – be? Danish high school mathematics has several levels and variations, but the core could be described as a study of concrete one variable functions and models based on such, up to practical uses of differential and integral calculus. Other mandatory domains are probability, statistics and geometry. Students' grades in mathematics depend largely on their ability to solve standard tasks, with or without the use of computer algebra systems. Deductive reasoning still appears, but recent curricula give reinforced attention to modelling and interactions with other high school disciplines, and to mathematical inquiry. Also, the use of computer tools – especially computer algebra systems – is strongly emphasised.

A bridge needs solid bases on both sides. It is not an aim for UvMat to connect all of the mathematical background of students to all of the high school mathematics they will have to deal with as teachers, but each task in UvMat must have solid connections to *both*, and in particular focus on important aspects of high school mathematics.

EXAMPLES AND ANALYSIS OF TASKS DESIGNED FROM P1-P4

In the following, the tasks we present come from the final exam in the course; former exam items (from an inventory of well over 100) are also used as exercises in the course. The students are informed, though, that exam items are never mere variations of former exam items; they always require the student to create new connections between course contents and university mathematics, and the high school object involved. Thus, the continuous development of such tasks form a central challenge of running UvMat. As exam tasks need to be relatively simple, the course also involves more involved assignments (see Huo, to appear, for an in-depth analysis of an example).

P1: Solving "advanced variations" of school mathematical tasks

Mathematics teachers naturally need to be able solve the tasks given to students, and some of the items worked on in UvMat are merely advanced variants of high school tasks (often involving non-trivial construction of a mathematical model, e.g. for a probability item). Here is an example:

> Britta participates in a multiple choice test with n questions. For every question, one can choose among 3 possible answers, of which only one is correct. Passing the test requires that one chooses correct answers for at least half of the questions. Britta knows nothing of the subject and answers randomly.
>
> a) If there are 10 questions, what is the probability that Britta passes the exam?
>
> b) If the test is to be made, so that students like Britta has less than a 5% chance to pass, how big must n then be? Explain your answer.
>
> (Exam June 2019, exercise 5)

Question a) is a standard application of the binomial distribution and, as such, is simply a high school level task. It merely prepares the second question b), which is technically harder, as the parameter n is unknown, rather than given. Students use tools to compute values of the binomial distribution function corresponding to given values of the parameters n and p (the latter being $\frac{1}{3}$ here). Another difficulty is that the meaning of "at least half" depends on whether n is even or odd. Many students will solve b) by computing, for increasing values of n, the binomial distribution function $F_{n,1/3}$ at something like $n/2$. Since $F_{n,1/3}$ is only defined on $\{0,1,\dots,n\}$, many students simply look for the first even n where $F_{n,1/3}(n/2) > 0.95$, which in this case is 30. However, the correct answer is, in fact, just 23 (we leave it to the readers to work out the details, using some software able to do compute binomial distributions).

To understand the kinds of ω elements to be drawn on here, we note that b) is quite similar to "inverse problems" related to probability distributions. The students have indeed encountered such problems in connection with subjects like confidence intervals, treated both in this course (from a high school perspective) and in the predecessor courses on statistics (at a technically more advanced level). Thus, solving b) certainly draws on very specific theoretical and technical elements ω, in addition to familiarity with a certain computer algebra system (Maple), which they have developed both in UvMat and in some of the prerequisite bachelor courses. This also illustrates that a variety of $R_U(\sigma,\omega)$ – even with ω being rather far from the school mathematical type of task O – could be relevant to such "extended" tasks of type O.

P2: Posing school mathematical tasks

Some of the mathematical tasks, which teachers need to solve regularly, are related to *preparing tasks for their students* – either by selection or construction. Here, more advanced mathematical work than merely solving tasks can be involved. The following UvMat item is an attempt to generate such a more advanced perspective:

> We say that a quadratic equation is *nice* if it is of form $x^2 + bx + c = 0$ where b and c are integers.
>
> a) If you are given two integers m and n, how can you construct a nice quadratic equation with m and n as solutions? Explain a method and give an example of how it works.
>
> b) Can you construct a nice quadratic equation that has both a rational and an irrational solution? Explain.
>
> (Exam August 2019, exercise 2)

The mathematical elements from university mathematics which need to be drawn on here, can be summarized as: knowing how to use a formal, *ad hoc* definition ("nice quadratic") without confounding the definition with an everyday conception of "nice"; a result about polynomials ("a is a root of $p(x)$ if and only if $x - a$ is a factor in $p(x)$"; and some experience with reasoning about irrational numbers. The latter may sound a bit vague, but in fact, b) can be solved in many ways – the most complicated probably being to use the quadratic formula. A better way is to use an observation easily made from a), namely that the product of the solutions is c, while this product will be irrational in the case described in b). Thus, very simple facts about polynomials and irrational numbers – not currently taught in high school, but certainly encountered at university – are activated here, to address what is clearly a relevant mathematical task for high school teachers, given that quadratic equations are taught and used there. We note that the kind of $R_U(\sigma,O)$ built here is typically more theoretical and less technical than what is aimed at in P1, corresponding to mobilizing more theoretical parts of ω.

P3: Assessing or imagining student techniques

Logarithm functions and their use form part of the core content in high school mathematics. Part of the difficulty is the "indirect" definition they are usually given, as inverse functions of exponential functions (cf. also P4 below). On the other hand, this theoretical definition is important in many frequent practices, such as solving equations involving exponents. The following item directly attacks such situation:

> Peter and Lise have to decide whether the equation $\ln y^2 = e^{-x}$ defines a function (with y as a function of x). Peter says: "Yes, for the equation can be rewritten as $2 \ln y = e^{-x}$, so $y = \exp\left(\frac{1}{2}e^{-x}\right)$." Lise says: "No, for the equation can be rewritten as $y^2 = \exp\,(e^{-x})$, so for each x-value there are two y-values".
>
> a) Who is right? Give a detailed explanation of the correct answer.
>
> b) One of the two answers is false. Explain where the error arises.
>
> (Exam June 2014, exercise 3)

We first note that this exercise involves a school mathematical task ("does the equation $\ln y^2 = e^{-x}$ define a function…") but the tasks given to the university students in the above item is at another level: consider some (imaginary) student solutions, decide whether they are correct, and explain why. In fact, it is a crucial teacher task to relate to students' mathematical work and provide feedback; items formed in this manner are therefore found occasionally (such in about 1 in 10 of the exercises proposed) throughout the course. As for the mathematical contents, students will know the identity $\ln x^a = a \ln x$ but may not have thought about that it is only valid, and meaningful, for positive values of x. In the exercise, Peter makes a mistake in his first "rewriting" of the equation, since the given equation is also meaningful for negative y, while the second is not.

The university knowledge, which the students could apply to solve this item, is not very advanced. They have certainly worked with more formal (set-theoretical) definitions of functions than what is seen in high school, but the informal definition ("to each value of x there must correspond exactly one y") suffices to realize that Lise is right, and then look for an error in Peter's rewritings. That identities such as $\ln x^a = a \ln x$ may have restricted validity (beyond what makes the expressions meaningful) is also something which university studies could increase students' awareness of. In particular, unlike in high school, they would often see qualifications like "for all $a \in \mathbb{R}, x > 0$", following an identity. So we can say that, in addition to the explicit treatment (in the UvMat course) of logarithms, the theoretical notion of function, and the logical subtleties related to equation solving (in particular, implications), the main university mathematical element (ω) to invest in this task is a more developed practice of applying identities only were they are valid. This specific $R_U(\sigma,\omega)$ turns out, in practice, not to be sufficiently developed for many students. Indeed, as observed by Winsløw et al. (2014), working with tasks designed to facilitate some passage $R_U(\sigma,\omega) \rightarrow R_U(\sigma,O$ often involves "repairing" dysfunctional $R_U(\sigma,\omega)$.

P4: Making new theoretical connections

A great deal of the work in UvMat – perhaps as much as half – concerns theoretical aspects of high school mathematics, like proofs of results or constructions which appear more informally in high school, such as the general meaning (and construction) of x^y for $x > 0, y \in \mathbb{R}$; see Winsløw et al., 2014, pp. 77-79). The strong focus on theory is in part a consequence of the aim to draw on university elements ω (where R_U(σ,ω) is often very limited when it comes to students' practical experience with the praxis level of ω. The general familiarity that students have gained with formal theory is frequently an asset they need to draw on, as they solve tasks based on P4.

Our last example relates to work carried out in the course with the theory behind linear regression. In high school mathematics, it is mainly taught as a practice carried out with some tool like excel, along with informal explanations that this provides "the best linear model" for a given 2d data set. In the course we revisit proofs which the students may have seen in statistics courses, along with more elementary approaches (see Winsløw et al., 2014, pp. 79-81). The following item links the theoretical problem of "minimizing least squares" to one-dimensional optimization as taught in high school:

> A simplified form of linear regression results from requiring that the regression line passes through the point (0,0), so that the equation of the line is of form $y = ax$ where a is a constant.
>
> a) Derive a formula for a corresponding to a data set $(x_1, y_1), \dots, (x_n, y_n)$, by determining a such that the sum $S(a) = \sum_{k=1}^{n} (y_k - ax_k)^2$ is minimized.
>
> b) Explain how this formula can be used to determine the resistance R in an electrical circuit, based on corresponding values of the current I and the voltage U, knowing that Ohm's law says that $U = RI$.
>
> (Exam January 2012, exercise 5)

It is a fact that optimization of two variable functions is not usually taught in Danish high school, and that alternative approaches to linear regression (as taught in UvMat) are also somewhat beyond what most high school classes would meet, or be able to cope with. While a) and b) rely in principle only on high school mathematics, the theoretical nature of the questions – and the requirements in terms of symbolic computation – certainly draws on R_U(σ,ω) with ω involving both practical and theoretical elements that are not strictly related to linear regression, but nevertheless supposed to be developed or strengthened at university.

The last question is not technically demanding (one should explain translate the formula from a) to give an estimate for R in terms of a set of measurements (I_k, U_k) of current and voltage). Nevertheless, it is important for mathematics teachers at high school to know and integrate models from neighbouring disciplines in their teaching, not least when it comes to statistics topics like the present one. Question b) requires one to make a connection between a simple model from physics and the mathematical result developed in a).

DISCUSSION AND CONCLUSIONS

To what extent can the principles and examples above be of interest outside of the context in which they arose? To most university teachers and researchers interested in the general problem we described in the introduction, the principles (derived from an ATD model of the problem) would remain arid speculations without the examples. Yet, it is fully possible that many of the same scholars would also not see the relevance of the examples for contexts familiar to them. Indeed, they are more or less arbitrary cases of efforts to link certain objects O and ω which occur in Danish high school mathematics and in the first two years of undergraduate mathematics at the University of Copenhagen. Most probably, many of these objects could in themselves be found in equivalent contexts elsewhere, but the emphasis on theoretical and technical aspects of them, reflected in the examples, could still be felt to be less relevant there. For instance, a recent study by Bosch et al. (2021) of external didactical transpositions in university mathematics suggests that in North America, the first years of undergraduate studies are often much more focused on technical aspects of calculus. This might mean that integrating such technical aspects would be seen as much more important than what is reflected by the examples given here, while the emphasis on proof (reflected by P4) would be considered less helpful. Similarly, for contexts in which probability distributions or linear regression do not feature centrally in the secondary curriculum, the example given for P1 and P4 would appear irrelevant.

Thus, to go beyond those examples of T – that certainly depend on the specific context – we really need to hold on to *principles* such as P1-P4. They can constitute a framework which could be adapted to such more or less different contexts: engaging students in work with O (centrally occurring in secondary mathematics) that is characterised by a focus on different praxeological levels of relevant to future teachers on the one side, and on drawing on similar levels of some central objects ω in students' university mathematical background.

But this conclusion merits other reservations. Applying the four principles take a certain inventory of O and ω as given conditions, and the potential as well as the feasibility of linking them as a working hypothesis (following Klein, 1908/2016). At least two major questions are left open by this: the question of *external didactical transposition*, both at university and in secondary school, resulting in the given O and ω; and the actual *importance of the potential for the subsequent professional practice* of t (a position which is somehow put aside by the non-examined "similarity" $R_U(\sigma,O) \cong R_S(t,O)$). The first major question contains in fact two separate aspects: the possibility of inadequacy (or at least needs for development) of secondary mathematics, and the problem of determining the adequate mathematical basis to be developed at university, and the extent to which this should consist in teaching conceived for more general publics than future teachers. The last question has been examined in more detail by Winsløw (to appear), linking it to recent quantitative studies of correlation between teachers' university mathematical background and the quality of their performance as secondary teachers. This kind of research may also have bearings on the second major question, as it involves identifying structural similarities of teacher education programmes that produce, according to such quantitative studies, teachers with high performance (measured by learning gains of the teachers' students, or by measures of teacher knowledge that correlate with high teaching performance). Among these similarities are, in fact, a combination of certain standard undergraduate mathematics units and units focusing on teachers' knowledge of central secondary mathematics. But we still need more direct and theoretically precise ways to investigate how students' participation in such courses affects their later performance as teachers, and in particular how principles for task design may contribute to enhanced effects.

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Coding in math learning: A ‘triple instrumental genesis’ approach to support the transition from university learner to school teacher

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Keywords: Transition to, across and from university mathematics; Novel approaches to teaching; Coding; ‘Triple instrumental genesis’.

INTRODUCTION AND RESEARCH QUESTION

We are witnessing a rapidly changing landscape of coding and computational thinking integration in compulsory education in many parts of the world. In particular, in their 2022 math assessment framework, PISA states that “students should possess and be able to demonstrate computational thinking skills as they apply to mathematics” and that they anticipate “a reflection by participating countries on the role of computational thinking in mathematics curricula” (OECD, 2018, p. 5, para. 12). Such a push to rethink school mathematics creates a pressing need to rethink the preparation of math teachers.

Since 2001, the Mathematics Department at Brock University in Ontario (Canada) has implemented a sequence of three courses (MICA I-II-III) in which math students (including future math teachers) learn to use coding to investigate mathematical concepts, conjectures, theorems, and applications. In line with the international trend, the Grades 1-9 math curriculum in Ontario was revised in 2020-21 to include the expectation that students develop and use coding skills to learn math. This created a need to revisit the design of the MICA III section dedicated to future teachers.

Indeed, the challenges faced by teachers in the transition from university math learning to school math teaching have been known for a long time and have been addressed by the INDRUM community (see, e.g., an ongoing international seminar; Grenier-Boley, n.d.). To address the second of “Klein’s double discontinuity,” we based our redesign on the following research question: *How can a mathematics content course for future teachers assist them in gaining skills and attitudes needed for making the transition to their future role of teachers, specifically in the case of using coding for mathematics learning?* In this poster, we present our redesign of MICA III as our initial attempt at responding to this question, explore a potential theoretical lens for reflecting on the redesign, and provide some illustrations using student work. We hope to have input from the INDRUM community as we prepare to give the course again in 2023.

COURSE DESIGN AND PROPOSED THEORETICAL FRAMEWORK

Sporadic meetings between the two first authors took place over 8 months in order to revise and/or develop new course objectives and elements using an “experiential” education (Kolb, 1984) perspective to provide opportunities for future teachers to make productive links between their learning of university math and their future profession as school teachers (thus helping to bridge Klein’s second discontinuity).

The three course objectives are: O1) to further one's experience of using coding to learn math (including conducting investigations); O2) to develop an understanding of that experience (the learning), including affordances of coding for math; and O3), a new objective, to develop an understanding of teaching (supporting the learning) and curriculum. The course continues to be structured around four individual coding-based math inquiry projects (similar to MICA I-II; O1), complemented with a posteriori revised guided reflections based on selected new readings (O2-3). Two new lab activities were also introduced on learning and comparing coding languages (O2-3). The course concludes with a revised collaborative project, where future teachers work in pairs with a local school teacher to prepare and implement a coding and math activity in the classroom, and to reflect on the experience (O3).

We propose to frame learning in MICA III with a triple instrumental genesis approach that aligns with the three objectives: teachers undergo a *personal genesis* (developing schemes to use coding in their own math learning; O1) and a *professional genesis* (developing schemes to use coding for didactic purposes in math classrooms; Haspekian, 2011; O3). As part of the latter, the teacher must also support *school students' geneses* of coding for math learning (Gueudet et al., 2020; O1-3).

NEXT STEPS

Our next steps include analysing student data (student work and perceptions collected through pre-/post-questionnaires and interviews) to evaluate the design of the course and prepare for the next iteration of course design refinement.

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The reform of mathematics teacher education in Hungary

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Keywords: preparation and training of university mathematics teachers, curricular and institutional issues concerning the teaching of mathematics at university level.

RESEARCH TOPIC

From September 2022, teacher training in Hungary will be reformed in both form and content. As part of this, the training of mathematics teachers will also be transformed [1].

Since 2013, teacher training has been organised in 10 (for primary school teachers) and 12 (for secondary school teachers) semester-long courses. (Eurydice, 2022) Students follow a two-subject teacher training programme, leading to a master's degree. Teacher training consists essentially of subject-specific preparation and teacher preparation. The credits for teacher preparation include credits in pedagogical, psychological theory and practice, in subject methodology, credits for teaching practice in parallel with and following the training, and credits for the preparation of a portfolio, which is compulsory during the training. The highest credit value of the teacher preparation is the continuous individual teaching practice for 50 credits. (Novotná et al., 2021)

Every year for the past few years, I have conducted a survey of graduate students, asking for their views on teacher training. The three problems most frequently reported were: (1) the training is too long (6 years); (2) students meet too late school pupils in the teaching practice during their training (usually only in year 5); (3) the mathematics curriculum is too much and difficult to learn.

RESEARCH QUESTION

What are the structural changes in teacher training programs and what difficulties these changes can correspond to?

RESEARCH RESULTS

The new teacher training lasts 10 semesters (5 years) and the qualification allows the students to teach both in primary and secondary schools. The training leads to a master's degree, but the diploma does not allow teachers to prepare for the higher level final exam. To do so, one must also complete a one-year course of training to become a so-called "master teacher".

The biggest change from the previous training is the completely reformed system of teaching practices. Whereas before, students had 15 hours of teaching practice first in their 5th year of university, the new system will be as followed:

in semesters 2 to 4, there will be socialisation exercises in schools, which are not yet linked to the subject area, students will be introduced to different aspects of school life; in semesters 6 to 7, students will participate in a group teaching practice of their subject, a completely new part of the course, which will be accompanied by a methodology course at the university; in the 8th and 9th semesters, the individual teaching practice mentioned above will take place and the continuous teaching practice in the 10th semester is not a new element in the training neither (but half the length of the recent version).

	Primary school teacher training (5 years)		Secondary school teacher training (6 years)	
	Mathematics	Teaching	Mathematics	Teaching
Credits	100	100	130	100

Table 1: Distribution of credits in parallel teacher training programmes before 2022

	Teacher training (primary and secondary school teachers, 5 years)		Master teacher training (higher level education, 1 year)	
	Mathematics	Teaching	Mathematics	Teaching
Credits	105	90	50	10

Table 2: Distribution of credits in consecutive teacher training programmes after 2022

The answers to the three problems raised are thus: (1) the course will be one year shorter, but those who want to teach students at higher level will still have to spend a total of 6 years at university; (2) there will be a definite positive change, students will meet students and will have the opportunity to try teaching more often and much sooner than before; (3) the same situation as in (1): those who do not want to teach at higher level will have a significantly reduced set of mathematical skills to learn (105 credits instead of 130).

NOTES

1. On the poster the structure of the new mathematics teacher training programme will be presented.

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Design principles for intertwining local and nonlocal mathematics - The case of relating registers and representations in abstract algebra

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Keywords: Transition to, across and from university mathematics, Teaching and learning of linear and abstract algebra, Pre-service teachers, Design principles.

Pre-service teachers often see no relevance in their mathematical content courses for their mathematics teaching (Ticknor, 2012). In this context, mathematical ideas can be distinguished in the local content that will be taught at school and the nonlocal content that is often addressed in university mathematics (Wasserman, 2018). The nonlocal content is supposed to help restructure and deepen the understanding of the local content from a higher standpoint, but many pre-service teachers are not able to make these connections on their own (Hefendehl-Hebeker, 2013). One manifestation of this restructuring is the unification of seemingly unrelated mathematical objects so that they are grasped more coherently under an overarching mathematical idea (Serbin, 2021).

This project aims at supporting pre-service teachers in relating local and nonlocal mathematical content by the design of profession-specific teaching-learning arrangements for abstract algebra learning. Wasserman, Fukawa-Connelly, Villanueva, Mejia-Ramos and Weber (2017) proposed an instructional model for the overall structure of such intertwined learning trajectories, but further empirical research is needed on a more specific task design level and the design principles.

For task design, *relating registers and representations* has proven as a fruitful principle for developing conceptual understanding of mathematical concepts at school level (Prediger & Wessel, 2013). In addition, also in university mathematics teaching relating registers and representations is an activity central to developing understanding of mathematical content (Moreno-Arotzeno, Pombar-Hospitaler & Barragués, 2021). On a more specific task design level, comparing and contrasting as cognitive activities are often used to identify characteristic properties of a concept and are shown to have positive effects on the learning process (Lipowsky et al., 2019). We see a coordination of relating registers and representations with contrasting and comparing as a potential in regards to intertwining local and nonlocal mathematics (e.g. solving $x + 5 = 12$ and solving $d_{120} \circ x = s_1$ in the dihedral group). This leads us to the question: "Which potential and conditions of success can be identified for the design principle *relating registers and representations* in terms of intertwining local and nonlocal algebra?"

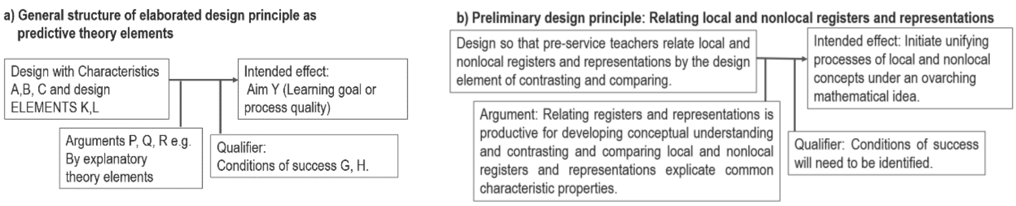

Figure 1: General logical structure (Prediger, 2019, p.6) and project-specific realization

Methodologically, this project follows the design research approach according to Prediger (2019). The formulation of design principles with a general logical if-then structure (see figure 1a) is a central predictive theory element. The poster will focus on the design principle *relating registers and representations* that arose within the first cycle of the design experiments (see figure 1b). Design experiments with secondary pre-service teachers will be conducted, transcribed and analysed with qualitative content analysis. The poster will present insight into the learning pathways and the resulting implications for the effect of the design principle and its conditions of success.

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Promoting favourable beliefs of prospective math teachers concerning the nature of mathematics by using Interactive Mathematical Maps

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Keywords: teacher education, mathematical maps, digital and other resources in university mathematics education, transition to and from university mathematics

INTRODUCTION AND RESEARCH QUESTION

The poster contains information about a study conducted in a recent course among prospective Swedish mathematics teachers at Karlstad University evaluating "Interactive Mathematical Map(s)1" developed at the University of Passau, Germany. In this study, we analyse *to what extent does the use of the digital tool Interactive Mathematical Map(s)*[1] *promote students' favourable beliefs concerning the nature of mathematics*, as described in Felbrich et al. (2008). Such beliefs "are a crucial part of the professional competence of mathematics teachers" (ibid, p. 763). The interactive three-dimensional mathematical map is intended to "offer the student an optimal solution for establishing successful learning processes" (Brandl, 2009, p. 106) by integrating the historical origin of mathematical concepts as well as interdependencies between them. The design-based research and development process of the digital didactical tool Interactive Mathematical Map(s) is described in detail in Przybilla et al. (2022) and Datzmann et al. (2020).

THEORETICAL FRAMEWORK

The study is based on Felbrich's understanding of the belief structure as a one-dimensional space containing four principal orientations: "the *formalism*-related ", "the *scheme*-related", "the *process*-related", and "the *application*-related" orientations (cf. Felbrich, 2008, p. 764). Two "mutually exclusive and antagonistic" (ibid.) alignment poles can be distinguished: a *dynamic* one, represented by *process* and *application* and a *static* one, defined by *formalism* and *scheme*. This means, a "person favours either a dynamic or a static view on mathematics" (ibid.). While the static perspective gives a non-accurate picture of mathematics, which is *unfavourable* for teachers and learners (cf. ibid.), the dynamic view on mathematics emphasizes the subject as an *emerging science* containing and allowing for failures. The latter is considered as *favourable beliefs* related to the nature of mathematics (cf. ibid.). By analysing students' feedback to working with the Maps, we try to understand if favourable beliefs are promoted.

METHODOLOGY AND RESULTS

In the study, the digital tool Interactive Mathematical Map(s) was used in a Geometry course for Swedish mathematics teacher education. In total, 44 students (in-service and

[1] The tool is freely available at the web address https://math-map.fim.uni-passau.de/ under the tab Interactive Map.

prospective teachers) participated in the study. For each of the components of the map a short explanatory video was created. The participants tested the functionalities of the map and produced own texts as application of the mathematical-historical content in the mathematical map during work assignments[2]. The assignments concluded with a technical and content evaluation of the individual components of the map.

As result of qualitative content analysis the students' answers contain clear indications that the map seems to open up for viewing mathematics as an *emerging science*. The considerations about the usefulness of the map included reflections upon the relations to the structure of mathematics and students' opportunities to learn. *Dynamic* understanding of mathematics can be seen in, for instance, students' reasoning about the struggle when mathematicians develop mathematical concepts. This was expressed in, for example, terms of how the historically authentic development of mathematics gives students a deeper knowledge of mathematics and its nature. Most students show a *process*-related and *application*-related orientation in relation to their future role as teachers when using the map. The tool is also seen as a way to motivate their future students to see the relevance of mathematics. Overall, students' reflections suggest that the use of the digital tool Interactive Mathematical Map(s) promotes *favourable beliefs* related to the nature of mathematics.

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TWG6: Students' practices and assessment

TWG6: Students' practices and assessment

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INTRODUCTION AND DISCRIPTION OF THE ORGANIZATION

This report summarizes the work of TWG6 "students' practices and assessment" of the third INDRUM conference in Hannover (Germany). It had 19 registered participants from nine different countries, and 14 contributions – nine papers and five posters.

The themes addressed in the contributions of our group mostly focused on students' practices. These especially covered the following activities: self-regulated and informal learning, problem-solving, programming, communicating about mathematics, and note-taking. But we also had contributions on possible learning barriers as well as on students' self-perceptions and their emotions towards mathematics.

The sessions of TWG6 were organized as follows. In two 90-minute presentation sessions, the authors of the papers were asked to present the main ideas of their paper within 20 minutes, followed by a short slot for essential questions. The authors of a poster could present their poster within 5 minutes. In subsequent discussion sessions the whole group discussed the papers presented intensively. These sessions had been originally planned in three phases: a phase with questions to the authors of each paper in the plenary, followed by small group discussions about these questions, and a final reporting phase. However, our group spontaneously decided to stay in the plenary discussion mode, because already in the first phase with questions to the authors of the papers presented, intensive debates evolved – not only with the authors but within the whole group. Discussions around the posters took place in a separate poster session.

SYNTHESIZED SUMMARY OF THE CONTRIBUTIONS

The group leaders assigned the contributions to one of the two following overarching themes: students' practices in traditional settings and students' practices in innovative courses or related to innovative course elements.

Students' practices in traditional settings

Five papers and four posters had been assigned to this overarching theme. An overview of the topics of the corresponding papers can be seen in Table 1.

The first three papers in Table 1 focus on students' learning behavior in their self-study phase. *Tim Kolbe & Lena Wessel* investigated the learning strategies used by engineering students in a mathematics service course. They specifically found that the students relied much on rehearsal strategies, and used many course-external resources such as YouTube videos. *Robin Göller et al.* compared the self-regulated learning behavior of four students from two different countries: two students from Finland and

Authors	Topic
Tim Kolbe & Lena Wessel	Self-regulated learning by engineering students in a mathematics service course
Robin Göller, Juulia Lahdenperä, & Lara Gildehaus	Self-regulated learning of two German and two Finish students with the common goal of getting though the exam
Lukas Günther, Nico Marten, & Katharina Berendes	Development of an analytical framework for describing informal learning situations in mathematical study programs
Johanna Ruge	Dynamic barriers in students' learning processes
Jocelyn Rios	Multilingual students' experiences in introductory college mathematics courses

Table 1: Papers on students' practices in traditional settings

two students from Germany. They found that although the students shared the goal of passing the exam, their learning behavior differed and was probably influenced by the institutional context, for instance, the possibility to receive institutionalized help in the case of difficulties. The paper by *Lukas Günther et al.* focused on informal learning situations as opposite to institutionalized learning. They developed a framework for analyzing students' activities in such situations in detail, which can make way to suggestions for supporting them in their self-study phase. *Nico Marten* then presented a poster on a project in which this framework is used to analyze how engineering students at a German university gather information for finding answers to mathematical problems, and then to offer appropriate support.

The other two papers from Table 1 focus on barriers that might inhibit students' learning activities. *Johanna Ruge* reflected on so-called "dynamic learning barriers". In such dynamic learning barriers, students inhibit their own learning although they actually want to learn, for example, because of constraints impressed by the institutional environment such as examinations, but also by a curriculum prescribing precisely the subject matter to be learned. Finally, *Jocelyn Rios* showed that also language might be a learning barrier in mathematics classes. She, for instance, found that multilingual students who prefer to do mathematics in a language other than the language of the classroom are less likely to feel comfortable speaking in class.

In addition to the contributions above, we had three posters focusing on students' affective state. *Aaron Gaio at el.* investigated the development of students' self-efficacy with regard to mathematics in their first year at a university in Italy, and found that students tend to overestimate themselves at the beginning of their studies. *Sophia Pantelaki* investigated the development of students' emotions towards mathematics, and found that social interaction might influence these emotions positively. Finally, *Takuo Oguro at el.* focused on a special emotion, namely math anxiety. They found that the math-anxiety of students majoring in social and human environment at a university in Japan did not increase – against their hypotheses. A reason might be that the mathematics courses observed were activity-oriented.

Students' practices with innovative course elements or in innovative courses

An overview over the four papers assigned to this theme is shown in Table 2. In addition, *Johanna Rämö et al.* presented a poster proposing an innovative teaching model that promotes cooperative learning, which builds upon group work facilitated by the teachers in primetime meetings.

Authors	**Topic**
Frank Feudel & Anja Panse	Students' perspectives on suitable positions of blanks in guided notes
Elena Nardi	Students' narratives on exponential growth in colloquial situations
Irene Biza	Students' usage of digital resources for problem-solving
Laura Broley et al. (presented by Chantal Buteau)	Effective orchestration features of a project-based learning course on programming for mathematics investigation

Table 2: Papers on students' practices with respect to innovative course elements or in innovative courses

The foci of the four papers in Table 2 were rather individual. *Frank Feudel* & *Anja Panse* investigated at which elements of a mathematics lecture students appreciate blanks in guided notes (notes with blanks students fill in during the lecture) and why, and found that the extent of appreciation and the reasons for students' preferences varied between different elements of a mathematics lecture. *Elena Nardi* investigated in a mathematics education course that especially aimed at making mathematics visible in daily life and society how students communicate about exponential growth in a colloquial situation related to the Covid-19 Pandemic. She found that many students spoke rather sloppy about exponential growth in this situation although an accurate description might be important for being able to judge the political decisions made.

The latter two papers focused on students' usage of digital tools. *Irene Biza* investigated students' usage of digital resources when solving a problem on divisibility. She found that the availability of such resources influenced students' problem-solving activities, for example because they used these to search for solutions to the problems posed that had been found by others. She then rose the question of how the availability of such resources could be used to create productive learning activities. Finally, *Laura Broley et al.* identified in a project-based course on programming for mathematics investigation what effective orchestration features of such programming courses could be from the students' point of view, for instance, teaching assistants that can push the students to the next step in the case of problems, and a pleasant class atmosphere.

IMPORTANT POINTS FROM THE DISCUSSION

Our group had intensive discussions on each paper. Instead of presenting details of all these discussions, we want to summarize some important general points that emerged during them. These mainly evolved around four themes:

1) Impact of students' practices on teaching and vice versa
2) Methodological issues in empirical research – especially on students' practices
3) More general theoretical issues that arose from the research presented
4) Our identities as researchers

Impact of students' practices on teaching and vice versa: We want to highlight three issues discussed. One refers to digital resources. Their availability has a great impact on students' practices, e.g., on their problem-solving activities. In our discussions, it became obvious that further research is necessary to specify this impact with the goal to design learning situations in which the usage of such recourses and especially the process of seeking and processing information provided by online resources becomes a productive learning activity. A second important issue refers to the tendency to guide students' practices in teaching innovations too much, which might take away their own responsibility for their learning. Therefore, a balance is needed between providing freedom and help for designing productive learning situations. A third important issue that came up in our discussions was the possible effect of active learning on students' practices. There was an agreement that active learning does not only have the potential to change students' practices like their participation in class, but also their view of mathematics, for example about the relevance of mathematics in society.

Methodological issues in empirical research – especially in research on students' practices: We want to mention two important issues here that came up in our discussions several times. One is the problem that research on students' practices often relies on self-reports. On the one hand, activities/phenomena reported do not necessarily coincide with the actual state. Especially a non-reference of certain phenomena does not allow conclusions about these. For example, if students do not mention certain experiences in a course does not mean that these did not occur. Maybe other experiences were just considered more relevant. A further problem of self-reports is that students might have a different perspective on educational notions than researchers, for example, on "understanding". Both issues should be considered when interpreting research results relying on self-reports. A second important issue we want to mention here is the influence of the context on research results. Several of our contributions indicated that the institutional setting and the design of courses research takes place in have a substantial influence on the results. This should be considered when interpreting such research result, and a replication of studies in other contexts might help to specify the influence of the context.

Theoretical issues that arose from the research presented: We want to mention three important issues. One was the issue of using general theories or frameworks in mathematics education research, for example a general framework on informal learning situations or a general note-taking framework. We discussed benefits and problems of using such frameworks. On the one hand, they might allow conclusions for more students (in all kinds of settings), but they might disregard specificities of mathematics, for instance, of mathematical reasoning. A second issue that came up in the discussions was the role of the mathematical content in research on students' practices. As it is one aspect of the research context, it certainly influences the results. However, the actual role the mathematical content plays in a specific study probably depends much on the question(s) investigated. The third important theoretical issue we want to mention here refers to basic educational notions from mathematics education research such as learning, but even the notion "mathematics". We recognized that it is hard to define them and questioned whether a definition is really necessary, in particular, because students also have their own interpretations of these notions.

Our identities as researchers: This was the most general theme that came up in our discussions. Formerly, most of our participants had originally been socialized in the discipline of mathematics. This might influence practices carried out as mathematics education researchers, for instance, the desire to define notions used precisely. However, since mathematics education research is multidisciplinary, we asked ourselves whether our identities might have changed. One view that many of us agreed on is that we have multi identities that we call upon in different situations.

CONCUDING REMARKS AND FUTURE PERSPECTIVE FOR TWG6

Our group had lots of interesting contributions focusing on all kinds of students' activities like self-regulated and informal learning, problem-solving, programming, communicating about mathematics, and note-taking, as well as on barriers that might inhibit students from carrying out such activities and from engaging with mathematics. The contributions were good starting points for fruitful discussions which brought up lots of general issues that are relevant for the mathematics education community at large – and for research on students' practices in particular.

However, some important themes were underrepresented in our group this year:

- We only had few contributions focusing on students' affective and emotional state although these probably highly influence the activities students finally carry out. We therefore hope for more contributions focusing on these themes in the future.
- We only had one study that touched upon the problem of equity although inequity is a big problem in current education – also at university.
- The "assessment" part of our TWG's name "students' practices and assessment" was only touched upon this year in two studies in which portfolio assessments were used for gathering data, although assessment is a very important issue that highly influences students' study behavior.

Therefore, research focusing on these themes might enrich this group in the future much, but, of course, also more research on students' activities considered this year in other contexts to find out to what extent the results presented might be generalizable.

Problem solving in the digital era: Examples from the work of mathematics students on a divisibility problem

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This paper draws on the discussion around the impact of digital resources on problem-solving activities and presents findings from the analysis of eight undergraduate mathematics students' responses to a problem on divisibility that was part of their summative assessment through a portfolio of learning outcomes. The analysis indicates that the availability of digital resources impacts students' problem-solving activity: digital resources provide information useful for problem-solving; they provide answers to the problem; and, they facilitate hypothesis building or execution of time-consuming procedures. A digital resource might be used mainly for a procedural performance of a repetitive task (e.g., trial and error) or it may also include modelling (e.g. programming an algorithm underpinning a repetitive task).

Keywords: Students' practices, problem-solving, digital resources, exploration, divisibility.

PROBLEM-SOLVING AND RESOURCES

Problem-solving is an activity related to tasks students do not know how to approach in advance. What makes a task a 'problem' depends on the context in which this task is encountered, on the available tools and on solvers' precedent experiences (Bosch & Winsløw, 2015; Schoenfeld, 1992). A routine task for secondary students may become a problem for exploration for primary school students. Problem-solving appears in all human activities. Especially in mathematics, problems might be *pure mathematical* problems within a mathematical theory (e.g., proving a conjecture that will lead to a new theorem); *application* problems that are related to real-life situations (e.g., calculating the volume of a 3D shape); or, *modelling* problems that are application problems, in which a transformation of a real-life situation to the mathematical structure is required (e.g., modelling the spread of a virus) (Verschaffel et al. 2014).

Problem-solving activity has been seen through approaches that can guide and organise mathematical explorations – see, for example, the four problem-solving steps proposed by George Pólya (1945): *understand the problem*, *devise a plan*, *carry out the plan* and *look back on your work*. In mathematics education, problem-solving is seen also as a vehicle for students' learning. Teaching through problem-solving opens opportunities for mathematical learning as well as for appreciation of mathematics and its value (e.g., Schoenfeld, 1992; Liljedahl et al., 2016). Problem-solving activities have been also connected to mathematical intuition and affect (Liljedahl et al., 2016). Also, problem-solving activities trigger, or are followed by, discussions and reflection *about* problem-solving. Such discussions have been seen by researchers as: a metacognitive activity (Verschaffel et al. 2014); a *synthesis* of selection, organisation and connection of results obtained during problem-solving that will be useful for future problems (Bosch & Winsløw, 2015); or, as an opportunity for meta-level learning (Sfard, 2008). In the work I am discussing in this paper, I am interested in the last of these, as I see problem-solving activity as an opportunity for reflection on solvers' ways to engage with exploration, conjecturing and verification.

Of relevance to the discussion in this paper is the role of resources in problem-solving activities. Problem-solving is strongly related to the tools that are used and the environment in which the problem-solving takes place (Bosch & Winsløw, 2015). Amongst these tools we can consider analogue tools (e.g., rulers, textbooks, physical models etc.) as well as digital resources such as educational, or other general use software (e.g., Dynamic Geometry Software - DGS, online blogs, etc.) Students and teachers can access platforms with affordances to seek information, to participate in discussions, to ask questions or to experiment with ideas (Santos-Trigo, 2020). The use of digital resources has enhanced the range of mathematical investigations with quick and accurate calculations, reliable drawings, dynamic manipulations of objects and affordances for modelling. Also, digital resources offer more opportunities, in comparison to the analogue world, for conjecturing (abductive reasoning) through exploration of a wide range of cases (e.g., through experimentations in DGS environments or trials with repeated calculations in spreadsheets) before proceeding to a deductive proof of what looks like a plausible response to the problem. Bosch and Winsløw (2015) discuss the dialectic relationship between *questions* and *answers* as an essential component of knowledge development. Answers to questions, when established, become resources for the investigation of further questions "through a variety of *media* (books, journal articles, conference talks, teachers, web tutorials and so on)" (p. 363). Media are not seen in abstract: they are part, and contribute to, knowledge development, in interaction with the institutional context (*milieu*) where questioning and answering are taking place (ibid). Problem-posing and problem-solving that consider media together with "an appropriate experimental milieu" (ibid, p. 21) are essential for students' self-sustaining work with questions and answers.

Recently, some university mathematics programmes have introduced programming courses in which students investigate mathematical ideas, solve problems, and discuss real-life applications of mathematics (e.g., Buteau et al. 2019). In those courses, programming is a means for mathematical investigation as well as for mathematical learning. Gueudet et al (2020) report that programming mediates mathematical enquiry activity in the social context of those who are involved. Very often, programming is one of the range of resources available to solvers that mediate problem-solving. Thus, it is plausible to claim that such spread of available resources (mostly digital) has changed our way of solving problems. This interaction of the problem-solving activity with the available (digital) resources is the focus of the investigation presented in this paper.

For this investigation, I draw on the documentational approach (Gueudet et al., 2014) that has been developed to discuss the interaction of the resources with teachers. In this paper, the attention is on the interaction of problem solvers, and not necessarily teachers, with available resources when they deal with the problem which is not familiar to them. A resource can be anything that informs problem-solving activity, it can be an online blog, a piece of software, a textbook or interactions with others (Trouche et al., 2019; Kayali & Biza, 2021). In contrast to other studies that discuss problem-solving in the mediation with a specific digital technology, here I do not refer to any specific type of digital technology. Problem-solvers would use any resource at their discretion for their investigation. Thus, the choice of the resources, their use and appropriation to the problem-solving activity, as well as the mediation of these resources to the problem-solving discourse are seen together and in interaction. With this conceptual frame in mind, in this paper, I investigate the question: *How does the use of digital resources influence the problem-solving work on an unfamiliar divisibility problem*? I do so through the analysis of undergraduate students' written work on a problem of divisibility, with a particular focus on their use of resources (digital or not). I now present the context of the study, the participants and the problem before discussing examples from students' work.

CONTEXT, PROBLEM, PARTICIPANTS AND METHODS

The examples I discuss in this paper are from the work of eight students who attended a Mathematics Education course for Mathematics (also, occasionally Engineering or Science) undergraduate students. The course is offered as optional to finalist (Year 3) students of Bachelor of Science courses in a research-intensive university in the UK.

The aim of the course (entitled *The Learning and Teaching of Mathematics*) is to introduce students to the study of the teaching and learning of mathematics typically included in the secondary and post compulsory curriculum (Biza & Nardi, 2020). The learning objectives of the course include: to become familiar with Research in Mathematics Education (RME) theories; to be able to critically appraise RME literature and use it to compose arguments regarding the learning and teaching of mathematics; to become familiar with the requirements (professional, curricular and other) for teaching mathematics; to engage with findings from research into the use of digital resources in the learning and teaching of mathematics; and, to practise problem-solving. Contact time is four hours per week (two for lectures and two for seminars) for twelve weeks. Lectures are teacher-led and partly interactive. Seminars are student-led (see details about the course in Biza & Nardi, in press; Nardi & Biza, in press).

"Problem-Solving" is one of the topics discussed in the sessions. Students are introduced to literature on problem-solving (e.g., Verschaffel et al. 2014; Pólya, 1945) in the lectures. Also, students have the opportunity to practise with mathematical problems and reflect on their solution in the seminars. The course is assessed through a *Portfolio of Learning Outcomes* that involves: nutshell accounts of RME theoretical constructs; reflection on students' own learning experiences in mathematics; solving a mathematical problem and reflecting on the problem-solving approach; and, responding to fictional classroom situations (see Biza & Nardi, in press). The examples presented in this paper are from students' responses to the problem-solving item of the portfolio (Figure 1) and their reflections on their problem-solving approach.

If possible, construct a 10-digit number, which is divisible by all natural numbers up to 18, including 18, by using ALL digits 0, 1, 2, … , 9 only ONCE.

Figure 1: A divisibility problem

The problem in Figure 1, is an adaptation of similar problems on divisibility found online in a blog for mathematics teachers and students (https://www.algebra.com/). The problem was chosen because it can be approached with different methods, it requires simple divisibility rules and does not require a known algebraic approach. Also, the problem requires a level of exploration of what the target number might be, without knowing whether such a number exists or not. Such exploration can be done through the use of divisibility rules (e.g., the digits of a number divisible by 9 add up to a number which is divisible by 9 and vice versa), finding the Lower Common Multiple (LCM) of all the divisors of the target number (LCM of 1, 2, …, 18 is 12252240) and, then, finding a multiple of LCM that has ALL the 0, 1, 2, … , 9 digits only ONCE, if this number exists. The last step involves the time-consuming process of checking all the multiples of 12252240 with 10-digits. In fact, there are four numbers that satisfy the conditions of the problem: 2438195760, 3785942160, 4753869120 and 4876391520. Any of those numbers is a sufficient response to the problem that asks to "construct a 10-digit number". As the description in the portfolio indicates, students had the liberty to follow their own way with the problem and use any available resources (including digital tools):

> Any mathematically correct and accurately justified response will receive full marks. In your investigation, you may consider using digital tools (e.g., computer or scientific calculators) and software (e.g., Excel, MATLAB®[1], etc.). In addition to your solution to the problem, you will attach your working on the problem. This is not going to be marked […] It does not need to be tidy or correct; a scanned version of your handwriting suffices.

Data include students' solutions to the problem, their working on the problem and their reflection on their problem-solving approach. Although there was no access to the actual problem-solving activity of the students, I analyse the submitted responses as evidence of what the students chose to report and how they self-reported their approaches to the problem.

EXAMPLES OF STUDENTS' WORK ON THE PROBLEM

Of the eight responses I discuss here, only Student H (for simplicity S-H), followed a deductive approach to the problem. S-H wrote that he accessed the divisibility rules from the *Brilliant* platform of resources for STEM[2]. They named the target number *ABCDEFGHIJ* (where each letter represents a digit of the number) and they applied the divisibility rules to create a set of simultaneous equations, see an excerpt from the

[1] MATLAB®, https://uk.mathworks.com/products/matlab.html

[2] Brilliant, https://brilliant.org/wiki/divisibility-rules/

response (not the entire response) in Figure 2. A logical fault in the steps led to a contradiction that made S-H to conclude that such a number does not exist.

Suppose we had a 10-digit number ABCDEFGHIJ consisting of 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 exactly once each. It must be divisible by every natural number from 1 to 18.

All integers are divisible by 1, so ABCDEFGHIJ is divisible by 1. Also, as it is divisible by 10, the number ends in a 0.

The divisibility rules state that all multiples of 3 have the total sum of their digits equal to a multiple of 3, and ditto with multiples of 9.

0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45, which is divisible by both 3 and 9, so we can construct a number divisible by both 3 and 9 regardless of arrangement.

All multiples of 10 are divisible by both 2 and 5, so ABCDEFGHIJ is divisible by both 2 and 5.

[…]

Divisibility rules for 7, 11, 13 and 17 are a little more complicated. The divisibility rule for 14 is dependent on divisibility by 7.

Anything divisible by both n and 10 divided by 10 is divisible by n, so it is sufficient to find ABCDEFGHI, which is divisible by 7, 11, 13 and 17.

According to Brilliant (the source used), the following must be true at the same time:

- $ABCDEFGH - 2I = 7n$
- $A - B + C - D + E - F + G - H + I = 11k$
- $ABCDEFGH + 4I = 13p$

[…]

Figure 2: An excerpt from Student H's response to the problem

Another student, S-C, calculated how many numbers with 10 different digits exist ("We start with 10! different numbers") and started narrowing down the choices of numbers:

> We start with 10! different numbers. We can identify that the final digit of the number must be 0, otherwise the number would not be divisible by 10. This now leaves 9! different numbers.
>
> To be divisible by 4, the last 2 digits must form a number that is also divisible by 4. This gives us the choices of 2,4,6,8 as possibilities for the 9th digit. By checking through each possibility for the 8th digit and eliminating repeat numbers, we can reduce this to 7! × (32). (S-C's response, original copy)

Then, S-C created a Java program that uses the Heap algorithm (Heap, 1963) to produce and check 10-digit numbers that satisfy the conditions of the problem:

> Having narrowed the choices down and being unable to get any further with the problem, I made a computer program that used Heap's algorithm to check every possibility for the 10-digit number that met the constraints of the problem. This gave four solutions: 4876391520, 4753869120, 3785942160, 2438195760. (S-C's response, original copy)
>
> Since the question only asks for one solution, I chose 4876391520 and checked it was divisible by each number 1-18 manually. This was indeed a solution to the problem. (S-C's response, original copy)

Although S-C started the exploration by narrowing down the range of 10-digit numbers, they did not manage to produce a small enough set of numbers. As a result, their course of action changed and they programmed an algorithm that produces and checks all the 10-digit numbers (10^{10}).

The remaining students calculated the LCM and then tried to find the appropriate multiple of the LCM that satisfies the condition of the problem. One of them, S-A, identified the LCM correctly but could not work out an approach, other than trial and error in a range of numbers as they describe below:

> I must admit that I was unable to come across this number on my own mathematical ability alone as I could not work out a way, other than a simple trial and error approach, to complete the problem without assistance. […]
>
> From here I looked to find what ballpark number [roughly estimated number] would be needed to multiply my LCM to get a 10-digit number. It was clear that some value in between roughly 100 and 1000 would give me the required result. Other than plugging some very random values into my calculator, this is where I hit a wall. I eventually crumbled and resorted to researching online to find a method or some sort of answer by anyone who had done [on a] similar problem [sic]. After some searching, I found a website in which people submit different problem solving questions and people try and give their answers. Someone had already submitted this question [the problem in Figure 1] and people had gone about it in a similar way to myself. One person had written a computer program which gave back several 10 digit numbers constructed from the digits 0,…, 9 which supposedly were divisible by the natural numbers up to and including 18.
>
> I checked that this number, **2438195760** [their emphasis], was divisible by my LCM, which it was meaning that this 10-digit number is indeed divisible by the natural up to and including 18 using each digit only once. (S-A's response, original copy with my additions in square brackets)

S-A found the LCA, but "hit a wall" in their effort to find the right number. They could not see any option other than "plugging some very random values". So, they felt that they cannot solve the problem on their "own mathematical ability" and sought help from somebody who has solved a similar problem. So, with appropriate search, they found a webpage[3] that includes a discussion on, and a proposed solution to, the problem. In this webpage, S-A found a response to the problem by somebody who had "written a computer program". It is not clear whether S-A attempted the computer program or not and how they ended up with the right number (which they then checked whether it was divisible by the LCM). If S-A took the number from the website, as the outcome of the work somebody else "had done [on a] similar problem", their role as problem-solver was to verify whether this number satisfies the given conditions or not.

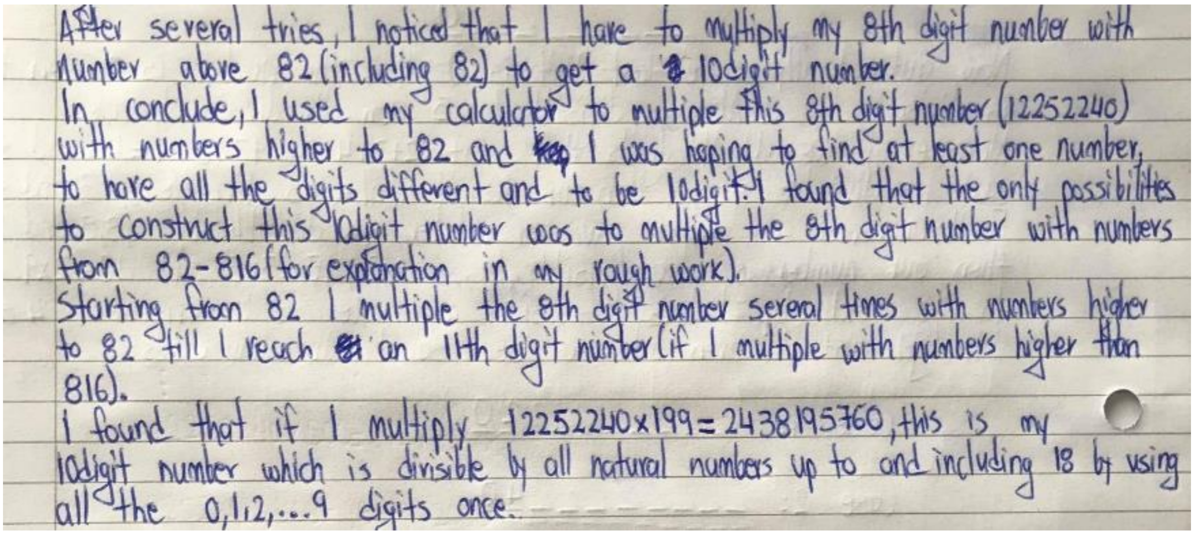
[3] The website S-A mentions in their response is: https://www.algebra.com/algebra/homework/word/misc/Miscellaneous_Word_Problems.faq.question.58446.html

So, instead of exploring whether a number with certain properties exists, S-A ended up confirming whether a number found by somebody else has these properties or not.

S-D, S-F and S-G calculated the LCM as well and then identified the target number with trial and error. S-D, for example, used the (ANS+NUMBER) functionality of the calculator and checked the answers one by one:

> Solving the LCM as 12,252,240 I then used trial and error on my calculator (ANS + 12,252,240) and visually checked each answer for one that met the conditions of the problem. (S-D's response, original copy)

S-F and S-G identified a range where the multiplier might be located (in the interval 82-816 for S-F and in the interval 101-199 for S-G) before performing their trials (Figure 3).



After several tries, I noticed that I have to multiply my 8th digit number with number above 82 (including 82) to get a ~~4~~ 10digit number.
In conclude, I used my calculator to multiple this 8th digit number (12252240) with numbers higher to 82 and ~~keep~~ I was hoping to find at least one number, to have all the digits different and to be 10digit. I found that the only possibilities to construct this 10digit number was to multiple the 8th digit number with numbers from 82-816 (for explanation in my rough work).
Starting from 82 I multiple the 8th digit number several times with numbers higher to 82 till I reach ~~a~~ an 11th digit number (if I multiple with numbers higher than 816).
I found that if I multiply 12252240×199=2438195760, this is my 10digit number which is divisible by all natural numbers up to and including 18 by using all the 0,1,2,...9 digits once.

Figure 3: An excerpt from Student F's response to the problem

The choice of 199 as the upper boundary by S-G (199 is the first multiplier that gives a target number) sounds quite precise. This boundary might have chosen retrospectively after S-G had found the target number, but this is a speculation that cannot be verified. However, it seems that S-G is not convinced that the trial and error is the best approach to the problem, as they acknowledge:

> Although I am happy that I found the correct solution, I feel that my approach was not the most efficient. If I had a better comprehension of [a] mathematical programming tool such as MatLab I could have produced a code that would have eliminated a lot of the tedious calculation that took up a lot of time. (S-G's response, original copy)

S-B and S-E overcame the tedious part of calculating possible numbers by using a spreadsheet. As S-B wrote:

> To begin I found that the lowest common multiple of all of these numbers is 12252240. Then, as the number must be divisible by 10, it must end in zero. This means the smallest and largest possible numbers are 1234567890 and 9876543210, respectively, so dividing both by 12252240 give about 100 and 708. I then made a spreadsheet of the multiples of

> 12252240 from 100 and 708. I then checked all of these numbers to see if they satisfied the requirements. I found the numbers 2438195760, 3785942160, 4753869120 and 4876391520. (S-B's response, original copy)

S-E took a creative step by narrowing down possible numbers in the spreadsheet:

> I used Excel to calculate multiples of the LCM in the required range. Some blocks that could be ignored were easily identifiable and shaded out (those with the first and second digits the same, and with a 0 as second digit as well as last). I then scanned the remaining numbers and ignored those with digits repeated. (S-E's response, original copy)

It seems that the spreadsheet facilitated the generation of LCM multiples, similarly to the repeated additions (or multiplications) other students did with the calculator. However, in the spreadsheet, the whole range of numbers was provided, instead of producing one number after another in a calculator. In a spreadsheet, the identification of patterns is easier, as is the elimination process – exactly as S-E did.

DISCUSSION

Findings presented in this paper aim to contribute to the discussion around the impact the availability of digital resources may have on problem-solving work. Specifically, I draw on the work of eight undergraduate mathematics students on a problem to investigate the question: *How does the use of digital resources influence the problem-solving work on an unfamiliar divisibility problem*? The examples indicate three observations.

First, online resources might be used as a source of information (e.g., definitions, rules, etc.) that feeds the problem-solving activity (e.g., S-H search online to find divisibility rules). Such resources become *documents* (Gueudet et al., 2014) for solvers and influence their approach to the problem. The accuracy of those resources, and whether such accuracy was checked by students, is not discussed in this paper. However, personal experience has indicated that uncritical use of information may mislead problem-solving activity. For example, one result of a Google search for *what a polynomial is* might be the inaccurate statement: "an expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable(s)".

Second, an online search may aim to identify responses to a problem provided by others; searching for *answers* to questions (Bosch & Winsløw, 2015). This is well connected to everyday practices of seeking responses to enquiries through a search to the web for what other people have done in a similar situation (e.g., Yeoman et al., 2017). Finding what other solvers have done to a similar problem shifts the nature of problem-solving activity from explorative to confirmatory (e.g., S-A confirmed whether the number they found online meets the criteria instead of identifying such number). Solvers search with appropriate keywords, interpret a solution they have found and confirm that the proposed solution is right. Thus, exploratory routines of problem-solving activity – for example, conjecturing and testing – change to routines such as: unpacking the problem for search purposes; interpreting others' work; or, accepting the work of others, sometimes after verification or sometimes uncritically.

Third, digital resources might facilitate hypothesis building or execution of time-consuming procedures. This may lead to a less productive engagement with procedural performance of a repetitive task (e.g., pressing the button in a calculator) or to creative engagement with mathematical modelling (e.g., programming an algorithm that can produce and examine range of cases effectively).

I note that the students worked on the problem for the purpose of summative assessment with the liberty of using any resource available to them. The examples discussed in this paper draw on students' self-reported responses and not on the observation of students working on the problem. As a result, the examples reflect what students have chosen to report. For example, students might have found the right number through an online search and then constructed a narrative about the process through which they reached a solution retrospectively. Future research should draw on the observation of students' actual activity with consideration of the resources that are available and the context in which this activity takes place (media-milieu interaction, Bosch & Winsløw, 2015).

In conclusion, as the availability of digital resources impacts problem-solving activity, further research should provide more insight into such impact first, and then propose problem design that factors in this impact. It is plausible to assume that solvers will keep seeking help from digital resources and keep looking for what others have done in similar situations. A question is how we make sure that solvers are prepared to manage such abundance of resources productively and to their learning benefit.

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Effective orchestration features of a project-based approach to learning programming for mathematics investigation

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In this exploratory study, we examine the teaching of programming for mathematics investigation in an undergraduate project-based learning environment. Using instrumental orchestration as our theoretical framework, we explore the orchestration features that students consider to be the most effective for supporting their learning. A qualitative analysis of 43 students' questionnaire responses led to the identification of such features, regrouped in 5 main themes (help and support, organization of the course, instructor interventions, instructor characteristics, and class atmosphere). Results suggest that students recognize the need for their instrumental geneses to be steered and highlight the importance of individualized interventions and a supportive learning environment.

Keywords: Teachers' and students' practices at university level, digital and other resources in university mathematics education, instrumental orchestration, project-based learning, programming

INTRODUCTION

Research has documented many affordances of programming for student learning of mathematics at the university level. Studies have shown how programming can support students' activity and understanding in specific areas, such as calculus, abstract algebra, combinatorics, statistics, and probability (Buteau et al., in press). It has also been argued that programming activities can engage students in crucial mathematical disciplinary practices (Broley et al., 2017).

In their recent call to the international community of research on university mathematics education, Lockwood and Mørken (2021) argue that much more needs to be investigated concerning the integration of programming, including effective instructional interventions, teaching practices, and didactic models:

> there are already varying models around the world for how computing is being integrated into mathematics classes and programs, and we see opportunities for systematically studying different ways for this integration to occur. […] the RUME community can explore what kinds of programs are effective and why. (p. 6)

This paper addresses the above area of need by exploring effective features of a project-based learning (PBL) model for integrating programming in university mathematics. As an integration approach, PBL engages students in actively constructing knowledge by working through an inquiry process that is structured by authentic and complex tasks (Shpeizer, 2019). PBL has been implemented in various academic institutions and fields, with the literature generally arguing for its positive effects on learning outcomes (Thomas, 2000). In the context of programming-based math education, PBL has deep roots: it has been over 50 years since Seymour Papert pointed to the potential of engaging students in meaningful projects in which they construct (i.e., program) computer environments to learn and do mathematics like mathematicians do. Papert (1980) emphasized that such a "constructionist" approach requires a fundamental change to "traditional" teaching methods: from presenting mathematical ideas to students, to creating didactic conditions that foster students' own pursuit of ideas.

In the current study, we explore students' perspectives on what those conditions might be: in particular, how university math instructors may support students' learning to use programming for math investigations in a project-based approach. Our exploration takes place within a "natural constructionist environment" (Buteau et al., 2015) that has been implemented for 20 years at Brock University (Canada) and as part of a larger 5-year (2017-2022) iterative design research that uses that environment to study how students learn to use programming in authentic pure or applied mathematics projects, if and how that use is sustained over time, and how instructors support that learning.

THEORETICAL FRAMEWORK

In our larger research, we have shown that the instrumental approach (Artigue, 2002) can be useful in studying the teaching and learning of using programming for math investigation projects at the university level. In Gueudet et al. (2020), we used the notion of instrumental genesis to better understand how students transform a programming language (an artefact) into a math investigation tool (an instrument for accomplishing the goals involved in math investigation projects) by developing instrumented action schemes. In this study, we are interested in looking at features of instructional practice that can support this instrumental genesis. Following Buteau et al. (in press), we frame our study using the notion of instrumental orchestration.

Trouche (2004) introduced the notion of instrumental orchestration to highlight the necessity of an external steering of students' individual and collective geneses and to describe the instructional decisions and actions involved. More precisely,

> an *instrumental orchestration* is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a—in this case computerised—learning environment in a given mathematical task situation, in order to guide students' instrumental genesis (Trouche, 2004). (Drijvers et al., 2010, p. 214)

Building on the work of Trouche (2004), who introduced "didactical configuration" and "exploitation mode" as two key components of an instrumental orchestration, Drijvers et al. (2010) introduced a third component, the "didactical performance," and defined the three components as follows:

> A *didactical configuration* is an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it. …
>
> An *exploitation mode* is the way the teacher decides to exploit a didactical configuration for the benefit of his or her didactical intentions. This includes decisions on the way a task

> is introduced and worked through, on the possible roles of the artefacts to be played, and on the schemes and techniques to be developed and established by the students. …
>
> A *didactical performance* involves the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode: what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals. (p. 215)

Studies in math education have employed the notion of instrumental orchestration with various technologies (graphing calculators, dynamic geometry software, spreadsheets, …), mainly at the school level. For instance, Drijvers et al. (2010) used the notion to study the use of applets with eighth-grade students and began cataloguing different orchestration types (e.g., *Technical-demo* and *Explain-the-screen*). At the time of writing this paper, we are unaware of other research on the orchestration of programming in investigation projects, i.e., in a PBL approach (Buteau et al., in press).

PBL is an instructional model that organizes learning around projects. According to Thomas (2000), such projects, "as well as the activities, products, and performances that occupy [students'] time, must be orchestrated in the service of an important intellectual purpose" (p. 3). In terms of teaching, PBL redefines the traditional role of the teacher as one of collating sources, facilitating thinking, and inspiring students to impact the world, with class time used to probe students about their sense-making and skills acquisition (Prince & Felder, 2007). This changing role of the teacher is seen as a key challenge in successfully implementing PBL in the classroom (Shpeizer, 2019).

In light of the framework outlined above, we pose the following research question: What are the features of an effective orchestration of programming for mathematics investigation in a project-based approach?

METHODOLOGY

Our study is situated in the context of three semester-long project-based math courses (MICA I-II-III) offered at Brock University. In these courses, math majors and co-majors (including future math teachers) learn to use programming to investigate math concepts, theorems, conjectures, and real-world applications (Buteau et al., 2015).

Our past work has examined the instrumental orchestration of programming in MICA courses primarily from the institution's and instructors' points of view (e.g., Buteau et al., in press). Using institutional documents and interviews with an experienced MICA instructor (who was also involved in the development of the MICA courses), key orchestration features were highlighted and discussed: e.g., the teaching format (part of the didactic configuration), which includes weekly 2-hour lectures (where the instructor introduces the math) and 2-hour labs (where the students work on projects); the assessment structure (part of the exploitation mode), the heart of which (70-80% of students' grades) are 4-5 investigation projects developed by the instructor; and the kind of help (both individual and collective) given to students to support their work in labs (part of the didactic performance). In this study, we complement this past work by examining key features of the orchestration from students' points of view.

Part of years 2-4 of our larger research included students being invited to voluntarily respond to questionnaires administered before (pre) and after (post) each MICA course. There were various sections featured in the questionnaire, including demographics, students' perceptions of the importance of programming, confidence in programming, etc. The question we consider in this study is taken from the post-questionnaire, where students in years 3-4 were asked to indicate and elaborate on (by writing a text) what their instructors or teaching assistants (TAs) did that had the most impact on their assignment work or learning in any of the MICA courses they had taken so far.

In this study, we analyzed the responses of 43 MICA students from years 3-4 (2019-20) of the research (25 from MICA I, 5 from MICA II, and 13 from MICA III). Responses were coded independently by two coders and then codes were consolidated. Codes were then grouped into themes and sub-themes using an emerging theme approach. Finally, we reflected on our results using our theoretical framework. We note that some participants' responses were coded with several codes, possibly within more than one sub-theme or theme. Also, our findings are representative of every participant's voice: even if a sub-theme emerged from only one participant response, we considered it valuable to include it in our results since, in this initial study, we aim to identify possible features of an effective orchestration (from students' points of view). Given the relatively small sample size and voluntary participation, we also note that participants are not necessarily representative of all MICA students.

RESULTS: MOST IMPACTFUL ORCHESTRATION FEATURES

After coding and consolidation, 5 main themes and 16 sub-themes emerged, as synthesized in Tables 1-5. These themes characterize orchestration features that, according to students, had the most impact on their assignment work and learning. Given the context in which we work (as described above), we interpret these as being features that may contribute to an effective orchestration of programming for mathematics investigation in a project-based approach. In the following, we describe the themes and sub-themes, including several illustrative quotes from students.

Help and support

Many students indicated the most impactful thing the instructors or TAs did included providing help and support (see Table 1, with sub-themes and descriptive quotes). Some students spoke in a general sense, simply highlighting that they were given (lots of) help, while some (also) specified the part of the course on which they received the help or support (assignments, programming, and/or mathematics).

When mentioning "help with programming," some students specified further the part of the programming process with which they received help (e.g., "I was able to … get some help *debugging some minor errors*"). Other students emphasized impactful approaches used by instructors or TAs to provide support for programming, including:

- *giving one-on-one help with programming* ("When I would come for help, the TAs would look through my program and then let me know where the error may be and would hint at how to fix it");
- *providing additional coding information to the class, on-demand, just-in-time* ("Some help hours TAs or instructors realized everyone was having the same issue and would provide some extra coding information to incorporate in the assignment that would help everyone understand the project further");
- *displaying and explaining example codes* (e.g., "It was very helpful when example code was displayed on the screen and explained. It helped to see how it could actually be used"); and
- *making example codes available* ("I was able to download codes from lectures and play around with them and change things so that I was able to better understand how the different codes worked").

Sub-Themes	**Descriptive Quotes**
General help	Helping me when I was stuck.
Help on assignments	With the help of the instructors/TAs I was able to understand the assignments better.
Help with programming	… they provided help for me when I was stuck and didn't know what to do next while I was programming for an assignment.
Help with math	it is also helpful when … [they] know how to help you when you have a problem with both the mathematics and the programming.

Table 1: Sub-themes and descriptive quotes for the Help and Support theme

We interpret this theme as an implicit recognition by students that their instrumental genesis needs to be steered. The "help with programming" sub-theme highlights students' views about different ways in which an effective steering may occur, including certain orchestration types: e.g., *Discuss-the-screen* is connected to *displaying and explaining example codes*.

Organization of the course

Some students' responses pointed to features associated with the format of the course; in particular, the different modes by which help was made available to them by the instructors or TAs, and the general organization of course content (see Table 2). Students explicitly mentioned several ways in which help was made available to them outside lectures, including labs, help hours, emails, and (extra) office hours.

In terms of instrumental orchestration, we interpret this theme as mainly describing elements of the didactic configuration and exploitation mode. The various modes of making help available extends the opportunities for students to experience the same kinds of interventions as in a *Work-and-walk-by* (Drijvers, 2012) classroom orchestration. This suggests that students' instrumental geneses require readily available individualized help/interaction with a mentor.

Sub-Themes	Descriptive Quotes
Modes of making help available	I really like the format of *labs*; it gave me a chance to talk to the instructor or TA about my assignment and also to get help if I needed it. Proving *help hours* so I didn't have to struggle on my own. I was able to access help easily during *help hours* and reach my instructor through *email*. He gave very helpful tips even through *email.* They added *extra office hours* for assignments when needed…
Organization of course material	…the [online] course work was all nicely in one spot. It was easy to find and easy to understand what had to be done. The assignments were also nicely broken up into questions and parts of questions. It was just really organized, and I appreciate that.

Table 2: Sub-themes and descriptive quotes for the Organization of the Course theme

Instructor characteristics

Another theme that emerged expresses students' views on certain impactful "ways of being" of instructors or TAs: according to students, they were not only knowledgeable, but also available, kind, and supportive (see Table 3).

Sub-Themes	Descriptive Quotes
Available	Their anytime response to our doubts irrespective of their schedules.
Knowledgeable	It is also helpful when they're knowledgeable and actually know what they're talking about and know how to help you…
Kind	He was so incredibly kind.
Supportive	Programming was brand new to me in [MICA I] and it was extremely intimidating (still often is) so it is nice to have helpful and supportive instructors and TAs.

Table 3: Sub-themes and descriptive quotes for the Instructor Characteristics theme

Some of these instructor characteristics could be interpreted as linked to the exploitation mode. In particular, being available and supportive towards students aligns with the expectation that students will need a lot of individualized support. Instructors may plan these "ways of being" in order to offer this support.

Instructor interventions

Some students also mentioned specific and effective (according to them) ways in which instructors or TAs facilitate their work or learning (See Table 4).

Sub-Themes	Descriptive Quotes
Ways of providing help	My TA would break the projects down for me to a level that I would understand. Which allowed me to be successful in the course.
Feedback on assignments	The TA and professor would let me know where I lost marks which made me improve those things for future assignments.
Intervention for high-achieving students	I found that I was able to complete most assignments rather quickly. As such, the prof. would often give me ideas that would be difficult to implement and allowed me to brainstorm how I would implement these tasks … These difficult tasks allowed me to learn concepts and think outside the box far more than if I was to just complete assignments as they are written.

Table 4: Sub-themes and descriptive quotes for the Instructor Interventions theme

The "ways of providing help" sub-theme was associated to a rich collection of responses, which specified different methods that instructors or TAs used while offering them the help they needed. In addition to *breaking down content to a student's level of understanding* (exemplified in Table 4), students described the following impactful ways of providing help:

- *re-explaining multiple times when needed* ("taking the time to go through it with me multiple times when I didn't understand something");
- *explaining what a student is doing wrong and why* ("they would inform me what I was doing wrong and WHY it was wrong. By doing this, I can grow and learn from the experience");
- *providing a full explanation* (*"*The professor would always fully explain the issue rather then giving a half-hearted cryptic help response. Sometimes teachers try to give a little hint in hopes you'll figure it out yourself. But I wouldn't so getting a lesson about what went wrong is more helpful");
- *guiding towards rather than telling the answer* (e.g., "They never said 'figure it out' but they helped guide us to the correct answer without fully saying 'here it is'"); and finally
- *giving meaningful answers* ("they didn't give vague answers, they truly did help you").

We interpret this theme as describing elements of the didactic performance. The "ways of providing help" sub-theme suggests effective (from students' points of view) individualized interactions that may occur, for example, in the *Work-and-walk-by* orchestration during labs.

Class atmosphere

Finally, some students' responses pointed to the kind of environment created by the instructors or TAs to foster students' learning (see Table 5). Students indicated that the class was a space where they felt safe to ask questions, encouraged to make contributions, and able to work on their own if they wanted.

Sub-Themes	Descriptive Quotes
Safe to ask questions	Everyone in the class knew they could ask [the TA or instructor] any question and receive a helpful and cheerful answer. They created an environment where students weren't afraid to ask questions and that is what was most needed to fully understand the content.
Encouraged to contribute	Instructor encouraged us to attempt to formulate our own theorems before resorting to finding a pre-existing one to study. Instructor took an interest in the fact that I had recently heard of steps towards cracking the Collatz conjecture.
Can work on own if want	I didn't ask for a lot of help on assignments since I enjoy working things out myself and any time that I got stuck, I was able to get unstuck again.

Table 5: Sub-themes and descriptive quotes for the Class Atmosphere theme

We interpret this theme as being part of an instructor's exploitation mode, which includes how they will present tasks, how students will work through the tasks, and the atmosphere that will surround that work. The fact that students need individualized support as they work through tasks appears to provide guiding principles to how the instructor creates the atmosphere (by explicitly inviting students to ask questions, responding to students' questions in a "kind and cheerful" manner, etc.).

DISCUSSION

In this paper, we answer the call from Lockwood and Mørken (2021) for more research about effective instructional models for integrating computing in university math education by exploring features of an effective orchestration of programming for mathematics investigation in a project-based approach. Our study contributes to the literature involving the instrumental orchestration frame by (1) using it to examine PBL, a particular instructional approach that has not yet been examined using the frame (Buteau et al., in press); and (2) identifying most impactful features from students' points of view, which, to our knowledge, has not been done. Our study also contributes to literature on PBL, in which there is a lack of studies specifying the required teacher's role for a successful implementation (Shpeizer, 2019). Our results align with some key elements that have been identified: e.g., the creation of a safe learning environment, the encouragement of students (including to ask questions), and the importance of formative and summative assessment (Pan et al., 2021).

In this study, we analyzed questionnaire responses from 43 students using an emerging theme approach, which led to 16 sub-themes organized by 5 main themes: help and support, organization of the course, instructor interventions, instructor characteristics and class atmosphere. These themes characterize orchestration features that, according to students, had the most impact on their work and learning and, therefore, may be inferred to contribute to an effective orchestration of programming for math investigation in a PBL approach. Interpreting the themes using the instrumental orchestration frame points to features that were not made explicit in the description of its three components (Drijvers et al., 2010): e.g., "class atmosphere" as a feature of the "exploitation mode," or "feedback on assignments" (ad hoc decisions occurring outside the classroom) as a feature of the "didactic performance." Responses from students also highlight a need, specific to the university level, of considering TAs as additional players, who have orchestrations of their own, which are shaped by and situated within an instructor's orchestration. Our interpretation of the identified themes also suggests some elements that may be specific to a PBL approach (in comparison to a "traditional" one): e.g., the "organization of the course" theme suggests that for a PBL instructor, it may not always be helpful to instruct the entire class based on one person's issue (they may expect students to require individualized support). In relation to this, we propose a new orchestration type (Drijvers, 2012): *Work-and-reach-out- when-needed*.

This initial exploratory study sets the ground for future work examining more deeply the impact (or effectiveness) of the different features we have identified. Some students elaborated on their perception of the impact of instructors' or TAs' actions on their learning or completion of projects: e.g., with respect to the creation of a learning environment where it is safe to ask questions, one student said that "that is what was most needed to fully understand the content." Future work could look more systematically at the impacts of different orchestration features on students' learning and project work. This could inform recommendations for practice, especially at a time when computing is becoming more integrated across mathematics education.

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What are suitable positions for blanks in guided notes? – An investigation of students' perspectives

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One approach to address the problem that in mathematics lectures students are often busy writing, and cannot pay attention to the instructor's oral explanations might be the use of guided notes. These are a modified version of the instructor's notes that contain blanks the students have to fill in during the lecture. However, the extent to which the use of guided notes actually supports students during their note-taking probably depends on the positions of the blanks. We investigated students' perspectives on where to put the blanks by exploring in which elements of a mathematics lecture they appreciate blanks and why. The results yield some suggestions on where to put blanks and preprinted parts in a guided notes script, so that many students might benefit from guided notes for their note-taking in mathematics lectures.

Keywords: students' practices, novel approaches to teaching, note-taking, guided notes, mathematics lectures.

INTRODUCTION AND EMBEDDING OF THE RESEARCH

In traditional mathematics lectures, the instructor usually writes the definitions, theorems, and proofs on the board, and provides additional oral explanations (Artemeva & Fox, 2011). In such lectures, students are often busy copying everything correctly, and have trouble paying attention to the instructor's explanations (Freitag, 2020). This could make it hard for them to gain an understanding of the content during the lecture. However, making sense of the content after class solely on the basis of the notes taken is also a hurdle for many students, because they often only note down the definitions, theorems, and proofs the lecturer has written on the board, and not the additional oral explanations that are especially important for making sense of the content (Fukawa-Connelly et al., 2017), e.g., ideas behind the formal proofs.

One approach to address these problems might be the use of *guided notes.* These are preprinted lecture notes with blanks at certain places that the students are required to fill in as the lecture progresses (Austin et al., 2004). There are some contributions in the mathematics education literature on the effect of guided notes on students' note-taking and their learning. Cardetti et al. (2010), for instance, investigated the effect of guided notes on students' learning in a calculus course by comparing the exam results of the years 2006 and 2007. In 2006, the students had to take notes solely by themselves, while in 2007, they received parts of the instructor's notes as a script, and only had to fill in blanks (the instructor was the same in both years). Cardetti et al. (2010) then found that the students in the guided notes group attained better results although they scored lower in the SAT entry test. Furthermore, students in this group particularly liked about the guided notes that they could follow the lecturer better, and did not have to concentrate on writing exclusively. Similar results were found by Iannone and Miller (2019) who investigated students' attitudes towards guided notes in a qualitative study. Furthermore, Iannone and Miller (2019) found that some students also noted down the instructor's oral comments or added own comments, which can particularly help to recall and to make sense of the content covered later on the basis of the notes taken. We also confirmed these findings in a course with a larger sample in a study using a mixed-methods design (Feudel & Panse, 2022). Hence, overall, the literature suggests that guided notes can be on the one hand an aid for processing the information noted – the so-called *encoding function* of note-taking (Di Vesta & Gray, 1972; Kiewra, 1989) – and for creating notes that are useful in a later review phase (the so-called *storage function*).

The extent to which guided notes actually support students during their note-taking, however, possibly depends on the positions of blanks and preprinted parts. Instructors using guided notes, for instance, often leave blanks in examples or in problems students might be able to solve by themselves or with the help of a discussion in class (Alcock, 2018; Cardetti et al., 2010; Feudel & Panse, 2022; Tonkes et al., 2009). This can foster students' active engagement with the corresponding content, and help them process the information presented already during the lecture.

But at which text modules of a guided notes script do *students* consider blanks as useful and why? Their opinions on this matter add an important further perspective on where to put blanks in a guided notes script, which can especially help to derive empirically grounded recommendations for the design of such a script. Research on this issue is, however, currently limited. In the study by Cardetti et al. (2010) mentioned above, one student appreciated that definitions had already been preprinted, and did not have to be written down. Further results on the question of where to put blanks were found by Tonkes et al. (2009). They explored to what extent students of a large first-year mathematics course appreciated blanks within a distributed script, by surveying them about the perceived usefulness of the blanks and about the extent to which the students actually filled in the blanks by themselves. The corresponding questionnaire especially contained Likert items asking whether blanks should be left in examples, definitions, or when introducing new mathematical techniques. While there was no clear preference about whether blanks should be left at the latter two positions, many students strongly agreed that blanks should be left in examples. However, the studies by Cardetti et al. (2010) and Tonkes et al. (2009) did not systematically investigate which parts of a mathematics lecture should be preprinted or left blank within a guided notes script. Furthermore, these studies did not provide reasons for students' preferences that might help to find out why students consider blanks as helpful at certain positions for their note-taking. The research we present here attempts to fill these gaps.

THEORETICAL FRAMEWORK

For investigating why students might consider blanks or preprinted material at certain positions in a guided notes script as helpful for their note-taking, we used for our data analysis a theoretical framework that particularly considers students' goals of note-taking: the college students’ theory of note-taking by Van Meter et al. (1994). On the basis of interviews with 252 students from different disciplines about their perceptions of the functions of note-taking, their regulations of note-taking, and their beliefs regarding factors that influence their note-taking, Van Meter et al. (1994) developed a framework describing different aspects of students’ note-taking behavior. According to them, students have the following goals of note-taking (p. 338):

1) Increase attention,
2) Increase comprehension and memory of the lecture content,
3) Organize the material presented,
4) Inform about the content of the exam, and
5) Inform about solutions to practice problems and provide information relevant for written assignments.

Although Van Meter et al. (1994) restricted the last point to homework assignments in their framework (they called it “homework aid”), we want to view the term “written assignment” broader as any assignment students are required to work on in a course – also in the exam. This also fits to the data Van Meter et al. (1994) presented to goal 5, as the students they cited only mentioned that writing down worked examples helps to find solutions to similar problems, and did not refer to homework explicitly.

Besides the goals of note-taking, Van Meter et al. (1994) describe further aspects of students’ note-taking in their framework: content-structure of students’ notes, contextual factors affecting students’ note-taking, and students’ post-class processing of the notes. Since in the study presented here, we wanted to investigate at which positions of a lecture students appreciate blanks or preprinted parts in a guided notes script, and tried to explore reasons for their choices, we particularly referred to the goals of note-taking when analyzing our data. Therefore, we refrain from describing the other three aspects of students’ note-taking in the framework by Van Meter et al. (1994) in detail here (for details, see Van Meter et al. (1994), p. 337).

EXTERNAL SETTING OF THE STUDY

The study took place in an introductory mathematics course called “Introduction to mathematical thinking and working” for teacher students in Germany, who will teach mathematics up to the end of secondary level. The course consisted of one 90-minute lecture and one 90-minute tutorial per week, and the students had to complete written assignments each week. The topics were elementary number theory (divisibility, prime factorization, residue classes, Euclidean algorithm), sets and relations, basic algebraic structures (groups, rings, fields), and the construction of the number systems.

The instructor taught the course on the basis of a guided notes script following the idea of Alcock (2018). Before each session, she distributed a paper script with blanks at certain positions. A sample page can be seen in Fig. 1.

7.2 Rings

Definition 7.3 (Commutative ring). A set $R \neq \emptyset$ with an addition $+ : R \times R \to R$ and a multiplication $\cdot : R \times R \to R$ is called *communative ring*, if

- $(R, +)$ is
- $(R, \cdot)$ is
- (D) Distributive property:

If $(R, \cdot)$ furthermore has a neutral element e, $(R, +, \cdot, e)$ is called commutative ring with *one*.

Example 7.19. $(\mathbb{Z}, +, \cdot, 1)$ is

Example 7.20. $(2\mathbb{Z}, +, \cdot)$ is

Example 7.21. $(\mathbb{Q}, +, \cdot, 1)$ is

Example 7.22. For each $m \in \mathbb{N}$ the set $\mathbb{Z}/_{m\mathbb{Z}}$ of the residue classes modulo m with the corresponding addition and multiplication is

Proposition 7.5. *Let* $(R, +, \cdot)$ be a commutative ring and 0 the zero of $(R, +)$. Then the following holds:

(i) $\forall x, y, z \in R: \ (x + y)z =$

(ii) $\forall x, y, z \in R: x(y - z) =$

(iii) $\forall x \in R: \ 0x =$

(iv) $\forall x, y \in R: (-x)y =$

Proof. (left blank)

Figure 1: Sample page of the guided notes script used (see also Feudel and Panse, 2022, p. 15)

The instructor made the decisions for the positions of the blanks on the basis of her teaching experience from former years, in which the course was taught traditionally. She especially left blanks at positions,

1) at which she assumed that students could complete the blanks by themselves,
2) at which she considered statements as particularly important, or
3) at which she suspected particular problems in making sense of the content.

During the lecture, she initiated activities at the blanks that involved the students, e.g., discussions about different options for filling the blanks (for further details see Feudel and Panse (2022)). After the lecture, she made the completed script available.

METHODS USED IN THE STUDY

For finding out at which positions students appreciate blanks and why, we surveyed the course participants during a lecture four weeks before the final exam. In this survey, we asked them to complete the following assignment:

The instructor of the course "Introduction to mathematical thinking and working" wants to use guided notes in the next semester. She informs you about this idea and adds that she is not sure about where to put the blanks. A mathematics lecture consists of different elements (definitions, theorems, proofs, explanations, and examples). Please give the instructor an advice on where to put the blanks on the basis of your experience with guided notes. Justify your advice.

61 students took part in the survey. Four of these did not provide answers to the task, i.e., they either did not refer to guided notes at all, or just wrote something general about guided notes. We excluded these four from the further analysis.

Our data analysis consisted of three steps. First, we categorized for each of the lecture elements mentioned above (definitions, theorems, proofs, explanations, and examples) if our participants appreciated blanks in this element or not. In a second step, we categorized the extent to which they appreciated blanks in the different lecture elements inductively. Hereby, we first developed the corresponding category system on the basis of the responses of the first ten cases. It was the same for each lecture element and is shown in Table 1. Then we coded the whole data with this system separately, compared the degree of agreement afterwards, and finally resolved disagreements in a discussion.

Category	Description of the category
Leave the element fully blank	A student mentions explicitly that the whole element (e.g., whole definitions) should be left blank.
Leave blanks at some positions of the element	A student mentions that parts of the lecture element should be left blank while other parts should be printed.
Leave blanks in some elements of this type	A student mentions that blanks should be left in some of the elements of a certain type (e.g., in some definitions).
Print the element	A student mentions that the element should be printed.

Table 1: Categories for the extent the students appreciated blanks in the different lecture elements (definitions, theorems, proofs, explanations, examples)

Finally, we searched within the answers for reasons why blanks in the different lecture elements were considered as helpful by the students for their note-taking. For this, we categorized their responses with content analysis (Mayring, 2015), and used the different parts of our theoretical framework (the different aspects of note-taking in the college students' theory of note-taking, especially the goals of note-taking) as categories. Whenever a student referred to one of these aspects, e.g. a goal of note-taking, we assigned the corresponding part of the response to the respective category.

RESULTS OF THE STUDY

Table 2 shows for each of the lecture elements mentioned in the survey if our participants appreciated blanks in it or not. The extent to which they appreciated the blanks in these elements (if they mentioned an extent) is shown in Table 3.

Element	Definitions	Theorems	Proofs	Examples	Explanations
Leave blanks	70.2%	66.7%	78.9%	71.9%	24.6%
Print	22.8%	22.8%	10.5%	12,3%	40.4%
No statement	7.0%	10.5%	10.5%	15.8%	35.1%

Table 2: The share of our participants appreciating blanks in the different elements of a mathematics lecture (*N* = 57)

Element	Definitions	Theorems	Proofs	Examples	Explanations
Leave fully blank	22.8%	8.8%	12.3%	8.8%	3.5%
Leave blanks at some positions	10.5%	24.6%	14.0%	7.0%	3.5%
Leave blanks in some elements of this type	12.3%	15.8%	24.6%	14.0%	-

Table 3: The extent to which our participants appreciated blanks in the different elements of a mathematics lecture – if mentioned (*N* = 57)

We will now discuss the results for each element in detail, including possible reasons why the students appreciated blanks in the corresponding element. Since our focus lay on students' perceptions towards blanks in the different elements of a mathematics lecture, we will especially present those reasons for which the students presented a clear connection to specific facets of the respective element in their responses.

Definitions

Table 2 indicates that the majority (over 70%) of our participants appreciated blanks in definitions. Table 3 shows that many of these even wished definitions to be left completely blank. The reason most often mentioned – namely by 14 students – was that blanks in definitions can be helpful for memorizing these (memorization is one of the students' goals of note-taking), as the following quote shows:

> *Definitions should be written for yourself, as students should memorize these. And if you write something for yourself, it better remains in your memory than if you only read it.*

The other aspect of the college students' theory of note-taking many students referred to in their statements related to definitions was the note-taking goal "understanding". Four students, for instance, argued that blanks help to gain an understanding because they are required to think when writing for themselves, as the following quote shows:

> *Definitions are in my opinion suitable for blanks, because by writing for yourself, you are required to think more than if you only have to listen.*

Others argued that blanks in definitions provide opportunities for further explanations:

> *In my opinion, blanks should be left in definitions, so that these are explained in more detail, and not just passed over.*

13 students wished definitions to be fully preprinted. One important reason mentioned was that mistakes may occur during the writing process in the lecture. These might be especially problematic for definitions because definitions are some kind of basis, as the following quote indicates:

> *Print definitions, because students still need to acquire the knowledge, and mistakes induced by writing can cause a chain reaction.*

Theorems

Table 2 indicates that the majority also appreciated blanks in theorems (although the numbers might suggest that these were the same students as for definitions, this was not the case). However, Table 3 indicates that many of our participants (24.6%) wished blanks only in parts of a theorem, for instance in the claim, while the framing and the assumptions should be printed. Concerning reasons why blanks should be left in theorems, some students referred to the note-taking goal of memorization again, but much fewer than in the case of definitions (only 8 versus 14 students). Instead, in the case of theorems, the students more often referred to the goal of understanding. Besides the explanations how blanks might help to gain an understanding of the content that had been already mentioned in the case of the definitions, several students now argued that blanks provide opportunities for working with the theorems covered during the lecture. Six participants, for instance, pointed out that students might be able to deduce the claim of a theorem by themselves or together during the lecture:

> *You should leave blanks for the core assertion of a theorem (a mathematical formula, a deduction, an equivalence, ...), since with the help of the remaining part of the theorem, students could try to deduce this core, so that the idea behind it becomes tangible.*

13 students wanted theorems to be printed (not the same 13 students as for definitions), and these mentioned several reasons. For instance, one reason was again that mistakes might occur when having to write for yourself. Another one was that *not having to write* and being able to listen instead helps to make sense of the theorems covered.

Proofs

Table 2 shows that the majority of our participants also appreciated blanks in proofs. Besides the reason that writing for yourself encourages thinking that had already been mentioned for definitions and theorems, the following reason for blanks in proofs was mentioned by several participants (6 students): having to write proofs for yourself helps to practice or might be helpful for written assignments (informing about solutions relevant for written assignments is also one of the note-taking goals by Van Meter et al. (1994)). This was, for instance, expressed in the following quote:

> *Perhaps, it* [filling in the blanks during the lecture] *helps for other assignments. You get used to proving because you never did this at school.*

Nevertheless, many students only wished blanks in parts of the proofs or in some proofs (see Table 3). Ten students, for instance, mentioned that long or complicated proofs should be printed or should contain *only some blanks*, as the following quote indicates:

It depends on the situation. Long and complicated proofs should be printed, but with blanks for additional comments. Small proofs that are also useful for practicing, and which are required by the students, should contain blanks/ be left blank. This helps to internalize the proof structure.

The quote above also shows the aspect “writing for practicing” again. Other students argued that complicated proofs should be printed, so that one could work through them step by step during the lecture, or that one does not have to concentrate on writing exclusively. Those answers are again related to the note-taking goal “understanding”.

Students who wanted proofs to be fully printed argued, for example, again that *not having to write these* allows them to be attentive towards the instructor’s explanations and to reflect on the proofs, which can help to gain an understanding of these.

Examples

Just like in the study by Tonkes et al. (2009), most of our participants also appreciated blanks in examples (see Table 2). Many of them argued again that the blanks encourage thinking and/or that these offer opportunities for working actively with them during the lecture, which can again help to reach the note-taking goal “understanding”. Furthermore, just as in the case of proofs, there were students who mentioned that blanks in examples provide opportunities for practice.

Also in the case of examples, some students wished these to be already fully printed (7 students). The reasons mentioned in our survey were that examples would not require thorough thinking, that printing them allows the treatment of more examples, or again that having to write for yourself may lead to mistakes. However, one of these seven students wished an additional blank for own examples instead:

Explanations and examples can be given. But afterward, there may be a blank for own examples.

Hence, this student nevertheless wanted blank space for being able to adapt the content-structure of the notes (this wish was also expressed in general by 8 students).

Explanations

40.4% of our participants wanted explanations to be preprinted, and 35.1% did not refer to this lecture element at all in the survey (see Table 2). Since over half of the ones who did not make a statement on the element “explanations” only referred in their survey responses to those lecture elements at which they appreciated blanks, it is probably even the majority who wanted explanations to be preprinted. Reasons mentioned were, for example, that the explanations given in the guided notes script of the course were understandable immediately, that explanations do not need extra attention, that writing them down requires too much time, or that students cannot be attentive towards the explanations if they also need to write these down. Overall, the reasons why our participants wished explanations to be printed were rather diverse.

SUMMARY AND DISCUSSION

We investigated at which elements of a mathematics lecture students appreciate blanks in a guided notes script, and explored why students might consider blanks in these elements as helpful for their note-taking from the perspective of the college students' theory of note-taking by Van Meter et al. (1994). This research especially *specified perceived benefits* of blanks and preprinted material in a guided notes script for students' note-taking in a mathematics lecture (that have also been found in previous research, see section 1) *for different elements of such a mathematics lecture*.

First, our data showed that the extent to which our participants appreciated blanks in a guided notes script differed for different elements of a mathematics lecture. Just like in the study by Tonkes et al. (2009), most of our participants appreciated blanks in examples. Reasons mentioned were that having to write these for yourself encourages thinking about the content, and that blanks provide opportunities for working with them actively during the lecture. Unlike in the study of Tonkes et al. (2009), our participants also mostly appreciated blanks in definitions, especially because having to write these helps to memorize them, as further content often builds upon the definitions.

Our participants were more selective about whether blanks should be left in theorems and proofs. In theorems, several of our participants wished assertations to be printed, while claims should be left blank, as these could be conjectured by the students by themselves, which can help to make deeper sense of the theorems covered. Concerning proofs, our participants appreciated blanks especially in short proofs that they are also required to carry out in an exam, because the blanks provide opportunities for practicing, and because having to write encourages thinking about the proofs. But several students mentioned that long or complicated proofs should be printed, so that one could go through them step by step during the lecture. Finally, concerning explanations, the majority of our participants wished these to be printed. Reasons were, for example, that students could then be more attentive towards these explanations, or that writing these down would require too much time.

Our findings finally also provide some practical consequences for the implementation of guided notes into mathematics lectures. First, our data yield some suggestions on where to put blanks and preprinted parts into a guided notes script, so that many students might benefit from guided notes for their note-taking. They especially suggest that leaving blanks in examples, definitions, partly in theorems, and in short proofs might be beneficial for many students, while it might be advisable to preprint parts of longer or more complicated proofs and additional explanations. But our data also indicate that there are students who prefer those elements of a mathematics lecture to be fully printed in which the majority of our participants appreciated blanks. Reasons found were that having to concentrate on writing can make it difficult to grasp the instructor's explanatory comments, or that mistakes might occur during the writing process in the lecture. Hence, it might be advisable to also offer students the instructor's full notes after the lecture. If one follows these suggestions, many students might benefit from the use of guided notes for their note-taking in mathematics lectures.

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Getting through the exam: A case study of four Finnish and German students' self-regulated learning of university mathematics

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The institutional settings at universities offer students a lot of freedom to shape their own learning, which is also associated with many difficulties, especially in mathematics. From a self-regulated learning perspective and based on interviews with four students from two different countries, we qualitatively describe similarities and differences in students' learning of mathematics at university who all shared the common goal of getting through the exam. Results show that despite this common goal, the self-regulated learning of mathematics of the four analysed students differed heavily. However, similarities could be found in the importance of exercise tasks as well as in social strategies. Especially the comparison of the two countries contributes to a discussion of possible impacts of pedagogical interventions.

Keywords: Teachers' and students' practices at university level, assessment practices in university mathematics education, self-regulated learning.

STUDENTS SELF-REGULATED LEARNING IN THE TRANSITION FROM SCHOOL TO UNIVERSITY MATHEMATICS

According to Finnish and German study regulations of mathematics study programmes, about two thirds of the time scheduled for mathematics modules is assigned to students' self-study. However, there is remarkably low evidence on how mathematics students self-regulate their learning in such self-study phases. In light of students' difficulties in transitioning from school to university in mathematics (Gueudet & Thomas, 2020), universities have developed various support measures and curricular adaptations to assist students' mathematical learning and to reduce the transition difficulties (Biehler et al., 2021; Lawson et al., 2020; Rämö et al., 2021). There is evidence that such support structures have an influence on students' self-regulated learning (Lahdenperä et al., 2022). But still, "getting through the exam" seems to be a main and sometimes the only goal for some students – instead of understanding the mathematical content (Göller, 2022; Lahdenperä et al., 2021).

In this paper, we qualitatively describe the learning of mathematics of four students, whose main goal was to "get through the exam", in terms of Boekaerts' (2011) dual processing model of self-regulated learning. In doing so we hope to understand how students could be supported in applying self-regulated learning in the undergraduate mathematics context.

THEORETICAL BACKGOUND

Self-regulated learning can be defined as "an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behaviour, guided and constrained by their goals and the contextual features in the environment" (Pintrich, 2000, p. 453). This definition shows the importance of students' goals as well as their strategies – which we define as goal-directed behaviours, thoughts, or activities – to achieve these goals in models of self-regulated learning. In terms of goals, we refer to the prominent distinction between *learning goals* which focus on increasing competence (e.g. understanding a specific proof), and *performance goals* which focus on the attainment of positive judgments (e.g. good grades, Pintrich, 2000). Self-regulated learning driven by learning and performance goals is positively linked to learning outcomes (for a meta-analysis see Schneider & Preckel, 2017). Furthermore, it is central in building mathematical competence as it is essential in problem-solving (de Corte et al., 2011) and proof-based mathematics (Talbert, 2015). With regard to students who aim at "just getting through the exam", Boekaerts' (2011) model which additionally takes the goal to prevent threat to well-being into account seems a promising theoretical approach and will be introduced in the following.

Boekaerts' dual processing model of self-regulated learning

The dual processing self-regulation model of Boekaerts (2011) identifies three purposes of self-regulation: (a) expanding knowledge and skills, (b) preventing threat to the self, and (c) protecting one's commitment. This model theorizes that learners constantly appraise learning situations and tasks for their congruence with their personal goals, values, and needs. If a student appraises the learning situation to be congruent with their personal goals, values, and needs, characterized by trust, confidence, and interest towards the learning task, they will be encouraged to commit to the task and to activate strategies that ensure the expansion of knowledge and skills. Students' learning on this *mastery pathway* can be guided by learning as well as by performance goals.

If a student appraises the learning situation as not being congruent with their personal goals, values, and needs, the learning situation poses a (potential) threat to well-being. Such a mismatch with the learning environment can occur if a task is perceived as too difficult, ambiguous, or as impairing autonomy. Accordingly, the task will be seen as an obstacle to achieving important goals such as performance or well-being and the student might activate strategies (e.g., avoidance, denial, giving up, distraction) to prevent threat and harm to the self and try to restore well-being (*well-being pathway*).

Students' constant appraisals of the learning environment can reroute their pathways. Students may be (e.g., at first) committed to a learning task and (afterwards) experience obstacles that threaten their well-being e.g., by causing a loss of confidence or interest. Negative emotions, such as disappointment, worry, stress, anticipated embarrassment, or hopelessness will direct them towards the well-being pathway, however, they may activate strategies (e.g., suppressing these emotions, increasing effort, working harder, re-appraising the situation, focusing on the positive, seeking social support) to protect their commitment to the task and switch (back) on the mastery pathway (see Boekaerts, 2011, for a more detailed presentation of the model).

Research questions

This paper aims at qualitatively reconstructing mathematics students' self-regulated learning in terms of Boekaerts' (2011) dual processing self-regulation model in different social and cultural contexts. We thus pose the following research questions:

RQ 1: Which goals guide or constrain students' learning of mathematics?

RQ 2: Which (appraised) threats, obstacles, or mismatches in the learning environment to the achievement of these goals can be identified?

RQ 3: Which strategies do students use to achieve their goals?

While addressing these three questions, we hope to provide some ideas on how to support students in learning the skills needed in self-regulated learning and shifting from the well-being pathway to the mastery pathway (cf. Boekaerts, 2011).

METHODS

To answer these questions, we analyse interview data of four different students from two different universities (one in Finland, one in Germany) from three different research projects (Göller, 2020; Lahdenperä et al., 2021; Liebendörfer, 2018) all of whom have explicitly stated the goal of "surviving the course" or "getting through the exams" (no matter how well). The interviews had different foci and a length of approximately one hour. At the time of the interviews all four students had already participated in at least one math exam at university. To analyse the data we used qualitative content analysis (Kuckartz, 2019) with "goals", "threats", "obstacles", "mismatches", and "strategies" as concept-driven ('deductive') categories, which were then data-driven ('inductively') further developed. We report here the 'case-oriented analysis' (for details see Kuckartz, 2019). The given quotations were translated from Finnish or German by the authors. We shortly introduce the four students and the institutional settings they studied in.

Kim and Luca (all names changed; we use gender-neutral names, as gender is not the focus here) were higher secondary (Gymnasium) pre-service teachers at a German university with mathematics as one of two (compulsory) subjects. In line with the proposed curriculum, Kim attended two five-credits mathematics modules (elementary linear algebra, introduction to mathematical reasoning) together with mathematics majors in their first semester. Luca, who studied two years above Kim, attended two nine-credits mathematics modules (linear algebra, analysis; also in line with the proposed curriculum) together with mathematics majors in their first semester. At the time of the analysed interview in their second semester, both attended a nine-credits proof-based linear algebra (Kim) respectively analysis (Luca) course together with mathematics majors as well as a five-credits course on elementary geometry for preservice teachers (for details see Göller, 2020). All these modules consisted of lectures, where mathematical theory was introduced (i.e., definitions, examples, theorems and their proofs were presented) and exercises were handed out weekly. Students had to work on these exercises in self-study and submit their solutions which then were corrected, graded, and discussed in a separate lesson. To pass such a module, 50 % of all exercises must have been solved correctly and a written exam had to be passed. In their first semester, Kim was among the best 10 % of those who reached points for the exercises, and an average grade in the written exams. Luca reached the 50 % of the points for the exercises in linear algebra and analysis but failed both exams. One year after the interview was conducted, Luca was no longer enrolled in mathematics.

Kuura is a mathematics major, and Tuisku is a statistics major studying a compulsory minor subject in mathematics, both studying in the Finnish university. They are both first-year students who attended a five-credit proof-based linear algebra and matrices course with mathematical content such as general vector spaces, subspaces, linear mappings, and scalar products (for more details see Lahdenperä et al., 2022). The course was implemented with Extreme Apprenticeship combining inquiry-based mathematics education with a flipped learning approach (see Rämö et al., 2021). The students started to study a new topic by solving introductory tasks. After submitting them, they attended lectures that were based on student discussions and focused on the main contents and their connections. After the lectures, the students solved more challenging problems and a new set of introductory tasks. To support students to solve the problems, they were offered guidance in an open learning space for several hours a day. Students received bonus points from completing the tasks. In Finland, exams are low stakes as students can retake them as many times as they want. Both Kuura and Tuisku received the maximum amount of bonus points (≥ 90 % of tasks completed) and got the grade 3 (out of 5, 'good') from the course exam.

RESULTS

In the following, we describe the four students' self-regulated learning of mathematics at university in terms of their *goals* (RQ 1), their perceived *threats*, *obstacles*, or *mismatches* (RQ 2), and their *strategies* (RQ 3) for achieving those goals.

The case of Kim

Kim wanted to become a teacher (RQ 1). They wanted to pass the exams, wanted to "get through" (RQ 1, performance goals). They wanted to understand content which they appraised useful for them as a teacher (RQ 1, learning goal). Linear Algebra and especially proofs were not appraised as useful.

> In elementary geometry I notice now at least that there is some content that I can actually use for school. And there I want to get ahead somehow. Because then I realize that I will really need it at some point. With linear algebra it's more like, I deal with it as much as I can, and try to pass the exam somehow, but I honestly don't care how quickly I forget it, as long as I don't need it again for the later modules. Because I'm just interested in my profession and not really in any proofs that I'll never need again.

Consequently, there was a mismatch between Kim's perception of what they needed as a teacher and the contents of the linear algebra course, especially its emphasis on proofs (RQ 2). They sometimes tried to understand the proofs of the lecture, but "failed with every proof that was longer than half a page" (RQ 2). There were several obstacles for Kim in the mathematics contents ("there are things that I can look at the definitions ten times and still don't get anywhere") and exercises of the linear algebra course (RQ 2). They sometimes tried to solve single exercises by themselves, but most of the work on them they did with their "math crew" of four students (RQ 3). Together they could solve less than half of the exercises (RQ 2). There were always tasks where they "haven't found any approach at all on [their] own", or "don't know how to write it down." When writing proofs, Kim was "never really sure if they're correct", and "with some tasks [they] don't even understand the task." So, before submission deadline they exchanged solutions with other students and eventually copied them (RQ 3).

These strategies were on the one hand a response to the requirements of the extent of the exercises that in their "opinion, it's just too much time that's required" (RQ 2):

> If I wanted to solve the exercises completely by myself, let's say the present one, I would probably need 15 hours. And I would say that's true for 80 percent of my fellow students. And that's just time that you can't spend in a week like that.

On the other hand, these obstacles and strategies were not appraised as a threat for their goal to pass the exam: From their experience of their first exams, they concluded that they rather needed to comprehend the solutions of the exercises than to solve them:

> it's really not so necessary that you bring the knowledge to solve all the exercise sheets yourself, depending on what aspiration you have. But if you just want to get through like I do and you have the right people or the right ambition to kind of try to solve the exercise sheets, you can kind of get through.

In summary, although Kim experienced mismatches of the learning environment and their own goals, values, and needs as well as obstacles especially regarding proofs and proofing, they managed to stay committed to their long-time performance goals and found a way to participate in university mathematics which they were yet rather not interested in.

The case of Luca

Luca wanted to pass the exams to become a teacher (RQ 1, performance goal), describing mathematics at university as "necessary evil". They thus needed "to get the points" (RQ 1, performance goal) on the exercises to participate in the exam. While Luca did not struggle to get these points ("Not that I get to my percentages, so that's not a problem"), because they could rely on peers ("Someone always has something for the exercises"), their learning goals were less clear. They first described "I had planned to rework the exercises (…) and try to understand them" but experienced struggle and too much effort on doing so (RQ 2). Instead of deep understanding, Luca thus tried to understand "how to do things", to pass the exam (RQ 1):

> But now, with regard to the next exam, I definitely think that I have to practice, learn, and repeat a lot more tasks. […] I learn for example how to calculate a path integral. [...] that

> is in principle, simply for different equations yes always the same. So, I'm just looking for any patterns, because you need something, where you can hold on somehow.

Their most important strategy therefore was focusing the "how to" exercises and cutting out the proofs (RQ 3) which they appraised as being less relevant for the exam, for their understanding of theorems, and for their future goal of becoming a teacher:

> It's always like that on the exercise sheet anyway. So, one task you can always cross out directly, that's the proof. Then there are three more tasks that you can solve. These are then mostly calculations and there is also usually an example somewhere.

Still facing struggle with this, Luca learned together with peers (RQ 3), what – however – mostly did not help them to better understand things (RQ 2):

> So, we exchange information and also discuss with each other how we would do it. Or also explain things to each other when one has understood something that the other has not. But most of the time no one understands it anyways.

Experiencing frustration while learning, Luca also reported strategies that aimed at well-being:

> I had moments where I said: now you're not in the mood anymore. Uh, then I just left it there and then at some point later I sat down to it again. […] After two hours we just went to the canteen. Sometimes we did something afterwards. But we did something for two hours and then it was enough for us.

Following this, Luca was rarely on the mastery path i.e., the learning environment at university was mostly not congruent with their goals, values, and needs. However, the future goal of becoming a teacher led to a commitment by them to some tasks (mainly non-proof, calculation), within the bounds of compatibility with their well-being goals.

The case of Kuura

Kuura's goal for the course was to "see how it goes, as long as I pass, that's the thing" (RQ 1). However, this performance goal can be part of their first-year experience, as they later explained that "it has been very difficult to set any goals because I have […] never studied at university before".

Kuura's main strategy to meet their performance goal was solving the weekly tasks (RQ 3). They explained:

> Solving the problems, that's it, like if you don't do them, nothing will work out. You need to work on the tasks.

They found the bonus points received from solving the tasks motivating, as they relieved performance stress in the exam and "pushed you to work hard on the tasks" found essential in passing the course. With their peers, Kuura made a weekly schedule for working on the tasks in all courses. Kuura skipped about half of the lectures (the ones in the morning) but worked actively on the tasks in the open learning space with peers and often asked help from the tutors. Kuura found the guidance supportive, as "it was easy to ask for help […] and you didn't have to be alone with your problems".

Interestingly, despite stating only performance goals explicitly, Kuura reported on guidance preferences that supported learning (RQ 1):

> It was very important that the tutors don't give you the ready answers but more like guide you to the right direction, like show where one could start thinking […] or so.

Later, Kuura continued:

> In mathematics, you need to truly understand what's going on. And if you readily get an answer to a task then you haven't necessarily understood what was happening in the […] proof […]. So, it doesn't develop your own competence at all. And it is of utmost […] on first-year courses, to you get some kind of basic knowledge and develop your mathematical thinking. And it doesn't develop through ready-made answers.

To conclude, Kuura had challenges in setting concrete learning goals, but within their performance goals, reported on activities that indicate shifting towards the mastery pathway. This is supported by the fact that Kuura reported that there was nothing hindering their studying and learning in the course. Furthermore, they described:

> I got ['needs fixing' feedback] from a task and I looked at the it and I had missed one small but very central assumption […]. And somehow, I became aware of how very important it is to do the things properly, […] like I started to realise why mathematical proofs are just the way they are.

These types of eureka moments can be central in constructing the way towards setting learning goals and reaching the mastery pathway.

The case of Tuisku

Tuisku acknowledged that there is a possibility for setting learning goals, as they stated that "[this course] is compulsory for statistics students and it's obviously compulsory for a reason". However, they reported mainly performance goals, as "the aim was of course to pass the course", and as they found collecting the bonus points motivating, they had set a goal for the percentage of completed tasks (RQ 1).

In addition, engaging in social interaction can be considered Tuisku's goal for the course and for their university studies in general (RQ 1). However, Tuisku's general experience at the university was nothing but individualised (RQ 2). They stated that at university, "it feels like a student is just a person in an enormous mass that is just transferred through the courses". For this reason, they greatly valued and engaged in social interaction while learning. For example, they said that "for me, it is important to work together with peers […] as after all, the studies provide very little sense of community".

On the one hand, their goal for engaging in social interaction supported certain strategies for achieving their performance goals: Tuisku found the tasks challenging (RQ 2) and relied on peers and tutors to solve them (RQ 3). They stated:

> We sat in the [open learning space with friends] and for almost every task, we asked help from the tutor. […] The tutors were really nice, […] and the atmosphere very supportive.

On the other hand, the goal for engaging in social interaction also hindered certain strategies for achieving their performance goals: For example, Tuisku had recognized the tasks as difficult, and they realised there was a need to attend the lectures. However, “when my closest friends didn’t attend them, I didn’t go either”. Also, Tuisku could have missed skills to work on the tasks individually, as “if you didn’t finish [the tasks in the open learning space], you didn’t get back to them so easily at home” (RQ 3).

For reaching the goal of certain amount of bonus points, Tuisku was optimising their limited time. For example, they stated:

> The one point you get from a task is of different value in different tasks in terms of time. Like right away when you see that this is an easier but laborious task […], it gives you this feeling that timewise, this doesn’t pay off.

To conclude, the performance goal for bonus points prevented them from learning as it was more convenient to optimize their time management and stay on the well-being pathway. In contrast, Tuisku’s performance goals for passing the course supported them in finding ways to complete the tasks perceived as difficult. This supported them in shifting from the well-being pathway to the mastery pathway. The goal for social interaction was acting in both directions; on the one hand, it had a central supportive role in learning as it helped Tuisku in solving the tasks, and on the other, a central role in hindering learning as they sometimes chose peers over what could have been more beneficial for their own learning. Overall, Tuisku was on the well-being pathway trying to shift to the mastery pathway. It can be hypothesised that they were eventually successful in shifting as they already here recognised that “perhaps my way of going through the things was in the end not optimal”.

DISCUSSION

In order to qualitatively describe mathematics students’ self-regulated learning, we analysed interviews with four students from two different institutional settings in Finland and Germany. A central result of this analysis is that, although we focused on four students who share the (seemingly similar) common goal of simply getting through the exams, the self-regulated learning of the analysed students differed in many ways. Kim’s and Luca’s primary goal was to become a teacher. Both experienced mismatches in the learning environment with their personal goals (proofs, appraised usefulness of the content for schoolteachers). However, while Kim’s ambition helped them to stay committed to their long-time performance goals and to (at least partly) participate in university mathematics, Luca did not seem to get beyond the goal and strategy of looking for patterns of “how to do things”. Kuura valued strategies aimed at deeper learning and (possibly therefore) seemed to experience less mismatches in the learning environment with their personal goals, values, and needs. For Tuisku, the goal for engaging in social interaction was central which supported strategies that involved collaboration with peers and tutors, but also hindered them to engage in strategies with less social interaction. In summary, these examples highlight the importance of the orchestration of different goals, individually appraised mismatches, and activated strategies (beyond the shared performance goal of passing the exam) for the qualitative details of self-regulated learning of mathematics at university which are anticipated in Boekaerts' (2011) model, but not specified in such detail.

Beneath these differences, there were some commonalities found which apply to all four students: Firstly, the exercise tasks take the most prominent part in the regulation of students' learning of mathematics (cf. Göller, 2022). In particular, it can be observed that some students select specific tasks in accordance with their performance goals, which underlines the importance of exam tasks for self-regulated learning processes. Secondly, all four students worked with others to solve the exercises. However, while it seems that in the Finnish setting it was rather easy to find suitable (institutionalised) support from tutors and peers to overcome obstacles in the tasks, the two students in the German setting mostly relied on peers who often could not help.

Implications, limitations, and outlook

Even though the self-regulated learning perspective – which we took here – focuses on individual learning processes, the influence of contextual features on students' self-regulated learning is obvious from the data (cf. Lahdenperä et al., 2022). For example, this is evident in the already discussed importance of exercise tasks which thus entail the practical potential to guide and scaffold students' learning of mathematics. On the other hand, considering the shown differences in students' self-regulated learning, especially their differences in appraised obstacles and mismatches which were found on the cognitive but also the motivational and behavioural level, institutionalised individual support structures (e.g. Lawson et al., 2020) seem a promising approach to support students' learning of mathematics. They seem even more promising if they do not only address students' cognitive but also their motivational and behavioural obstacles (for hints see Göller, 2020) and enable students to co-create their learning environment in a way, they appraise to be congruent with their goals, values, and needs.

When interpreting the results, it must be kept in mind that only four students with a common goal were considered here, who studied in specific contexts which influenced their self-regulated learning. However, the results show the potential of qualitative comparisons of different institutional (and cultural) settings to reveal similarities and differences in students' self-regulated learning, also allowing insights into the potential impact of pedagogical interventions.

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Informal learning situations in the context of mathematics studies - development of an analysis framework

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Abstract: In mathematics-related university study programmes, self-study has a special importance. This gives rise to individual and self-directed learning situations that trigger phases of informal learning within the formal learning contexts of mathematics studies. For a differentiated description of such learning situations, an analysis framework is presented that enables an analysis of the subphases of individual learning actions in a gradual spectrum between formal and informal. Using an exemplary learning situation, the framework presented allows a detailed view of independent learning in mathematics studies and is intended to identify starting points for the promotion of individual, self-directed learning processes.

Keywords: Teachers' and students' practices at university level, Transition to, across and from university mathematics, digital and other resources in university mathematics, informal learning, analysis framework.

INTRODUCTION

In the formal context of university mathematics, self-study takes on a special significance, whereby various parts of student learning actions take place in individual and self-designed learning situations, both within and outside of curricular and didactic frameworks (Liebendörfer, 2018). Such learning situations are often referred to as *informal learning*. According to Jadin et al. (2008) this term describes individual, self-initiated and self-regulated acquisition of knowledge, which is generally distinguished from *formal learning*. The latter designates institutionalised and structured learning that takes place within the framework of educational institutions and leads to certification.[ii]

In recent decades, the benefits of informal learning contexts for sustainable knowledge acquisition have been repeatedly demonstrated empirically (e.g. BMBF, 2001; Cedefop, 2003; Chrishol et al., 2005). However, it remains unclear how specific learning situations (such as the understanding of a certain mathematical concept) are formed within different learning contexts (such as lectures, tutorials, study groups) (Jadin et al., 2008, p. 170). Moreover, recent publications express that the dichotomous view on formal vs. informal learning, as described above, is “less informative than the differentiated analysis of the various dimensions in which learning activities vary”. (Callanan et al., 2011, p. 648, author's transl.). In this sense, concepts have been proposed which describe the tension between formal and informal as gradual and do not refer macroscopically to entire learning contexts, but rather to individual learning situations and their specific learning actions (e.g. Arnold, 2016; Callanan et al., 2011; Jadin et al., 2008).

Aiming towards analysing such individual, mathematics-related learning situations and the inherent informal learning, it's necessary to classify independent learning actions of mathematics students in the field of tension between formal and informal. With the help of a differentiation between learning contexts (macro-level) and individual learning situations embedded in them (micro-level), an analysis framework will be presented in the following. It looks at learning situations and their sub-phases and enables a classification of those within a gradual spectrum between formal and informal. Such an analysis can reveal new, individual starting points for support measures. This approach ties in with recent publications that highlight educational successes as well as positive social and personal developments through the promotion of informal learning in formal contexts (Peeters et al., 2014).

Derived from the primary research concern three subordinate questions arise: How can learning contexts and learning situations be distinguished from each other? In which sub-phases do learning situations proceed (especially in mathematics studies)? Which gradual spectrum describes the field of tension between formal and informal learning in a useful way?

Based on this line of thought, in the following chapters a gradual spectrum between informal and formal learning in the context of expansive learning processes will be discussed before a subdivision of learning situations into analysable sub-phases is proposed. Afterwards the application of the proposed framework will be exemplary presented and possible implications for support measures in mathematics-related studies will be discussed.

INFORMAL LEARNING

Proceeding from the first description by Dewey in 1899 (cf. Archambault, 1966), over the last century numerous definitions of and perspectives on formal and informal learning were stated. The development of the terms and their interrelationship has already been described in several review articles (e.g. Harring et al., 2018; Rohs, 2016), although there is still no general and comprehensive definition (Jadin et al., 2008). However, Jadin et al. (2008) summarise the demarcation as follows:

> "Formal learning is institutional, highly structured, takes place within the framework of educational institutions and is concluded with a certificate. [...] Informal learning can [...] take place in institutions but is characterised by low structuring and does not lead to a certificate of completion. The initiative and control of learning is not dependent on an institution, but lies in the hands of the learners themselves." (Jadin et al., 2008, pp. 170-171, author's transl.)

Depending on the discipline, the descriptions of the terms are more open- or closed-ended and the used terminologies vary. These differences can usually be explained by the diverse contexts of application within the disciplines and refer primarily to the inclusion or exclusion of the following characteristics: Intentionality, structure and structuredness, institutional dependency, learning for certification, self-direction as well as randomness of learning (cf. e.g. Livingstone, 1999; Molzberger & Overwien, 2004; Münchhausen & Seidel, 2016; Rohs, 2016; for informal learning in the context of mathematics learning see e.g. Pattison et al., 2016). Nevertheless, the definitions have a conceptual dichotomy between formal and informal in common, which can be traced back to the focus on learning contexts. These are understood as the overarching, macroscopic frameworks and environments of learning, e.g. university studies themselves, lectures, tutorials, student interactions or exercises (see below, Jadin et al., 2008).

LEARNING CONTEXTS AND SITUATIONS IN MATHEMATICS STUDIES

Studying mathematics, whether as a stand-alone course or as a (sub)module of another, is fundamentally different to the forms of teaching and learning in school. In addition to participation in curricular courses (e.g. lectures, seminars, exercise sheets, examinations), university students are required to learn in "substantial self-study" (Liebendörfer, 2018, p. 342). Its intensity and scope differs considerably from preparation and follow-up work in school (e.g. homework), since it includes not only the training of familiar procedures, but rather independent problem-solving and special precision through formalism and abstraction (Hochmuth et al., 2021; Liebendörfer, 2018). In addition, the high complexity and barriers of comprehension make independent learning university mathematics indispensable (Liebendörfer, 2018; Pritchard, 2015).

However, the self-study of mathematics often does not proceed in an unproblematic and straight forward manner. Due to the large amounts of complex and formalised subject content and methods, learning meanders between extrinsic and intrinsic motivation (Bauer et al., 2020). From the perspective of a subject-scientific theory of learning, such learning actions can be described as expansive and defensive. Defensive learning is "primarily externally controlled and [...] fact-bound" (Marvakis & Schraube, 2016, p. 212) and focussed on the achievement of an extrinsic goal (e.g. successful examination performance). In expansive learning the learning problem and thus the learning object are in the foreground of the learning action (Holzkamp, 1993). At the same time, expansive learning is at least indirectly influenced by interaction between teachers and learners. Marvakis and Schraube (2016) refer to this as the fluidity of learning:

> "The learning process of individual subjects is always a social process and situated in relation to others, unfolding as a constant back-and-forth between learning and teaching in and between persons. This fluidity of learning and teaching forms a basic element of expansive learning and the nucleus of a productive and lively learning practice." (Marvakis & Schraube, 2016, author's transl.)

If such an understanding of learning – from the perspective of the individual – is assumed, it is not sufficient to consider only its learning contexts for the detailed analysis of mathematics learning. Rather, individual learning situations, which as emergent processes contain the concrete moments of learning within those contexts, would have to be analysed in more detail: Solving a single problem, understanding a practice task or an unknown concept, getting stuck in a proof… Furthermore, the analysis would have to be done from the perspective of the individual (cf. Göller, 2020; Jadin et al., 2008).

Inspired by models of inquiry-based and self-regulated learning in the study of mathematics (Göller, 2020; Roth & Weigand, 2014; Wildt, 2009; Zimmerman, 2000) six inherent sub-phases of a concrete learning situation can be described: *learning occasion*, *goal*, *(subject-related) content, methodology*, *feedback* and *reflection*. Based on Holzkamp's "learning problem" (1993) the *learning occasion* describes the trigger, the problem, the call to action of a learning situation, e.g. concrete contents of a course, an exercise or a statement by a teacher or a peer. It leads to a *learning objective,* which names the desired gain of knowledge or the final state of the learning situation. The learning action, which can be expansive or defensive, can be divided into a content-related and a methodological component. The *subject content* refers to all subject-mathematical terms and procedures that are needed during the learning action to achieve the objective. The strategies used for this are summarised within the *methodology.* This includes conscious and unconscious, independent and group-based as well as assimilated, accumulated and acquired methods. *Feedback* is any kind of response to the four previous sub-phases by teachers, peers, media, experience or oneself (Hattie and Timperley, 2007, p. 81; cf. Pepin, 2014). Finally, *reflection* is the personal metacognitive perception, discussion and, if necessary, future adaptation of the learning process and/or its individual sub-phases.

Although the learning contexts of mathematics studies can be usefully described by a dichotomous demarcation of formal and informal learning, this is not possible with learning situations and their inherent subphases as described above: They move between defensive and expansive phases of learning, on the one hand through individual motivations and actions, on the other open to fluid interaction with learners and teachers inside and outside courses and sometimes have more, sometimes less, sometimes no reference to certification. Therefore, the analysis of learning situations requires less rigid, dichotomous criteria, but more "gradual characteristics" (Jadin et al., 2008, p. 171) in a field of tension between formal and informal. Besides Decius et al. (2021), Jadin et al. (2008) and Callanan et al. (2011) also describes Arnold (2016) such a scale.

DIFFERENTIATED ANALYSIS OF LEARNING SITUATIONS

In reference to Holzkamp (1993) Arnold describes informal learning as "the self-organised, often accidental biographical learning in which the person intentionally strives for the transformative search for new and more functional solutions" (Arnold, 2016, p. 483) and differentiates it into three intertwined degrees of informalisation:

Implicit learning describes unconscious and inherent learning. It takes place in everyday situations as well as interactions between people. In this sense, it is the least self-determined, rather accidental and unavoidable learning. *Reactive learning* describes a conscious learning process that is experienced by the individual in response to a new problem or challenge. Finally, there is metacognitive *reflective learning* "in the aftermath or in preparation of experiences and actions" (Arnold, 2016, p. 486), which is aimed at optimising one's own behaviour. It represents the transition between informal and formal learning and can be observed in both (Arnold, 2016).

Together with the sub-phases described above and supplemented by the perspective on formal learning of Jadin et al. (2008), these degrees of informalisation create a possible basis for the differentiated analysis of individual learning situations in mathematics studies. Independent of the assignment of the superordinate learning context, concrete learning situations can thus be described in their subphases between *formal, reflexive informal, reactive informal* and *implicit informal.*

APPLICATION TO A MATHEMATICAL LEARNING SITUATION

In the following, the proposed analysis framework will be applied to a real learning situation in the context of university mathematics studies. Since empirical studies based on this framework will not be conducted until the winter semester 2022/2023, a sufficiently described situation from a study conducted by Heinrich and Hattermann (2021) will be used here as an example to enable an authentic and realistic application.

In the learning context observed by Heinrich and Hattermann, two fellow students deal with descriptive statistics by means of instructional texts and tasks in the digital learning environment Moodle as part of an assessed study and examination performance. Firstly, the arithmetic and harmonic means were introduced through definitions and explanations. The learners now work on application tasks, of which they only enter the solution into the digital learning environment and receive a binary evaluation ("correct"/"incorrect"). They have several attempts to solve the tasks.

In the learning situation relevant to this paper, one of the learners (L1) asks the fellow learner (L2) for support (Heinrich & Hattermann, 2021, pp. 182-183, author's transl.):

L1: But why do I need the harmonic mean and the arithmetic mean? I don't understand that.

L2: There [pointing to the screen] he always drives the same time and there [pointing to the other example] he always drives the same distance but takes different lengths of time.

Based on this information L1 looks at the definitions and explanations in the learning environment and tries to establish a connection between the means and the clues received. L1 starts a new attempt to solve, the result of which is marked as “correct” by the digital learning environment. L1 makes notes.

Based on Jadin et al. (2008) the learning context can be described as formal, since the learners work on a task given by the teacher in connection with a curricular course, which is relevant for completing the course itself (part of the course work and content-related preparation for examination work), but consequently also relevant for the success of studying mathematics on university level.

Nevertheless, characteristics of informal learning are also recognisable: By implementing the tasks in a digital learning environment, the duration, pace, and location of learning as well as the process and methodology of knowledge acquisition are determined by the learners or the learning group. Self-organised learning or self-study is thus initiated. Within this framework, the two learners L1 and L2 decide to learn and work on the given contents and tasks at a common place and at the same time.

Based on this, the learning situation that arose between L1 and L2 will be classified in the following within the field of tension between formal-informal based on an interpretation of the speech acts using the proposed analysis framework.

The learning situation begins with L1's speech act. It can be assumed that it was preceded by a discussion about the task. In this, L2 seems to have assigned the arithmetic or harmonic mean to the two sub-elements of the task. This represents an incomprehensible step for L1, which can be classified as a learning occasion. Since, in summary, this is an affinitive, reactive and self-directed cognition on the part of L1, the occasion can be classified as reactively informal.

It is important to note that the learning occasion in this particular situation does not arise from the task itself, but only from the non-understanding of the steps of L2. Nevertheless, understanding the application of the theory and subsequently solving the task is the learning goal. This can be seen on the one hand from the concrete work on the task itself, and on the other hand from the exclusive use of the given material. Since the learning objective is curricularly specified and didactically prepared in the context of the course and the digital learning environment, it can be described as formal.

In order to achieve the goal based on the learning occasion, L1 first decides to enquire L2. This seems to be based on the situation analysed above, which also led to the learning occasion. The questioning here is L1's methodology to advance his learning. This is a self-directed reaction to the learning occasion and can be assigned to the reactive-informal.

However, the technical content chosen for comprehension is no longer controlled by L1. It is the speech act or the answer of L2 that determines what the content focus of L1 is directed to, here: The relationship between time, distance and the two means. This dimension is externally controlled – in relation to the prepared task – and can be described as formal with regard to L1.

From L1's reaction to L2's speech act, an indirect conclusion can be drawn regarding the dimensions feedback and reflection: Since L1 does not focus on the solution of the task after L2 has answered, but again looks at the technical contents of the digital learning environment, L2's explanations do not seem to have been sufficient to solve L1's difficulties in understanding. It can be assumed that intrinsic feedback (e.g. "I still haven't understood what he means.") led to the reflection to turn to the subject content again. Both steps can be understood as a follow-up to the first learning action, leading to an attempt at optimisation and are thus reflexive-informal actions.

Which intrinsic process is taking place in L1 can only be extracted indirectly from the example. Based on his action, it is reasonable to assume that L1 enters a secondary learning action and tries to link L2's statement about the dependence of the means on time and distance with the given subject content. Subphases can be derived from this, which on the one hand describe their own learning situation, but at the same time try to achieve the original learning goal.

Consequently, the secondary reactive-informal learning occasion just described arises as well as the secondary reflexive-informal learning goal of wanting to precisely establish this link to achieve the primary learning goal. The learning methodology, the multiple reading of the given learning material and the independent testing of knowledge on the task, represents a reactive-informal action, whereby further processes, which cannot be read from the situation description, may be running internally here. Since L1 refers exclusively to the digitally given materials and L2's statements, the subject content is formal. Also, L1 does not seem to be convinced of his result by his own attempt to solve the problem, but only by the feedback from the learning platform. Therefore, this sub-phase can also be described as formal. Based on L1's action of taking notes after the feedback, a reflexive-informal process seems to take place.

RÈSUMÉ

Based on the high significance of self-study and independent learning actions in mathematics-related study programmes, an analytical approach was presented in this paper that classifies the different subphases of individual learning situations in a gradual spectrum between formal and informal learning. Based on a subject-scientific approach, learning in mathematics studies was seen as an interplay between internal, intrinsic processes, interactions with learners and teachers as well as learning dispositions and materials.

The exemplary application based on a concrete learning situation shows the potential of such a detailed framework. Through its step-by-step breakdown, learning actions and the strategies applied in the process can be worked out and considered in a differentiated and detailed way. This enables a deeper understanding of learning strategies in dealing with subject content, methodologies, obstacles, and coping strategies. Based on this, new or adapted teaching-learning settings as well as indirect support possibilities for independent learning, e.g. through strategies of research-based learning, can be developed. A possible starting point would be, for example, the primary learning occasion of L1, by looking more closely at the reason for L1's hurdle or comprehension problem. As a result, the learning material could be adapted to either be more detailed or to offer opportunities for independent research.

However, the application also highlights the need for a sufficiently detailed description of the situation and the effort required for the proposed hermeneutic approach. The planned empirical application will thus also be used to optimise and validate the framework as well as to elaborate an analysis methodology.

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[i] Website of the Research Training Group LernMINT: https://lernmint.org (last visited: July 1st 2022)

[ii] In this article, the discussion of formal and informal learning does not include non-formal learning processes, primarily because this term can contribute little to the learning situations considered here. For a definition of the term and its classification in the context of (in)formal learning, please refer to Rohs (2016).

Self-regulated learning in a mathematics course for engineers in the first semester: insights into students´ reported resource management and cognitive strategies

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The secondary-tertiary transition still poses a lot of problems for students in mathematical courses. To gain insights what causes these problems, we used the methodological approach of ambulatory assessment and asked students (N=6) to report on their mathematical learning on a weekly basis and draw on the theoretical approach of self-regulated learning. The aim of our study is to better understand the learning behaviour of students along the semester. Our results show that students use a lot of non-course-related resources, such as Internet resources. Additionally, rehearsal strategies seem to be the dominant method to process the course contents.

Keywords: Teachers´ and students´ practices at university level, Transition to, across and from university mathematics, Teaching and learning of mathematics for engineers.

INTRODUCTION

The transition from high-school mathematics to university mathematics still proves to be challenging for many students. We see this in high drop-out rates (up to 41%) in STEM-subjects in Germany, especially in study programs with high proportions of mathematics (Heublein & Schmelzer, 2018). Furthermore, in a national study from the U.S., almost half of the freshman students reported that they had a hard time understanding and solving complex mathematical problems (Noel-Levitz, 2015).

One explanation for this could be that due to the different teaching structure at university more self-regulated learning is required in learning mathematics than it was at school. In contrast to mathematics learning at school, university mathematics requires students to work on topics more self-responsively in order to prepare well for exams or finish homework successfully (Göller, 2020). Additionally, there is also a lot more freedom for students, as most lectures, tutorials, etc. are voluntary. Students decide which parts they attend and how much time they allow for them in their own learning.

University students have to organize their learning by setting goals, using learning strategies and evaluating the learning process. Especially, situations where complex tasks have to be mastered on their own are seen as demanding in terms of self-regulation (Dresel et al., 2015). This is particularly evident in mathematics courses at the university, since more than half of the scheduled study time is allocated to self-study (Liebendörfer et al., 2022). In consequence, self-regulated learning becomes even more important for mathematics students' learning.

SELF-REGULATED LEARNING

The theoretical foundation of this study is the model of self-regulated learning which has already been conceptualized by several authors with different foci. However, most self-regulated learning models have one commonality: they are conceptualized as cyclic processes that encompass three phases: 1. Pre-Action phase, 2. Action phase, 3. Post-action Phase (Zimmermann & Moylan, 2009).

In the *pre-action phase* (*forethought phase* in Zimmermann & Moylan, 2009) the learner analyses situational and task demands, sets goals and plans how to reach those goals. It also contains motivational beliefs, which for example have influence on the activation on learning strategies. The *action phase* (*performance phase,* ibid.) is divided into two processes, self-control and self-observation. Self-control ensures that the learning strategies used are in line with the previously set goals, which also benefits motivation and goal-oriented use of (cognitive) resources. Self-observation is used to metacognitively monitor learning strategies, perceive emotional reactions and maintain motivation as well as positive emotions. The *post-action phase* (*self-reflection phase,* ibid.) can again be divided into two processes, self-judgement and self-reaction. The learner uses self-judgement to evaluate their performance by comparing the achieved performance with the previously set goals. Reasons and causes are sought which can be attributed to one`s own performance. Self-reaction refers to the affective evaluation of learning success and conclusions for one`s own learning, for example adapting learning strategies in the following learning cycle.

The focus of these process models is the coordination, control and regulation of cognitive, meta-cognitive and motivational processes in the consecutive phases of learning (Dresel et al., 2015). These process models of self-regulated learning are complemented by the component models (e.g. Pintrich & Garcia, 1994), which identify the types of strategies that are involved in self-regulated learning – they include cognitive strategies that regulate the process of knowledge acquisition, metacognitive strategies that control and monitor cognitive strategies and the resource management strategies, which are understood as self-management activities to organise learning actions. They are divided into internal (e.g. time investment, effort) and external (e.g. peer learning, literature) strategies. The components are understood as the learner´s characteristics for learning, regardless of the phases of the learning process they can occur in (Wirth & Leutner, 2008).

It has been shown that self-regulated learning and their underlying components are a prerequisite for successful studying in higher education (e.g. meta-analysis by Robbins et al., 2004). For example, looking at cognitive strategies, we know that deep processing strategies (e.g. elaboration) are more useful than surface strategies for learning and have positive correlations to grades per average (Karagiannopoulou Milienos & Athanasopoulos, 2018).

Specificities of mathematical self-regulated learning

University mathematics has some specificities which differentiate them from other subjects. That is for example conceptual understanding in form of the multifaceted role of proof (Weber, 2014), the role of procedural knowledge, e.g. in performing calculations (Bergsten, Engelbrecht & Kågesten, 2017) and requires a high level of time commitment. These specifics also have an influence on self-regulated learning.

In German mathematics courses it is quite common that students have to complete weekly homework assignments in addition to taking part in lectures, working on exercises for and in tutorials. The successful completion of these homework assignments is usually required for taking the final course exam. Therefore, students have to deal with the current mathematical content every week in order to complete the assignments successfully (Göller, 2020). Furthermore, weekly assignments are a good way to monitor learning and also provide a regular opportunity to review your own learning. Afterwards, the learning process can be evaluated and adjusted for the next week, if necessary. According to the self-regulated learning theory of Zimmerman & Moylan (2009), every week in itself can be a self-regulated learning cycle.

There also has been some research on self-regulated learning and their underlying components in undergraduate mathematics, but it mainly focuses on the relationship to performance in exams (e.g. Liebendörfer et al., 2022 Johns, 2020). Johns (2020) observed differences in self-regulation strategies between under and over achievers as well as low and high achievers in a calculus course. From this, he concluded that self-regulation strategies do play a role in students´ performance in the exam. Liebendörfer et al. (2022) also support that some strategies, for example practicing certain types of tasks, help performance in exams in mathematics course for engineers. They concluded that cognitive strategies within mathematics need to be assessed subject-specifically, since (among other specificities) in mathematics with regard e.g. to rehearsal, some strategies seem to have positive (practicing) as well as negative (repeating) links to learning. In addition, external resources are also an important factor for learning mathematics, as the majority of students have a study group or study partner for learning undergraduate mathematics (Alcock, Hernandez-Martinez, Godwin Patel & Sirl, 2020).

Weekly learning of mathematics is an important element for deepening and applying the learning content. Due to the importance of homework assignments and exercises as well as the high proportion of self-allocated study time (e.g. working on these weekly tasks), it is particularly interesting to better understand which *actions* students take for their everyday learning. This information would help lecturers at university to understand what students typically do in their learning and e.g. might redirect students from using weaker to more useful strategies to achieve the intended learning goals. Due to the high self-study time in mathematics courses, strategies from the action phase are rather important. We believe that strategies in the action phase are important factors for students' learning, however in this paper we are focussing on cognitive and external resource management strategies. Referring to cognitive strategies, it is interesting to see how students integrate and process the contents from the course and referring external resource management strategies as well as to see what materials and other help students seek for during their learning.

RESEARCH QUESTIONS

For the purpose of this paper, we are focussing on external resource management and cognitive strategies (elaboration and rehearsal) in the action phase of self-regulated learning. This leads us to the following research questions:

RQ1 Which external resources reported by students in the action phase can be reconstructed in their learning for a mathematics course for engineers?

RQ2 Which elaboration and rehearsal strategies reported by students in the action phase can be reconstructed in their learning for a mathematics course for engineers?

METHOD

In order to answer the research questions, students' self-regulated learning was assessed through self-reports. In general, self-report instruments are very well-suited for assessing learner's intended cognitive, meta-cognitive, motivational as well as resource management strategy use. In addition, self-report instruments are an appropriate tool for higher education (Roth, Ogrin & Schmitz, 2016).

Ambulatory Assessment

To be more specific, we used the method of ambulatory assessment, whereby this type of methodological approach is also known under different names, for example experience sampling or diary methods (Conner & Mehl, 2015).

> At their core, these methods allow researchers to study individuals (1) in their natural settings, (2) in real time (or close to real-time), and (3) on repeated occasions (Conner & Mehl, 2015, p. 5f).

This approach is suitable for the above research questions since they focus on a better understanding of the individual student's self-regulated learning (1) in a natural setting (learning mathematics in a first semester course), (2) in real-time (in this study: close to real-time) and (3) on repeated occasions (weekly insights). Boekaerts and Corno (2005) also concluded that these types of methodological approaches lead students to be more open than in other forms of assessment for self-regulated learning.

The *mathematics for engineers I* (Mathematik für Maschinenbau I) course, in which we conducted our study, already provides a weekly structure with lectures, exercises and tutorials, all of which were carried out in face-to-face teaching. In addition, there are weekly homework assignments that students can complete on their own to gain some extra points for the exam. The focus of the lecture was providing the theoretical mathematical background, while the homework assignments and exercises were mainly application-oriented. We also know that mathematics majors base their learning around weekly homework assignments (Göller, 2020). Although homework assignments do not have the same importance in mathematics for engineers as in mathematic major modules, we still believe that weekly insights are important and also reflect the natural situation of the learning. Students were asked to submit the weekly report after they submitted the homework assignment.

The structure of our self-report is reminiscent of a learning diary. We chose a more structured approach, because studies using structured diaries indicate a higher validity for documenting students' self-regulated learning (Roth et al., 2016). The typical structure of the self-report instrument consists of five parts:

- Part 1 and Part 5 deal with students' goals, first from a retrospective perspective ("looking back at last week") and last from a prospective perspective ("looking forward to next week")
- Part 2 focuses on what happened during the preceding week, for example, which teaching opportunities students took advantage of and how exactly they went about doing mathematics
- Part 3 addresses different aspects of self-regulated learning (e.g. motivation, independent study time, self-efficacy, satisfaction), which should be answered on a scale from 1-10 accompanied by a short explanation of their classification
- In part 4, we asked students about their emotions and feelings in the preceding week in specific learning situations

We analysed all five parts, and in terms of external resource management and cognitive strategies we were able to reconstruct the most strategies from the second part, although the other parts also provided useful information.

We started to hand out the self-reports regarding self-regulated learning to students (N=6; 2 females, 4 males) in the 4^{th}-6^{th} week of the semester and it ended just one week before the exam. In total, we received 56 self-reports divided between 9-11 self-reports from each participant.

Audio diary

Students were asked to submit their self-reports as an audio file. Those so-called audio diaries create richer and more natural reporting compared to written submissions (Williamson, Leeming, Lyttle & Johnson, 2015). One disadvantage compared to interviews is that you cannot follow up on students´ statements directly. With the weekly insights we were able to change small details of our structure for each participant so that possible follow-up questions could be asked, though only one week later.

Overall, the audio files were between 1:41 minutes and 13:27 minutes long with an average of 5:50 minutes. Participation in the study was remunerated with 30 euro.

Coding the data

The self-reports were transcribed shortly after they were sent to us. For the purpose of categorization, we used a hierarchical model with three levels. The top two levels were developed deductively from self-regulated learning theory. The top level consists of cognitive learning strategies, and resource management strategies (Pintrich & Garcia, 1994). On the second level these two categories were subdivided, again based on the literature. Within cognitive strategies and resource management strategies, we used the subdivisions from the LimSt (Questionnaire on learning strategies in mathematical studies) questionnaire (Liebendörfer et al., 2022): Cognitive strategies differentiate between rehearsal and elaboration strategies while resource management strategies differentiate internal and external resources. Further subcategories on the third level were created inductively from the data material.

Inter-rater agreement was ensured by training both coders with data coding. To estimate the inter-rater agreement, a random sample of 10% of the self-reports was coded twice with maxQDA. MaxQDA`s analysis tool allowed us to check both coders work for consistency. For that purpose, we used the function ´code overlaps on segments´ with an overlap interval of 95%, resulting in a Cohen´s kappa of .59. After reviewing, we realised that we often assigned the same codes, but the overlap interval of specific coding segments was often smaller than 95%. We then looked at these inconsistencies and revised them, finally achieving a Cohen`s kappa of .91.

RESULTS

With our procedure of coding the data, we found that students reported various strategies of their weekly self-regulated learning in their *action phase*. This is why we focus on that phase in particular in the following.

External resource management strategies

Regarding external resource management strategies (RQ1), we differentiated between course-related, non-course-related, working on homework assignments and exercises, and no (additional) external resources (see Table 1). Although students reported, with the exception of one, that they worked on almost every homework assignment and exercise, they reported making very little use of the course-related materials (e.g. lecture script/notes, tutors, professor), but instead using a lot of non-course-related resources, especially YouTube videos

> Jonas: There are often tutorials by Daniel Jung [German YouTuber] on YouTube, for example, where he better explains how it works, and usually in a much shorter time and much better than the professor ever could. So, I found that incredibly good.

and the Google search function to help them solve those problems.

> Milena: I did a Google search and typed in radius of convergence example problems and Cauchy sequence. I wanted to look at example problems and see if they were similar to the homework in order to solve them in the same way or to understand how it works. And yes, the Cauchy sequence, I looked at things on Google.

Non-course-related resources were not only reported to be used for working on exercises or homework assignments, but were also very popular for better understanding the topics presented in the lectures.

Luca: As far as theory is concerned, I am most likely to go to YouTube for it, because just reading mathematical theorems is rather difficult for me if it is a topic I am not yet familiar with.

Another aspect reported at higher priority was learning with classmates, as this was mentioned in almost half (25 of 56) of the self-reports. In most cases (19 of 25), classmates are then called-in when one got stuck with working on tasks oneself or needs an advice.

In a few cases, students reported that no (additional) external resources were used while working on exercises or homework assignments to create an exam-like situation.

Tom: For the homework assignments, I tried to solve them without looking at the script and doing as little research as possible on the internet so that it was a bit more like exam preparation […], because you can not look at the script in the exam.

External resource management strategies	**Frequency of code**
Non-course related	126 (in 45 self-reports)
Course-related	12 (in 11 self-reports)
Homework assignments and/or exercises	65 (in 39 self-reports)
No (additional) resources	3 (in 3 self-reports)

Table 1: Frequency of codes for external resource management strategies

Elaboration and rehearsal strategies

With regard to elaboration and rehearsal strategies (RQ2), we could identify only a few reports of elaboration strategies, while the focus was on rehearsal (see Table 2).

Except for one student, everyone reported having used elaboration strategies (18 of 56) every so often. And if so, they were mainly used when they were related to the completion of homework or exercises

Tom: So, for example, when it comes to convergence, you try the different methods and think about it, go through the different criteria in your head and see if you can find any similarities to previously solved tasks or examples in the lecture. And then, if you find something, you try it and then it often works.

and rarely to understand mathematical concepts.

For rehearsal strategies we identified different patterns throughout the semester. Firstly, rehearsal was the main strategy for students to process the content presented in the course. We identified in 39 of the 56 weekly self-reports some kind of rehearsal strategy. Secondly, students focussed on rehearsing the contents of the course differently. In this context, students reported three different main patterns:

1. Focus on the content which was presented in the lecture

2. Focus on repeating and practicing the tasks from the homework assignments or exercises
3. Weekly focus on repeating either the lecture notes or the exercises and homework assignments

However, when students reported about repeating or reworking, they often did not go into detail about how exactly they were doing that. Regarding the exercises and homework assignments, students reported that they mostly just solved the tasks once again or compared their own work with the sample solution which the course provided.

Luca: And now I am always solving exercises and homework assignments once again, so I do not have to think, so to speak, and I immediately recognise patterns and my hand just sort of writes away.

Although students rarely reported using the script as a resource, they still described detailed strategies, for example memorization, reading lecture notes (multiple times) and writing down the lecture again.

Cognitive strategies	**Frequency of code**
Elaboration strategies	39 (in 18 self-reports)
Rehearsal strategies	93 (in 39 self-reports)

Table 2: Frequency of codes for cognitive strategies

DISCUSSION

This paper aims at giving a deeper insight into the strategies that students report making use of in their action phase of self-regulated learning.

The results indicate that the students in this study prefer to use non-course-related resources which seems counter-intuitive, since the course usually provides lots of material for working on exercises and homework assignments and later on for preparing for exams. Instead, we have seen that non-course-related resources were rather popular and were frequently addressed when referring to homework assignments, exercises, to understand certain topics or to rework contents from the lecture. Although the study was conducted during the return to face-to-face teaching, this could possibly be due to the previous online semesters, which naturally promoted digital learning. In a similar study, Kempen & Liebendörfer (2021) observed during a digital semester that Linear Algebra students rated the usefulness of traditional resources, such as tutorials, lecture and lecture notes slightly higher than digital resources, such as videos or webpages from the Internet. We did not assess students' ratings on the usefulness of resources in our study, but the reported frequencies show a tendency towards non-course related resources in our context. The high use of non-course-related resources raises the question of why students use them more often than the course-related resources. Videos in particular seem to play a big role in this, possibly because students can work through the videos at their own liking and speed. Thus, the use of self-produced videos adapted to one´s own course could offer further potential for teaching mathematics. Additionally, it might be important to reflect with students about the use of external digital resources, such as critical thinking and questioning the resources as well as compare them with the course-related material.

The results also show that students hardly report any use of elaboration strategies during the semester to process the content of the course and prefer to use rehearsal strategies. Here, one would expect that especially during the semester, elaboration strategies would predominate in order to understand the new content. Due to our more general and not specific content-related questioning of the strategies, it could also be that students had difficulties reporting elaboration, which was also noticed in the study by Göller (2020). It may be that students do use elaboration strategies while learning mathematics, but they may lack the skills to express them in the self-reports. However, this claim needs further investigation. Otherwise, mathematics courses for engineers are often focused more deeply on applying and executing mathematical methods. Rehearsal strategies like practicing and repeating exercises or homework assignments exactly do this, consolidate such mathematical methods which also might prepare students better for the exams (Liebendörfer et al., 2022).

In future research, it might be useful to take a closer look, which is directly involved in the learning processes of students, e.g. when repeating or working on the homework assignments and exercises or the lecture. This way, more information can be gained on students self-regulated learning of mathematics.

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“exponential” as just “another word” to say “fast”: When colloquial discourse tells only part of a story

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As part of a research and development programme that examines intra- and extra-mathematical visibility of mathematics especially for students enrolled on a Bachelor’s in Education, most of whom will soon train as primary teachers, I made “exponential growth” the focus of one part of the summative, portfolio-based assessment in a Mathematics Education course. Commognitive analyses of students’ portfolio entries revealed a significant majority of student narratives about “exponential” as roughly synonymous to “very fast”. Here, I show evidence of how students navigate across literate and colloquial narratives about exponential growth, and I discuss how intra-mathematical deficits in these narratives may play out adversely in the students’ capacity to recognise and engage with its manifestations in colloquial situations.

Keywords: Mathematics Education courses, social significance of Mathematics, commognition, exponential growth, literate and colloquial discourses

> “If you look at the number of in-patients […] on 7 September there were 536 cases. By the time you get to the beginning of October, it is over 2,500. As of today, it has breached 10,000 people in hospital. You do not need too much modelling to tell you that you are on an exponential upward curve of beds.”[1]; “it’s amazing how many clever people don’t know what exponential growth means”[2] Chris Whitty, Chief Medical Officer for England

INTRODUCTION

Throughout the pandemic, public discourse about Covid-19 – in the UK, led primarily by daily, televised conferences of the Government’s Chief Medical Officer and his Deputy as well as the Chief Scientific Adviser – brimmed with mathematical references. Amongst those was the abundantly used reference to “exponential growth”. As debates raged about whether and how to convince the public of the utter necessity for the personal, social and economic sacrifices that tackling the virus implied, a stark realisation started to emerge: that many of these references may not have the impact that the scientists who were making them were hoping to achieve. In tandem with findings from research that indicated how invisible mathematicians and mathematics often seem to be (Yeoman, Bowater & Nardi, 2017; Nardi, in press), I conjectured that relentless exposure of the public to said mathematical references may make some difference. To explore this conjecture, I made “exponential growth” one focus of the summative assessment in a Mathematics Education course I teach to final year BA Education students. Here, I report analyses of data I collected during this assessment.

[1] Prof. Whitty’s evidence to Parliamentary Committee, 3 November 2020: transcript at https://t.co/PjjRQBlT3L

[2] Plenary at LGA/ADPH Annual Public Health Conference 2021: Rising to the challenge parts 1 and 2, 23 March 2021 | Local Government Association,

In what follows, I discuss the background and rationale of the research and development work I am conducting in the context of the course's design, implementation and evaluation. I then outline the commognitive underpinnings of this work – with a focus particularly on pertinent constructs such as *literate* (as in, e.g., mathematical) and *colloquial* (as in everyday, public) *discourses* (Sfard, 2008; p.118). I then introduce the context, participants and data collection methods I use in this work. Subsequently, I present data and analysis relating to students' narratives about exponential growth as they emerged from aforementioned summative assessment. I conclude with reflections on what this analysis may imply for university mathematics pedagogy, especially for students on the cusp of entering the teaching profession.

INTRA- AND EXTRA- MATHEMATICAL VISIBILITY OF MATHEMATICS

Visibility of mathematics and mathematicians – whether, for example, in popular culture (Nardi, 2017) or in the narratives of secondary school students (Yeoman *et al.*, 2017) – is becoming a focus for mathematics education researchers as well as science communication scholars. "What does a mathematician do?" was one question that Yeoman *et al.* (2017) asked during focused group interviews with secondary students in a study that investigated student narratives about how research in various disciplines is conceived, conducted – and where its utility and significance lie. Evidence from the study (group interviews with 100 students aged 11-19, questionnaire responses from 2634 students) on whether "Mathematicians do a lot of research" – and examples of said research – was alarmingly scarce: examples of research in mathematics which participants considered worthwhile elicited *zero* responses (Nardi, in press).

Yeoman *et al.* (2017)'s evidence corroborates the claim that mathematics and mathematicians are relatively invisible and that public narratives about mathematics (including those of young people still in their schooling years) remain poor (Nardi, 2017). These findings also resonate with my experience as the faculty member in charge of *Children, teachers and mathematics: Changing public discourses about mathematics*, an introductory, optional course in Research in Mathematics Education (RME), to final-year BA Education students that I designed and have been delivering since 2012. The course aims to welcome Education undergraduates (of whom about three quarters will soon enter primary teacher preparation courses) into RME. Particularly, the course aims to trigger revisiting the students' own, often traumatic experiences of learning mathematics and to help them overcome their reticence about their mathematical ability. Over the years, from a relatively narrow exercise in tackling disaffection with the subject, the course has evolved into a platform on which student narratives about what mathematics is and what it is for are regularly challenged. The course is assessed through a portfolio of learning outcomes (see Course section of this paper): this invites responses to ten tasks that include biographical accounts of their mathematical experiences, responses to *mathtasks* (Nardi & Biza, in press) and "maths pitches", snappy narratives which present an important piece of mathematics briefly and clearly to a member of the general public.

With Research Ethics Committee permission and student consent, student portfolio responses are analysed via a frame consisting of a typology of four characteristics informed by the Theory of Commognition (see next section of this paper).

As reported in (Nardi, in press), over the years, I observed that students' choices of topic for their "maths pitch" invariably focused on classical topics from mathematics created in the distant past. Consequently, I started requesting two pitches, one entitled "mathematics over time" and one entitled "mathematics today". In the dataset collected from the first student cohort with which I trialled the request for a dual pitch, the vast majority, over 80%, ignored the "today" in the brief and simply presented a bit of ancient mathematics, sometimes disguised as having contemporary relevance (say, Pythagoras' Theorem in the construction of buildings). This, I label a "fossil" narrative about mathematics. And, I label the few concretely described uses of mathematics in engineering, environmental science, economics, medicine and technology evidenced in a handful of portfolio entries a "fuel" narrative about mathematics.

What this, and other, portfolio evidence suggests resonates with Yeoman *et al*.'s (2017) conclusion that, to these undergraduates – now at the cusp of university graduation and entering the world of work, including the world of the classroom – mathematics remains largely invisible. I concur on this matter with (Herbst *et al.*, 2021) in wondering: should we not, as university mathematics educators, do better at enriching these professionals-soon-to-be narratives about what mathematics is, and what it is for?

Spurred on by the aforementioned omni-presence of references to "exponential growth" in pandemic-related public announcements, I invited students' "pitches" on this quintessential mathematical object in one recent portfolio task. I did so in awareness of research findings that alert us to documented student difficulties with this topic. For example, Ellis *et al*. (2016) recap findings about: university students' struggle with rules of exponentiation and with connecting these to rules for logarithms; and, about secondary students' struggle with the transition from linear to exponential representations and with identifying what makes data exponential. They also report that teachers find instruction about exponential growth challenging, particularly of how exponential growth as repeated multiplication connects with the closed-form equation.

Crucially to the focus of this paper, Ellis *et al.* (2016) also include in these challenges the recognition of growth as exponential in nature and the generalization of rules such as the multiplication and power properties of exponents. Overall, however, the bulk of prior research into the challenges posed by the learning and teaching about exponentiation has an intra-mathematical focus. The focus of the discussion I propose in this paper is on the boundaries of intra/extra-mathematical consideration of the mathematical object "exponential growth": how may manifestations of exponential growth feature in colloquial situations? How may students / teachers / members of the public navigate across literate and colloquial narratives about exponential growth? And, how may intra-mathematical deficits in narratives about exponential growth play out adversely in our capacity to recognise and engage with its manifestations in said colloquial situations?

COMMOGNITION, MathTASK AND FOUR CHARACTERISTICS

The theoretical perspective of this work is discursive and draws on the commognitive framework (Sfard, 2008) that sees Mathematics, Education and Mathematics Education as distinctive discourses and learning as communication acts within these discourses. In resonance with this discursive lens, the design of the Mathematics Education course that is the focus of this paper aims, *inter alia*: to attend to discursive differences, and potential conflicts, between Mathematics and colloquial discourses that students may experience; and, to assist students towards smooth navigation across these discourses.

A further, also commogntively-informed, element of the theoretical perspective of this work comes from the pre- and in- service mathematics teachers focussed *MathTASK* programme. In MathTASK, we engage teachers with fictional but realistic classroom situations, which we call *mathtasks* (Biza et al., 2018). Mathtasks are presented to teachers as short narratives that comprise a classroom situation where a teacher and students deal with a mathematical problem and a conundrum that may arise from the different responses to the problem put forward by different students.

Towards the analysis of student data collected during the delivery of the course, I deploy a typology of four interrelated characteristics that emerged from themes identified in prior MathTASK research (see detailed rationale, definitions and examples in Biza et al., 2018; Nardi & Biza, in press) and is tailored to aforementioned commognitive underpinnings: *consistency*; *specificity*; *reification of RME discourse*; *reification of mathematical discourse*. This typology is the basis for the assessment frame deployed towards the formative and summative assessment of the students' work during the course. Each one of the typology's four characteristics encapsulates features that can be traced firmly and concretely in the students' writing. So, for example, for reification of mathematics discourse, I scrutinize student portfolio entries in terms of how specific, relevant, reliable and accurate the use of mathematical terminology is. I ask questions such as: are mathematical terms used accurately? Is there direct relevance of a mathematical utterance to the claim being made? Is the utterance consistent with mathematical theory? Is the link between the utterance and the overall claim made explicit? Are any credible sources for the mathematical utterance quoted? Etc.

Seeking such evidence in the students' responses not only secures a verifiable (by course moderators and external examiners) route to a student's course mark; it also paves the way for identifying which student narratives about the teaching and learning of mathematics (and RME) subsequent versions of the course (and research thereof) need to challenge, and, hopefully, change. As part of my larger investigation into the visibility of mathematics in extra-mathematical situations, the portfolio item (the mathtask in Figure 1) and the analysis of corresponding portfolio entries that are the focus of this paper aimed to trace how students navigate colloquial and literate narratives on one mathematical object, "exponential growth", and whether, and how, students recognise mathematics in colloquial situations.

COURSE, PARTICIPANTS AND PORTFOLIO-BASED DATA COLLECTION

The course is part of the BA Education (BEd) programme's suite of optional courses. Contact time is four hours per week (two for lectures and two for seminars) for a period of twelve weeks. Lectures are teacher-led and partly interactive. Seminars are student-led. As about three quarters of the programme's graduates continue into training to become primary teachers, the course is designed to address directly the widely reported reticence of those students towards mathematics and their generally low self-esteem in mathematics. Its aim is to equip these students with the means to tackle the disaffection that often tantalises the relationship with mathematics experienced by themselves as well as the young people many of them will soon be preparing to teach (Nardi, 2017).

The inception of the course stems from acknowledging that the preparation of teachers rarely equips them for this complex task – and its twelve weeks of lectures and seminars are organized to address in-school (curriculum, classroom) and out-of-school (media, popular culture, arts) discourses on mathematics. In the Portfolio of Learning Outcomes, the course's single item of summative assessment, students are asked to: return to ten activities they prepared for in the weekly seminars; study the materials accumulated during the twelve weeks of the course; and, compose a revised contribution to each one of the ten activities, written in the light of what they learnt during those twelve weeks. The headings of the ten activities are:

> **1**. *Mathematics and I*: A biographical account of your relationship with mathematics **2**. *Mathematics in the media*: A brief analysis of a mathematics-related media excerpt (paper press or online) **3.** *School mathematics and I*: Reflections on one aspect of the primary or secondary mathematics curriculum **4.** *Mathematics over time*: A 2-minute Maths Pitch from the history of mathematics **5.** *Mathematics today*: A 2-minute Maths Pitch on a contemporary application of mathematics **6.** *Mathematics in the classroom*: A brief analysis of a classroom incident (with mathematical, social, affective, meta-mathematical elements) **7.** *Mathematics in art and popular culture*: A brief analysis of a mathematics-related art or popular culture excerpt (film, TV, theatre, literature, arts, music) **8**. *Mathematical ability on film*: A brief analysis of the portrayal of a mathematically able character on film **9.** *Myths about maths*: A brief essay, with evidence, debunking myths about maths (such as Innate, Male, Introvert, Burn Out, Uncreative) **10.** *Mathematics lesson plan*: A plan for a mathematics lesson on a topic (of each student's own choice).

Students are asked to deploy RME theoretical constructs introduced during the course: see (Nardi & Biza, in press) for examples of these across developmental, sociocultural, anthropological, embodied and discursive theories. They are also expected to refer to a small number of research papers (and, where needed, other publications such as policy documents, reports or media excerpts) in each part. An example of a mathtask students were asked to respond to in Part 6 of a recent portfolio is in Figure 1. In this paper, I focus on student responses to question 1, mainly the "exponential growth" part.

Ms Jones is about to start a mathematics lesson in a Year 6 class (student age 10-11). As she walks into class, she notices two students arguing about whether wearing a mask during the pandemic makes sense. One of the students, Neil, complains as follows:

Neil: Ms, I hate these! Why do we even care? Only one percent of people will end up in hospital anyway. Let's do the maths! One percent is almost nothing!

Ms Jones: Ok, Neil, yes, let's do the maths. [*to the class*] Any ideas anyone?

Anna: Well, I never liked percentages. They are so vague. Ok, here is my bag of crisps [*she takes a Kettle© chips 30 gr bag out of her backpack*]. And here is the one my dad asked me to bring him on my way back from school for tonight's game on TV [*takes Kettle© chips bag of 150gr out of her backpack*]. I will take one percent of this bag [*rattles 150gr bag*], any day, Neil!

Barack: Yeah, ok, unless the big one is almost empty and the little one is sealed and full [*laughs*]. And, Neil said let's do the maths. He didn't say let's talk about crisps ... Neil, you know what? In her silly crisps and the like kind of way, Anna has a point. One percent of what? If one hundred people get infected, then one person will go to hospital. We can sort of deal with that, right? [*writes 1 on his whiteboard*]. If one hundred thousand people get infected, then one thousand people go to hospital [*writes 100,000 and 1,000 on his whiteboard*]. Hm... and if one million people get infected [*writes 1,000,000 on his whiteboard*] … You see where I am going? Oh, plus infections are doubling every day... I heard my big sister say something about a thing called expo. ... something growth, sounded like a tumour ... Scary. I am telling you. If we don't do something, we are doomed to be stuck with these masks and all for ever!

Clive: OMG, Barack! You and your depressing speeches! Too many zeros are giving me a headache by the way. I just want to ask Anna: do you *really* think this big bag of crisps in your backpack will still be there by lunchtime!?

[*The class erupts with laughter.*]

You are the teacher and you just heard what Neil, Anna, Barack and Clive said….

1. Here is a sentence that sounds like ones that we have been hearing in the news in the last twenty months or so: "*About 1% of infected people will need hospitalisation and the growth of infections is exponential*". How do you explain this sentence to someone who does not know what "percentage" and "exponential growth" mean?
2. How would you respond to Anna?
3. How would you respond to Barack?
4. How would you respond to Clive?
5. How would you respond to the whole class – also in the light of Neil's initial comment – and conclude the lesson?

Figure 4. The *Percentages, exponential growth and masks* mathtask from a Portfolio of Learning Outcomes. The focus of this paper is on Question 1.

"PLAIN" AND "FASTER" NARRATIVES ON EXPONENTIAL GROWTH

The focus of this paper is on 25 student entries (2 were non-responses). I grouped the students' narratives on exponential growth in the remaining 23 as follows (Table 1).

"plain growth" narratives (5 entries). In these entries, exponential growth is described as "a quantity increasing as time goes by" or as "something [that] constantly increases over time". 3 entries contextualise the response in the case of a virus, noting that exponential growth indicates "the growing rates in infections" (with 2 specifying "over time"). Entries in this group tend to be low in specificity: none, bar one, includes reference to a particular numerical example, dataset or graph. The one that does ("…it is a *growth rate that becomes larger in relation to the total growing number* (a), [my emphasis] […] 1% become infected and this percentage is *growing every day* (b) *as the rate is exponential* (c)") sets out with two promising utterances – a and b – but is let down by a circular warrant, c, where the term "exponential" is used to define itself.

'faster and faster growth" narratives were discerned in the remaining 18 entries: 15 were solely verbal; and, 3 were reliant on closely connected graphical and verbal realisations of "exponential growth". Amongst the 15, there were 5 consisting of exclusively general statements and 10 included specific examples. Entries in each category (of 5 and 10) were differentiated as per type of growth each described.

'faster and faster growth" narratives (*verbal, general:* 5 entries) sometimes stand on an ambiguous boundary with *"plain growth"* narratives. For example, in "[t]he number of infected people will *increase* (a), as they will likely pass the infection to *several people* (a), leading to a faster and faster increase (b) in case numbers", I see (a) as alluding to a "plain" and (b) to a "faster and faster" narrative. Another student quotes the *Organisation for Economic Co-operation and Development* (OECD)'s glossary ("a circumstance in which growth compounds continuously at every instant of time") but concludes their sentence with the "plain growth" utterance "meaning rates of infection are continuously growing". Another student, even though in principle she would set out from asking her interlocutors for examples of "when we see [exponential growth] in everyday life" and she would "provide a definition and examples", she then simply quotes the *Centre for Evidence-Based Medicine*'s glossary ("when the speed of growth is proportional to the size of the population") and qualifies no further. A similar, if convoluted, attempt is in another student's "…a *specific way* (a) that numbers can increase over time. Described as a function, a quantity that undergoes *exponential growth is an exponential function of time* (b), that is, the variable representing time is exponential (*in contrast to other types of growth, such as quadratic growth*) (a).": I see (a) as worthy attempts to highlight that exponential growth is not just any growth let down by (b)'s circular reasoning. All, bar one of these five entries, make no reference to any other realisations of "exponential growth". The one that does ("… when the *rate of increase becomes faster as the population increases* (a). The *more people with covid* (b) the faster the rate of infection therefore a sharp curve would be *seen on a graph* (c).") includes: a verbal, but of low specificity, reference to a graph (c); an attempt to capture the speed of increase (a); and, an attempt at contextualising (b).

’faster and faster growth” narratives (*verbal, example-based:* 10 entries) were a significant majority. Most of these 10 entries evidence a narrative that exponential growth is a particular type of growth (“another word which can be used to describe something that is fast. More so, it describes how something can grow more quickly over time”). I saw variable attempts to locate this particularity. In “…when an event increases more and more so the increase gets faster over time”, the student captures the difference of speed in the increase as “increasing at a faster rate than time is passing” and “the speed it will infect people will get faster” (two other students wrote similarly). These efforts are however followed by examples such as: “that on day 1 one person is in hospital day 2, two more people are in hospital and on day 3 ten more people are in hospital”; or, “if 5 people were reported as infected today and tomorrow 10 people were infected”. Such responses are quite far from identifying the exponential by which the growth takes place (noted in 4 of the 10 entries). What kind of increase exponential growth means is captured a bit more in another student’s entry who notes that “exponential growth refers to the power of the epidemic” and specifies with the example “[w]hen there are two people infected, then the number of infected people is no longer 2 but 4 people”. Other attempts at capturing “exponential” include: the syntactically challenged – and potentially conflating quadratic and exponential growth – “if the factor were n^2 starting at 2, value 1 would be 4 (2^2,) value 2 would be 16 (4^2,) value 3 would be 256 (16^2,) etc.”; and, the elegant “say at the beginning of the pandemic we had an increase of 2 cases per day for the first week, but then the second week it is 4 per day, the third week it is 8, and so on”. Explicitly differentiating from other types of growth, another student writes: “if 1 person was infected, the next day that one person could infect maybe around 5 more. The next day those 5 people could infect 25 more, then 125 then 625 then 3,125 then 15,625 so you see that the more people get infected the more it spreads. It does not just go from 1 to 2, it has larger gaps between each jump”.

<table>
<tr><th>“plain” (5)</th><th colspan="3">“faster and faster” (18)</th></tr>
<tr><td rowspan="2">general (4)</td><td rowspan="2">general (5)</td><td colspan="2">verbal only (4)</td></tr>
<tr><td colspan="2">verbal with reference to graph (1)</td></tr>
<tr><td rowspan="5">specific (1)</td><td rowspan="5">specific (13)</td><td rowspan="3">verbal only (10)</td><td>imprecise (5)</td></tr>
<tr><td>ambiguous (3)</td></tr>
<tr><td>precise (2)</td></tr>
<tr><td rowspan="2">verbal with graph (3)</td><td>graph illustrative yet generic (1)</td></tr>
<tr><td>graph illustrative, informative (2)</td></tr>
</table>

Table 1: “plain” and “faster” narratives on exponential growth in portfolio entries

Finally, 3 entries evidenced *'faster and faster growth" narratives (verbal, with graph)* in which a graphical realisation of exponential growth has significance that varies from minimal ("…growing or increasing really fast and is often shown in a graph as a really steep gradient or line. Like in the graph below", Figure 1a) to quite substantial (Figures 1b, 1c). The student entry that corresponds to Figure 1b

> "[…] speed of growth proportional to the population. I would first explain linear growth (1 more person is infected each day, so 1 on day1, 2 on day2, 3 on day3, 4 on day4 etc). Then use doubling and compare to exponential growth (1 person is infected on day1, then 2 on day2, 4 on day3, 8 on day4 etc. A simple graph would help highlight the difference between exponential growth and linear growth."

is purposefully embedded into a "doubling" example of exponential growth ($f(n)=2^{n-1}$) as distinct from linear growth ($f(n)=n$). Figure 1b captures the difference between the two with the annotation "disruption". The student entry that corresponds to Figure 1c ("Something doubling each day […] something becoming more and more rapid […] a process that increases quantity over time […] a relatively small number can become very large very quickly […]) evokes a different but also helpfully transparent realisation of "exponential growth".

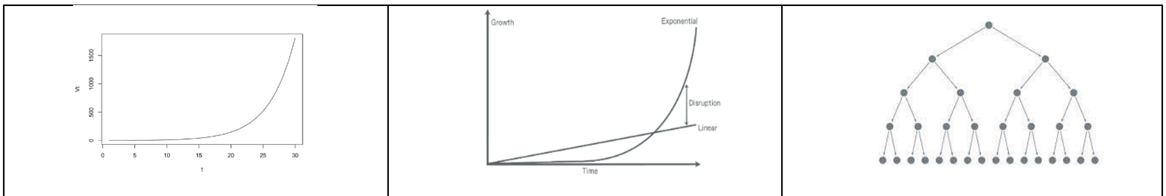

Fig. 1a	**Fig. 1b**	**Fig. 1c**

Figure 1: "exponential growth" images in 3 portfolio entries.

LOSING URGENCY OF "EXPONENTIAL" IN COLLOQUIAL DISCOURSE

Missing the exponential nature of the virus' transmission proved to be a key casualty of governmental policy – and its endorsement by the public – during the pandemic in the UK. The bulk of the student portfolio entries discussed in this paper equate "exponential" more or less with "very fast": this equating obfuscates the urgency of the exact sort of "fast" that exponentiality carries (as Professor Whitty's quotations at the start of this paper amply demonstrate). While mathematics education research into the learning and teaching of exponentials has mainly focused on its intra-mathematical significance and precision, its extra-mathematical relevance and importance has been left as a matter to be dealt with outside the classroom: see, for example, Chapter 1 in Yates (2021) for a lucid demonstration of exponential growth in the context of algal bloom and nuclear reactions; or, the Centre for Evidence-Based Medicine's nutshell account in the context of viral transmission. In "picturing a crisis" mathematics is here to help with "formatting solutions" (Skovsmose, 2021; p. 371) to it – crucially though, this can only happen in tandem with civic appreciation for mathematics. In preparing citizens for the workplace, especially long-term influencers such as teachers, our role in fostering this appreciation as university mathematics educators is key.

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Analyzing multilingual students' experiences in introductory college mathematics courses

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The goal of this paper is to better understand the experiences of multilingual students in undergraduate math courses, particularly when active learning is used. Despite significant advances at the K-12 level, little research at the post-secondary level has examined language in mathematics education. To address this, this paper analyzes survey data from undergraduate students in introductory math courses. Using a quantitative critical lens, the analysis focuses on identifying salient features of students' identities that impacted their course experiences, particularly their comfort speaking in class, comfort being oneself in class, and recognition by others as being good at mathematics. In addition, the analysis examines the effect of different frequencies of active learning on these outcomes.

Keywords: curricular and institutional issues concerning the teaching of mathematics at university level, students' practices at university level, multilingual mathematics classrooms, active learning

INTRODUCTION

At the post-secondary level, math classrooms are becoming more reflective of the multicultural landscape that we live in. Because of political and economic global shifts, college mathematics classrooms serve increasingly more students whose home language differs from the language of instruction (Durand et al., 2016). It is becoming more common for college math classrooms to be rich, multilingual spaces where students use different cultural, linguistic, and experiential knowledge to make sense of mathematics.

At the same time, language is often over-looked in mathematics education because of the common viewpoint that math is a universal language. However, research at the K-12 level has shown that students' linguistic and cultural backgrounds significantly shape how students learn math (Planas & Civil, 2013). Furthermore, language is not neutral, as some languages – like English – are assigned higher social status over others – like Spanish. The literature on K-12 math education has documented examples of how multilingual students (whose home language differed from the language of instruction) experienced less access to classroom participation (Planas & Civil, 2013) and were less likely to be positioned as mathematically competent (Takeuchi, 2016). Despite significant advances at the K-12 level, little research at the post-secondary level has focused on better understanding multilingual students' classroom learning experiences and best practices for supporting equity in language diverse classrooms.

Active learning

Active learning is becoming more commonly implemented in post-secondary mathematics classrooms and the literature has generally considered active learning to be an equitable teaching approach. Some scholars, however, have demonstrated that not all students experience active learning in the same way (Henning et al. 2019; Voigt et al., 2022). For example, Henning et al. (2019) examined the different facets of students' identities (like race, gender, socioeconomic status, etc.) that were most significant in shaping their classroom participation, STEM self-efficacy, and sense of belonging in active learning introductory college biology courses. For instances, findings indicated that black students reported "higher pressure to conform to the views of their peers" during active learning (p. 7). In introductory college math courses, Voigt et al. (2022) found that women and Indigenous students experienced a larger decrease in their mathematics identity over the course of the semester compared to white male students.

Multilingual students may also have differential experiences with active learning. Active learning is a talk-intensive pedagogy and participating in active learning generally requires students to communicate verbally, engage in interpersonal interactions, and make sense of mathematics collectively. Given that these practices are all mediated by language, language becomes a more visible tool for learning and participating in active learning settings. More research is needed to better understand multilingual students' experiences in undergraduate mathematics courses, particularly those which utilize active learning pedagogies.

Research Question

Using critical quantitative methods, this paper seeks to answer the following research question: *How does active learning and different aspects of multilingual students' identities influence their perceived experiences in undergraduate math courses?*

THEORETICAL FRAMEWORK

Sociopolitical Theory

This paper draws from sociopolitical theory (Gutiérrez, 2013), by foregrounding issues of identity and power in the classroom. Particularly in active learning spaces, students' identities become more visible as they negotiate social interactions (Henning et al., 2019). Furthermore, students' identities can impact classroom power and participation structures (Takeuchi, 2016). For example, given the high status that English is often afforded in the college classroom, students' language identities can shape how others perceive their mathematical abilities (Rios, 2022).

In this study, sociopolitical perspectives were used to identify significant aspects of student experiences. Instead of analyzing students' classroom experiences from a normative lens (i.e., examining access or achievement), this study focuses on understanding them using a critical lens that centralizes issues of power and identity in the classroom (Gutiérrez, 2013). For instance, focusing on students' comfort speaking and comfort being themselves in the classroom can inform whether multilingual students' identities were affirmed by classroom practices (Henning et al, 2019). Similarly, examining whether students were positioned as being *good at mathematics* by their peers can inform how power and participation were distributed in classroom discourses (Takeuchi, 2016).

QuantCrit Theory

QuantCrit is an emerging field in education research which applies the core tenets of critical race theory (CRT) to traditional quantitative methods (Garcia, López, & Vélez, 2018). QuantCrit studies often use statistical modeling to explore the impact of discrimination (e.g., racism or sexism) on outcomes (e.g., shifts in math identity), by including social markers like race and gender as explanatory variables (e.g., Voigt et al., 2022). Garcia, López, & Vélez (2018) discuss the five main tenets of QauntCrit theory: (1) the centrality of racism in society and that racism is difficult to capture using quantitative methods, (2) numbers are not neutral and can be used to perpetuate deficit narratives, (3) human categories are often socially constructed and require a critical interpretation, (4) data cannot speak for itself and must be informed by experiential knowledge, and (5) results from statistical analyses are not valuable unless they are used to promote social justice.

A situated interpretation of data is fundamental to QuantCrit analysis (Garcia, López, & Vélez, 2018). For instance, variables that capture social makers like race or gender must be viewed as measuring the impact of racism (not race) or sexism (not sex) on outcomes. Similarly, variables encompassing language identities should be viewed as capturing the impact of language bias on students' experiences. Furthermore, Van Dusen and Nissen (2019) argue that by using p-values as sole indicators of significance, statistical models can ignore potentially meaningful effects that pertain to underrepresented groups. Instead, they advocate for the use of other metrics like including standard errors and confidence intervals to indicate significance. Finally, quantitative findings should be supplemented with rich qualitative data that capture students' voices and lived experiences (Garcia, López, & Vélez, 2018).

METHODS

Data Source

The data presented in this paper is part of a larger, mixed-methods study which explores the experiences of multilingual students in undergraduate pre-calculus and calculus courses. This study took place at a large, public, research university in the United States that has been designated as a Hispanic-Serving Institution. The qualitative portion of this study consisted of semi-structured interviews with twenty-eight multilingual undergraduate students. The data for this paper stemmed from the quantitative portion of this study. In this phase of the study, students from 45 sections of pre-calculus, calculus I, calculus II, and calculus III were surveyed about their experiences in their math courses ($n = 579$). The survey included several existing instruments from the literature which measured critical aspects of students' course experiences, like *comfort speaking in class* (Henning et al., 2019), *comfort being oneself in class* (Eddy et al., 2015), *mathematics identity* (Cass et al., 2011), and *recognition,* which is defined as "perceived recognition by others as being a good mathematics student" (Cass et al., 2011, p. 2). The survey also included student demographic questions about their race/ethnicity, gender identity, and language identity.

The also survey gauged the frequency that active learning was used in students' math courses. Although active learning can encompass many different approaches, at the university where this study was conducted, the most common approach for implementing active learning was using group work. Therefore, this study, operationalizes active learning to mean classroom learning that takes place through peer collaborations. To measure the frequency of this, the survey included the *student interaction and collaboration scale* (Walker & Fraser, 2005).

Data Analysis

General linear and logistic regression models were used for the preliminary analysis of survey data. All analyses were done using the R statistical software package. Any incomplete surveys were omitted, leaving $n = 464$ surveys that were analyzed.

To operationalize facets of students' course experiences from a Sociopolitical lens, which considers power and identity, numerous variables were included in the survey. These variables assessed students' comfort in class, sense of belonging, and mathematics identity. Findings from the study's qualitative analysis of student interviews were then used to inform the selection of a subset of explanatory and response variables to be included in the regression models. The explanatory variables that were selected included: *comfort speaking in class*, *mathematics identity*, *course level* and *frequency of active learning*. Social markers were also included such as race, gender, international student status, first generation status. Language identity markers included students' home language and the language students' felt most comfortable doing math in. Confirmatory factor analysis was used to assess the fit of instruments from the survey by examining how well items loaded.

General linear regression models

Two general linear regression models were fit to the data. The first model had *comfort being oneself in class* as the response variable and the second model had *recognition as a good mathematics student* as the response variable. The same explanatory variables described in the previous section were included in both models. Following the recommendation of QuantCrit scholars (Van Dusen & Nissen, 2019), AICc was used to perform model selection. The best-fit model for each response variable is shown in equation (1) and (2) respectively.

$$\begin{array}{c} \textit{comfort being} \\ \textit{oneself} \end{array} = \begin{array}{l} \text{intercept} + \textit{course level} + \textit{math identity} \\ \quad + \textit{active learning} \times \textit{comfort speaking} \end{array} \tag{1}$$

$$\boxed{\begin{array}{ccc} \textit{recognition} & & \text{intercept} + \textit{course level} + \textit{math identity} + \\ \textit{as a good} & = & \textit{race} + \textit{comfort speaking} \\ \textit{math student} & & + \textit{active learning} \times \textit{home language} \end{array}} \quad (2)$$

Logistic regression model

The explanatory variable *comfort speaking during class* was retained in both linear regression models and was a main theme in the qualitative analysis. Therefore, a separate analysis of this variable seemed warranted. A logistic regression model was fit to the data, using *comfort speaking during class* as the response variable with the same explanatory variables used in the previous models. AICc was also used for model selection and the best-fit model is shown in equation (3).

$$\boxed{\begin{array}{ccc} \textit{comfort} & & \text{intercept} + \textit{course level} + \textit{math identity} + \\ \textit{speaking} & = & \textit{active learning} + \textit{gender} + \textit{language} \\ \textit{in class} & & \textit{preference for doing math} \end{array}} \quad (3)$$

RESULTS

This section will first discuss the results of the logistic regression model fitted to predict comfort speaking in class and then provide results from the linear regression models for predicting *comfort being oneself in class* and *recognition as a good math student*. These variables represent critical aspects of students' classroom experiences.

Logistic regression model: comfort speaking in class

The factor variable for *comfort speaking in class* had four levels: very uncomfortable, uncomfortable, comfortable, and very comfortable. Table 1 provides the model's logistic regression estimates, expressed in odds ratios. The model suggests that students who preferred doing mathematics in a language other than English were on average, almost half as likely (OR = 0.44) to report feeling comfortable speaking during class (95%CI [0.20, 0.95]), while holding all other predictors constant. This suggests that classroom learning environments may have privileged students who previously learned mathematics in English, allowing them to feel more comfortable communicating.

Women were also less likely to report feeling comfortable speaking in class compared to men (OR = 0.59, 95%CI [0.40, 0.88]). Drawing from QuantCrit theory, this result demonstrates the impact of sexism in undergraduate mathematics classrooms, and its impact on women students' comfort speaking during class.

Active learning was associated with a higher probability of students' reporting to be comfortable speaking in class (OR = 1.46, 95%CI [1.18, 1.81]). The OR estimate indicates that when the frequency of active learning was increased by one unit, students were on average approximately one and half times more likely to report feeling comfortable speaking in class, while holding all other predictors constant.

Estimates:	Confidence intervals				
	Odds ratio estimate	2.5%	97.5%	SE	p-value
Course features					
Calculus I	0.86	0.53	1.38	0.24	0.53
Calculus II-III	0.52	0.31	0.87	0.26	<0.01
Frequency of active Learning	1.46	1.18	1.81	0.11	<0.001
Social markers					
Cis woman	0.59	0.40	0.88	0.20	<0.01
Gender non-Conforming	0.88	0.22	3.80	0.72	0.86
Math identity markers					
Preferring to do math in a language other than English	0.44	0.20	0.95	0.39	<0.05
Math identity	4.06	2.81	5.93	0.19	<0.001

Table 1: Logistic regression estimates for comfort speaking in class, in odds ratios with confidence intervals

In addition, the model suggests that students in Calculus II, and III were less likely to report feeling comfortable speaking during class, compared to students in pre-calculus (OR = 0.52, 95%CI [0.31, 0.87]). Mathematical identity was also strongly associated with a higher probability of reported comfort speaking in class (OR = 4.06, 95%CI [2.81, 5.93]) For instance, when students' math identities increased by one unit, they were four times more likely to feel comfortable speaking.

Linear regression models: measuring other aspects of students' experiences

Comfort being oneself in class

The continuous response variable *comfort being oneself in class* ranged from [0, 4], where scores greater than two reflect positive comfort being oneself in class. The intercept estimate (β = 2.72, 95%CI [1.92, 2.67]) suggests that on average monolingual students who were very comfortable speaking in class and were not exposed to active learning reported having slightly positive comfort being oneself in class.

Being comfortable speaking was a salient predictor of comfort being oneself in class. For example, students who reported being very uncomfortable speaking, on average scored 1.05 units lower on the comfort being oneself scale than students who reported being very comfortable (β = -1.05, 95%CI [-1.37, -0.73]).

Figure 1 shows the interaction effects for active learning and comfort speaking in class. For students who reported being very comfortable, comfortable, and uncomfortable speaking, the slope estimates are positive. Therefore, students in these groups experienced an average increase in their *comfort being oneself* score when the frequency of active learning increased. On the other hand, students who reported being very uncomfortable speaking in class experienced a further decrease in their *comfort being oneself* score when active learning was used (β = 0.10−0.25= -0.15, 95%CI [-0.34, 0.05].

Estimates:	Confidence intervals				
	Coefficients	2.5%	97.5%	SE	p-value
(intercept)	2.72	2.47	2.99	0.13	<0.001
Course features					
Calculus I	-0.10	-0.2	0.006	0.06	<0.1
Calculus II-III	-0.23	-0.35	-0.11	0.06	<0.001
Frequency of active Learning	0.10	0.003	0.21	0.05	<0.05
Comfort speaking					
Comfortable	-0.41	-0.64	-0.19	0.11	<0.05
Uncomfortable	-0.79	-1.04	-0.54	0.13	<0.001
Very uncomfortable	-1.05	-1.37	-0.73	0.16	<0.01
Math identity markers					
Math identity	0.33	0.24	0.41	0.04	<0.001
Interactions					
Active learning × comfortable	-0.03	-0.14	0.09	0.06	0.62
Active learning × uncomfortable	-0.05	-0.18	0.91	0.07	0.51
Active learning × very uncomfortable	-0.25	-0.47	-0.02	0.11	<0.05

Table 2: Regression estimates for comfort being oneself in class with confidence intervals

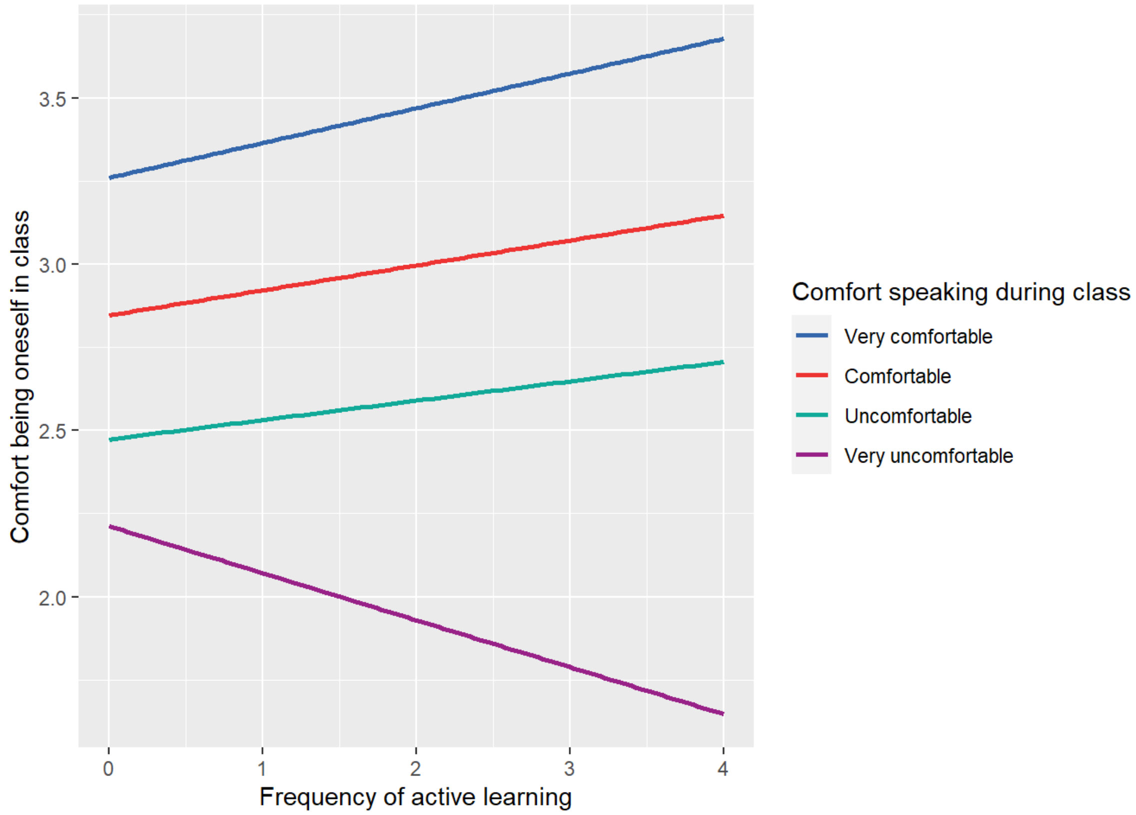

Figure 1: Interaction effects of active learning and comfort speaking during class on predicting comfort being oneself

Recognition as a good math student

The continuous response variable *recognition as a good math student* ranged from [0, 4], where scores greater than 2 reflect positive recognition by others. Students who reported having a home language other than English were less likely to report feeling like their peers recognized them as being good at math (β = -0.23, 95%CI [-0.42, -0.04]). As expect, *math identity* was also strongly correlated with higher *recognition* scores (β = 1.02, 95%CI [0.90, 1.13]). Students that identified as being middle eastern or Asian also on average reported being less likely to be recognized by others as a good math student, holding all other variables, like math identity, constant. This was most significant for Middle Eastern students (β = -0.50, 95%CI [-0.81, -0.18]).

Estimates: (significant only)	Confidence intervals				
	Coefficients	2.5%	97.5%	SE	p-value
(intercept)	0.03	-0.19	0.25	0.11	0.80

Course features					
Calculus I	0.17	0.004	0.33	0.08	<0.05
Calculus II-III	0.36	0.17	0.50	0.08	<0.001
Math identity markers					
Math identity	1.02	0.90	1.13	0.06	<0.0001
Language identity					
Home language other than English	-0.23	-0.42	-0.04	0.09	<0.01
Social markers					
Asian	-0.20	-0.40	0.01	0.10	<0.1
Middle Eastern	-0.50	-0.81	-0.15	0.16	<0.05

Table 3: Regression estimates for recognition as a good math student with confidence intervals (only significant estimates are included)

DISCUSSION

Using a QuantCrit approach, this study examines the experiences of students in undergraduate math courses, focusing on their comfort speaking in class, comfort being themselves in class, and recognition by peers as being *good* mathematics students. The study sought to identify aspects of students' identities that shaped their experiences in the classroom, particularly students' language identities. Findings indicate that students who preferred to do math in a language other than English were less likely to feel comfortable speaking in class. Additionally, students' whose home languages were not English felt less likely to be identified by peers as being *good at mathematics*. These findings suggest that classroom language biases can impact students' comfort in the classroom and access to being positioned as mathematically competent.

Active learning also influenced students' experiences. On average, higher frequencies of active learning were associated with students reporting that they feel more comfortable communicating in the classroom. However, the model indicates that students who reported being very uncomfortable speaking in the classroom, felt even less comfortable when active learning was used. This suggests that speaking is a significant part of navigating the active learning classroom and that students experienced active learning differently based on their comfort speaking during class. Further analysis is needed to investigate multilingual undergraduate mathematics classrooms and best practices for implementing active learning in these spaces to affirm and support all students.

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To want and not want to learn

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This contribution is based on a phenomenon that is well known in the research community: An initial impression that in research interviews mathematics is not talked about at all, that specific subject content is not addressed, or at a bare minimum in a quite superficial manner. The contribution traces what such superficial thematisation of mathematical content can tell us about learning barriers that are specific to mathematics teacher education. This theoretical reflection takes descriptions of my interview partner Anna as a starting point and analyses them using the vocabulary of the subject-scientific approach. The focus is on dynamic learning barriers which are characterised by the simultaneity of wanting and not wanting to learn.

Keywords: curricular and institutional issues concerning the teaching of mathematics at university level, teachers' and students' practices at university level, subject-scientific approach, learning barriers.

THE PHENOMENON OF NON-THEMATISATION OF MATHEMATICS

In an interview study on the learning experiences of pre-service mathematics teachers at university, I came across a phenomenon that is well known in the research community, which I will refer to in the following as "non-thematisation of mathematics": When reading the interview texts, the initial impression is that mathematics is not talked about at all, that specific subject content is not addressed, or at a bare minimum in a quite superficial manner. Mathematics seems to take a secondary place to pedagogical concerns or the thematisation of specific teaching-learning arrangements. I was assured by colleagues, that this observation at the surface level is not specific to the interviews for my research work and similar observations are reported on a regular basis. On the part of the teacher educators and also researchers within the field, a lack of or only superficial thematisation of mathematics is interpreted as a motivational issue: A lack of interest in the subject matter. However, solid analyses of this phenomenon are rare.

Brown & McNamara (2011) share their experiences made in interviews with prospective teachers: They describe that it was difficult for the prospective teachers to talk explicitly and exclusively about mathematical content, or the nature of the subject (ibid., p. 11). Understandings of and experiences with mathematics were articulated by the prospective teachers primarily in affective terms (ibid., 2011, p. 25) and school mathematics experiences, mostly, could not be named concretely with reference to the mathematical content (ibid., p.121/122). If "teaching mathematics" was addressed, it was mainly pedagogical materials that stood in for mathematics - the mathematical ideas behind the material were not presented -, or mathematics was subsumed into the articulation of teaching practices and addressed through an administrative lens, such as classroom management (ibid., p. 25). Their analysis is based on a psychoanalytic perspective and focuses on the concurrence of negative experiences and affect with a local reform initiative in mathematics education.

Furthermore, the non-thematisation poses a challenge when it comes to generating data about learning at universities and typical hurdles that are specific to mathematics. This leads researchers, among other things, to use methodical techniques to create situations in which respondents have to talk about mathematics more explicitly. However, even in these situations, it is sometimes difficult to talk about mathematics and learning mathematics with the respondents. Bibby (2002) reports that talking about mathematics and learning mathematics can be a shame-related topic for student-teachers, or adult learners in general. It is worth mentioning, that the prospective teachers in both studies did not explicitly choose to study mathematics as a subject, but needed to do so to become a teacher.

This contribution takes up this challenge and traces what even a superficial thematisation in interview texts can tell us about learning barriers that are specific to mathematics teacher education.

Situating the theoretical reflection on the phenomenon of non-thematisation

The following theoretical reflection is situated within the field of mathematics teacher education and focuses on its university part. The role that the university should play as an institution in mathematics teacher education is certainly contested. There is agreement that it should offer learning opportunities that go beyond the acquisition of mathematical knowledge, but it is debated what defines this "beyond". A possible "beyond" that the institution university can offer to teacher education is not only to provide specialist knowledge but also to offer a space for reflection that is relieved from the burden of practice (Wenzl et al., 2018), resp. the necessity of immediate action. Within such a space for reflection alternative approaches to current practices are open for debate. This includes, among other things, approaching mathematical knowledge from different angles and in its interconnections, e.g. to be able to discuss alternatives to current curricula.

Situated within this specific context and stance towards mathematics teacher education, the focus of my research work is on the learning experiences of student-teachers that are associated with the socialisation into the academic field of mathematics education – and, therefore, not on a specific mathematical topic or domain. As outlined in Ruge (in press), socialisation is understood as a reciprocal task, not a one-way affair. It involves not only an acquisition of current or desired practices but also participation in the formation of the field.

In contrast to other international studies focusing on student-teachers (e.g. Brown & McNamara, 2011; Bibby, 2002), all my interview partners choose to study mathematics as one of two school subjects they want to teach in the future, - so mathematics was not a subject imposed on them to pursue a career as a teacher. They explicitly but not exclusively chose mathematics. Furthermore, most of my interview partners did not describe school mathematics as having been a thoroughly negative experience for them and reported positive experiences and, partially, confidence in their mathematical ability.

In the following, I will, first, introduce my theoretical background – the subject-scientific-approach – with a focus on learning theory and categories to articulate obstacles to learning and learning barriers. Second, I provide an example of the non-thematisation of mathematics in interview texts by referring to the account of my interview partner Anna. Her account also makes it possible to contextualise the phenomenon within her learning experiences. Third, I will present a theoretical reflection on what can be learned from Anna's accounts. In the discussion, I will, fourth, contrast my findings with the above-mentioned studies and perspectives within the field of university mathematics education.

THEORETICAL LENSE

The theoretical reflection presented in the following follows Hochmuth's (2018) demand to take up the subject-scientific approach for the further development of mathematics education theory. The subject-scientific approach provides analytical categories that allow explicating current and persisting restrictions within societal and institutional arrangements (Holzkamp, 1995, 1985), e.g. university teacher education programmes. It will be apparent in the following that the subject-scientific approach is not a mathematics-specific theory development. This contribution thus also addresses the *lack of research that relates to mathematical learning in general and in university in particular* (Hochmuth, 2018, p. 517/518) within the strand.

Within the subject-scientific approach the relation between individuals and social conditions – which includes institutional arrangements - is regarded as being dialectical (Holzkamp, 1985). Therefore, human actions are societal-mediated and not determined by the structure of society, or immediate circumstances. Furthermore, it is acknowledged and conceptualised as a central characteristic of human actions that these comprise a *twofold possibility* [doppelte Möglichkeit] to either reproduce restrictive conditions or the possibility to extend established practices and alter obstructive conditions.

The object being studied is approached starting from the standpoint of the subject, and the research endeavour *does not aim to classify nor evaluate individuals but to gain an understanding of their actions from their specific standpoint - e.g. as learners* (Ruge, in press). The analyses thus concern the characteristics of the learning process and the embeddedness of the learning object in the social/societal context on the one hand and in concrete teaching-learning arrangements on the other - i.e. situating the object at stake. The approach makes it possible to explore all varieties of learning - also ambivalent, recalcitrant, restrained or fractured learning processes.

Directionality of the learning process

The twofold possibility is also reflected in the learning theory and is theoretically captured in its categories. A distinction is made on the basis of the directionality of the learning process (Holzkamp, 1987, 1985): Defensive learning is *grounded in trying to avert an experienced or anticipated threat*, which leads to primarily directing one's learning process *towards dealing with this threat* and not *towards a deep understanding of the subject matter* (Ruge, in press). It does not lead to a complete refusal of learning, rather a contradictory brokenness of the learning process is characteristic. This can appear in many different forms: e.g. as reticence, lack of commitment, or half-heartedness - sometimes with corresponding rationalisations as to why it is "not worth it" to be more committed, or backing off because of anticipated social isolation (e.g. by being considered a swot). But also, the strict following of a pre-given path can be a technique of the learner to deal with the threat of e.g. continual monitoring and assessment of outcomes. The occurrence of such defensive strategies is quite common in educational institutions and should not be pathologised. Analysing such learning processes from a situated perspective, these strategies rather point to a contradictory constellation of interests that manifest themselves, among other things, in contradictory demands of educational institutions (for a comprehensive analysis, see Holzkamp 1995, chp. 4). The learning object, resp. knowledge is considered in its societal-mediatedness and situatedness in a specific context. Thus, the learning object is not just an entity for itself but can be regarded as a shared object of/ within practice (Nissen, 2012). Approaching an object thus also means (re-)connecting the object with social practices (Nissen, 2012), which always entails tackling its reference to the ambiguous, conflictual and contradictory nature of social/societal reality (Marvakis & Schraube, 2016, p. 205) and the broader field of connected meanings. This situated perspective on knowledge is reflected in the characteristics of expansive learning:

> Expansive learning (…) is characterised by a deeper processing of the object of learning, which transcends one's own immediate experience and looks beyond the superficial appearance of a phenomenon (or empirical observation), trying to understand it in its societal-mediatedness. (Ruge, in press).

This distinction provides information about resistances to learning. It allows to situate learning resistances and thus to investigate what they might ground on. The identification of learning barriers – points at which learning processes come to a standstill - can then in turn be the starting point for didactic considerations on how to address them.

Learning barriers

Holzkamp (1987) makes an analytical distinction between structural learning barriers and a dynamic self-restrainment of the learning process [in short: dynamic learning barrier] (pp. 23/24). Both barriers are closely interrelated. If learners get stuck at a certain point in the learning process because they still lack specific knowledge components, or necessary learning principles to progress independently, it is termed a structural learning barrier. The task of mathematics education would not be to eradicate structural barriers to learning but to address them. An example of how structural learning barriers are addressed in a university mathematics course is discussed in Specovius-Neugebauer et al. (2022). Also, an ATD analysis of the subject matter (cf.

Hochmuth, 2018) can be informative for approaching structural learning barriers. It is noteworthy, that overcoming structural learning barriers requires so-called affinitive phases in which the focus of the learning process is not only exclusively on the object to be learned, but also the broader field of connected meanings (objects, techniques, areas of application, contested interpretations etc.) is explored to grasp the object (Holzkamp, 1995, pp. 324-337; Specovius-Neugebauer et al., 2022).

Such affinitive phases can be quite precarious, especially in educational institutions (Holzkamp, 1995, pp. 324-337). The mastering of the respective learning problem must be anticipatable for the learner so that s/he can engage in the necessary learning process. Following a pre-given path might seem to reconcile time pressure with the insecurity within a learning process, while this path might not fit or are even counter to the foreground and interests towards the subject matter of the respective learner.

It is characteristic of dynamic learning barriers that the learner wants to learn and does not want to learn at the same time, and it is precisely this simultaneity of wanting and not-wanting that leads to a learner not being able to pursue the learning endeavour further or in a more profound manner (Holzkamp, 1987). The continuation of a learning process is interrupted because aspects of the learning object or its embeddedness emerge during learning and discourage the learner from wanting to learn this learning object - as it presents itself. Generally, this is not a complete rejection of the learning object. The question for mathematics education theory is: What discourages learners from wanting to progress in their learning process? What could be the reasons that prevent a prospective mathematics teacher from pursuing mathematical content in more depth?

ANNA'S ACCOUNT OF HER LEARNING

The account of my interview partner Anna serves as a basis for my further theoretical reflections, which will lead us beyond her case. There was a good atmosphere during the interview, both interviewee and interviewer were laughing. She reports positive experiences with mathematics at the school level and displays confidence in her abilities to become a mathematics teacher. Mathematical content is only scratched at best and in some cases, there is a rather offhand reference (e.g.: "*such strange interest rate percentages*" (Quote 1)) in her descriptions.

Anna: ... And I think that in maths it's actually (...) not difficult if you can explain it well. So if the teacher makes an effort to explain it well and present it in a comprehensible way, then (..) even such strange interest rate percentages and who knows what else are so difficult for many people in the seventh or eighth grade, um, actually simple (Quote 1, translated by the author)

The statements become more specific when it comes to mathematics-specific teaching-learning arrangements. Here it is striking that her descriptions, in which she refers to mathematics-specific forms of teaching and assessment (for further analyses see Ruge & Hochmuth (2017)), are quite ambivalent. Anna positions herself against passively being taught mathematics (e.g. "*only get heard the new material*" (Quote, 2)) and emphasises that mathematics must actively be understood. The adoption of this position is demonstrated with reference to the didactic arrangement of the exercise sheet and the common assessment form of the written exam:

> Anna: Um, I have to say that in the course of my studies I also learned to let things go a bit. (.) And (sighs) for example, I also copied out the exercise sheets or let my partner do it, so to speak. (...) Because I was already very EXTREME, so I ALWAYS did it and ALWAYS did everything and on the other hand, um (.) there/ so, especially in maths you always have these exercise sheets. And just because of that, I think it's good that they exist, because you always have to work on them. Because if you only get heard the new material (.) and then wrote an exam at the end of your studies, I think many exams would have been much worse because you didn't really work on it, because, as I said, you have to understand maths, and you can only do that if you work on it more closely and can understand these ways and steps (Quote 2, translated by the author)

She describes that examination phases can also trigger learning blockades for her and how she, together with her learning group, quite creatively constructs a “huge” memory game to break through this blockade - an attempt which is directed towards mastering the exam. In addition, she describes a quite structured approach to learning and the processing and safekeeping of the knowledge taught:

> Anna: For example, I have a folder from each semester with the description of the seminar. Then all the lectures and notes and such are in there. And at the front is my summary. So I (laughing) still have everything (Quote 3, translated by the author)

Anna also voices her position on the structure of teacher training and formulates alternatives to the currently prevailing institutional design, but these are not formulated in relation to the subject of mathematics.

In summary, Anna’s case has the following characteristics:

- Rather offhand descriptions of mathematical content
- Ambivalent valuation of mathematics-specific teaching-learning arrangements, which are described in relation to the examination system
- Social resources and creativity in overcoming learning blockades
- A structured approach to mastering learning requirements

THEORETICAL REFLECTION

The vocabulary of the subject-scientific approach makes it possible to conflate the above-mentioned characteristics with the phenomenon of non-thematisation, which is also present in Anna's account. What can be learnt from it about learning barriers specific to mathematics teacher education?

The direction of following and staying within safe and existing pathways is characteristic of the learning process Anna describes. This is partly transcended concerning education, where she reflects on the institutional design of the teacher education programme and expresses alternatives to the existing one. With regard to mathematics, the importance of dealing with the contents for understanding is emphasised. However, the engagement with the material provided seems to focus on the comprehension of the pre-given knowledge arrangement; existing curricular paths are not left behind. This learning approach seems to encounter a dynamic learning barrier. It is apparent that she wants to learn, that she wants to engage with the subject matter in more depth, but she struggles to do so: Her statement "*I also learned to let things go a bit*" (Quote, 2) reflects a dissatisfaction with not being able to move forward after a certain point, but she learned to deal with it during her studies. Her ways of dealing with the teaching-learning arrangement include aspects of self-activation and self-mobilisation (see Kaindl, 2006) - here she draws on social resources (her learning group)-, working through the given material and compiling folders. In these folders, knowledge is fixed as it is arranged in curricula. This safekeeping of knowledge certainly includes the option of postponing a renewed or even more extensive engagement with knowledge objects. Thus, there is no lack of interest in the subject matter as such, but this interest is not pursued in-depth.

This brings us back to the following questions: What discourages learners from wanting to progress in their learning process? What could be the reasons that prevent a prospective mathematics teacher from pursuing mathematical content in more depth?

Anna's account contains circumstantial indications that bring us closer to answering these questions. Striking is her reference to the relation between the teaching-learning arrangement and the examination system, which is in its specific formation (exercise sheets & written exams) quite specific to mathematics (see also Ruge & Hochmuth, 2017). Her expressed ambivalence relates to "how" mathematics is learned. In her work, Skog (2014) looked at what is taken up as negotiable by student teachers within a mathematics teacher education programme and which discourses appear as non-negotiable. If the discourse appears to be non-negotiable, the space for reflecting upon alternative horizons and pathways is foreclosed. Interestingly, her work also shows that the so-called "institutional discourse" - which refers to restrictions, rules and conditions within educational institutions - appears to be non-negotiable. Especially time restrictions and exams limit productive discussions about the respective mathematical content (ibid.).

If one focuses on the "what", it is noticeable that the engagement does not seem to go beyond a certain point, or is not worthy of thematisation within the interview. The described active engagement with the subject matter is oriented towards a recapitulation of required content. An engagement with the contents beyond the curricular paths does not seem to be conceivable in her learning process within the teacher education programme. This "beyond" – affinitive phases - always entails the risk, that distancing oneself from a pre-given path leads to a temporary loss of cohesion, which may not be manageable within the strict time schedule. The learning object then does not appear to be masterable outside of the predefined pathway. Therefore, exploring alternative pathways and connections becomes a dangerous endeavour.

This perspective also makes a non-thematisation or only superficial thematisation of mathematics intelligible: If it is obvious what is to be taught and learned, and it is more a question of "how", then there is no reason to elaborate on the "what". The possibility of change, resp. alterability of the established teaching-learning arrangement is ignored, and it is accepted as an existing fact.

If one relates this to the context of teacher education, it is noticeable that the process of anticipation implicates a double barrier: The mastering of the subject matter has to be anticipated for oneself and, additionally, the extent to which this subject matter can be made accessible to others (e.g. the future pupils) is part of the anticipation process. Furthermore, the subject matter shall be made accessible to others, at least to a certain degree, within the scope of school mathematics, which in turn is a quite specific context that entails further possibilities and restrictions. Thus, it is also about the "usefulness" of the respective subject matter, in the sense of the position of the mathematical subject matter within social reality and associated educational goals. A learning process that takes these anticipations into account requires maintaining familiar and well-established perspectives on the subject matter from school, while at the same time it may be necessary to detach from them to be able to delve deeper into the subject matter.

Situating the dynamic learning barrier reveals that dynamic aspects – such as an ambivalent affective-motivational positioning within an ongoing learning process – are more than just a mere personal irrationality, but is grounded in the institutional arrangement of university mathematics teacher education. From the perspective of student-teachers, it touches on the question of what formation of mathematics one shall identify with: Mathematics as it presents itself at university or school mathematics? And what formation of mathematics does the learner – as a prospective teacher – want to be identified with – as being a representative of mathematics at school?

DISCUSSION

Anna's account and the subsequent theoretical reflection can be related to the existing discussions within the mathematics education community of a non- or only superficial thematisation of mathematics as follows. Brown & McNamara (2011) focus on negative experiences that can lead to obstacles to learning. They describe how reform discourses and the identity formation of student-teachers are pushed in a direction of an appropriation of reform discourses, which also serves to avoid having to deal with one's own insecurities about the subject matter. This description provides insight into political and institutional constraints in addressing learning barriers, maybe even about an infusion of further layers of dynamic barriers by reform movements. Bibby's (2002) analysis points to a relevant social component of a dynamic learning barrier: social images of mathematics. Shame can be associated with the anticipation of not being an adequate representative of mathematics as it currently presents itself. In her analysis, an important point is the possible display of vulnerability in referring to mathematics.

Especially in examination phases, the disclosure of a vulnerability is a precarious aspect, but disclosure of vulnerability might also be undesirable if one anticipates being a representative of mathematics at school.

Often the focus on affective-motivational aspects of mathematics education leads to a plea for motivating students to deepen their engagement with mathematics and suggestions on how students can be motivated and guided in their learning processes. A seemingly superficial account of the subject matter is mostly misjudged as a motivation problem on the part of the students. But, if we consider dynamic learning barriers, it is more than just a matter of motivating students for the mathematical content of their studies. It is more than just a problem of motivating students to delve into the subject matter in more depth because it is characteristic of such a learning barrier that learners want to learn. For the field of mathematics education, the question arises as to how this desire to learn can be taken up constructively without this being transferred in the direction of defensive strategies that are directed more towards passing exams than towards a more profound exploration of the subject matter and, thus, enable understanding:

Certainly, it is not helpful to have an increasingly strict pre-structuring of learning processes. A situated approach promises more insight here. If one takes these situated approaches to knowledge and learning seriously, then growing into a community of practice, into the mathematics education community, also means participating in (co-)constructing the field. In my interview with Anna, obstacles to this aspect of becoming involved in the (co-)construction became apparent concerning the structure of the mathematics teacher education programme: Concerning the institutional design she engages in its construction by voicing critique, however, with regard to mathematics, she seems to arrange herself with the given path, learns to adapt to it and follows it. Even though this arrangement is accompanied by dissatisfaction with her own progress in delving into the subject matter. The topics and learning paths pre-structured by the institution appear in her narrative as fixed and not alterable, alternatives to current practices and pathways seem to be non-negiotable. From this vantage point, the following questions arise: Why should mathematics be addressed at all when talking about learning? Why should specific characteristics and relations of the content be elaborated on if it is clear and predetermined what it is about anyway? Within such a non-negiotable discourse, there is only the question of “how” best to work through this and not the question of “what”. But, also the negotiation of the “what” is important, if teacher education shall provide a space for reflecting upon current practices and possible alternative approaches. Thus, the following question must be addressed: How can we create spaces within mathematics teacher education that allow student-teachers to (co-)construct the field of mathematics education and thus feel entitled to participate in the negotiation of the “what”?

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UNIVERSITY STUDENTS' SELF-EFFICACY BELIEFS CHANGES DURING FIRST-YEAR COURSES: A PRELIMINARY ANALYSIS

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Keywords: Transition to, across and from university mathematics, Assessment practices in university mathematics education, students' perception, self-perception, self-efficacy.

MOTIVATION AND CONTEXT

In the current post-pandemic situation, transition to university, for students coming from one to two years of distance learning, seems particularly worth consideration. Teachers' informal observation during first-year university courses, confirmed by students' achievement in exams, seem to suggest a portion of average-scoring students is struggling more than before. Our research aims in general to analyse the difficulties encountered during the first year of university and, more in details here, the role of self-perception of mathematical ability in students. Also, is there a relationship between students' self-efficacy beliefs entering university after secondary school studies and effective students' achievement in first year exams? Does this self-perception change in the first months of university?

LITERATURE

In psychological literature, academic self-perception has been long studied (Bem, 1972), also from a mathematics education perspective (Hannula et al., 2016). It has been documented that there can be a weak correlation between perceived performance and the one then assessed by teachers (i.e., exam results – e.g., Chemers et al., 2001). Self-efficacy beliefs are considered following Chemers et al.' interpretation (2001). Students generally over-estimate their own ability (Falchikov & Boud, 1989). On the other hand, academic self-efficacy has been found to influence students' persistence, self-regulated strategies, and effort in mathematics, and to be correlated with persistence (Hannula et al., 2016).

RESEARCH AND METHOD

Two long questionnaires were distributed among students: one in October, during the first week of the courses, and one in February, after the first two exam sessions. Students involved are enrolled in university in mathematics, physics or engineering. In this analysis, only students who filled in both questionnaires were considered. A couple questions of both questionnaires asked students to evaluate their general attitude and self-perception towards their path of studies (e.g., "evaluate the following sentence: I am convinced that I will achieve my goals in the chosen university path"). To avoid considering students whose motivation drastically changed, we choose to analyse here answers of students who did not change their self-evaluation about the above by more than one point (out of a 5-point Likert scale). Our final observation group consists of

144 first-year students. The sections of the questionnaires we are interested in, for the purpose of this presentation, were about students' self-efficacy beliefs and usual behaviours during study time.

OBSERVATION AND RESULTS

Some preliminary facts about some of the questions are worth noticing. While there is a correlation between self-efficacy beliefs and exam results, there seems to be a self-perception change during the first semester. Self-evaluation about the students' general preparation in mathematics went from 3.12 out of 4 to 2.55 out of 4 (same question in pre- and post-course data collection, in a 4-point Likert scale). Another related question was to self asses the following: "I happen to think that I have understood a topic but then discover (at exams, ...) that it is not so" where the after-secondary-school answer was averaging at 2.21 while the after-the-first-semester answer arrived at an average of 2.57, in a 4-point Likert scale. In a direct question about the relation between expected and obtained results in the final Calculus exam, 47% of the students admit they were expecting a higher grade and, in a final question regarding obstacles encountered during the course, 32.9% point out that their initial preparation was not sufficient to properly face the course demands. There is also evidence of behavioral changes from the intended method (declared in the October questionnaire) and the review questionnaire, such as a need to implement new study methods, to quit "last minute" studying, to discuss more with peers and tutors and to rely on the teachers' availability. Some first conclusions tell us that, even though not discouraged in going on with their STEM studies, students are faced in this first semester with more challenges than they expected and, during the semester, gain some deeper insight into self-evaluation and what is expected from them and intend to change some behaviors to better adapt to the new education level. Further research is expected in this direction, together with some proposal of activities to alleviate these obstacles and facilitate students' transition to university, which we are working on in the present.

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Analysis of informal learning situations at the Ostfalia University to support digital competences of first semester students

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Keywords: Digital and other resources in university mathematics education, Teaching and learning of mathematics for engineers, Informal learning situations.

RESEARCH TOPIC

The analysis of the use of digital media in informal learning situations is a part of the dissertation project "Using informal media to address knowledge gaps in basic mathematics " of the LernMINT doctoral program.

The use of media by students has changed significantly in recent years, not least due to the Covid19 pandemic. Students do less research in the library than on YouTube or other learning offerings on the internet. (Steffens et al., 2017)

For studying mathematics, the field of digital media has a diverse offer from explanatory videos on portals to internal university learning management systems and websites that present mathematical content online. The didactic as well as the professional quality of these digital support offerings for explaining mathematical content vary depending on the creator and content (Ehlers, 2004).

The starting point of the analysis is the informal learning situations of students in basic mathematics. According to Arnold (2015), these situations are a search for functional solutions that are self-organised by the student.

To support students in this self-organised study, the first step is an analysis of current student information gathering processes. Experience-based qualitative interviews with students from the Faculty of Mechanical Engineering in the 1st to 4th semesters will be conducted. They describe their individual experiences of research and information gathering, especially for mathematical questions.

The interview guide for the qualitative research was constructed according to the SPSS method developed by Helfferich (2011). The analysis of the experience-based student interviews is carried out using the evaluation method of Mayring (2015).

The first preliminary results of the analysis of qualitative student interviews show that the students develop and apply very individual research paths. These lead to qualitatively different information while searching with digital media, which is then used to solve mathematical questions. This first hypothesis will be verified through further interviews.

Approaches to support the use of digital media in basic mathematics will be developed and tested based on the identified hypotheses. These new support offers should increase the students digital competences according to the DIGCOMP model by Vuorikari et al. (2022).

CONTENT PRESENTATION

In the first step, the research topic and design are explained. Then the underlying theories of the different fields of competence in the context of informal learning processes with digital media in the field of university mathematics are visually linked. The second part of the poster is to present the results from the qualitative interviews. Hypotheses are formulated and explained using examples from the student interviews. The last section of the poster describes initial approaches of support services to improve the digital competences.

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Change of math anxiety in mathematics course for non-STEM university students in Japan

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Keywords: teachers' and students' practices at university level, teaching and learning of mathematics in other fields, math anxiety, reciprocal theory.

INTRODUCTION

Carey et al. (2016) proposed the reciprocal theory of math anxiety and math performance. According to this theory, math anxiety decreases math performance and poor performance increases math anxiety. In this one-year longitudinal study, we examined the relationships between gender, course selection, math performance, non-STEM students' interest in each topic, and math anxiety. Based on the reciprocal theory, we hypothesized that (1) non-STEM university students in Japan had higher math anxiety, and (2) their math anxiety would increase by studying university-level mathematics. Math anxiety was measured with the abbreviated math anxiety scale (AMAS; Hopko et al., 2003), which had two subscales: learning math anxiety (LMA) and math evaluation anxiety (MEA).

METHODS

Participants

The participants of this study were 129 first-year students (53 males, 68 females, and 8 non-respondents; mean age 18.52 ± 1.48 years) majoring in social and human environment at a Japanese university. They had taken the course "Basic Math I" in the first semester, and 75 of them continued to take the advanced course "Basic Math II" in the second semester. Both are activity-oriented courses in mathematical modelling, developed for non-STEM students (Kawazoe & Okamoto, 2017).

Measures

The survey was conducted three times: at the beginning (T1), at the end (T2) of the first semester, and at the end of the second semester (T3). The AMAS, translated into Japanese for this study, was used to measure math anxiety. The participants responded to items on a five-point Likert scale ranging from 1 (low anxiety) to 5 (high anxiety) (T1, T2, T3). They also responded to questionnaires about their math scores in entrance examinations (T1), attitudes toward learning (T1, T3), as well as their understanding and interest in each content (T2, T3). We also collected the scores of the online exercises conducted almost every week during the semesters. All data were collected on a learning management system.

RESULTS AND DISCUSSION

As hypothesized, math anxiety in Japanese students (LMA: M = 12.9, SD = 4.9, MEA: M = 15.0, SD = 4.0 at T1) was higher than Italian students (LMA: M = 8.4, SD = 3.4, MEA: M = 13.1, SD = 3.8; Primi et al., 2014), especially in LMA.

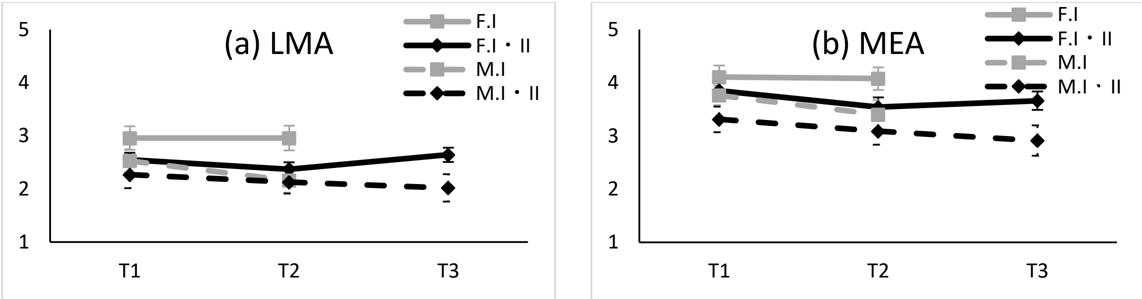

Figure 1: Math anxiety (points/item) as a function of time, gender, and course-taking.

Figure 1 presents the changes in math anxiety. A four-way ANOVA, 2 (gender: male/female) x 2 (course taking: Basic Math I/Basic Math I and II) x 2 (time: T1/T2) x 2 (math anxiety: LMA/MEA) indicated that the main effects of math anxiety ($F_{(1, 99)} = 254.08$, $p = 0.00$, partial $\eta^2 = 0.72$), gender ($F_{(1, 99)} = 7.77$, $p = 0.01$, partial $\eta^2 = 0.07$), course taking ($F_{(1, 99)} = 4.62$, $p = 0.03$, partial $\eta^2 = 0.04$), and time ($F_{(1, 99)} = 16.69$, $p = 0.00$, partial $\eta^2 = 0.14$) were significant. All interactions were not significant. A three-way ANOVA, which included anxiety, gender, and time (T1/T2/T3) for the participants who took Basic Math II, indicated significant main effects of anxiety and gender, and a marginally significant main effect of time ($F_{(1.56, 85.88)} = 2.86$, $p = 0.08$, partial $\eta^2 = 0.05$). A paired comparison for time revealed that math anxiety in T2 was lower than in T1 and other pairs were not significant. Our results revealed that the MEA was higher than the LMA, and in females, it was higher than in male university students, which was consistent with the findings of Hopko et al. (2003). Contrary to our hypothesis, math anxiety did not increase from T1 to T2. This result might be due to the features, activity-oriented and mathematical modeling, of the mathematics course.

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Emotions in Undergraduate Mathematics: A Distance Education Perspective

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Keywords: emotions in undergraduate mathematics, distance education at the tertiary level, university linear algebra, online learning, transactional distance.

Little research has been done on the interpersonal aspects of the learning process and particularly on how several aspects of mathematics learning impact the academic-related emotions of students (Eligio, 2017). The emotions experienced in Linear Algebra courses are still underestimated (Stewart et. al, 2019); however, it seems that social factors, such as peer-perceived course difficulty, can impact students' emotions experienced in Linear Algebra (Martinez-Sierra & Garcia-Gonzalez, 2016). Thus, it is important to further study the factors that influence students' emotions in Linear Algebra, especially under the socio-educational context of Distance Education.

THEORETICAL FRAMEWORK

Thinking of education in a simplistic way, interaction underlies education. In a DE setting, due to the geographical and/or temporal distance between students and their teacher, how students interact can reflect the "*psychological and communications space (that has) to be crossed, a space of potential misunderstanding between the inputs of instructor and those of the learner*" (Moore, 1993, p.20) and is named '*Transactional Distance*' (TD). According to the Theory of Transactional Distance, a student's perception of TD is measured by three types of interactions they have: *learner-content* (LC), *learner-instructor* (LI), and *learner-learner* (LL) *interaction* (also met as SC, ST, SS interaction correspondingly) (Moore, 1989).

METHODOLOGY

The survey was conducted online at a Greek university. A total of 97 (out of 832) undergraduates enrolled in an online Linear Algebra course participated voluntarily and anonymously. The course was based on video lectures for both the theory and the exercise-related hours. The online questionnaire was specially designed, consisting of two parts: in the first part student-perceived TD was measured by using the *Revised Scale of Transactional Distance* (Paul et al., 2015); in the second part students' positive and negative emotions experienced in the course were measured using a combination of the *Achievement Emotions Questionnaire* (Pekrun et al., 2011) and the *Positive and Negative Affect Schedule* Scale (Watson et al., 1988). Descriptive data analysis was conducted for identifying students' perceptions of TD and experienced emotions. Independent samples t-tests and correlation analyses were run in mean scores (ST_{mean}, SC_{mean}, SS_{mean}, $Positive_{mean}$, $Negative_{mean}$). Finally, two regression analyses were run to identify if ST_{mean}, SC_{mean} and SS_{mean} can be predictors of $Positive_{mean}$ and $Negative_{mean}$.

RESULTS

Overall, undergraduates in the sample felt TD at a fairly low level and experienced both positive and negative emotions at a moderate level. Also, all three types of interactions were positively correlated with positive emotions and negatively correlated with negative emotions. The most important predictor for both positive and negative emotions was identified to be the Student-Content interaction.

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Promoting co-operative learning in undergraduate mathematics via primetime instruction

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In our poster, we describe a new teaching model in undergraduate mathematics that combines an existing student-centred teaching model with co-operative learning facilitated by "primetime" meetings. We also analyse student reflections (n=63) on what supported their co-operation.

Keywords: Teachers' and students' practices at university level, Novel approaches to teaching, Co-operative learning, Group work, Mathematics education.

INTRODUCTION AND THEORETICAL BACKGROUND

Teamwork and social skills are important working life skills that are rarely developed enough in university curricula. This has remained so, even though small-group co-operative learning has shown its effectiveness in education from primary school to universities (Johnson & Johnson, 2009). In mathematics, meta-analyses by Springer et al. (1999) for undergraduate mathematics, and Capar and Tarim (2015) for all ages, show that small-group learning enhances mathematics achievement, mathematics self-esteem and attitudes towards mathematics.

In this report, we describe how we have introduced co-operative learning to an already established student-centred teaching model. We also analyse students' perceptions of those aspects that have supported their group work. We base our analysis on the five criteria of effective co-operative learning by Johnson and Johnson (2009): positive interdependence, individual accountability, promotive interaction, appropriate use of social skills, and group processing.

PRIMETIME-FACILITATED GROUP WORK

The course investigated in this study is an undergraduate course in abstract algebra that was taught in a Finnish university in spring 2020. The course was taught on campus and had 83 students. The teaching practices of the course were built on a student-centred model of Extreme Apprenticeship (Rämö et al., 2020), and co-operative learning was promoted using primetime learning (Koskinen et al., 2018). Individually, the students completed weekly tasks using the textbook and the help of tutors who taught in an open learning space. For co-operative learning, the students worked in groups of six people that completed two projects during the course. Promotive interaction was facilitated by requiring that the groups meet regularly to work on the projects and by giving feedback on their work process. Positive interdependence and appropriate use of social skills were built by making visible the roles of group work and letting the students try out different roles in their group. Group processing was facilitated by giving the groups tasks which required reflecting on the group's actions. Individual accountability was supported by creating an assessment system in which students' individual achievements influenced the group's grade. The teacher supported the groups' working in primetime meetings, in which each group had a 15-minute one-on-one meeting with the teacher.

ANALYSIS AND RESULTS

The data for this study consists of students' reflections they wrote at the end of the course (n=63). The students were asked to describe what supported their group's work, and their answers were analysed using qualitative content analysis based on the criteria for effective co-operative learning by Johnson and Johnson (2009). Of the dimensions in Johnson and Johnson's framework, the most frequently mentioned were *Promotive interaction* (30 mentions, e.g., "Common meetings in which everyone participated") and *Appropriate use of social skills* (21 mentions, e.g., "Open and supportive atmosphere"). A few instances of *Group processing* (4 mentions, e.g., "We needed an organised method for solving the tasks") and *Positive interdependence* (4 mentions, e.g., "Different learners in the group noticed different things") were found. *Individual accountability* was not identified in any of the answers. The results indicate that either the learning environment supported promotive interaction and appropriate use of social skills more than the other dimensions of effective co-operative learning, or alternatively, those dimensions were easier to identify for the students than other dimensions.

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