Planning, teaching and reflecting on how to explain inverse rational numbers: The case of Ana

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To promote understanding of rational numbers is challenging to teachers and prospective teachers, and this led us to study the planning, teaching, and reflection processes of prospective teachers (grades 5-6) on that topic. The aim of the paper is to analyse how Ana, a prospective teacher, prepared, developed, and reflected about communication, with special focus on instructional explanations and on mathematics ideas about the unit on multiplication of fractions. Ana prepared and developed different explanations to reach pupils' ideas. We discuss the nature of her didactic knowledge as she seeks to promote pupils' understanding of the multiplication of inverse rational numbers.

Keywords: Prospective teachers, Teaching practice, Communication, Rational numbers, Multiplication.

INTRODUCTION

Rational numbers are a fundamental topic in the mathematics curriculum. Teaching it presents a strong challenge to teachers' knowledge and practice and they must use different representations and meanings in order to promote pupils' understanding of this concept. However, we know little about how teachers use different representations and meanings, for what purpose and facing what difficulties (Mitchell, Charalambous & Hill, 2013). We especially want to understand grade 5-6 prospective teachers' knowledge and practice during supervised teaching practice, as this enables a close look at the nature of their knowledge. In analysing teaching practice we focus on tasks, classroom communication and prospective teachers' actions, striving to understand the nature of their didactics knowledge. In this paper *the aim is to analyse how a prospective teacher, prepared, developed, and reflected about communication, with special focus on instructional explanations and on mathematics ideas about the unit on multiplication of fractions.*

PROSPECTIVE TEACHERS' KNOWLEDGE AND COMMUNICATION

Prospective teachers' knowledge may be considered from different perspectives. Both mathematics and didactics knowledge are critically important and deeply interconnected within teaching practice. Besides mapping both kinds of knowledge, it is important to understand their nature and how they relate to teaching practice. Didactics knowledge has two essential dimensions: knowledge about both tasks and pupils (Ponte & Chapman, 2015). Teachers must be able to select, design, and sequence tasks and to explore pupils' strategies, establishing learning sequences and recognizing learning opportunities. They also must anticipate pupils' common mistakes and misconceptions, listen and interpret their ideas, anticipate their solutions, and know what they will consider challenging, interesting, or confusing (Son & Crespo, 2009). A fundamental idea about rational numbers is that they "have multiple interpretations, and making sense of them depends on identifying the unit" (Barnett-Clarke et al., 2010, p. 17). Regarding representations, teachers should know how pupils deal with pictorial and symbolic representations (verbal, fractions, decimal and percentages) and how to relate them, making sense of the numerical set as a whole (Barnett-Clarke et al., 2010). So, when preparing tasks, teachers should recognize the pros and cons of using certain representations and know how to take advantage of pupils' strategies and representations to promote mathematics ideas (Ball et al., 2008; Stylianou, 2010). However, the use of particular representations may raise challenges to teachers since they may induce pupils into mistakes or incomplete conversions and may be far from pupils' initial knowledge.

Tasks and communications are essential aspects of teaching practice (Ponte, Quaresma & Branco, 2012) and they need special attention from teachers when preparing, teaching and reflecting about teaching practice. Communication is a fundamental element of teaching practice and it is inherent to the process of building knowledge (Menezes et al., 2014). Communication involves sharing something and, to do so, we make use of gestures, images and symbolic representations, explanations and questions. Communication may be oral or written, and it includes both linguistic and mathematics representations (Ponte & Serrazina, 2000). One important aspect of communication is questioning using confirmation, focus, and inquiry questions (Ponte & Serrazina, 2000). Communication also includes those representations that are used to aid in solving a task, such as building or illustrating objects, concepts, and mathematics situations. These representations may arise from pupils or not (Mitchell et al., 2013). Instructional explanations are another important aspect of communication. Far from being mere "transmissions of content," instructional explanations support the establishment of relationships between mathematics concepts. Active, pictorial (iconic and drawings), and symbolic representations (Bishop & Goffree, 1986; Bruner, 1999), together with verbal communication (Ponte & Serrazina, 2000) may be used to convey concepts and procedures to pupils. Instructional explanations may have different purposes and characteristics, focusing on procedures and/or concepts, and may be carried out at different times during a lesson. According to Charalambous, Hill, and Ball (2011), explanations can be used to introduce new content, answer pupil questions, or support pupils with difficulties. A good explanation may eliminate erroneous ideas, meanings and processes. Charalambous et al. (2011), in a study of prospective teacher education looked at the issue of the quality of explanations and concluded that an incoherent, incomplete, or unclear explanation may affect pupils' learning. On the other hand, a "good explanation" is meaningful and easy to understand. Thus, prospective teachers must: (i) keep the audience in mind, using language suitable for pupils; (ii) define appropriately the key terms and concepts; (iii) highlight the main mathematics ideas while explaining the process step-by-step; (iv) use appropriate examples and representations while also modelling procedures and concepts; and (v) clarify the issue in question, showing how it should be answered.

RESEARCH METHODOLOGY

This is a qualitative and interpretative case study. Ana is a 24-year old prospective teacher of a School of Education, doing supervised teaching practice on rational numbers (grade 5) in her last year of studies. She studied mathematics 12 years before coming to teacher education and is regarded as a good student. She is visibly insecure about what she intends to carry out in her practice and feels torn between direct and exploratory teaching. Ana was interviewed at the beginning (IE) and end (FE) of her supervised teacher practice. Ana's classes were observed and video taped for later analysis and video-stimulated recall interviews before and after each lesson (BCiE, ACiE) (Nguyen, McFadden, Tangen & Beutel, 2013). Her lessons plans and personal notes were also analysed. Data analysis is descriptive and interpretative searching to understand the processes of planning, teaching, and reflecting about the product of two inverse fractions. During planning, we analysed the strategies for solving tasks and representations that she prepared to support her explanations. During teaching, we focused on how Ana provided explanations and highlighted several aspects. During reflection, we emphasized her view of the explanations that she gave and the mathematics ideas that emerged. The categories used for analysis were taken from framework presented formerly in this paper.

PLANNING, TEACHING AND REFLECTING ON EXPLANATIONS

Class preparation

Ana taught several lessons about rational numbers, four of which introduced new concepts. In the first, the key idea was to explore the inverse of a rational number. As the pupils already knew how to multiply rational numbers, the aim of the task was that they pictorially represent expressions to visualize that "the product of a number by its inverse is 1." For Ana, it was essential that the pupils understand the rule:

> So that there might be a logical sequence to the classes. Because when they were doing multiplication, they identified the rule themselves and when they were adding as well… That way they will really realize what they are doing instead of memorizing…. I wanted to try to use the process of understanding rather than memorizing. As this is a new attempt, for me, let's say... I'll just experiment to see what'll happen. To see what works better, let's say. (BC1E)

Ana prepared a plan which briefly described what she intended to accomplish. She defined objectives and general ideas about the activity that was supposed to happen in different moments of the class. The plan was sent to her supervisors and did not anticipate possible pupils' solutions to the tasks. Therefore she did not discuss with her supervisors her potential reaction to pupils' answers. However, Ana solved the task in a personal notebook. She did not include the answers in the lesson plan as she was reluctant to expose possible weaknesses to her "supervisors/evaluators." Our analysis of the different records found that she solved the problems in different ways (symbolically and pictorially), as shown in Figure 1.

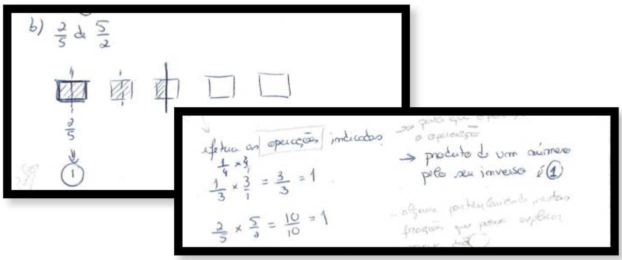


Figure 1. Solution of the proposed task (1st version).

Doing symbolic computations, she answered the problems using the algorithm *a/b×c/d = ac/cd* and when she obtained integers she turned them into fractions. When solving the problem “2/5 of 5/2”, using a pictorial representation, she used a unit of five that she divided into two equal parts by colouring two rectangles and a half. She divided the new unit into five parts and coloured two of them in the same manner as the rectangle. Thus, Ana began to interpret the fraction 5/2 as a quotient when she divided 5 rectangles in half. She then thought of “2/5” as an operator when she took one fifth to be a new whole and then she took two parts. Thus the product of 2/5 by 5/2 were two parts of five, which were equivalent to a rectangle.

Class 1: Solving 2/5 of 5/2

In class, Ana asked the pupils to represent the expressions 1/4 of 4 and 1/3 of 3. In these expressions, the fractions have the operator meaning, which posed no problems to the pupils or to her. She then wrote “2/5 of 5/2” on the board and again asked the pupils to represent the expression “with a drawing or a diagram.” The pupils began to solve the task and Ana moved around the room, emphasizing the need to illustrate the expression pictorially. At a certain point, she realized that there were recurrent questions from pupils, and she decided to discuss the task with the whole class. She began to focus pupils’ attention on the fraction 5/2, focusing their attention on 5, as the starting unit, and dividing the unit into two parts. A pupil proposed to divide each of the 5 rectangles into halves and Ana drew Figure 2 and explained:

Ana: Gabi wanted to divide each of the 5 units in half. But is that what they want us to do?... We have five units… And we’re going to divide them into two parts... [Gabi’s idea] will help us find out where the half of our five parts is... What will it be? [draws a line through the middle] Why? We have here two and a half units and another 2 units and a half... We will now have five halves. And now these 2/5? Now we have to represent 2/5 of 5/2. That is, when we represent the 5/2 we find our unity for 2/5. In other words, when we will represent 2/5. Why? Because now our universe will now turn into just one part. That's where we'll represent 2/5… What shall we colour? 2 parts of 5...

Gabi: I thought about putting just 5/2 (five halves).

Ana:	You're right. These 2/5 will only belong to this part. We don't need to represent 2/5 of our entire universe, of the 5 units. It's just this part here. And now what will these 2/5 be? They will be these two parts, these 5. What will that give us?
Gabi:	One.

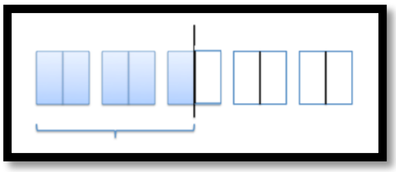


Figure 2. Ana's representation in the 1st class.

Ana had planned to discuss the pupils' answers but during class she rushed ahead and explained the problem herself. She explained that the improper fraction 5/2 is a quotient where the unit is 5 and is divided into two parts. She defined the terms of the expression emphasizing the importance of indicating the reference unit and stressing the word *of* to know that a multiplication was involved. She then told pupils that they had gotten to a new unit and considered two fifths of this new unit. Thus, using step-by-step modelling, she explained the main mathematics idea that she intended to illustrate. However, the pupils seemed to interpret the fractions as a part-whole relationship. Ana did not reject this view, nor did she explore these two perspectives, missing the opportunity to clarify how her explanation fitted the pupils' previous ideas.

Reconsidering the explanation

After class, at the request of the school supervisor, Ana tried to recast her explanation. In her notebook, we found another attempt at solution where we realize that she had not decided on the way she wanted to represent 5/2 (Figure 3):

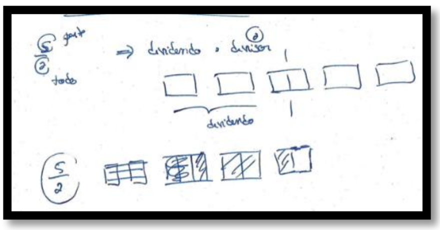


Figure 3. Second solution of 2/5 to 5/2.

When we analysed her notes, we realized that Ana thought of 5/2 as a fraction meaning a part-whole as well as a quotient where the numerator is the dividend and the denominator is the divisor. Thus 5 is the dividend and 2 is the divisor and that is why she divided the five rectangles in half. In the second illustration (Figure 4) we see that she scratched out two rectangles as being extra and used only two rectangles and a half or five half rectangles. She seemed to be satisfied with the solution about the fraction 5/2. In this second solution she thought of the rectangle as a unit and interpreted the fraction as part-whole. So, in trying to decide on how to illustrate the expression 2/5 of 5/2 she hesitated between the part-whole and quotient meanings depending on the unit identified and the representation used. From a didactics point of view, this question may have an impact on pupils' understanding of the concept.

To review the explanations for pupils, Ana wrote down in her notebook some ideas to point out. We note that she had anticipated potential ways to illustrate the concept and to clarify the issue. Figure 4 appeared in notes that she produced before making a PowerPoint that the showed in the second class about inverse rational numbers.

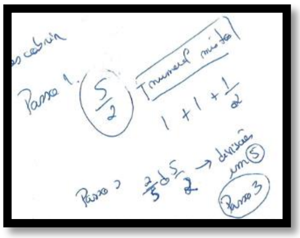
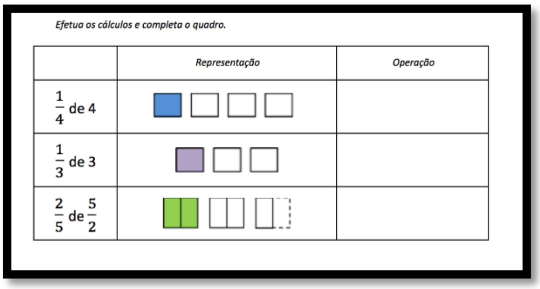


Figure 4. Solution of the proposed task for class 1 and 2 (2nd version).

When we analysed her log, we realized that Ana had kept the final idea and considered the fractions to mean part-whole, drawing three rectangles. It is worth noting that the right side of the figure predicted the explanation she later gave. She had anticipated identifying the unit 5/2 (improper fraction) and intended to decompose it into 1 + 1 + 1/2. In the second step she planned to split the five halves into five parts and then taking two parts of this whole.

Class 2: Continued discussion of the solutions focusing on procedures

In the next day, Ana began by handing out a form to systematize the ideas explored in the previous lesson and reviewed the work done. She projected the hand out on the whiteboard (Figure 5) and explored the expressions 1/4 of 4 and 1/3 of 3. In this explanation the fractions took on the meaning as operators.



Efetua os cálculos e completa o quadro.

	Representação	Operação
$\frac{1}{4}$ de 4		
$\frac{1}{3}$ de 3		
$\frac{2}{5}$ de $\frac{5}{2}$		

Figure 5. Summary table projected on the whiteboard.

Ana asked a pupil how to multiply using calculation procedures. Thus, although the proposed expressions are the same we have a new task with a different nature focused in procedural skills. Note that pupils were asked to solve the expressions using the multiplication calculation procedures.

At the end of the solving process, Ana repeated the operation and the respective product, leading the pupil to verbalize that 4/4 is equal to 1. Next a pupil went to the board to solve "1/3 of 3" and another pupil "2/5 of 5/2" using the same multiplication procedures. Finally, and to systematise the operation of 2/5 of 5/2, Ana used a PowerPoint and explained again the solution (Figure 6).

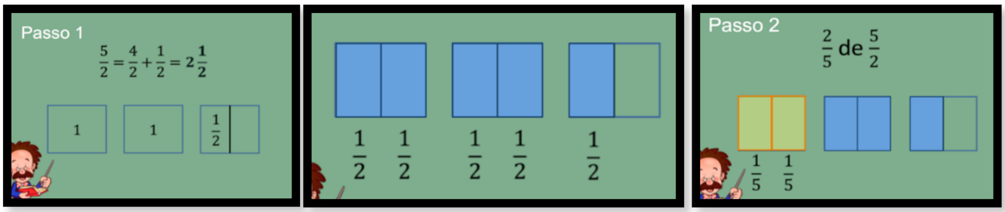


Figure 6. PowerPoint situation 2/5 of 5/2.

Ana: We figured it out by the calculation. But yesterday how did we do it? By a diagram. So let's clarify what we were doing yesterday. What do we see in 5/2? If we use the mixed number numeral, what are we going to get? We'll have 2/2 + 2/2 + 1/2. What does this tell us?

Pupil: 2 1/2.

Ana: 2 1/2, which is what we have here (pointing to the first slide). In other words, we have two units, which we have here, 1 plus 1 plus a half, which is what we have represented here, right? And what we want to know is 2/5 of 5/2. So how do we do? We have represented our whole. We have 2 units and then we have the 5 in total, don't we? Each unit is divided into two and what do we want? 2/5 The two parts of five. So what do we get? One, two... will correspond to how much? How much will 2/5 of 5/2 be?

Pupil: One.

Ana began her explanation by reviewing the work done previously in order to focus pupils' attention again on the expression 2/5 of 5/2. Then she said that 5/2 can be represented by a mixed numeral but she did not explain why. In order to explain the step-by-step process, she decomposed the mixed numeral so that pupils could see why the unit are two and a half rectangles. Finally, she represented pictorially each step, showing how the multiplication can be seen as two pieces of five (5/2) which corresponds to two rectangle halves, namely a unit. She then confirmed with the pupils that the issue in question was clarified.

Reflecting on how to explain 2/5 of 5/2

At the end of the process, in her reflection, Ana explained what had happened and what it meant, saying:

> [The explanation of the first class] goes beyond the procedure and may lead to a conflict of ideas. Ideas...Conflicting ideas is good for discussion. However, confusing pupils is something else. So, I think that's more what happened, the kids were confused. Why? . . . In this case, since I was asking something that went a bit beyond what we were working

> on, talking about, it demanded a little more and they ended up just feeling a little "what's just happened here ?!"… [In fact] it was halves of 5. This was the problem... At the time I clarified their confusion... (FE)

In this reflection Ana felt that her explanation did not consider the children's knowledge. The pupils began by interpreting the fraction 5/2 as a part-whole relation and she stuck to her plan and interpreted the fraction as a quotient. Maybe that is why she felt she confused the pupils and the meaning of the quotient might have become confusing. She described a dialog from the first class:

> Ana: She said we had five units, which were five square [rectangles] and we had to split them in half. And the question was "how to divide them in half?" Because I did not know if she was going to divide each unit in half or if she wanted to divide the set of units in half. She went to the board to divide each unit in half.
>
> Researcher: So she believed there were five units?
>
> Ana: Rather than five, only one unit. And that's what I think I failed to take advantage of. Because I wanted them to realize that the five was a unit that could be divided in half. And after this, they were going to be thinking of a unit for two-fifths, let's say... (AC1E)

In this reflection we realize that Ana had difficulty in understanding what unit the pupil was thinking of. In this interview, just after the first class, she appeared to be insecure about her explanation since the pupils had not fully understood her unit of reference. After this interview, she reflected with the school supervisor, who helped her to understand the pupils' perspective. As a result of this conversation and as described, Ana rethought her explanation. While reflecting on the second class and planning future attempts, she said:

> For me this process was so logical it didn't occur to me that they would be thinking of the five parts...I thought it would be easy for them to get here. Why? Because at the time I did not divide them into five, only at the second step did I divide them. I missed out on one of the strategies, so to speak… My way made more sense… Then, when the [new] proposal made sense… I think the kids realized where we wanted to go, but maybe the strategy should have been explored another way… [We could have] compared the two strategies, both proposals... That would have even been ideal. Perfect! (AC2E)

In this excerpt, Ana still did not feel completely confident about the explanation given in the second class. For her, it made more sense to focus on half of 5 as a reference of five halves. But she found that focusing on the meaning of the part of the whole, and thus on the reference unit "rectangle" made more sense for the pupils. So her final solution was to combine the two perspectives without explaining how.

Ana's reflections show that she did not question the purpose of these classes. She was confident in the tasks she had designed. However, this confidence did not extend to her explanations. She did not foresee alternative solutions to the first task and the difficulties the pupils might feel. So, when she faced an unexpected solution, she found it hard to understand the pupils' thinking.

Conclusion

This paper presents the case of Ana, who, as a part of her supervised teaching practice, was supposed to explore the concept of inverse rational numbers. To this end, she planned and carried out a lesson. So that the pupils might visualize the product obtained in a task and not be "given" a rule to memorize, she asked them to draw an illustration of the expressions. During planning, she solved the expressions both symbolically and pictorially but did not anticipate possible pupils' solutions. Relating what she planned with what she accomplished in the classroom we conclude that Ana shows weaknesses in her didactical knowledge about pupils because she did not anticipate that her pupils could identify one rectangle as the unit and interpret 2/5 as a part-whole relationship. Note that Ana did not talk with her school supervisor and did not realize that the pupils could interpret the fractions as a relation part-whole as a result of their mathematics experience. When Ana designed the task she proposed an expression that could have multiple interpretations. This issue raises questions about her didactical knowledge about tasks.

When Ana was faced with an unexpected interpretation from her pupils she chose to stick to her plan. So, she did not consider the pupils' perspective nor did she compare the two views. In this situation, she was not able to apprehend the pupils' understanding and adapt her approach. She planned to discuss the pupils' solutions but rushed ahead and took "control" of communication and built an instructional explanation supported in her ideas of the reference unit and her interpretation of 2/5 of 5/2. To convey concepts and procedures to pupils she used pictorial and symbolic representations (Bishop & Goffree, 1986) connected with verbal communication (Ponte & Serrazina, 2000). According to Charalambous, Hill, and Ball (2011) Ana gave a "good explanation" but, as these authors highlight, she did not offer a meaningful and easy to understand explanation because she did not take in account her pupils' previous experience and knowledge. Ana later planned a second, 45-minute class. After talking to the cooperating teacher and reflecting, she rethought her explanation. In this second stage, her explanation was more confident and she was able to deal with unforeseen conceptual issues and was more attentive to pupils' ideas. Throughout the process, Ana reflected on these issues, became aware of the complexity of teaching rational numbers, and developed her didactic knowledge about pupils.

It is not our aim to analyse issues related to the supervision process, but some reflections may be made. Ana's case illustrates the complexity of teachers' knowledge for teaching rational numbers (Barnett-Clarke et al., 2010). As a teaching practice that focuses on understanding concepts is complex and requires careful planning (Serrazina, 2012), both prospective teachers and their educators must be aware of issues related to planning and carrying out such teaching practice. During planning, prospective teachers need to discuss alternative solutions with their supervisors to be able to deal with unforeseen situations. Such issues are related to didactics knowledge regarding pupils and tasks and about classroom communication. Just as Mitchell et al. (2013) indicate, prospective teachers sometimes think that pictorial representations illustrate concepts by themselves. However, this does not always happen and it is important to reflect on the most appropriate representations for which purposes and how they might support pupils' learning.

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