

Pedagogical context and its role in articulating mathematical knowledge for teaching

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This paper examines the design of tasks for developing and assessing mathematical knowledge for teaching, in particular the role of pedagogical context. It argues that pedagogical context plays a vital role in shaping the reasoning involved in generating correct responses and in the articulation of mathematical knowledge for teaching more generally. It concludes with suggestions for more fully specifying the design of tasks to developed and assess mathematical knowledge for teaching.

Keywords: pedagogical content knowledge, mathematical knowledge for teaching.

Compelling examples of mathematical knowledge for teaching (MKT) (Ball, Lubienski, & Mewborn, 2001; Ma, 1999) and evidence associating it with improved mathematics teaching and learning (Baumert et al., 2010; Hill, Rowan, & Ball, 2005; Hill, Umland, Litke, & Kapitula, 2012) have sparked interest in making it central in the mathematical education of teachers. Despite this interest, programs still focus mostly on disciplinary knowledge rather than MKT. What is needed for a more solid shift in teacher education and professional development are robust tasks for developing and assessing MKT. This observation leads to a basic challenge: although an initial set of MKT tasks have supported the development of some measures, large numbers of compelling new tasks have not been readily forthcoming.

Several challenges hamper progress. To better understand these challenges, we have found it useful to reflect on the design of effective MKT assessment tasks used in large-scale evaluation projects. According to Hill, Sleep, Lewis, and Ball (2007), a key innovation in assessment studies that yielded demonstrable effects was the inclusion of pedagogical scenarios that frame mathematical problems situated in practice. From an examination of items on the National Teacher Examination (NTE) used in the United States in the 1980s, which described pedagogical context yet failed to measure consequential knowledge, they point out two ways in which the inclusion of pedagogical context can go awry. First, for many items, they found that the pedagogical context was merely “window dressing” (p. 119) — because the item would measure essentially the same knowledge if the pedagogical context were stripped away. Second, at another extreme, they discussed items that lacked a defensible solution because ambiguity in the pedagogical context allowed more than one professionally defensible answer.

To understand the design and functioning of pedagogical context in MKT tasks, we conducted talk-aloud interviews with research mathematicians and mathematically knowledgeable and experienced teachers.[1] Based on analysis and our continued efforts to support people in writing MKT tasks, this paper extends the observations of Hill et al. (2007) to explain how pedagogical context matters in MKT tasks and argue that it plays a fundamental role in articulating MKT. We begin by describing our interview study and what we learned about how pedagogical context shapes the mathematical work of responding to MKT tasks and the MKT assessed. Finally, we argue that our analysis, together with our experiences in supporting others in writing MKT tasks, suggests that pedagogical context is essential to articulation of MKT, both in MKT tasks and more generally in the identification of MKT.

LEARNING FROM PERFORMANCE AND MIS-PERFORMANCE

In previous work validating MKT assessment items, our group found that research mathematicians, often from their missteps, revealed much that otherwise might be presumed trivial or remain tacit in our understanding of the work involved in responding to MKT items. Additionally, highly experienced and knowledgeable teachers often expressed aspects of the work that otherwise might have remained unrecognized and unnoted. In this study, we interviewed about 60 experts with 26 items from the Learning Mathematics for Teaching (LMT) (Hill et al., 2005), Measuring Effective Teaching (MET) (Phelps Weren, Croft, & Gitomer, 2014), and Diagnostic Teacher Assessment in Mathematics and Science (DTAMS) (Saderholm, Ronau, Brown, & Collins, 2010) instruments.[3] We analyzed both the text of items and interview data. In these analyses, we engaged in a logical analysis and professional vetting of the work of teaching (Hoover, Mosvold, Ball, & Lai, 2016).

For a lesson on comparing fractions, Mr. Howard wants to choose a model that will make it easy for his students to compare a wide range of fractions, in problems such as:

Which is larger, $\frac{2}{3}$ or $\frac{3}{5}$?

Which is larger, $\frac{1}{6}$ or $\frac{3}{16}$?

Which is larger, $\frac{2}{7}$ or $\frac{3}{10}$?

Of the following models, which would best serve his purpose?

- a) Drawings of round pizzas
- b) Drawings of rectangles
- c) Pattern blocks
- d) Money
- e) These models would work equally well to compare a wide range of fractions

Figure 1: An example choosing representations item.

In an initial stage of the project, 46 interviewees read aloud and talked through the solution of 11 LMT items. (See Figure 1 for an example item.) The content of the items was from upper elementary and middle school topics in areas of whole number operation, rational number, and proportional reasoning. Twenty-seven interviewees were research mathematicians selected from the participant list of the annual Joint

Mathematics Meeting of the Mathematical Association of America and the American Mathematical Society. Many were eminent mathematicians from highly ranked research mathematics departments. Nineteen interviewees were expert teachers identified by nationally recognized leaders of professional development as most likely (within the United States) to know MKT. All had at least ten years of teaching experience. Many were themselves in leadership positions, but all had taught or been actively engaged with students and teachers in schools within the last five years.

After coding the pedagogical context of the items, we recorded how each element of the pedagogical context might be used in relation to decision points produced from analysis of the interviews (Figure 2). (An “X” indicates that the element of pedagogical context plays a role in making the mathematical observation in such a way that deleting it would remove the grounds for making the observation. The “X” is bolded for an element that is primary for the observation.)

Mathematical observations contributing to the solution	Purpose	Problems	Models
1. Patterns blocks and money work well only for restricted sets of fractions (halves, thirds, fourths and halves, fifths, tenths respectively), while circles and rectangles are more flexible.	X	--	X
2. Drawings introduce construction issues (imprecision, more room for error, ...), circles even more problematic than rectangles (especially with denominators that are odd or multiples of odd numbers).	X	--	X
3. Rectangles are readily aligned for easy comparison (or sub-divided vertically for one fraction and horizontally for the other, yielding comparable pieces).	X	--	X
4. Other fractions may present difficulty or require special consideration (e.g., large or prime denominators, pairs of fractions that are not evaluated by other means, ...).	X	X	X

Figure 2: Coding of the use of elements of pedagogical context for decision points identified from the narrative of competent performance for the example item.

We then replicated this process with 4-6 interviewees for each item for 10 items from the MET and 6 items from the DTAMS instruments. Analyses of interviews formed the basis for writing narratives for the role of pedagogical context in shaping the MKT assessed by each item. We found that items across all instruments had a teaching purpose/task and provided some form of instructional materials, records, and examples (such as example problems, student work, manipulatives or instructional representations, student explanations, and classroom dialogue) and that these elements of pedagogical context played a prominent role in supporting the mathematical observations that contributed to answering items correctly. We then analyzed the work across items to produce generalized narratives for items with the same task of teaching. (See Figure 3 for two examples.)

SHAPING THE MATHEMATICAL KNOWLEDGE MEASURED

We found that, in items designed to measure MKT, pedagogical context is needed in reasoning toward correct answers. It is used in making mathematical observations which, taken as a whole and vetted professionally, provide evidence for defensible answers. Together, the pedagogical context and supported observations provide a cogent characterization of the MKT reasoning involved in responding to the item.

More than this, the teaching purpose, a crucial element of the pedagogical context, often provides important orientation and sense of direction for the mathematical work involved. For instance, the task of choosing a model to compare fractions leads one to noticing which fractions are and are not easy to represent with different materials and what is involved in using the model for comparison. Instead of being asked simply to decide which of two fractions is larger, this MKT item asks for comparing the complexity of using different models. To persist with this task and to have a sense of how to judge the complexity of using a model, what it means to compare, and how to know when sufficient distinctions have been made, one needs to know the purpose — in this case, choosing a model that makes it easy for students to compare a range of different kinds of fractions. The example comparison problems provided in the item give a sense of the range of comparisons to consider. The set of models constrains the scope of the work and frames the set of issues to be considered.

Looking across items with similar tasks of teaching provides further generalization. Below is the generalized description for the task of choosing representations and for a second task of choosing examples (Figure 3). Notice that for choosing representations the first step involves sizing up a range of issues that might be pertinent, which then serves as a guide for knowing what to pay attention to when experimenting with the use of different representations.

Choosing representations

1. Recognizing the features and relationships prominent in the design of the objects being considered.
2. Considering how to use each representation for the purposes.
3. Considering and running through sensible test cases.

Choosing examples – selecting a problem for an exercise

1. Tracking on the instructional purpose for the exercise (e.g., introduce a procedure, assess student understanding, provoke error, highlight a special case, encourage multiple approaches, etc.).
2. Considering the features of or what happens with particular numbers or examples by working through the given problems, playing with different ways that students might solve them, and determining what is different mathematically about the examples and how these differences might impact students' thinking, their approaches to solving the problems, or the mathematical issues that might arise.
3. Identifying what feature of the example addresses that instructional purpose and whether aspects of the examples obscure or get in the way of the instructional purpose.

Figure 3: Narrative of MKT reasoning involved for two example tasks of teaching

Our analysis across items led to three observations about the role of pedagogical context in shaping knowledge measured by MKT items, which we will use to support our argument for the role of pedagogical context in articulating MKT as a domain. Our first observation is that pedagogical context shifts tasks from being disciplinary mathematics tasks to being pedagogical mathematics tasks. Figure 4 summarizes key characteristics of disciplinary mathematics tasks as compared to pedagogical mathematical tasks. In the example task, the pedagogical context shifted the nature of the task from that of comparing fractions, which is the students' mathematical task in the context, to a pedagogical mathematics task of comparing models, where comparing fractions is a subordinate task carried out in the service of comparing models. Comparing models is not a pedagogical task just because it may have a pedagogical aim; it is a pedagogical task because the chosen model should work on a set of comparisons such as those given and should be easy to use. The implicated pedagogical mathematical work is figuring out how to use the given mathematical representations to carry out the example mathematical comparisons and deciding which numbers might pose thornier mathematical challenges. In the same way that two numbers shape the work of comparing fractions, the pedagogical context of a collection of fractions to be compared shapes the pedagogical mathematics task of choosing representations. A different set of fractions to be compared might have changed the work of which representation to choose.

<i>Disciplinary mathematics tasks</i>	<i>Pedagogical mathematics tasks (or mathematical tasks of teaching)</i>
Compare fractions Compute Solve a problem Justify a solution Identify structure	Choose representations Analyze errors Appraise nonstandard work Solve in different ways Follow others' thinking Size up incomplete reasoning

Figure 4: Contrasting disciplinary tasks and pedagogical mathematics tasks

Second, pedagogical context situates pedagogical mathematics tasks in contexts that require doing mathematics while holding onto and coordinating with pedagogical purpose. For instance, a person engaged with the pedagogical mathematics task of *choose representations* might specialize the purpose to *choose a model that will be easy for students to use*. Or, *appraise nonstandard work* might specialize to *which potential interpretation of thinking best fits with nonstandard student work*. The pedagogical purpose provides the basis for doing the work of the item; this basis is not mathematically determined. Exactly which fractions need to be compared? Which approaches to comparison might students find easy or hard?

Instead of changing the problem from a mathematics problem requiring mathematical knowledge to a pedagogical problem requiring pedagogical knowledge, the pedagogical purpose shifts the nature of the cognitive demand associated with the mathematical problem. It introduces potentially competing agendas and a need to

track on purpose while engaging in mathematical work. In our analysis, this was particularly evident in the contrast between mathematician and teacher interviews, where mathematicians often lost track of pedagogical purpose in ways that led them astray, while teachers facilely tracked on and used pedagogical purpose to navigate decisions. Mathematicians would worry about not having determinant information when experienced teachers would have a sense of what is sufficient for answering underlying mathematical questions for the purpose at hand, even if it is not complete information. In addition, mathematicians often struggled to hold on to the question being asked, drifting off to other questions, often back into doing the mathematics problem given to students or exploring mathematical ideas seen as related to those problems but not related to the MKT question being asked. Our point here is that frequent missteps, despite displaying sophisticated disciplinary mathematical knowledge, made more apparent the distinctive character of the mathematical work required when carrying out that work with regard for pedagogical context.

Third, the pedagogical context establishes a basis for an orientation and character for mathematical reasoning distinctive to teaching as professional work. We have mentioned the way in which the pedagogical purpose of choosing a model to compare fractions provides an orientation for the mathematical work, giving it purpose and a sense of direction. Consistent with the two examples characterized in Figure 3, many of the items required doing mathematical work while heeding pedagogical purpose. It is as if, more than pedagogical knowledge or skill, pedagogical heed is required in responding to MKT items.

ARTICULATING CONTENT KNOWLEDGE FOR TEACHING

Our observations demonstrate ways in which the pedagogical context provided in well-designed MKT items shapes the MKT being assessed. In this section, we argue that the formulation of the pedagogical context is what *articulates* content knowledge for teaching – it gives expression to MKT.

The word *articulate* comes from the notion of dividing into distinct parts, which taken together convey a more complex sense of the whole. It can mean to pronounce clearly, but also to joint something — to formulate in an article or articles or to express or convey (a thought) by means of language. It is this sense that pedagogical context provides a means of expressing or conveying MKT that we mean here.

The analysis in the previous section demonstrated the role of pedagogical context in shifting the focus from disciplinary tasks to pedagogical mathematics tasks, associating tasks with pedagogical purpose, and establishing a particular orientation and character for mathematical reasoning distinctive to teaching. One way to interpret this is that without the pedagogical context these items would be limited to the domain of other mathematical tasks that typify disciplinary work (e.g., traits in Figure 4) and would fail to assess the distinctive knowledge and skill known to be associated with increased learning. In other words, the doing of mathematical work (such as comparing fractions) while keeping in mind a purpose (of choosing a representational

model) and attending to what is involved (in using a model, as one uses it or talks about it) is common in teaching, but uncommon in the discipline of mathematics. For instance, a disciplinary impulse can lead one to focus on the mathematics problem given to students or to explore variations or generalizations of a mathematical problem (a distraction that played out in many of the interviews with mathematicians), losing track of the need to interpret the mathematical validity of a student's confusing approach or generate a mathematical problem with a solution satisfying specific criteria. These latter tasks typify MKT, and it is pedagogical context that allows for their expression and that thus makes visible the articulation of the task of teaching, such as shown in Figure 3.

Our analysis is limited to sampling from items that have been produced to date, with a set of features of pedagogical context that is likely narrow. For instance, student background is not a prominent feature and plays a minor role in the items analysed. This is likely a result of narrowness of existing items and likely to change as scholars continue to expand work in this arena. For instance, Goffney (2010) has pointed out the mathematical demands of equitable teaching and Wilson (2016) has explored the development of assessment items to measure such knowledge in relation to dual language learners. Despite these limitations, we propose that the role of pedagogical context is important in the development of tasks to support the development of equitable teaching and that lessons from the above analysis can provide valuable guidance.

We close by offering a suggestion about how MKT might be articulated in the work of specifying the design of MKT tasks, in line with an approach developed by Illustrative Mathematics. Their approach requires not only writing a problem, but providing a commentary (and sample solutions). Consider the item in Figure 5.

Ms. Seidel is introducing the distributive property. To motivate her students, she wants to give them an example that will focus their attention on how using the distributive property can simplify computation. In which of the following examples will the use of the distributive property most simplify the computation?

a) $12 \times 29 + 12 \times 38 = \underline{\quad}$

b) $17 \times 37 + 17 \times 63 = \underline{\quad}$

c) $13 \times 13 + 15 \times 15 = \underline{\quad}$

d) $16 \times 24 + 16 \times 24 = \underline{\quad}$

Figure 5: Choosing examples item

Based on a narrative for doing the task (Figure 6), a commentary might be written (Figure 7), where the commentary characterizes the MKT that the task is intended to develop or assess, intended use or the task, and the pedagogical context provided in the scenario. The production and review of such a commentary provide powerful tools for collaborative efforts to develop MKT tasks, where explicit statements about rationale for pedagogical context significantly enhance development, review, and professional sanctioning.

1. Tracking on the fact that the instructional purpose for the example is to focus students' attention on how using the distributive property can simplify computation.
2. Considering different ways of evaluating the expressions and of using the distributive property and what these imply about what it means to simplify the computation, including recognizing the following: the most reasonable way of using the distributive property in (a) yields $12(29 + 38) = (12)(67)$, which reduces the computation from two to one application of multiplication; the most reasonable way of using the distributive property in (b) yields $17(37 + 63) = (17)(100)$, which reduces the computation from two non-trivial applications of multiplication to one simple one; it is not clear how to use the distributive property in (c); and although there are numerous quantities that could be factored out of the two terms (to similar effect as in (a)), none significantly simplifies the complexity of the multiplication to be done (use of doubling can be made with or without the use of the distributive property).
3. Recognizing that in problems such as these the distributive property does not avoid multiplication, but does allow for regrouping quantities into powers of 10, which greatly simplifies multiplication in a base ten system, and that (b) is the only one that affords this opportunity.

Figure 6: Narrative for the MKT reasoning involved in choosing examples item.

Examples shape instructional opportunities, however crafting and choosing good examples requires mathematical dexterity and skill in doing mathematical problems while tracking on instructional goals. This task asks for an example in which the distributive property can be used to simplify computation significantly. The purpose of this task is to see whether teachers flexibly consider different ways of evaluating the expressions using the distributive property and, simultaneously, what these imply for efficiency of the computation. It requires recognizing that the distributive property does not avoid multiplication, but does allow for regrouping quantities into powers of 10, which greatly simplifies multiplication in a base ten system. The task is currently written as a multiple-choice item for assessment. But it also can be used for launching a discussion about the nature of examples for which the distributive property is useful.

The mathematical task of teaching is choosing examples, but the teaching scenario needs to create a realistic need for choosing an example that requires the distributive property. In this scenario, the pedagogical purpose is to motivate learning of the distributive property. In particular, the scenario proposes motivating the distributive property by giving an example that will focus students' attention on how using the distributive property can simplify computation. This means that the example needs to provide a sharp contrast in the extent to which the computation is simplified by using the property relative to not using it. The examples given in the options in this task are selected to create such a contrast, where only option (b) significantly reduces the complexity of the multiplication. The instructional setting of introducing the distributive property contributes to a sense that the scenario is realistic.

Figure 7: Commentary for choosing examples item.

Through this process, task developers can encode implicit hypotheses about what matters about the pedagogical context when teachers face particular content problems

of practice. Teachers who are able to use the pedagogical context in tasks as a resource for responding to tasks demonstrate knowledge in a way that simulates teacher knowledge use in teaching; their reasoning with the pedagogical context can be used to scrutinize and make visible the implicit hypotheses, iterate item development, and refine articulations of mathematical tasks of teaching and MKT assessed.

Bringing together content and pedagogy has been a persistent theme in conversations about the content-knowledge education of teachers over the last 50 years. However, taking stock of scholarship on content knowledge for teaching, Graeber and Tirosh (2008) remind us that, while the concepts of pedagogical content knowledge and content knowledge for teaching are useful, the union of content and pedagogy remains elusive. Beyond introducing complexity and challenge for writing MKT tasks, pedagogical information plays a non-trivial function in tasks designed to develop and assess professionally situated mathematical knowledge by articulating constrained instances of the relationship between content and teaching that is at the heart of the notion of MKT. Ball (2000) characterizes the “intertwining of content and pedagogy” as a continuation of Dewey’s (1964/1904) effort to find the “proper relationship” between theory and practice. Our growing understanding of the role of pedagogical context in the design of and reasoning within MKT tasks is beginning to give us a better understanding of the “*proper relationship*” between content and pedagogy in characterizations of content knowledge for teaching.

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2. We were denied access to COACTIV items and release of TEDS-M items occurred after we completed interviewing.

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