Growth/collapse and decay/surge Markov processes : from population dynamics to systems of interacting neurons

Branda Goncalves

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Branda Goncalves

Growth/collapse and decay/surge Markov processes : from population dynamics to systems of interacting neurons

Processus de Markov de type 'growth/collapse' et 'decay/surge' : de la dynamique des populations à un système de neurones en interaction

Présentée en vue de l'obtention du grade de DOCTEUR

Dirigée par : Eva Löcherbach, Thierry Huillet et Jean Avan

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À ma famille !

Résumé

Peut-on déterminer le temps moyen d'extinction d'une population en fonction de la taille de la population ? Une population peut-elle croître indéfiniment ? Pour tenter de répondre à ces questions, nous utilisons dans cette thèse les processus de Markov avec catastrophes qui sont des outils utilisés à la fois en modélisation de la dynamique des populations et en neurosciences.

Dans un premier temps, on a étudié ces modèles avec catastrophes en temps discret avec un flot aléatoire et en temps continu avec un flot déterministe. Des conditions sur l'existence de la mesure invariante, sur la récurrence et la transience de tels modèles sont trouvées. De même, des conditions sur la finitude du temps moyen de la première extinction et du temps de non-explosion de la population ont été étudiées. Dans le cas où il y a un risque de surpopulation, la loi du temps qu'il faut pour qu'une population dépasse une barrière seuil (assez grande) a été donnée.

Dans un second temps, le point de vue de modélisation a changé, la taille de la population est interprétée comme l'état d'un neurone et les catastrophes sont considérées comme des décharges de neurones (spikes). Ce processus est mis en réseau pour un nombre de neurones fini dans un premier cas et infini dans un second cas. Dans le cas où le nombre de neurones dans le réseau est fini, l'existence d'une fonction de Lyapunov permet d'obtenir une condition de non-évanescence du processus et les conditions locales de Doeblin permettent l'étude de la récurrence du processus. Lorsque le nombre de neurones dans le réseau est infini, un algorithme de simulation parfaite permet la construction graphique du processus. Cet algorithme est utilisé pour trouver des critères de récurrence du système et montrer l'existence d'un état stationnaire du processus.

Abstract

Can the mean time to extinction of a population be determined by its size ? Can a population grow indefinitely ? To try to answer these questions, we use in this thesis Markov processes with catastrophes which are tools used both in population dynamics and in neuroscience.

In a first step, we studied these models with catastrophes in discrete time with a random flow and in continuous time with a deterministic flow. Conditions on the existence of the invariant measure, on the recurrence and transience of such models are found. Similarly, conditions on the finiteness of the mean time of the first extinction and the time of non-explosion of the population have been studied. In the case where there is a risk of overpopulation, the law of the time it takes for a population to exceed a (large) threshold has been given.

In a second step, the point of view of the modelling has changed, the size of the population is now interpreted as the state of a neuron and disasters are considered as spikes of the neurons. This process is networked for a finite number of neurons in a first case and infinite in a second case. In the case where the number of neurons in the network is finite, the existence of a Lyapunov function allows us to obtain a non-evanescence condition for the process and the local Doeblin conditions allow the study of the recurrence of the process. When the number of neurons in the network is infinite, a perfect simulation algorithm allows the graphical construction of the process. This algorithm is used to find the recurrence criteria of the system and to show the existence of a stationary state of the process.

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Au début d'une thèse 👩🏾‍🎓, comme tout autre grand projet dans la vie, on ne sait pas combien de temps cela prendra, on voit l'horizon lointain. On pense avoir assez de temps, assez pour toutes les choses qu'on prévoit de faire, pour exploiter les opportunités qui se présenteront à nous. Et puis, avant de s'en rendre compte, c'est déjà la fin. Les articles, les voyages, les conférences et les séminaires, tout ce qu'on a appris sur la vie, le travail d'enseignant-chercheur et surtout, toutes les personnes qu'on a connues durant cette aventure nous donnent un aperçu du temps passé. On se rend compte que "le temps passe si vite". Et pour clôturer en beauté cette aventure, on se doit de remercier certaines personnes qui ont contribué de près ou de loin à la réalisation de ce projet de vie.

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Notations

$[z^x]f(z)$	The z^x-coefficient in the power series expansion of $f(z)$
X_n	Discret Markov chain
$c \circ X_n$	Bernoulli thinning of X_n
pmf	Probability mass function
X_t	Continuous process
$M(.,.)$	A Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with Lebesgue intensity.
$\mathbb{1}$	The indicator function. $\mathbb{1}_x(A) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$
δ_x	The Dirac mass at x, i.e, for all $A \in \mathcal{B}(\mathbb{R}_+),\ \delta_x(A) = \mathbb{1}_x(A)$.
pgf	Probability generating function.
SD	Self decomposable.
ID	Infinitely divisible.
$\mathcal{L}(X_t \vert X_0 = x)$	The law of X_t knowing $X_0 = x$.
$I_0(x)$	The time needed to reach some state $x > 0$ starting from 0.
$t_0(x)$	The time for the flow to hit 0 starting from x.
$I_\infty(x)$	The time needed to reach ∞ starting from some state $x > 0$.
T_x	The time at which a first jump occurs, when the process starts from x.
$\alpha : \mathbb{R} \to \mathbb{R}_+$	The deterministic drift function.
$\beta : \mathbb{R}_+ \to \mathbb{R}_+$	The jump rate function.
$a \wedge b$	The minimum between a and b.
$a \vee b$	The maximum between a and b.
∂_x	The partial derivative with respect to x.

$Z_n = X_{S_n}$	The embedded chain of the process.
$\tau_{x,y}$	The first time the process X comes back to y starting from x.
$\bar{\tau}_{x,y}$	The return time to y when the process starts from x.
π	The invariant measure of process.
Ei	Integral exponential.
$\mathcal{H}$	The height of an excursion.
$\mathcal{B}(\mathbb{R}_+)$	The Borel sets of $\mathbb{R}_+$.
x_j^k	The state of neuron k before the j-th jump.
$\mathcal{V}_{i\to\cdot}$	The incoming neighbourhood of neuron i .
$\mathcal{V}_{\cdot\to i}$	The outcoming neighbourhood of neuron i .
C_t^i	The clan of ancestors of neuron i at time t.
L^∞	The set of measurable and bounded functions.

1

Introduction

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Cette thèse s'inscrit dans l'étude théorique de processus aléatoires (processus avec catastrophes ou processus avec résurgence, décrivant l'évolution de populations). Le terme "catastrophe" désigne de manière générale des événements au cours desquels une partie ou la totalité de la population peut être anéantie. Ces processus avec catastrophes (bi-stochastiques ou semi-stochastiques) permettent la modélisation de nombreux phénomènes physiques, biologiques et de grandeurs économiques tels que : la naissance, la mort, l'évolution d'une maladie, la croissance d'un marché ou du capital d'une entreprise, l'évolution de populations vieillissantes et en déclin qui sont revigorées par l'immigration, le niveau d'eau d'un barrage qui diminue de façon

déterministe selon une programmation fixe de la libération de l'eau et qui est sujet à des augmentations soudaines dues à la pluie ou aux inondations, etc.

Les chapitres 2, 3 et 4 correspondent à la première partie de ce manuscrit. Elle vise à étudier ces modèles avec catastrophes en temps discret avec un flot aléatoire c'est-à-dire des modèles bi-stochastiques. Le chapitre 2 porte sur des chaînes de Markov impliquant des catastrophes totales en temps discret et en temps continu. Dans le chapitre 3 nous étudions des processus qui sont assujettis à des catastrophes dont la probabilité de survie suit une loi binomiale. Les résultats des chapitres 2 et 3 portent entre autre sur l'existence de la mesure invariante, la récurrence et la transience, les temps de retour, le temps de première extinction et la hauteur d'une excursion en se basant sur Neuts (1994) et Huillet (2011). Le chapitre 4 se base sur des processus qui sont confrontés à deux menaces majeures : l'extinction et la surpopulation. Les résultats portent sur le temps d'extinction de la population et le temps qu'il faut au processus pour dépasser une barrière donnée.

Les chapitres 5, 6 et 7 correspondent à la deuxième partie de ce manuscrit. Elle s'intéresse à des modèles avec catastrophes en temps continu avec flot déterministe c'est-à-dire des modèles semi-stochastiques. D'une part, les résultats des chapitres 5 et 6 portent sur la non-explosion, la non-extinction, la récurrence et l'existence d'une fonction d'échelle dans le cas unidimensionnel en se basant sur les articles de Eliazar and Klafter (2006b, 2007) et Gripenberg (1983, 1985). D'autre part, dans le chapitre 7, nous changeons de point de vue de modélisation. Au lieu de décrire des tailles de populations, nous nous intéressons à l'état de neurones et nous interprétons les catastrophes comme des instants de décharges de neurones (instants de spike). De plus, nous mettons ces processus en réseau avec un certain nombre (fini ou infini) de neurones ou de particules en étendant les articles de Cottrell (1992) et Fricker et al. (1993) au cas où les spikes (catastrophes) surviennent de manière aléatoire.

1.1 Le modèle

Les chaînes de Markov sont des processus de Markov à temps discret ou à temps continu et à espace d'états discret. Dans la première partie de cette thèse, nous nous intéressons principalement à des chaînes de Markov à temps discret et à espace d'états discret. Ces processus permettent de modéliser de nombreux phénomènes ; ils sont par exemple utilisés pour des problèmes de télécommunication, de théorie du signal, d'imagerie informatique, etc.

Un processus de Markov à temps continu est un processus aléatoire dont l'évolution future est indépendante du passé sachant l'état présent. Dans la deuxième partie de cette thèse, nous nous intéressons à des processus de Markov déterministe par morceaux (PDMP). Les PDMP ont été introduits par Davis (1984). Ils forment une famille de processus de Markov càdlàg suivant une évolution déterministe interrompue par des sauts aléatoires.

Nous commençons par donner quelques définitions et propriétés sur ces processus.

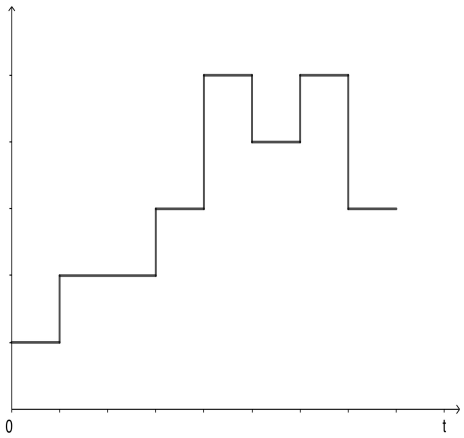
1.1.1 Quelques définitions utiles.

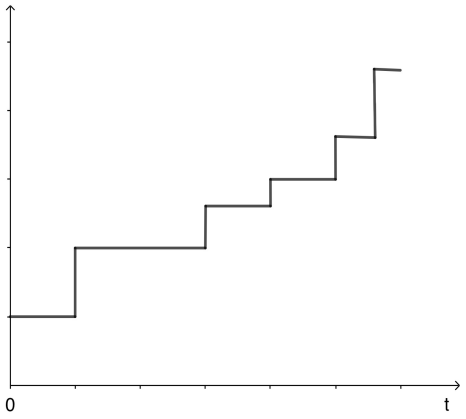
Définition 1.1. *Un processus stochastique est une collection* $(X_t)_t$ *de variable aléatoire indexée par un paramètre temps* t *prenant des valeurs dans* $\mathbb{N}$ *ou dans* $\mathbb{R}_+$.

Les processus stochastiques servent souvent à rendre compte de phénomènes dépendant à la fois du temps et de l'aléa (du hasard). X_t peut-être donc interprété en dynamique des populations comme la taille d'une population à la date t.

Example 1.1.

1. Modéliser l'évolution de la taille d'une population au cours du temps : elle peut augmenter (naissance, immigration) ou diminuer (mort, émigration). Schéma à gauche.
2. Modéliser la trajectoire au cours du temps des arrivées à un péage d'autoroutes à partir de l'instant initial $t = 0$. Schéma à droite.





Définition 1.2. *Soit $(X_n)_{n\geq 0}$ un processus stochastique à valeurs dans un espace d'états dénombrable $E \subset \mathbb{Z}$. Le processus $(X_n)_{n\geq 0}$ est appelé chaîne de Markov s'il existe une famille $P(x,y)_{x,y\in E}$ de réels appelés probabilités de transition, telle que, pour tout $n \geq 0$, $x_0, \cdots, x_n, x_{n+1} \in E$,*

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, \cdots, X_0 = x_0) = P(x_n, x_{n+1})$$

dès que $\mathbb{P}(X_n = x_n, \cdots, X_0 = x_0) > 0$.

Soit $(X_n)_{n\geq 0}$ une chaîne de Markov de probabilité de transition P sur l'espace d'états E et A un sous-ensemble A de E.

Définition 1.3. *Soit $A \subset E$. Le premier temps de retour de la chaîne dans A est la variable aléatoire*

$$\tau_A = \inf\{n \geq 1 : X_n \in A\}.$$

Le nombre de visites de la chaîne à A est la variable aléatoire

$$N_A = \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n \in A\}} \in \mathbb{N} \cup \{\infty\}.$$

Le théorème de convergence monotone pour les séries donne immédiatement la formule suivante pour tous $x, y \in E$:

$$\mathbb{E}_x(N_y) = \sum_{n=0}^{\infty} P^n(x,y).$$

Définition 1.4. *La fonction de Green de la chaîne de Markov X est la fonction $G : E \times E \to \mathbb{R} \cup \{\infty\}$ donnée par, pour tout $x, y \in E$,*

$$G(x, y) = \mathbb{E}_x(N_y) = \sum_{n=0}^{\infty} P^n(x, y).$$

Définition 1.5. *Soit $(S_n)_{n\geq 0}$ une suite strictement croissante de variables aléatoires réelles positives $(S_1, S_2 - S_1, \cdots, S_n - S_{n-1}, \cdots)$. Le processus $\{N(t), t \geq 0\}$ avec $N(t) = \sum_{n\geq 1} \mathbb{1}_{\{S_n \leq t\}}$ est un processus de comptage.*

Remarque 1.1. *On peut remarquer que*

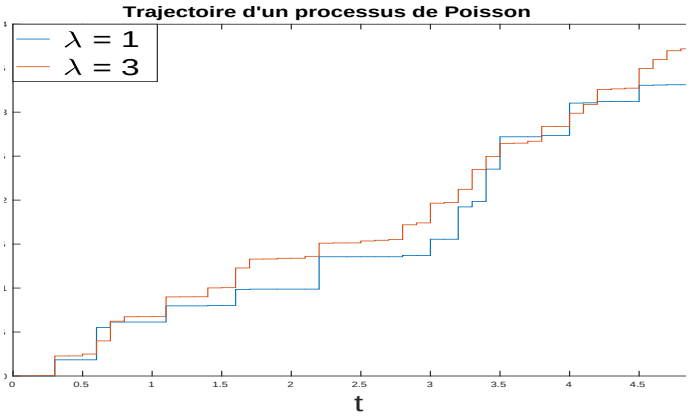
$$N(t) = \begin{cases} 0, & si\ t < S_1 \\ n, & si\ S_n \leq t < S_{n+1}. \end{cases}$$

Définition 1.6. *Un processus de Poisson est un processus de comptage $N(t)$ tel que $N(0) = 0$ qui vérifie les conditions suivantes :*

1. *Le processus est à accroissements indépendants, c'est-à-dire, pour tout $n \in \mathbb{N}^*$ et pour tous $s_1, \cdots, s_n$ tels que $s_1 < \cdots < s_n$, les accroissements $N(s_1) - N(0), \cdots, N(s_n) - N(s_{n-1})$ sont des variables aléatoires indépendantes.*
2. *Le processus est à accroissement stationnaire, c'est-à-dire, pour tout s et pour tout t, l'accroissement $N(t + s) - N(s)$ a la même loi que N_t.*
3. *Le processus est à événements rares c'est-à-dire, $\lim_{h\to 0_+} \mathbb{P}(N(h) > 0) = 0$ et $\lim_{h\to 0_+} \frac{\mathbb{P}(N(h)>1)}{\mathbb{P}(N(h)=1)} = 0$.*

Remarque 1.2. *On dit qu'un processus de Poisson est d'intensité λ, lorsqu'il vérifie les points 1 et 2 de la définition 1.6 et $\mathbb{P}(N(t + h) - N(t) = 0) = 1 - \lambda h + o(h)$.*

Le graphique ci-dessous est un exemple typique de la simulation des deux processus de Poisson d'intensités respectives de $\lambda = 1$ et $\lambda = 3$. Le processus d'intensité $\lambda = 1$ est représenté en bleu et le processus d'intensité $\lambda = 3$ est représenté en rouge. Dans cet exemple le nombre d'itérations est de 50.



Simulation. Pour simuler un processus de Poisson d'intensité λ, il suffit de simuler des variables aléatoires i.i.d suivant une loi exponentielle de paramètre λ.

Remarque 1.3. *Lorsque le point* 3 *de la définition 1.6 n'est pas vérifié, les événements peuvent se produire par 'grappes' c'est-à-dire beaucoup d'événements peuvent se passer aux mêmes instants, l'effectif de la grappe étant lui-même aléatoire.*

Exemples
→ *Arrivées d'avions dans un aéroport : chaque avion transporte un certain nombre de passagers*
→ *Trafic routier : chaque accident engendre un certain nombre de blessées…*

Soit $(A_n)_{n\in\mathbb{N}}$ *la famille des temps d'arrivées qui sont aléatoires. Soit* A_n *l'instant de la* n^{ime} *arrivée :* $A_n = \inf\{t \geq 0; N(t) = n\}$.
Les instants d'occurrence des grappes d'événements sont les A_n *d'un processus de Poisson de paramètre* λ.
À chaque instant A_n *est associé une variable aléatoire* Y_n *désignant le nombre d'événements se produisant à l'instant* A_n. *On suppose les* Y_n *indépendantes et de même loi donc*

$$Z_t = Y_1 + \cdots + Y_{N_t}$$

est le nombre d'événements apparus jusqu'à l'instant t. *Cette relation permet, en particulier, de déterminer la fonction génératrice de* Z_t. Z_t *ainsi défini est un processus de Poisson composé donc,* Z_t *est un processus de Markov et on peut le caractériser par son générateur infinitésimal.*

Cette remarque nous permet d'introduire deux notions qui sont celle de la fonction génératrice des moments et celui du générateur infinitésimal.

Définition 1.7. *Soit* X *une variable aléatoire, telle que la fonction* g_X *définie par :*

$$g_X(u) = \mathbb{E}[e^{uX}],$$

soit définie sur un voisinage de 0. g_X *est alors appelée fonction génératrice des moments de* X *et elle caractérise la loi de la variable aléatoire* X.

En littérature, dans les processus de Markov, il y a plusieurs notions de générateur infinitésimal qui sont utilisées. Dans la suite, on utilise celle liée à la convergence simple pour chaque x. Pour cela on a la définition suivante :

Définition 1.8. *Soit* $(X_t)_{t\geq 0}$ *un processus Markovien à valeurs dans un espace* E, *tel que* (X_t) *vérifie les propriétés de Markov faible et forte. Ce processus définit alors un semi-groupe d'opérateurs* $(P^t)_{t\geq 0}$ *agissant sur* $L^\infty(E)$ *par :*

$$P^t f(x) = \mathbb{E}[f(X_t) \mid X_0 = x]$$

où f *est une fonction test. On appelle générateur du processus markovien* X_t *l'opérateur* G *défini par :*

$$Gf(x) = \lim_{t\to 0} \frac{P^t f(x) - f(x)}{t}$$

où $Gf(x)$ *est défini seulement pour la classe des fonctions* f *pour lesquelles cette limite existe.*

Moralement, le générateur $Gf(x)$ exprime le changement infinitésimal de $f(X_t)$ lorsqu'on se trouve au point x à l'instant t.

Example 1.2. Soit un processus de Markov à valeurs dans un espace à deux points $\{x, y\}$: le processus saute de x vers y avec une loi exponentielle de paramètre α, et de y vers x avec une loi exponentielle de paramètre β. Pour une fonction $f : \{x, y\} \to \mathbb{R}$, le générateur G de ce processus est donnée par

$$\begin{cases} Gf(x) = \alpha(f(y) - f(x)); \\ Gf(y) = \beta(f(x) - f(y)). \end{cases}$$

Définition 1.9. *Soit une chaîne de Markov de matrice de transition P. Une loi de probabilité définie par un vecteur ligne stochastique π vérifiant $\pi P = \pi$ est appelée distribution stationnaire de la chaîne de Markov.*

Définition 1.10. *On appelle chaîne extraite (embedded chain), la chaîne à temps discret qu'on obtient aux différents instants de sauts (instants juste avant les sauts) d'un PDMP ou d'une chaîne de Markov à temps continu $(X_t)_t$. Autrement dit, soit*

$$T_{n+1} = \inf\{t > T_n : X_t - X_{t-} \neq 0\}.$$

Les chaînes $(X_{T_n})_n$ et $(X_{T_n-})_n$ sont deux chaînes extraites de $(X_t)_t$.

Remarque 1.4. *Une méthode pour trouver la distribution stationnaire π d'une chaîne de Markov ergodique à temps continu consiste d'abord à étudier sa chaîne extraite.*

Proposition 1.1. *Si π est une distribution stationnaire du processus X_t, alors pour toute fonction test f, on a :*

$$\int (Gf)(x) d\pi(x) = 0. \tag{1.1.1}$$

Définition 1.11. *Pour G le générateur d'un processus Markovien, on appelle fonction d'échelle ou fonction harmonique toute fonction s telle que $Gs = 0$.*

Une fonction d'échelle est donc une fonction vérifiant :

Lemme 1.1. *Si $(X_t)_{t\geq 0}$ est un processus de Markov de générateur G et s une fonction d'échelle, alors on a pour tous $0 \leq t \leq t'$, pour tout $x \in E$:*

$$\mathbb{E}[s(X_{t'}) \mid X_t = x] = s(x).$$

On dit que le processus aléatoire $s(X_t)$ est une martingale locale. Une fonction d'échelle est donc une fonction qui transforme le processus en une martingale locale. Notons que, toute fonction constante est une fonction d'échelle.

Example 1.3. Supposons que

$$X_t = X_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s$$

où b et σ sont des fonctions de $\mathbb{R}$ dans $\mathbb{R}$ bornées. La fonction

$$s(x) = \int_0^x \exp(-\int_0^y \frac{2b(v)}{\sigma^2(v)} dv) dy$$

est une fonction d'échelle de X_t (voir Khasminskii (1980), exemple 2 section III.8). En effet,

$$Gs(x) = \lim_{t\to 0} \frac{\mathbb{E}(s(X_t) \mid X_0 = x) - s(x)}{t} = b(x)s'(x) + \frac{1}{2}\sigma^2(x)s''(x) = 0$$

et donc $s(X_t)$ est une martingale locale donc s est une fonction d'échelle.

Définition 1.12. *Un processus X est appelé* Harris récurrent *s'il existe une mesure σ-finie π sur $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ telle que pour tout $A \in \mathcal{B}(\mathbb{R}_+)$,*

$$\pi(A) > 0 \Rightarrow \quad \mathbb{P}_x\left(\int_0^\infty 1_A(X_s)ds = \infty\right) = 1 \text{ pour tout } x \in \mathbb{R}_+.$$

Dans l'exemple 1.3, la mesure invariante de X est donnée par

$$\pi(dx) = \frac{2}{\sigma^2(x)} \exp\left(\int_0^x \frac{2b}{\sigma^2}(v)dv\right) dx, \quad x \in \mathbb{R} \tag{1.1.2}$$

et la diffusion est Harris récurrente si et seulement si la fonction s est une transformation de l'espace, c'est-à-dire

$$\lim_{x\to-\infty} s(x) = -\infty \text{ et } \lim_{x\to+\infty} s(x) = +\infty$$

1.1.2 Le modèle à valeurs discrètes

1.1.2.a Le modèle de catastrophe dépendant de la taille de la population

On considère ici un modèle de croissance de population sujet à des catastrophes totales ou partielles en temps discret.
Pour décrire ce modèle, soient $\beta > 0$, $\nu > -1$ et $0 < \alpha < \nu + 1$. On considère la chaîne de Markov homogène à temps discret $X := (X_n,\ n \geq 0)$ avec un espace d'état $\mathbb{N}_0 = \{0, 1, ...\}$. La dynamique de la chaîne de Markov $(X_n)_n$ dont il est question est la suivante :
Pour $X_n = x \in \{1, 2, ...\}$,

$$X_{n+1} = \begin{cases} x+1 & \text{avec probabilité } : p_x = 1 - \alpha/\left(\nu + x^\beta\right) \\ 0 & \text{avec probabilité } : q_x = \alpha/\left(\nu + x^\beta\right). \end{cases} \tag{1.1.3}$$

Pour $X_n = 0$,

$$X_{n+1} = \begin{cases} +1 & \text{avec probabilité } : p_0 \leq 1 \\ 0 & \text{avec probabilité } : q_0 = 1 - p_0. \end{cases} \tag{1.1.4}$$

Dans le modèle (1.1.3), le marcheur X_n est occasionnellement renvoyé vers l'origine et la probabilité de cet événement devient très faible lorsque le marcheur a déjà atteint une grande valeur x.

Supposons $p_0 = 0$. Dans ce cas l'état 0 est absorbant. Soit $x \in \mathbb{N}$. La probabilité $\mathbb{P}(X_n \to \infty | X_0 = x) = \prod_{y\geq x} p_y$ est égale à 0 si et seulement si $\sum_{y\geq x} q_y = \infty$, ce qui est vérifié si et seulement si $\beta \leq 1$. On conclut que :

- Si $\beta \leq 1$ la chaîne $(X_n, n \in \mathbb{N}_0)$, ayant pour état initial $X_0 = x$, finira par s'éteindre.

- Si $\beta > 1$ la chaîne issue de x, tend vers l'infini avec une probabilité positive $\prod_{y \geq x} p_y > 0$ et s'éteint avec la probabilité complémentaire $1 - \prod_{y \geq x} p_y > 0$.

Supposons maintenant $p_0 \in (0, 1]$. Dans ce cas, l'état 0 est réfléchissant et on conclut que :

- Si $\beta > 1$, la chaîne est transiente et n'a pas de mesure invariante non triviale ($\neq \mathbf{0}$).

- Si $0 < \beta < 1$, la chaîne est récurrente positive et sa mesure invariante a une queue exponentielle.

- Si $\beta = 1$ (cas critique). Lorsque x est grand, la dérive de cette chaîne de Markov est d'ordre $1 - \alpha + (\nu - \alpha)/x$. Ainsi, lorsque $\alpha > 1$, le marcheur est attiré par l'origine : la force de l'attraction est de $\alpha - 1$ pour x grand. Pour $\alpha < 1$, le marcheur est repoussé de l'origine de manière correspondante. Lorsque $\alpha = 1$, sa dérive est toujours attractive, mais d'ordre $(\nu - 1)/x$, et la dérive du marcheur s'annule lorsque x approche ∞ dans le cas où $\nu < 1$. En dépassant la valeur critique $\alpha = 1$ par le haut, le processus X passe de récurrent positif à récurrent nul.

On peut également décrire ce modèle lorsque les catastrophes se font en temps continu.
Désignons par $P = [P(x, y)]$ la matrice de transition de la chaîne discrète X_n et par $D_r = diag(r)$ la matrice diagonale formée par le vecteur de taux $r = (r_0, r_1, \cdots, r_x, \cdots)$ où pour tout $\lambda \in (-\infty, +\infty)$, $r_x = r_0(x+1)^\lambda$. La matrice

$$Q = D_r(P - I) \tag{1.1.5}$$

(où I est la matrice identité) est la matrice de taux de transition de la chaîne en temps continu. Ceci définit un processus de Markov en temps continu que l'on notera $\overline{X}_t$. On peut remarquer que la chaîne X_n est la chaîne de Markov extraite du processus $\overline{X}_t$.
Pour un tel modèle à temps continu, la transition de x à $x+1$ se passe au taux de croissance $r_x p_x$ où $r_x p_x \sim r_0 x^\lambda$ pour x grand, tandis que la transition de x à 0 se passe au taux d'effondrement $r_x q_x$ où $r_x q_x \sim \alpha r_0 x^{\lambda - \beta}$ pour x grand. Si $\lambda > 0, \alpha x^{\lambda-\beta} \ll x^\lambda$ et si $0 < \lambda < \beta$, $\overline{X}$ monte d'une unité fréquemment, tandis que son effondrement (la survenue d'une catastrophe) devient de plus en plus rare. Si $\lambda < 0$, le taux de croissance $r_x p_x \sim r_0 x^\lambda$ est petit pour x grand tandis que le taux d'effondrement $r_x q_x \sim \alpha r_0 x^{\lambda-\beta}$ est encore plus petit. Dans tous les cas, les taux d'effondrement sont faibles par rapport aux taux de croissance.

1.1.2.b Modèle de catastrophe binomiale

Au cours d'un processus de croissance simple, un organisme modifie au hasard la quantité de ses cellules constitutives. Au cours de sa vie, l'organisme alterne au hasard des périodes d'activité et de repos. Dans une période de croissance, il produit une quantité aléatoire ou fixe de nouvelles cellules qui s'ajoutent à son stock actuel de cellules. Dans une période d'inactivité, l'organisme risque d'être soumis, par exemple, à des attaques de virus/bactéries ou à des radiations avec une certaine probabilité de mortalité fixe, indépendamment les unes des autres.

Lorsque la taille de la population est réduite selon une distribution binomiale avec une probabilité de survie c, on dira que le processus ainsi défini est un processus de catastrophe binomiale.

La dynamique de la chaîne de Markov $(X_n)_n$ dont il est question ici (est un équilibre entre les événements de naissance et de mort) est donnée comme suit :

$$X_{n+1} = \begin{cases} X_n + \beta_{n+1}, & \text{avec probabilité } p \\ c \circ X_n, & \text{avec probabilité } q = 1 - p \end{cases} \tag{1.1.6}$$

où $c \in (0,1)$ et $\beta_n, n = 1, 2, \cdots$ une suite iid (de naissance) de variable aléatoire ayant des valeurs dans $\mathbb{N}$ et $c \circ X_n = \sum_{m=1}^{X_n} b_m(c)$ avec $b_n(c), n = 1, 2, \cdots$ une suite iid de variable aléatoire de Bernoulli de paramètre c.

Remarque 1.5. *Lorsque la taille de la population vaut* 0 *à l'instant* n *c'est-à-dire* $X_n = 0$, *alors à l'instant* $n+1$ *on a :*

$$X_{n+1} = \begin{cases} \beta_{n+1}, & \textit{avec probabilité } p \\ 0, & \textit{avec probabilité } q = 1 - p. \end{cases}$$

En d'autres termes, lorsqu'il y a une extinction de la population à l'instant n, *cette extinction est locale dans ce sens où au bout d'un temps il y a des migrants qui arrivent dans la population suivant la loi* β_n.

Lorsque le paramètre de survie $c = 1$, c'est-à-dire il n'y a pas de mort dans la population alors le processus X_n est croissant et tends vers l'infini. Par contre, lorsque le paramètre de survie $c = 0$, la catastrophe est totale dans le sens où on quitte une taille de population X_n pour 0. Si on note

$$\tau_{x_0,0} = \inf\left(n \geq 1 : X_n = 0 \mid X_0 = x_0\right)$$

le premier temps d'extinction du processus X_n, alors $\mathbb{P}(\tau_{x_0,0} = x) = qp^{x-1}, x \geq 1$ et $\mathbb{E}(\tau_{x_0,0}) = 1/q$. Donc $\tau_{x_0,0}$ suit une loi géométrique avec paramètre de succès q.
Si $\beta \sim \delta_1$ alors un mouvement vers le haut se traduit par l'ajout d'un seul individu dans la population. Ce modèle constitue une version discrète simple d'un modèle continu de croissance/décroissance semi-stochastique.
Une variante du modèle binomial que nous étudions est donnée par :
Pour une variable aléatoire U choisie, étant donné $X_n \geq 1$ on a

$$X_{n+1} = \begin{cases} X_n + \beta_{n+1} & \text{avec probabilité } p \\ U \circ (X_n - 1) & \text{avec probabilité } q \end{cases}$$

où le support de $U \circ (x-1)$ est $\{0, ..., x-1\}$ avec

$$U \circ x = \sum_{i=1}^{x} B_i\left(U\right), \tag{1.1.7}$$

et $\left(B_i\left(U\right), i \geq 1\right)$ est une suite iid de Bernoulli avec $\mathbb{P}\left(B_i\left(U\right) = 1\right) = U$ aléatoire.

Remarque 1.6. *1. Pour de telles chaînes de Markov, le temps* $T \geq 1$ *écoulé entre des événements catastrophiques consécutifs emportant des vies est géométrique avec* $\mathbb{P}(T = k) = qp^{k-1}$, $k \geq 1$ *et la croissance nette* B *du processus pendant ce laps de temps est*

$$B = \sum_{m=1}^{T-1} \beta_m, \tag{1.1.8}$$

où T et $(\beta_n;\ n \geq 1)$ sont indépendants.
*2. On peut remplacer U par une Dirac et donc $U \circ (X_n - 1)$ devient $c \circ (X_n - 1)$ avec $c \in (0, 1)$. Dans ce cas, étant donné un mouvement vers le bas avec une probabilité q : **un** individu sur X_n est systématiquement retiré de la population $(X_n \to X_n - 1)$; chaque individu parmi les $X_n - 1$ restants est soumis, indépendamment des autres individus à une question de survie ou de de mort (avec une probabilité c ou avec une probabilité $1 - c$ resp.) à la génération suivante.*

1.1.2.c Modèle de croissance aléatoire avec catastrophes aléatoires

Nous considérons aussi des modèles de chaînes de Markov généralisées de naissance et de mort sur les entiers positifs comme suit. Avec $x_+ = x \vee 0 = \max(x, 0)$ et $x \wedge 0 = \min(x, 0)$, on considère la chaîne de Markov X_n avec des catastrophes dont l'évolution temporelle est donnée par :

$$X_{n+1} = \begin{cases} X_n + \beta_{n+1}, & \text{avec probabilité } p \\ (X_n - \delta_{n+1})_+ = X_n - X_n \wedge \delta_{n+1}, & \text{avec probabilité } q = 1 - p \end{cases} \tag{1.1.9}$$

où $(\delta_n)_{n \geq 1}$ est une suite i.i.d et à valeurs dans $\mathbb{N}$. Dans cette définition, un événement catastrophique doit conduire à la mort d'au moins un individu, en évitant les événements catastrophiques ne produisant aucun décès.
On peut interpréter ce modèle comme suit : à chaque étape $n + 1$, un marcheur se déplace vers le haut avec une probabilité p et l'amplitude du mouvement ascendant est β_{n+1} (comme dans les marches aléatoires standard). Avec la probabilité q, un événement catastrophique se produit (interrompant le processus de croissance), et le nombre d'individus retirés par étapes (lorsqu'ils sont disponibles) est δ_{n+1} : la distribution commune des tailles des catastrophes δ ne dépend pas de la taille actuelle de la population, mais cette diminution régulière se produit dans la mesure où il y a actuellement suffisamment d'individus à enlever. Dit autrement, étant donné que la taille de la population est égale à x à l'instant n, l'ampleur du saut vers le bas est de $x \wedge \delta_{n+1}$, tout en tronquant δ_{n+1} à x. Une hypothèse réaliste dans certaines circonstances est de supposer q petit (dans une configuration d'événements catastrophiques rares : $q \ll p$).

1.1.3 Le modèle à temps continu et à valeurs réelles

1.1.3.a Modèle de croissance et de décroissance déterministe avec catastrophes aléatoires

Si la seule composante stochastique d'un processus de Markov X_t est donnée par ses sauts et entre les sauts, il y a un comportement déterministe, alors on l'appelle processus de Markov déterministe par morceaux (PDMP). Nous étudions ici une classe particulière de processus de Markov déterministes par morceaux (PDMP) X_t comme dans Boxma et al. (2006), Eliazar and Klafter (2006b, 2007), Gripenberg (1983, 1985) et Hanson and Tuckwell (1978) qui sont des versions catastrophes semi-stochastiques des modèles de population de type 'growth-collapse' (resp. 'decay-surge') où une évolution déterministe croissante (resp. décroissante) suivant un flot non linéaire est interrompue par des sauts de taille aléatoire survenant à des moments aléatoires.

Une note concernant la notation : Tout au long de l'introduction, quand les signes $\pm/\mp$ apparaissent, le premier signe renvoie au système growth-collapse et le second signe au système decay-surge.

L'évolution de la taille de la population a une dynamique déterministe qui est donnée par

$$\dot{x}_t(x) = \pm\alpha(x_t(x)), x_0 = x$$

où x est la taille de la population initiale et la dérive α une fonction continue et positive sur $(0, \infty)$. En biologie, le cas $\alpha(0) > 0$ se produira s'il y a une immigration dans la population. En prenant $\beta(x)$ une fonction positive et continue sur $(0, \infty)$ représentant le taux de saut du processus ayant pour état actuel x, le taux de saut du processus ne dépend que de sa position actuelle.

Modèles déterministes de population. Soit $x_t > 0$ la taille d'une certaine population au temps $t \geq 0$, ayant pour état initial $x := x_0 > 0$. Avec α_1, $a > 0$, considérons la dynamique de croissance

$$\dot{x}_t = \alpha_1 x_t^a, \, x_0 = x, \tag{1.1.10}$$

pour une certaine dérive $\alpha(x) := \alpha_1 x^a$. Notons que ce $\alpha(x)$ est croissant avec x**.** En intégrant lorsque $a \neq 1$ (le cas non linéaire), on obtient formellement

$$x_t(x) = \left(x^{1-a} + \alpha_1 (1-a)\, t\right)^{1/(1-a)}, \, x > 0. \tag{1.1.11}$$

En général, $\alpha(x)$ est supposé continu sur $[0, \infty)$, positif sur $(0, \infty)$. Alors

$$\int_x^{x_t(x)} \frac{dy}{\alpha(y)} = t.$$

On définit, $I_\infty(x) := \int_x^\infty \frac{dy}{\alpha(y)}$ comme le temps nécessaire pour atteindre l'état ∞ en partant d'un certain état x.
Si pour $x > 0$, $I_\infty(x) = \int_x^\infty \frac{dy}{\alpha(y)} < \infty$, (i.e, l'état ∞ est accessible), alors

$$x_t(x) = I_\infty^{-1}\left(I_\infty(x) - t\right),$$

et en général, si ce n'est pas le cas,

$$x_t(x) = I^{-1}\left(I(x) + t\right)$$

où $I(x) = \int^x \frac{dy}{\alpha(y)}$ est une intégrale indéterminée.

Example 1.4. Avec μ, $a > 0$, considérons la dynamique
$(\alpha(x) = \mu(1+x)(\log(1+x))^a)$

$$\dot{x}_t = \mu(1+x_t)(\log(1+x_t))^a, \, x_0 = x > 0. \tag{1.1.12}$$

En introduisant $z_t = \log(1+x_t)$ et $z = \log(1+x)$, z_t obéit à (1.1.10) avec la condition initiale z. En intégrant (1.1.12), on obtient formellement si $a \neq 1$

$$x_t(x) = \exp\left((\log(1+x))^{1-a} + \mu(1-a)\, t\right)^{1/(1-a)} - 1, \tag{1.1.13}$$

Nous concluons :

• Si $0 < a < 1$: la croissance de x_t est exp-algébrique au taux algébrique $1/(1-a) > 1$.

• Si $a > 1$: la taille de la population x_t atteint l'infini en temps fini $I_\infty(x) = \frac{(\log(1+x))^{1-a}}{\mu(a-1)}$. On obtient

$$x_t(x) = (1+x)^{\left(1-\frac{t}{I_\infty(x)}\right)^{1/(1-a)}} - 1.$$

• Si $a = 1$: alors (1.1.12) a une solution super-exponentielle $x_t(x) = (1+x)^{e^{\mu t}} - 1$ pour $t \geq 0$. La croissance se produit à un rythme super-exponentiel (ou double exponentiel). Avec $I(x) = \int^x \frac{dy}{\mu(1+y)\log(1+y)} = \frac{1}{\mu}\log(\log(1+x))$, on peut vérifier que

$$x_t(x) = I^{-1}(I(x) + t).$$

Modèles déterministes incluant l'immigration. On considère dans ce paragraphe deux cas incluant l'immigration, i.e., $\alpha_0 > 0$.

1. $\alpha(x) = \alpha_0 + \alpha_1 x^a$ (taux d'immigration constant α_0)
2. $\alpha(x) = \alpha_0 x + \alpha_1 x^a$ (taux d'immigration linéaire α_0).

Cas 1. : La solution de $\dot{x}_t = \alpha(x_t) = \alpha_0 + \alpha_1 x_t^a$, $x_0 = x$ est donnée par $x_t(x) = I^{-1}(I(x) + t)$ où

$$I(x) = \int^x \frac{dy}{\alpha_0 + \alpha_1 y^a} = \frac{x}{\alpha_0} F\left(1, \frac{1}{a}, \frac{1}{a}+1; -\frac{\alpha_1}{\alpha_0}x^a\right)$$

impliquant la fonction hypergéométrique de Gauss $F(a,b,c;z)$ où $F(a,b,c;z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!}$ et $(a)_n = a(a+1)\cdots(a+n-1)$. Lorsque $a = 1$,

$$x_t(x) = xe^{\alpha_1 t} + \frac{\alpha_0}{\alpha_1}\left(e^{\alpha_1 t} - 1\right), \tag{1.1.14}$$

correspondant à un modèle de croissance de Malthus renforcé par l'immigration.

Dans ce cas, $I_\infty(x) < \infty \Leftrightarrow a > 1$ (l'état ∞ est accessible en un temps fini).

Cas 2. : La solution à $\dot{x}_t = \alpha(x_t) = \alpha_0 x_t + \alpha_1 x_t^a$, $x_0 = x$ est :

$$x_t(x) = e^{\alpha_0 t}\left(x^{1-a} + \frac{\alpha_1}{\alpha_0}\left(1 - e^{-(1-a)\alpha_0 t}\right)\right)^{1/(1-a)},$$

pour tout $a \neq 1$. Lorsque $a = 1$, $x_t(x) = xe^{(\alpha_0+\alpha_1)t}$

Dans ce cas, $I_\infty(x) < \infty \Leftrightarrow a > 1$ (l'état ∞ est accessible en temps fini $I_\infty(x) = \frac{1}{(a-1)\alpha_0}\log\left(1 + \frac{\alpha_0}{\alpha_1}x^{1-a}\right)$).

Des modèles de croissance à des modèles de décroissance de population. Un simple changement de temps permet de passer des modèles de population de croissance à ceux de décroissance.

Si $\alpha(x) \to \overline{\alpha}(x) := -\alpha(x)$ où α est positif, alors la taille de la population ayant la dynamique $\dot{x}_t = -\alpha(x_t)$, $x_0 = x$ se réduit au fur et à mesure que le temps passe. Le flot de tels modèles de décroissance s'obtient simplement en effectuant la substitution $t \to -t$ dans $x_t(x)$ avec un taux de croissance $\alpha(x)$.

Example 1.5. 1. Si $\alpha(x) = \alpha_1 x^a$ et $\alpha(x) = \alpha_0(1+x)\log^a(1+x)$, sont les dérives associées aux flots $x_t(x) = (x^{1-a} + \alpha_1(1-a)t)^{1/(1-a)}$, $x > 0$ respectivement $x_t(x) = \exp\left((\log(1+x))^{1-a} + \mu(1-a)t\right)^{1/(1-a)} - 1$, alors

$$\begin{aligned} x_t(x) &= \left(x^{1-a} + \alpha_1(a-1)t\right)^{1/(1-a)} \text{ et} \\ x_t(x) &= \exp\left((\log(1+x))^{1-a} + \alpha_0(a-1)t\right)^{1/(1-a)} - 1, \end{aligned}$$

sont les flots associés à $\overline{\alpha}(x) = -\alpha_1 x^a$ et $\overline{\alpha}(x) = -\alpha_0(1+x)(\log(1+x))^a$.
Lorsque $a < 1$, chaque flot de ce type touche l'état 0 en temps fini.

2. Le modèle de population décroissante obtenu en inversant le temps du modèle de Malthus avec l'immigration (1.1.14) est le suivant

$$x_t(x) = \left(x + \frac{\alpha_0}{\alpha_1}\right) e^{-\alpha_1 t} - \frac{\alpha_0}{\alpha_1}.$$

Il correspond à un modèle à décroissance exponentielle renforcé par une émigration à taux constant ($\overline{\alpha}(x) = -\alpha_0 - \alpha_1 x$). Il atteint l'état 0 en un temps fini $t_0(x) = \frac{1}{\alpha_1}\log\left(\frac{\alpha_0+\alpha_1 x}{\alpha_0}\right)$.

Pour une population en déclin générée par $\overline{\alpha}(x) = -\alpha(x)$, avec $0 \le a < x$, l'intégrale

$$t_a(x) := \int_a^x \frac{dy}{\alpha(y)} \tag{1.1.15}$$

est le temps nécessaire au flot pour atteindre l'état a en partant de l'état x.
Si $t_0(x) := \int_0^x \frac{dy}{\alpha(y)} < \infty$, le flot déterministe $x_t(x)$ touche l'état 0 en temps fini et

$$x_t(x) = t_0^{-1}(t_0(x) - t).$$

En général,

$$x_t(x) = t^{-1}(t(x) + t)$$

où $t(x) := -\int^x \frac{dy}{\alpha(y)}$, est une intégrale indéterminée.

Les modèles incluant l'aléa. Aux instants de saut, la taille de la population dans un modèle de type 'growth-collapse' augmente (resp. dans un modèle de type 'decay-surge' diminue) d'une quantité aléatoire $\Delta(X_{t-}) \in (0, X_{t-}]$ (resp. $\Delta(X_{t-}) \in (0, \infty)$) de sa taille actuelle. Le PDMP $(X_t)_{t\ge 0}$ qui suit le flot déterministe ayant pour dérive $\alpha(x)$, et pour fonction de saut $\beta(x)$ a pour dynamique :

$$dX_t = \pm\alpha(X_{t-})dt \mp \Delta(X_{t-})\int_0^\infty \mathbb{1}_{\{z\le\beta(X_{t-})\}} M(dt,dz),\ X_0 = x. \tag{1.1.16}$$

Dans cette équation, $M(dt,dz)$ est une mesure de Poisson aléatoire sur $[0,\infty)\times[0,\infty)$ avec intensité $dtdz$.
Posons $Y(X_{t-}) := X_{t-} - \Delta(X_{t-})$ (resp. $Y(X_{t-}) := X_{t-} + \Delta(X_{t-})$ la position de X_t après le saut sachant que le processus était à x. On a donc :

$$\mathbb{P}(Y(x) \le y \mid X_- = x) = H(x,y)$$

respectivement,

$$\mathbb{P}(Y(x) > y \mid X_- = x) = K(x,y),$$

où H (resp. K) est le noyau de saut donnant la loi de l'amplitude du saut.

On sait que les modèles de type 'growth-collapse' décrivent une croissance déterministe de la population où à des moments de saut aléatoires, la taille de la population subit une catastrophe réduisant sa taille actuelle à une fraction aléatoire de celle-ci. Et donc plus précisément, le générateur infinitésimal d'un processus de type 'growth-collapse', ayant des paramètres (α, β, H) ; est donné pour toutes fonctions test lisses u par

$$Gu(x) = \alpha(x)u'(x) + \beta(x)\int_0^x H(x, dy)[u(y) - u(x)],\ x \geq 0. \tag{1.1.17}$$

Une relation naturelle entre les modèles de population de type 'decay-surge' et les modèles de population de type 'growth-collapse' est donnée par $\tilde{X}_t = 1/X_t$ où $\tilde{X}_t$ et X_t sont les processus de decay-surge et de growth-collapse. Plus précisément, le générateur infinitésimal de ce processus de type 'decay-surge', ayant des paramètres $(\tilde{\alpha}, \tilde{\beta}, K)$ est donnée par :

$$Gu(x) = -\tilde{\alpha}(x)u'(x) + \tilde{\beta}(x)\int_x^\infty K(x, dy)[u(y) - u(x)],\ x \geq 0 \tag{1.1.18}$$

avec $\tilde{\alpha}(x) = x^2\alpha(1/x),\ \tilde{\beta}(x) = \beta(1/x)$ et $K(x, y) = H(\frac{1}{x}, \frac{1}{y})$.

Remarque 1.7. *Dorénavant, les notations $\alpha(x),\ \beta(x)$ et $\Gamma(x)$ seront indépendantes du type de modèle utilisé.*

Le premier instant de saut du processus est donné par

$$T_x = \inf\{t > 0 :\ X_t \neq X_{t-} | X_0 = x\}.$$

La probabilité que le premier instant de saut du processus dépasse un temps $t > 0$ est égale à :

$$\mathbb{P}(T_x > t) = e^{-\int_0^t \beta(x_s(x))ds},\ \text{pour tout } t \geq 0$$

Tant que $t < I_\infty(x)$, (resp. $t < t_0(x)$) en faisant un changement de variables on obtient $\mathbb{P}(T_x > t) = e^{\mp[\Gamma(x_t(x)) - \Gamma(x)]}$, pour tout $t \geq 0$ avec $\Gamma(x)$ la primitive de $\gamma(x) = \beta(x)/\alpha(x)$.
La question naturelle qu'on se pose est de savoir sous quelle condition le premier instant de saut du processus est fini. Cette condition est donnée par :
- Dans le cas growth-collapse : $\Gamma(\infty) = \infty$
- Dans le cas decay-surge : $\Gamma(0) = -\infty$.

Dans le cas decay-surge, en observant que $t_y(x) := \int_y^x \frac{dz}{\alpha(z)}$ est le temps nécessaire au flot déterministe pour atteindre y pour la première fois en partant de $x > y$, les événements '$X_{T_{x-}(x)} \leq y$' et '$T_x > t_y(x)$' coïncident où la quantité est $X_{T_{x-}}$, est l'état de X_t juste avant son premier saut. Par conséquent,

$$\mathbb{P}\left(X_{T_{x-}} \leq y\right) = \mathbb{P}\left(T_x > \int_y^x \frac{dz}{\alpha(z)}\right) = e^{-[\Gamma(x) - \Gamma(y)]},\ y < x$$

c'est-à-dire que la probabilité qu'au moment du premier saut, le processus soit en deçà d'un certain seuil $y < x$ coïncide avec la probabilité que le premier saut se réalise au-delà $t_y(x)$.
En utilisant la relation entre les deux processus, on a dans le cas growth-collapse

$$\mathbb{P}\left(X_{T_{x-}} \geq y\right) = e^{-[\Gamma(y) - \Gamma(x)]},\ y > x$$

Sous la condition que $\mathbb{P}(T_x < \infty) = 1$ c'est-à-dire que le premier instant de saut du processus est fini presque sûrement, on obtient

- Dans le cas growth-collapse :

si $\Gamma(\infty) = \infty$, $\mathbb{P}(X_{T_{x-}} < \infty) = 1$ et si $\Gamma(\infty) < \infty$, $\mathbb{P}(X_{T_{x-}} = \infty) = e^{-(\Gamma(\infty)-\Gamma(x))}$

- Dans le cas decay-surge :

si $\Gamma(0) = -\infty$, $\mathbb{P}(X_{T_{x-}} > 0) = 1$ et si $\Gamma(0) > -\infty$, $\mathbb{P}(X_{T_{x-}} = 0) = e^{-(\Gamma(0)-\Gamma(x))}$.

La question du temps d'atteinte de l'état ∞ en temps fini se pose dans les modèles de type growth-collapse et par dualité des processus, celle du temps d'atteinte de l'état 0 en temps fini se pose dans les modèles de type decay-surge.

Le fait que $\Gamma(\infty) = \infty$ (resp. $\Gamma(0) = -\infty$) assure que le processus stochastique $X_t(x),\ x > 0$ saute nécessairement avant de toucher l'état ∞ (resp. l'état 0) dans le cas growth-collapse (resp. decay-surge). En particulier, pour $x > 0,\ X_t(x)$ n'atteint presque sûrement jamais l'état ∞ (resp. l'état 0) en temps fini.

Example 1.6. Nous donnons ici un exemple dans lequel le processus peut atteindre l'état ∞ en temps fini ou en temps infini suivant différents cas.

Pour cela, on rappelle que

$$\mathbb{P}(X_{T_{x-}} > y) = e^{(\Gamma(y)-\Gamma(x))},\ \text{si } y > x \text{ dans le cas growth-collapse.}$$

On pose $\alpha(x) = \alpha e^{-ax}$ et $\beta(x) = \beta$ avec $\alpha, \beta > 0$. $\Gamma(x) = \lambda e^{ax}$ avec $\lambda = \beta/(\alpha a)$ si $a \neq 0$ et $\lambda = \beta/\alpha$ si $a = 0$.

En changeant les signes du paramètre a, on constate que le processus subit une 'transition de phase' dans le sens de Eliazar and Klafter (2006b).

- Si $a > 0$ (cas sous-critique), $\Gamma(x) = \lambda e^{ax}$ et on a :

$$\mathbb{P}(X_{T_{x-}} > y) = e^{-\lambda(e^{ay}-e^{ax})}.$$

  Donc, l'état du processus avant le premier saut est toujours fini, c'est-à-dire que le processus ne peut pas atteindre l'état infini en temps fini.

- Si $a < 0$ (cas surcritique), $\Gamma(x) = -|\lambda|e^{-|a|x}$ et on a

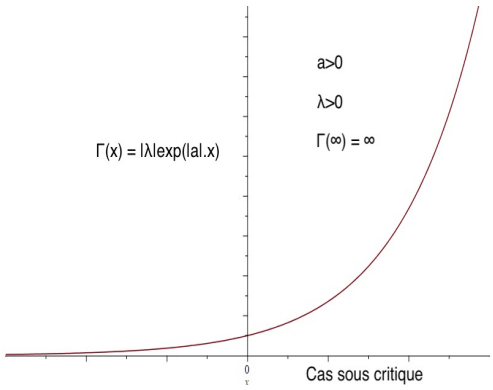
$$\mathbb{P}(X_{T_{x-}} > y) = e^{|\lambda|(e^{-|a|y}-e^{-|a|x})}.$$

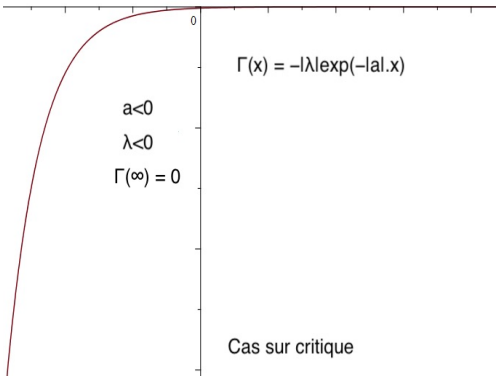
  Dans ce cas, avec une probabilité positive le processus touche l'état ∞ avant son premier saut et cette probabilité est donnée par :

$$\mathbb{P}(X_{T_{x-}} = \infty) = e^{-|\lambda|e^{-|a|x}} > 0.$$

En utilisant la dualité* reliant les deux processus, on s'attend à ce que dans le cas decay-surge il y ait aussi une transition de phase. Mais dans le cas decay-surge, le changement des signes du paramètre a ne permet pas au processus de subir une transition de phase. Quel que soit $a \in \mathbb{R}$, le processus stochastique peut toucher l'état 0 en temps fini.

∗. la dualité dont on parle s'applique uniquement quand le processus n'a pas encore atteint l'état 0.





Lorsqu'elle existe, la densité de la mesure invariante $\pi(x)$ du processus est solution de l'équation fonctionnelle suivante.
- Dans le cas growth-collapse :

$$\tilde{\pi}(x) = \int_x^\infty \gamma(z) H(z,x) \tilde{\pi}(z) dz$$

- Dans le cas decay-surge :

$$\tilde{\pi}(x) = \int_0^x \gamma(z) K(z,x) \tilde{\pi}(z) dz$$

où $\tilde{\pi}(x) = \alpha(x)\pi(x)$ et $\gamma(x) = \beta(x)/\alpha(x)$. Cette équation fonctionnelle a toujours une solution lorsque le processus est Harris récurrent.
Dans le cas séparable i.e. $H(x,y) = h(y)/h(x)$ (resp. $K(x,y) = k(y)/k(x)$), où h (resp. k) est une fonction continue et croissante (resp. décroissante), la densité de la mesure invariante du processus a une expression explicite qui est donnée par

$$\pi(x) = C \frac{h(x)}{\alpha(x)} e^{\mp \Gamma(x)} \tag{1.1.19}$$

où $h(x)$ peut-être remplacé par $k(x)$ et $\Gamma(x)$ est la primitive de $\gamma(x) = \beta(x)/\alpha(x)$.

Lorsque le processus $(X_t)_t$ est déterministe entre deux instants de sauts, la connaissance des instants de sauts du processus $(X_t)_t$ permet de le décrire sous condition de connaître le flot déterministe. On est donc naturellement amenés à travailler sur les chaînes extraites (embedded chain) aux instants juste avant les sauts et aux instants de (après) sauts du processus. On note $U_n = X_{S_n-}$ la chaîne extraite aux instants juste avant les sauts et $Z_n = X_{S_n}$ la chaîne extraite aux instants de sauts. Si l'état 0 n'est pas absorbant pour tout $x \geq 0$, alors dans le cas growth-collapse,

$$\mathbb{P}(U_n \in dy \mid U_{n-1} = x) = \int_0^x H(x, dz) \int_0^\infty dt \beta(x_t(z)) e^{-\int_0^t \beta(x_s(z))ds} \delta_{x_t(z)}(dy)$$

et

$$\mathbb{P}(Z_n \in dy \mid Z_{n-1}) = \int_0^\infty dt \beta(x_t(x)) e^{-\int_0^t \beta(x_s(x))ds} H(x_t(x), dy)$$

De manière analogue, on peut retrouver ces probabilités de transition dans le cas decay-surge.
Ces chaînes ainsi définies sont des chaînes de Markov discrètes. La proposition suivante donne la relation entre la mesure invariante du processus et celle des chaînes extraites du processus.

Proposition 1.2. *Supposons que le processus X est Harris récurrent et sa mesure invariante π est telle que $0 < \pi(\beta) < \infty$. Alors $(U_n)_n$ et $(Z_n)_n$ sont aussi Harris récurrents. Leurs mesures invariantes π^U et π^Z valent pour chaque fonction mesurable et bornée $g : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, :*

$$\pi^U(g) = \frac{1}{\pi(\beta)}\pi(\beta g) \text{ et } \pi^Z(g) = \frac{1}{\pi(\beta)}\pi(\beta H g)$$

où $\beta H g(x) = \beta(x) \int H(x, dy) g(y)$ avec H qui peut-être remplacé par K dans le cas decay-surge.

Pour finir cette partie de la description des modèles, nous allons changer de point de vue de modélisation.

- Au lieu de décrire des tailles de populations, nous allons nous intéresser à la modélisation de l'état de neurones ;
- On interprète les catastrophes comme des instants de décharges de neurones (instants de spike).
- On met ces processus en réseau avec un nombre (fini ou infini) de neurones.

1.1.3.b Le réseau de neurones

Avec plus de 100 milliards de neurones, le cerveau est de loin le composant de notre corps qui présente une telle concentration de neurones. Chaque neurone peut recevoir 10000 à 100000 décharges par unité de temps d'autres neurones voisins, plus précisément, les neurones présynaptiques (le neurone situé avant la synapse) envoient des décharges à leurs partenaires postsynaptiques (le neurone situé en aval de la synapse). Selon qu'il soit excitateur ou inhibiteur, le neurone présynaptique va imposer deux comportements opposés au neurone postsynaptique :

1. un neurone excitateur qui décharge va activer le neurone postsynaptique ;
2. un neurone inhibiteur qui décharge va empêcher le neurone postsynaptique de décharger.

Ces décharges sont appelées potentiels d'action ou spike. Dans notre étude, on ne considérera pas le neurone dans sa complexité, mais on s'intéressera à la modélisation de l'état d'inhibition d'un neurone. Par exemple on peut modéliser l'état d'inhibition d'un neurone par le temps d'attente du prochain spike et plus ce temps est long plus l'inhibition est grande.

Nous étudions comme dans Cottrell (1992) et Fricker et al. (1993) un modèle de réseau de neurones en interaction effectuant chacun un saut avec un taux de saut. Soit $\alpha_i : \mathbb{R}_+ \to \mathbb{R}$ et $\beta_i : \mathbb{R}_+ \to \mathbb{R}_+$ deux fonctions déterministes. Pour $i \in \{1, \cdots, N\}$, $X_t^{i,N}$ modélise l'état d'inhibition du neurone i au temps t dans le réseau. Ce neurone émet des spikes (émet des potentiels d'action) à des instants aléatoires. Le taux de saut du neurone i au temps t est $\beta_i(X_t^{i,N})$: il ne dépend que de l'état d'inhibition du neurone i. Quand le neurone i émet un spike au temps t alors son état d'inhibition est remplacé par Y^i qui suit la loi F^i et tous les autres neurones reçoivent le poids d'inhibition $W_{i\to j} \geq 0$ pour tout $j \neq i$, c'est-à-dire :

$$\forall j \neq i,\ X_t^{j,N} = X_{t-}^{j,N} + W_{i\to j}.$$

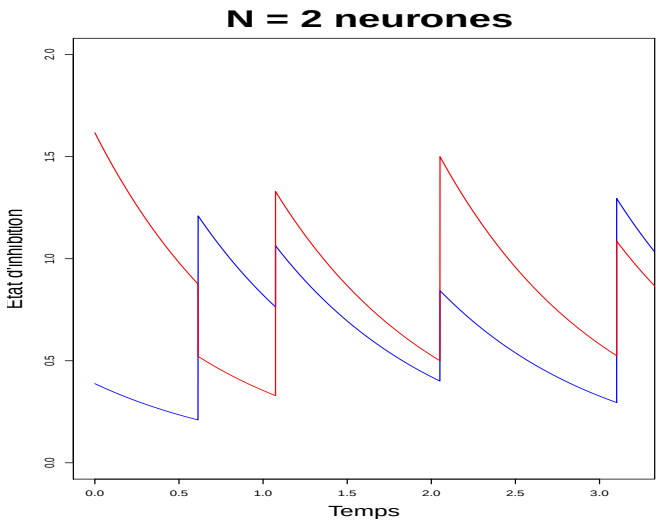
Entre deux sauts successifs du processus, chaque neurone i suit la dynamique déterministe

$$\dot{x}_t^i = -\alpha_i\left(x_t^i\right),\ x_0^i = x^i,$$

où la fonction de dérive α_i décrit comment l'état d'inhibition évolue entre deux sauts successifs.
Pour chaque $N \in \mathbb{N}$, considérons $X_t^N = (X_t^{1,N}, \cdots, X_t^{N,N}) \in \mathbb{R}_+^N$ comme le processus de Markov déterministe par morceaux (PDMP) associé aux états d'inhibition d'un neurone. $X_t^{i,N}$ a pour dynamique :

$$dX_t^{i,N} = -\alpha_i\left(X_{t-}^{i,N}\right)dt + \int_0^\infty \int_0^\infty (y^i - X_{t-}^{i,N})\mathbb{1}_{\left\{r\leq\beta_i\left(X_{t-}^{i,N}\right)\right\}} M^i\left(dt, dr, dy^i\right) + \sum_{j\neq i} W_{j\to i}\int_0^\infty \int_0^\infty \mathbb{1}_{\left\{r\leq\beta_j\left(X_{t-}^{j,N}\right)\right\}} M^j\left(dt, dr, dy^j\right), \quad (1.1.20)$$

où M^i est une mesure de Poisson aléatoire d'intensité $dtdrF^i(dy)$ et pour tout i, les M^i sont tous indépendants.
La simulation suivante est une trajectoire typique de $X_t^{i,N}$ avec $N = 2$ neurones, $\alpha(x) = x,\ \beta(x) = 2e^{-x},\ F^i \sim \exp(1)$ et $W_{j\to i} = 1$ pour tout $j \neq i$. Lorsqu'un des neurones émet un spike, l'autre augmente de 1. Dans cet exemple, c'est le neurone 2 (couleur rouge) qui fait le premier spike et il donne $W_{2\to 1} = 1$ au neurone 1 (couleur bleue). Au bout d'environ 0.5 unité de temps, le neurone 1 fait à son tour un spike et donne $W_{1\to 2} = 1$ au neurone 2 et ainsi de suite.



Pour résumer, le modèle est paramétré par la fonction de dérive α_i, la fonction de taux β_i et les poids d'inhibition $W_{i\to j}$. On s'intéresse aux propriétés de la récurrence de ce processus dans le cas d'un système fini et d'un système infini. Dans le cas de système infini, on cherche à avoir une idée de la distribution théorique de l'état du processus dans son régime stationnaire.

1.2 État de l'art

Dans cette section, un aperçu non exhaustif est donné sur des travaux qui ont déjà été réalisés et sur lesquels se basent notre travail.

Le cas d'une population ou d'une particule

Dans Neuts (1994), l'auteur considère un modèle dont la dynamique peut-être décrite comme suit. Un marcheur se déplace aléatoirement sur l'ensemble des entiers positifs. Si le marcheur est à $i \geq 0$ à l'instant n, il se déplace à l'instant $n+1$, avec une probabilité p, vers un emplacement choisi au hasard dans l'ensemble $\{0, 1, ..., i\}$. Ou bien, avec la probabilité qr_k, il se déplace de $k \geq 1$ pas vers la droite. La suite (r_k) est une densité de probabilité sur l'ensemble des entiers positifs et $0 < p = 1-q < 1$. Dans le modèle décrit dans Neuts (1994), les emplacements $(X_n)_n$ du marcheur aux instants successifs forment une chaîne de Markov dont la matrice de transition P est donnée par :

$$P_{i,j} = \begin{cases} \frac{p}{i+1}, & \text{si } i \geq 0, j \leq i \\ qr_{j-i}, & \text{si } i \geq 0, j > i. \end{cases}$$

L'auteur montre que si la densité de la probabilité invariante $\{\pi_i\}$ existe, alors sa fonction génératrice des probabilités $\Pi(z) = \sum_{k\geq 0} \pi_k z^k$ satisfait l'équation

$$(1-z)[1-qR(z)]\Pi(z) = \pi_0 - p\int_0^z \Pi(u)du$$

où $R(z)$ est la fonction génératrice des probabilités de la densité r_k. Dans le cas où π_i existe, l'auteur montre que la chaîne de Markov $(X_n)_n$ est récurrente positive si et seulement si

$$\sum_{k=1}^{\infty} \log kr_k < \infty.$$

L'auteur s'intéresse ensuite à une variante binomiale du modèle décrit précédemment. La matrice de transition P de ce modèle est donnée par :

$$P_{ij} = pC_i^j(1-\beta)^{i-j}\beta^j, \quad \text{pour } 0 \leq j \leq i,$$

pour tout $i \geq 0$ et β une constante satisfaisant $0 < \beta < 1$. Ce qui rend intéressant ce modèle, c'est que la condition de récurrence positive, les expressions analytiques pour les probabilités stationnaires et les arguments mathématiques nécessaires dépendent fortement de la forme spécifique des transitions du modèle.

Le modèle décrit dans Huillet (2011) est une version particulière du modèle décrit par 1.1.3. La chaîne de Markov $X := (X_n)_n$ utilisée dans Huillet (2011) a pour transition : si $X_n = x \in \{1, 2, \cdots\}$ alors

$$X_{n+1} = \begin{cases} x+1 & \text{avec probabilité } p_x = e^{\beta/x} \\ 0 & \text{avec probabilité } q_x = 1 - e^{\beta/x} \sim \beta/x \end{cases}$$

Dans Huillet (2011), l'auteur travaille dans un cas où, si à un instant n le marcheur vient à l'origine alors à l'instant $n+1$ il quitte l'origine avec une probabilité 1. Cela se traduit par le fait que si

$$X_n = 0 \text{ alors } X_{n+1} = 1 \text{ avec probabilité } p_0 = 1.$$

Or en utilisant 1.1.4, si à un instant n le marcheur vient à l'origine alors à l'instant $n+1$ il quitte l'origine avec une probabilité $p_0 \leq 1$ et avec une probabilité $q_0 = 1-p_0$ il reste à l'origine pour toujours. L'auteur montre également dans le cas récurrent positif que si à l'instant n, τ_n est le temps séparant la chaîne $(X_n)_n$ de la prochaine visite à 0, alors $\tau_n \xrightarrow{d} \tau$ (convergence en loi) lorsque $n \to \infty$ avec

$$\mathbb{P}(\tau = k) = \frac{k\mathbb{P}(\tau_{0,0} - 1 = k)}{\mathbb{E}(\tau_{0,0})}, \ k \geq 1.$$

Dans Artalejo et al. (2007), un modèle de population avec le processus d'immigration soumis à des catastrophes binomiales et géométriques a été étudié. Les auteurs ont obtenu la loi de la taille de la population après catastrophe et la loi de la taille maximale de la population entre deux extinctions (hauteur des excursions). Ils ont également étudié le temps de première extinction et le temps de survie d'un individu donné.

Dans Gripenberg (1983, 1985), l'auteur considère un modèle où la dérive $\alpha(x)$, le taux de saut $\beta(x)$ et la loi du saut $H(x,y)$ du processus sont généraux. Dans le modèle décrit dans Gripenberg (1983) la loi du saut n'est pas séparable. L'auteur se place sous l'hypothèse qu'il existe $x_0 \in [0,\infty)$ tel que, pour tout $y \in I_0 = (0,x_0)$, le temps qu'il faut au processus partant de y pour atteindre x_0 est infini. Si de plus, $I = [0,\infty)$ est le support d'une mesure invariante alors la densité f de cette dernière existe et vérifie l'équation

$$\lim_{t\to\infty} P_t(x,y) = \int_I f(z)P_s(z,y)dz,\ \ s>0,\ \ y \in I_0$$

où $P_t(x,y) = \mathbb{P}(X_t \leq y \mid X_0 = x)$. Cette mesure n'est pas explicite dans le cas où la loi du saut n'est pas séparable.

Le modèle décrit par l'équation (1.1.16) est une version assez générale des modèles décrits dans Eliazar and Klafter (2006b, 2007). Dans leur papier, la dynamique s'écrit comme suit :

$$x \mapsto \begin{cases} x + F(x)\Delta t & \text{avec prob. } 1 - R(x)\Delta t, \\ Mx & \text{avec prob. } R(x)\Delta t, \end{cases} \tag{1.2.1}$$

où F et R correspondent respectivement à la dérive α et à la fonction de saut β de l'équation (1.1.16). La dérive F est continue et positive et $0 < M < 1$ dans le cas growth-collapse, et F est continue et négative et $M > 1$ dans le cas decay-surge. C'est une version particulière de notre modèle avec $H(x,y) = \mathbb{P}(M \leq y/x)$ (resp. $K(x,y) = \mathbb{P}(M \geq y/x)$). En effet, dans (1.1.16), avec la probabilité de saut, le processus effectue un saut de x à $x \mp \Delta(x)$ et dans (1.2.1) le saut se fait de x à Mx. Le générateur infinitésimal associé à ce modèle dans le cas growth-collapse est donné par

$$Gu(x) = F(x)u'(x) + R(x)\int_0^x \mathbb{P}(Mx \in dy)[u(y) - u(x)],\ \ x \geq 0 \tag{1.2.2}$$

où F est continue et positive et l'expression du générateur associé aux modèles de type decay-surge est

$$Gu(x) = F(x)u'(x) + R(x)\int_x^\infty \mathbb{P}(Mx \in dy)[u(y) - u(x)],\ \ x \geq 0. \tag{1.2.3}$$

où F est continue et négative.
Dans Eliazar and Klafter (2006b), les auteurs travaillent dans le cas particulier où pour tout $\gamma > 0$, $\mathbb{P}(M < m) = m^\gamma$, avec $0 < m < 1$ dans le cas growth-collapse et $\mathbb{P}(M > m) = m^{-\gamma}$, avec $m > 1$ dans le cas decay-surge. Cette condition est équivalente dans Goncalves et al. (2020, 2022) au cas séparable où $h(x) = x^\gamma$ et $k(x) = x^{-\gamma}$ respectivement. En effet, dans Goncalves et al. (2020, 2022), la probabilité d'aller

d'un état x à un seuil y est donnée par :
Dans le cas Goncalves et al. (2022) i.e. growth-collapse,

$$\begin{aligned} H(x,y) &= \mathbb{P}(X_t \leq y | X_{t-} = x) = \mathbb{P}(\Delta(x) \geq x - y), \forall y \leq x \\ &= \mathbb{P}(M < y/x) = (\frac{y}{x})^{\gamma} = \frac{y^{\gamma}}{x^{\gamma}} = \frac{h(y)}{h(x)}, \text{ avec } h(x) = x^{\gamma}. \end{aligned}$$

De manière analogue, on obtient dans le cas decay-surge

$$K(x,y) = \mathbb{P}(\Delta(x) \leq y - x) = \mathbb{P}(Mx > y) = \frac{y^{-\gamma}}{x^{-\gamma}} = \frac{k(y)}{k(x)}, \text{ avec } k(x) = x^{-\gamma}.$$

Dans ce cadre particulier, les auteurs s'intéressent à différents cas de dérive et de taux de saut. Dans un premier cas, ils considèrent $\alpha(x) = \pm a x^{\alpha}$, $\beta(x) = b x^{\pm\beta}$ et dans un second cas le ratio $\beta(x)/\alpha(x) = \pm c x^{\delta}$ où $\delta \in \mathbb{R}$ et $c > 0$. Sous la condition $\Gamma(\infty) = \infty$ (resp. $\Gamma(0) = -\infty$) que le processus n'atteint pas l'infini en temps fini (resp. n'atteint pas 0 en temps fini) les résultats dans Eliazar and Klafter (2006b) et dans Goncalves et al. (2020, 2022) sont équivalents concernant le générateur infinitésimal et la mesure invariante du processus. Il suffit de remplacer la loi de Mx qui est donnée par $\gamma x^{\mp\gamma} m^{\pm\gamma-1}$ par $H(x, dy)$ dans le cas growth collapse et par $K(x, dy)$ dans le cas decay surge pour retrouver les expressions des générateurs (1.1.17) et (1.1.18). On retrouve également l'expression de la densité de la mesure invariante $\pi(x)$ du processus en remplaçant $x^{\pm\gamma}$ par $h(x)$ ou par $k(x)$.

Soit T_x le premier temps de saut de $X_t(x)$ partant de x et, soit $Z_x = X_{T_x}(x)$ sa position. Les questions suivantes ont été posées par Eliazar and Klafter (2006b)[pages 114-115] :
- Dans le cas growth-collapse : si $x_t(x)$ atteint l'état ∞ en temps $I_{\infty}(x) < \infty$, se peut-il que cet événement ('explosion' i.e. atteinte de l'infini) se produise avant T_x?
Puisque

$$\mathbb{P}(T_x > t) = e^{-(\Gamma(x_t(x)) - \Gamma(x))}, \ \forall t < I_{\infty}(x)$$

on a

$$\mathbb{P}(T_x \geq I_{\infty}(x)) = e^{-(\Gamma(\infty) - \Gamma(x))}$$

et cette probabilité ne peut être positive que si $\Gamma(\infty) < \infty$. Dans ce cas, puisque $\mathbb{P}(Z_x > z) = e^{-(\Gamma(z) - \Gamma(x))}$, $z > x$, on a bien $\mathbb{P}(Z_x = \infty) = e^{-(\Gamma(\infty) - \Gamma(x))} > 0$.

- Dans le cas decay-surge : si $x_t(x)$ atteint 0 en temps $t_0(x) < \infty$, se peut-il que cet événement ('extinction' c'est-à-dire atteinte de 0) se produise avant T_x?
Puisque pour tout $t < t_0(x)$,

$$\mathbb{P}(T_x > t) = e^{-(\Gamma(x_t(x)) - \Gamma(x))}, \ \forall t < t_0(x)$$

on a

$$\mathbb{P}(T_x \geq t_0(x)) = e^{-(\Gamma(0) - \Gamma(x))}$$

et cette probabilité ne peut-être positive que si $\Gamma(0) < \infty$. Dans ce cas, puisque $\mathbb{P}(Z_x < z) = e^{-(\Gamma(z) - \Gamma(x))}$, $z \leq x$, on a bien $\mathbb{P}(Z_x = 0) = e^{-(\Gamma(0) - \Gamma(x))} > 0$.

Les auteurs illustrent ce point pour les systèmes growth-collapse et decay-surge de type 'loi de puissance' (power law) et montrent l'existence d'une 'transition de phase' :

1. Dans le cas growth-collapse : l'événement 'explosion', c'est-à-dire accumulation d'une infinité de sauts en temps fini, ne peut se produire que dans un régime supercritique de la dynamique growth-collapse.
2. Dans le cas decay-surge : l'événement 'extinction' ne peut se produire que dans un régime supercritique de la dynamique decay-surge.

Dans Boxma et al. (2006), les auteurs se sont intéressés à un cas particulier des processus dont la dynamique est décrite par l'équation (1.1.16). En effet, dans ce papier les auteurs ont étudié le cas où le taux de saut $\beta(x)$ est linéaire (voir Pakes et al. (1979), Trajstman (1981)) et la dérive est égale à la constante 1. Dans ce cas, la condition de non-explosion du processus est donnée par $\int^{\infty} \beta(x) = \infty$ et son générateur infinitésimal pour une fonction test f est

$$Gf(x) = f'(x) + \beta(x) \int_0^x H(x, dy)[f(y) - f(x)], \ x \geq 0 \tag{1.2.4}$$

où $H(x, dy)$ est la loi du saut. $\beta(x)$ est remplacé par $r(x)$ et $H(x, dy)$ par $\mu_x(dy)$ dans Boxma et al. (2006). Lorsque la loi du saut a une densité $p(x, y)$ telle que $p(x, y) = x^{-1}\mathbb{1}_{(0,x)}(y)$ et $\beta(x) = \lambda x$, la densité stationnaire de X et la transformée de Laplace de $T_a = \inf\{t \geq 0 \mid X_t = a, \ a > 0\}$ sont données par $q(x) = \lambda x e^{-\lambda x^2/2}$ et $\mathbb{E}_x(e^{-\alpha T_a}) = \frac{1+\alpha \int_0^x e^{\alpha y + \lambda y^2/2} dy}{1+\alpha \int_0^a e^{\alpha y + \lambda y^2/2} dy}$, $0 \leq x < a$. (Boxma et al. (2006), théorème 1).

Le cas d'un système de particules.

Dans Cottrell (1992), l'auteur considère un réseau de $N \in \mathbb{N}$ neurones en interaction où un neurone $i \in \{1, \cdots, N\}$ spike (émet un potentiel d'action) lorsque son état d'inhibition touche la valeur 0. Lorsqu'un neurone i spike, ces neurones post-synaptiques reçoivent un état d'inhibition supplémentaire θ. On appelle $\mathcal{V}_i$ le voisinage du neurone i, c'est-à-dire tous les neurones postsynaptiques du neurone i. Dans son modèle l'état d'inhibition est juste le temps d'attente jusqu'au prochain spike. Sachant l'état d'inhibition à l'instant donné, ce temps est déterministe. L'auteur montre dans Cottrell (1992) qu'il existe un seuil $\theta_0 = \mathbb{E}(F)/v$, où $v = |\mathcal{V}_i|$ est le cardinal de $\mathcal{V}_i$ tel que, si $\theta < \theta_0$ et $\mathbb{E}(F^2) < \infty$ alors les processus X_t^i et $X_{T_n}^i$ sont ergodiques, irréductible, apériodique et récurrent positifs. Dans le cas $\theta > \theta_0$, le résultat principal de l'auteur est que dans le cas transient, le réseau de neurone est divisé en deux groupes : les neurones actifs et les neurones inactifs.

Le modèle étudié dans l'article Fricker et al. (1993) est une généralisation du modèle étudié dans Cottrell (1992). En effet, dans Fricker et al. (1993), le neurone i spike lorsque son état d'inhibition touche la valeur 0. Lorsqu'un neurone spike, l'état d'inhibition des autres neurones augmente d'une quantité aléatoire θ_i et le neurone qui spike reçoit immédiatement une inhibition aléatoire indépendante suivant une loi exponentielle de paramètre λ_i. Dans ce modèle, les auteurs montrent que la chaîne extraite $(X_{T_n}(i), 1 \leq i \leq N)_{n \in \mathbb{N}}$ est ergodique si $\max_{1 \leq i \leq N} \lambda_i \mathbb{E}(\theta_i) < 1$.

Dans Galves and Löcherbach (2013), les auteurs étudient un réseau de neurones qui peut paraître proche de notre modèle, mais le modèle considéré est un modèle en excitation et en temps discret tandis que le nôtre est un modèle inhibiteur et en temps continu. Les auteurs montrent qu'à un instant donné, la probabilité d'apparition d'un potentiel d'action (un spike), qui dépend (à priori) du passé de tout le

système, peut se décomposer à l'aide de la décomposition de type Kalikow (on ne s'intéresse pas à cette décomposition dans cette thèse). Cette décomposition permet cependant de construire graphiquement le processus à l'aide d'un algorithme de simulation parfaite. Nous utiliserons dans Goncalves (2021) cet algorithme de simulation parfaite pour montrer la récurrence du système sous certaines conditions et pour démontrer l'existence d'un état stationnaire du processus.

1.3 Contributions

Dans cette section, nous présentons les résultats importants obtenus dans cette thèse.

1.3.1 Temps de retour et noyau de Green de la chaîne : chapitre 2

Dans ce chapitre, nous étudions les modèles de population Markoviens impliquant des catastrophes totales (voir 1.1.3), à la fois en temps discret et en temps continu. On montre que la mesure invariante existe (dans le cas discret et dans le cas continu) si et seulement si $\sum_{y=1}^{\infty} q_y = \infty$. De plus la formule explicite de la mesure invariante est donnée par

$$\pi_x = \frac{\prod_{y=0}^{x-1} p_y}{1 + \sum_{x\geq 1} \prod_{y=0}^{x-1} p_y}, \forall x \geq 0.$$

Une fois à 0, le marcheur monte d'une étape avec la probabilité p_0 ou reste à 0 avec la probabilité q_0 (ce dernier événement étant considéré comme un retour d'une étape à 0). En posant $\tau_{0,0}$ le premier temps de retour à 0 partant de 0 la loi de $\tau_{0,0}$ est donnée par :

$$\mathbb{P}(\tau_{0,0} = x+1) = q_x \prod_{y=0}^{x-1} p_y.$$

Un de nos résultats principaux porte sur le temps de premier passage et le noyau de Green de la chaîne X_n.
En notant $\tau_{x,y}$ le temps de premier passage à $y \neq x$ quand la chaîne X_n est issue de $X_0 = x$, alors la fonction génératrice de la loi de $\tau_{x,y}$ est donnée par :

$$\phi_{x,y}(z) \equiv \sum_{k=1}^{\infty} z^k \mathbb{P}(\tau_{x,y} = k).$$

Posons

$$g_{x,y}(z) \equiv \sum_{n=0}^{\infty} z^n \mathbb{P}_x(X_n = y) = \sum_{n=0}^{\infty} z^n P^n(x,y) = (I - zP)^{-1}(x,y)$$

la fonction génératrice de $P^n(x,y)$ ou encore la fonction de Green de la chaîne X_n. On montre que la fonction génératrice de la loi de $\tau_{x,y}$ peut être écrite comme suit :

$$\phi_{x,y}(z) = \frac{g_{x,y}(z)}{g_{y,y}(z)}$$

où $g_{x,x}(z)$ est la fonction de Green de la chaîne en x. En se mettant dans le cas où la chaîne est récurrente positive, la formule explicite de la fonction de Green en x est donnée par

$$g_{x,x}(z) = (I - zP)^{-1}(x,x) = \frac{1 - \sum_{x'=0}^{x-1} z^{x'+1} q_{x'} \prod_{y=1}^{x'-1} p_y}{1 - \sum_{x'\geq 0} z^{x'+1} q_{x'} \prod_{y=0}^{x'-1} p_y}.$$

De plus, le noyau de Green de la chaîne s'écrit :

$$\begin{aligned} g_{x,y}(z) &= g_{x,x}(z) z^{y-x} \prod_{y'=x}^{y-1} p_{y'}, \text{ si } y \geq x \\ g_{x,y}(z) &= (g_{x,x}(z) - 1) z^{y-x} / \prod_{y'=y}^{x-1} p_{y'}, \text{ si } 0 \leq y < x. \end{aligned}$$

1.3.2 Existence d'une transition de phase : chapitre 3

Le chapitre 3 étudie la récurrence et la transience du modèle de catastrophe binomiale (voir 1.1.6). On montre que lorsqu'elle existe, la fonction génératrice de probabilité du processus dans son régime stationnaire est solution de l'équation du point fixe

$$\Phi_\infty(z) = pB(z)\Phi_\infty(z) + q\Phi_\infty(1 - c(1-z)) \tag{1.3.1}$$

où $\Phi_n(z) := \mathbb{E}_0(z^{X_n})$. De plus, on prouve que si la loi de X_∞ existe, alors X_∞ est auto-décomposable.
En posant $\phi_\Delta(z) := \frac{q}{1-pB(z)} = \mathbb{E}(z^\Delta)$, une condition d'existence de la mesure invariante est la suivante :

Proposition 1. *La mesure invariante existe pour tout $c \in (0,1)$ si et seulement si $\mathbb{E}(\log_+ \Delta) < \infty$.*

Pour $X_0 = 0$, rappelons que $\Phi_n(z) = \mathbb{E}(z^{X_n}) := \overline{\Phi}_n(z) + \Phi_n(0)$ avec $\Phi_0(z) = 1$. Un des résultats principaux de ce chapitre porte sur la transition de phase du processus et est donné par la proposition suivante :

Proposition 1.3. *(transition de phase).*

1. *Cas sous-critique : $\overline{\Phi}_\infty(1) + \Phi_\infty(0) = 1 \Rightarrow \Phi_\infty(0) = (q - p\rho)/(q - p\rho + p_0)$, définit bien une probabilité seulement si $p\rho < q$. Dans ce cas, la chaîne est récurrente positive. Le terme $p\rho$ est la taille moyenne d'un mouvement ascendant qui doit être inférieure à la taille moyenne q d'un mouvement descendant.*
2. *Cas critique : Si $q = p\rho$, alors $\Phi_\infty(0) = 0 \Rightarrow \overline{\Phi}_\infty(z) = 0$ pour chaque z. La chaîne est récurrente nulle et elle n'a pas de mesure invariante non triviale ($\neq \mathbf{0}$) π.*
3. *Cas supercritique : Si $q < p\rho \leq \infty$, la chaîne est transiente à l'∞.*

1.3.3 Temps de premier dépassement d'un seuil de la population : chapitre 4

Notation : $P := [P(x,y)], x, y \geq 0$ désigne la matrice de transition de la chaîne X_n, $P_{(x_0)}$ la matrice telle que, pour tout $0 \leq x, y \leq x_0$, $P_{(x_0)}(x,y)$ est la probabilité d'un déplacement en un pas de X_n lorsque $X_n = x \leq x_0$ et $X_{n+1} = y \leq x_0$ et $Q_{(x_0)}$ la restriction de P à ses $x_0 + 1$ premières lignes, en laissant les colonnes telles qu'elles sont dans P.

Dans ce chapitre (voir 1.1.9), on suppose $x^* > 0$ un seuil de taille de population important qui ne peut pas être dépassé sans mettre en danger l'ensemble de la population en raison de la limitation des ressources. Cette valeur peut être fixée par une politique publique de contrôle de la population (chaque fois que X_n dépasse

ce seuil, X_n a tendance à diminuer, ce qui est dû au manque de ressources). On s'intéresse au temps qu'il faut pour une telle chaîne de Markov X_n issue de $X_0 = x_0 \ll x^*$ pour atteindre x^* et quelle est alors l'ampleur du dépassement ?
Pour cela on pose, $X_0 = X_0^* = x_0 > 0$ et

$$X_{n+1}^* = \max(X_n^*, X_{n+1})$$

la chaîne extrémale de X_n. On définit également

$$T_1(x^*) = \inf(n \geq 1 : X_n^* > x^* \mid X_0 = x_0)$$

comme le premier instant où la chaîne extrémale X_n^* dépasse le seuil x^*.
La proposition suivante est un de nos résultats principaux et porte sur la loi de $T_1(x^*)$.

Proposition 2. *Pour tout* $x_0 < x^* < y$,

$$\mathbb{P}_{x_0}\left(T_1\left(x^*\right) = l, X_{T_1(x^*)} = y\right) = \left[P_{(x^*)}^{l-1} Q_{(x^*)}\right](x_0, y)$$

et

$$\mathbb{P}_{x_0}\left(X_{T_1(x^*)} = y\right) = \left[\left(I - P_{(x^*)}\right)^{-1} Q_{(x^*)}\right](x_0, y)\,.$$

Lorsqu'on introduit les records générés par la chaîne de Markov extrémale extraite $X_k^* = X_{R_k}$ où

$$R_k = \inf(r \geq 1 : r > R_{k-1}, X_r > X_{R_{k-1}}),$$

on trouve une formule assez explicite de la probabilité que le nouveau record soit atteint au bout d'un temps l et que la chaîne extrémale extraite dépasse le seuil y sachant que le dernier record est atteint à l'instant r et que la chaîne extraite vaut x.

1.3.4 Existence d'une fonction d'échelle particulière : chapitre 5

Pour les chapitres 5 et 6 nous faisons les hypothèses suivantes indépendamment du modèle utilisé.

Hypothèse 1. $\Gamma(\infty) = \infty$

Hypothèse 2. $\Gamma(0) > -\infty$

L'hypothèse 3 suivante ne concerne que le chapitre 5.

Hypothèse 3. $H(x, 0) > 0$ *pour un certain* $x > 0$ *et* 0 *est réfléchissant, c'est-à-dire que* $I_0(x) < \infty$.

Dans le chapitre 5 (voir 1.1.16), nous étudions une classe de PDMP très irrégulière dans le sens où la seule façon de monter est par un mouvement continu déterministe et donc les temps de sortie des intervalles $[0, b]$ seront toujours des temps d'atteinte du niveau b alors que le processus ne descend que par des sauts et donc les temps de sortie des intervalles $[a, \infty]$ seront toujours des sauts. La vraie difficulté dans ce modèle est qu'en général, les seules fonctions d'échelles du processus $(X_t)_t$ ne peuvent être que des constantes. Or, la récurrence des processus de Markov unidimensionnels réguliers est généralement étudiée au moyen de la fonction d'échelle et de la mesure de vitesse associée. Notre but est alors de trouver une fonction dont la bijectivité

est équivalente au fait que le processus est récurrent (on se met dans le cas où l'état 0 est réfléchissant et accessible et l'état ∞ inaccessible).
Pour cela, sous l'hypothèse 2, on pose pour tout $x \geq 0$,

$$s(x) = \int_0^x \frac{\gamma(y)}{h(y)} e^{\Gamma(y)} dy \tag{1.3.2}$$

où $h(0) > 0$ car l'état 0 est accessible.
Cette fonction s n'est pas une fonction d'échelle, car on peut voir facilement que sous l'hypothèse que l'état 0 est réfléchissant, le processus $X_t > 0$ presque sûrement pour tout $t > 0$, et $s(X_t) \geq s(0)$ ce qui contredit la propriété martingale. Mais, cette fonction s ainsi définie permet de caractériser la récurrence du processus et de calculer des probabilités de sortie sous certaines hypothèses.

On pose pour tout $x \geq 0$,

$$\tau_{x,0} = \inf\{t > 0 : X_t = 0 \mid X_0 = x\}$$

le premier temps de retour à 0 du processus X_t partant de x.
En posant

$$p(x, b) = \mathbb{P}_x(\tau_{x,0} < \tau_{x,b})$$

la probabilité que le temps de sortie du processus par l'état 0 soit inférieur au temps de sortie par l'état b partant de l'état $x \in [0, b)$, on a le résultat le suivant :

Théorème 1.1. *Soient les hypothèses 1 et 2 et soit $0 < x < b$. Supposons de plus que $H(x, y) = \frac{h(y)}{h(x)}$ avec $h(0) > 0$. Supposons que $\Gamma(0) = 0$ et posons*

$$\kappa := 1/h(0).$$

Alors

$$\mathbb{P}(\tau_{x,0} > \tau_{x,b}) = \frac{\kappa + s(x)}{\kappa + s(b)}. \tag{1.3.3}$$

De plus, si $\mathcal{H} = \sup\{X_t : t < \tau_{0,0} \mid X_0 = 0\}$ est la hauteur d'une excursion et $\tau_{0,0} = \inf\{t > 0 : X_t = 0 \mid X_0 = 0\} > 0$ est le premier temps de retour en 0 partant de 0, on obtient la probabilité que la hauteur d'une excursion soit plus petit que b, i.e, $\mathbb{P}(\mathcal{H} < b)$ et on a :

$$\mathbb{P}(\mathcal{H} < b) = \mathbb{P}(\tau_{0,0} < \tau_{0,b}) = p(0, b) = \frac{s(b)}{\kappa + s(b)}.$$

Les temps de retour successifs du processus X à 0 induisent un schéma de régénération de base et sont donc liés à la récurrence du processus.

Un de nos résultats principaux est la caractérisation de la récurrence du processus qui est équivalente au fait que $s(\infty) = \infty$ et $I_0(x) < \infty$ où la fonction $s(x)$ est donnée par (1.3.2). Dans ce cas $\tau_{x,0} < \infty$ presque sûrement et l'unique mesure invariante possède une densité de Lebesgue sur $\mathbb{R}_+$ qui est donnée par (1.1.19). De plus, le processus est récurrent positif si $\int^\infty \frac{h(x)}{\alpha(x)} e^{-\Gamma(x)} dx < \infty$, récurrent nul sinon.

Dès lors que l'on connaît les conditions de récurrence (positive ou nulle) du processus, on veut savoir au bout de quel temps le processus partant de x reviendra en 0 dans le cas de la récurrence positive. On sait que, si $x > 0$ la loi de $\tau_{x,0}$ est donnée par

$$\tau_{x,0} \overset{d}{=} T_x \mathbb{1}(X_{T_x} = 0) + \mathbb{1}(X_{T_x} > 0) \left(T_x + \tau'_{X_{T_x},0}\right), \tag{1.3.4}$$

où $\tau'_{X_{T_x}}$ est indépendant de $\mathcal{F}_{T_x}$ et distribué comme $\tau_{X_{T_x}}$.
Cependant, il n'est pas possible de calculer $\tau_{x,0}$ mais il est possible de calculer son espérance. Notre second résultat principal permet d'établir explicitement $\mathbb{E}(\tau_{x,0})$ i.e, le temps moyen de retour en 0 partant de $x > 0$.

Théorème 1.2. *Soient les hypothèses 1, 2, 3 ainsi que $\pi(\beta) < \infty$, et supposons que $H(x,y) = h(y)/h(x)$, où h est différentiable, croissante, avec $h(0) > 0$ et $\alpha(0) > 0$. On pose $\Gamma(0) = 0$. Alors,*

$$\mathbb{E}(\tau_{0,0}) = u(0) = \frac{1}{Ch(0)}. \tag{1.3.5}$$

et $\mathbb{E}(\tau_{x,0}) := u(x)$ est donné par

$$\begin{aligned} u(x) &= u(0) + \int_0^x dy \frac{\gamma(y)\, e^{\Gamma(y)}}{h(y)} \int_y^\infty e^{-\Gamma(z)} \frac{h(z)}{\alpha(z)} dz - \int_0^x \frac{1}{\alpha(y)} dy \\ &= u(0) + s(x) \int_x^\infty \pi(y) dy + \int_0^x s(y)\pi(y) dy - \int_0^x \frac{1}{\alpha(y)} dy, \end{aligned} \tag{1.3.6}$$

où $u(0)$ est donné par (1.3.5).

1.3.5 Récurrence de Harris en utilisant une fonction d'échelle : chapitre 6

Nous étudions ici la récurrence de Harris du processus défini en 1.1.16. On cherche une fonction s solution de l'équation $\mathcal{G}s(x) = 0$ où $\mathcal{G}$ est le générateur associé au processus. Contrairement au chapitre 5, les fonctions d'échelles du processus existent ici et permettent d'avoir une caractérisation classique de la récurrence.

Soit,

$$s(x) = \int_1^x \gamma(y) e^{-\Gamma(y)} / k(y) dy, \ ; x \geq 0. \tag{1.3.7}$$

Supposons que $s(\infty) = \infty$. On a deux cas possibles :

1. L'hypothèse 1 est vérifiée. Alors, la fonction s définit dans 1.3.7 est une fonction d'échelle du processus. Si de plus $k(0) < \infty$, alors la fonction s est une transformée d'espace $[0, \infty) \to [-\infty, \infty)$.
2. L'hypothèse 1 n'est pas vérifiée. Alors les seules fonctions d'échelle sont les constantes.

Remarque 1.8. *Lorsque $s(\infty)$ peut prendre des valeurs finies, alors la fonction s peut ne pas être une transformée d'espace.*

Cette fonction d'échelle joue un rôle clé dans la compréhension des probabilités de sortie du processus. Plus précisément, si $s(\infty) < \infty$, nous avons la formule explicite suivante pour les probabilités de sortie. Pour tout $0 < a < x < b$, soient $\tau_{x,a}$ et $\tau_{x,b}$ les temps de sortie respectifs par a et b du processus partant de x.
En supposant que $\tau_{x,a} \wedge \tau_{x,b} < \infty$ presque sûrement, on a

$$\begin{aligned} \mathbb{P}(X_t \text{ entre } (0,a] \text{ avant d'entrer } [b,\infty) \mid X_0 = x) &= \mathbb{P}(\tau_{x,a} < \tau_{x,b}) \\ &= \frac{s(x) - s(a)}{s(b) - s(a)}. \end{aligned} \tag{1.3.8}$$

De plus, si l'hypothèse 1 est vérifiée et $s(\infty) < \infty$, soit le processus explose (dans le sens où il y a une infinité de sauts) en temps fini avec une probabilité positive, soit il est transient en $+\infty$ c'est-à-dire pour tout $a < x$, $\tau_{x,a} = \infty$ avec une probabilité positive.

Passons maintenant à l'étude du cas $s(\infty) = \infty$. Sous l'hypothèse 1, le théorème 6.1 établit alors une borne inférieure locale de Doeblin sur l'opérateur de transition du processus. Cette minoration de la probabilité de transition a pour conséquence notre principal résultat :

Théorème 1.3. *Supposons que $\beta(0) > 0$, $k(0) < \infty$, que $\Gamma(\infty) = \infty$ et que $s(x)$ soit définie comme dans 1.3.7 et $s(\infty) = \infty$. Alors le processus est récurrent au sens de Harris et sa mesure invariante unique est donnée par π.*

1.3.6 Récurrence d'un réseau de neurones (infini) en utilisant la simulation parfaite : chapitre 7

Dans le chapitre 7 (voir 1.1.20), nous étudions la récurrence d'un système fini et d'un système infini de neurones en interaction. Notre premier résultat principal porte sur l'existence d'une probabilité invariante du processus dans le cadre d'un nombre fini de neurones.

Nous avons construit une fonction de Lyapunov pour le processus. Pour cela, on introduit la matrice H avec

$$H_{ij} = W_{j\to i}\|\gamma_i\|_\infty,\ i \neq j,\ H_{ii} = \|\gamma_i\|_\infty \int_0^\infty y^i F^i(dy^i)$$

telle que ρ soit son rayon spectral et κ le vecteur propre associé à ρ. La fonction $V : \mathbb{R}_+^N \to \mathbb{R}$ telle que $V(x) = \sum_{i=1}^N m_i x^i$ où $m_i = \kappa_i\|\gamma_i\|_\infty$ est une fonction de Lyapunov pour le processus et on a :

$$G^N V(x) \leq -\sum_i \alpha_i(x^i)\|\gamma_i\|_\infty \kappa_i \left(1 - \frac{\gamma_i(x^i)}{\|\gamma_i\|_\infty}\rho\right)$$

où $G^N V$ est le générateur associé au processus.

En effet, si $\rho < 1$ alors $G^N V(x) < 0$ et donc V est une fonction de Lyapunov du processus. Ce résultat nous montre qu'il n'y a pas d'explosion possible du processus en temps fini. Cependant, il n'est pas suffisant pour prouver l'existence d'une mesure invariante du processus. Le résultat suivant nous donne une condition pour la récurrence du processus.

Théorème 1.4. *Supposons qu'il existe un ensemble compact $K \subset (0,\infty)^N$ tel que pour tout $x \in K$, pour tout $1 \leq i \leq N$, $\beta_i(x^i + \sum_{j=1}^{i-1} W_{j\to i}) > 0$, $\|\beta_i\|_\infty < \infty$ pour tout i et que $F^i(dy)$ est absolument continu. Alors, il existe $d \in (0,1)$ et une mesure de probabilité ν sur $(\mathbb{R}_+^N, \mathcal{B}(\mathbb{R}_+^N))$, telle que*

$$Q^N(x, dy) \geq d1_K(x)\nu(dy) \tag{1.3.9}$$

où Q est l'opérateur de transition de la chaîne extraite $Z_n = X_{S_n}$ et Q^N est son $N-$ième itéré. Le processus est donc récurrent.

Les critères de récurrence classiques ne fonctionnent plus dans le cas de systèmes infinis. Le second résultat principal dans ce chapitre porte sur les critères de récurrence du système infini en utilisant la simulation parfaite.

On construit un algorithme de simulation parfaite pour montrer la récurrence du système infini sous certaines conditions. Notre algorithme de simulation parfaite sera scindé en deux procédures. La procédure inverse qui consiste au retournement du temps dans le sens où on quitte le temps présent pour explorer des événements du passé qui ont possiblement influencé l'état du présent. La procédure directe quant à elle consiste à décider lesquels parmi tous ces possibles événements sera effectivement accepté (procédure d'acceptation et de rejet).

Dans la suite, on se place dans le cas où chaque neurone i a exactement deux voisins de sorte que le neurone i interagit uniquement avec les neurones $i+1$ et $i-1$. En d'autres termes, le voisinage de i est $\mathcal{V}_i = \{i+1, i-1\}$. On suppose que pour tout i, le taux de saut $\beta_i(x^i)$ est borné, c'est-à-dire que $\beta_i(x^i) \in [\beta_*, \beta^*]$ pour tous $x^i > 0$, où $0 < \beta_* < \beta^* < \infty$.

Le but de cette première procédure est d'explorer le passé de tout neurone $i \in \mathbb{Z}$ fixé pour déterminer tous les ensembles de neurones et de temps qui ont affecté la valeur du neurone i au temps 0. L'ensemble de tous ces couples (j, s) de neurones et de temps seront appelés le clan des ancêtres du neurone i au temps t et on le note C_t^i. Par convention, on note $C_0^i = \{i\}$ le clan des ancêtres du neurone i au temps 0.

Les variables suivantes seront utilisées pour écrire l'algorithme de simulation parfaite :

- T est le vecteur temps
- P est la matrice des positions où chaque ligne de cette matrice représente les différentes positions des neurones à un instant fixe
- I est le vecteur qui représente l'indice du neurone qui spike.

Algorithme (procédure inverse)

1. Nous simulons, $\forall\ l \in \mathbb{Z}$, $N_t^{l,s}$ et $N_t^{l,p}$ deux processus de Poisson avec des intensités respectives β_* et $\beta^* - \beta_*$. Les temps de saut de $N_t^{l,s}$ et $N_t^{l,p}$ sont respectivement de $T_n^{l,s}$ et $T_n^{l,p}$ pour le neurone l après le $n-$ième saut. Les temps de saut $T_n^{l,s}$ seront considérés comme des temps de sauts sûrs (comptés par le processus $N_t^{l,s}$) et les temps de saut $T_n^{l,p}$ seront considérés comme des temps de sauts possibles (comptés par le processus $N_t^{l,p}$).

2. Soit $i \in \mathbb{Z}$ fixé et $T_1 = \inf\{T_1^{i\pm1,s}, T_1^{i\pm1,p},\ ; T_1^{i,s}\}$ le premier instant de saut du réseau.
   - Si $T_1 = T_1^{i\pm1,s}$, alors $C_{T_1}^i = \{i\}$
   - Si $T_1 = T_1^{i\pm1,p}$ alors $C_{T_1}^i = \{i, i \pm 1\}$
   - Si $T_1 = T_1^{i,s}$, on pose $C_{T_1}^i = \emptyset$ et on arrête l'algorithme.

3. Supposons que T_n est le $n-$ième temps de saut de $C_{T_n}^i$. Nous avons :

$$T_{n+1} = \inf\{T_m^{j,s}, T_m^{j,p} > T_n : |j - C_{T_n}^i| \leq 1,\ T_m^{k,s} > T_n,\ k \in C_{T_n}^i\}.$$

   - Si $T_{n+1} = T_m^{j,p}$ on pose :

$$\begin{cases} \text{Si } j \in C_{T_n}^i, & C_{T_{n+1}}^i = C_{T_n}^i \\ \text{Si } j \notin C_{T_n}^i, & C_{T_{n+1}}^i = C_{T_n}^i \cup \{j\} \end{cases}$$

- Si $T_{n+1} = T_m^{k,s}$ on pose :

$$\begin{cases} \text{Si } k \in C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \setminus \{k\} \\ \text{Si } k \notin C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \end{cases}$$

Nous implémentons C^i_t et recommençons la procédure. Nous arrêtons la procédure au temps T^i_{stop} où $T^i_{stop} = \inf\{t : C^i_t = \emptyset\}$.

Maintenant que nous avons déterminé tous les neurones et tous les temps qui ont pu interagir avec le clan des ancêtres du neurone i, nous sommes à présent capables en appliquant la procédure directe de trouver la valeur du neurone i dans son régime stationnaire lorsque T^i_{stop} est fini.
Pour cela nous définissons :

$$N^i_{stop} = \inf\{n > 0 : C^i_{T_n} = \emptyset\}, \bar{\mathcal{C}}^i = \cup_{n=0}^{N^i_{stop}} C^i_{T_n} \text{ et } \partial_{ext}(C^i_t) = \{j \notin C^i_t : \exists k \in C^i_t, \mathcal{V}_{j\to k} > 0\}$$

où N^i_{stop} est le nombre d'étapes de la procédure inverse (la procédure de retour en arrière), $\bar{\mathcal{C}}^i$ est l'union de tous les clans d'ancêtres jusqu'à N^i_{stop} et $\partial_{ext}(C^i_t)$ est l'ensemble des neurones n'appartenant pas au clan d'ancêtre du neurone i mais ayant une interaction avec au moins un neurone dans le clan d'ancêtre du neurone i au temps t.

Dans cette procédure, nous nous appuierons sur les réalisations a priori des processus $N^{i,s}_t$ et $N^{i,p}_t$.

Algorithme (procédure directe)

1. Nous initialisons l'ensemble des sites pour lesquels la décision d'accepter peut être prise indépendamment de tout par

$$\mathcal{S}^i = \{(I_m, T_m) \in \bar{\mathcal{C}}^i \times \mathbb{R}_+, C^{I_m}_{T_m} = \emptyset\}$$

Pour $n = N^i_{stop}$ nous avons $P^{I_n}_n \sim F^{I_n}$. En partant de $n \to n-1$:

2. Si $(I_{n-1}, T_{n-1}) \in \mathcal{S}^i$ alors $P^{I_{n-1}}_{n-1} \sim F^{I_{n-1}}$.
- Si pour $j \in \mathcal{V}_{I_{n-1}} = \{I_{n-1} \pm 1\}$, on a $j \in C^i_{T_{n-1}}$ alors

$$P^j_{n-1} = x^j_{T_n - T_{n-1}}(P^j_n) + W_{I_{n-1}\to j}$$

- Si pour $j \notin \mathcal{V}_{I_{n-1}}$, on a $j \in C^i_{T_{n-1}}$ alors

$$P^j_{n-1} = x^j_{T_n - T_{n-1}}(P^j_n)$$

- Si $I_{n-1} \in \partial_{ext}(C^i_{T_{n-1}})$, nous avons

$$P^l_{n-1} = x^l_{T_n - T_{n-1}}(P^l_n) + W_{I_{n-1}\to l}$$

où il existe $l \in C^i_{T_{n-1}}$ tel que $I_{n-1} \in \mathcal{V}_l$.

3. Si $(I_{n-1}, T_{n-1}) \in (\bar{\mathcal{C}}^i \times \mathbb{R}_+) \setminus \mathcal{S}^i$ alors :
- On décide avec la probabilité

$$p = \frac{\beta(x^{I_{n-1}}_{T_n - T_{n-1}}(P^{I_{n-1}}_n)) - \beta_*}{\beta^* - \beta_*}$$

d'accepter la présence d'un spike du neurone I_{n-1}.

Dans ce cas, on implémente

$$\mathcal{S}^i \leftarrow \mathcal{S}^i \cup \{(I_{n-1}, T_{n-1}) \in \bar{\mathcal{C}}^i \times \mathbb{R}_+, C^{I_{n-1}}_{T_{n-1}} \subset \mathcal{S}^i\}$$

et on retourne à l'étape 2.

- Sinon, avec la probabilité $1-p$, on rejette la présence d'un spike du neurone I_{n-1} et on a $P^{I_{n-1}}_{n-1} = x^{I_{n-1}}_{T_n - T_{n-1}}(P^{I_{n-1}}_n)$.

Toute cette procédure n'a de sens que si $T^i_{stop} < \infty$ presque sûrement. Notre second résultat principal est que, pour un paramètre $\delta = \frac{\beta_*}{\beta^* - \beta_*}$ donné, il existe une valeur critique $\delta_c \in (0, \infty)$ telle que, si $\delta < \delta_c$ le temps d'extinction T^i_{stop} du processus est infini avec probabilité non nulle et si $\delta > \delta_c$ le temps d'extinction T^i_{stop} du processus est fini presque sûrement.

2

Scaling features of two special Markov chains involving total disasters

Abstract. Catastrophe Markov chain population models have received a lot of attention in the recent past. We herewith consider two special cases of such models involving total disasters, both in discrete and in continuous-time. Depending on the parameters range, the two models can show up a recurrence/transience transition and, in the critical case, a positive/null recurrence transition. The collapse transition probabilities are chosen in such a way that the models are exactly solvable. The study includes : existence and shape of the invariant measure, return time to the origin, extinction probability, height and length of the excursions, scale function, first time to collapse and first-passage times, divisibility properties.

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2.1 Introduction

We consider two particular instances of both discrete and continuous-time Markov chains on the integers subject to state-dependent total disasters probabilities, as particular cases of similar models with partial catastrophes. The chosen disaster transition probabilities ensure that the models are exactly solvable. In the discrete-time version of these models, there is a possibility to move up by one unit with some state-dependent probability and a complementary collapse probability to return back instantaneously to state zero (total disaster). Partial reflection at the origin is assumed. The collapse probability will be a decreasing function of the state, in contrast with the class of "house-of-cards" processes where adding a card to an already large house is more likely to lead to a breakdown. In both models, we are able to give necessary and sufficient conditions for the existence and integrability of the invariant measures and characterize their shape when they exist. We obtain the precise expression of the laws of the first time to collapse, first-passage times, return time to the origin and height and length of the excursions.
The discrete-time models are used as building blocks of their continuous-time versions, which are obtained after a state-dependent Poissonization. The jump rates are chosen to be algebraic in the state, leading again to explicit transience/recurrence and positive/null recurrence criteria.

2.2 A first special Markov chain model with total disaster : results and background

Discrete-time integral-valued growth-collapse processes where long periods of linear growth alternate with rare catastrophic events occur in a large variety of systems. A collapse or catastrophic event is when the size of the system shrinks by a random number of units, not exceeding the current system's size. A total disaster is when size of the system shrinks instantaneously to zero (a massive extinction event). Disastrous growth-collapse models occur as models for population growth subject to rare catastrophic extinction events.
A one-parameter version of such discrete-time models was investigated in Huillet (2011). Here, holding probabilities were allowed (with some probability the system's size can be left unchanged) and pure reflection at the origin was assumed (once in state zero, the system's size grows by one unit with probability 1). Whenever zero is a reflection/absorption barrier, pomp periods will alternate with periods of

scarcity (the Joseph and Noah effects). We herewith focus on discrete-time disastrous growth-collapse models with no holding probability and with zero standing for a reflection/absorption barrier. The probabilities of either growth or disastrous events will be chosen to be suitably dependent one the current state, so as to favor large populations in the long run. Also, we shall consider a continuous time version of this process with jump rates algebraic in the current state. In this setup, transient and equilibrium issues will be studied.
We herewith summarize the obtained results for a first Markov chain model with total disaster, both in discrete and continuous times.

2.2.1 The model and outline of the results

Our first model can precisely be described as follows.
Let $\beta > 0, \nu > -1$ and $0 < \alpha < \nu + 1$. Consider the discrete time-homogeneous Markov chain $X := (X_n,\ n \geq 0)$ with state-space $\mathbb{N}_0 = \{0, 1, ...\}$ and non-homogeneous spatial transition probabilities characterized by :

- given $X_n = x \in \{1, 2, ...\}$, the increment of X_n is

$$\begin{cases} x+1 & \text{with probability } : p_x = 1 - \alpha/\left(\nu + x^\beta\right) \\ 0 & \text{with probability } : q_x = \alpha/\left(\nu + x^\beta\right). \end{cases} \tag{2.2.1}$$

- given $X_n = 0$, the increment of X_n is

$$\begin{cases} +1 & \text{with probability } : p_0 \leq 1 \\ 0 & \text{with probability } : q_0 = 1 - p_0. \end{cases}$$

This defines the transition matrix $P = [P(x, y)]$ of the discrete-time (DT) Markov chain X_n as :

$$P = \begin{bmatrix} q_0 & p_0 & & & & \cdots \\ q_1 & 0 & p_1 & & & \cdots \\ \vdots & \vdots & \ddots & \ddots & & \cdots \\ q_x & 0 & \cdots & \ddots & p_x & \cdots \\ \vdots & 0 & \cdots & & 0 & \ddots \\ \vdots & \vdots & & & & \ddots \end{bmatrix}.$$

Note that, defining $P^c(x, y) = \sum_{z=0}^{y} P(x, z)$, $x' > x \Rightarrow P^c(x', y) < P^c(x, y)$ for all y if and only if $q_{x'} < q_x < 1$ which is the case when $q_x = \alpha/\left(\nu + x^\beta\right)$. The chain X is stochastically monotone.

With $(U_n,\ n \geq 1)$ a sequence of independent identically distributed (iid) uniform random variables (rvs), the dynamics of X_n reads

$$X_{n+1} = (X_n + 1)\, \mathbb{1}(U_{n+1} > q_{X_n}).$$

With $x \geq 1$, we have :

$$\begin{aligned} f(x) &= \mathbb{E}(X_{n+1} - X_n \mid X_n = x) = 1 - \alpha/\left(\nu + x^\beta\right) - (\alpha x)/\left(\nu + x^\beta\right) \\ \sigma^2(x) &= \mathbb{E}\left((X_{n+1} - X_n)^2 \mid X_n = x\right) = 1 - \alpha/\left(\nu + x^\beta\right) + \left(\alpha x^2\right)/\left(\nu + x^\beta\right) \\ \frac{f(x)}{\sigma^2(x)} &= \frac{\nu - \alpha + x^\beta - \alpha x}{\nu - \alpha + x^\beta + \alpha x^2} < 1 \end{aligned}$$

giving the local drift and variance of X_n. Note that, when x is large, $f(x)$ is convex (concave) when $\beta < 1$ ($\beta \geq 1$), with :

$$\begin{aligned} f(x) &\sim -\alpha x^{1-\beta} \to -\infty \text{ when } \beta < 1 \\ f(x) &\sim 1 - \alpha x^{-(\beta-1)} \to 1 \text{ when } \beta > 1 \\ f(x) &= 1 - \alpha \frac{1+x}{\nu+x} \to 1-\alpha \text{ when } \beta = 1. \end{aligned}$$

When x is large also,

$$\begin{aligned} \frac{f(x)}{\sigma^2(x)} &\sim -\frac{1}{x} \text{ if } \beta < 1; \sim \frac{1-\alpha}{1+\alpha}\frac{1}{x} \text{ if } \beta = 1 \\ &\sim \frac{1}{x^{2-\beta}} \text{ if } 2 > \beta > 1; \sim \frac{1}{1+\alpha} \text{ if } \beta = 2 \\ &\sim 1^- \text{ if } \beta > 2. \end{aligned}$$

In model (2.2.1), the walker X_n is occasionally bounced back to the origin and the probability of this event becomes very small once the walker has already reached a large value x. We will check that :

- If $\beta > 1$, the chain is transient with no non-trivial ($\neq \mathbf{0}$) invariant measure.

- If $0 < \beta < 1$, the chain is positive recurrent, the invariant measure of which has stretched exponential behaviour.

- If $\beta = 1$ (critical case). When x is large, the drift of this MC is of order $1 - \alpha + (\nu - \alpha)/x$. So when $\alpha > 1$, the walker is attracted to the origin : The strength of the attraction goes like $\alpha - 1$ for large x. For $\alpha < 1$, the walker is repelled from the origin correspondingly. When $\alpha = 1$, its drift is still attracting but of order $(\nu - 1)/x$, and the drift that the walker feels vanishes when x approaches ∞ if $\nu < 1$. We will see that while crossing the critical value $\alpha = 1$ from above, the process X switches from positive recurrent to null-recurrent.

Remark 2.1. *The collapse probability q_x may be a decreasing or an increasing function of the current population size x. In the former case, large populations are getting more and more stable, after having survived the early growth stages. This is the case under study here in model (2.2.1) with $q_x \underset{x\to\infty}{\sim} \alpha x^{-\beta}$, $(\alpha, \beta > 0)$. In the latter opposite case, large populations would be more susceptible and vulnerable to collapse and so quite unlikely to grow large and develop too much. This would be the case for a model with $q_x \underset{x\to\infty}{\sim} 1 - \alpha x^{-\beta}$ (while switching the role of p_x and q_x). If one also thinks of the process of building a house of cards, clearly adding a new card to a house of cards of size x is more likely to lead to a collapse of the whole structure if x is already large (Fernández et al. (2001), p. 47 and Ferrari and Galves (2000), p. 9). In such a situation, the process is always positive recurrent with light-tailed invariant measure (stretched exponential).*

We now come to a natural continuous-time version of the process (2.2.1) which will be one of our main process of interest.
Let $\lambda \in (-\infty, +\infty)$ and consider the transition rate matrix Q of a continuous-time MC process $\overline{X}(t)$:

$$Q = D_{\mathbf{r}}(P - I) \tag{2.2.2}$$

with $D_{\mathbf{r}}$ =diag($\mathbf{r}$) the diagonal matrix formed from the rate vector $\mathbf{r} = (r_0, r_1, ..., r_x, ...)$ with $r_x = r_0 (x+1)^{\lambda}$, $x \geq 0$, $r_0 > 0$. Clearly, X_n is the embedded MC of $\overline{X}(t)$.

For such a continuous-time model, the growth rate of the transition $x \to x+1$ is $r_x p_x$ with $r_x p_x \sim r_0 x^\lambda$ for large x, while the one of the collapse transition $x \to 0$ is $r_x q_x$ with $r_x q_x \sim \alpha r_0 x^{\lambda-\beta}$ for large x. If $\lambda > 0$, $\alpha x^{\lambda-\beta} \ll x^\lambda$ always and if $0 < \lambda < \beta$, $\overline{X}$ moves up by one unit frequently, while its collapse becomes increasingly rare. If $\lambda < 0$ (the lazy chain), the growth rate $r_x p_x \sim r_0 x^\lambda$ is small for large x while the collapse rate $r_x q_x \sim \alpha r_0 x^{\lambda-\beta}$ is still smaller. In any case, the collapse rates are small compared to the growth ones.

We can view the process $\overline{X}(t)$ as follows. Let $P(t)$ be a standard Poisson process with intensity $t \geq 0$ and $P(0) = 0$. Let $Z(t) = X_{P(t)}$ be the chain X_n subordinated to $P(t)$. Clearly $Z(t)$ is a CT Markov chain with transition rate matrix $P - I$. Then the process $\overline{X}(t)$ turns out to be

$$\overline{X}(t) = Z\left(\int_0^t r_{\overline{X}_s} ds\right).$$

It has infinitesimal backward generator $G_{\overline{X}}$ whose action on real-valued bounded functions h on $\mathbb{N}_0 = \{0, 1, 2, ...\}$ is

$$G_{\overline{X}} h(x) = r_x \left\{\left(h(x+1) - h(x)\right) p_x + \left(h(0) - h(x)\right) q_x\right)\right\},$$

meaning

$$\mathbb{E}_x h\left(\overline{X}(t)\right) = \mathbb{E}_x h\left(\overline{X}(0)\right) + \int_0^t \mathbb{E}_x \left(G_{\overline{X}} h\right)\left(\overline{X}(s)\right) ds.$$

When $h(x) = x$, with $x(t) := \mathbb{E}_x \overline{X}(t)$, $x(0) = x$, by Jensen inequality

$$\begin{aligned}
\dot{x}(t) &= \mathbb{E}_x\left(r\left(\overline{X}(t)\right)\left(p_{\overline{X}(t)} - \overline{X}(t) q_{\overline{X}(t)}\right)\right) =: \mathbb{E}_x \overline{f}\left(\overline{X}(t)\right) \\
&= r_0 \mathbb{E}_x\left(\left(\overline{X}(t)+1\right)^\lambda \left(\frac{\nu - \alpha\left(1 + \overline{X}(t)\right) + \overline{X}(t)^\beta}{\nu + \overline{X}(t)^\beta}\right)\right) \\
&\geq r_0 (x+1)^\lambda \left(\frac{\nu - \alpha(1+x) + x^\beta}{\nu + x^\beta}\right) = r_0 (x+1)^\lambda f(x)
\end{aligned}$$

if the latter function to the right of the inequality (the drift $\overline{f}(x) = r_0 (x+1)^\lambda f(x)$ of the continuous-time MC) is a convex function of x, at least for large x.

The range $\lambda > 0$ ($\lambda < 0$) accounts for the fact that the moves of $\overline{X}$ get frequent (respectively rare) when the height x of $\overline{X}$ gets large, and given a move has occurred and x is large, $\overline{X}$ grows by one unit with large probability $\sim 1 - \alpha x^{-\beta}$ or undergoes a catastrophic event with small complementary probability $\sim \alpha x^{-\beta}$. Such transition mechanisms favor large values of $\overline{X}$. This chain is irreducible and aperiodic, either transient or recurrent (possibly then either positive or null recurrent). We will show that :

(i) When it is transient ($\beta > 1$), the process $\overline{X}$ is either explosive ($\lambda > 1$) or non-explosive ($\lambda \leq 1$). When $\beta > 1$, after a finite number of returns to 0, $\overline{X}$ drifts to ∞. And (from an argument on Yule processes to appear in the proof) it explodes if and only if $\lambda > 1$. The process $\overline{X}$ has no non-trivial ($\neq \mathbf{0}$) invariant measure.

(ii) When it is recurrent ($\beta \leq 1$), the process $\overline{X}$ is :
- recurrent positive if $\beta < 1$, with invariant measure showing a stretched exponential behavior.

- recurrent positive if $\beta = 1$ and $\alpha + \lambda > 1$, recurrent null if $\beta = 1$ and $\alpha + \lambda \leq 1$. The invariant measure of $\overline{X}$ is of power-law type with index $\alpha + \lambda$. The critical value $\beta = 1$ separates a recurrent phase ($\beta < 1$) from a transient phase ($\beta > 1$).

The birth and collapse probabilities depend on the current state of the population in the specific way just described, together with the jump rates. Our study will include, among other topics, first-return time probabilities to the origin (excursion length), eventual return (contact) probability to the origin, excursion height, time to failure and the fraction of time spent in the catastrophic state.
The continuous-time version of model (2.2.1) is skip-free to the right, with p_x, q_x and r_x dependent on x as specified. To a large extent therefore, the Markov chain model under study here is one of the simplest possible in the vast family of growth-collapse models. As we will show, it turns out that it is an exactly solvable case. Using ideas stemming from excursion theory, we will make precise here to what extent growth is (algebraically) slow when catastrophic events are rare under our over-simplistic model hypothesis. It is hoped that models in the same class of universality could share similar behaviors as one reasonably can expect some sort of robustness. This will be the purpose of the last Section 4 involving a Pareto-Zipf paradigm.

2.3 Detailed analysis of the special Markov chain with total disaster : proofs

In this Section, we will supply a detailed study of the general DT catastrophe Markov chain (MCC), without specifying the disaster probabilities q_x. For each item under study, we will subsequently particularize the detailed expression of the results for the special MC (2.2.1). We end up this Section with the study of its continuous-time version with algebraic rates r_x.

2.3.1 The discrete-time MCC chain

Consider a general catastrophe process X_n for which both p_x and $q_x > 0$, for all $x \geq 0$, with $p_x + q_x = 1$ and so with associated stochastic transition matrix : $P = [P(x, y)]$, $(x, y) \in \mathbb{N}_0^2$ with $P(x, 0) = q_x$ and $P(x, x + 1) = p_x$, $x \geq 0$.

2.3.1.a Existence and shape of the invariant measure

Let $\pi' \equiv (\pi_0, \pi_1, ..)$ be the row-vector of the invariant measure, whenever it exists. Then π should solve $\pi' = \pi' P$, whose formal solution is :

$$p_0 \pi_0 = \sum_{x \geq 1} \pi_x q_x \text{ and } \pi_x = \pi_0 \prod_{y=0}^{x-1} p_y,\ x \geq 1. \tag{2.3.1}$$

Let $u_x = \prod_{y=1}^{x-1} p_y$ with $u_1 \equiv 1$. Using the second equation, the first equation is satisfied whenever

$$\sum_{x \geq 1} q_x \prod_{y=1}^{x-1} p_y = \sum_{x \geq 1} (u_x - u_{x+1}) = 1,$$

so also when $u_\infty = \prod_{y=1}^{\infty} p_y = 0$ which is fulfilled if and only if

$$C_1 \equiv \sum_{y=1}^{\infty} q_y = \infty.$$

We first conclude that there exists an invariant measure if and only if $C_1 = \infty$. If in addition,

$$C_2 \equiv \sum_{x\geq 1} \prod_{y=0}^{x-1} p_y < \infty,$$

then $\pi_0 = \frac{1}{1+C_2} \in (0,1)$ and the invariant measure is unique and is a proper invariant probability measure. In this case, with the empty product being 1, we have

$$\pi_x = \frac{\prod_{y=0}^{x-1} p_y}{1+C_2},\ x \geq 0.$$

When $C_2 = \infty$, the measure π_x solution to (1.1.9) exists but it is not a probability measure as there is no way to normalize it so as to have a probability measure. We will see below that this corresponds to a case where the first return time to 0 of the chain has infinite mean.

APPLICATION : When dealing with the special MC, we conclude :

- $C_1 < \infty$ if and only if $\beta > 1$: in this case the MC is transient with no invariant measure.

- If $\beta < 1$, then $C_1 = \infty$ and $C_2 < \infty$: the MC is positive recurrent. Furthermore, for large x :

$$\pi_x \sim \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y^\beta}\right) \sim e^{-\alpha \sum^x y^{-\beta}} \sim e^{-\frac{\alpha}{1-\beta} x^{1-\beta}}$$

with stretched exponential behaviour.

Remark 2.2. *The simpler case $\beta = 0$ was excluded from the study. Here, the transition probabilities (p_x, q_x), $x \geq 1$, are homogeneous and may be set to (p, q), $p + q = 1$, where $q = \alpha/(\nu+1) < 1$. The invariant measure in this case exists, is summable and takes the simple geometric form $\pi_x = \pi_0 p_0 p^{x-1}$ if $x \geq 1$. The value of $\pi_0 = 1/(1+C_2)$ is found to be : $\pi_0 = q/(p_0+q)$. The corresponding chain is positive recurrent.*

- In the critical case $\beta = 1$:

$$\pi_x \sim \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y}\right) \sim e^{-\alpha \sum^x y^{-1}} \sim x^{-\alpha}$$

with power-law(α) behaviour. The chain is recurrent. We will compute the exact shape of π_x later. But from this one can guess that the DT chain is positive recurrent if $\alpha > 1$, null recurrent if $\alpha \leq 1$.

2.3.1.b Return time to the origin

Let X_n be the Markov chain with transition probability matrix P and state-space $\mathbb{N}_0$. Starting from $X_0 = x \geq 1$, the walker moves one step up with probability p_x or goes back to the origin (the catastrophic event) with probability q_x. Once at 0, the

walker moves one step up with probability p_0 or stays at 0 with probability q_0 (the latter event being considered as a one-step return to 0). Clearly of interest are the times $\tau_{0,0} \geq 1$ between consecutive visits to 0 (the first return times to 0). With the convention that the empty product is 1, with $u_0 \equiv 1$, we have

$$\mathbb{P}(\tau_{0,0} = x+1) = q_x \prod_{y=0}^{x-1} p_y = p_0 (u_x - u_{x+1}), \; x \geq 0. \tag{2.3.2}$$

Equivalently, $\mathbb{P}(\tau_{0,0} > x) = \prod_{y=0}^{x-1} p_y = p_0 u_x$. We note that, conventionally, $\tau_{0,0} = 1$ with probability q_0 (the holding probability at state 0).

Note also that $p_x = \mathbb{P}(\tau_{0,0} > x+1) / \mathbb{P}(\tau_{0,0} > x)$ and $q_x = \mathbb{P}(\tau_{0,0} = x+1) / \mathbb{P}(\tau_{0,0} > x)$: If the law of the lifetime $\tau_{0,0}$ is known in the first place, this gives the survival probability p_x given the age (backward recurrence time) X_n of the current machine is x.
Note also that $\mathbb{P}(\tau_{0,0} < \infty) = 1$ if and only if $u_\infty = 0$ ($C_1 = \infty$) which is the recurrence condition for X_n.

We conclude that :

- If $C_1 \equiv \sum_{y=1}^{\infty} q_y < \infty$, X_n is transient with $\mathbb{P}(\tau_{0,0} < \infty) < 1$.
- If $C_1 = \infty$, X_n is recurrent.
  - If $C_1 = \infty$ and $C_2 \equiv \sum_{x \geq 1} \prod_{y=0}^{x-1} p_y < \infty$, X_n is positive recurrent with $\mu := \mathbb{E}(\tau_{0,0}) = 1/\pi_0 = 1 + C_2 < \infty$.
  - If $C_1 = C_2 = \infty$, X_n is null recurrent with $\tau_{0,0} < \infty$ almost surely (a.s.) and $\mathbb{E}(\tau_{0,0}) = \infty$.

Note that in the positive recurrent case,

$$\pi_x = \frac{\prod_{y=0}^{x-1} p_y}{1 + C_2} = \frac{\mathbb{P}(\tau_{0,0} > x)}{\mu}, \; x \geq 0.$$

APPLICATION :

- If $\beta > 1$: the chain is transient with $(0 < u_\infty < 1)$

$$\mathbb{P}(\tau_{0,0} = \infty) = u_\infty = \prod_{y=0}^{\infty} \left(1 - \frac{\alpha}{\nu + y^\beta}\right) > 0.$$

- If $\beta < 1$: the chain is positive recurrent with

$$\mu := \mathbb{E}(\tau_{0,0}) = 1 + C_2 = 1 + \sum_{x \geq 1} \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y^\beta}\right) < \infty.$$

- Critical case : $\beta = 1$, $0 < \alpha < \nu + 1$. In this case, let

$$\psi_0(z) := \sum_{x \geq 1} q_x z^x \prod_{y=1}^{x-1} p_y = \sum_{x \geq 1} z^x \frac{\alpha}{\nu + x} \prod_{y=1}^{x-1} \left(1 - \frac{\alpha}{\nu + y}\right)$$
$$= \frac{\alpha z}{\nu + 1} \cdot F(1, \nu + 1 - \alpha; \nu + 2; z)$$

involving a Gauss hypergeometric function (by F we mean here ${}_2F_1$). The function $\psi_0(z)$ is the probability generating function (pgf) of an extended Sibuya rv $S \geq 1$ with parameters (α, ν), Kozubowski and Podgórski (2018). The shifted rv $S - 1$

(with pgf $z^{-1}\psi_0(z)$) is discrete-self-decomposable (SD), Kozubowski and Podgórski (2018) (see the Appendix 8.1 for a reminder on this concept). When $\nu = 0$, the law of S is also known as the standard Sibuya distribution, Sibuya (1979), while when $\nu = \alpha$ the Yule-Simon distribution is recovered, Yule (1925), Simon (1955).
When $\alpha = 1$, an easy computation shows that

$$\psi_0(z) = z - \nu L_\nu(z)(1-z)$$

where

$$L_\nu(z) = \sum_{n\geq 1} z^n/(\nu+n) \underset{z\downarrow 1}{\sim} -\log(1-z).$$

Using Stirling formula :

$$\begin{aligned}\mathbb{P}(S=x) &:= s_x = [z^x]\psi_0(z) \underset{x\to\infty}{\sim} \alpha\Gamma(\nu+1)x^{-(\alpha+1)}/\Gamma(\nu+1-\alpha) \text{ if } \alpha\neq 1\\ \mathbb{P}(S>x) &= \frac{1}{\nu+1}\frac{1}{\nu+x} \text{ if } \alpha = 1 \ (\nu>0)\end{aligned}$$

and for all $x \geq 1$

$$s_{x+1}/s_x = (\nu-\alpha+x)/(\nu+x+1) < 1,\ s_1 = \alpha/(\nu+1) \tag{2.3.3}$$

(s_x is monotone decreasing). As a result, s_1 is the maximal value of the s_x : the probability mass function (pmf) s_x has its mode at $x = 1$.
Equivalently,

$$\begin{aligned}\psi_0(z) &\underset{z\downarrow 1}{\sim} 1 - \tfrac{\Gamma(\nu+1)\Gamma(1+\alpha)}{\Gamma(\nu+1-\alpha)}(1-z)^\alpha \text{ if } 0<\alpha<1\\ \psi_0(z) &\underset{z\downarrow 1}{\sim} 1 - \tfrac{\nu}{\alpha-1}(1-z) + O(1-z)^\alpha \text{ if } \alpha>1\\ \psi_0(z) &\underset{z\downarrow 1}{\sim} 1-(1-z)-\nu(1-z)\log(1-z) \text{ if } \alpha=1\end{aligned}$$

The pgf of $\tau_{0,0}$ itself therefore reads,

$$\begin{aligned}\phi_{0,0}(z) &= \sum_{x\geq 0} q_x z^{x+1}\prod_{y=0}^{x-1} p_y = z\left(q_0 + p_0\sum_{x\geq 1} q_x z^x \prod_{y=1}^{x-1} p_y\right)\\ &= z\left(q_0 + p_0\frac{\alpha z}{\nu+1}\cdot F(1,\nu+1-\alpha;\nu+2;z)\right) = z(q_0+p_0\psi_0(z)).\end{aligned}$$

Using $F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ and $F'(a,b;c;1) = \frac{ab}{c}F(a+1,b+1;c+1;1)$ we find the mean persistence time

$$\phi'_{0,0}(1) = \mathbb{E}(\tau_{0,0}) := \mu = \frac{1}{\pi_0} = 1 + p_0\frac{\nu}{\alpha-1}.$$

It can be checked that if $\mathbb{E}(\tau_{0,0})$ is to exist, then necessarily $\mathbb{E}(\tau_{0,0}) > 1+p_0$.
This condition forces $1<\alpha<\nu+1$ which is the positive recurrence condition for the critical MCC. If $p_0 = 1$, $\mathbb{E}(\tau_{0,0}) = \frac{\nu+\alpha-1}{\alpha-1} > 2$ and there are no trivial excursions.
In this positive recurrent case, we also have

$$\begin{aligned}
\pi_x &= \frac{\prod_{y=0}^{x-1} p_y}{\mu} = \pi_0 p_0 \prod_{y=1}^{x-1}\left(1-\frac{\alpha}{\nu+y}\right),\ x\geq 0\\
\frac{\pi_x}{1-\pi_0} &= \frac{\pi_0 p_0}{1-\pi_0}\prod_{y=1}^{x-1}\left(1-\frac{\alpha}{\nu+y}\right) = \frac{\alpha-1}{\nu}\prod_{y=1}^{x-1}\left(1-\frac{\alpha}{\nu+y}\right)\\
&= \frac{\alpha-1}{\nu-1+x}\prod_{y=1}^{x-1}\left(1-\frac{\alpha-1}{\nu-1+y}\right),\ x\geq 1
\end{aligned}$$

an identity showing that $Y_\infty := X_\infty \mid X_\infty \geq 1 \overset{d}{\sim}$ Sibuya$(\alpha-1,\nu-1)$ so with

$$\begin{aligned}
\mathbb{E}\left(z^{Y_\infty}\right) &= \frac{(\alpha-1)\,z}{\nu}\cdot F\left(1,\nu+1-\alpha;\nu+1;z\right) =: \psi_\infty(z)\\
\mathbb{E}\left(z^{X_\infty}\right) &= \pi_0+(1-\pi_0)\,\psi_\infty(z).
\end{aligned}$$

This gives an explicit expression of the pgf of X_∞ (in the positive recurrent case) in terms of a Gauss hypergeometric function. Note that X_∞ is the Bernoulli mixture of the two rvs $Y_\infty := X_\infty \mid X_\infty \geq 1$ and (say) Y_0 which is degenerate at 0 so with $Y_0 \overset{d}{\sim} \delta_0$. We also observe that π_x is unimodal with mode at the origin ($\pi_{x+1}/\pi_x = \left(1-\frac{\alpha}{\nu+x}\right) < 1$, $x\geq 1$, $\pi_1/\pi_0 = p_0 < 1$).
As an extended Sibuya$(\alpha-1,\nu-1)$ rv, the shifted Sibuya rv

$$Y_\infty - 1 := (X_\infty - 1 \mid X_\infty \geq 1) \text{ is discrete self-decomposable.}$$

We now raise the following question : Is X_∞ itself infinitely-divisible, say ID (meaning compound-Poisson) ? discrete self-decomposable ? Discrete-SD rvs constitute a remarkable sub-class of ID rvs, Steutel and van Harn (2004) ; see the Appendix 8.1. It turns out that X_∞ is ID if p_0 is small enough (else π_0 large enough).
Observing $\pi_x^2 \leq \pi_{x-1}\pi_{x+1}$ for $x\geq 2$, a sufficient condition for X_∞ to be log-convex and so ID (see Steutel and van Harn (2004) Example 11.9, page 84) is that

$$\pi_1^2 \leq \pi_0\pi_2 \text{ which is } p_0 \leq 1-\alpha/(\nu+1).$$

But this condition is not necessary although, from the unimodality of π_x, it could be a necessary and sufficient condition for X_∞ to be self-decomposable in some cases. Whenever X_∞ is ID or SD, it can be produced as limit laws of alternative Markov processes (see Appendix 8.2). We illustrate this result by the following explicit example showing that this can happen. We were not able to treat the general case.

Example 2.1. Suppose $\nu = 1$, so with $\pi_0 = 1-p_0/(\alpha-1+p_0)$. Then, with $\alpha\in(1,2)$:

$$\begin{aligned}
\psi_\infty(z) &= (\alpha-1)\,zF\left(1,2-\alpha;2;z\right) = 1-(1-z)^{\alpha-1}\\
\mathbb{E}\left(z^{X_\infty}\right) &= 1-(1-\pi_0)\,(1-z)^{\alpha-1}
\end{aligned}$$

The rv X_∞ is obtained while thinning (scaling) a Sibuya$(\alpha-1)$ distributed rv (with pgf $\psi_\infty(z)$), Schreiber (1999). Note that X_∞ has infinite mean. Furthermore, (see Theorem 1 and 2 of Christoph and Schreiber (2000)) :

- X_∞ is ID if and only if : $1-\pi_0 \leq 1-(\alpha-1) = 2-\alpha$, else : $p_0 \leq 2-\alpha$ (compare with the sufficient condition which here is $p_0 \leq 1-\alpha/2$).

- X_∞ is discrete-SD if and only if : $1-\pi_0 \leq \frac{1-(\alpha-1)}{1+(\alpha-1)} = \frac{2-\alpha}{\alpha}$, else : $p_0 \leq 1-\alpha/2$ (the right condition).

Note that in this example

$$\begin{aligned}
\psi_0(z) &= \frac{\alpha z}{2} \cdot F(1, 2-\alpha; 3; z) = \frac{1}{\alpha-1}\left(\alpha - \frac{1}{z}\left(1-(1-z)^\alpha\right)\right) \\
\phi_{0,0}(z) &= z\left(q_0 + p_0\psi_0(z)\right) = z\left(1 + \frac{p_0}{\alpha-1}\right) - \frac{p_0}{\alpha-1}\left(1-(1-z)^\alpha\right) \\
&= 1 - \mu(1-z) + (\mu-1)(1-z)^\alpha
\end{aligned}$$

with $\tau_{0,0}$ having finite mean but infinite variance.

If the condition $1 < \alpha < \nu+1$ is not met, the special MCC is null recurrent with $\mathbb{E}(\tau_{0,0}) = \infty$.

2.3.1.c The scale (or harmonic) function.

In the recurrent case, the sample paths of X_n are made of i.i.d. excursions (the pieces of the sample paths between consecutive visits to 0). The lengths of the excursions are $\tau_{0,0}$. Let us look at their heights H. As observed just before, $H \overset{d}{=} \tau_{0,0} - 1$, because X grows linearly between consecutive visits to 0 (if it grows). Let us rapidly check this with the use of the scale function. The scale function idea will appear useful.

Assume $X_0 = x$. Let $X_{n\wedge\tau_{x,0}}$ stopping X_n when it first hits 0. Let us define the scale (or harmonic) function φ of X_n as the function which makes $Y_n \equiv \varphi\left(X_{n\wedge\tau_{x,0}}\right)$ a martingale. The function φ is important because, as is well-known, for all $0 < x < h$, with τ_x the first hitting time of $\{0, h\}$ starting from x (assuming $\varphi(0) \equiv 0$)

$$\mathbb{P}(X_{\tau_x} = h) = \mathbb{P}(\tau_{x,h} < \tau_{x,0}) = \frac{\varphi(x)}{\varphi(h)}.$$

Using this remark, the event $H = h$ is realized when $\tau_{0,h} < \tau_{0,0}$ and $\tau_{h,h+1} > \tau_{h,0}$, the latter two events being independent. Thus (recalling $\mathbb{P}(H=0) = q_0$) :

$$\mathbb{P}(H = h) = p_0 \frac{\varphi(1)}{\varphi(h)}\left(1 - \frac{\varphi(h)}{\varphi(h+1)}\right),\ h \geq 1. \tag{2.3.4}$$

We clearly have $\sum_{h\geq 1}\mathbb{P}(H=h) = p_0$ because partial sums are part of a telescoping series. But this is also $\mathbb{P}(H \geq h) = 1/\varphi(h)$. It remains to compute φ with $\varphi(0) = 0$. We wish to have : $\mathbb{E}_x(X_{n+1} \mid X_n = y) = y$, leading to

$$\varphi(x) = p_x\varphi(x+1) + q_x\varphi(0) = p_x\varphi(x+1),\ x \geq 1.$$

Thus, the searched 'harmonic' (increasing) function is

$$\varphi(x) = \frac{1}{\prod_{y=0}^{x-1} p_y},\ x \geq 1,\ \varphi(0) \equiv 0. \tag{2.3.5}$$

Note $\varphi(1) = 1/p_0$ and $\varphi(x)$ is diverging whenever the chain is recurrent. Equations (2.3.4) and (2.3.5) characterize the law of the excursion height of the random walker in the recurrent case. Note

$$\mathbb{P}(H \geq h) = 1/\varphi(h) = \prod_{y=0}^{h-1} p_y = \mathbb{P}(\tau_{0,0} > h),$$

showing, as expected from the beginning, that $H \stackrel{d}{=} \tau_{0,0} - 1$.

Remark 2.3. *(Doob transform) : Let $\overline{P}$ be obtained from P by removing its first and last column. With $\varphi := (\varphi(1), \varphi(2), ...)$ and $D_\varphi = diag(\varphi)$, the Markov chain with stochastic transition matrix :*

$$\overline{P}_\varphi = D_\varphi^{-1} \overline{P} D_\varphi$$

is X (with transition matrix P) conditioned to hit first ∞ before 0. In our context :

$$\overline{P}_\varphi = \begin{bmatrix} 0 & 1 & & & & \cdots \\ 0 & 0 & 1 & & & \cdots \\ \vdots & \vdots & \ddots & \ddots & & \cdots \\ 0 & 0 & \cdots & \ddots & 1 & \cdots \\ \vdots & 0 & \cdots & & 0 & \ddots \\ \vdots & \vdots & \cdots & \cdots & & \ddots \end{bmatrix}.$$

2.3.1.d Probability of extinction

Consider now the same Markov chain but assume now that $p_0 = 0$, $q_0 = 1$. In this case, the state 0 is absorbing. Consider then the restriction $\overline{P}$ of matrix P to the states $\{1, 2, ...\}$. Let ϕ_x, $x \geq 1$ be the probabilities that state 0 is hit in finite time given the chain started originally at x. Let $\phi \equiv (\phi_1, \phi_2, ..)'$ be the column-vector of these absorption probabilities. Let $\mathbf{q} \equiv (q_1, q_2, ..)'$. Then ϕ is the smallest non-negative solution to $\phi = \mathbf{q} + \overline{P}\phi$ whose formal solution is : $\phi = \left(I - \overline{P}\right)^{-1} \mathbf{q}$. The ϕ_xs obeys the recurrence : $\phi_x = q_x + p_x \phi_{x+1}$, else $1 - \phi_{x+1} = \frac{1}{p_x}(1 - \phi_x)$. All ϕ_x can therefore be expressed in terms of ϕ_1, leading simply to :

$$1 - \phi_x = \frac{1}{\prod_{y=1}^{x-1} p_y}(1 - \phi_1).$$

The formal solution is also $\phi = \left(I - \overline{P}\right)^{-1} \mathbf{q}$, involving the resolvent of $\overline{P}$. Because $\left(I - \overline{P}\right)^{-1}$ is computable with upper triangular structure,

$$\left(I - \overline{P}\right)^{-1} = \begin{bmatrix} 1 & p_1 & p_1p_2 & p_1p_2p_3 & p_1p_2p_3p_4 & \cdots \\ & 1 & p_2 & p_2p_3 & p_2p_3p_4 & \cdots \\ & & 1 & p_3 & p_3p_4 & \cdots \\ & & & 1 & p_4 & \cdots \\ & & & & 1 & \cdots \\ & & & & & \ddots \end{bmatrix},$$

ϕ_x thus takes the alternative form

$$\phi_x = \sum_{y \geq x} q_y \prod_{y'=x}^{y-1} p_{y'}.$$

If $C_1 \equiv \sum_{y\geq 1} q_y = \infty$, then $u_x = \prod_{y=1}^{x-1} p_y \to 0$: the restriction $\phi_x \in [0,1]$ forces $\phi_1 = 1$ and so $\phi_x = 1$ for all $x < \infty$: The state 0 is hit with probability 1, starting from x, for all $x < \infty$. The Markov chain is recurrent.
But, if $C_1 < \infty$, then we can take $\phi_1 < 1$ so long as $\phi_x \geq 0$ for all $x \geq 1$. The minimal solution occurs when $1 - \phi_1 = \mathbb{P}(\tau_{1,0} = \infty) = \prod_{y\geq 1} p_y > 0$, leading to the alternative expression of ϕ_x :

$$\phi_x = 1 - \frac{\prod_{y\geq 1} p_y}{\prod_{y=1}^{x-1} p_y} = 1 - \prod_{y\geq x} p_y. \tag{2.3.6}$$

In this case, $\phi_x < 1$ for $x \geq 1$ and the absorbed random walker started at x avoids 0 with positive probability (a transience case for the original reflected Markov chain). Note that $x < x' \Rightarrow \phi_x > \phi_{x'}$. We can extend Eq. (2.3.6) to $x = 0$, because, by first-step analysis,

$$\phi_0 := \mathbb{P}(\tau_{0,0} < \infty) = \phi_{0,0}(1) = p_0 \mathbb{P}(\tau_{1,0} < \infty) + q_0 = p_0\phi_1 + q_0,$$

leading to

$$\phi_0 = 1 - p_0(1 - \phi_1) = 1 - \prod_{y\geq 0} p_y.$$

To summarize, we have :

(i) If $C_1 < \infty$, the MC is transient and, with

$$\tau_{x,0} = \inf(n \geq 1 : X_n = 0 \mid X_0 = x), \ x \geq 0,$$

$\mathbb{P}(\tau_{x,0} = \infty) = \prod_{y\geq x} p_y > 0$. The chain X started at $x \geq 0$ has probability $\phi_x = 1 - \prod_{y\geq x} p_y < 1$ to undergo a first extinction.

(ii) If $C_1 = \infty$, the MC is recurrent with $\mathbb{P}(\tau_{x,0} = \infty) = 0$. Moreover, it is : null recurrent if $C_2 = \infty$, positive recurrent if $C_2 < \infty$.
Due to irreducibility (because p_x and $q_x > 0$, for all $x \geq 1$), states are either all transient or recurrent

APPLICATION : for the model (2.2.1),

$$\mathbb{P}(\tau_{x,0} = \infty) > 0 \Leftrightarrow \beta > 1.$$

2.3.1.e Times to collapse (first extinction)

How long does it take, starting from $x \geq 1$, to first hit 0? We give here some insight on the way to compute the law of these first times to collapse. With $x \geq 1$, let thus $\tau_{x,0}$ be the time it takes to first hit 0, starting from $X_0 = x \geq 1$. With $\tau'_{x+1,0}$ a statistical copy of $\tau_{x+1,0}$, from first-step analysis, we clearly have :

$$\tau_{x,0} \stackrel{d}{=} (1 - B_x) \cdot 1 + B_x \cdot \left(1 + \tau'_{x+1,0}\right),$$

where B_x is a Bernoulli random variable with $\mathbb{P}(B_x = 1) = p_x$. Therefore with $\phi_{x,0}(z) = \mathbb{E}(z^{\tau_{x,0}})$, $\phi_x(z)$ obeys the recurrence $\phi_{x,0}(z) = q_x z + p_x z \phi_{x+1,0}(z)$, with initial condition $\phi_{1,0}(z)$: again, $\phi_{x,0}(z)$ can easily be deduced once $\phi_{1,0}(z)$ is known. The recurrence is also

$$1 - \phi_{x+1,0}(z) = \frac{z-1}{p_x z} + \frac{1 - \phi_{x,0}(z)}{p_x z}.$$

The full pgf of $\tau_{x,0}$ follows by recurrence. When $z=1$, $\phi_{x,0}(1)=\phi_x=\mathbb{P}(\tau_{x,0}<\infty)$ are the absorption probabilities already computed.
With $\phi(z)=(\phi_{1,0}(z),\phi_{2,0}(z),...)'$ the column-vector of the $\phi_{x,0}(z)$, and $\mathbf{q}=(q_1,q_2,...)'$ the column-vector of the q_x, $\phi(z)$ solves :

$$\phi(z)=z\mathbf{q}+z\overline{P}\phi(z), \tag{2.3.7}$$

whose formal solution is $\phi(z)=z\left(I-z\overline{P}\right)^{-1}\mathbf{q}$, involving the resolvent of $\overline{P}$ which is

$$\left(I-z\overline{P}\right)^{-1}=\begin{bmatrix} 1 & zp_1 & z^2p_1p_2 & z^3p_1p_2p_3 & z^4p_1p_2p_3p_4 & \cdots \\ & 1 & zp_2 & z^2p_2p_3 & z^3p_2p_3p_4 & \cdots \\ & & 1 & zp_3 & z^2p_3p_4 & \cdots \\ & & & 1 & zp_4 & \cdots \\ & & & & 1 & \cdots \\ & & & & & \ddots \end{bmatrix}.$$

We get

$$\phi_{x,0}(z)=\sum_{y\geq x}q_yz^{y-x+1}\prod_{y'=x}^{y-1}p_{y'}.$$

Equivalently, with $x\geq 1$, the pmf of $\tau_{x,0}$ reads

$$\mathbb{P}(\tau_{x,0}=k)=q_{k+x-1}\prod_{y'=x}^{k+x-2}p_{y'},\ k\geq 1.$$

The above first-step analysis clearly also makes sense if one starts from $x=0$ and the recurrence $\phi_{x,0}(z)=q_xz+p_xz\phi_{x+1,0}(z)$ also holds when $x=0$. We conclude

$$\phi_{0,0}(z)=q_0z+\sum_{y\geq 1}q_yz^{y+1}\prod_{y'=0}^{y-1}p_{y'}=\sum_{y\geq 0}q_yz^{y+1}\prod_{y'=0}^{y-1}p_{y'},$$

consistently with (4.2.52).

APPLICATION : for the model (2.2.1),

$$\mathbb{E}(\tau_{x,0})=\infty\Leftrightarrow\beta>1\text{ (transience) or }\beta=1\text{ and }\alpha\leq 1\text{ (null recurrence).}$$

If $\beta<1$ or $\beta=1$ and $\alpha>1$ (positive recurrence), $\mathbb{E}(\tau_{x,0})=\phi'_{x,0}(1)<\infty$.

2.3.1.f Transience versus recurrence

We here discuss the criterion for recurrence or transience of the general catastrophe Markov chain.

- When $C_1=\infty$, the recurrent chain started at x first hits 0 with probability 1 and returns infinitely often to 0. Given $X_0=x$, with $\mathcal{N}_{x,y}\equiv\sum_{n\geq 0}\mathbb{1}(X_n=y)$, the number of visits to state y, then $\mathcal{N}_{x,y}=\infty$, $\mathbb{P}_x-$almost surely. If $\tau_{x,x}$ is the first return time at x, then $\mathbb{P}(\tau_{x,x}<\infty)=1$. Furthermore, with $\mathcal{N}_{x,y}\equiv\sum_{n=0}^{\tau_{x,x}}\mathbb{1}(X_n=y)$ the number of visits to state y before the first return time to state x, then : $\mathbb{E}(\mathcal{N}_{x,y})=\frac{\pi_y}{\pi_x}$ and by the Chacon-Ornstein limit ratio ergodic theorem (see Chacon and Ornstein (1960) :

$$\frac{\sum_{n=0}^{N}\mathbb{1}(X_n=y)}{\sum_{n=0}^{N}\mathbb{1}(X_n=x)}\underset{N\nearrow\infty}{\rightarrow}\frac{\pi_y}{\pi_x},\ \mathbb{P}_x-\text{almost surely.}$$

Starting in particular from $x = 0$, a recurrent chain is made of infinitely many independent and identically distributed (i.i.d.) excursions which are the sample paths of $(X_n; n \geq 0)$ between consecutive visits to state 0. We have : $\mathbb{E}(\mathcal{N}_{0,x}) = \frac{\pi_x}{\pi_0} = \prod_{y=0}^{x-1} p_y$. When the chain is positive recurrent ($C_2 < \infty$) the expected time elapsed between consecutive visits to 0 is finite and equal to $\mathbb{E}(\tau_{0,0}) \equiv \mu = 1/\pi_0 = 1 + C_2$, whereas this expected time is infinite when the chain is null recurrent.

- When $C_1 < \infty$, the state $x \geq 0$ is transient.
Thus, $\mathcal{N}_{x,x} < \infty$, $\mathbb{P}_x-$almost surely and $\mathbb{P}(\mathcal{N}_{x,x} = k) = (1 - \alpha_x)\,\alpha_x^{k-1}$ where $\alpha_x = \mathbb{P}(\tau_{x,x} < \infty) < 1$.
Also, starting from $x \geq 1$, the walker hits at once 0 with probability $\phi_x = \mathbb{P}(\tau_{x,0} < \infty)$ and given this occurred it undergoes a number $\mathcal{N}_e = k$ excursions with $\mathbb{P}(\mathcal{N}_e = k) = (1 - \phi_{0,0})\,\phi_{0,0}^k$, $k \geq 0$ before drifting to ∞ for ever. With probability $1 - \phi_x$, the walker drifts to ∞ without ever visiting 0 and $\mathcal{N}_e = 0$. The time $\tau_x^{(d)}$ at which the walker (initially at x) starts drifting linearly to ∞ for ever is thus :

$$\tau_x^{(d)} := \left(\tau_{x,0} + \sum_{k=1}^{\mathcal{N}_e} \tau_{0,0}^{(k)}\right) \mathbb{1}(\tau_{x,0} < \infty)$$

where the $\tau_{0,0}^{(k)}$s are iid copies of $\tau_{0,0}$.

2.3.1.g First passage times and Green kernel

Let $\tau_{x,y}$ be the first passage time at $y \neq x$ when the process X is started at x. We wish here to briefly derive an exact formal formula for the law of $\tau_{x,y}$, making use of the Green function of a MC. Let

$$\phi_{x,y}(z) \equiv \sum_{k=1}^{\infty} z^k \mathbb{P}(\tau_{x,y} = k)$$

be the generating function of the law of $\tau_{x,y}$. Then, with

$$g_{x,y}(z) \equiv \sum_{n=0}^{\infty} z^n \mathbb{P}_x(X_n = y) = \sum_{n=0}^{\infty} z^n P^n(x,y) = (I - zP)^{-1}(x,y)$$

the generating function of $P^n(x,y)$ (the Green potential function of the chain), using $P^n(x,y) = \sum_{m=0}^{n} \mathbb{P}(\tau_{x,y} = m)\, P^{n-m}(y,y)$, we easily get the expression :

$$\phi_{x,y}(z) = \frac{g_{x,y}(z)}{g_{y,y}(z)}.$$

In particular,

$$\phi_{x,0}(z) = \frac{g_{x,0}(z)}{g_{0,0}(z)} \text{ and } \phi_{0,x}(z) = \frac{g_{0,x}(z)}{g_{x,x}(z)}$$

are the generating functions of $\tau_{x,0}$ and $\tau_{0,x}$.
The pgf $\phi_{x,x}(z)$ of the first-return time $\tau_{x,x}$ to state x satisfies

$$\phi_{x,x}(z) = \frac{g_{x,x}(z) - 1}{g_{x,x}(z)} = 1 - \frac{1}{g_{x,x}(z)}$$

where $g_{x,x}(z) = \sum_{n=0}^{\infty} z^n \mathbb{P}_x(X_n = x) = \sum_{n=0}^{\infty} z^n P^n(x,x)$ is the Green function at x. Recall (in the positive recurrent case with $\phi'_{x,x}(1) < \infty$)

$$\mathbb{E}(\tau_{x,x}) = \phi'_{x,x}(1) = 1/\pi_x.$$

In the specific MCC the resolvent can easily be computed. We find :

$$g_{x,x}(z) = (I - zP)^{-1}(x,x) = \frac{1 - \sum_{x'=0}^{x-1} z^{x'+1} q_{x'} \prod_{y=1}^{x'-1} p_y}{1 - \sum_{x' \geq 0} z^{x'+1} q_{x'} \prod_{y=0}^{x'-1} p_y}$$

giving the Green kernel as

$$\begin{aligned} g_{x,y}(z) &= g_{x,x}(z) z^{y-x} \prod_{y'=x}^{y-1} p_{y'} \text{ if } y \geq x \\ g_{x,y}(z) &= (g_{x,x}(z) - 1) z^{y-x} / \prod_{y'=y}^{x-1} p_{y'} \text{ if } 0 \leq y < x. \end{aligned}$$

Note that the denominator term of $g_{x,x}(z)$ is $1 - \phi_{0,0}(z)$ where $\phi_{0,0}(z)$ is the pgf of $\tau_{0,0}$ computed above in the special case. This yields

$$\phi_{x,x}(z) = \frac{\sum_{x' \geq x} z^{x'+1} q_{x'} \prod_{y=0}^{x'-1} p_y}{1 - \sum_{x'=0}^{x-1} z^{x'+1} q_{x'} \prod_{y=0}^{x'-1} p_y}.$$

In particular,

$$\alpha_x = \mathbb{P}(\tau_{x,x} < \infty) = \phi_{x,x}(1).$$

Note also that

$$g_{0,0}(z) = 1/(1 - \phi_{0,0}(z)) = \sum_{n=0}^{\infty} z^n \mathbb{P}_0(X_n = 0),$$

where $\mathbb{P}_0(X_n = 0)$ is the contact probability at 0 at time n.

APPLICATION (contact probability at 0) : With $\phi_{0,0}(z) = z(q_0 + p_0 \psi_0(z))$ in the critical special case.

- If $0 < \alpha < 1$, using $\psi_0(z) \underset{z \downarrow 1}{\sim} 1 - \frac{\Gamma(\nu+1)\Gamma(1+\alpha)}{\Gamma(\nu+1-\alpha)} (1-z)^{\alpha}$, we have

  $$1 - \phi_{0,0}(z) \underset{z \downarrow 1}{\sim} p_0 (1 - \psi_0(z)) \sim p_0 \frac{\Gamma(\nu+1)\Gamma(1-\alpha)}{\Gamma(\nu+1-\alpha)} (1-z)^{\alpha}$$

  showing by singularity analysis that, in the null recurrent case (algebraic decay of the contact probability)

  $$\mathbb{P}_0(X_n = 0) \underset{n \uparrow \infty}{\sim} \frac{\Gamma(\nu+1-\alpha)}{p_0 \Gamma(\nu+1)\Gamma(1-\alpha)\Gamma(\alpha)} n^{-(1-\alpha)}.$$

  When $\alpha = 1 - \varepsilon$ ($\varepsilon > 0$ small), the constant in front of $n^{-(1-\alpha)}$ vanishes like $\varepsilon/(p_0 \nu)$.
- When $\alpha = 1$, using $\psi_0(z) \underset{z \downarrow 1}{\sim} 1 - (1-z) - \nu (1-z) \log(1-z)$

  $$1 - \phi_{0,0}(z) = 1 - z(q_0 + p_0 \psi_0(z)) \underset{z \downarrow 1}{\sim} (1-z)(1+p_0) + p_0 \nu (1-z) \log(1-z),$$

  with logarithmic singularity, showing by singularity analysis, that

  $$\mathbb{P}_0(X_n = 0) \underset{n \uparrow \infty}{\sim} \frac{1}{p_0 \nu \log n}.$$

- When $\alpha > 1$ (positive recurrence case), using $\psi_0(z) \underset{z\downarrow 1}{\sim} 1 - \frac{\nu}{\alpha-1}(1-z) + O(1-z)^{\alpha}$,

$$1 - \phi_{0,0}(z) \underset{z\downarrow 1}{\sim} 1 - z\left(q_0 + p_0\left(1 - \frac{\nu}{\alpha-1}(1-z)\right)\right) \sim (1-z)\left(1 + p_0\frac{\nu}{\alpha-1}\right)$$

showing, as required that $\mathbb{P}_0(X_n = 0) \underset{n\uparrow\infty}{\to} \pi_0 = 1/\left(1 + p_0\frac{\nu}{\alpha-1}\right)$. When $\alpha = 1+\varepsilon$ ($\varepsilon > 0$ small) : $\pi_0 \sim \varepsilon/(p_0\nu)$, just like when $\alpha < 1$.

Remark 2.4. *(spectral aspects of P) : for the choice of (p_x, q_x) as from (2.2.1), in the critical case $\beta = 1$, the operator P can easily be checked not to be compact, nor even quasi-compact, Gosselin and Lebreton (2009). Looking for sequences $\mathbf{u}' = (u_0, u_1, u_2, ...)$ solving*

$$(I - zP)\mathbf{u} = \mathbf{0}$$

yields

$$u_x = \frac{u_0}{z^x \prod_{y=0}^{x-1} p_y}\left(1 - \sum_{x'=0}^{x-1} z^{x'+1} q_{x'} \prod_{y=0}^{x'-1} p_y\right),\ x \geq 0$$

which are well-defined in the Banach space $c_0 = \{\mathbf{u} : u_x \to 0 \text{ as } x \to \infty\}$ whenever $Re(z) \geq 1$. We conclude that for each z obeying $Re(z) \geq 1$, there are solutions $\mathbf{u}$ (eigen-states), vanishing at ∞, to $(I - zP)\mathbf{u} = 0$, defined up to a multiplicative constant. Letting $\lambda = z^{-1}$, we get that the closed disk of $\mathbb{C}$ centered at $(1/2, 0)$ with radius $1/2$ (which is :$Re(\lambda^{-1}) \geq 1$) constitutes the point spectrum of P. When λ belongs to the latter disk with radius $1/2$, $(\lambda I - P)^{-1}$ does not exist. Because there is a continuum of eigenvalues in the latter disk, the corresponding operator P has no spectral gap. The points λ belonging to the complementary of the latter disk in the unit disk centered at 0 constitute the continuous spectrum where $(\lambda I - P)^{-1}$ exists but is not defined on the whole space c_0. The λs belonging to the open domain $|\lambda^{-1}| < 1$ (the complementary of the unit disk of $\mathbb{C}$ centered at 0) are regular points of P for which $(\lambda I - P)^{-1}$ exists, is defined on the whole space c_0 and is bounded.

2.3.2 Including rates and time change : continuous-time

Consider now the continuous-time catastrophe MC $\overline{X}(t)$, so with $\overline{X}(t)$ generated by the transition rate matrix : $Q = D_{\mathbf{r}}(P - I)$ defined in (2.2.2). We now specify the results of the latter Section to this special MC in continuous-time, including the opportunity of a holding rate in the transition matrix of the DT catastrophe process. This special MC deserves special interest in particular because it is, to a large extent, amenable to exact analytic computations. The study of $\overline{X}$ should take into account the three parameters $\alpha, \beta > 0, \nu > -1$, $\alpha < \nu + 1$, together with the rates $r_x = r_0(x+1)^{\lambda} > 0$, $x \geq 0$.

2.3.2.a Invariant measure

We now investigate the way the invariant measure is modified by the adjunction of holding rates. Let $D_{\mathbf{r}} = \text{diag}(r_0, r_1, r_2, ...)$. We have $Q = D_{\mathbf{r}}(P - I)$. Let $\overline{\pi}$ be the invariant measure associated to Q, when it exists. It should solve $\mathbf{0}' = \overline{\pi}'Q$, and we get :

$$r_0 p_0 \overline{\pi}_0 = \sum_{x\geq 1} \overline{\pi}_x q_x r_x \text{ and } \overline{\pi}_x = \overline{\pi}_0 r_0 \frac{\prod_{y=0}^{x-1} p_y}{r_x},\ x \geq 1.$$

Using the second equation, the first equation is satisfied whenever

$$\sum_{x\geq 1} q_x r_x \frac{\prod_{y=1}^{x-1} p_y}{r_x} = \sum_{x\geq 1} (u_x - u_{x+1}) = 1,$$

so again when $u_x = \prod_{y=1}^{x-1} p_y \underset{x\to\infty}{\to} 0$ which is fulfilled if and only if $C_1 \equiv \sum_{y=1}^{\infty} q_y = \infty$ (the recurrence condition for X) : the time change leading from X to $\overline{X}$ does not change the road map of X so the recurrence criterion is identical for both X and $\overline{X}$. However, the criterion for positive recurrence is modified.
Indeed, if in addition $\overline{C}_2 \equiv \sum_{x\geq 1} \left(\frac{r_x}{r_0}\right)^{-1} \prod_{y=0}^{x-1} p_y < \infty$, then $\overline{\pi}_0 = \frac{1}{1+\overline{C}_2} \in (0,1)$ and then the MC is positive recurrent with invariant probability measure

$$\overline{\pi}_x = \overline{\pi}_0 r_0 \frac{\prod_{y=0}^{x-1} p_y}{r_x},\ x \geq 0.$$

Else if $\overline{C}_2 = \infty$, the MC $\overline{X}$ is null recurrent.

APPLICATION : When dealing with the special continuous-time Markov chain (CTMC), we conclude :

- $C_1 < \infty$ if and only if $\beta > 1$: in this case the MC is transient with no invariant measure.
- If $\beta < 1$, then $\overline{C}_2 < \infty$: the MC is positive recurrent. Furthermore, for large x :

  $$\overline{\pi}_x \sim x^{-\lambda} \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y^\beta}\right) \sim x^{-\lambda} e^{-\alpha \sum^x y^{-\beta}} \sim x^{-\lambda} e^{-\frac{\alpha}{1-\beta} x^{1-\beta}}$$

  with stretched exponential behaviour.
- In the critical case $\beta = 1$:

  $$\overline{\pi}_x \sim x^{-\lambda} \prod_{y=0}^{x-1} \left(1 - \frac{\alpha}{\nu + y}\right) \sim x^{-(\lambda+\alpha)}$$

  with power-law$(\alpha+\lambda)$ behaviour. The chain is recurrent. From this one can guess that the CT chain is positive recurrent if $\alpha + \lambda > 1$, null recurrent if $\alpha + \lambda \leq 1$. In particular, the chain is always positive recurrent if $\lambda > 1$.

2.3.2.b Scale function

$\overline{X}(t)$ is again the size at time t of the current population before the last catastrophic event. In the recurrent case ($C_1 = \infty$), $\overline{X}$ is again made of i.i.d. excursions, whose height $\overline{H}$ has the same law as the one for X : indeed, one can easily check that the scale function $\overline{\varphi}$ of $\overline{X}$ solving $Q\overline{\varphi} = 0$, $\overline{\varphi}_0 = 0$, coincide with the scale function φ of X solving $P\varphi = \varphi$, $\varphi_0 = 0$. Because the scale function determines the height's law, we have the claimed statement that the height's law is left unchanged by time substitution.

2.3.2.c Excursions lengths

However, because of the time change, the times $\overline{\tau}_{0,0}$ between consecutive visits to 0 (the excursions lengths of $\overline{X}$) are of course very different from $\tau_{0,0}$ (statistically longer). With $H \overset{d}{=} \tau_{0,0} - 1$ and $E_y \sim \text{Exp}(r_y)$

$$\overline{\tau}_{0,0} = \sum_{y=0}^{H} E_y$$

with the E_xs mutually independent and independent of H. [Anderson (2012), page 21].
Given the height of an excursion is h, with

$$c_{h,x} = \prod_{y=0:y\neq x}^{h} \frac{r_y}{r_y - r_x}$$

we have

$$\mathbb{P}\left(\overline{\tau}_{0,0} > t \mid H = h\right) = \mathbb{P}\left(\sum_{x=0}^{h} E_x > t\right) = \sum_{x=0}^{h} c_{h,x} e^{-r_x t}.$$

and with $x_* = \arg\min\left(r_x : x = 0, ..., h\right) : \mathbb{P}\left(\overline{\tau}_{0,0} > t \mid H = h\right) \sim c_{h,x_*} e^{-r_{x_*} t}$ as t is large. Note $x_* = 0$ if $\lambda > 0$, $= h$ if $\lambda < 0$.
Consider the critical case ($\beta = 1$) with $\lambda < 1$. Estimating the Laplace-transform of $\overline{\tau}_{0,0}$ yields

$$\mathbb{E}\left(e^{-p\overline{\tau}_{0,0}}\right) = \sum_{h\geq 0} \mathbb{P}\left(H = h\right) \mathbb{E}\left(e^{-p\sum_{x=0}^{h} E_x}\right) = \sum_{h\geq 0} \mathbb{P}\left(H = h\right) \prod_{x=0}^{h} \frac{1}{1 + p/r_x}$$

$$\underset{p\to 0}{\sim} \sum_{h\geq 0} \mathbb{P}\left(H = h\right) e^{-p\sum_{x=0}^{h} 1/r_x} \sim \sum_{h\geq 0} \mathbb{P}\left(H = h\right) e^{-\frac{p}{r_0(1-\lambda)} h^{1-\lambda}} = \mathbb{E}\left(e^{-\frac{p}{r_0(1-\lambda)} H^{1-\lambda}}\right)$$

suggesting $\overline{\tau}_{0,0}$ is tail-equivalent to $\frac{1}{r_0(1-\lambda)} H^{1-\lambda}$. Recalling $\mathbb{P}\left(H \geq h\right) \approx h^{-\alpha}$, we get

$$\mathbb{P}\left(\overline{\tau}_{0,0} > t\right) \approx t^{-\alpha/(1-\lambda)}.$$

Clearly, $\mathbb{E}\left(\overline{\tau}_{0,0}\right) < \infty$ if and only if $\alpha/\left(1 - \lambda\right) > 1$ which, when $\lambda < 1$, is the positive-recurrence criterion.
If $\lambda > 1$,

$$\sum_{x=0}^{h} 1/r_x \sim \frac{1}{r_0\left(\lambda - 1\right)} \left(1 - h^{-(\lambda-1)}\right) \nearrow \frac{1}{r_0\left(\lambda - 1\right)}$$

suggesting $\overline{\tau}_{0,0} \sim \frac{1}{r_0(\lambda-1)} \left(1 - H^{-(\lambda-1)}\right)$ with finite mean, always. With $\mu_h = \sum_{x=0}^{h} 1/r_x$ ($\nearrow \mu = \frac{1}{r_0(\lambda-1)}$ as $h \to \infty$), this is consistent with [Janson (2018), Theorem 5.1, (ii)]

$$\mathbb{P}\left(\overline{\tau}_{0,0} > t \mid H = h\right) = \mathbb{P}\left(\sum_{x=0}^{h} E_x \geq t\right) \leq e e^{-t/\mu_h} < e e^{-t/\mu} \text{ when } t \text{ is large.}$$

2.3.2.d Time to first extinction

In the continuous-time setting, with $(E_y,\ y \geq x)$ mutually independent rvs with distribution $\text{Exp}(r_y)$, independent of $\tau_{x,0}$, we clearly have

$$\overline{\tau}_{x,0} = \sum_{y=x}^{\tau_{x,0}+x-1} E_y,$$

consistently with the first-step analysis giving :

$$\overline{\tau}_{x,0} \overset{d}{=} (1 - B_x) \cdot E_x + B_x \cdot \left(E_x + \overline{\tau}'_{x+1,0}\right).$$

2.3.3 Continuous state-space

Some of the results for the discrete-space, continuous-time, model $\overline{X}$ have analogues for a continuous-time Markov chain on the non-negative reals.
Suppose that given the process X moves up and is in a small neighborhood of state $x > 0$, it moves to a location $y > x$ with probability $f(y-x)\,dx$ where f is a probability density on $\mathbb{R}_+$. The continuous-time, continuous-state version of the process $\overline{X}$, say $\underline{X}$, has backward generator $G_{\underline{X}}$ whose action on real-valued bounded functions h on $[0,\infty)$ now is

$$G_{\underline{X}}h(x) = r_x \left\{ \left(p_x \int_0^\infty (h(x+z) - h(x)) f(z)\,dz + q_x (h(0) - h(x)) \right) \right\}.$$

Whenever it exists, the invariant density measure $(\underline{\pi}_x, x \geq 0)$ of $\underline{X}$ obeys

$$r_y \underline{\pi}_y = \int_0^y r_x p_x \underline{\pi}_x f(y-x)\,dx + \delta_{y,0} \int_0^\infty r_x p_x \underline{\pi}_x dx,$$

which is the continuous-state version of the equation $\overline{\pi}'Q = \mathbf{0}'$ characterizing $\overline{\pi}_x$, $x \in \mathbb{N}_0$.

2.4 Appendix

2.4.1 Discrete infinite-divisibility and self-decomposability

For the sake of completeness, let us briefly introduce the notion of self-decomposability.

Definition 2.1. *Let $X \geq 0$ be an integer-valued random variable. The probability generating function (pgf) $\phi(z) := \mathbb{E}\left(z^X\right)$ is the one of a discrete self-decomposable (SD) variable X if for any $u \in (0,1)$, there is a pgf $\phi_u(z)$ (depending on u) such that (see Steutel and van Harn (1979)),*

$$\phi(z) = \phi(1 - u(1-z)) \cdot \phi_u(z). \tag{2.4.1}$$

Define the u-thinned version of X, say $u \circ X$, as the random sum

$$u \circ X \overset{d}{=} \sum_{x=1}^{X} B_x(u), \tag{2.4.2}$$

with $(B_x(u))_{x\geq 1}$ a sequence of iid Bernoulli variables such that $\mathbb{P}(B_x(u) = 1) = u$, independent of X.

This binomial thinning operator, acting on discrete rvs, has been defined in Steutel and van Harn (1979) ; it stands as the discrete version of the change of scale $X \to u \cdot X$ for continuous rvs X. If $\phi(z)$ is the pgf of the SD random variable X obeying Eq. (2.4.1), then X can be additively (self-)decomposed as

$$X \overset{d}{=} u \circ X' + X_u. \tag{2.4.3}$$

Here, X and X' have the same distribution and $u \circ X'$ is independent of the remaining random variable X_u with pgf, say $\phi_u(z)$.

Remark 2.5. *The self-decomposability idea also (pre-)exists for continuous rvs on* $\mathbb{R}_+$ *:* $X > 0$ *is said to be SD if, for any* $u \in (0,1)$

$$X \overset{d}{=} u \cdot X' + X_u. \tag{2.4.4}$$

with X *and* X' *having the same distribution and* $u \cdot X'$ *being independent of the remaining random variable* $X_u > 0$. *If* $\Phi(\lambda) = \mathbb{E}\left(e^{-\lambda X}\right)$, $\lambda \geq 0$, *is to be the Laplace-Stieltjes transform (LST) of* X, *SD, then for any* $u \in (0,1)$, *there is a LST* $\Phi_u(\lambda)$ *(depending on* u*) such that*

$$\Phi(\lambda) = \Phi(\lambda u) \cdot \Phi_u(\lambda).$$

The two notions of self-decomposability are related as follows : let $Y > 0$ *be a continuous rv. Then* Y *is self-decomposable if and only if the discrete random variable supported by* $\mathbb{N}_0$ *defined by :* $X = P(Y)$ *(where* $P(Y)$ *is a Poisson rv with random intensity* Y*) is discrete self-decomposable (Corollary* 1 *of Sapatinas (1995)). Indeed,*

$$\phi_X(z) = \mathbb{E}\left(z^X\right) = \Phi_Y(1-z)$$

and, with $\phi_{X_u}(z) = \mathbb{E}\left(z^{X_u}\right) = \Phi_u(1-z) = \mathbb{E}\left(e^{-(1-z)Y_u}\right)$, *the pgf of* $X_u = P(Y_u)$

$$\phi_X(z) = \phi_X(1-u(1-z)) \cdot \phi_{X_u}(z) \Leftrightarrow \Phi_Y(\lambda) = \Phi_Y(\lambda u) \cdot \Phi_{Y_u}(\lambda).$$

In such cases, the pmf of X is related to the density f of Y by :

$$\mathbb{P}(X = x) = \frac{1}{x!} \int_0^\infty y^x e^{-y} f(y)\, dy, \ x \in \mathbb{N}_0.$$

This raises the question of which Y are SD ? We first need to recall the notion of an HCM rv. Following (Steutel and van Harn (2004), (5.15) p. 371), a HCM positive rv is one whose density f obeys that

$$\forall y > 0 : \text{ the function } x \to f(xy)\, f(y/x),\ x > 0$$

is completely monotone on $(2, \infty)$ as a function of $z = x + 1/x$. Completely monotone functions h obey : $(-1)^k h^{(k)}(x) \geq 0$ for all $k \geq 0$ in some range of x.
With $a > 0$, consider now the rv $Y = G(a)^{1/\beta}$, the $1/\beta-$power of $G(a) \overset{d}{\sim} \text{Gamma}(a,1)$. This rv is hyperbolically completely monotone (HCM) if and only if $|\beta| \leq 1$ (ex. 12.8 of Steutel and van Harn (2004)). This is also true of the so-called Generalized Inverse Gaussian rvs Halgreen (1979), with density

$$f_{\lambda,\delta_1,\delta_2}(x) = \left(\frac{\delta_1}{\delta_2}\right)^{\alpha/2} \frac{1}{2K_\alpha\left(\sqrt{\delta_1\delta_2}\right)} x^{\alpha-1} \exp\left\{-\frac{1}{2}\left(\delta_1 x + \delta_2/x\right)\right\},\ x > 0 \tag{2.4.5}$$

in the parameter range : $\delta_2 \geq 0$, $\delta_1 > 0$ if $\lambda > 0$, $\delta_2 > 0$, $\delta_1 \geq 0$ if $\lambda < 0$ and $\delta_2 > 0$, $\delta_1 > 0$ if $\lambda = 0$. Such densities include Gamma distributions ($\delta_2 = 0$) and inverse Gamma distributions ($\delta_1 = 0$) as particular cases. K_α is the modified Bessel function of the second kind.

HCM rvs constitute a subclass of Generalized-Gamma-Convolution (GGC) rvs (Proposition 2 of Bondesson (2015)) and GGC rvs are SD (Theorem 1 of Bondesson (2015)). We refer to Steutel and van Harn (2004), Section 5 and Bondesson (2012), for the precise definition of GGC rvs. So, whenever Y is a GGC rv, it is SD and $X = P(Y)$, as a Poisson-mixture with respect to a SD distribution, is discrete-SD. Self-decomposable distributions are unimodal.

Coming back to discrete self-decomposability itself, the following representation result is also known to hold true, see Steutel and van Harn (1979). Let $R(z)$ (with $r_0 = R(0) > 0$) be the canonical function defined through

$$\phi(z) = \mathbb{E}\left(z^X\right) = e^{-\int_z^1 R(z')dz'}. \tag{2.4.6}$$

The random variable X is discrete SD if and only if the function $h(z) := 1 - (1-z)R(z)/r_0$ defines a pgf such that $h(0) = 0$ (see Schreiber (1999), Lemma 2.13). Consequently, X is discrete SD if and only if, for some $r_0 > 0$, its pgf can be written in the form

$$\phi(z) = e^{-r_0 \int_z^1 \frac{1-h(z')}{1-z'}dz'}. \tag{2.4.7}$$

This means that the series coefficients $(r_x = [z^x]R(z),\ x \in \mathbb{N}_0)$ of $R(z)$ constitute a non-negative, non-increasing sequence of x (Theorem 4.13 p. 271 of Steutel and van Harn (1979)). As a result, the associated probability system $\mathbb{P}(X = x) := \pi_x$, $x \in \mathbb{N}_0$ of X, if SD, is unimodal, with mode at the origin if and only if $r_0 = \frac{\pi_1}{\pi_0} \leq 1$. The SD subclass of infinitely divisible distributions (ID) therefore consists of unimodal distributions, with mode possibly at the origin (Theorem 2.3 of Steutel and van Harn (1979)).
Note that X is ID if and only if the sequence $(r_x, x \in \mathbb{N}_0)$ is non-negative only, with (as can be checked from (2.4.6)) the sequences $(\pi_x,\ r_x,\ x \in \mathbb{N}_0)$ related by the convolution formula

$$(x+1)\pi_x = \sum_{y=0}^{x} \pi_y r_{x-y},\ x \in \mathbb{N}_0.$$

We recall that for rvs with integral support $\mathbb{N}_0$, the notion of an infinitely divisible rv coincides with the one of a compound Poisson rv.

Definition 2.2. *A compound Poisson rv is one which is obtained as an independent Poisson sum of positive iid rvs which are called compounding rvs.*

A random variable with probability mass 0 at 0 cannot be ID.
With $r > 0$, the pgf of ID rvs takes the form : $\phi(z) = e^{-r(1-h(z))}$ where $h(z)$ is the pgf of the compounding rvs, obeying $h(0) = 0$.

Remark 2.6. *Let $Y_\infty \geq 1$ be a rv such that $Y_\infty - 1$ is SD. In the study of the MCC we encountered the delicate problem of deciding whether or not the mixed rv $X_\infty \overset{d}{\sim} \pi_0\delta_0 + (1-\pi_0)Y_\infty$ was ID or SD. Consider the simpler case where $Y_\infty - 1$ is*

Geometric(p), *which is SD. Then*

$$\begin{aligned}
\phi_{X_\infty}(z) &= \pi_0 + (1-\pi_0)\frac{pz}{1-qz} = \frac{\pi_0 - (\pi_0 - p)z}{1-qz} \\
\log \phi_{X_\infty}(z) &= \log(\pi_0 - (\pi_0 - p)z) - \log(1-qz) \\
R(z) &= \log \phi_{X_\infty}(z)' = -\frac{\pi_0 - p}{\pi_0 - (\pi_0 - p)z} + \frac{q}{1-qz} \\
r_x &= [z^x] R(z) = q^{x+1} - \left(1 - \frac{p}{\pi_0}\right)^{x+1}
\end{aligned}$$

Observing $q \geq 1 - \frac{p}{\pi_0}$, *a necessary and sufficient condition for* X_∞ *to be ID is* $1 - \frac{p}{\pi_0} \geq 0$ *(*$\pi_0 \geq p$*), leading to* $r_x \geq 0$ *for all* $x \geq 0$. *Under this condition, it will be SD if and only if* $r_{x+1} \leq r_x$, *meaning*

$$\left(\frac{1 - \frac{p}{\pi_0}}{q}\right)^x \leq \pi_0 \text{ for all } x.$$

This will always be the case if in addition $1 - \frac{p}{\pi_0} \leq q\pi_0$, *equivalently if* $\pi_0 \leq p/q$. *If* $p \geq q$ *(*$p \geq 1/2$*),* X_∞ *is SD if* $\pi_0 \geq p$. *If* $p < q$ *(*$p < 1/2$*),* X_∞ *will be SD only if* $p \leq \pi_0 \leq p/q$. *In the range* $\pi_0 \in (p/q, 1)$ *it is only ID.*

Complete monotonicity : A rv X with support $\mathbb{N}$ is completely monotone if, for some probability measure π on $[0,1]$, the following Hausdorff representation holds

$$\begin{aligned}
\overline{F}(x) &:= \mathbb{P}(X > x) = \int_0^1 u^x \pi(du), \; x \in \{0,1,2,...\} \\
\mathbb{P}(X = x) &= \pi_x = \int_0^1 u^x (1-u)\pi(du), \; x \in \{1,2,...\}.
\end{aligned}$$

If this is the case, for all $x \in \{0,1,2,...\}$ (see p. 77 of Steutel and van Harn (2004))

$$(-1)^k \Delta^{(k)} \overline{F}(x) \geq 0, \text{ equivalently } (-1)^k \Delta^{(k)} \pi_x \geq 0$$

where $\Delta : \Delta h(x) = h(x+1) - h(x)$ is the right-shift operator and $\Delta^{(k)}$ its $k-$th iterate. If this is the case, the rv $X - 1$, with support $\mathbb{N}_0$, is completely monotone, log-convex and therefore infinitely divisible (see Theorem 10.4 p. 77 of Steutel and van Harn (2004)).

2.4.2 On simple Markov realizations of ID and SD distributions

The catastrophe Markov chains that have been studied in this draft showed up invariant measures that can be either ID or SD (such as $Y_\infty - 1$ always or X_∞ itself in some parameter range on p_0). Different Markov processes can have the same invariant equilibrium measure and here are natural ones whenever the latter is either ID or SD :

- **The ID case :** Consider a time-inhomogeneous Poisson process $P(R_t)$ with decaying rate function re^{-t} and intensity $R_t = r(1 - e^{-t})$, $r > 0$. Consider the compound-Poisson process (with independent increments) :

$$X_t = \sum_{k=1}^{P(R_t)} \Delta_k$$

where $(\Delta_k; k \geq 1)$ is the iid sequence of the positive jumps occurring at the jump times of $P(R_t)$. If $h(z) = \mathbb{E}\left(z^{\Delta_1}\right)$,

$$\phi_t(z) = \mathbb{E}\left(z^{X_t}\right) = e^{-R_t(1-h(z))} \underset{t\to\infty}{\to} e^{-r(1-h(z))}$$

which is the pgf of an ID rv. A mechanism responsible of the decay of the population, when balanced by incoming immigrants with sizes Δ, will produce an ID limiting population size.

- **The SD case** (subcritical branching with immigration) : Consider now a time-homogeneous compound Poisson process $P(rt)$, $t \geq 0$, $P(0) = 0$, so with pgf

$$\mathbb{E}\left(z^{P(rt)}\right) = \exp\left\{-rt\left(1 - h(z)\right)\right\}, \tag{2.4.8}$$

where $h(z)$, with $h(0) = 0$, is the pgf of the jumps arriving at the jump times of $P(rt)$ having rate $r > 0$. Let now

$$\varphi_t(z) = 1 - e^{-t}(1 - z), \tag{2.4.9}$$

be the pgf of a pure-death (rate-1) Greenwood branching process started with one particle at $t = 0$, Greenwood (1931). This expression of $\varphi_t(z)$ is easily seen to be the solution to $\dot{\varphi}_t(z) = f(\varphi_t(z)) = 1 - \varphi_t(z)$, $\varphi_0(z) = z$, as is usual for a pure-death continuous-time Bellman-Harris branching processes with affine branching mechanism $f(z) = r_d(1 - z)$ and fixing the death rate to be $r_d = 1$, Harris et al. (1963). The distribution function of the lifetime of the initial particle is thus $1 - e^{-t}$. Let X_t with $X_0 = 0$ be a random process counting the current size of some population for which a random number of individuals (determined by $h(z)$) immigrate at the jump times of $P(rt)$, each of which being independently and immediately subject to the latter pure death Greenwood process. We have

$$\phi_t(z) := \mathbb{E}\left(z^{X_t}\right) = \exp -r\int_0^t \left(1 - h\left(\varphi_{t-s}(z)\right)\right) ds,\ \phi_0(z) = 1, \tag{2.4.10}$$

with $\phi_t(0) = \mathbb{P}(X_t = 0) = \exp -r\int_0^t \left(1 - h\left(1 - e^{-s}\right)\right) ds$, the probability that the population is extinct at t. It holds

$$\phi_t(z) = e^{-r\int_0^t (1-h(1-e^{-s}(1-z)))ds} = e^{-r\int_{\varphi_0(z)=z}^{\varphi_t(z)} \frac{1-h(z')}{1-z'}dz'} \underset{t\to\infty}{\to} \phi_\infty(z) = e^{-r\int_z^1 \frac{1-h(z')}{1-z'}dz'}. \tag{2.4.11}$$

So, $X := X_\infty$, as the limiting population size of this pure-death process with immigration, is a SD rv, Van Harn et al. (1981). As in the ID case, a mechanism responsible of the decay of the population is balanced by incoming immigrants.

Remark 2.7. *If instead of a pure-death Greenwood branching process, the immigrants shrink, more generally, according to any* subcritical *branching process with branching mechanism f, as those whose pgf obeys $\dot{\varphi}_t(z) = f(\varphi_t(z))$, then, with $f'(1) < 0$, $\varphi_t(z) \to 1$ as $t \to \infty$ and*

$$\phi_t(z) = e^{-r\int_0^t (1-h(\varphi_s(z)))ds} = e^{-r\int_{\varphi_0(z)=z}^{\varphi_t(z)} \frac{1-h(z')}{f(z')}dz'} \to \phi_\infty(z) = e^{-r\int_z^1 \frac{1-h(z')}{f(z')}dz'}.$$

The obtained limiting pgf is the one of a self-decomposable rv induced by the subcritical semigroup $\varphi_t(z)$ generated by $f(z)$, Van Harn et al. (1981). Recall $f(z) = \varphi(z) - z$ where $\varphi(z)$ is the pgf of the branching number per capita in a Bellman-Harris process.

3

A generating function approach to Markov chains undergoing binomial catastrophes

Abstract. In a Markov chain population model subject to catastrophes, random immigration events (birth), promoting growth, are in balance with the effect of binomial catastrophes that cause recurrent mass removal (death). Using a generating function approach, we study two versions of such population models when the binomial catastrophic events are of a slightly different random nature. In both cases, we describe the subtle balance between the two birth and death conflicting effects.

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3.1 Introduction, motivations and background

In some simple growth process, some organism changes the amount of its constitutive cells at random as follows. In the course of its lifetime, the organism alternates at random busy and idle periods. In a busy growth period, it produces a random or fixed amount of new cells which are being added to its current stock of cells (say, the incremental or batch cells). In an idle catastrophic period, the organism stands at risk being subject say to virus/bacteria attacks or radiation, resulting in each of its constitutive cells being either lysed or lost after mutations with some fixed mortality probability, independently of each other. Under such a catastrophic event, the population size is thus reduced according to a binomial distribution with survival probability say c; hence the name binomial catastrophes. It will happen that the current size of the organism is reduced to zero at some random time. In a worst disaster scenario for instance, all cells can be lysed in a single idle period leading instantaneously to a first disastrous extinction event. From a first extinction event, the organism can then either recover taking advantage of a subsequent busy epoch and starting afresh from zero, or not, being stuck to zero for ever. In the first case, we shall speak of a local extinction, state zero being reflecting, while in the latter case eventual or global extinction is at stake, state zero being absorbing.

As just described in words, the process under concern turns out to be a Markov chain on the non-negative integers, displaying a subtle balance between generalized birth and death events. Whenever it is ergodic, there are infinitely many local extinctions and the pieces of sample paths separating consecutive passages to zero are called excursions, the height and length of which are of some relevance in the understanding of the population size evolution. Excursions indeed form independent and identically distributed (iid) blocks of this Markov chain.

Stochastic models subject to catastrophes have a wide application in different fields viz. bioscience, marketing, ecology, computer and natural sciences, etc. For instance,

- In recent years, the study of catastrophe models has attracted much attention due to their wide application in computer-communication networks and digital telecommunication systems where interruptions due to various types of virus attacks are referred to as catastrophes. Here, unfriendly events, like virus attacks, result in abrupt changes in the state of the system caused by the removal of some of its elements (packets), posing a major threat to these queueing systems. Continuous-time queueing models subject to total disasters have also been analyzed by a few researchers such as Baumann and Sandmann (2012), Boudali and Economou (2013). A detailed survey of catastrophic events occurring in communication networks has been carried out in Dabrowski (2015).

- In oversimplistic earthquakes' physics models, the state of the process may be viewed as the total energy embodied in the earth's crust system, obtained as the sum of the energies of its constitutive blocks, each carrying say a fixed unit quantum of energy. A busy period corresponds to an accumulation of energy epoch in the system, while an idle period yields a stress release event. More realistic growth/collapse or decay/surge models in the same spirit, although in the continuum, were considered in Eliazar and Klafter (2006a), Eliazar and Klafter (2007) and Eliazar and Klafter (2009), with physical applications in mind, including stress release mechanisms in the physics of earthquakes.

In forestry, a good management of the biomass depends on the prediction of how

steady growth periods alternate with tougher periods, due to the occurrence of cataclysms such as hurricanes or droughts or floods, hitting each tree in a similar way.

In Economy, pomp periods often alternate with periods of scarcity when a crisis equally strikes all the economic agents (the Joseph and Noah effects).

Some aspects of related catastrophe models were recently addressed in Brockwell et al. (1982a), Artalejo et al. (2007), Economou (2004), Economou and Fakinos (2008) and Cairns and Pollett (2004), chiefly in continuous times. For instance, Economou (2004) analyzed a continuous- time binomial catastrophe model wherein individuals arrive according to a compound Poisson process while catastrophes occur according to a renewal process. He obtained the distribution of the population size at post-catastrophe, arbitrary and pre-catastrophe epochs. In Artalejo et al. (2007), a population model with the immigration process subject to binomial and geometric catastrophes has been investigated; the authors obtained the removed population size and maximum population size between two extinctions (height of excursions). They also discussed the first extinction time and the survival time of a tagged individual. In Economou and Fakinos (2008) also, several approaches for the transient analysis of a total disaster model were introduced with an extension of the methodologies to the binomial catastrophe model. The authors of Kapodistria et al. (2016) carried out the transient analysis of a binomial catastrophe model with birth/immigration- death processes and obtained explicit expressions for factorial moments, which are further used to develop the population size distribution. Results for the equilibrium moments and population size distributions are also given. Finally, recently, Yajima and Phung-Duc (2019) obtained the stationary queue length of the $M^X/M/\infty$ queue with binomial catastrophes in the heavy traffic system, using a central limit theorem.

Let us finally mention that not all aspects of binomial catastrophes are included in the above formulation. In Wilcox and Elderd (2003), a density-dependent catastrophe model of threatened population is investigated. Density-dependence is shown here to have a significant effect on population persistence (mean time to extinction), with a decreasing mean persistence time at large initial population sizes and causing a relative in- crease at intermediate sizes. A density-dependent model involving total disasters was designed in Huillet (2011).

Discrete-time random population dynamics with catastrophes balanced by random growth has a long history in the literature, starting with Neuts (1994). Mathematically, binomial catastrophe models are Markov chains (MCs) which are random walks on the non-negative integers (as a semigroup), so differing from standard random walks on the integers (as a group) in that a one-step move down from some positive integer cannot take one to a negative state, resulting in transition probabilities being state-dependent, Dette (1996). Such MCs may thus be viewed as generalized birth and death chains. The transient and equilibrium behaviour of such stochastic population processes with either disastrous or mild binomial catastrophes is one of the purposes of this work. We aim at studying the equilibrium distribution of this process and deriving procedures for its approximate computation. Another issue of importance concerns the measures of the risk of extinction, first extinction time, time elapsed between two consecutive extinction times and maximum population size reached in between.

The detailed structure of the manuscript, attempting to realize this program, can

be summarized as follows :

- Section 2 is designed to introduce the model in probabilistic terms. We develop three particular important cases :

  - Survival probability $c = 1$. In this case, on a catastrophic event, the chain remains in its current state with no depletion of cells at all.

  - Survival probability $c = 0$. This is a case of total disasters for which, on a catastrophic event, the chain is instantaneously propelled to state 0.

  - the semi-stochastic growth/collapse scenario when the adjunction of incremental cells, on a growth event, is deterministic being reduced to a single element, and $c \neq \{0, 1\}$ so that binomial mortality is not degenerate.

- In Section 3, we discuss the conditions under which the chain is recurrent (positive or null) or transient. We emphasize that positive recurrence is generic, unless the batch random variables have unrealistic very heavy tails. We use a generating function approach which is well-suited to this analysis. When, as in the positive recurrent case, it is non trivial, we discuss the shape of the invariant probability mass function of the chain. Specifically,

  - When $c \in (0, 1)$, we show that the invariant probability mass function is the one of a compound Poisson (infinitely divisible) random variable, which is moreover discrete self-decomposable.

  - When $c = 0$, (total disasters), the invariant probability mass function is the one of a geometric sum of the incremental (batch) random variables, so in the compound Poisson class, but not necessarily self-decomposable.

- In Section 4, we analyze some important characteristics of the chain in the positive recurrent regime, making use of its Green (potential) kernel., including :

  - from the Green (potential) kernel at $(0, 0)$: the contact probability at state 0 and first return time to 0 (length of an excursion).

  - from the Green (potential) kernel at $(x_0, 0)$: the first extinction time at 0 when starting from $x_0 > 0$. We show that this random variable has geometric tails. Some representation results on the mean (persistence) time to extinction are supplied.

  - Height of an excursion making use of the idea of the scale or harmonic sequence of the chain.

- In Section 5, we sketch some brief insight on the way the parameters of the chain could be estimated from an $N-$sample of observations.

- Finally, a variant of the binomial catastrophe model is introduced and studied in Section 6. It was motivated by the following observation : suppose at each step of its evolution, the size of some organism either grows by a random number of batch cells (not reduced to a fixed number) or shrinks deterministically by only **one** unit (a semi-stochastic decay/surge scenario 'dual' to the semi-stochastic growth/collapse one). The above standard binomial catastrophe model is not able to represent a scenario where at least one individual is removed from the population at catastrophic events. To remedy this, we therefore define and analyze a variant of the standard binomial model where, on a catastrophic event, the binomial shrinking mechanism applies to all currently alive cells but one which is systematically removed. We also take control on the future of the population once it hits state zero. Despite the apparent small changes in the definition of this modified binomial catastrophe Markov chain, the impact on its asymptotic behavior is shown to be drastic.

3.2 The binomial catastrophe model

We first describe the model as a toy model one.

3.2.1 The model

Consider a discrete-time MC X_n taking values in $\mathbb{N}_0 := \{0, 1, 2, ...\}$. With $b_n(c)$, $n = 1, 2, ...$ an independent identically distributed (iid) sequence of Bernoulli random variables with success parameter $c \in (0, 1)$, let $c \circ X_n = \sum_{m=1}^{X_n} b_m(c)$ denote the Bernoulli thinning of X_n *. Let β_n, $n = 1, 2, ...$ be an iid birth sequence of random variables with values in $\mathbb{N} = \{1, 2, ...\}$. The dynamics of the Markov chain under concern here is a balance between birth and death events according as $(p + q = 1)$:

$$X_{n+1} = \begin{cases} 1 \circ X_n + \beta_{n+1} = X_n + \beta_{n+1}, & \text{with probability } p \\ c \circ X_n, & \text{with probability } q = 1 - p. \end{cases}$$

This model was considered by Neuts (1994), Ben-Ari et al. (2019). We have $c \circ X_n \sim \text{bin}(X_n, c)$ hence the name binomial catastrophe. The binomial effect is appropriate when, on a catastrophic event, the individuals of the current population each die (with probability $1 - c$) or survive (with probability c) in an independent and even way, resulting in a drastic depletion of individuals at each step. Owing to : $c \circ X_n = X_n - (1 - c) \circ X_n$, the number of stepwise removed individuals is $(1 - c) \circ X_n$ with probability (wp) q. This way of depleting the population size (at shrinkage times) by a fixed random fraction c of its current size is very drastic, especially if X_n happens to be large. Unless c is very close to 1 in which case depletion is modest (the case $c = 1$ is discussed below), it is very unlikely that the size of the upward moves will be large enough to compensate depletion while producing a transient chain drifting at ∞. We will make this very precise below.
Note also $X_n = 0 \Rightarrow X_{n+1} = \beta_{n+1}$ wp p, $X_{n+1} = 0$ wp q (translating reflection of X_n at 0 if $q > 0$).

Let $B_n(p)$, $n = 1, 2, ...$ be an iid sequence of Bernoulli random variables with $\mathbb{P}(B_1(p) = 1) = p$, $\mathbb{P}(B_1(p) = 0) = q$. Let $C_n(p, c) = B_n(p) + c\overline{B}_n(p) = c^{\overline{B}_n(p)}$ where $\overline{B}_n(p) = 1 - B_n(p)$. The above process' dynamics (driven by $B_n(p)\beta_n$) is compactly equivalent to

$$\begin{aligned} X_{n+1} &= c^{\overline{B}_{n+1}(p)} \circ X_n + B_{n+1}(p)\beta_{n+1}, \ X_0 = 0 \\ &= \overline{B}_{n+1}(p) \cdot c \circ X_n + B_{n+1}(p)(X_n + \beta_{n+1}), \end{aligned}$$

(X_n, β_{n+1}) being mutually independent. The thinning coefficients are now $c^{\overline{B}_{n+1}(p)}$, so random.

With $b_x := \mathbb{P}(\beta = x)$, $x \geq 1$ and $d_{x,y} := \binom{x}{y} c^y (1 - c)^{x-y}$ the binomial probability mass function (pmf), the one-step-transition matrix P of the MC X_n is given by :

$$P(0, 0) = q, P(0, y) = p\mathbb{P}(\beta = y) = pb_y, y \geq 1$$

$$P(x, y) = \begin{cases} q\binom{x}{y} c^y (1 - c)^{x-y} = qd_{x,y}, & x \geq 1 \text{ and } 0 \leq y \leq x \\ p\mathbb{P}(\beta = y - x) = pb_{y-x}, & x \geq 1 \text{ and } y > x. \end{cases} \tag{3.2.1}$$

∗. In words, the thinning operation acting on a discrete random variable is the natural discrete analog of scaling a continuous variable, i.e., multiplying it by a constant in $[0, 1]$.

If β has first and second moment finite, with $\overline{c} = 1 - c$, $\overline{c} \circ x \sim \text{bin}(x, \overline{c})$, as x gets large

$$\begin{aligned} m_1(x) &= \mathbb{E}\left((X_{n+1} - X_n) \mid X_n = x\right) = p\mathbb{E}(\beta) - q\overline{c}x \sim -q\overline{c}x \\ m_2(x) &= \mathbb{E}\left((X_{n+1} - X_n)^2 \mid X_n = x\right) = p\mathbb{E}\left(\beta^2\right) + q\left(\overline{c}x + \overline{c}^2 x(x-1)\right) \sim q\overline{c}^2 x^2 \end{aligned}$$

with

$$\frac{m_1(x)}{m_2(x)} \sim -\frac{1}{\overline{c}x}, \ x \text{ large.}$$

Note the variance of the increment is

$$\sigma^2\left(X_{n+1} - X_n \mid X_n = x\right) = p\sigma^2(\beta) + qc\overline{c}x \sim qc\overline{c}x.$$

3.2.2 Special cases

(i) **When** $c = 1$**,** the lower triangular part of P vanishes leading to

$$\begin{aligned} P(0,0) &= q, \ P(0,y) = p\mathbb{P}(\beta = y) = pb_y, \ y \geq 1 \\ P(x,y) &= 0, \ x \geq 1 \text{ and } 0 \leq y < x, \ P(x,x) = q, \ x \geq 1 \\ P(x,y) &= p\mathbb{P}(\beta = y - x) = pb_{y-x}, \ x \geq 1 \text{ and } y > x. \end{aligned}$$

The transition matrix P is upper-triangular with diagonal terms. The process X_n is non-decreasing, so it drifts to ∞.

(ii) **When** $c = 0$ (total disasters),

$$\begin{aligned} P(0,0) &= q, \ P(0,y) = p\mathbb{P}(\beta = y) = pb_y, \ y \geq 1 \\ P(x,y) &= 0, \ x \geq 1 \text{ and } 0 < y \leq x, \ P(x,0) = q, \ x \geq 1 \\ P(x,y) &= p\mathbb{P}(\beta = y - x) = pb_{y-x}, \ x \geq 1 \text{ and } y > x. \end{aligned}$$

When a downward move occurs, it takes instantaneously X_n to zero (a case of total disasters), independently of the value of X_n. This means that, defining $\tau_{x_0,0} = \inf(n \geq 1 : X_n = 0 \mid X_0 = x_0)$, the first extinction time of X_n, $\mathbb{P}(\tau_{x_0,0} = x) = qp^{x-1}$, $x \geq 1$, a geometric distribution with success parameter q, with mean $\mathbb{E}(\tau_{x_0,0}) = 1/q$, independently of $x_0 \geq 1$. Note that $\tau_{0,0}$, as the length of any excursion between consecutive visits to 0, also has a geometric distribution with success parameter q and finite mean $1/q$. In addition, the height H of an excursion is clearly distributed like $\sum_{x=1}^{\tau_{0,0}-1} \beta_x$ (with the convention $\sum_{x=1}^{0} \beta_x = 0$). Finally, this particular Markov chain clearly is always positive recurrent, whatever the distribution of β. Consecutive excursions are the iid pieces of this random walk on the non-negative integers.

Some Markov catastrophe models involving total disasters are described in Swift (2001), Huillet (2011).

(iii) If $\beta \sim \delta_1$ a move up results in the addition of only one individual, which is the simplest deterministic drift upwards. In this case, the transition matrix P is lower- Hessenberg. This model constitutes a simple discrete version of a semi-stochastic growth/collapse model in the continuum, Eliazar and Klafter (2006a).

3.3 Recurrence versus transience

Using a generating function approach, we start with the transient analysis before switching to the question of equilibrium.

3.3.1 The transient analysis

Let $B(z) = \mathbb{E}\left(z^{\beta}\right)$ be the pgf of $\beta = \beta_1$, as an absolutely monotone function on $[0,1]$ [†]. Let $\pi'_n = (\pi_n(0), \pi_n(1), ...)$ where $\pi_n(x) = \mathbb{P}_0(X_n = x)$ and $'$ denotes the transposition. With $\mathbf{z} = (1, z, z^2, ...)'$, a column vector obtained as the transpose $'$ of the row vector $(1, z, z^2, ...)$, define

$$\Phi_n(z) = \mathbb{E}_0\left(z^{X_n}\right) = \pi'_n \mathbf{z},$$

the pgf of X_n. With $D_{\mathbf{z}} = \text{diag}(1, z, z^2, ...)$, $\mathbf{z} = D_{\mathbf{z}}\mathbf{1}$, the time evolution $\pi'_{n+1} = \pi'_n P$ yields

$$\Phi_{n+1}(z) = \pi'_{n+1}\mathbf{z} = \pi'_n P\mathbf{z} = \pi'_n P D_{\mathbf{z}}\mathbf{1},$$

leading to the transient dynamics

$$\Phi_{n+1}(z) = pB(z)\Phi_n(z) + q\Phi_n(1 - c(1-z)), \Phi_0(z) = 1. \tag{3.3.1}$$

The fixed point pgf of X_∞, if it exists, solves

$$\Phi_\infty(z) = pB(z)\Phi_\infty(z) + q\Phi_\infty(1 - c(1-z)). \tag{3.3.2}$$

(i) **When $c = 1$,** there is no move down possible. The only solution to $\Phi_\infty(z) = pB(z)\Phi_\infty(z) + q\Phi_\infty(z)$ is $\Phi_\infty(z) = 0$, corresponding to $X_\infty \sim \delta_\infty$. Indeed, when $c = 1$, combined to $\Phi_\infty(1) = 1$,

$$\begin{aligned} \Phi_{n+1}(z) &= (q + pB(z))\Phi_n(z), \Phi_0(z) = 1 \\ \Phi_n(z)^{1/n} &= q + pB(z), \end{aligned}$$

showing that, if $1 \leq \rho := B'(1) = \mathbb{E}(\beta) < \infty$, $n^{-1}X_n \to q + p\rho \geq 1$ almost surely as $n \to \infty$. The process X_n is transient in that, after a finite number of passages in state 0, it drifts to ∞.

(ii) **When $c = 0$** (total disasters), combined to $\Phi_\infty(1) = 1$, (3.3.2) yields

$$\Phi_\infty(z) = \frac{q}{1 - pB(z)} =: \phi_\Delta(z), \tag{3.3.3}$$

as an admissible pgf solution. In (3.3.3), we introduced the integral-valued random variable Δ whose pgf is $\phi_\Delta(z) = \mathbb{E}\left(z^{\Delta}\right) = q/(1 - pB(z))$ obtained while compounding a shifted-geometric pgf $q/(1-pz)$ with the pgf $B(z)$ of the βs [‡]. We conclude that in the total disaster setup when $c = 0$, the law of X_∞ is obtained as a compound shifted-geometric sum Δ of the βs, whatever the distribution of β. Note that the random variable Δ clearly is the height of any total disaster excursion, as the sample path between any two consecutive visits of X_n to 0. The length of such excursions clearly is geometric with success probability q, in that case.

†. A function B is said to be absolutely monotone on $(0,1)$ if it has all its derivatives $B^{(n)}(z) \geq 0$ for all $z \in (0,1)$. Pgfs are absolutely monotone and the composition of two pgfs is a pgf.

‡. A geometric(q) random variable with success probability q takes values in $\mathbb{N} = \{1, 2, ...\}$. A shifted geometric$(q)$ random variable with success probability q takes values in $\mathbb{N}_0 = \{0, 1, 2, ...\}$. It is obtained while shifting the former one by one unit.

3.3.2 Existence and shape of an invariant pmf ($c \in [0,1)$)

We shall distinguish two cases.

- **The case** $c \in (0,1)$.

From (3.3.2) and (3.3.3), the limit law pgf $\Phi_\infty(z)$, if it exists, solves the functional equation

$$\Phi_\infty(z) = \phi_\Delta(z)\,\Phi_\infty(1-c(1-z)), \tag{3.3.4}$$

so that, formally

$$\Phi_\infty(z) = \prod_{n\geq 0}\phi_\Delta(1-c^n(1-z)), \tag{3.3.5}$$

as an infinite product pgf.

Proposition 3.1. *The invariant measure exists for all* $c \in (0,1)$ *if and only if* $\mathbb{E}(\log_+ \Delta) < \infty$.

Proof. (Theorem 2 in Neuts (1994)) : By a comparison argument, we need to check the conditions under which $\pi(0) = \Phi_\infty(0)$ converges to a positive number. We get

$$\begin{aligned}\Phi_\infty(0) &= \prod_{n\geq 0}\phi_\Delta(1-c^n) > 0 \Leftrightarrow \sum_{n\geq 0}(1-\phi_\Delta(1-c^n)) < \infty \\ &\Leftrightarrow \int_0^1 \frac{1-\phi_\Delta(z)}{1-z}dz < \infty \Leftrightarrow \sum_{x\geq 1}\log x\,\mathbb{P}(\Delta = x) = \mathbb{E}(\log_+\Delta) < \infty,\end{aligned}$$

meaning that Δ has a finite logarithmic first moment. □

For most βs therefore, the process X_n is positive recurrent, in particular if β has finite mean.
When β has finite first and second order moments, so do Δ and X_∞ which exist. Indeed :
If $B'(1) = \mathbb{E}\beta = \rho < \infty$, (with $\mathbb{E}\Delta = (p\rho)/q$)

$$\Phi'_\infty(1) = q\left(\frac{p\rho}{q^2} + \frac{c}{q}\Phi'_\infty(1)\right) \Rightarrow \Phi'_\infty(1) = \mathbb{E}(X_\infty) =: \mu = \frac{p\rho}{q(1-c)} < \infty$$

If $B''(1) < \infty$, X_∞ has finite variance :

$$\Phi''_\infty(1) - \Phi'_\infty(1)^2 = \frac{p}{q}\frac{1}{1-c^2}\left(B''(1) + 2\frac{p}{q}\frac{2c-1}{1-c}\rho^2\right)$$

Counter-example : With $\beta, C > 0$, suppose that $\mathbb{P}(\beta > x) \sim_{x\uparrow\infty} C(\log x)^{-\beta}$ translating that β has very heavy logarithmic tails (any other than logarithmic slowly varying function would do the job as well). Then $\mathbb{E}\beta^q = \infty$ for all $q > 0$ and β has no moments of arbitrary positive order. Equivalently, $B(z) \sim_{z\downarrow 1} 1 - \frac{C}{(-\log(1-z))^\beta}$.
Therefore, with $C' = pC/q$,

$$\phi_\Delta(z) = \frac{q}{1-pB(z)} \sim_{z\downarrow 1} 1 - \frac{C'}{(-\log(1-z))^\beta}$$

translating that $\mathbb{P}(\Delta > y) \sim_{y\uparrow\infty} C'(\log y)^{-\beta}$ shares the same tail behavior as β. From this, $\mathbb{P}(\log\Delta > x) \sim_{j\uparrow\infty} C'x^{-\beta}$ so that $\log\Delta$ has a first moment if and only if

$\beta > 1$. For such a (logarithmic tail) model of β, we conclude that X remains positive recurrent if $\beta > 1$ and starts being transient only if $\beta < 1$. The case $\beta = 1$ is a critical null-recurrent situation.
Being strongly attracted to 0, the binomial catastrophe model exhibits a recurrence/transience transition but only for such very heavy-tailed choices of β. Recall that :

- When positive recurrent, the chain visits state 0 infinitely often and the expected return time to 0 has finite mean.
- When null recurrent, the chain visits state 0 infinitely often but the expected return time to 0 has infinite mean.
- When transient, the chain visits state 0 a finite number of times before drifting to ∞ for ever after an infinite number of steps (no finite time explosion is possible for discrete-time Markov chains).

Corollary 3.1. *If the process is null recurrent or transient, no non-trivial ($\neq \mathbf{0}'$) invariant measure exists.*

Proof. This is because, $\Phi_\infty(z)$ being an absolutely monotone function on $[0,1]$ if it exists,

$$\pi(0) = \Phi_\infty(0) = 0 \Rightarrow \Phi_\infty(1-c) = 0 \Rightarrow \pi(x) = 0 \text{ for all } x \geq 1.$$

□

Clustering (sampling at times when thinning occurs, time change) :
Let $G = \inf(n \geq 1 : B_n(p) = 0)$, with $\mathbb{P}(G = k) = p^{k-1}q$, $\mathbb{E}\left(z^{G-1}\right) = \frac{q}{1-pz}$. G is the time elapsed between two consecutive catastrophic events. So long as there is no thinning of X (a catastrophic event), the process grows of $\Delta = \sum_{k=1}^{G-1} \beta_k$ individuals. Consider a time-changed process of X whereby one time unit is the time elapsed between consecutive catastrophic events. During this laps of time, the original process X_n grew of Δ individuals, before shrinking to a random amount of its current size at catastrophe times. We are thus led to consider the time-changed integral-valued Ornstein-Uhlenbeck process [also known as an Integer-Valued Autoregressive of Order 1 (in short INAR(1)) process**,** see McKenzie (2003)] **:**

$$X_{k+1} = c \circ X_k + \Delta_{k+1},\ X_0 = 0,$$

with Δ_k, $k = 1, 2, ...$ an iid sequence of compound shifted geometric random variables. In this form, X_k is also a pure-death subcritical branching process with immigration, Δ_{k+1} being the number of immigrants at generation $k+1$, independent of X_k. With $\Phi_k(z) = \mathbb{E}\left(z^{X_k}\right)$, we have

$$\Phi_{k+1}(z) = \frac{q}{1 - pB(z)} \Phi_k(1 - c(1-z)),\ \Phi_0(z) = 1.$$

The limit law (if it exists) $\Phi_\infty(z)$ also solves (3.3.4). Thus

$$\Phi_\infty(z) = \prod_{n \geq 0} \phi_\Delta(1 - c^n(1-z)),$$

corresponding to

$$X_\infty \stackrel{d}{=} \sum_{n \geq 0} c^n \circ \Delta_{n+1}.$$

As conventional wisdom suggests, the time-changed process has the same limit law as the original binomial catastrophe model, so if and only if the condition $\mathbb{E}\log_+\Delta<\infty$ holds.

Proposition 3.2. *When the law of X_∞ exists, it is discrete self-decomposable (SD).*

Proof. This follows, for example, from Theorem 3.1 of Bouzar et al. (2008) and the INAR(1) process representation of (X_k). See Steutel and van Harn (1979) for an account on discrete SD distributions, as a remarkable subclass of compound Poisson ones. □

The random variable X_∞ being SD, it is unimodal, with mode at the origin if $\pi(1)<\pi(0)$, or with two modes at $\{0,1\}$ if $\frac{\pi(1)}{\pi(0)}=1$ (see Steutel and van Harn (2004), Theorem 4.20).
With $\mathbb{P}(\Delta=1)=\phi'_\Delta(0)=pq\mathbb{P}(\beta=1)$, we have

$$\begin{aligned}\pi(0) &= \Phi_\infty(0)=q\Phi_\infty(1-c)\\ \pi(1) &= \Phi'_\infty(0)=\mathbb{P}(\Delta=1)\Phi_\infty(1-c)+qc\Phi'_\infty(1-c)\\ &= \frac{\mathbb{P}(\Delta=1)}{q}\pi(0)+qc\Phi'_\infty(1-c)>p\mathbb{P}(\beta=1)\pi(0)\end{aligned}$$

A condition for unimodality at 0 is thus

$$(\log\Phi_\infty)'(1-c)<\frac{1-p\mathbb{P}(\beta=1)}{c}. \tag{3.3.6}$$

Note also

$$\begin{aligned}\pi(1) &= \Phi'_\infty(0)=\sum_{m\geq0}c^m\phi'_\Delta(1-c^m)\prod_{n\neq m}\phi_\Delta(1-c^n)\\ &= \pi(0)\sum_{m\geq0}c^m(\log\phi_\Delta)'(1-c^m)\\ &= \pi(0)\sum_{m\geq0}c^m\frac{pB'(1-c^m)}{1-pB(1-c^m)}\end{aligned}$$

giving a closed-form condition for unimodality at 0. For instance, if $B(z)=z$, $\pi(1)<\pi(0)$ if and only if

$$\sum_{m\geq0}\frac{pc^m}{q+pc^m}<1.$$

Tails of X_∞.
The probabilities $\pi(x)=[z^x]\Phi_\infty(z)$, $x\geq1$ (the z^x-coefficient in the power series expansion of $\Phi_\infty(z)$) are hard to evaluate. However, some information on the large x tails $\sum_{y>x}\pi(y)$ can be estimated in some cases.

- Consider a case where $B(z)\sim C\cdot(1-z/z_c)^{-1}$ as $z\to z_c>1$ so that $\mathbb{P}(\beta>x)\sim C\cdot z_c^{-x}$ has geometric tails. As detailed below,

$$\phi_\Delta(z)\sim C'\cdot(1-z/z'_c)^{-1}\ \text{ as } z\to z'_c>1$$

so that, with $C'=pqC/(1-pC)<1$, $\mathbb{P}(\Delta>x)\sim C'\cdot z_c'^{-x}$ also has geometric (heavier) tails but with modified rate $z'_c=z_c(1-pC)<z_c$.

Then

$$\Phi_\infty(z) \sim C'' \cdot (1 - z/z'_c)^{-1} \text{ as } z \to z'_c > 1$$

so that $\mathbb{P}(X_\infty > x) \sim C' \cdot z_c'^{-x}$ also has geometric tails with rate z'_c. Indeed, with $z''_c = (z'_c - (1-c))/c > z'_c$,

$$\begin{aligned} C'' \cdot (1 - z/z'_c)^{-1} &\sim C'C'' \cdot (1 - z/z'_c)^{-1} (1 - (1 - c(1-z))/z'_c)^{-1} \\ &= C'C'' \left(\frac{z'_c}{cz''_c}\right) \cdot (1 - z/z'_c)^{-1} (1 - z/z''_c)^{-1} \end{aligned}$$

showing that

$$\Phi_\infty(z) \sim C'C'' \left(\frac{z'_c}{cz''_c}\right) (1 - z'_c/z''_c)^{-1} \cdot (1 - z/z'_c)^{-1}, \text{ as } z \to z'_c > 1.$$

- Consider a positive recurrent case with $B'(1) = \mathbb{E}\beta = \infty$. This is the case if, with $\alpha \in (0,1)$, $B(z) \sim 1 - (1-z)^\alpha$ as $z \to 1$, or if (unscaled Sibuya, see Sibuya (1979)) $B(z) = 1 - (1-z)^\alpha$. In this case $\phi_\Delta(z) \sim 1 - \frac{p}{q}(1-z)^\alpha$ as a scaled Sibuya random variable (with scale factor $\frac{p}{q} = \mathbb{E}(G-1)$). Indeed,

$$\frac{q}{1 - p(1 - (1-z)^\alpha)} = \frac{1}{1 + \frac{p}{q}(1-z)^\alpha} \underset{z \to 1}{\sim} 1 - \frac{p}{q}(1-z)^\alpha$$

Then, X_n being recurrent, in view of $\Phi_\infty(z) = \frac{q}{1-pB(z)} \Phi_\infty(1 - c(1-z))$,

$$\Phi_\infty(z) \underset{z \to 1}{\sim} 1 - \gamma(1-z)^\alpha$$

where $\gamma = p/[q(1-c)^\alpha]$. Indeed, as $z \to 1$,

$$1 - \gamma(1-z)^\alpha \sim \left[1 - \frac{p}{q}(1-z)^\alpha\right] [1 - \gamma c^\alpha (1-z)^\alpha]$$

allowing to identify the scale parameter γ. The three random variables β, Δ and X_∞ have power law tails with index α.

• **The case $c = 0$ (total disasters).**
In that case, combined to $\Phi_\infty(1) = 1$,

$$\Phi_\infty(z) = \frac{q}{1 - pB(z)} = \phi_\Delta(z)$$

is an admissible pgf solution. The law of X_∞ is a compound shifted-geometric of the βs, whatever the distribution of β.

The probabilities $\pi(x) = [z^x]\Phi_\infty(z)$, $x \geq 1$ are explicitly given by the Faa di Bruno formula for compositions of two pgfs, Comtet (1970). Let us look at the tails of X_∞.

- If in particular, with $C \in (0,1)$, $B(z) \sim C \cdot (1 - z/z_c)^{-1}$ as $z \to z_c > 1$ so that $\mathbb{P}(\beta > x) \sim C \cdot z_c^{-x}$ has geometric tails, then

$$\Phi_\infty(z) \sim C' \cdot (1 - z/z'_c)^{-1} \text{ as } z \to z'_c > 1$$

so that, with $C' = pqC/(1-pC) < 1$, $\mathbb{P}(X_\infty > x) \sim C' \cdot z_c'^{-x}$ also has geometric (heavier) tails but with modified rate $z'_c = z_c(1-pC) < z_c$.

- If $B(z) = \left(e^{\theta z} - 1\right) / \left(e^{\theta} - 1\right)$ (β is Poisson conditioned to be positive) is entire. $\Phi_\infty(z)$ has a simple pole at $z_c > 1$ defined by

$$\left(e^{\theta z_c} - 1\right) / \left(e^{\theta} - 1\right) = 1/p$$

and X_∞ has geometric tails with rate z_c.

- If, with $\alpha \in (0,1)$, $B(z) \sim 1 - (1-z)^\alpha$ as $z \to 1$, or if (unscaled Sibuya) $B(z) = 1 - (1-z)^\alpha$, then $\Phi_\infty(z) \sim 1 - \frac{p}{q}(1-z)^\alpha$ scaled Sibuya (with scale factor $\frac{p}{q} = \mathbb{E}(G-1)$:

$$\frac{q}{1 - p\left(1 - (1-z)^\alpha\right)} = \frac{1}{1 + \frac{p}{q}(1-z)^\alpha} \underset{z \to 1}{\sim} 1 - \frac{p}{q}(1-z)^\alpha$$

- Suppose, with $z_0 > 1$,

$$B(z) = \frac{1 - (1 - z/z_0)^\alpha}{1 - (1 - 1/z_0)^\alpha}$$

If there exists $z_c > 1 : B(z_c) = 1/p$ else if $B(z_0) = 1/\left(1 - (1-1/z_0)^\alpha\right) > 1/p$ ($(1-1/z_0)^\alpha > q$ or $z_0 > 1/\left(1 - q^{1/\alpha}\right)$), then $\Phi_\infty(z)$ has a simple algebraic pole at z_c. If no such z_c exists, $\Phi_\infty(z)$ is entire. This is reminiscent of a condensation phenomenon. We finally obserandom variablee that

$$\frac{1 - \phi_\Delta(z)}{1 - z} = \frac{p(1 - B(z))}{(1-z)(1 - pB(z))} = \frac{p}{q}\frac{1 - B(z)}{1 - z}\phi_\Delta(z),$$

$$\begin{aligned}
\mathbb{P}(\Delta > n) &= \frac{p}{q}\sum_{m=0}^{n} \mathbb{P}(\beta > n - m)\,\mathbb{P}(\Delta = m), \\
\mathbb{P}(\Delta = n) &= \frac{p}{q}\left(\sum_{m=0}^{n-1} \mathbb{P}(\beta = n - m)\,\mathbb{P}(\Delta = m) + \mathbb{P}(\Delta = n)\right).
\end{aligned}$$

When is Δ with $\phi_\Delta(z) = \frac{q}{1 - pB(z)}$ itself SD ? In any case, Δ is at least infinitely divisible (ID) else compound Poisson because $\phi_\Delta(z) = \exp -r(1 - \psi(z))$ where $r > 0$ and $\psi(z)$ is a pgf with $\psi(0) = 0$. Indeed, with $q = e^{-r}$,

$$\psi(z) = \frac{-\log(1 - pB(z))}{-\log q}$$

is a pgf (the one of a Fisher-log-series random variable).

Proposition 3.3. *With $b_x = [z^x]\,B(z)$, $x \geq 1$, the condition*

$$\frac{b_{x+1}}{b_x} \leq \frac{x - pb_1}{x+1} \text{ for any } x \geq 1, \tag{3.3.7}$$

entails that Δ is self decomposable (SD).

Proof. If Δ is self decomposable then (see Schreiber (1999), Lemma 2.13)

$$\phi_\Delta(z) = e^{-r\int_z^1 \frac{1 - h(z')}{1 - z'} dz'},$$

for some $r > 0$ and some pgf $h(z)$ obeying $h(0) = 0$. We are led to check if

$$\frac{pB'(z)}{1 - pB(z)} = r\frac{1 - h(z)}{1 - z},$$

for some pgf h and $r = pb_1$ where

$$h(z) = 1 - \frac{1}{b_1}(1 - z)\frac{B'(z)}{1 - pB(z)} = \frac{1}{b_1}\frac{b_1(1 - pB(z)) - (1 - z)B'(z)}{1 - pB(z)}.$$

Denoting the numerator $N(z)$, a sufficient condition is that

$$[z^x] N(z) \geq 0 \text{ for all } x \geq 1$$

But

$$N(z) = \sum_{x \geq 1} z^x \left[(x - pb_1) b_x - (x + 1) b_{x+1}\right].$$

□

Let us show on four examples that these conditions can be met.

1. Suppose $B(z) = z$. Then $0 = \frac{b_{x+1}}{b_x} \leq \frac{x-p}{x+1}$ for all $x \geq 1$. The simple shifted-geometric random variable Δ is SD.

2. Suppose $B(z) = b_1 z + b_2 z^2$ with $b_2 = 1 - b_1$. We need to check conditions under which $\frac{b_2}{b_1} \leq \frac{1-pb_1}{2}$. This condition is met if and only if the polynomial $pb_1^2 - 3b_1 + 2 \leq 0$ which holds if and only if $b_1 \geq b_1^*$ where $b_1^* \in (0, 1)$ is the zero of this polynomial in $(0, 1)$.

3. Suppose $B(z) = \overline{\alpha}z/(1 - \alpha z)$, $\alpha \in (0, 1)$, the pgf of a geometric$(\overline{\alpha})$ random variable, with $b_x = \overline{\alpha}\alpha^{x-1}$. The condition reads : $\alpha \leq \frac{x-p\overline{\alpha}}{x+1}$. It is fulfilled if $\alpha \leq \frac{x-p}{x+q}$ for all $x \geq 1$ which is $\alpha \leq q/(1 + q) < 1$ (or $\overline{\alpha} = b_1 \geq b_1^* = 1/(1 + q)$).

4. Sibuya. Suppose $B(z) = 1 - (1 - z)^\alpha$, $\alpha \in (0, 1)$, with $b_x = \alpha [\overline{\alpha}]_{x-1} / x!$, $x \geq 1$ (where $[\overline{\alpha}]_x = \overline{\alpha}(\overline{\alpha} + 1) ... (\overline{\alpha} + x - 1)$, $x \geq 1$ are the rising factorials of $\overline{\alpha}$ and $[\overline{\alpha}]_0 := 1$). The condition reads : $\frac{\overline{\alpha}+x-1}{x+1} \leq \frac{x-p\alpha}{x+1}$ which is always fulfilled. The shifted-geometric random variable with Sibuya distributed compounding random variable is always SD.

Proposition 3.4. *Under the condition that X_∞ is SD and so unimodal, X_∞ has always mode at the origin.*

Proof. The condition is that, with $\pi(0) = \Phi_\infty(0) = q$ and $\pi(1) = \Phi'_\infty(0) = pqB'(0) = pqb_1$, $\pi(1)/\pi(0) = pb_1 < 1$ which is always satisfied. □

3.4 Green (potential) kernel analysis

In this Section, we analyze some important characteristics of the chain making use of its Green (potential) kernel.

3.4.1 Green (potential) kernel at $(0,0)$: Contact probability at 0 and first return time to 0

Suppose $X_0 = 0$. With then $\Phi_0(z) = 1$, define the double generating function $\Phi(u,z) = \sum_{n\geq 0} u^n \Phi_n(z)$, obeying $\Phi(u,1) = 1/(1-u)$, see Woess (2009). [Note that $\log z$ was Laplace-conjugate to X_n and now $\log u$ is Laplace-conjugate to n]. Then,

$$\frac{1}{u}(\Phi(u,z) - 1) = pB(z)\Phi(u,z) + q\Phi(u, 1 - c(1-z))$$

$$\Phi(u,z) = \frac{1 + qu\Phi(u, 1-c(1-z))}{1 - puB(z)} \tag{3.4.1}$$

$$\Phi(u,0) = 1 + qu\Phi(u, 1-c).$$

With $H(u,z) := 1/(1 - puB(z))$, upon iterating, with $\Phi(0,z) = \Phi_0(z) = 1$

$$\Phi(u,z) = \sum_{n\geq 0} (qu)^n \prod_{m=0}^{n} H(u, 1 - c^m(1-z)). \tag{3.4.2}$$

In particular

$$\Phi(u,0) = \sum_{n\geq 0} (qu)^n \prod_{m=0}^{n} H(u, 1-c^m) = 1 + \sum_{n\geq 1} \prod_{m=1}^{n} \frac{qu}{1 - puB(1-c^m)}. \tag{3.4.3}$$

Note that

$$G_{0,0}(u) := \Phi(u,0) = \sum_{n\geq 0} u^n \mathbf{P}_0(X_n = 0) = (I - uP)^{-1}(0,0)$$

is the Green kernel of the chain at $(0,0)$ (the matrix element $(0,0)$ of the resolvent of P). Consequently,

Proposition 3.5. *The Green kernel $G_{0,0}(u)$ is given by (3.4.3).*
With $h_{m'} := [u^{m'}] \prod_{m=0}^{n-1} (1 - puB(1-c^m))^{-1}$, we have the following expression for the contact probability at 0 :

$$[u^n]\Phi(u,0) = \Phi_n(0) = \mathbb{P}_0(X_n = 0) = \sum_{m'=0}^{n-1} q^{n-m'} h_{m'}. \tag{3.4.4}$$

Remark 3.1. *(i) Let us have a quick check of this formula. When $n = 1$, this leads to $\mathbb{P}_0(X_1 = 0) = q$, and, if $n = 2$ to $\mathbb{P}_0(X_2 = 0) = q^2 + q[u^1](1 - puB(1-c))^{-1} = q^2 + pqB(1-c)$. The second part is not quite trivial because it accounts for any movement up in the first step (wp p) immediately followed by a movement down to 0. This is consistent however with the binomial formula $\mathbb{P}(X_1 = 0 \mid X_0 = x) = q\mathbb{P}(c \circ x = 0) = q(1 - c(1-z))^x \mid_{z=0} = q(1-c)^x$ so that the second part is*

$$p\sum_{x\geq 1} \mathbb{P}(\beta = x)\mathbb{P}(X_1 = 0 \mid X_0 = x) = pq\sum_{x\geq 1} b_x (1-c)^x = pqB(1-c).$$

(ii) With $B_m := pB(1-c^m)$, $m \geq 1$, an increasing $[0,1]-$ valued sequence converging to p, decomposing the product $\prod_{m=1}^{n-1}(1 - uB_m)^{-1}$ into simple fraction elements, we get if $n > 1$

$$h_{m'} = \sum_{m=1}^{n-1} A_{m,n} B_m^{m'},$$

where $A_{m,n} = B_m^{n-2} / \prod_{m' \in \{1,..,n-1\} \setminus \{m\}} (B_m - B_{m'})$. *The term* $A_{n-1} B_{n-1}^{m'}$ *contributes most to the sum. Hence,* $\mathbb{P}_0 (X_1 = 0) = q$ *and if* $n > 1$

$$\begin{aligned} \mathbb{P}_0 (X_n = 0) &= q^n \sum_{m'=0}^{n-1} q^{-m'} \sum_{m=1}^{n-1} A_{m,n} B_m^{m'} = q^n \sum_{m=1}^{n-1} A_{m,n} \sum_{m'=0}^{n-1} \left(\frac{B_m}{q} \right)^{m'} \\ &= q \sum_{m=1}^{n-1} A_{m,n} \frac{q^n - B_m^n}{q - B_m} \end{aligned}$$

is an alternative representation of (3.4.4).
In the positive recurrent case, as $n \to \infty$,

$$\mathbb{P}_0 (X_n = 0) \to \pi (0) = \prod_{n \geq 0} \phi_\Delta (1 - c^n) > 0.$$

The Green kernel at $(0,0)$ is thus $G_{0,0} (u) = \Phi (u, 0)$.
If $n \geq 1$, from the recurrence $\mathbb{P}_0 (X_n = 0) = P^n (0,0) = \sum_{m=0}^n \mathbb{P} (\tau_{0,0} = m) P^{n-m} (0,0)$, we see that the pgf $\phi_{0,0} (u) = \mathbb{E} (u^{\tau_{0,0}})$ of the first return time to 0, $\tau_{0,0}$ and $G_{0,0} (u)$ are related by the Feller relation (see Bingham (2001) pp $3-4$ for example)

$$G_{0,0} (u) = \frac{1}{1 - \phi_{0,0} (u)} \text{ and } \phi_{0,0} (u) = \frac{G_{0,0} (u) - 1}{G_{0,0} (u)}.$$

Hence,

Proposition 3.6. *The pgf* $\phi_{0,0} (u)$ *of the first return time* $\tau_{0,0}$ *is*

$$\phi_{0,0} (u) = 1 - \frac{1}{G_{0,0} (u)}, \tag{3.4.5}$$

where $G_{0,0} (u)$ *is given by (3.4.3).*

Note

$$G_{0,0} (1) = \sum_{n \geq 0} \mathbb{P}_0 (X_n = 0) = 1 + \sum_{n \geq 1} q^n \prod_{m=0}^{n} H (1, 1 - c^m) = \infty$$

if and only if X is recurrent, Neveu (1964), Sato (2020). And in that case, $\phi_{0,0} (1) = \mathbb{P} (\tau_{0,0} < \infty) = 1 - \frac{1}{G_{0,0}(1)} = 1$. Positive-(- null) recurrence is when $\phi'_{0,0} (1) = \mathbb{E} (\tau_{0,0}) = 1/\pi_0 < \infty \ (= \infty)$, Kac (1947). Note finally $G_{0,0} (0) = 1$ so that $\phi_{0,0} (0) = \mathbb{P} (\tau_{0,0} = 0) = 0$.

3.4.2 Starting from $x_0 > 0$: Green kernel at $(x_0, 0)$ and first extinction time $\tau_{x_0,0}$

Suppose now $X_0 = x_0 > 0$. After shifting X_n of x_0, with $\Phi (u, z) = \sum_{n \geq 0} u^n \mathbb{E} \left(z^{x_0 + X_n} \right)$, we now get

$$\frac{1}{u} (\Phi (u, z) - z^{x_0}) = pB (z) \Phi (u, z) + q \Phi (u, 1 - c (1 - z)).$$

Then

$$\Phi (u, z) = \frac{z^{x_0} + qu \Phi (u, 1 - c (1 - z))}{1 - puB (z)}, \tag{3.4.6}$$

entailing

$$\begin{aligned}\Phi(u,z) &= \sum_{n\geq 0}(qu)^n\left(1-c^n(1-z)\right)^{x_0}\prod_{m=0}^{n}H\left(u,1-c^m(1-z)\right),\\ \Phi(u,0) &= \sum_{n\geq 0}(qu)^n\left(1-c^n\right)^{x_0}\prod_{m=0}^{n}H\left(u,1-c^m\right).\end{aligned}$$

We obtained :

Proposition 3.7. *The contact probability at* 0 *for the chain started at* $x_0>0$ *is given by*

$$[u^n]\,\Phi(u,0)=\Phi_n(0)=\mathbb{P}_{x_0}(X_n=0)=\sum_{m'=0}^{n-1}\left(1-c^{n-m'}\right)^{x_0}q^{n-m'}h_{m'}. \tag{3.4.7}$$

Let us give the first two terms as compared to when $x_0=0$. As required, when $n=1$, $\mathbb{P}_{x_0}(X_1=0)=q(1-c)^{x_0}$ and when $n=2$,

$$\mathbb{P}_{x_0}(X_2=0)=q^2\left(1-c^2\right)^{x_0}+pq(1-c)^{x_0}B(1-c)$$

a weighted sum of the two terms appearing in the above expression of $\mathbb{P}_0(X_2=0)$.

Corollary 3.2. *(i) When* x_0 *is large and* n *fixed, the small but dominant term is when* $m'=0$ *which is* $q^n(1-c^n)^{x_0}$. *So* $\mathbb{P}_{x_0}(X_n=0)$ *decays geometrically with* x_0. *This expression quantifies the probability that the population is in an early state of extinction given the initial population size was large. Early is when* $c^n\gg 1/x_0$ *(so that* $(1-c^n)^{x_0}\ll(1-1/x_0)^{x_0}\ll e^{-1}$*), so when* $n\ll-\log_c x_0$.

(ii) In the transient case, when n *is large and* x_0 *is fixed, the dominant term is when* $m'=n-1$*which is* $q(1-c)^{x_0}(pB(1-c))^n$. *So* $\mathbb{P}_{x_0}(X_n=0)$ *decays geometrically with* n *at rate* $pB(1-c)$. *In the positive recurrent case,* $\mathbb{P}_{x_0}(X_n=0)\to\pi(0)>0$ *as* $n\to\infty$*, independently of* x_0.

The Green kernel at $(x_0,0)$ is thus $G_{x_0,0}(u)=[z^0]\,\Phi(u,z)$ (the matrix element $(x_0,0)$ of the resolvent of P). It is related to the pgf of the first extinction time $\tau_{x_0,0}$ by the Feller relation

$$\phi_{x_0,0}(u)=\mathbb{E}\left(u^{\tau_{x_0,0}}\right)=\frac{G_{x_0,0}(u)}{G_{0,0}(u)}. \tag{3.4.8}$$

Therefore,

Proposition 3.8. *With* $x_0>0$*, the pgf of the first extinction time* $\tau_{x_0,0}$ *is*

$$\phi_{x_0,0}(u)=\mathbb{E}\left(u^{\tau_{x_0,0}}\right)=\frac{\sum_{n\geq 1}(qu)^n\left(1-c^n\right)^{x_0}\prod_{m=1}^{n}H\left(u,1-c^m\right)}{\sum_{n\geq 0}(qu)^n\prod_{m=0}^{n}H\left(u,1-c^m\right)}. \tag{3.4.9}$$

In the recurrent case, state 0 is visited infinitely often and so both $G_{0,0}(1)$ and $G_{x_0,0}(1)=\infty$, and

$$\mathbb{P}\left(\tau_{x_0,0}<\infty\right)=\phi_{x_0,0}(1)=\frac{G_{x_0,0}(1)}{G_{0,0}(1)}=1.$$

We finally note that, because state 0 is reflecting, $\tau_{x_0,0}$ is only a local extinction time, followed by subsequent extinction times after $\tau_{0,0}$. In the sequel, we shall let $\overline{P}$ stand for the substochastic transition matrix obtained from P while deleting its first row and column.

There are two alternative representations of $\phi_{x_0,0}(u)$.

• One alternative representation follows from the following classical first-step analysis :
Let $X_1(x_0)$ be the position of X_n started at x_0 .
Let $X_+(x_0)$ be a positive random variable with $\mathbb{P}(X_+(x_0)=y)=\overline{P}(x_0,y)/\sum_{y\geq 1}\overline{P}(x_0,y)$, $y\geq 1$. With $\tau'_{X_+(x_0),0}$ a statistical copy of $\tau_{X_+(x_0),0}$, first-step analysis yields, (see Norris (1998) , Woess (2009)) :

$$\tau_{x_0,0}\overset{d}{=}1\cdot\mathbb{1}_{\{X_1(x_0)=0\}}+\mathbb{1}_{\{X_1(x_0)>0\}}\cdot\left(1+\tau'_{X_+(x_0),0}\right).$$

Clearly, $\mathbb{P}(X_1(x_0)=0)=P(x_0,0)=q(1-c)^{x_0}=:q_{x_0}$, $\mathbb{P}(X_1(x_0)>0)=:p_{x_0}=\sum_{y\geq 1}\overline{P}(x_0,y)=1-q_{x_0}$. Therefore $\phi_{x_0,0}(u):=\mathbb{E}(u^{\tau_{x_0,0}})$, obeys the recurrence $\phi_{x_0,0}(u)=q_{x_0}u+up_{x_0}\mathbb{E}\phi_{X_+(x_0),0}(u)$.
With $\phi(u)=(\phi_{1,0}(u),\phi_{2,0}(u),...)'$ the column-vector of the $\phi_{x_0,0}(u)=\mathbb{E}u^{\tau_{x_0,0}}$, and $\mathbf{q}=(q_1,q_2,...)'$ the first column-vector of the matrix P, $\phi(u)$ then solves :

$$\phi(u)=u\mathbf{q}+u\overline{P}\phi(u), \tag{3.4.10}$$

whose formal solution is (compare with the explicit expression (3.4.9))

$$\phi(u)=u\left(I-u\overline{P}\right)^{-1}\mathbf{q}=:u\overline{G}(u)\,\mathbf{q}, \tag{3.4.11}$$

involving the resolvent matrix $\overline{G}(u)$ of $\overline{P}$. Note $\phi(1)=\left(I-\overline{P}\right)^{-1}\mathbf{q}$ gives the column-vector of the probabilities of eventual extinction $\phi(1):=(\phi_{1,0}(1),\phi_{2,0}(1),...)'$, so with $\phi_{x_0,0}(1)=\mathbb{P}(\tau_{x_0,0}<\infty)$ if $\overline{G}(1)\,\mathbf{q}<\infty$. Clearly $\phi(1)=\mathbf{1}$ (the all-one column vector) in the recurrent case. In that case, from (3.4.10), introducing the column vector $\mathbb{E}(\tau_{.,0}):=(\mathbb{E}(\tau_{1,0}),\mathbb{E}(\tau_{2,0}),...)'$ where $\mathbb{E}(\tau_{x_0,0})=\phi'_{x_0,0}(1)$ and obse-random variableing $\mathbf{q}+\overline{P}\phi(1)=\mathbf{1}$, we get

$$\phi'(1):=\mathbb{E}(\tau_{.,0})=\mathbf{1}+\overline{P}\mathbb{E}(\tau_{.,0}),\ \text{equivalently}\ \mathbb{E}(\tau_{.,0})=\left(I-\overline{P}\right)^{-1}\mathbf{1}=\overline{G}(1)\,\mathbf{1},\ \text{or}$$

$$\mathbb{E}(\tau_{x_0,0})=\sum_{y\geq 1}\overline{G}_{x_0,y}(1),$$

where $\overline{G}_{x_0,y}$ is the matrix element (x_0,y) of the resolvent of $\overline{P}$.

• Yet another alternative representation $\phi(u):=(\phi_{1,0}(u),\phi_{2,0}(u),...)'$ is as follows. From the identity

$$\overline{P}^n(x_0,y)=\mathbb{P}_{x_0}(X_n=y,\tau_{x_0,0}>n),$$

we get $\mathbb{P}(\tau_{.,0}>n)=\overline{P}^n\mathbf{1}$,as the column vector of $(\mathbb{P}(\tau_{1,0}>n),\ \mathbb{P}(\tau_{2,0}>n),...)'$, and so,

$$\sum_{n\geq 0}u^n\mathbb{P}(\tau_{.,0}>n)=\overline{G}(u)\,\mathbf{1}.$$

This leads in particular, as expected, to $\mathbb{E}(\tau_{.,0})=\overline{G}(1)\,\mathbf{1}$ and (compare with (3.4.11) and the explicit expression (3.4.9)) to

$$\phi(u)=\sum_{n\geq 0}u^n\mathbb{P}(\tau_{.,0}=n)=\mathbf{1}-(1-u)\,\overline{G}(u)\,\mathbf{1}. \tag{3.4.12}$$

We obtained :

Proposition 3.9. *We have* $\phi(1) = \mathbb{P}(\tau_{\cdot,0} < \infty)$ *and so* $\overline{G}(1)\mathbf{1} < \infty \Rightarrow \mathbb{P}(\tau_{\cdot,0} < \infty) = \mathbf{1}$*, meaning recurrence of* X_n*. In fact, positive recurrence is precisely when* $\mathbb{E}(\tau_{\cdot,0}) = \overline{G}(1)\mathbf{1} < \infty$*. If* $\overline{G}(1)\mathbf{1} = \infty$*, the chain is null-recurrent if* $(1-u)\overline{G}(u)\mathbf{1} \to \mathbf{0}$ *as* $u \to 1$*, transient if* $(1-u)\overline{G}(u)\mathbf{1} \to \mathbb{P}(\tau_{\cdot,0} = \infty)$ *as* $u \to 1$*, a non-zero limit.*

The matrix $\overline{P}$ is substochatic with spectral radius $\rho \in (0,1)$. With $\mathbf{r}$ and $\mathbf{l}'$ the corresponding right and left positive eigenvectors of $\overline{P}$, so with $\overline{P}\mathbf{r} = \rho\mathbf{r}$ and $\mathbf{l}'\overline{P} = \rho\mathbf{l}'$, $\overline{P}^n \sim \rho^n \cdot \mathbf{r}\mathbf{l}'$ (as n is large) where $\mathbf{r}\mathbf{l}'$ is the projector onto the first eigenspace. By Perron-Frobenius theorem, Vere-Jones (1967), Glynn and Desai (2018), we can normalize $\mathbf{l}$ to be of l_1- norm one to get

Proposition 3.10. *In the positive recurrent case for* X_n *(*$\mathbb{E}\log_+ \Delta < \infty$*):*
(i) With $r(x_0)$ *the* x_0-*entry of* $\mathbf{r}$*,*

$$\rho^{-n}\mathbb{P}(\tau_{x_0,0} > n) \to r(x_0), \text{ as } n \to \infty, \tag{3.4.13}$$

showing that $\mathbb{P}(\tau_{x_0,0} > n)$ *has geometric tails with rate* ρ *(extinction is fast).*
(ii) With $l(y)$ *the* $y-$ *entry of* $\mathbf{l}$*, for all* $x_0 > 0$*,*

$$\mathbb{P}_{x_0}(X_n = y \mid \tau_{x_0,0} > n) \to l(y), \text{ as } n \to \infty, \tag{3.4.14}$$

showing that the left eigenvector $\mathbf{l}$ *is the quasi-stationary distribution of* X_n *(or Yaglom limit, Yaglom (1947)), Collet et al. (2013).*

Proof. In this case, with $R = \rho^{-1} > 1$, the convergence radius of $\overline{G}$, $\overline{G}(R) = \infty$ and $\overline{P}$ is $R-$positive recurrent.
(i) follows from $\mathbb{P}(\tau_{\cdot,0} > n) = \overline{P}^n\mathbf{1}$ and
(ii) from $\mathbb{P}_{x_0}(X_n = y \mid \tau_{x_0,0} > n) = \overline{P}^n(x_0, y)/\overline{P}^n\mathbf{1}$. □

Remark 3.2. *The full Green kernel at* (x_0, y_0) *is* $G_{x_0,y_0}(u) = [z^{y_0}]\,\Phi(u,z)$*. Hence*

$$G_{x_0,y_0}(u) = \sum_{n\geq 0}(qu)^n [z^{y_0}](1 - c^n(1-z))^{x_0} \prod_{m=0}^{n} H(u, 1 - c^m(1-z)) \tag{3.4.15}$$

$$= \sum_{n\geq 0}(qu)^n \sum_{y=0}^{y_0} h_{n,y}(u)\, g_{n,y_0-y}$$

where

$$g_{n,y} = [z^y](1 - c^n(1-z))^{x_0} = \binom{x_0}{y} c^{ny}(1-c^n)^{x_0-y}$$

and

$$h_{n,y}(u) = [z^y] \prod_{m=0}^{n} H(u, 1 - c^m(1-z)),$$

which can be obtained from a decomposition into simple elements of the inner product. Using $P^n(x_0, y_0) = \sum_{m=1}^{n} \mathbb{P}(\tau_{x_0,y_0} = m) P^{n-m}(y_0, y_0)$*,* $n \geq 1$ *we easily get the expression of the pgf of the first hitting times* $\tau_{x_0,y_0} = \inf(n \geq 1 : X_n = y_0 \mid X_0 = x_0)$*, as*

$$\phi_{x_0,y_0}(z) = \frac{G_{x_0,y_0}(z)}{G_{y_0,y_0}(z)}. \tag{3.4.16}$$

3.4.3 The height H of an excursion

Assume $X_0 = x_0$ and consider a version of X_n which is absorbed at 0. Let $X_{n\wedge\tau_{x_0,0}}$ stopping X_n when it first hits 0. Let us define the scale (or harmonic) function or sequence φ of X_n as the function which makes $Y_n \equiv \varphi\left(X_{n\wedge\tau_{x_0,0}}\right)$ a martingale. The function φ is important because, for all $0 \leq x_0 < h$, with $\tau_{x_0} = \tau_{x_0,0} \wedge \tau_{x_0,h}$ the first hitting time of $\{0, h\}$ starting from x_0 (assuming $\varphi(0) \equiv 0$)

$$\mathbb{P}\left(X_{\tau_{x_0}} = h\right) = \mathbb{P}\left(\tau_{x_0,h} < \tau_{x_0,0}\right) = \frac{\varphi(x_0)}{\varphi(h)},$$

resulting from

$$\mathbb{E}\varphi\left(X_{n\wedge\tau_{x_0}}\right) = \varphi(x_0) = \varphi(h)\,\mathbb{P}\left(\tau_{x_0,h} < \tau_{x_0,0}\right) + \varphi(0)\,\mathbb{P}\left(\tau_{x_0,h} > \tau_{x_0,0}\right).$$

• **The case $\beta_1 \sim \delta_1$.** Let us consider the height H of an excursion of the original MC X_n first assuming $\beta_1 \sim \delta_1$ (a birth event adds only one individual). With probability q, $H = 0$ and with probability p, starting from $X_1 = 1$, it is the height of a path from state 1 to 0 of the absorbed process X_n. Using this remark, the event $H = h$ is realized when $\tau_{1,h} < \tau_{1,0}$ and $\tau_{h,h+1} > \tau_{h,0}$, the latter two events being independent. Thus (with $\mathbb{P}(H = 0) = q$) :

$$\mathbb{P}(H = h) = p\frac{\varphi(1)}{\varphi(h)}\left(1 - \frac{\varphi(h)}{\varphi(h+1)}\right),\ h \geq 1. \tag{3.4.17}$$

We clearly have $\sum_{h\geq 1}\mathbb{P}(H = h) = p$ because partial sums form a telescoping series. But (3.4.17) is also

$$\mathbb{P}(H \geq h) = 1/\varphi(h),\ h \geq 1, \tag{3.4.18}$$

with $\varphi(1) = 1/p$.

It remains to compute φ with $\varphi(0) = 0$ and $\mathbb{P}(H \geq 1) = 1/\varphi(1) = p$. We wish to have : $\mathbb{E}_{x_0}(Y_{n+1} \mid Y_n = x) = x$, leading to

$$\varphi(x) = p\varphi(x+1) + q\sum_{y=1}^{x}\binom{x}{y}c^y(1-c)^{x-y}\varphi(y),\ x_0 \geq 1.$$

The vector φ is the right eigenvector associated to the eigenvalue 1 of the modified version P^* of the stochastic matrix P having 0 as an absorbing state : $(P^*(0,0) = 1)$, so with : $\varphi = P^*\varphi$, $\varphi(0) = 0$, Norris (1998). The searched 'harmonic' function is increasing and given by recurrence, $\varphi(1) = 1/p$ and

$$\varphi(x+1) = \frac{1}{p}\left(\varphi(x)\left[1 - qc^x\right] - q\sum_{y=1}^{x-1}\binom{x}{y}c^y(1-c)^{x-y}\varphi(y)\right),\ x \geq 1 \tag{3.4.19}$$

The first two terms are

$$\begin{aligned}
\varphi(2) &= \frac{1}{p}(1-qc)\varphi(1) = \frac{1}{p^2}(1-qc),\\
\varphi(3) &= \frac{1}{p}\left(\varphi(2)\left[1-qc^2\right] - 2qc(1-c)\varphi(1)\right),\\
&= \frac{1}{p^3}(1-qc)\left(1-qc^2\right) - 2\frac{q}{p}c(1-c).
\end{aligned}$$

The sequence $\varphi(x)$ is diverging when the chain X is recurrent.

Proposition 3.11. *When $\beta \sim \delta_1$, equations (3.4.18) and (3.4.19) characterize the law of the excursion height H of the random walker in the recurrent case. In the transient case, $\varphi(x)$ converges to a value φ^* and $\mathbb{P}(H=\infty) = 1/\varphi^* = \mathbb{P}(\tau_{0,0} = \infty)$.*

• **General β_1.** Whenever the law of β is general, the matrix P^* is no longer lower Hessenberg and the harmonic vector $\varphi = P^*\varphi$, with $\varphi(0) = 0$, cannot be obtained by a recurrence. However, the event $H \geq h \geq 1$ is realized whenever a first birth event occurs with size $\beta_1 \geq h$ or, if $\beta_1 < h$, whenever for all states $h' \geq h$ being hit when the amplitude β of a last upper jump is larger than $h' - h$, then $\tau_{\beta_1,h'} < \tau_{\beta_1,0}$. Hence,

Proposition 3.12. *For a recurrent walker with general β, $\mathbb{P}(H=\infty) = 0$ where, when $h \geq 1$,*

$$\begin{aligned}\mathbb{P}(H \geq h) &= p\mathbb{P}(\beta_1 \geq h) + p\sum_{x=1}^{h-1}\mathbb{P}(\beta_1 = x)\sum_{h'\geq h}\frac{\varphi(x)}{\varphi(h')}\mathbb{P}(\beta > h'-h)\\ &= p\mathbb{P}(\beta_1 \geq h) + p\cdot\sum_{x=1}^{h-1}\mathbb{P}(\beta_1 = x)\varphi(x)\cdot\sum_{h''\geq 0}\frac{1}{\varphi(h+h'')}\mathbb{P}(\beta > h'')\end{aligned}$$

generalizing (3.4.18).

3.5 Estimation from an N-sample of X, say $(x_0, x_1, ..., x_N)$

We briefly sketch here how (in presence of real data which are suspected to be in the binomial catastrophe framework), to estimate its constitutive parameters.
From the transition matrix $P(x,y)$, the log-likelihood function of the N-sample is

$$L(x_0, x_1, ..., x_N) = \sum_{n=1}^{N}\left[\log\left(qd_{x_{n-1},x_n}\right)\mathbb{1}(x_n \leq x_{n-1}) + \log\left(pb_{x_n - x_{n-1}}\right)\mathbb{1}(x_n > x_{n-1})\right].$$

If one knows that some population grows and decays according to the binomial catastrophe model with $\mathbb{E}\beta = \rho < \infty$, we propose the following estimators : the Maximum-Likelihood-Estimator of p while setting $\partial_p L = 0$ is

$$\widehat{p} = \frac{1}{N}\sum_{n=1}^{N}\mathbb{1}(x_n > x_{n-1}). \tag{3.5.1}$$

With $\rho = \mathbb{E}\beta < \infty$, if the law of β is a known one-parameter $\rho-$family of pmfs,

$$\widehat{\rho} = \frac{1}{\sum_{n=1}^{N}\mathbb{1}(x_n > x_{n-1})}\sum_{n=1}^{N}(x_n - x_{n-1})\,\mathbf{1}(x_n > x_{n-1}). \tag{3.5.2}$$

Note that if

$$\widehat{\rho p} = \frac{1}{N}\sum_{n=1}^{N}(x_n - x_{n-1})\,\mathbb{1}(x_n > x_{n-1}),$$

then $\widehat{\rho}\widehat{p} = \widehat{\rho p}$.
Also, in view of $\mathbb{P}(X_{n+1} = x \mid X_n = x) = qc^x$, $x \geq 0$,

$$\widehat{c} = \frac{1}{\sum_{n=1}^{N}\mathbb{1}(x_n = 1)}\sum_{n=1}^{N}\mathbb{1}(x_n = 1, x_{n-1} = 1). \tag{3.5.3}$$

3.6 A variant of the binomial catastrophe model

Suppose we are interested in the following simple semi-stochastic decay/surge model : at each step of its evolution, the size of some population either grows by a random number of individuals or shrinks by only **one** unit. The above binomial catastrophe model is not able to represent this scenario where at least one individual is removed from the population at catastrophic events. To remedy this, we therefore define and study a variant of the above binomial model whereby the transition probabilities in the bulk and at 0 are slightly modified in order to account for the latter decay/surge situation, Eliazar and Klafter (2009).

- If $X_n \geq 1$, define

$$X_{n+1} = \begin{cases} 1 \circ X_n + \beta_{n+1} = X_n + \beta_{n+1}, & \text{wp } p \\ c \circ (X_n - 1), & \text{wp } q \end{cases}$$

Given a move down wp q : **One** individual out of X_n is systematically removed from the population $(X_n \to X_n - 1)$; each individual among the $X_n - 1$ remaining ones being independently subject to a survival variable/death issue (with probability c / $1 - c$) in the next generation.

- If $X_n = 0$ $(p_0 + q_0 = 1)$ then,

$$X_{n+1} = \begin{cases} \beta_{n+1}, & \text{wp } p_0 \\ 0, & \text{wp } q_0. \end{cases}$$

Unless $p_0 = p$, our model yields some additional control on the future of the population once it hits 0 (extinction event).

With $b_x := \mathbb{P}(\beta = x)$, $x \geq 1$, the one-step-transition matrix P of the modified MC X_n now is given by :

$$\begin{aligned} P(0,0) &= q_0,\ P(0,y) = p_0 \delta_1, \text{ if } y \geq 1, \\ P(x,x) &= 0 \text{ if } x \geq 1, \\ P(x,y) &= q \binom{x-1}{y} c^y (1-c)^{x-1-y}, \text{ if } x \geq 1 \text{ and } 0 \leq y < x, \\ P(x,y) &= p\mathbb{P}(\beta = y - x) = p b_{y-x}, \text{ if } x \geq 1 \text{ and } y > x. \end{aligned}$$

Note that because at least one individual dies out in a shrinking event, the diagonal terms $P(x,x)$ of P are now 0 for all $x \geq 1$.

Remark 3.3. *One can introduce a holding probability r_x to stay in state x given $X_n = x$, filling up now the diagonal of P. This corresponds to a time change while considering a modified transition matrix $\widetilde{P}$ where : $p \to p_x = p(1 - r_x)$ and $q \to q_x = q(1 - r_x)$ $(p_x + q_x + r_x = 1)$. So, with $\rho := \mathbf{1} - \mathbf{r}$, a column vector with entries $\rho_x = 1 - r_x$,*

$$P \to \widetilde{P} = I + D_\rho (P - I).$$

If the invariant measure π obeying $\pi' = \pi' P$, (the fixed point of $\pi'_{n+1} = \pi'_n P$) exists, then the one of $\widetilde{P}$ also exists and obeys : $\widetilde{\pi}' D_\rho = \pi'$.

Suppose first $X_0 = 0$. Let $\mathbb{E}z^{X_n} := \Phi_n(z) = \overline{\Phi}_n(z) + \Phi_n(0)$ with $\Phi_0(z) = 1$ translating $X_0 = 0$. Then, with $p + q = 1$,

$$\begin{aligned}
\Phi_{n+1}(z) &= (q_0 + p_0 z)\,\Phi_n(0) + pB(z)\,\overline{\Phi}_n(z) + \frac{q}{1 - c(1-z)}\overline{\Phi}_n(1 - c(1-z)),\ \overline{\Phi}_0(z) = 0,\\
\Phi_{n+1}(0) &= q_0\Phi_n(0) + \frac{q}{1-c}\overline{\Phi}_n(1-c),\ \Phi_0(0) = 1.
\end{aligned}$$

Note $\Phi_n(0) = \mathbb{P}(X_n = 0)$ and $\overline{\Phi}_n(0) = 0$ for each $n \geq 0$.
Thus, if these fixed point quantities exist

$$\begin{aligned}
\overline{\Phi}_\infty(z) &= p_0(z-1)\,\Phi_\infty(0) + pB(z)\,\overline{\Phi}_\infty(z) + \frac{q}{1 - c(1-z)}\overline{\Phi}_\infty(1 - c(1-z)),\\
\Phi_\infty(0) &= \frac{q}{p_0(1-c)}\overline{\Phi}_\infty(1-c).
\end{aligned}$$

We shall iterate the first fixed point equation which makes sense only when $c \neq 0$.
With $C(z) = \frac{p_0(z-1)}{1-pB(z)}$ and $D(z) = \frac{q}{1-c(1-z)}\frac{1}{1-pB(z)}$, we get

$$\begin{aligned}
\overline{\Phi}_\infty(z) &= \Phi_\infty(0)\sum_{n\geq 0} C(1 - c^n(1-z))\prod_{m=0}^{n-1} D(1 - c^m(1-z))\\
&= \Phi_\infty(0)\left[C(z) + D(z)\sum_{n\geq 1} C(1 - c^n(1-z))\prod_{m=1}^{n-1} D(1 - c^m(1-z))\right]\\
&= \frac{p_0\Phi_\infty(0)(z-1)}{1 - pB(z)}\left[1 + \sum_{n\geq 1}(cq)^n \prod_{m=1}^{n}\frac{1}{1 - c^m(1-z)}\frac{1}{1 - pB(1 - c^m(1-z))}\right].
\end{aligned}$$

Except when $c = 1$, the term inside the bracket has no pole at $z = 1$. Then $\overline{\Phi}_\infty(1) = 0$ and so, assuming $\overline{\Phi}_\infty(0) = 0$, $\overline{\Phi}_\infty(z) = 1$ for all $z \in [0,1]$ is the only possible solution to the first fixed point equation. Recalling from the second one that $\Phi_\infty(0) = \frac{q}{p_0(1-c)}\overline{\Phi}_\infty(1-c)$, we conclude that $\Phi_\infty(0) = 0$ and so $\Phi_\infty(z) = 0$ for all $z \in [0,1]$. The only solution $\Phi_\infty(z)$ is the trivial null one.

It remains to study the cases $c = 1$ and $c = 0$.

• If $c = 1$. In this case, only a single individual can stepwise be removed from the population ; the transition matrix P is upper- Hessenberg. This constitutes a simple discrete version of a decay/surge model (some kind of time-reversed version of the simple growth/collapse model).

$$\begin{aligned}
\overline{\Phi}_\infty(z) &= \frac{p_0 z(z-1)}{z(1 - pB(z)) - q}\Phi_\infty(0)\\
&= \frac{p_0 z(z-1)}{z - 1 + p(1 - zB(z))}\Phi_\infty(0).
\end{aligned}$$

with $\overline{\Phi}_\infty(0) = 0$. Letting $\overline{B}(z) = (1 - B(z))/(1-z)$ the tail generating function of β, this is also

$$\begin{aligned}
\overline{\Phi}_\infty(z) &= \frac{p_0 z}{1 - p\left(1 + z\overline{B}(z)\right)}\Phi_\infty(0)\\
\Phi_\infty(z) &= \left(1 + \frac{p_0 z}{1 - p\left(1 + z\overline{B}(z)\right)}\right)\Phi_\infty(0).
\end{aligned}$$

With $\overline{B}(1) = B'(1) = \mathbb{E}\beta =: \rho$, $\overline{\Phi}_\infty(1) = \frac{p_0}{q-p\rho}\Phi_\infty(0) = \frac{1}{1-c}\overline{\Phi}_\infty(1-c)$.

We conclude :

Proposition 3.13. *(phase transition).*
- Subcritical case : $\overline{\Phi}_\infty(1) + \Phi_\infty(0) = 1 \Rightarrow \Phi_\infty(0) = (q-p\rho)/(q-p\rho+p_0)$*, well-defined as a probability only if* $p\rho < q$*. In this case, the chain is positive-recurrent. The term* $p\rho$ *is the average size of a move-up which has to be smaller than the average size* q *of a move-down.*

- Critical case : If $q = p\rho$*, then* $\Phi_\infty(0) = 0 \Rightarrow \overline{\Phi}_\infty(z) = 0$ *for each* z*. The chain is null-recurrent and it has no non-trivial (*$\neq \mathbf{0}$*) invariant measure* π*.*

- Supercritical case : If $q < p\rho \leq \infty$*, the chain is transient at* ∞*.*

Example 3.1. (i) In addition to $c = 1$, assume $B(z) = \overline{\alpha}z/(1-\alpha z)$ (a geometric model for β with success probability $\overline{\alpha}$). Then, if $p < q\overline{\alpha}$, $\Phi_\infty(0) = \pi(0) = (q\overline{\alpha}-p)/(\overline{\alpha}(q+p_0)-p)$ is a probability and

$$\overline{\Phi}_\infty(z) = \frac{p_0 z}{1-p(1+z/(1-\alpha z))}\Phi_\infty(0) = \frac{p_0 z(1-\alpha z)}{q-z(\alpha+p\overline{\alpha})}\Phi_\infty(0).$$

Thus, with $(\alpha < a := (\alpha+p\overline{\alpha})/q < 1)$, we obtain that

$$\pi(x) = [z^x]\,\overline{\Phi}_\infty(z) = \pi(0)\frac{p_0}{q}(a-\alpha)a^{x-2},\ x \geq 1$$

is the invariant probability measure of the chain, displaying geometric decay at rate a.

(ii) If in addition to $c = 1$, we assume $B(z) = z$, X is reduced to a simple birth and death chain (random walk) on the non-negative integers, reflected at the origin. In this case, we get $\overline{\Phi}_\infty(z) = \frac{p_0 z}{1-p(1+z)}\Phi_\infty(0)$ with $(\rho = 1) : \Phi_\infty(0) = (q-p)/(q-p+p_0)$. The corresponding chain is positive recurrent if $p < 1/2$, null-recurrent if $p = 1/2$ and transient at ∞ when $p > 1/2$. In the positive-recurrent case, with $\pi(0) = (1-2p)/(1-2p+p_0)$, we have

$$\pi(x) = [z^x]\,\overline{\Phi}_\infty(z) = \pi(0)\frac{p_0}{q}\left(\frac{p}{q}\right)^{x-1},\ x \geq 1,$$

a well-known result, Feller (2008).

In both examples, whenever the process is positive recurrent, the invariant measure having geometric decay at different rates a and $p/q < 1$ respectively.

- If $c = 0$ (total disasters)

$$\begin{aligned}\overline{\Phi}_\infty(z) &= p_0(z-1)\Phi_\infty(0) + pB(z)\overline{\Phi}_\infty(z) + q\overline{\Phi}_\infty(1)\\ \Phi_\infty(0) &= \frac{q}{p_0}\overline{\Phi}_\infty(1),\end{aligned}$$

leading to $\Phi_\infty(0) = q/(p_0+q) = \pi(0)$ and

$$\overline{\Phi}_\infty(z) = \frac{p_0 z}{1-pB(z)}\Phi_\infty(0),\ \text{or}$$

$$\Phi_\infty(z) = \left(1 + \frac{p_0 z}{1-pB(z)}\right)\Phi_\infty(0). \tag{3.6.1}$$

We conclude :

Proposition 3.14. *In the total disaster case, the modified chain is always positive recurrent with invariant measure having pgf (3.6.1), as a shifted compound geometric distribution.*

Example 3.2. (i) In addition to $c = 0$, assume $B(z) = \overline{\alpha}z/(1-\alpha z)$ (a geometric model for β with success probability $\overline{\alpha}$). Then

$$\Phi_\infty(z) = \left(1 + \frac{p_0 z}{1 - pB(z)}\right)\Phi_\infty(0).$$

(ii) In addition to $c = 0$, assume $B(z) = z$. Then, with $\pi(0) = q/(p_0 + q)$,

$$\begin{aligned}\Phi_\infty(z) &= \left(1 + \frac{p_0 z}{1 - pz}\right)\Phi_\infty(0), \\ \pi(x) &= [z^x]\,\Phi_\infty(z) = \pi(0)\,p_0 p^{x-1},\ x \geq 1,\end{aligned}$$

a geometric distribution with decay rate p.

4

Random walks facing life-taking disasters : discerning extinction and overcrowding

Abstract. Discrete-time Markov chain population models where random stationary growth alternates with random disastrous events, either with moderate stationary amplitudes or with massive depletion are analyzed ; disasters are taken life-taking. The recurrence versus transience conditions are explicited and relies on the existence or not of a finite weak carrying capacity. The analysis was carried over to disaster cases showing size-dependent probability of occurrence. Such populations face two major threats : extinction and overcrowding. For the threat of extinction, both probability of eventual extinction and law of the time to eventual extinction are explicited . For the threat of overcrowding, the time to reach some high threshold, possibly the weak carrying capacity, and the amount of the corresponding overshoot are analyzed using extreme value theory for Markov chains. The analysis relies on the control of the spectral properties of the north-west truncation of the transition matrix of the original Markov chain with disasters.

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4.1 Introduction

We consider discrete-time models where random stationary population growth alternates with random catastrophic events, either with moderate stationary amplitudes or leading to massive depletion ; they are generalized birth and death Markov chains on the non-negative integers. A catastrophic event in our life-taking definition must lead to the death of at least one individual, avoiding catastrophic events producing no death toll.
Continuous-time Markov chains with (geometric and binomial) disasters, in the spirit of the ones developed here, were considered in Artalejo et al. (2007) and references therein.
Disasters are characterized by their probability of occurrence and by their intensity or amplitude. The probability of a disaster is the chance that it will occur at some time (the chance of an extreme drought or a pest outbreak in some day or year depending on time scales). Independently, the amplitude is the amount of the resulting death toll. By its nature, the amplitude is population-size dependent, the death toll being bounded above by the current population size. Concerning the probability of occurrence, it is or not population-size dependent, depending on the existence or not of a feedback effect on it (either positive or negative) of the current population size. Growth is modeled by the random daily adjunction of newborns or immigrants to the current population size and there will be a competition between growth and depletion effects due to disasters. State zero where local extinctions occur will be initially chosen as being partially reflecting so that the population can start afresh from here. Populations eventually of infinite size are unrealistic, so the conditions under which the above Markov chains with growth punctuated by disasters are recurrent, either positive or null, is a critical issue. A signature of positive recurrence is the existence of a finite weak carrying capacity as a level of the population size above which the population process necessarily drifts to the origin.
Extinction is a critical threat in the control of populations but overcrowding can be a problem as well. When the Markov chain is recurrent and state zero is forced to be absorbing, its first local extinction event becomes definitive. The probability of eventual extinction and the law of the time to eventual extinction for the above Markov chains with disasters are made explicit using first-step analysis. To tackle the overcrowding problem, we try to answer the following questions : How long does it take for a Markov chain with disasters to reach some high threshold, possibly the

weak carrying capacity, and what is then the amount of the overshoot ? How long does it take to observe a new population size record and what is the value of this new record ? These are problems of extreme value theory for Markov chains. We show that the answer to both questions rely on the control of the spectral properties of the north-west truncation of the transition matrix of the original chain with disasters. Both the time to extinction and the time to over-cross a barrier have tails with geometric decay.

4.2 Growth models with disasters

We herewith consider two main classes of growth models alternating with disasters, each of a very different nature. In contrast with previous similar models in the literature, disasters with no deaths will be ignored.
In the sequel, we shall use the abbreviations "iid" for independent and identically distributed, "pgf" for probability generating function and "rv" for random variable. Also, a "hatted" variable, say $\widehat{u}$, will represent a column-vector so that its transpose, say $\widehat{u}'$, will be a row-vector. With x integer, the x-th entry of $\widehat{u}$ will be denoted by $\widehat{u}\,(x)$. And $P\,(x,y)$ will denote the (x,y) entries of the matrix P.

4.2.1 Disasters with moderate amplitudes

Birth (growth) : Let $(\beta_n)_{n\geq 1}$ be an iid sequence taking values in $\mathbb{N} := \{1,2,...\}$, with $b_x := \mathbb{P}(\beta_1 = x)$, $x \geq 1$. We shall let $\phi_\beta\,(z) := \mathbb{E}\big(z^\beta\big)$ be the common pgf of the β' s. With $\phi'_\beta\,(z)$ the derivative with respect to z of $\phi_\beta\,(z)$, we shall assume $\lambda := \phi'_\beta\,(1) = \mathbb{E}(\beta) \in [1,\infty]$ (with $\mathbb{E}(\beta) = 1$ if and only if $\beta \overset{d}{\sim} \delta_1$, the Dirac distribution at 1).
Death (depletion) : Let $(\delta_n)_{n\geq 1}$ be an iid sequence also taking values in $\mathbb{N} := \{1,2,...\}$ and with pgf $\phi_\delta\,(z) := \mathbb{E}\big(z^\delta\big)$. We let $d_x := \mathbb{P}(\delta = x)$, $x \geq 1$.
With $x_+ = x \vee 0 = \max(x,0)$ and $x \wedge 0 = \min(x,0)$, consider the Markov chain X_n with disasters whose temporal evolution is given by :

$$X_{n+1} = \begin{cases} X_n + \beta_{n+1} \text{ with probability } p \\ (X_n - \delta_{n+1})_+ = X_n - X_n \wedge \delta_{n+1} \text{ with probability } q = 1-p. \end{cases} \tag{4.2.1}$$

We assume the initial number of ancestors is fixed : $X_0 = x_0$ (typically $x_0 = 1$, one founder) and that q does not depend on $X_n = x$ at this stage.
At each step $n+1$, the walker moves up with probability p and the amplitude of the upward move is β_{n+1} (as in standard random walks). With probability q a catastrophic event occurs (interrupting the growth process), and the number of step-wise removed individuals (whenever available) is δ_{n+1} : the common distribution of the sizes of the disasters δ does not depend on the current size of the population, but this steady depletion occurs to the extent that there are currently enough individuals to be depleted. Stated differently, given the population size is x at n, the magnitude of the downward jump is $x \wedge \delta_{n+1}$, while truncating δ_{n+1} at x. A realistic hypothesis in some circumstances is to assume q small (in a rare catastrophic events setup : $q \ll p$).
If $X_n = 0$ (a local extinction event), then $X_{n+1} = \beta_{n+1}$ with probability p (reflection at 0) and $X_{n+1} = 0$ with probability q (absorption at 0). While making this assumption, possibly after some time delay, immigrants are allowed to refresh the

population, even after a (local) extinction event.

Remark 4.1. (*i*) *This family of Markov chains is time-homogeneous, irreducible and aperiodic. As a result, all states are either recurrent or transient. It is not time-reversible (detailed balance does not hold). It may be viewed as a balance between immigration and emigration.*

(*ii*) *its properties rely on the choice of the (im-)emigration distributions of both* (β, δ) *which have no particular symmetry reason of being the same ; such Markov chains are highly skewed. We will discuss two illustrative examples emphasizing the sensitivity to* δ*' s distribution. In Eq. (4.2.1), the* β_n*' s stand for a steady random input of newborns while the* δ_n*' s account for a steady output of daily deaths.*

(*iii*) *if both* $\beta \overset{d}{\sim} \delta_1$ *and* $\delta \overset{d}{\sim} \delta_1$*, the transition matrix* P *of model Eq. (4.2.1), is the one of the simple birth and death Markov chain with tri-diagonal Jacobi transition matrix.*

(*iv*) *we can define a dual chain as the one obtained under the substitutions*

$$p \to q,\ q \to p$$
$$\beta \to \delta,\ \delta \to \beta.$$

We now point out an important quantity that will be of interest in the sequel. We have

$$m(x) := \mathbb{E}\left[(X_{n+1} - X_n) \mid X_n = x\right] = p\mathbb{E}(\beta) - q\mathbb{E}(x \wedge \delta) \tag{4.2.2}$$

where

$$\mathbb{E}(x \wedge \delta) = \sum_{y=1}^{x} y d_y + x\mathbb{P}(\delta > x) \tag{4.2.3}$$

and

$$\sum_{y=1}^{x} y d_y = [z^x] \frac{z\phi'_\delta(z)}{1-z} \text{ and } \mathbb{P}(\delta > x) = [z^x] \frac{1-\phi_\delta(z)}{1-z} \tag{4.2.4}$$

where $[z^x]\, h(z)$ denotes the z^x-coefficient in the power series expansion of $h(z)$. Note $m(0) = p\mathbb{E}(\beta)$.
In most cases when the chain is recurrent, there exists an $x_c > 0$ such that $\mathbb{E}[(X_{n+1} - X_n) \mid X_n = x] < 0$ for all $x > x_c$ (equivalently because x is an integer : $x > [x_c]$, the integral part of x_c). The quantity $[x_c]$ may then be viewed as a weak carrying capacity ; $[x_c]$ is not a categorical barrier that cannot be crossed, rather if the population size exceeds $[x_c]$ it will then tend to decrease. Above (below) $[x_c]$ indeed, X_n is a super-martingale (sub-martingale). We illustrate this on examples.

4.2.2 Examples

4.2.2.a The truncated geometric case

Suppose $(\delta_n)_{n\geq 1}$ is geometric, with success parameter $d \in (0,1)$. With $\overline{d} := 1 - d$, then $\mathbb{P}(\delta = x) = d_x = d\overline{d}^{x-1}$, $x \geq 1$ and $\phi_\delta(z) = dz/\left(1 - \overline{d}z\right)$. We put $\mu := \mathbb{E}(\delta) = \phi'_\delta(1) = 1/d > 1$. This is a slightly different model to the one appearing in Huillet (2011) and Neuts (1994) where the δ' s took values in $\mathbb{N}_0 := \{0, 1, 2, ...\}$ and the variations resulting from this new modification need to be checked. By choosing the support of δ to be $\mathbb{N}$ instead of $\mathbb{N}_0$, a disaster occurring with probability q results in an effective decline of the population size of at least one individual. So here, we

ignore disasters with no death and the original geometric model slightly changes in a way that we now exhibit.
The one-step stochastic transition matrix P (obeying $P\widehat{1} = \widehat{1}$, where $\widehat{1}$ is a column vector of ones) of the Markov chain X_n is :

$$\begin{array}{l} P(0,0) = q,\ P(0,y) = pb_y,\ y \geq 1 \\ P(x,y) = qd_{x-y} = q\overline{d}\overline{d}^{x-1-y},\ x \geq 1 \text{ and } 1 \leq y \leq x-1 \\ P(x,0) = q\sum_{y \geq x} d_y = q\overline{d}^{x-1},\ x \geq 1 \text{ and } 0 = y < x \\ P(x,y) = pb_{y-x},\ x \geq 1 \text{ and } y > x. \end{array} \tag{4.2.5}$$

Note $P(x,x) = 0$, $x \geq 1$. With $\widehat{\pi}_n := (\mathbb{P}_{x_0}(X_n = 0), \mathbb{P}_{x_0}(X_n = 1), ...)'$ the column vector of the states' occupation probabilities at time n, $\widehat{\pi}'_{n+1} = \widehat{\pi}'_n P$, $X_0 \overset{d}{\sim} \delta_{x_0}$, is the master equation of its temporal evolution.
An equivalent recurrence for the pgf of X_n started at $X_0 = x_0$ is :

Lemma 4.1. *With* $\Phi_n(z) := \mathbb{E}_{x_0}\left(z^{X_n}\right) = \sum_{x \geq 0} z^x \mathbb{P}_{x_0}(X_n = x)$, *such that* $\Phi_0(z) = z^{x_0}$, *it holds,*

$$\Phi_{n+1}(z) = \Phi_{n+1}(0) + \left(p\phi_\beta(z) - \frac{dq}{\overline{d} - z}\right)\Phi_n(z) \tag{4.2.6}$$

$$+\frac{dq}{\overline{d} - z}\left(\Phi_n(0) + \frac{z}{\overline{d}}\left(\Phi_n\left(\overline{d}\right) - \Phi_n(0)\right)\right), \tag{4.2.7}$$

with boundary condition

$$\Phi_{n+1}(0) = \frac{q}{\overline{d}}\left(\Phi_n\left(\overline{d}\right) - d\Phi_n(0)\right). \tag{4.2.8}$$

Proof. From Eq. (4.2.5), upon reversing the sums between x and y in the sum below, and after elementary algebraic manipulations,

$$\Phi_{n+1}(z) = \widehat{\pi}_{n+1}(0) + p\phi_\beta(z)\Phi_n(z) + q\sum_{y \geq 1} z^y \sum_{x \geq y+1} \widehat{\pi}_n(x)\, d\overline{d}^{x-y-1} \tag{4.2.9}$$

yields Eq. (4.2.6). Taking Eq. (4.2.6) at $z = 1$, $\Phi_n(1-) = 1$ yields the recurrence Eq. (4.2.8).

□

Theorem 4.1. *The chain* X_n *with geometric disaster is ergodic if and only if* $\lambda := \mathbb{E}(\beta_1) < \lambda_c := (q\mu)/p < \infty$. *The pgf of* X_∞ *is then given by Eq. (4.2.16). It is infinitely divisible (equivalently compound Poisson).*

Proof : (for comparison, we adapt the line of proof appearing in Huillet (2011), dealing with the case $\delta \in \mathbb{N}_0$). If a limiting random variable X_∞ exists, from Eq. (4.2.6) at $z = 0$ and letting $n \to \infty$,

$$\Phi_\infty(0) = \frac{q}{\overline{d}}\left(\Phi_\infty\left(\overline{d}\right) - d\Phi_\infty(0)\right), \text{ thus} \tag{4.2.10}$$

$$\Phi_\infty(0) = \frac{q}{dq + \overline{d}}\Phi_\infty\left(\overline{d}\right) < \Phi_\infty\left(\overline{d}\right). \tag{4.2.11}$$

If a limiting random variable X_∞ exists, it has pgf $\Phi_\infty(z) := \mathbb{E}\left(z^{X_\infty}\right)$ obeying from Eq. (4.2.6) as $n \to \infty$ and making use of the relationship between $\Phi_\infty(0)$ and $\Phi_\infty\left(\overline{d}\right)$ in Eq. (10)

$$\Phi_\infty(z) = \frac{(1-z)(1-dp)\Phi_\infty(0)}{\left(\overline{d}-z\right)(1-p\phi_\beta(z))+dq} =: \frac{N(z)}{D(z)}. \tag{4.2.12}$$

Suppose $\lambda := \phi'_\beta(1) = \mathbb{E}(\beta_1) < \infty$. Only in such a case does the numerator N and denominator D both tend to 0 as $z \to 1$ while the ratio Φ_∞ tends to 1, as required for $\Phi_\infty(z)$ to be a pgf. By L'Hôpital's rule $\frac{N'(z)}{D'(z)} \to 1$ as $z \to 1$, leading to the explicit expression

$$\Phi_\infty(0) = \widehat{\pi}_\infty(0) = \frac{q-pd\lambda}{1-dp} = \frac{q\mu - p\lambda}{\mu - p}. \tag{4.2.13}$$

Because $\mu > p$, for $\widehat{\pi}_\infty(0)$ to be the limiting probability that $X_\infty = 0$, it is required that $q\mu - p\lambda > 0$ which is the ergodicity condition stated in the Theorem.
Under this condition, with $\Phi_B(z) := q/(1-p\phi_\beta(z))$ the pgf of the random variable B representing the amount of individuals that came to life between two consecutive catastrophic events (Eq. (4.2.42) below), the limiting pgf of X_∞ reads

$$\Phi_\infty(z) = \frac{(1-z)(q-pd\lambda)}{\left(\overline{d}-z\right)(1-p\phi_\beta(z))+qd} = \frac{\Phi_B(z)(1-z)(q-pd\lambda)}{\left(\overline{d}-z\right)q+qd\Phi_B(z)}. \tag{4.2.14}$$

This pgf takes the alternative equivalent form

$$\Phi_\infty(z) = \frac{q-pd\lambda}{q}\Phi_B(z)\left[1-d\frac{1-\Phi_B(z)}{1-z}\right]^{-1}. \tag{4.2.15}$$

With $\Phi_A(z) := \frac{1-\Phi_B(z)}{\mathbb{E}(B)(1-z)}$ the probability generating function of some random variable A (with probability mass function $\mathbb{P}(A=y) = \frac{\mathbb{P}(B>y)}{\mathbb{E}(B)}$), with $\zeta = \lambda/\lambda_c < 1$, this is also the factorized form

$$\Phi_\infty(z) = \Phi_B(z)\frac{1-\zeta}{1-\zeta\Phi_A(z)}, \tag{4.2.16}$$

as a product of the probability generating function of B times the one of a shifted compound geometric$(1-\zeta)$ random variable with compounding probability generating function $\Phi_A(z)$ (the pgf of the stationary-excess version of B, Abate and Whitt (1996). The fact that X_∞ is infinitely divisible follows from the arguments in Huillet (2011).
Due to $\mathbb{E}(B) = (p\lambda)/q$, $\Phi_A(0) = p/\mathbb{E}(B) = q/\lambda$ and we can consistently check

$$\Phi_\infty(0) = \Phi_B(0)\frac{1-\zeta}{1-\zeta\Phi_A(0)} = q\frac{1-\zeta}{1-dp} = \frac{q-dp\lambda}{1-dp} \in (0,1). \tag{4.2.17}$$

Finally,

$$\zeta < 1 \Leftrightarrow p\mathbb{E}(\beta_1) = p\lambda < q/d = q\mathbb{E}(\delta_1) = q\mu. \tag{4.2.18}$$

This is a sub-criticality condition stating that the one-step unlimited average move down $q\mu$ must exceed the average move up $p\lambda$ of the geometric chain. As conventional wisdom suggests, the condition on λ becomes stringent if q is assumed small. □
Finally, when $\lambda > \lambda_c = (q\mu)/p$, the chain is transient at infinity (X_n drifts at ∞ with probability 1), while it is null-recurrent in the critical case $\lambda = \lambda_c$, with $\widehat{\pi}_\infty(0) = \Phi_\infty(0) = 0$.

Remark 4.2. (*i*) *With* $\widehat{\pi}'_\infty = \widehat{\pi}'_\infty P$ *defining the invariant probability mass of* X_n*, the expression of* $\widehat{\pi}_\infty(x) := [z^x]\,\Phi_\infty(z)$ *is quite involved even though* $\Phi_\infty(z)$ *is explicit from Eq. (4.2.16).*

(*ii*) *From Eq. (4.2.16),* $\mathbb{E}(\beta^2) < \infty \Rightarrow \mathbb{E}(B^2) < \infty$ *and*

$$\mathbb{E}(X_\infty) = \Phi'_\infty(1) = \mathbb{E}(B) + \frac{\zeta}{1-\zeta}\mathbb{E}(A) = \mathbb{E}(B) + \frac{\zeta}{2(1-\zeta)}\frac{\mathbb{E}(B^2)}{\mathbb{E}(B)} < \infty. \quad (4.2.19)$$

A geometric ergodic disaster model (necessarily in particular with $\mathbb{E}(\beta) < \infty$) for which the variance of β would be infinite, would result in $\mathbb{E}(X_\infty) = \infty$; X_∞ is then heavy-tailed. The shape of the invariant measure thus depends on the statistical features of β.

Corollary 4.1. *If the chain with geometric disasters* β *is positive recurrent, its dual chain is transient at* ∞*, so long as* $\mathbb{E}(\delta) < \infty$*.*

Proof. this is because under the substitutions to obtain the dual chain, $p\mathbb{E}(\beta) < q\mathbb{E}(\delta)$ becomes $q\mathbb{E}(\delta) < p\mathbb{E}(\beta)$. □

Proposition 4.1. *We have*

$$m(x) := \mathbb{E}[(X_{n+1} - X_n) \mid X_n = x] = p\lambda - qd^{-1}\left(1 - \overline{d}^x\right) \quad (4.2.20)$$

showing that if the chain is positive recurrent so with $\lambda < \lambda_c = (q\mu)/p$*, there exists an* x_c *such that* $m(x) < 0$ *for all* $x > x_c$*. The value of the weak carrying capacity is*

$$x_c = \frac{-\log(1 - \lambda/\lambda_c)}{-\log \overline{d}}. \quad (4.2.21)$$

Proof : This results from

$$\sum_{y=1}^{x} y d_y = d^{-1}\left(1 - \overline{d}^x(1 + dx)\right) \text{ and } \mathbb{P}(\delta > x) = \overline{d}^x \quad (4.2.22)$$

so that

$$\mathbb{E}(x \wedge \delta) = \sum_{y=1}^{x} y d_y + x\mathbb{P}(\delta > x) = d^{-1}\left(1 - \overline{d}^x\right) \quad (4.2.23)$$

$$m(x) = p\mathbb{E}(\beta) - q\mathbb{E}(x \wedge \delta) = p\lambda - qd^{-1}\left(1 - \overline{d}^x\right). \quad (4.2.24)$$

If $\lambda = p\lambda_c < \lambda_c$, then

$$x_c = \frac{-\log q}{-\log \overline{d}} \quad (4.2.25)$$

and x_c becomes large when q becomes small (rare catastrophic events).
Observe that in the null-recurrent case when $\lambda = \lambda_c$, $m(x) = q/d\overline{d}^x = p\lambda\overline{d}^x$ which is monotone decreasing and positive. There is no x_c in that case. So here, the existence of an x_c is a signature of positive recurrence.

4.2.2.b The truncated Sibuya case

Choosing a δ with $\mathbb{E}(\delta) < \infty$ and a distribution which is not geometric, $p\mathbb{E}(\beta) < q\mathbb{E}(\delta)$ clearly remains the condition of positive recurrence. In case δ is not geometric with $\mathbb{E}(\delta) < \infty$, the shape of the invariant measure is not explicitly known however. Model Eq. (4.2.1) is thus sensitive to the law of δ, what we now illustrate, while departing from the classical geometric model for δ with $\mathbb{E}(\delta) < \infty$. This point raised seems to be new. Suppose now $\phi_\delta(z) = 1-(1-z)^\alpha$, $\alpha \in (0,1)$, the pgf of a heavy-tailed Sibuya random variable, Sibuya (1979). Note $\mathbb{E}(\delta) = \phi'_\delta(1) = \infty$. Let $[a]_x = a(a+1)...(a+x-1)$. The transition matrix P corresponding to Eq. (4.2.1) with this choice of δ could be made explicit.

Proposition 4.2. *We have*

$$m(x) := \mathbb{E}[(X_{n+1} - X_n) \mid X_n = x] = p\lambda - q[2-\alpha]_{x-1}/(x-1)! \tag{4.2.26}$$

showing that the chain is positive recurrent if $\lambda = \mathbb{E}(\beta) < \infty$ *and then there exists an* x_c *such that* $m(x) < 0$ *for all* $x > [x_c]$. *The value of the weak carrying capacity is*

$$x_c = u^{-1}(p\mathbb{E}(\beta)/q), \tag{4.2.27}$$

where u^{-1} *is the inverse of the increasing sequence* $u(x) = [2-\alpha]_{x-1}/(x-1)!$. *When* q *is small (rare catastrophic events),*

$$x_c \sim (\Gamma(2-\alpha)\, p\mathbb{E}(\beta)/q)^{1/\overline{\alpha}}. \tag{4.2.28}$$

Proof :

Proof. We have $d_y = \alpha[\overline{\alpha}]_{y-1}/y!$

$$\sum_{y=1}^{x} y d_y = [z^x]\frac{z\phi'_\delta(z)}{1-z} = \alpha[2-\alpha]_{x-1}/(x-1)! \tag{4.2.29}$$

$$\mathbb{P}(\delta > x) = [z^x]\frac{1-\phi_\delta(z)}{1-z} = [\overline{\alpha}]_x/x! \sim x^{-\alpha}/\Gamma(\overline{\alpha}) \tag{4.2.30}$$

$$\mathbb{E}(x \wedge \delta) = [2-\alpha]_{x-1}/(x-1)! = [\overline{\alpha}]_x/(\overline{\alpha}\Gamma(x)) \tag{4.2.31}$$

Letting $u(x) = \mathbb{E}(x \wedge \delta)$, we observe $u(x+1)/u(x) = 1 + \overline{\alpha}/x > 1$ so that $u(x)$ is increasing. Then

$$m(x) = p\mathbb{E}(\beta) - qu(x) < 0 \tag{4.2.32}$$

entails $x > x_c = u^{-1}(p\mathbb{E}(\beta)/q)$ where $u^{-1}(y) = \inf(x : u(x) \geq y)$. When x is large, $u(x) \sim x^{\overline{\alpha}}/\Gamma(2-\alpha)$ and $u^{-1}(y) \sim (\Gamma(2-\alpha)\, y)^{1/\overline{\alpha}}$. □

Remark 4.3. *For such a depletion random variable* δ *with Sibuya heavy tails, although still moderate, the strength of the attraction to* 0 *of the walker* X_n *is stronger than when* δ *is geometric (or any other finite mean distribution). Therefore the positive recurrence condition* ($\lambda = \mathbb{E}(\beta) < \infty$) *is less drastic in the Sibuya case than in the geometric one* ($\lambda < \lambda_c$).

4.2.3 Massive depletion : the binomial disaster case

4.2.3.a Preliminaries

For any fixed x integer and U a $[0,1]$-valued rv, first consider the rv

$$U \circ x = \sum_{i=1}^{x} B_i(U), \tag{4.2.33}$$

where $(B_i(U), i \geq 1)$ is an iid sequence of Bernoulli rv' s with $\mathbb{P}(B_i(U) = 1) = U$, random ($U \circ x$ is the Bernoulli thinning of x, Steutel and van Harn (1979)). Clearly $U \circ 0 = 0$ and

$$\mathbb{E}\left(z^{U \circ x}\right) = \mathbb{E}\left[(1 - U(1-z))^x\right], \text{ equivalently} \tag{4.2.34}$$

$$\mathbb{P}(U \circ x = y) = \binom{x}{y} \mathbb{E}\left[U^y (1-U)^{x-y}\right], \ 0 \leq y \leq x \tag{4.2.35}$$

and the support of $U \circ x$ is $\{0, ..., x\}$. With $U \overset{d}{\sim}$ fixing the probability distribution of U, examples are :

$$\mathbb{P}(U \circ x = y) = \binom{x}{y} u^y (1-u)^{x-y} \text{ if } U \overset{d}{\sim} \delta_u, \ u \in (0,1) \tag{4.2.36}$$

$$\mathbb{P}(U \circ x = y) = \delta_{y,0} \text{ if } U \overset{d}{\sim} \delta_0 \tag{4.2.37}$$

$$\mathbb{P}(U \circ x = y) = \delta_{y,x} \text{ if } U \overset{d}{\sim} \delta_1 \tag{4.2.38}$$

$$\mathbb{P}(U \circ x = y) = \frac{1}{x+1} \text{ if } U \text{ is uniform.} \tag{4.2.39}$$

$$\mathbb{P}(U \circ x = y) = \binom{x}{y} \frac{B(\alpha + y, \beta + x - y)}{B(\alpha, \beta)} \text{ if } U \overset{d}{\sim} \text{Beta}(\alpha, \beta), \alpha, \beta > 0. \tag{4.2.40}$$

For the first binomial example with U concentrated in a point u of $(0,1)$, both mean and variance of $U \circ x$ are of order x for large x (xu and $xu(1-u)$ respectively). For the second and third examples with U concentrated on the extreme points of $[0,1]$, $U \circ x$ is concentrated on the extreme points of its support, 0 and x, respectively. For the last two examples with U truly random and Beta(α, β) distributed, the mean is of order x and the variance of order x^2 for large x ($x\mathbb{E}(U)$ and $x^2\sigma^2(U) + x(\mathbb{E}(U) - \mathbb{E}(U^2))$ respectively). The Bernoulli thinning operator allows for the definition of the size of the population immediately after a catastrophic event.

Let $(\beta_n;\ n \geq 1)$ be a sequence of iid rv' s taking values in $\mathbb{N} = \{1, 2, 3, ...\}$. Consider now the discrete time-homogeneous Markov chain $(X_n; n \geq 0)$ with state-space $\mathbb{N}_0$ and non-homogeneous spatial transition probabilities characterized by :

- given $X_n \geq 1$,

$$X_{n+1} = \begin{cases} X_n + \beta_{n+1} \text{ with probability } p \\ U \circ (X_n - 1) \text{ with probability } q \end{cases} \tag{4.2.41}$$

and the support of $U \circ (x-1)$ is $\{0, ..., x-1\}$. Note $X_n = 1 \Rightarrow X_{n+1} = 0$ with probability q and $U \circ (X_n - 1) = X_n - \left(1 + \overline{U} \circ (X_n - 1)\right)$ where $\overline{U} := 1 - U$. This Markov chain is a new growth model as it involves life-taking binomial disasters.

- given $X_n = 0$, the increment of X_n is β_{n+1} with probability p and 0 with probability q so that X_n is reflected at 0.

For such Markov chains, the time $T \geq 1$ elapsed between consecutive life-taking catastrophic events is geometric with $\mathbb{P}(T = k) = qp^{k-1}$, $k \geq 1$ and the net growth B of the process during this laps of time is

$$B = \sum_{m=1}^{T-1} \beta_m, \tag{4.2.42}$$

where T and $(\beta_n;\ n \geq 1)$ are independent.
When dealing with such a growth model with life-taking binomial disasters, whenever a catastrophic event occurs, one individual systematically dies out and each remaining individual present in the current population, independently of the others, is subject to survival (death) with possibly random probability U (respectively $\overline{U}$). There is thus a weak ever dying individual at the onset of depletion. As a result, the strength of the depletion of individuals in a catastrophic event is very strong (massive) compared to the one occurring in model Eq. (4.2.1). This situation occurs when a catastrophic event strikes simultaneously and independently all currently alive members of some population. Think of an extreme blast striking a forest : each tree, independently of its neighbors, will have to face a chance U of survival, and considering U random can be a natural issue to take into account the variability of the trees in their struggle against the blast (binomial disasters in random environment). To the best of the authors' knowledge, the randomness feature of U has not been fully considered in the literature.

Remark 4.4. *Again, this model is a slight modification of the model discussed in Goncalves and Huillet (2021a) and Neuts (1994), in that on a catastrophic event, at least one individual is removed from the population. We shall not give here the changes resulting from this modification on the invariant measure computed there, leaving this important point to a future work. As a result of its massive depletion feature, it is known however Neuts (1994) that the original chain with binomial disaster is ergodic if and only if the logarithmic moment conditions [*$\mathbb{E}(\log \beta) < \infty$ *when U is random uniform or $\mathbb{E}\left(\log_+ B\right) < \infty$ when $U \overset{d}{\sim} \delta_u$] hold, so, under these conditions, even if $\mathbb{E}(\beta) = \infty$: the condition for positive recurrence of the growth model with binomial disaster is extremely weak. The chain with life-taking binomial disaster being even more attracted to 0 than the original one, the condition for positive recurrence is necessarily possibly even weaker, so the logarithmic moment conditions are sufficient conditions in that case as well. And so positive recurrence is the rule for such chains as well.*

The stochastic transition matrix of the model Eq. (4.2.41) is $P = [P(x, y)]$ where $P(x, x) = 0$, $x \geq 1$ and

$$P(x, y) = \begin{cases} q\mathbb{P}(U \circ (x-1) = y) & \text{if } 0 \leq y < x \\ p\mathbb{P}(\beta = y - x) & \text{if } y > x \end{cases} \tag{4.2.43}$$

and the shape of the invariant measure was discussed in Goncalves and Huillet (2021a) when $U \overset{d}{\sim} \delta_u$, for some fixed $u \in (0, 1)$.

4.2.3.b Special extreme cases

If, in the life-taking binomial model, both $\beta \overset{d}{\sim} \delta_1$ and $U \overset{d}{\sim} \delta_1$, P reduces to the transition matrix of the simple birth and death Markov chain with tri-diagonal Jacobi transition matrix (a birth event is reduced to one individual and all individuals

but the weak ever dying individual survive). When U is uniform, $P(x,y) = q/x$ if $0 \leq y < x$ (the original uniform model of Neuts (1994) but with disaster with no death ruled out) whereas when $U \sim \delta_u$, $P(x,y) = q\binom{x-1}{y} u^y (1-u)^{x-1-y}$ if $0 \leq y < x$ (the binomial model of Neuts (1994) with no death ruled out). In the uniform case, a catastrophic event takes $X_n = x$ to any state $y \in \{0, ..., x-1\}$ with uniform probability $1/x$.
If $\beta \overset{d}{\sim} \delta_1$ and $U \overset{d}{\sim} \delta_0$, $U \circ (X_n - 1) = 0$, a catastrophic event takes instantaneously $X_n = x \geq 1$ to state 0, a total disaster event, Goncalves and Huillet (2020) and Swift (2001). In that extreme case, the transition matrix P is primitive, with its first column consisting of the q' s and an upper diagonal filled with the p' s.
For the above general binomial disaster model, we have

$$m(x) := \mathbb{E}\left[(X_{n+1} - X_n) \mid X_n = x\right] = p\mathbb{E}(\beta) - q\mathbb{E}\left(\overline{U} \circ (x-1)\right) \tag{4.2.44}$$

$$= p\mathbb{E}(\beta) - q\mathbb{E}\left(\overline{U}\right)(x-1), \tag{4.2.45}$$

showing that if $\mathbb{E}(\beta) < \infty$, the binomial chain is recurrent and

$$x_c = 1 + \frac{p\mathbb{E}(\beta)}{q\mathbb{E}\left(\overline{U}\right)}. \tag{4.2.46}$$

If $\mathbb{E}(\beta) = \infty$, the binomial chain can still be recurrent (under the logarithmic moment conditions) but x_c is then sent to ∞. This shows that the existence of an $x_c < \infty$ in this context is a sufficient condition of recurrence, but it is not, strictly speaking, necessary. The choice of a β with an infinite mean is mathematically interesting but it is not physically realistic though.

Remark 4.5. *The three chains with life-taking disaster considered here can be slowed down while introducing $P(x,x) = r_x \in (0,1)$ if $x \geq 1$ and while substituting $p \to p(1-r_x)$ and $q \to q(1-r_x)$ in the original transition matrix $P(x,y)$, $x \geq 1$. The number r_x is the probability that no move (either birth or death) occurs in one step given $X_n = x \geq 1$. When $r_x > 0$, the chain becomes lazy and idle periods are made possible. The holding probabilities were chosen possibly size-dependent. In the process, $m(x) \to (1-r_x)m(x)$ but x_c remains unchanged, being dependent on the ratio p/q. The transition matrix $\widetilde{P}$ of the new lazy chain becomes*

$$P \to \widetilde{P} := D_{\mathbf{r}} + D_{\widehat{1}-\widehat{r}}P = I + D_{\widehat{1}-\widehat{r}}(P - I), \tag{4.2.47}$$

where $D_{\widehat{r}} = diag(\widehat{r})$ is the diagonal matrix formed from the holding probability vector $\widehat{r} = (0 = r_0, r_1, ..., r_x, ...)$. The fact that we ignored here disasters with no death clarifies the construction of the chain made lazy by filling the diagonal null terms of the original P : a holding event in some state is not polluted by the possibility of a disaster producing no dead. Finally if $\widehat{\pi}'_\infty = \widehat{\pi}'_\infty P$ defines the invariant probability mass of X_n, the expression of $\widehat{\pi}_\infty$ for the lazy chain is $\widetilde{\pi}'_\infty = \widehat{\pi}'_\infty D^{-1}_{\widehat{1}-\widehat{r}}$, solving $\widetilde{\pi}'_\infty = \widetilde{\pi}'_\infty \widetilde{P}$.

Choosing for instance $r_x = r^x$, $x \geq 1$, $r \in (0,1)$ yields $\widetilde{\pi}_\infty(x) = (1-r^x)^{-1}\widehat{\pi}_\infty(x)$, $x \geq 1$ which is summable at ∞ if $\widehat{\pi}_\infty$ is. Choosing $r_x = 1 - x^{-\lambda}$, $x \geq 1$, $\lambda > 0$ yields $\widetilde{\pi}_\infty(x) = x^\lambda \widehat{\pi}_\infty(x)$, $x \geq 1$, which may ($\alpha > \lambda + 1$) or not ($\alpha \leq \lambda + 1$) be summable at ∞ if $\widehat{\pi}_\infty(x) \sim x^{-\alpha}$ in a power-law way for some $\alpha > 1$ (below and Goncalves and Huillet (2020) for an occurrence of such a situation in the total disaster model with size-dependent disaster probabilities).

4.2.4 Population-size dependent disaster probability

Assume $\lambda = \mathbb{E}(\beta) < \infty$. Suppose that in the above models of random growth with disaster, the disaster probability q is size-dependent, so q_x is the probability of a disaster given $X_n = x$. The question of the existence of such a feedback was raised in Wilcox and Elderd (2003) who study its effect on the persistence time (time to extinction) of the population, Hanson and Tuckwell (1978).
A rare catastrophic model could result from q_x decreasing with $q_x \to 0$ as $x \to \infty$. But then X_n is then expected to drift to ∞ (transience). For such models, populations with large size are indeed favored being less subject to disasters (positive feedback of population size on the strength of disasters).
On the contrary, if q_x is increasing with x, with $q_x \to 1$ as $x \to \infty$, one expects X_n to be recurrent. For such models, populations with large size are disadvantaged being more subject to disasters (negative feedback of population size and loss of immunity to disasters by size). Think of a house of cards formation where the collapse probability is increasing with the number of cards, Goncalves and Huillet (2020). When it exists, the computation of the invariant measure in this more general setup can be very involved. However, recurrence follows from the following simple arguments.

- *First geometric model.* Here $m(x)$ becomes

$$m(x) = p_x \lambda - (q_x/d)\left(1 - \overline{d}^x\right) \tag{4.2.48}$$

and $m(x) < 0$ is achieved when

$$\lambda = \mathbb{E}(\beta) < \frac{q_x}{p_x d}\left(1 - \overline{d}^x\right). \tag{4.2.49}$$

The right hand-side is increasing with x, starting from 0. So, if $\mathbb{E}(\beta) < \infty$, there is an x_c for which the latter inequality holds for all $x > x_c$. This model is positive recurrent.

- *Second model (Sibuya).* With $u(x) = [\overline{\alpha}]_x / (\overline{\alpha}\Gamma(x))$ increasing,

$$m(x) = p_x \mathbb{E}(\beta) - q_x u(x) < 0 \tag{4.2.50}$$

and the same arguments apply.

- *Third model (binomial).*

$$m(x) = p_x \mathbb{E}(\beta) - q_x \mathbb{E}\left(\overline{U}\right)(x-1) < 0 \tag{4.2.51}$$

The same argument applies.
We conclude that in the three cases, if q_x is increasing with $q_x \to 1$ as $x \to \infty$, X_n is (positive) recurrent.

Let us give an example (with $q_x \to 0$ as $x \to \infty$) for which the existence or not of an x_c does not always permit to conclude on recurrence.
Let $\beta > 0$ and $0 < \alpha < \nu$. Consider the discrete time-homogeneous Markov chain X_n with state-space $\mathbb{N}_0 = \{0, 1, ...\}$ and non-homogeneous spatial transition probabilities characterized by (total disaster model) : given $X_n = x$,

$$X_{n+1} = \begin{cases} x+1 \text{ with probability : } & p_x = 1 - \alpha/\left(\nu + x^\beta\right) \\ 0 \text{ with probability : } & q_x = \alpha/\left(\nu + x^\beta\right). \end{cases} \tag{4.2.52}$$

Then $m(x) = p_x - x q_x$, with $m(x) < 0$ if and only if

$$\nu - \alpha + x^\beta - \alpha x < 0. \tag{4.2.53}$$

When $\beta > 1$, there is no x_c, whereas there is one if $\beta < 1$. When $\beta = 1$, there is an x_c only if $\alpha > 1$. Following Goncalves and Huillet (2020), the latter Markov chain with total disasters is transient if $\beta > 1$, positive recurrent if $\beta < 1$. In the critical case $\beta = 1$, the chain is positive recurrent if $\alpha > 1$ and null recurrent if $\alpha \in (0, 1]$, α being the exponent of the power-law probability mass function of X_∞. The latter is thus a case of recurrence (actually of null-recurrence) with no existence of an x_c. So here again, the existence of an x_c is a signature of positive recurrence, not of recurrence.

4.3 The question of extinction (local or global)

So far, due to the reflection condition imposed when X_n hits 0, the chains with disasters X_n were shown to be recurrent under some conditions, with the existence of x_c in most relevant cases. Therefore, given $X_0 = x_0 > 0$, X_n visits infinitely often state 0 (local extinctions), in particular the time to the first visit to 0 is finite with probability 1. If we now consider state 0 as being absorbing ($X_n = 0 \Rightarrow X_{n+1} = 0$) while considering $P(0, x) = \delta_{x,0}$, then (global) extinction occurs with probability 1, as is the case for members of an endemic species.

Proposition 4.3. *Suppose the reflected chain with disasters X_n is recurrent. By forcing state 0 to be absorbing, the new process becomes eventually extinct.*

We now develop some of the main statistical features of the time to extinction

$$\tau_{x_0,0} = \inf(n \geq 1 : X_n = 0 \mid X_0 = x_0), \ x_0 > 0. \tag{4.3.1}$$

With $\Phi_n(z) := \mathbb{E}_{x_0}\left(z^{X_n}\right) = \sum_{y\geq 0} z^y \mathbb{P}_{x_0}(X_n = y) = \sum_{y\geq 0} z^y P^n(x_0, y)$, define the bivariate generating function $\Phi(u, z) = \sum_{n\geq 0} u^n \Phi_n(z)$. The Green kernel at (x_0, y) is

$$G_{x_0,y}(u) = [z^y]\,\Phi(u, z), \tag{4.3.2}$$

the matrix element (x_0, y) of the resolvent $G(u) := (I - uP)^{-1}$ of P.

• The Green kernel at $(x_0, 0)$ is thus $G_{x_0,0}(u) = [z^0]\,\Phi(u, z)$. It is related to the pgf of the first extinction time $\tau_{x_0,0}$ by the Feller relation (Bingham (2001) pp $3-4$ for example)

$$\phi_{x_0,0}(u) = \mathbb{E}(u^{\tau_{x_0,0}}) = \frac{G_{x_0,0}(u)}{G_{0,0}(u)}. \tag{4.3.3}$$

In the recurrent case, state 0 is visited infinitely often and so both $G_{0,0}(1)$ and $G_{x_0,0}(1) = \infty$, and

$$\mathbb{P}(\tau_{x_0,0} < \infty) = \phi_{x_0,0}(1) = \frac{G_{x_0,0}(1)}{G_{0,0}(1)} = 1. \tag{4.3.4}$$

We stress that, if state 0 is assumed reflecting, $\tau_{x_0,0}$ is only a local extinction time, followed by subsequent extinction times after $\tau_{0,0}$ (the first return times to 0). If state 0 is forced to be absorbing, then $\tau_{x_0,0}$ becomes a global (eventual) extinction time.
In the sequel, we shall let $\overline{P}$ stand for the sub-stochastic transition matrix obtained from P while deleting its first row and column.

There are two alternative representations of $\phi_{x_0,0}(u)$.

• One alternative representation follows from the following classical first-step analysis :
Let $X_1(x_0)$ be the position of X_n at time $n=1$, started at $x_0>0$.
Let $X_+(x_0)$ be a positive rv with $\mathbb{P}(X_+(x_0)=y)=\overline{P}(x_0,y)/\sum_{y\geq 1}\overline{P}(x_0,y)$, $y\geq 1$.
With $\tau'_{X_+(x_0),0}$ a statistical copy of $\tau_{X_+(x_0),0}$, first-step analysis yields, (Norris (1998) and Woess (2009)) :

$$\tau_{x_0,0}\overset{d}{=}1\cdot I_{\{X_1(x_0)=0\}}+I_{\{X_1(x_0)>0\}}\cdot\left(1+\tau'_{X_+(x_0),0}\right). \tag{4.3.5}$$

Clearly,

$$\mathbb{P}(X_1(x_0)=0)=P(x_0,0)=:\mathfrak{q}_{x_0},$$
$$\mathbb{P}(X_1(x_0)>0)=:\mathfrak{p}_{x_0}=\sum_{y\geq 1}\overline{P}(x_0,y)=1-\mathfrak{q}_{x_0}.$$

Therefore $\phi_{x_0,0}(u):=\mathbb{E}(u^{\tau_{x_0,0}})$, obeys the recurrence $\phi_{x_0,0}(u)=\mathfrak{q}_{x_0}u+u\mathfrak{p}_{x_0}\mathbb{E}\phi_{X_+(x_0),0}(u)$, $x_0>0$. $\mathfrak{q}_{x_0}$ is a collapse probability.
With $\widehat{\phi}(u)=(\phi_{1,0}(u),\phi_{2,0}(u),...)'$ the column-vector of the $\phi_{x_0,0}(u)=\mathbb{E}u^{\tau_{x_0,0}}$, and $\widehat{q}:=(\mathfrak{q}_1,\mathfrak{q}_2,...)'$ the first column-vector of the matrix P, $\widehat{\phi}(u)$ then solves :

$$\widehat{\phi}(u)=u\widehat{q}+u\overline{P}\widehat{\phi}(u), \tag{4.3.6}$$

whose formal solution is (compare with the expression Eq. (4.3.3))

$$\widehat{\phi}(u)=u\left(I-u\overline{P}\right)^{-1}\widehat{q}=:u\overline{G}(u)\widehat{q}, \tag{4.3.7}$$

involving the resolvent matrix $\overline{G}(u)=\left(I-u\overline{P}\right)^{-1}$ of $\overline{P}$. $\widehat{\phi}(1)=\left(I-\overline{P}\right)^{-1}\widehat{q}$ gives the column-vector of the probabilities of eventual extinction $\widehat{\phi}(1):=(\phi_{1,0}(1),\phi_{2,0}(1),...)'$.
Clearly $\widehat{\phi}(1)=\widehat{1}$ (the all-one column vector) in the recurrent case. In that case, from Eq. (4.3.6), introducing the column vector $\mathbb{E}(\tau_{.,0}):=(\mathbb{E}(\tau_{1,0}),\mathbb{E}(\tau_{2,0}),...)'$ where $\mathbb{E}(\tau_{x_0,0})=\phi'_{x_0,0}(1)$ and observing $\widehat{q}+\overline{P}\widehat{\phi}(1)=\widehat{1}$, we get

$$\widehat{\phi}'(1):=\mathbb{E}(\tau_{.,0})=\widehat{1}+\overline{P}\mathbb{E}(\tau_{.,0}), \tag{4.3.8}$$

equivalently,

$$\mathbb{E}(\tau_{.,0})=\left(I-\overline{P}\right)^{-1}\widehat{1}=\overline{G}(1)\widehat{1}, \tag{4.3.9}$$

or

$$\mathbb{E}(\tau_{x_0,0})=\sum_{y\geq 1}\overline{G}_{x_0,y}(1), \tag{4.3.10}$$

where $\overline{G}_{x_0,y}(u)$ is the matrix element (x_0,y) of the resolvent of $\overline{P}$.

• An alternative representation of $\phi(u)$ is as follows. From the identity

$$\overline{P}^n(x_0,y)=\mathbb{P}_{x_0}(X_n=y,\tau_{x_0,0}>n), \tag{4.3.11}$$

we get $\mathbb{P}(\tau_{.,0}>n)=\overline{P}^n\widehat{1}$, as the column vector of $(\mathbb{P}(\tau_{1,0}>n),\ \mathbb{P}(\tau_{2,0}>n),...)'$, and so,

$$\sum_{n\geq 0}u^n\mathbb{P}(\tau_{.,0}>n)=\overline{G}(u)\widehat{1}. \tag{4.3.12}$$

This leads in particular, as expected, to $\mathbb{E}(\tau_{\cdot,0}) = \overline{G}(1)\widehat{1}$ and (compare with Eq. (4.3.3) and Eq. (4.3.7)) to

$$\widehat{\phi}(u) = \sum_{n\geq 0} u^n \mathbb{P}(\tau_{\cdot,0} = n) = \widehat{1} - (1-u)\overline{G}(u)\widehat{1}. \tag{4.3.13}$$

We obtained :

Proposition 4.4. *We have* $\widehat{\phi}(1) = \mathbb{P}(\tau_{\cdot,0} < \infty)$ *and so* $\overline{G}(1)\widehat{1} < \infty \Rightarrow \mathbb{P}(\tau_{\cdot,0} < \infty) = \widehat{1}$, *meaning recurrence of* X_n. *In fact, positive recurrence is precisely when* $\mathbb{E}(\tau_{\cdot,0}) = \overline{G}(1)\widehat{1} < \infty$. *If* $\overline{G}(1)\widehat{1} = \infty$, *the chain is null-recurrent if* $(1-u)\overline{G}(u)\widehat{1} \to \mathbf{0}$ *as* $u \to 1$, *transient if* $(1-u)\overline{G}(u)\widehat{1} \to \mathbb{P}(\tau_{\cdot,0} = \infty)$ *as* $u \to 1$, *a non-zero limit.*

The matrix $\overline{P}$ is sub-stochastic with spectral radius $\rho \in (0,1)$. With $\widehat{r}$ and $\widehat{l}'$ the corresponding right and left positive Perron eigenvectors of $\overline{P}$, so with $\overline{P}\widehat{r} = \rho\widehat{r}$ and $\widehat{l}'\overline{P} = \rho\widehat{l}'$, $\overline{P}^n \sim \rho^n \cdot \widehat{r}\widehat{l}'$ (as n is large) where $\widehat{r}\widehat{l}'$ is the projector onto the first eigenspace of $\overline{P}$. By Perron-Frobenius theorem, Vere-Jones (1967), we can normalize $\widehat{l}$ to be of l_1- norm one to get

Proposition 4.5. *In the positive recurrent case for* X_n :
(*i*) *With* $\widehat{r}(x_0)$ *the* x_0-*entry of* $\widehat{r}$,

$$\rho^{-n}\mathbb{P}(\tau_{x_0,0} > n) \to \widehat{r}(x_0), \text{ as } n \to \infty, \tag{4.3.14}$$

showing that $\mathbb{P}(\tau_{x_0,0} > n)$ *has geometric tails with parameter* ρ *(extinction is fast).*
(*ii*) *With* $\widehat{l}(y)$ *the* $y-$*entry of* $\widehat{l}$, *for all* $x_0 > 0$,

$$\mathbb{P}_{x_0}(X_n = y \mid \tau_{x_0,0} > n) \to \widehat{l}(y), \text{ as } n \to \infty, \tag{4.3.15}$$

showing that the left eigenvector $\widehat{l}$ *is the quasi-stationary distribution of* X_n *or Yaglom limit, (Yaglom (1947), Collet et al. (2013)).*

Proof : In this case, with $R = \rho^{-1} > 1$, the convergence radius of $\overline{G}$, $\overline{G}(R) = \infty$ and $\overline{P}$ is $R-$positive recurrent.
(*i*) follows from $\mathbb{P}(\tau_{\cdot,0} > n) = \overline{P}^n\widehat{1}$ and
(*ii*) from $\mathbb{P}_{x_0}(X_n = y \mid \tau_{x_0,0} > n) = \overline{P}^n(x_0, y)/\overline{P}^n\widehat{1}$.

4.4 The extremal chain of Markov chains with disasters

Extinction is a critical issue in the control of populations but overcrowding can be a problem as well. To tackle this problem, we try to answer the following questions : How long does it take for a Markov chain with disasters X_n, as above, started at $X_0 = x_0$ to reach some high population barrier x^* (possibly x_c) and what is then the amount of the overshoot ? How long does it take to observe a new population size record and what is the value of this new record ? These are problems of extreme value theory for Markov chains. We show that the answer to both questions rely on some control of the spectral properties of the north-west truncation of the transition matrix of X_n.

4.4.1 Preliminaries

With $X_0 = X_0^* = x_0 > 0$, with U_n an iid sequence of uniformly distributed random variables, with f defined in Eq. (4.4.4) below, consider

$$\begin{aligned} X_{n+1} &= f(X_n, U_{n+1}) && (4.4.1)\\ X_{n+1}^* &= \max(X_n^*, X_{n+1}) && (4.4.2)\\ K_{n+1} &= K_n + I(X_{n+1} > X_n^*) && (4.4.3) \end{aligned}$$

where it is assumed that the Markov chain X_n takes values in $\mathbb{N}_0 := \{0, 1, 2, ...\}$ and that its transition matrix P is irreducible. The Markov chain X_n is assumed to model the size of some population as from the above models involving disasters. With $P^c(x, y) = \mathbb{P}(X_{n+1} \leq y \mid X_n = x)$, we always have

$$f(x, U_{n+1}) = \sum_{y \geq 1} y \cdot I(P^c(x, y-1) < U_{n+1} \leq P^c(x, y)). \tag{4.4.4}$$

From this, the extremal Markov chain X_n^* is a sequence driven by a Markov chain, namely X_n, so that (X_n, X_n^*) is a bivariate Markov chain. The auxiliary process K_n (where by convention, we have set $K_0 = 0$) counts the number of records by time n of X_n^* with,

$$\mathbb{E}(K_n) = \sum_{m=1}^{n} \mathbb{P}\left(X_m > X_{m-1}^*\right) = \sum_{m=0}^{n-1} \mathbb{P}(f(X_m, U_{m+1}) > X_m^*). \tag{4.4.5}$$

The marginal sequence X_n^* is not a Markov chain as both (X_n, X_n^*) are needed to produce X_{n+1}^*. The Markov chain (X_n, X_n^*) is easily amenable to numerical simulations, and

Proposition 4.6. *Given $(X_n, X_n^*) = (x, x^*)$, $x \leq x^*$, the one-step transition matrix Q that the next day $\left(X_{n+1}, X_{n+1}^*\right) = (y, y^*)$, $x^* \leq y^*$, is*

$$Q((x, x^*), (y, y^*)) \tag{4.4.6}$$

$$= \delta_{y,y^*} \mathbb{P}(f(x, U_{n+1}) = y)\, I_{y > x^*} + \delta_{x^*,y^*} \mathbb{P}(f(x, U_{n+1}) = y)\, I_{y \leq x^*} \tag{4.4.7}$$

$$= \delta_{y,y^*} P(x, y)\, I_{y > x^*} + \delta_{x^*,y^*} P(x, y)\, I_{y \leq x^*}, \tag{4.4.8}$$

where $P(x, y) = \mathbb{P}(f(x, U_{n+1}) = y) = \mathbb{P}(X_{n+1} = y \mid X_n = x)$ is the transition matrix of X_n.

Proof : If indeed when X_n is moving from x to y in one-step, y exceeds x^* then $y = y^*$ is the new extremal value, while if y is smaller than x^*, then $y^* = x^*$ as the extremal value is left unchanged.
Given $(X_n, X_n^*) = (x, x^*)$, $x \leq x^*$, the probability that the next day is a record day regardless of its amplitude is

$$\mathbb{P}(f(x, U_{n+1}) > x^*) = \sum_{y > x^*} P(x, y). \tag{4.4.9}$$

It is a decreasing function of x^* and an increasing function of x.
We can base an alert when this probability becomes larger than some critical value.

Example 4.1. With (β_n), (δ_n) iid sequences of integral-valued rv' s in $\mathbb{N} := \{1, 2, ...\}$, defining the birth and death Markov chain Eq. (4.2.1), then

$$\mathbb{P}\left(f\left(x, U_{n+1}\right) > x^*\right) = p\mathbb{P}\left(\beta > x^* - x\right) \tag{4.4.10}$$

is a decreasing function of the difference $x^* - x \geq 0$. This function only depends on the distribution of β. The probability that the next day is a record day and that the value of the record is y is

$$p\mathbb{P}\left(\beta = y - x\right) I\left(y > x^*\right). \tag{4.4.11}$$

It is a decreasing function of y if $b_x := \mathbb{P}(\beta = x)$ is a decreasing function of x.

4.4.2 First jump time A_1 over $x_0 = X_0^*$ of X_n^* and corresponding overshoot at $x^* > x_0$

Suppose $X_0 = X_0^* = x_0 > 0$. Then, with $A_1 = \inf\left(n \geq 1 : X_n > x_0 \mid X_0 = x_0\right)$ and $x^* > x_0$,

$$\mathbb{P}_{x_0}\left(K_l = 1, X_l^* = x^*\right) = \mathbb{P}\left(A_1 = l, X_{A_1}^* = x^*\right) \tag{4.4.12}$$

$$= \sum_{x_1=0}^{x_0} ... \sum_{x_{l-1}=0}^{x_0} \mathbb{P}_{x_0}\left(X_1 = x_1, .., X_{l-1} = x_{l-1}, X_l = x^*\right) \tag{4.4.13}$$

$$= P\left(x_0, x^*\right) \text{ if } l = 1 \tag{4.4.14}$$

$$= \sum_{x_1=0}^{x_0} ... \sum_{x_{l-1}=0}^{x_0} \prod_{m=1}^{l-1} P\left(x_{m-1}, x_m\right) P\left(x_{l-1}, x^*\right) \text{ if } l \geq 2 \tag{4.4.15}$$

Due to $x_0 = X_0^* > 0$, the rv A_1 is the first record time of X_n above x_0 and $X_{A_1}^*$ the corresponding first record value.
Clearly, given $X_0 = X_0^* = x_0 > 0$, the events "$X_l^* \leq x_0$" and "$A_1 > l$" coincide, together with "$K_l = 1$" and "$A_1 = l$".
Let $P_{(x_0)}$ be the truncated upper-left corner with size $(x_0 + 1, x_0 + 1)$ of the full transition matrix $P := [P(x, y)]$, $x, y \geq 0$, assumed irreducible, of X_n (the north-west part of P). The matrix $P_{(x_0)}$ is such that, for all $0 \leq x, y \leq x_0$, $P_{(x_0)}(x, y)$ is the probability of a one-step move of X_n on the events $X_n = x \leq x_0$ and $X_{n+1} = y \leq x_0$. It is sub-stochastic with $\left(P_{(x_0)}\widehat{1}\right)(x) \leq 1$ for all but at least one x for which $\left(P_{(x_0)}\widehat{1}\right)(x) < 1$. Then,

$$\mathbb{P}_{x_0}\left(A_1 = l, X_l^* = x^*\right) = \sum_{x=0}^{x_0} P_{(x_0)}^{l-1}\left(x_0, x\right) P\left(x, x^*\right) = \left[P_{(x_0)}^{l-1} Q_{(x_0)}\right]\left(x_0, x^*\right) \tag{4.4.16}$$

where $Q_{(x_0)}$ is the restriction of P to its first $x_0 + 1$ rows, leaving the columns as they are from P.
Let us detail the marginals.
Observing that with $\widehat{1}' = (1, ..., 1)$ and $\widehat{e}_{x_0}' = (0, ..., 0, 1)$ row vectors with size $x_0 + 1$, $\sum_{x^* > x_0} Q_{(x_0)}\left(x, x^*\right) = 1 - \left(P_{(x_0)}\widehat{1}\right)(x)$,

$$\mathbb{P}_{x_0}\left(X_{A_1}^* = x^*\right) = \sum_{x=0}^{x_0} \left(I - P_{(x_0)}\right)^{-1}\left(x_0, x\right) P\left(x, x^*\right) = \tag{4.4.17}$$

$$\left[\left(I - P_{(x_0)}\right)^{-1} Q_{((x_0))}\right](x_0, x^*). \tag{4.4.18}$$

Similarly,

$$\mathbb{P}_{x_0}(A_1 = l) = \sum_{x=0}^{x_0} P_{(x_0)}^{l-1}(x_0, x)\left(1 - \left(P_{(x_0)}\widehat{1}\right)(x)\right) \tag{4.4.19}$$

$$= \widehat{e}'_{x_0} P_{(x_0)}^{l-1}\widehat{1} - \widehat{e}'_{x_0} P_{(x_0)}^{l}\widehat{1} \tag{4.4.20}$$

with

$$\sum_{l\geq 1} \mathbb{P}_{x_0}(A_1 = l) = \mathbb{P}_{x_0}(A_1 < \infty) = \widehat{e}'_{x_0}\widehat{1} = 1 \tag{4.4.21}$$

$$\mathbb{P}_{x_0}(A_1 > l) = \widehat{e}'_{x_0} P_{(x_0)}^{l}\widehat{1}. \tag{4.4.22}$$

Corollary 4.2. *The rv A_1 is proper ($\mathbb{P}_{x_0}(A_1 < \infty) = 1$) and it has asymptotically geometric tails with parameter ρ_{x_0}, the spectral radius of $P_{(x_0)}$.*

Proof : The matrix $P_{(x_0)}$ has a spectral radius $\rho_{x_0} < 1$ and, for large l if $\widehat{u}'\widehat{v} = 1$, Horn and Johnson (2013),

$$P_{(x_0)}^{l} \sim \rho_{x_0}^{l}\widehat{u}\widehat{v}', \text{ as } l \to \infty \tag{4.4.23}$$

where $\rho_{x_0}\widehat{v}' = \widehat{v}'P_{(x_0)}$ and $P_{(x_0)}\widehat{u} = \rho_{x_0}\widehat{u}$ are the left (row) and right (column) positive eigenvectors of $P_{(x_0)}$ associated to $\rho_{x_0} = \lim_{l\to\infty}\left\|P_{(x_0)}^{l}\right\|^{1/l}$ for any matrix norm $\|\cdot\|$. Hence, for large l,

$$\mathbb{P}_{x_0}(A_1 > l) = \widehat{e}'_{x_0} P_{(x_0)}^{l}\widehat{1} \sim \widehat{u}(x_0)\,\rho_{x_0}^{l}\sum_{x=0}^{x_0}\widehat{v}(x) \text{ and} \tag{4.4.24}$$

$$\mathbb{P}_{x_0}(A_1 = l, X_l^* = x^*) \sim \rho_{x_0}^{l-1}\left(\widehat{u}\widehat{v}'Q_{(x_0)}\right)(x_0, x^*) \tag{4.4.25}$$

$$= \widehat{u}(x_0)\,\rho_{x_0}^{l-1}\sum_{x=0}^{x_0}\widehat{v}(x)\,Q_{(x_0)}(x, x^*) \tag{4.4.26}$$

where $\widehat{u}(x)$ (respectively $\widehat{v}(x)$) is the x-entry of $\widehat{u}$ (respectively $\widehat{v}$).
If x_0 is small, then ρ_{x_0} can easily be computed numerically (analytically if $x_0 = 1$ (or 2 or 3) because $P_{(x_0)}$ is a 2×2 (or 3×3, or 4×4) matrix), together with $\widehat{u}$ and $\widehat{v}$. Then a large l estimate of $\mathbb{P}(A_1 = l, X_l^* = x^*)$ follows from the expression of $Q_{(x_0)}(x, x^*)$.
To estimate this probability, simulate $(X_m, m = 1, ..., l-1)$ (started at $x_0 > 0$) and only keep those realizations $\overline{x}_m$ for which all $\overline{x}_m \leq x_0$. Then

$$\widehat{\mathbb{P}}(A_1 = l, X_l^* = x^*) = \prod_{m=1}^{l-1} P(\overline{x}_{m-1}, \overline{x}_m)\,P(\overline{x}_{l-1}, x^*). \tag{4.4.27}$$

is an estimate of $\mathbb{P}(A_1 = l, X_l^* = x^*)$, which is known if P is known. A better estimate of $\mathbb{P}(A_1 = l, X_l^* = x^*)$ would be obtained while taking the average of $\widehat{\mathbb{P}}(A_1 = l, X_l^* = x^*)$ over several realizations of $\overline{x}_m \leq x_0$.

4.4.3 First crossing time $T_1(x^*)$ of x^* from below and value $X_{T_1(x^*)}$ of the overshoot of X_n

Let $x^* > 0$ be a large population size threshold that cannot be over-crossed without endangering the whole population due to limitation of resources. This value may be fixed by public population control policy (whenever X_n is positive recurrent, one possible choice is $x^* = [x_c]$ because whenever X_n exceeds this threshold, X_n tends to decrease, a trustful signature of lack of resources). How long does it then take for a Markov chain with disasters X_n started at $X_0 = x_0 \ll x^*$ to reach x^* and what is then the amount of the overshoot ? First-crossing times problems are of a different nature than first hitting or first passage times (like first extinction time), Bertoin (1999) .
Denoting $T_1(x^*) = \inf(n \geq 1 : X_n^* > x^* \mid X_0 = x_0)$, with $x_0 < x^* < y$,

Proposition 4.7.

$$\mathbb{P}_{x_0}(T_1(x^*) = l, X_l = y) = \mathbb{P}_{x_0}\left(T_1(x^*) = l, X_{T_1(x^*)} = y\right) \tag{4.4.28}$$

$$= \left[P_{(x^*)}^{l-1} Q_{(x^*)}\right](x_0, y) \tag{4.4.29}$$

Proof : The latter probability is given by

$$\sum_{x_1=0}^{x^*} ... \sum_{x_{l-1}=0}^{x^*} \mathbb{P}_{x_0}(X_1 = x_1, .., X_{l-1} = x_{l-1}, X_l = y) \tag{4.4.30}$$

$$= P(x_0, y) \text{ if } l = 1 \tag{4.4.31}$$

$$= \sum_{x_1=0}^{x^*} ... \sum_{x_{l-1}=0}^{x^*} \prod_{m=1}^{l-1} P(x_{m-1}, x_m) P(x_{l-1}, y) \text{ if } l \geq 2 \tag{4.4.32}$$

$$= \sum_{x=0}^{x^*} P_{(x^*)}^{l-1}(x_0, x) P(x, y) = \left[P_{(x^*)}^{l-1} Q_{(x^*)}\right](x_0, y) \tag{4.4.33}$$

Considering the two marginals, we have

Proposition 4.8.

$$\mathbb{P}_{x_0}(T_1(x^*) > l) = \widehat{e}'_{x_0} P_{(x^*)}^l \widehat{1} \tag{4.4.34}$$

$$\mathbb{P}_{x_0}\left(X_{T_1(x^*)} = y\right) = \left[\left(I - P_{(x^*)}\right)^{-1} Q_{(x^*)}\right](x_0, y). \tag{4.4.35}$$

Clearly, given $X_0 = x_0 < x^*$, the events $X_l^* \leq x^*$ and $T_1(x^*) > l$ coincide with

$$\mathbb{P}_{x_0}(T_1(x^*) > l) = \mathbb{P}_{x_0}(X_l^* \leq x^*) = \widehat{e}'_{x_0} P_{(x^*)}^l \widehat{1} \sim \widehat{u}(x_0) \rho_{x^*}^l \sum_{x=0}^{x^*} \widehat{v}(x) \tag{4.4.36}$$

where $\rho_{x^*} \widehat{v}' = \widehat{v}' P_{(x^*)}$ and $P_{(x^*)} \widehat{u} = \rho_{x^*} \widehat{u}$ are now the left (row) and right (column) positive eigenvectors of $P_{(x^*)}$ associated to $\rho_{x^*} = \lim_{l \to \infty} \left\| P_{(x^*)}^l \right\|^{1/l}$. The rv $T_1(x^*)$ has asymptotic geometric tails with parameter ρ_{x^*}.

4.4.4 The embedded extremal record chain : upper record times and values sequences of X_n

Let X_n be a Markov chain taking values in $\mathbb{N}_0$. With $R_0 := 0$, define

$$R_k = \inf\left(r \geq 1 : r > R_{k-1}, X_r > X_{R_{k-1}}\right) \tag{4.4.37}$$

$$X_k^\star = X_{R_k}. \tag{4.4.38}$$

The sequences $(R_k, X_k^\star)$ constitute the upper record times and values sequences of X_n. Unless X_n goes extinct, $X_k^\star$ is a strictly increasing sequence tending to ∞. The sequence $X_k^\star$ is thus X_n (and also X_n^*) sampled at its record times R_k.
Following Adke (1993), with $(R_0 = 0, X_0^\star = x_0)$, $(R_k, X_k^\star)_{k\geq 0}$ clearly is a Markov chain with transition probabilities for $y > x$,

Proposition 4.9.

$$\overline{P}^\star(l, x, y) := \mathbb{P}\left(R_k = r + l, X_k^\star > y \mid R_{k-1} = r, X_{k-1}^\star = x\right) \tag{4.4.39}$$

$$= \overline{P}(x, y) \text{ if } l = 1 \tag{4.4.40}$$

$$= \sum_{x_1=0}^{x} \cdots \sum_{x_{l-1}=0}^{x} \prod_{m=1}^{l-1} P(x_{m-1}, x_m)\, \overline{P}(x_{l-1}, y) \text{ if } l \geq 2 \tag{4.4.41}$$

$$= \sum_{x_{l-1}=0}^{x} P_{(x)}^{l-1}(x, x_{l-1})\, \overline{P}(x_{l-1}, y) = \left[P_{(x)}^{l-1} \overline{Q}_{(x)}\right](x, y), \tag{4.4.42}$$

where $P(x, y) = \mathbb{P}(X_n = y \mid X_{n-1} = x)$, $\overline{P}(x, y) = \mathbb{P}(X_n > y \mid X_{n-1} = x)$ *and where* $\overline{Q}_{(x)}$ *is the restriction of* $\overline{P}$ *to its first* $x + 1$ *rows, leaving the columns as they are from* $\overline{P}$.

Clearly the marginal sequence $(X_k^\star)_{k\geq 0}$ is now Markov with transition matrix

$$\overline{P}^\star(x, y) := \mathbb{P}\left(X_k^\star > y \mid X_{k-1}^\star = x\right) = \sum_{l\geq 1} \overline{P}^\star(l, x, y) \tag{4.4.43}$$

$$= \sum_{x_{l-1}=0}^{x} \left(I - P_{(x)}\right)^{-1}(x, x_{l-1})\, \overline{P}(x_{l-1}, y) = \left[\left(I - P_{(x)}\right)^{-1} \overline{Q}_{(x)}\right](x, y), \tag{4.4.44}$$

but the record times marginal sequence $(R_k)_{k\geq 0}$ is non-Markov. However,

$$\mathbb{P}\left(R_k = r + l \mid R_{k-1} = r, X_{k-1}^\star = x\right) = \overline{P}^\star(l, x, x) = \left[P_{(x)}^{l-1} \overline{Q}_{(x)}\right](x, x), \tag{4.4.45}$$

showing that the law of $A_k := R_k - R_{k-1}$ (the age of the k-th record) is independent of R_{k-1} (although not of $X_{k-1}^\star$):

$$\mathbb{P}\left(A_k = l \mid X_{k-1}^\star = x\right) = \overline{P}^\star(l, x, x), \; l \geq 1. \tag{4.4.46}$$

Hence, for large l, given $X_{k-1}^\star = x$,

$$\mathbb{P}_x(A_k > l) = \widehat{e}_x' P_{(x)}^l \widehat{1} \sim \widehat{u}(x)\, \rho_x^l \sum_{z=0}^{x} \widehat{v}(z) \text{ and} \tag{4.4.47}$$

$$\mathbb{P}_x\left(A_k = l, X_k^\star = y\right) \sim \rho_x^{l-1}\left(\widehat{u}\widehat{v}' Q_{(x)}\right)(x,y) = \widehat{u}(x)\,\rho_x^{l-1}\sum_{z=0}^{x}\widehat{v}(z)\,Q_{(x)}(z,y) \quad (4.4.48)$$

where $\widehat{u}$ (respectively $\widehat{v}$) are the right and left eigenvectors of $P_{(x)}$, associated to the spectral radius ρ_x.
Suppose $X_{k-1}^\star = x$ with x already large. From Seneta (1968), $\rho_x \uparrow 1$ as $x \to \infty$. Letting $\widehat{p}_x := P_{(x)}\widehat{1}$ with entries $\widehat{p}_x(y) < 1$ for at least one y (the defective row-sum vector), it is also well-known that

$$\min_y \widehat{p}_x(y) \leq \rho_x \leq \max_y \widehat{p}_x(y) \quad (4.4.49)$$

and many attempts to bound ρ_x can be found in the literature, Horn and Johnson (2013).

Conjecture. This suggests that when the record value x is already large and $\widehat{p}_x := P_{(x)}\widehat{1}$ approaches $\widehat{1}$ from below, the geometric mean

$$\overline{\rho}_x := \left(\prod_{y=0}^{x}\widehat{p}_x(y)\right)^{1/(x+1)} \quad (4.4.50)$$

could serve as a reliable estimate of ρ_x which determines the tail behavior of the time to next record, starting from record x.

Remark 4.6. (*i*) *The extremal process X_n^* is non-Markov while the embedded extremal process $X_k^\star$ is.*

(*ii*) *Of interest is $(R_1 = A_1, X_1^\star = X_{R_1})$, the first upper record time and value because R_1 is the first time X_n exceeds the initial condition x_0, and X_{R_1} the corresponding overshoot at $y > x_0$. Its joint distribution was computed in Section 4.2.*

(*iii*) *Defining $K^* = \inf(k \geq 1 : X_k^\star > x^*)$, then $(R_{K^*}, X_{K^*}^\star)$ coincide with the time to first overcross the threshold x^* and corresponding overshoot, the laws of which were computed in Section 4.3.*

(*iv*) *Of interest finally is also the number K_n of records in the time lag $\{0, ..., n\}$. It is*

$$K_n := \#\{k \geq 1 : R_k \leq n\} = \sum_{k=1}^{n} I_{\{R_k \leq n\}} \quad (4.4.51)$$

and little can be said on the way this sequence tends to ∞.

Example 4.2. Let us illustrate our conjecture for $\overline{\rho}_x$ for various disaster models under study in this note, while specifying the row-sum vector of $P_{(x)}$, with $\widehat{p}_x(y) = q + p\sum_{z=1}^{x-y}\mathbb{P}(\beta = z)$ depending only on the distribution of β.
With $\overline{b}_y = \sum_{z>y}\mathbb{P}(\beta = z)$, for $y \in \{0, ..., x\}$, we have

$$\widehat{p}_x := P_{(x)}\widehat{1} = \left(1 - p\overline{b}_x, ..., 1 - p\overline{b}_{x-y}, .., 1 - p\right)', \quad (4.4.52)$$

so $\widehat{p}_x(y) = 1 - p\overline{b}_{x-y}$, $y = 0, ..., x$.

- If $\overline{b}_y = b^y$, $b \in (0,1)$, $y \geq 0$ (a geometric model for β), as $x \to \infty$,

$$\overline{\rho}_x := \left(\prod_{y=0}^{x}\widehat{p}_x(y)\right)^{1/(x+1)} \sim \left(\prod_{y=0}^{x}(1 - pb^y)\right)^{1/x} = e^{\frac{1}{x}\sum_{y=0}^{x}\log(1-pb^y)} \to 1- \quad (4.4.53)$$

- If $\beta \sim \delta_1$ (a birth event is the birth event of only one individual), $P_{(x)}\widehat{1} = (1, ..., 1, 1-p)'$ and as $x \to \infty$,

$$\overline{\rho}_x := \left(\prod_{y=0}^{x} \widehat{p}_x(y) \right)^{1/(x+1)} \sim q^{1/x} \to 1- \tag{4.4.54}$$

Example 4.3. Total disasters (model Eq. (4.2.52)) : the transition matrix has $(q_0, q_1, ..., q_x, ...)'$ as a first column and a super-diagonal with entries $(p_0, p_1, ..., p_x, ...)$, $p_x + q_x = 1$. Then

$$\widehat{p}_x := P_{(x)}\widehat{1} = (1, ..., 1, .., 1, q_x)'. \tag{4.4.55}$$

Suppose that x is an (already large) record value. If, with $\beta > 0$, the collapse probability at x is $q_x = \alpha / \left(\nu + x^\beta\right)$, decreasing with x, then for large x,

$$\overline{\rho}_x := \left(\prod_{y=0}^{x} \widehat{p}_x(y) \right)^{1/(x+1)} = \left(\frac{\nu + x^\beta}{\alpha} \right)^{-1/(x+1)} \sim x^{-\beta/x} = e^{-\beta \log(x)/x} \to 1- \tag{4.4.56}$$

If, as in the house of cards setup, $q_x = 1 - \alpha \left(\nu + x^\beta\right)^{-1}$ is increasing with x,

$$\overline{\rho}_x := \left(\prod_{y=0}^{x} \widehat{p}_x(y) \right)^{1/(x+1)} = \left(1 - \alpha \left(\nu + x^\beta\right)^{-1} \right)^{1/(x+1)} \sim e^{-\alpha x^{-(\beta+1)}} \to 1- \tag{4.4.57}$$

We have consistently $e^{-\beta \log(x)/x} \ll e^{-\alpha x^{-(\beta+1)}}$. In the first case indeed, moves up are favored for large x, so large records are more likely to occur than in the house of cards setup ; the time to the next record starting from a large record value x is shorter in the first case than in the second house-of-cards one.

4.5 Conclusion

There has been increasing recognition that disasters constitute an important factor in modelling threatened populations. Simple Markov chain models of population growth interrupted by disasters, leading either to moderate or massive depletion of individuals, have been considered. Disasters were chosen life-taking and in the first simplest case, the probability of occurrence of a disaster was chosen independent of the current population size. When dealing with moderate depletion, the truncated geometric Markov model was shown to be explicitly solvable : the condition of positive recurrence was the existence of a weak carrying capacity and, when positive recurrent, the limit law of the chain was shown explicit. This proved useful to grasp the positive recurrence condition for the truncated Sibuya model with the depletion random variable having infinite mean and also for disaster cases showing size-dependent probability of occurrence. The analysis was carried over to population growth models interrupted by life-taking disasters leading to massive binomial depletion. Such populations face two major threats : extinction and overcrowding. For the threat of extinction, the probability of eventual extinction and the law of

the time to eventual extinction were explicited using first-step like analysis. For the threat of overcrowding, the time to reach some high threshold, possibly the weak carrying capacity, and the amount of the corresponding overshoot were analyzed using extreme value theory for Markov chains.

5

On population growth with catastrophes

Abstract. In this chapter we study a particular class of Piecewise deterministic Markov processes (PDMP's) which are semi-stochastic catastrophe versions of deterministic population growth models. In between successive jumps the process follows a flow describing deterministic population growth. Moreover, at random jump times, governed by state-dependent rates, the size of the population shrinks by a random amount of its current size, an event possibly leading to instantaneous local (or total) extinction. A special separable shrinkage transition kernel is investigated in more detail, including the case of total disasters. We discuss conditions under which such processes are recurrent (positive or null) or transient. To do so, we introduce a modified scale function which is used to compute, when relevant, the law of the height of excursions and to decide if the process is recurrent or not. The question of the finiteness of the time to extinction is investigated together with the evaluation of the mean time to extinction when the last one is finite. Some information on the embedded jump chain of the PDMP is also required when dealing with the classification of states 0 and ∞ that we exhibit.

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5.1 Introduction

In this chapter we study population growth models subject to random catastrophes, designed to describe for instance the evolution of a disease, the growth of a market or of the capital of a company, etc. In our model, catastrophes appear after unpredictable random times. These random times are generalized exponentially distributed times having rate $\beta(x)$ whenever the current size of the process is x. When a catastrophe happens, the process shrinks by a random amount of its size, an event possibly leading to instantaneous local extinction (extinction of the disease, collapse of the market, failure of the company, etc.). The successive catastrophe events are the only jumps of the system. In between these jumps, the process follows a flow describing deterministic growth, given by

$$x_t = x_0 + \int_0^t \alpha(x_s)ds,$$

with a locally Lipschitz continuous drift function α which is strictly positive on $(0, \infty)$. This leads to a resulting strong Markov process $X = (X_t)_{t\geq 0}$. All features of its dynamic are gathered in its infinitesimal generator given for smooth test functions u by

$$Gu(x) = \alpha(x)u'(x) + \beta(x)\int_{[0,x]} H(x, dy)[u(y) - u(x)],\ x \geq 0, \tag{5.1.1}$$

where $H(x, dy)$ is the jump kernel giving the after-jump position $y \in [0, x]$, provided the current size of the population before undergoing the catastrophe is x.

Such processes are a particular instance of piecewise deterministic Markov processes (PDMP's) as introduced in Davis (1984); models of a similar flavor were considered in Brockwell et al. (1982a), Brockwell et al. (1983), Brockwell et al. (1982b), Hanson and Tuckwell (1978), Hanson and Tuckwell (1981), Pakes et al. (1979) and Trajstman (1981), see also Boxma et al. (2006), Gripenberg (1983), Gripenberg (1985) and Neuts (1994).

The process $(X_t)_t$ takes values in $[0, \infty]$, and both boundaries 0 and $+\infty$ have to be added to the state space. Indeed, since α is only locally Lipschitz continuous, the deterministic flow may reach $+\infty$ in finite time. Moreover, $\alpha(0)$ may equal 0, such that it is possible that the process gets stuck in 0. We therefore propose in a first step the classification of the two boundaries together with necessary and sufficient conditions ensuring that the process is of finite activity, that is non-explosive in the sense that the number of jumps per finite time interval is finite almost surely.

The main part of our chapter is devoted to the study of the return times to 0 of the process. The question whether the process almost surely returns to 0 and how long it takes to do so is of course of tremendous importance in any application. On the other hand, under the assumption that 0 is reflecting, the return times to 0 induce a basic regeneration scheme and therefore trivially imply the recurrence of the process.

Recurrence of one-dimensional Markov processes which are regular such as diffusions is usually studied by means of the associated scale function and speed measure. For example, one-dimensional elliptic diffusions are known to be recurrent if and only if their associated scale function (that is, a function transforming the process into a local martingale) is a space-transform, that is, a bijection, see e.g. Khasminskii (1980), Example 2 in Section 3.8. Once the scale function is explicitly known, exit probabilities of bounded intervals can be computed. Hitting time moments are also known, expressed in terms of the Green function and on the speed measure.

Contrarily to the case of one-dimensional elliptic diffusions, the class of PDMP's we study in this chapter is very irregular. A very particular feature of our process is the following intrinsic asymmetry. The only way to go up is by deterministic continuous motion – therefore exit times of intervals $[0, b]$ will always be hitting times of level b - while the process only goes down by jumps – exit times of intervals $[a, \infty]$ will always be jumps. Another difficulty comes from the fact that in general the process $(X_t)_t$ cannot possess other bijective scale functions $s(\cdot)$ than the constant ones. Indeed, starting from 0 and under the assumption that 0 is reflecting, $X_t > 0$ almost surely for any $t > 0$, such that $s(X_t) > s(0)$ as well – which obviously contradicts the martingale property. We are however able to define a *modified scale function* of the process that does not transform the trajectory into a martingale but allows to completely characterize the recurrence of the process and to compute certain exit probabilities. This function exists in the situation when the jump kernel is separable, that is, $H(x, y) := \int_{[0,y]} H(x, dz)$ is of the form $H(x, y) = h(y)/h(x)$ for some positive, non-decreasing function $h(\cdot)$, for any $0 \leq y \leq x$.

In this separable case, our first main result, Theorem 5.1, gives an explicit formula for $p(x, b)$, the probability of hitting 0 before hitting b, starting from $x \in (0, b)$, under suitable conditions on the coefficients of the system. Theorem 5.1 also gives the explicit value of $p(0, b) = \lim_{x\to 0} p(x, b)$. Unlike in the diffusion case, $p(0, b)$ does not equal 1 but is related to the distribution function of the height of an excursion, that is, the maximal value of the process in between two successive visits to 0. More precisely,

$$p(x, b) = \frac{s(b) - s(x)}{s(b) + \frac{1}{h(0)}},$$

where s is the *modified scale function* of the process, given by

$$s(x) = \int_0^x \frac{\gamma(y)}{h(y)} e^{\Gamma(y)} dy, \text{ with } \Gamma(x) = \int_0^x \frac{\beta(y)}{\alpha(y)} dy. \tag{5.1.2}$$

Although s is not a true scale function of X, the recurrence of X is however equivalent to the fact that s is a bijection, that is, $s(\infty) = \infty$. Therefore we recover the same characterization of recurrence as in the case of one-dimensional elliptic diffusions, at least if 0 is reflecting and accessible and $+\infty$ inaccessible. This result is stated in Proposition 5.9.

Our second main result, Theorem 5.2, gives then the expected length of an excursion out of $0-$ that is, of the expected time it takes the process to come back to 0, starting from there. To obtain this result, we rely on the fundamental formula relating the invariant measure of a process to the expected occupation time of a given set in between successive visits to a recurrent state (here, 0). This allows to recover the length of an excursion by means of the speed density π and the *expected local time in* 0 of the process during one excursion.

Organization of the chapter. In Section 2 we introduce our model and discuss some first properties, including the distribution of the first jump time, the classification of the boundary states 0 and ∞ and a discussion of the non-explosion of the stochastic process in Proposition 5.2 and Proposition 5.3. Section 3 is devoted to the study of some basic regularity properties of the associated transition semigroup. In particular, we show that the "noise" which is present in the random choices of the jump times regularizes in the sense that $\mathcal{L}(X_t|X_0 = x)$ is absolutely continuous with respect to the Lebesgue measure on $[0, x_t(x))$, see Proposition 5.4. We also provide an explicit formula for the speed measure and its density in (5.3.3). Section 4 contains the main results of the chapter related to the recurrence and the return times to 0. Finally, in Section 5 we present some simulation results.

5.2 Model definition and first results

We consider a piecewise deterministic Markov process X_t taking values in $[0, \infty]$, describing the random size of a population. The dynamic of the process is given by two main ingredients. Firstly, in between successive jumps, the size of the population grows in a deterministic way, described by a deterministic flow. Secondly, at some random jump times, catastrophic events occur at which the current size of the population shrinks by a random amount.

We start by discussing the deterministic growth part in between the successive jumps.

5.2.1 Deterministic population growth models

The evolution of the population size in between successive jumps follows the dynamic $\dot{x}_t = \alpha(x_t)$, $x_0 = x \geq 0$. Throughout this chapter, the drift function α is supposed to be continuous on $[0, \infty)$ and positive on $(0, \infty)$.

5.2.1.a Algebraic growth models

With α_1, $a > 0$, consider the growth dynamics

$$\dot{x}_t = \alpha_1 x_t^a,\ x_0 = x, \tag{5.2.1}$$

for some growth field $\alpha(x) = \alpha_1 x^a$. Note that in this case $\alpha(x)$ is increasing with x. Integrating when $a \neq 1$ (the non linear case), we get formally

$$x_t(x) = \left(x^{1-a} + \alpha_1 (1-a)\, t\right)^{1/(1-a)}. \tag{5.2.2}$$

In principle, such growth models are considered for some positive initial condition $x > 0$. In the sequel we will deal with catastrophic events that can send the population to state 0. Therefore it is important to consider such growth models when started at $x = 0$. Either after hitting state 0, the population remains stuck to 0, and in this case 0 is absorbing, or the population can regenerate starting afresh from 0, and 0 is reflecting.

Three cases arise :

1. $0 < a < 1$: then $x \geq 0$ makes sense and in view of $1/(1-a) > 1$, the growth of x_t is algebraic at rate larger than 1. When $x = 0$, the dynamics has two solutions, one $x_t(0) \equiv 0$ for $t \geq 0$ and the other $x_t(0) = (\alpha_1 (1-a)\, t)^{1/(1-a)}$ because the velocity field $\alpha(x)$ in (5.2.1) with $\alpha(0) = 0$ is not Lipschitz as x gets close to 0, having an infinite derivative. The solution $x_t(0) = (\alpha_1 (1-a)\, t)^{1/(1-a)}$ with $x = 0$ reflects some spontaneous generation phenomenon : following this path, the mass at time $t > 0$ is not 0, although initially it was. Whenever the spontaneous generation phenomenon holds, we shall say that state 0 is reflecting. In what follows, without explicitly mentioning it, we shall always choose this second, maximal solution describing spontaneous generation of mass.
2. $a > 1$: then $x > 0$ only makes sense and $x(t)$ reaches state $+\infty$ in finite time $I_\infty(x) = x^{1-a}/[\alpha_1 (a-1)]$. We get
$$x_t(x) = x\left(1 - t/I_\infty(x)\right)^{1/(1-a)},$$
with algebraic singularity. Whenever a growth process reaches state $+\infty$ in finite time, we shall say that state ∞ is accessible.
3. $a = 1$: this is a simple special case not treated in (5.2.2), strictly speaking. However, expanding the solution (5.2.2) in the leading powers of $1-a$ yields consistently, as $a \to 1$,
$$\begin{aligned} x_t(x) &= e^{\log\left(x^{1-a} + \alpha_1 (1-a) t\right)/(1-a)} \\ &= e^{\log\left[x^{1-a}\left(1 + \alpha_1 x^{a-1}(1-a)t\right)\right]/(1-a)} \\ &\sim x e^{(1/(1-a))\alpha_1 x^{a-1}(1-a)t} \sim x e^{\alpha_1 t}. \end{aligned} \tag{5.2.3}$$
Here $x \geq 0$ makes sense for (5.2.1) with $x_t(x) = x e^{\alpha_1 t}$ for $t \geq 0$ if $x \geq 0$. This is the simple Malthus growth model.

5.2.1.b The role of 0 and of $+\infty$

In general, α being positive on $(0, \infty)$, we have
$$\int_x^{x_t(x)} \frac{dy}{\alpha(y)} = t.$$
Notice that in particular $t' > t \geq 0$ entails $x_{t'}(x) > x_t(x)$, provided $x > 0$ and $x_{t'(x)} < \infty$. If for $x > 0$, $I_0(x) := \int_0^x \frac{dy}{\alpha(y)} < \infty$, then we have
$$x_t(x) = I_0^{-1}\left(I_0(x) + t\right).$$
If for $x > 0$, $I_0(x) = \infty$ and $I_\infty(x) := \int_x^\infty \frac{dy}{\alpha(y)} < \infty$, then
$$x_t(x) = I_\infty^{-1}\left(I_\infty(x) - t\right).$$

Finally we have in all cases

$$x_t(x) = I^{-1}(I(x) + t),$$

where $I(x) = \int^x \frac{dy}{\alpha(y)}$ is an indeterminate integral. This occurs for example when $\alpha(x) = x^a e^{-bx}$ with $a > 1$ and $b > 0$.
Clearly, $I_0(x)$ is the time needed to reach some state x inside the domain $(0, \infty)$ starting from 0, and $I_\infty(x)$ the time needed to reach ∞ starting from some x inside the domain. Thus

$$\begin{array}{rcl} I_0(x) & < & \infty \Longleftrightarrow \text{ state 0 is reflecting, } I_\infty(x) < \infty \Longleftrightarrow \text{ state } \infty \text{ is accessible,} \\ I_0(x) & = & \infty \Longleftrightarrow \text{ state 0 is absorbing, } I_\infty(x) = \infty \Longleftrightarrow \text{ state } \infty \text{ is inaccessible.} \end{array}$$

5.2.2 Adding catastrophes

We now consider the stochastic process X_t that follows the deterministic flow with drift α and jumps at position dependent rate β which is a continuous function on $[0, \infty)$, positive on $(0, \infty)$. At the jump times, the size of the population shrinks by a random amount $\Delta(X_{t-}) \in (0, X_{t-}]$ of its current size X_{t-}. Up to the next jump time, X grows following the deterministic dynamics started at $Y(X_{t-}) := X_{t-} - \Delta(X_{t-})$.

Let

$$\mathbb{P}(X \leq y \mid X_- = x) = \mathbb{P}(\Delta(x) \geq x - y) = H(x, y), 0 \leq y \leq x,$$

be the kernel H which fixes the law of the jump amplitude. $H(x, y)$ is a non-decreasing function of y with $H(x, y) = 1$ for all $y \geq x$. We shall also write

$$H(x, dy) = H(x, 0)\,\delta_0 + \overline{H}(x, dy), \; H(x, y) = \int_{[0,y]} H(x, dy') = H(x, 0) + \overline{H}(x, y),$$

with $\overline{H}(x, 0) = 0$, $\overline{H}(x, x) = 1 - H(x, 0)$. If $H(x, 0) > 0$, there is a positive probability of disasters (instantaneous local extinction).

A special (separable) interesting case is when

$$H(x, y) = \frac{h(y)}{h(x)} = \frac{h(0)}{h(x)} + \frac{h(y) - h(0)}{h(x)},$$

for some positive non-decreasing right-continuous function h.

Our main concern will deal with this particular separable structure of H. In this case, necessarily $x \mapsto H(x, y)$ is non-increasing in x for all y (because $y \mapsto H(x, y)$ is non-decreasing in y for all x entailing h non-decreasing).

Example 5.1. 1. Examples for the separable case are :

a. $h(x) = e^x$ in which case $H(x, 0) = e^{-x} > 0$ (instantaneous disaster can occur with some positive probability). This is the continuous version of the truncated geometric model defined in Neuts (1994).
Letting $Z > 0$ random, with ccdf $\overline{F}_Z(z) = \mathbb{P}(Z > z)$, $H(x, y) = \frac{\overline{F}_Z(x)}{\overline{F}_Z(y)}$ is also in this class, with $H(x, 0) = \overline{F}_Z(x) > 0$.

b. $h(x) = x$ in which case $H(x, 0) = 0$ (no instantaneous disaster).
In the latter two examples $H(\infty, y) = 0$ and there is no way to come down from infinity.

c. Let $Z > 0$ be random and proper, with ccdf $\overline{F}_Z(z) = \mathbb{P}(Z > z)$. Suppose $H(x, y) = h(y)/h(x)$ with $h(x) = h(\infty) - (h(\infty) - h(0))\overline{F}_Z(x)$, for some constants $\infty > h(\infty) > h(0) > 0$. Then, $h(x)$ being bounded from above, $H(\infty, y) = 1 - \frac{h(\infty)-h(0)}{h(\infty)}\overline{F}_Z(y)$ and there is a possibility to come down from infinity. Note that $H(x, 0) = h(0)/h(x) > 0$.

2. Examples for non separable kernels are :

a. $H(x, dy) = \delta_{ux}(dy)$, for some $u \in (0, 1)$. After each catastrophe a fixed fraction u of the previous population is kept.

b. Let $U \in (0, 1)$ random, with cdf $F_U(u) = \mathbb{P}(U \leq u)$. Define $H(x, y) = F_U\left(\frac{y}{x}\right)$. After each catastrophe a random fraction U of the previous population is kept.

5.2.3 Representing the process as a solution of a stochastic differential equation driven by a Poisson random measure

Introducing a Poisson random measure $M(dt, dz)$ on $[0, \infty) \times [0, \infty)$ with intensity $dtdz$, we are thus led to consider the piecewise deterministic Markov process (PDMP) $(X_t)_{t\geq 0}$ with state-space $[0, \infty]$ obeying

$$dX_t = \alpha(X_{t-})\, dt - \Delta(X_{t-}) \int_0^\infty \mathbb{1}_{\{z\leq\beta(X_{t-})\}} M(dt, dz), \tag{5.2.4}$$

$X_0 = x \geq 0$. The associated infinitesimal generator is given for any smooth test function u by

$$Gu(x) = \alpha(x)u'(x) + \beta(x) \int_{[0,x]} [u(y) - u(x)]H(x, dy), x \geq 0. \tag{5.2.5}$$

In the separable case $H(x, y) = h(y)/h(x)$ for all $0 \leq y \leq x$, this reads

$$Gu(x) = \alpha(x)u'(x) - \beta(x)/h(x) \int_0^x u'(y)h(y)dy, x \geq 0. \tag{5.2.6}$$

Notice that $t \to X_t$ is non-decreasing in between successive jumps such that the only possibility for the process to go down is by jumping. The underlying jump counting process is

$$dN_t = \int_0^\infty 1_{\{z\leq\beta(X_{t-})\}} M(dt, dz), \quad \text{with } \mathbb{E}(N_t) = \mathbb{E}\int_0^t \beta(X_s)\, ds. \tag{5.2.7}$$

As usual, to emphasize the dependence on the starting position, we shall write $\mathbb{P}_x$ and $\mathbb{E}_x$ for the probability and its associated expectation on the event when $X_0 = x$. Defining

$$T_x = \inf\{t > 0 : X_t \neq X_{t-} | X_0 = x\} \tag{5.2.8}$$

(with the convention that $\inf \emptyset = \infty$), T_x is the time at which a first jump occurs, when the process starts from x. In what follows, we shall write $S_0 = 0 \leq S_1 \leq S_2 \leq$

$\ldots \leq S_n$ for the successive jump times of the process X_t. Notice that $S_1 = T_x$, if $X_0 = x$. Moreover, conditionally on $X_{S_1} = x_1$, $S_2 - S_1 \overset{\mathcal{L}}{=} T_{X_{x_1}}$, etc.

We shall also consider for all $x \geq 0$

$$\tau_{x,0} = \inf\{t > 0 : X_t = 0 | X_0 = x\}, \inf \emptyset := +\infty, \tag{5.2.9}$$

which is the first time to local extinction in case $x > 0$. We are led to the following distinctions :

1. Total catastrophes (disasters) :

$$H(y, 0) = 1 \text{ for all } y > 0,$$

which means that $\mathbb{P}(X_{T_x} = 0 \mid X_{T_x -} = y) = \mathbb{P}(\Delta(y) = y) = 1$.

Given $x > 0$, state 0 is reached with probability 1, provided $T_x < \infty$ almost surely.

- If 0 is absorbing for x_t, then $X_t = 0$ for all $t \geq T_x$. Moreover T_x coincides with the first time to extinction $\tau_{x,0}$.

- If 0 is reflecting for x_t, X_t possibly visits 0 a finite or an infinite number of times depending on weather $T_x < \infty$ almost surely or not.

2. Partial catastrophes (catastrophes without disasters) :

$$H(x, 0) = 0 \text{ for all } x > 0,$$

which is equivalent to $\mathbb{P}(\Delta(x) < x) = 1$ for all $x > 0$.

Given $x > 0$, state 0 is never visited. The reflecting/absorbing status of state 0 is unimportant, being never reached. Formally, $\tau_{x,0} = \infty$.

3. General catastrophes :

$$H(x, 0) \in (0, 1),$$

which means that $\mathbb{P}(\Delta(x) < x) \in (0, 1)$ for all $x > 0$. Then $\mathbb{P}(X > 0 \mid X_- = x) = \mathbb{P}(x - \Delta(x) > 0) = 1 - H(x, 0) \in (0, 1)$.

- If 0 is absorbing for x_t, $X_t = 0$ for all $t \geq \tau_{x,0}$, where $\tau_{x,0}$ is stochastically larger than T_x.

- If 0 is reflecting for x_t, X_t possibly visits 0 a finite or an infinite number of times.

5.2.4 First jump distribution in case of $I_\infty(x) = \infty$

In this subsection we suppose that $I_\infty(x) = \infty$ such that the deterministic flow does not reach state $+\infty$ in finite time. Supposing $X_0 = x$, since $X_t = x_t(x)$ on $t < T_x$, we have

$$\mathbb{P}(T_x > t) = \mathbb{P}_x(N_t = 0) = \mathbb{P}_x\left(\int_0^t \int_0^\infty \mathbb{1}_{\{z \leq \beta(x_s(x))\}} M(ds, dz) = 0\right),$$

where N_t was defined in (5.2.7) above. With $\gamma(x) := \beta(x)/\alpha(x)$ and $\Gamma(x) := \int^x \gamma(y)\, dy$, an increasing function defined as an indefinite integral, we get, since $\alpha > 0$ on $(0, \infty)$,

$$\mathbb{P}(T_x > t) = e^{-\int_0^t \beta(x_s(x))ds} = e^{-[\Gamma(x_t(x)) - \Gamma(x)]}, \text{ for all } t \geq 0. \tag{5.2.10}$$

This leads naturally to the introduction of the following two conditions.

Assumption 1. $\Gamma(\infty) = \infty$.

Assumption 2. $\Gamma(0) > -\infty$.

Notice that imposing Assumption 1 ensures $\mathbb{P}(T_x < \infty) = 1$ for all $x > 0$. Indeed, since $\alpha > 0$ on $(0, \infty)$, for any $x > 0$, $x_t(x) \to \infty$ and thus also $\Gamma(x_t(x)) \to \Gamma(\infty) = \infty$ as $t \to \infty$. This implies that $\lim_{t\to\infty} \mathbb{P}(T_x > t) = 0$, whence $\mathbb{P}(T_x < \infty) = 1$.

Moreover, imposing Assumption 2 implies that for all $t \geq 0$, $\lim_{x\to 0} \mathbb{P}(T_x > t) > 0$ (this is condition 2.6 in Gripenberg (1983)). If 0 is reflecting, the definition of T_0 in (5.2.8) makes sense replacing x by 0, and (5.2.10) remains valid, since $t \mapsto x_t(0)$ is invertible. In this case, Assumption 2 is automatically satisfied.

Under Assumption 1, since $I_\infty(x) = \infty$ by assumption, we obtain for $x > 0$

$$\mathbb{E}(T_x) = \int_0^\infty e^{-\int_x^{x_t(x)} \gamma(y)dy} dt = \int_x^\infty \frac{1}{\alpha(z)} e^{-\int_x^z \gamma(y)dy} dz = e^{\Gamma(x)} \int_x^\infty \frac{1}{\alpha(z)} e^{-\Gamma(z)} dz. \tag{5.2.11}$$

Notice that the above expression is finite if we assume that β is lower-bounded in a neighborhood of ∞, say by a strictly positive constant $c > 0$. Then for x sufficiently large,

$$\mathbb{E}(T_x) = e^{\Gamma(x)} \int_x^\infty \frac{1}{\alpha(z)} e^{-\Gamma(z)} dz = e^{\Gamma(x)} \int_x^\infty \frac{dz}{\beta(z)} \gamma(z) e^{-\Gamma(z)}$$
$$\leq \frac{1}{c} \int_x^\infty \gamma(z) e^{-\Gamma(z)} dz < \infty.$$

Example 5.2. We take $\alpha(x) = \alpha_1 x^a$ with $a \leq 1$ such that state ∞ is inaccessible. Moreover we choose $\beta(x) = \beta_1 x^b$ with $b > a - 1$, implying that $\gamma(x) = \gamma_1 x^{b-a}$, $\Gamma(x) = \int_0^x \gamma(y)\, dy = \frac{\gamma_1}{b-a+1} x^{b-a+1}$, where we have chosen $\Gamma(0) = 0$. Notice that $\Gamma(\infty) = \infty$ and

$$\Gamma(x_t(x)) - \Gamma(x) = \frac{\gamma_1}{b-a+1} \left[y^{b-a+1}\right]_x^{x_t(x)} = \frac{\gamma_1}{b-a+1} \left(x_t(x)^{b-a+1} - x^{b-a+1}\right)$$
$$= \frac{\gamma_1}{b-a+1} \left(\left(x^{1-a} + \alpha_1 (1-a)\, t\right)^{(b-a+1)/(1-a)} - x^{b-a+1}\right).$$

In this case, T_x has a shifted Weibull distribution, with mean

$$\mathbb{E}(T_x) = \frac{e^{\frac{\gamma_1}{b-a+1} x^{b-a+1}}}{\alpha_1 (b-a+1)} \int_{x^{b-a+1}}^\infty u^{\frac{1-a}{b-a+1}-1} e^{-\frac{\gamma_1}{b-a+1} u} du < \infty.$$

5.2.5 First jump time when $+\infty$ is accessible

If $x_t(x)$ reaches state $+\infty$ in finite time $I_\infty(x) < \infty$, then we still have for all $t \geq 0$ the equality

$$\mathbb{P}(T_x > t) = e^{-\int_0^t \beta(x_s(x))ds}$$

which equals, for all $t < I_\infty(x)$,

$$\mathbb{P}(T_x > t) = e^{-\int_0^t \beta(x_s(x))ds} = e^{-[\Gamma(x_t(x)) - \Gamma(x)]}.$$

Letting $t \uparrow I_\infty(x)$ in the above equation, we get

$$\mathbb{P}(T_x \geq I_\infty(x)) = e^{-[\Gamma(\infty) - \Gamma(x)]}$$

by monotone convergence, since Γ is increasing, whence the necessary and sufficient condition

$$\mathbb{P}\left(T_x \geq I_\infty(x)\right) = 0 \Longleftrightarrow \Gamma\left(\infty\right) = \infty. \tag{5.2.12}$$

Notice that under Assumption 1, the representation (5.2.11) remains valid for all $x > 0$, and also for $x = 0$ if 0 is reflecting. Notice finally that $\mathbb{E}\left(T_x\right) < \infty$ since $T_x < I_\infty(x)$ almost surely.

Example 5.3. We consider $\alpha\left(x\right) = \alpha_1 x^a$ with $a > 1$ such that the solution

$$x_t\left(x\right) = \left(x^{1-a} + \alpha_1\left(1-a\right)t\right)^{1/(1-a)}$$

explodes in finite time at $I_\infty(x) = x^{1-a}/\left[\alpha_1\left(a-1\right)\right]$. Taking $\beta\left(x\right) = \beta_1 x^b$, we have for $b \neq a-1$,

$$\Gamma\left(x\right) = \frac{\gamma_1}{b-a+1}x^{b-a+1} \text{ such that } \Gamma\left(\infty\right) = \infty \Longleftrightarrow b > a-1.$$

If $b > a-1$, then $T_x < I_\infty(x)$ almost surely.

If $0 < b < a-1$, then $\Gamma\left(\infty\right) = 0$ and $\beta(\infty) = \infty$, and T_x has an atom at $I_\infty(x)$ with mass $e^{\frac{\gamma_1 x^{b-a+1}}{b-a+1}}$. If $b = 0$, the process jumps at constant rate independently of its value (finite or infinite). Finally, if $b < 0$, then $\beta(\infty) = 0$ and $T_x = +\infty$ with probability $e^{\frac{\gamma_1 x^{b-a+1}}{b-a+1}}$.

5.2.6 Joint distribution of (T_x, X_{T_x})

Under the assumption $I_\infty(x) = +\infty$, we have for all $y \in [0, x_t\left(x\right))$,

$$\begin{aligned}\mathbb{P}\left(T_x \in dt, X_{T_x} \in dy\right) &= dt\beta\left(x_t\left(x\right)\right)e^{-\int_0^t \beta(x_s(x))ds}H\left(x_t\left(x\right), dy\right)\\ &= dt\beta\left(x_t\left(x\right)\right)e^{-\int_x^{x_t(x)} \gamma(z)dz}H\left(x_t\left(x\right), dy\right).\end{aligned}$$

Moreover,

$$\mathbb{P}\left(T_x > \tau, X_{T_x} \in dy\right) = e^{\Gamma(x)}\int_{x_\tau(x)}^\infty dz\gamma\left(z\right)e^{-\Gamma(z)}H\left(z, dy\right)$$

and

$$\mathbb{P}\left(X_{T_x} \in dy\right) = e^{\Gamma(x)}\int_x^\infty dz\gamma\left(z\right)e^{-\Gamma(z)}H\left(z, dy\right).$$

We close this section with an important remark that we shall rely on later.

Proposition 5.1. *Under Assumptions 1 and 2, the jump chain $(Z_k)_k$, given by $Z_k = X_{S_k}$ is strong Feller.*

Proof. Let g be any bounded and measurable function. Then $x \mapsto \mathbb{E}_x(g(Z_1))$ is continuous, since

$$\mathbb{E}_x(g(Z_1)) = e^{\Gamma(x)}\int_x^\infty dz\gamma\left(z\right)e^{-\Gamma(z)}\int_{[0,z]} g(y)H(z, dy),$$

which is continuous in x by dominated convergence. □

5.2.7 Classification of state 0

The classification of state 0 is of utmost importance since return times to 0 allow to decompose the trajectory of the process $(X_t)_t$ into independent excursions out of 0, implying recurrence of the process – under the condition that the process comes back to 0 almost surely and is not stuck there.

With $x > 0$, state 0 is non-absorbing or reflecting if and only if

$$I_0(x) = \int_0^x \frac{dy}{\alpha(y)} < \infty.$$

If $I_0(x) = \infty$, then state 0 is absorbing. $I_0(x)$ is the time necessary for x_t to move from 0 to $x > 0$. In particular, if $I_0(x) < \infty$, then state 0 is a reflecting boundary.
We can get IN from some $x \in (0, \infty)$ to the boundary point 0 iff $H(x, 0) > 0$.
We can get OUT from the boundary point 0 iff $I_0(x) < \infty$ for some $x \in (0, \infty)$.
This leads to four possible combinations for the boundary state 0 :

1. $H(x, 0) > 0$ and $I_0(x) < \infty$: regular (accessible and reflecting).
2. $H(x, 0) > 0$ and $I_0(x) = \infty$: exit (accessible and absorbing).
3. $H(x, 0) = 0$ and $I_0(x) < \infty$: entrance (inaccessible and reflecting).
4. $H(x, 0) = 0$ and $I_0(x) = \infty$: natural (inaccessible and absorbing).

The first case is called regular because we can get in to 0 and we can start the process afresh from there. The second case is called exit because we can get in to 0 but cannot get out. The third is called an entrance boundary because we cannot get in to 0 but we can start the process there. Finally, in the fourth case the process can neither get to nor start afresh from 0, so it is reasonable to exclude 0 from the state space.

5.2.8 Reaching state ∞ and explosion of the stochastic process

As usual in the theory of jump processes, we say that the process possesses a finite explosion time S_∞ if

$$\lim_{n\to\infty} S_n = S_\infty < \infty \tag{5.2.13}$$

with positive probability, where $S_1 < S_2 < \dots$ is the sequence of successive jump times of the process. Explosion of the process $(X_t)_t$ therefore refers to the event that we observe an accumulation of an infinite number of jumps within a finite time interval.

Clearly, $I_\infty(x) = \infty$ implies that the process does not explode in finite time. Indeed, the upper bound $X_t \le x_t(x) < \infty$ (if $X_0 = x$) implies that the maximal jump rate of the process up to time t is given by $\max\{\beta(x_s(x)) : s \le t\}$ which is finite by continuity of β. The following proposition discusses the case $I_\infty(x) < \infty$.

Proposition 5.2. *Suppose that* $\Gamma(\infty) = \infty$ *and* $I_\infty(x) < \infty$ *for some (and hence all)* $x > 0$. *Let* $T_\infty(x) = \inf\{t > 0 : X_{t-} = \infty | X_0 = x\}$. *Then*

$$\mathbb{P}(T_\infty(x) < S_\infty) = 0.$$

The above result implies that the process is not able to reach the state $+\infty$ before the time of explosion S_∞.

Proof. Suppose that $T_\infty(x) < S_\infty$ with positive probability and write $T = T_\infty(x)$. Let $S_T = \sup\{S_n : S_n < T\}$ be the last jump of the process strictly before hitting the state $+\infty$. $T < S_\infty$ implies that there is only a finite number of jumps on $[0, T]$, such that, almost surely, $S_T < T$ and $X_{S_T} < \infty$. Moreover, conditionally on $X_{S_T} = y < \infty$,

$$X_{S_T+t} = x_t(y), \text{ for all } t < T - S_T \text{ and } T - S_T \stackrel{d}{=} I_\infty(y).$$

In particular, X does not jump in (S_T, T). However, since $\Gamma(\infty) = \infty$, by (5.2.12), almost surely,

$$T_y < I_\infty(y),$$

implying that X does indeed jump strictly before time T, which is a contradiction. □

We complete the above statement with the following observation. On the event of explosion $\{S_\infty < \infty\}$, the process approaches the state ∞ in finite time, namely, we have that $\lim_{n\to\infty} X_{S_n} = \infty$ almost surely. This also follows from the following result which extends the classical explosion criterion for pure Markov jump processes without drift (see e.g. Kersting and Klebaner (1995)) to the present frame of PDMP's.

Proposition 5.3. *Grant Assumptions 1 and 2 and suppose moreover that $I_0(x) < \infty$. Then almost surely*

$$(S_\infty < \infty) \iff \left(\sum_n e^{\Gamma(X_{S_n})} \int_{X_{S_n}}^\infty \frac{1}{\alpha(z)} e^{-\Gamma(z)} dz < \infty \right).$$

Proof. Let us write for short

$$e(x) := \mathbb{E}(T_x) = e^{\Gamma(x)} \int_x^\infty \frac{1}{\alpha(z)} e^{-\Gamma(z)} dz. \tag{5.2.14}$$

Then the process

$$A_n = \sum_{k=1}^n \mathbb{E}\left(S_k - S_{k-1} | \mathcal{F}_{S_{k-1}}\right) = \sum_{k=1}^n e(X_{S_{k-1}})$$

is the predictable increasing compensator of S_n, that is, $M_n := S_n - A_n$ is a martingale. Putting $\tau_a := \inf\{n : A_{n+1} > a\}$ it follows that $M^-_{n\wedge\tau_a} \le a$, and the martingale convergence theorem implies that $\{A_\infty < \infty\} \subset \{S_\infty < \infty\}$ almost surely. To prove the opposite inclusion, suppose $S_\infty < \infty$ with positive probability. Then necessarily $I_\infty(x) < \infty$. In particular, recalling (5.2.12),

$$\sup_n (S_n - S_{n-1}) \le \sup_n I_\infty(X_{S_{n-1}}) \le \int_0^\infty \frac{1}{\alpha(y)} dy < \infty$$

since 0 is reflecting by assumption and since $I_\infty(x) < \infty$. Introducing the stopping time $\sigma_a := \inf\{n : S_n > a\}$, it follows from the above that $\sup_n \mathbb{E}(M^+_{n\wedge\sigma_a}) < \infty$. Classical arguments then allow to conclude that $\{S_\infty < \infty\} \subset \{A_\infty < \infty\}$ almost surely. □

In what follows, we give conditions ensuring that the process reaches state $+\infty$ starting from any point $x \in (0,\infty)$. We also exhibit conditions implying that the process comes down from infinity to $y \in (0,\infty)$.

We can get IN from some $x \in (0,\infty)$ to the boundary point ∞ iff $\Gamma(\infty) < \infty$ and $I_\infty(x) < \infty$.

We can get OUT from the boundary point ∞ iff $H(\infty, y) > 0$ for some $y \in (0,\infty)$.

This leads to the following relevant combinations for the boundary state ∞. To classify them, we introduce $\Sigma = \sum_{n\geq 1} e(X_{S_n})$, where X_{S_n} is the embedded chain of X. Then we have :

1. $\Sigma < \infty$ and $H(\infty, y) > 0$: regular (accessible and reflecting).
2. $\Sigma < \infty$ and $H(\infty, y) = 0$: exit (accessible and absorbing).

5.3 Regularity of the transition operator and speed measure

We describe the infinitesimal generators of the process X.

Backward : With $u_t(x) := \mathbb{E}_x u(X_t)$, $u_0(x) = u(x)$, we have (Kolmogorov backward equation)

$$\partial_t u_t(x) = (G u_t)(x),$$

where G is given in (5.2.5).

Forward : With $\Pi_{t,x}(dy) = \mathbb{P}_x(X_t \in dy)$, $\Pi_{0,x}(dy) = \delta_x$, this also means

$$\frac{d}{dt}\int_0^\infty u(y)\,\Pi_{t,x}(dy) = \int_0^\infty (Gu)(y)\,\Pi_{t,x}(dy).$$

Notice that the measure $\Pi_{t,x}(dy)$ has support $[0, x_t(x)]$ with an atom at $x_t(x)$ with mass $\mathbb{P}(T_x > t)$. Considering the family of test functions $u(y) = e_\lambda(y) := e^{-\lambda y}$, $\lambda \geq 0$, for which

$$(Ge_\lambda)(x) = -\lambda\alpha(x)\,e_\lambda(x) + \lambda\beta(x)\int_0^x H(x,y)\,e_\lambda(y)\,dy,$$

we get, using Fubini's theorem and putting $\Pi_{t,x}(y) = \int_{[0,y]} \Pi_{t,x}(dz)$,

$$\begin{aligned}
&\frac{d}{dt}\int_0^\infty dy e_\lambda(y)\,\Pi_{t,x}(y) = \frac{d}{dt}\frac{1}{\lambda}\int_0^\infty e_\lambda(y)\,\Pi_{t,x}(dy)\\
&= -\int_0^\infty e_\lambda(y)\,\alpha(y)\,\Pi_{t,x}(dy) + \int_0^\infty dy e_\lambda(y)\int_{[y,\infty)} \beta(z)\,H(z,y)\,\Pi_{t,x}(dz). \quad (5.3.1)
\end{aligned}$$

Writing $\mathcal{D}'_+(\mathbb{R})$ for all distributions having support in $[0,\infty)$, we define the distribution $\delta_t\Pi_{t,x}$ by

$$< \delta_t\Pi_{t,x}, u >:= \frac{d}{dt}\int u(y)\Pi_{t,x}(y)dy$$

for any smooth test function u having compact support. Notice that $\delta_t\Pi_{t,x}$ is of compact support.

Therefore, Laplace transforms characterizing distributions with compact support in $\mathbb{R}_+$, by duality (Kolmogorov forward equation)

$$\delta_t\Pi_{t,x} = -\alpha(y)\,\Pi_{t,x}(dy) + dy\int_{[y,\infty)} \beta(z)\,H(z,y)\,\Pi_{t,x}(dz). \quad (5.3.2)$$

Proposition 5.4. *Suppose either that α is strictly positive on $[0,\infty)$ or, in case that $\alpha(0)=0$, either that $I_0(x)<\infty$ or that $H(x,0)=0$ for all $x>0$. Then for all $x>0$, $\Pi_{t,x}$ is absolutely continuous on $[0,x_t(x))$.*

Proof. Let g be a smooth test function having compact support in $[0,x_t(x))$. Then $\mathbb{E}_x(g(X_t)) = \mathbb{E}_x(g(X_t)\mathbb{1}_{\{t\le T_x\}})$. Recall that $S_1 < S_2 < \dots$ denote the successive jumps of X_t. Then we have

$$\mathbb{E}_x(g(X_t)) = \sum_{n=1}^{\infty} \mathbb{E}_x(g(X_t)\mathbb{1}_{\{N_t=n\}}).$$

The joint law of $Y_n := (S_1,\dots,S_{n+1},X_{S_1},\dots,X_{S_n})$ under $\mathbb{P}_x$ is given by

$$f_Y(s_1,\dots,s_{n+1},dx_1,\dots,dx_n)ds_1\dots ds_{n+1} = \beta(x_{s_1}(x))e_{s_1}(x)ds_1 \int_{\mathbb{R}_+} H(x_{s_1}(x),dx_1)\beta(x_{s_2}(x_1))e_{s_2}(x_1)ds_2\dots \int_{\mathbb{R}_+} H(x_{s_n}(x_{n-1}),dx_n)\beta(x_{s_{n+1}}(x_n))e_{s_{n+1}}(x_n)ds_{n+1},$$

where

$$e_t(x) := e^{-\int_0^t \beta(x_s(x))ds}.$$

Therefore,

$$\mathbb{E}_x(g(X_t)\mathbb{1}_{\{N_t=n\}}) = \int_{[0,t]^n\times[t,\infty[}\int_{\mathbb{R}_+^n} f_Y(s_1,\dots,s_{n+1},dx_1,\dots,dx_n)g(x_{t-s_n}(x_n))ds_1\dots ds_{n+1}.$$

Notice that under our condition, $x_{t-s_n}(x_n) > 0$ for all $s_n < t$. In particular we also have that $\alpha(x_{t-s_n}(x_n)) > 0$. Using the change of variables $s_n \mapsto z(s_n)$ with $z(s_n) := x_{t-s_n}(x_n) \in [x_n, x_t(x_n)]$, for fixed x_n, with $s_n = z^{-1}(z,x_n)$, we then have

$$\frac{dz}{ds_n} = -\alpha(x_{t-s_n}(x_n)) = -\alpha(z),$$

such that

$$\mathbb{E}_x(g(X_t\mathbb{1}_{\{N_t=n\}}) = \int_{\mathbb{R}_+} dz\frac{g(z)}{\alpha(z)}\Big(\int_{[0,t]^{n-1}\times[t,\infty[}\int_{\mathbb{R}_+^n} \mathbb{1}_{\{x_n\le z\le x_t(x_n)\}}f_Y(s_1,\dots,z^{-1}(z,x_n),s_{n+1},dx_1,\dots,dx_n)ds_1\dots ds_{n-1}ds_{n+1}\Big).$$

Summing over n implies the result. □

Let us come back to equation (5.3.2) together with the preceding considerations. We now know that under the conditions of Proposition 5.4, $\Pi_{t,x}(dy)$ admits a density $\pi_{t,x}(y)$ on $[0,x_t(x))$ and we have

$$\Pi_{t,x}(dy) = \mathbb{P}(T_x > t)\,\delta_{x_t(x)}(dy) + \pi_{t,x}(y)\,\mathbb{1}_{(y\in[0,x_t(x)))}dy.$$

(5.3.2) implies that on $[0,x_t(x))$, the distribution $\delta_t\Pi_{t,x}$ has a density $\delta_t\Pi_{t,x}(y)$ given by

$$\begin{aligned}\delta_t\Pi_{t,x}(y) &= -\alpha(y)\,\pi_{t,x}(y) + \int_y^\infty \beta(z)\,H(z,y)\,\Pi_{t,x}(dz)\\ &= -\alpha(y)\,\pi_{t,x}(y) + \int_y^\infty \beta(z)\,H(z,y)\,\pi_{t,x}(z)\,dz + \beta(x_t(x))H(x_t(x),y)\mathbb{P}(T_x>t).\end{aligned}$$

In the separable case $H(x,y) = h(y)/h(x)$, this can be rewritten as

$$\delta_t \Pi_{t,x}(y) = -\alpha(y)\,\pi_{t,x}(y) + h(y)\int_y^\infty \frac{\beta(z)}{h(z)}\pi_{t,x}(z)\,dz + \beta(x_t(x))\frac{h(y)}{h(x_t(x))}\mathbb{P}(T_x > t).$$

If $\widetilde{\pi}_{t,x}(y) := \alpha(y)\,\pi_{t,x}(y)$, recalling that $\gamma(x) := \beta(x)/\alpha(x)$, we have for all $y \in [0, x_t(x))$,

$$\delta_t \Pi_{t,x}(y) = -\widetilde{\pi}_{t,x}(y) + \int_y^\infty \gamma(z)\,H(z,y)\,\widetilde{\pi}_{t,x}(z)\,dz + \beta(x_t(x))H(x_t(x),y)\mathbb{P}(T_x > t)$$

In the separable case, this reads

$$\delta_t \Pi_{t,x}(y) = -\widetilde{\pi}_{t,x}(y) + h(y)\int_y^\infty \frac{\gamma(z)}{h(z)}\widetilde{\pi}_{t,x}(z)\,dz, + \beta(x_t(x))\frac{h(y)}{h(x_t(x))}\mathbb{P}(T_x > t).$$

Clearly, under the conditions of Proposition 5.4, $\Pi_{t,x}(0) = 0$. We conclude for $y = 0$: if $h(0) = 0$, $\widetilde{\pi}_{t,x}(0) = 0$. If $h(0) > 0$, then

$$\widetilde{\pi}_{t,x}(0) = h(0)\int_0^\infty \frac{\gamma(z)}{h(z)}\widetilde{\pi}_{t,x}(z)\,dz + \beta(x_t(x))\frac{h(0)}{h(x_t(x))}\mathbb{P}(T_x > t),$$

and the value of $\widetilde{\pi}_{t,x}(0)$ requires the knowledge of the whole $\widetilde{\pi}_{t,x}(z)$, for all $z \in (0, x_t(x))$.
We close this section with the following observation.

Proposition 5.5. *Suppose that $I_\infty(x) < \infty$ and that $\mathbb{P}(T_x < I_\infty(x)) = 1$. Grant moreover the assumptions of Proposition 5.4. Then $\Pi_{t,x}$ is absolutely continuous on $\mathbb{R}_+$ for all $t \geq I_\infty(x)$.*

Whenever an invariant measure π exists which is not equal to δ_0, the same argument leading to (5.3.1) implies that $\alpha(x)\pi(dx)$ admits a Lebesgue density $\widetilde{\pi}(x)$ solving the functional equation

$$\widetilde{\pi}(y) = \int_y^\infty \gamma(z)\,H(z,y)\,\widetilde{\pi}(z)\,dz$$

for $\lambda-$almost all $y > 0$. In the separable case $H(z,y) = \frac{h(y)}{h(z)}$, this yields the explicit expression

$$\pi(y) = C\frac{h(y)}{\alpha(y)}e^{-\Gamma(y)}, \tag{5.3.3}$$

up to a multiplicative constant $C > 0$. Notice that under Assumption 2, π is integrable in $0+$ if and only if $\int_0 h(x)/\alpha(x)dx < \infty$ which is equivalent to 0 reflecting in case $h(0) > 0$.

5.4 Harris Recurrence

5.4.1 Recurrence of X and of the embedded chain

In what follows we rely on the notion of Harris recurrence for Markov processes which we recall here for the convenience of the reader.

Definition 5.1 (see Azéma et al. (1969)). *X is called* Harris recurrent *if there exists some σ-finite measure m on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that for all $A \in \mathcal{B}(\mathbb{R}_+)$,*

$$m(A) > 0 \quad \textit{implies} \quad \mathbb{P}_x\left(\int_0^\infty 1_A(X_s)ds = \infty\right) = 1 \textit{ for all } x \in \mathbb{R}_+.$$

If is well-known (see again Azéma et al. (1969)) that if X is Harris recurrent, then there is a unique (up to constant multiples) invariant measure π for X, and the above property holds with π in place of m. X is then called *positive recurrent* (or also sometimes *ergodic*) if $\pi(\mathbb{R}_+) < \infty$, *null recurrent* if $\pi(\mathbb{R}_+) = \infty$.

Notice that whenever our process is Harris with invariant measure $\pi \neq \delta_0$, then its explicit density is necessarily given by (5.3.3) (in the separable case).

Example 5.4. If $h(x) \sim e^{\Gamma(x)}$ as $x \to \infty$, we have $\pi(x) \sim \frac{1}{\alpha(x)}$, as $x \to \infty$. In particular, $\int^{\infty} \pi(y)dy < \infty$ if and only if $I_\infty(x) < \infty$ for some (and thus all) $x > 0$. This means that the deterministic flow hits state $+\infty$ in finite time. Thus, the fact that the deterministic flow hits state ∞ in finite time helps the process be positive recurrent (compare also to (5.2.12)).

Let us now come back to our general framework. The following result establishes a relation between π and the invariant measure of the jump chains $(U_k)_k$ and $(Z_k)_k$ where $U_k = X_{S_k-}$ and $Z_k = X_{S_k}$, with $(S_k)_{k\geq 1}$ the sequence of successive jump times of the process.

Proposition 5.6. *Suppose that X is Harris recurrent having invariant measure π such that $0 < \pi(\beta) < \infty$. Then $(U_k)_k$ and $(Z_k)_k$ are both Harris recurrent. Their invariant measures π^U and π^Z are respectively given by*

$$\pi^U(g) = \frac{1}{\pi(\beta)}\pi(\beta g),\ \pi^Z(g) = \frac{1}{\pi(\beta)}\pi(\beta H g),$$

for any $g : \mathbb{R}^N \to \mathbb{R}$ measurable and bounded, where

$$\beta H g(x) = \beta(x) \int H(x, dy)g(y).$$

Proof. We just give the proof for $(Z_k)_k$, the case of $(U_k)_k$ is treated analogously. Let $g \geq 0$ be a bounded positive test function. It is sufficient to prove that $\frac{1}{n}\sum_{k=1}^n g(Z_k) \to \pi^Z(g)$ as $n \to \infty$, P_x-almost surely, for any fixed starting point x. But

$$\frac{1}{n}\sum_{k=1}^{n} g(Z_k) = \frac{1}{n}\sum_{k=1}^{n} g(X_{S_k}).$$

Introduce the jump measure

$$\mu(ds, dy, dz) = \sum_{n\geq 1} 1_{\{S_n<\infty\}}\delta_{(S_n, X_{S_n-}, X_{S_n})}(dt, dy, dz).$$

Its compensator is given by

$$\nu(ds, dy, dz) = \beta(X_{s-})ds\delta_{X_{s-}}(dy) \int H(y, dz).$$

Putting $N_t = \sup\{n : S_n \leq t\}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} g(X_{S_k}) = \lim_{t\to\infty} \frac{t}{N_t}\frac{1}{t}\sum_{k=1}^{N_t} g(X_{S_k}) = \lim_{t\to\infty} \frac{t}{N_t}\frac{A_t}{t},$$

where $A_t = \int_0^t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} g(z)\mu(ds, dy, dz)$ and N_t are additive functionals of the process X. By the ergodic theorem for the process X (which holds thanks to the

Harris recurrence of X_t), $N_t/t \to E_\pi(N_1)$ and $A_t/t \to E_\pi(A_1)$, and this convergence holds almost surely, for every starting point x. But $E_\pi(N_1) = E_\pi(\hat{N}_1)$ and $E_\pi(A_1) = E_\pi(\hat{A}_1)$, where

$$\hat{N}_t = \int_0^t \int\int \nu(ds, dy, dz) = \int_0^t \beta(X_s) ds$$

and

$$\hat{A}_t = \int_0^t \int\int g(z)\nu(ds, dy, dz) = \int_0^t \beta(X_s) \int H(X_s, dz) g(z) ds = \int_0^t \beta H g(X_s) ds.$$

Therefore, $E_\pi(N_1) = \pi(\beta)$ and $E_\pi(A_1) = \pi(\beta H g)$, and this finishes the proof. □

We use the above considerations to discuss rapidly that explosion of the process X (in the sense that $S_\infty < \infty$ with positive probability) is only possible if the jump chain Z_n is transient.

Proposition 5.7. *If Z_n is recurrent, explosion of X_t (that is, $\lim S_n = S_\infty < \infty$ with positive probability) is not possible.*

Proof. If $\pi^Z = \delta_0$, then non-explosion of the continuous time process is trivially implied. Let us therefore suppose that $\pi^Z \neq 0$. We know that explosion of X is equivalent to $\sum_{n\geq 1} e(Z_n) < \infty$ (recall the definition of e in (5.2.14)). But, if Z_n is recurrent (possibly null-recurrent), we know that for any function $g > 0$ such that $\pi^Z(g) \in (0, \infty)$,

$$\frac{\sum_{k=1}^n e(Z_k)}{\sum_{k=1}^n g(Z_k)} \to \pi^Z(e)/\pi^Z(g)$$

almost surely. Since $\sum_{k=1}^n g(Z_k) \uparrow \infty$ as $n \to \infty$, explosion implies that $\pi^Z(e) = 0$, whence $e = 0$ π^Z-almost surely. e being strictly positive on $(0, \infty)$, this yields a contradiction. □

Corollary 5.1. *In particular, if Z_n is recurrent (positive or null), then X is also recurrent (positive or null).*

Proof. Z_n recurrent implies $S_n \uparrow \infty$ almost surely, thanks to Proposition 5.7. Now let $A \in \mathcal{B}(\mathbb{R}_+)$ be such that $\pi^Z(A) > 0$ implying that $1_A(Z_n) = 1$ infinitely often. Then $\limsup_{t\to\infty} 1_A(X_t) \geq \limsup_{n\to\infty} 1_A(X_{S_n}) = \limsup_{n\to\infty} 1_A(Z_n) = 1$, whence the recurrence of X_t. □

5.4.2 Exit probabilities and excursions

In this section we propose a thorough study of the return times to 0 that enable us to state sufficient conditions for positive recurrence. Throughout this section we impose Assumptions 1 and 2. Recall that for any $x \geq 0$,

$$\tau_{x,0} = \inf\{t > 0 : X_t = 0 \mid X_0 = x\}$$

is the first time the process comes back to 0.
In what follows we fix $0 \leq x < b$ and are interested in establishing explicit formulae for

$$p(x, b) = \mathbb{P}_x\left(\tau_{x,0} < \tau_{x,b}\right),$$

where for any $0 \leq x < b$,

$$\tau_{x,b} = \inf\{t > 0 : X_t = b \mid X_0 = x\}. \tag{5.4.1}$$

Notice that it follows from the properties of our process that $\lim_{x\uparrow b} p(x,b) =: p(b,b) = 0$. However, we do not have that $\lim_{x\downarrow 0} p(x,b) =: p(0,b) = 1$. In general, $p(0,b) < 1$ is related to the height of an excursion between two successive visits to 0, see below.

A first step analysis implies that for all $0 < x < b$

$$p(x,b) = \int_0^{I_x(b)} \mathcal{L}(T_x)(ds) \left(H(x_s(x),0) + \int_{0+}^{x_s(x)} H(x_s(x),dy)p(y,b) \right),$$

with $I_x(b) = \int_x^b \frac{dy}{\alpha(y)}$ the time needed to go from x to b. A simple change of variables implies that

$$p(x,b) = \int_x^b \gamma(v)e^{-(\Gamma(v)-\Gamma(x))}H(v,0)dv + \int_x^b \gamma(v)e^{-(\Gamma(v)-\Gamma(x))} \left(\int_{0+}^v H(v,dy)p(y,b) \right) dv.$$

In the sequel we shall only consider the separable case $H(x,y) = \frac{h(y)}{h(x)}$ with

$$h(0) > 0.$$

In this case, the above formula implies that $x \mapsto p(x,b) \in C^1([0,b])$. Recalling that $p(b,b) = 0$, we rewrite

$$p(y,b) = -\int_y^b p'(z,b)dz,$$

where $p'(x,b) = \partial_x p(x,b)$ denotes partial derivative with respect to the initial position. We obtain

$$\begin{aligned} p(x,b) = (1-p(0,b)) \int_x^b \gamma(v)e^{-(\Gamma(v)-\Gamma(x))}\frac{h(0)}{h(v)}dv + \int_x^b \gamma(v)e^{-(\Gamma(v)-\Gamma(x))}p(v,b)dv \\ - \int_x^b \frac{\gamma(v)}{h(v)}e^{-(\Gamma(v)-\Gamma(x))} \int_0^v h(z)p'(z,b)dzdv. \end{aligned}$$

Taking derivatives, we obtain

$$p'(x,b)h(x) = \gamma(x)\int_0^x h(z)p'(z,b)dz - \gamma(x)(1-p(0,b))h(0).$$

Let

$$\kappa(x) := \int_0^x h(z)p'(z,b)dz - (1-p(0,b))h(0), 0 \leq x \leq b,$$

then we have $\kappa'(x) = h(x)p'(x,b)$ and $\kappa(0) = -(1-p(0,b))h(0)$. The above equation reads

$$\kappa'(x) = \gamma(x)\kappa(x)$$

leading to

$$\kappa(x) = Ce^{\Gamma(x)},$$

where we choose Γ such that $\Gamma(0) = 0$ and where C is such that

$$C = -(1-p(0,b))h(0); \text{ that is, } C = -h(0)(1-p(0,b)).$$

We deduce from this that

$$p'(x,b) = C\frac{\gamma(x)}{h(x)}e^{\Gamma(x)},$$

and thus, using once more that $p(b,b) = 0$,

$$p(x,b) = -C\int_x^b \frac{\gamma(y)}{h(y)}e^{\Gamma(y)}dy = h(0)(1-p(0,b))\int_x^b \frac{\gamma(y)}{h(y)}e^{\Gamma(y)}dy.$$

Finally, the value of $p(0,b)$ is deduced from

$$p(0,b) = h(0)(1-p(0,b))\int_0^b \frac{\gamma(y)}{h(y)}e^{\Gamma(y)}dy.$$

Let

$$s(x) = \int_0^x \frac{\gamma(y)}{h(y)}e^{\Gamma(y)}dy,\ \Gamma(y) = \int_0^y \gamma(t)dt. \tag{5.4.2}$$

Notice that under Assumption 2 and supposing that $h(0) > 0$, $s(x)$ is well-defined for any $x \geq 0$.

We obtain

$$p(0,b) = \frac{h(0)s(b)}{1+h(0)s(b)} \text{ and } \mathbb{P}(\tau_{x,0} < \tau_{x,b}) = p(0,b)[1-\frac{s(x)}{s(b)}]. \tag{5.4.3}$$

We have just proven the following

Theorem 5.1. *Grant Assumptions 1 and 2 and let $0 < x < b$. Suppose moreover that $H(x,y) = \frac{h(y)}{h(x)}$ with $h(0) > 0$. Suppose that $\Gamma(0) = 0$ and put*

$$\kappa := 1/h(0).$$

Then

$$\mathbb{P}(\tau_{x,0} > \tau_{x,b}) = \frac{\kappa + s(x)}{\kappa + s(b)}. \tag{5.4.4}$$

Notice that in case $h(x) = 1$ (total disaster), we obtain

$$\mathbb{P}(\tau_{x,b} < \tau_{x,0}) = e^{-(\Gamma(h)-\Gamma(x))}.$$

Discussion of the role of 0. Theorem 5.1 holds true in both cases 0 reflecting or absorbing. However what follows only makes sense in case 0 is reflecting, that is, $I_0(x) < \infty$. In this case we may introduce the height $\mathcal{H}$ of an excursion by

$$\mathcal{H} = \sup\{X_t : t < \tau_{0,0} | X_0 = 0\},$$

where $\tau_{0,0} = \inf\{t > 0 : X_t = 0 \mid X_0 = 0\} > 0$ is the first return time to 0. Since we have convergence $\tau_{x,0} \xrightarrow{\mathcal{L}} \tau_{0,0}$ as $x \to 0$, we may interpret $p(0,b)$ by means of the distribution function of the height of an excursion.

Proposition 5.8. *Grant the assumptions of Theorem 5.1 and suppose that $I_0(x) < \infty$. Then*

$$\mathbb{P}(\mathcal{H} < b) = \mathbb{P}(\tau_{0,0} < \tau_{0,b}) = p(0,b) = \frac{s(b)}{\kappa + s(b)}. \tag{5.4.5}$$

The successive return times of the process X to 0 induce a basic regeneration scheme and are thus related to the recurrence of the process.

Proposition 5.9. *Grant Assumptions 1 and 2 and suppose moreover that* $H(x,y) = \frac{h(y)}{h(x)}$ *with* $h(0) > 0$, *that* $I_\infty(x) = \infty$ *and* $I_0(x) < \infty$. *Then the process is recurrent if and only if* $s(\infty) = \infty$, *where the function* $s(x)$ *is given by* (5.4.2). *In this latter case,* $\tau_{x,0} < \infty$ *almost surely, and the unique invariant measure possesses a Lebesgue density on* $\mathbb{R}_+$ *which is given by* (5.3.3). *The process is positive recurrent if* $\int^\infty \frac{h(x)}{\alpha(x)} e^{-\Gamma(x)} dx < \infty$, *null-recurrent else.*

Proof. Suppose $s(\infty) = \infty$. We let $b \to \infty$ in (5.4.3) and notice that $\lim_{b\to\infty} p(0,b) = 1$ such that

$$\mathbb{P}(\tau_{x,0} < \tau_{x,\infty}) = 1.$$

This implies that $\tau_{x,0} < \infty$ almost surely.
On the other hand, suppose that the process is recurrent. It is straightforward to show that the recurrence implies that $\tau_{0,0} < \infty$ almost surely (recall that 0 is reflecting by assumption and that β is positive on $(0,\infty)$.) Since $\mathcal{H} \leq x_{\tau_{0,0}}(0)$ and since $I_\infty(x) = \infty$, this implies that $\mathcal{H} < \infty$ almost surely, i.e., $\lim_{b\to\infty} \mathbb{P}(\mathcal{H} < b) = \lim_{b\to\infty} p(0,b) = 1$. Under our assumptions, this is only possible if $s(\infty) = \infty$, since $\kappa \neq 0$. □

Remark 5.1. *We impose all assumptions of proposition 5.9 except that now we consider the absorbing case* $I_0(x) = \infty$. *In this case we still have that* $\tau_{x,0} < \infty$ *almost surely if and only if* $s(\infty) = \infty$ *: the process gets absorbed in* 0 *after a finite time almost surely and then stays there forever.*

When $h(x) = 1$ (total disasters), the event $\tau_{x,b} < \tau_{x,0}$ coincides with the event $T_x > I_b(x)$ where $I_b(x) = \int_x^b dy/\alpha(y)$ is the time needed for the flow to reach level b starting from x.

Example 5.5. Consider a growth model with $\alpha(x) = \alpha_1 x^a$, $\beta(x) = \beta_1$, $\gamma(x) = \gamma_1 x^{-a}$ and assume $h(x) = 1$. Assuming $a < 1$ for which boundary 0 is reflecting, then

$$x_t(x) = \left(x^{1-a} + \alpha_1(1-a)t\right)^{1/(1-a)} = b \Rightarrow I_b(x) = \frac{b^{1-a} - x^{1-a}}{\alpha_1(1-a)}.$$

Thus,

$$\begin{aligned}\mathbb{P}(\tau_{x,b} < \tau_{x,0}) &= \mathbb{P}(T_x > I_b(x)) \\ &= \mathbb{P}\left(T_x > \frac{b^{1-a} - x^{1-a}}{\alpha_1(1-a)}\right) = e^{-[\Gamma(x_t(x)) - \Gamma(x)]}\Big|_{t=\frac{b^{1-a}-x^{1-a}}{\alpha_1(1-a)}} = \frac{e^{\Gamma(x)}}{e^{\Gamma(b)}}\end{aligned}$$

with $\Gamma(x) = \frac{\gamma_1}{1-a} x^{1-a}$. As $x \to 0$,

$$\mathbb{P}(\tau_{0,b} < \tau_{0,0}) = \mathbb{P}(\mathcal{H} \geq b) = \mathbb{P}\left(T_0 > \frac{b^{1-a}}{\alpha_1(1-a)} = I_0(b)\right) = e^{-\Gamma(b)},$$

where $\mathcal{H}$ denotes the height of an excursion, which makes sense because boundary 0 is reflecting and the chain is recurrent ($s(\infty) = \infty$). So here

$$\mathcal{H} \overset{d}{=} \left(\alpha_1(1-a)T_0\right)^{1/(1-a)},$$

showing how height and length of excursions scale.

Example 5.6. Consider a growth model with $\alpha(x) = \alpha_0 + \alpha_1 x$ (Malthus growth with immigration), $\beta(x) = \beta_1$, $\gamma(x) = \beta_1/(\alpha_0 + \alpha_1 x)$ and assume $h(x) = e^x$. We have

$$\Gamma(x) = \frac{\beta_1}{\alpha_1} \log(\alpha_0 + \alpha_1 x)$$

satisfying Assumptions 1 and 2. State 0 is reflecting and the process X is transient at ∞. Here $\kappa = \frac{\beta_1}{\alpha_1} \log \alpha_0$, and

$$s(x) = \beta_1 \int_0^x (\alpha_0 + \alpha_1 y)^{\beta_1/\alpha_1 - 1} e^{-y} dy = \frac{\beta_1 e^{\alpha_0/\alpha_1}}{\alpha_1} \int_{\alpha_0}^{\alpha_0 + \alpha_1 x} z^{\beta_1/\alpha_1 - 1} e^{-z/\alpha_1} dz,$$

involving an integral Gamma function. It holds that

$$\mathbb{P}(\mathcal{H} \geq b) = \frac{\kappa}{\kappa + s(b)},$$

with $\mathbb{P}(\mathcal{H} = \infty) = \kappa/(\kappa + s(\infty)) > 0,\ s(\infty) < \infty$.

Remark 5.2. *Under the assumptions of Theorem 5.1, let us discuss the situation $s(\infty) < \infty$. In this case we have $\mathbb{P}(\tau_{x,0} < \tau_{x,\infty}) < 1$.*

In the case $\tau_{x,\infty} = \infty$, with positive probability the process never comes back to 0 and thus is transient, that is, converges to $+\infty$ as $t \to \infty$.

In the case $\tau_{x,\infty} < \infty$, the process hits state $+\infty$ even in finite time. Proposition 5.2 implies that in this case $S_\infty < \infty$ such that the jump chain $Z_n = X_{S_n}$ is transient. However in case ∞ is regular, we can add state $+\infty$ to the state space. In this particular situation the process X_t is even recurrent having $+\infty$ as recurrent state.

5.4.3 Classification of the recurrence/transience of state 0 in the separable case

We close this section with a classification of the recurrence/transience of state 0 in the separable case with $h(0) > 0$. Under Assumptions 1 and 2, we have :

1. $s(\infty) = \infty, I_0(x) < \infty$: 0 is recurrent, positive recurrent iff $\int^\infty \frac{h(x)}{\alpha(x)} e^{-\Gamma(x)} dx < \infty$.
2. $s(\infty) = \infty, I_0(x) = \infty$: The process is transient in 0 (almost surely hits 0 in finite time and stays there forever).
3. $s(\infty) < \infty, I_\infty(x) = \infty$: The process is transient (converges to $+\infty$ with positive probability).
4. $s(\infty) < \infty, I_\infty(x) < \infty$: The process is either transient (converges to $+\infty$ with positive probability) or hits state ∞ in finite time ($\tau_{x,\infty} < \infty$ with positive probability). If state $+\infty$ is REGULAR, we can add it to the state space, and it will become a recurrent state. If it is EXIT the process hits $+\infty$ in finite time and then stays there forever with positive probability.

5.4.4 Expected return times to 0

This section is devoted to obtaining an explicit formula for $u(x) = \mathbb{E}(\tau_{x,0})$ in the case of positive recurrence. In case of total disaster when $H(x, 0) = 1$ for all x, we

have $\tau_{x,0} = T_x$ which has already been discussed. So we suppose $H(x,0) < 1$ for all x in this subsection. If $x > 0$, we have

$$\tau_{x,0} \overset{d}{=} T_x \mathbb{1}\left(X_{T_x} = 0\right) + \mathbb{1}\left(X_{T_x} > 0\right)\left(T_x + \tau'_{X_{T_x},0}\right), \tag{5.4.6}$$

where $\tau'_{X_{T_x}}$ is independent of $\mathcal{F}_{T_x}$ and distributed as $\tau_{X_{T_x}}$. This implies

$$u(x) = \mathbb{E}\left(\tau_{x,0}\right) = \mathbb{E}\left(T_x\right) + \int_{0+}^{\infty} \mathbb{P}\left(X_{T_x} \in dy\right) \mathbb{E}\left(\tau_{y,0}\right), \; x > 0,$$

where we recall that

$$u_0(x) = \mathbb{E}\left(T_x\right) = e^{\Gamma(x)} \int_x^{\infty} \frac{dz}{\alpha(z)} e^{-\Gamma(z)}. \tag{5.4.7}$$

Imposing Assumptions 1 and 2 and moreover that $u_0(x) < \infty$, u_0 solves

$$\alpha(x)\, u_0'(x) - \beta(x)\, u_0(x) = -1, \text{ with } u_0(0) = e^{\Gamma(0)} \int_0^{\infty} \frac{dz}{\alpha(z)} e^{-\Gamma(z)},$$

which is finite under Assumption 2, if 0 is reflecting.
In what follows we shall always choose $\Gamma(0) = 0$, and we also impose

Assumption 3. *1. X is positive recurrent having 0 as recurrent point. In particular, $H(x,0) > 0$ for some $x > 0$ and 0 is reflecting, that is $I_0(x) < \infty$.*
2. The function $\mathbb{R}_+ \ni x \mapsto \int_0^x g(y)\overline{H}(x,dy)$ is continuous for all bounded measurable test functions g.

Proposition 5.10. *Suppose that Assumptions 1, 2 and 3 hold. Suppose moreover that $u(x) = \mathbb{E}(\tau_{x,0})$ is locally bounded, that is, $\sup\{u(y), 0 \le y \le x\} < \infty$ for all $x > 0$. Then $u \in C^1((0,\infty))$, and it solves*

$$\mathcal{G}u(x) = -1 \text{ on } (0,\infty), \tag{5.4.8}$$

where for all $x > 0$,

$$\mathcal{G}u(x) = \alpha(x)u'(x) - \beta(x)H(x,0)u(x) + \beta(x)\int_{0+}^{x} \overline{H}(x,dy)[u(y) - u(x)].$$

Notice that $u(0+) := \lim_{x\to 0} u(x) \neq 0$, implying that in general, $Gu(x) - \mathcal{G}u(x) = \beta(x)H(x,0)u(0) \neq 0$.

Proof. From (5.4.6), we have

$$\mathbb{E}\left(\tau_{x,0}\right) = \mathbb{E}\left(T_x\right) + \int_{0+}^{\infty} \mathbb{P}\left(X_{T_x} \in dy\right) \mathbb{E}\left(\tau_{y,0}\right).$$

If $y > 0$, $\mathbb{P}\left(X_{T_x} \in dy\right) = \int_x^{\infty} dz\gamma(z)\, e^{-\int_x^z \gamma(z')dz'} \overline{H}(z,dy)$. Therefore

$$\begin{aligned}
\mathbb{E}\left(\tau_{x,0}\right) &= \mathbb{E}\left(T_x\right) + \int_x^{\infty} dz\gamma(z)\, e^{-\int_x^z \gamma(z')dz'} \int_{0+}^{z} \overline{H}(z,dy)\, \mathbb{E}\left(\tau_{y,0}\right) \\
&= u_0(x) + e^{\Gamma(x)} \int_x^{\infty} dz\gamma(z)\, e^{-\Gamma(z)} \int_{0+}^{z} \overline{H}(z,dy)\, \mathbb{E}\left(\tau_{y,0}\right),
\end{aligned}$$

where u_0 is given in (5.4.7) and differentiable on $(0,\infty)$.

Since $z \mapsto \gamma(z)\, e^{-\Gamma(z)} \int_0^z H(z, dy)\, \mathbb{E}(\tau_{y,0})$ is continuous, $u(x) = \mathbb{E}(\tau_{x,0})$ is differentiable on $(0, \infty)$ and obeys

$$u'(x) = u_0'(x) + \gamma(x)\,(u(x) - u_0(x)) - \gamma(x) \int_{0+}^{x} \overline{H}(x, dy)\, u(y).$$

Recalling $u_0'(x) = \gamma(x)\, u_0(x) - 1/\alpha(x)$, this is

$$u'(x) = -1/\alpha(x) + \gamma(x) \left[u(x) - \int_{0+}^{x} \overline{H}(x, dy)\, u(y) \right]. \tag{5.4.9}$$

Finally,

$$u(x) - \int_{0+}^{x} \overline{H}(x, dy)\, u(y) = H(x, 0)u(x) + \int_{0+}^{x} \overline{H}(x, dy)(u(x) - u(y)),$$

which implies the assertion. □

In what follows, $\pi(y)$ designs the speed density with integration constant C introduced in (5.3.3) above. By our assumptions, $\pi(y)$ is integrable. We also recall the definition of the modified scale function s in (5.4.2). We suppose that H is separable with $h(0) > 0$. In this case it is possible to obtain an explicit formula for $u(x)$ as we shall show now.

We start with the following observation that allows us to determine the value of u at 0, $u(0+) := \lim_{x \to 0} u(x) = \mathbb{E}(\tau_{0,0})$.

Theorem 5.2. *Grant the assumptions of Proposition 5.10. Suppose moreover that H is separable with $h(0) > 0$. Let π the unique invariant measure given in (5.3.3), where the constant C is chosen such that π is tuned to a probability. Then for any Borel subset B of $\mathbb{R}_+$,*

$$\pi(B) = \frac{1}{u(0+)} \mathbb{E}_0 \int_0^{\tau_{0,0}} 1_B(X_s) ds. \tag{5.4.10}$$

Moreover, suppose now that $\pi(\beta) \in (0, \infty)$ and that $\alpha(0) > 0$. Then

$$\mathbb{E}(\tau_{0,0}) = u(0+) = \frac{1}{Ch(0)}. \tag{5.4.11}$$

Proof. Representation (5.4.10) is classical and follows from decomposing the trajectory of X into successive excursions out of 0 (see e.g. Proposition 2.8 in Löcherbach et al. (2011)). Applying (5.4.10) with $B = [0, \varepsilon]$, we obtain

$$\frac{1}{\varepsilon} \int_0^{\varepsilon} \pi(y) dy = \frac{1}{u(0+)} \mathbb{E}_0 \left(\frac{1}{\varepsilon} \int_0^{\tau_{0,0}} 1_{\{X_s \le \varepsilon\}} ds \right).$$

Letting $\varepsilon \to 0$, clearly the left hand side converges to $\pi(0) = C(h(0)/\alpha(0))e^{-\Gamma(0)} = Ch(0)/\alpha(0)$, since we have chosen $\Gamma(0) = 0$. The remainder of the proof is devoted to showing that

$$\lim_{\varepsilon \to 0} \mathbb{E}_0 \left(\frac{1}{\varepsilon} \int_0^{\tau_{0,0}} 1_{\{X_s \le \varepsilon\}} ds \right) = 1/\alpha(0).$$

Clearly,

$$\mathbb{E}_0 \int_0^{\tau_{0,0}} 1_{\{X_s \le \varepsilon\}} ds = \mathbb{P}(T_0 > I_0(\varepsilon)) \cdot I_0(\varepsilon) + R(\varepsilon), \tag{5.4.12}$$

with $I_0(\varepsilon) = \int_0^\varepsilon \frac{1}{\alpha(y)} dy$ the time needed for the deterministic flow to reach ε, starting from 0. In the sequel we will show that the remainder term $R(\varepsilon)$ is actually of the order $R(\varepsilon) = o(\varepsilon)$. Then the assertion follows from

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{P}(T_0 > I_0(\varepsilon)) \cdot I_0(\varepsilon) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} e^{-\Gamma(\varepsilon)} \int_0^\varepsilon \frac{1}{\alpha(y)} dy = \frac{1}{\alpha(0)}.$$

Step 1. In what follows, we rely on the fact that for any $y \geq \varepsilon$, we have that

$$\mathbb{E}_y \int_0^{\tau_{y,0}} 1_{\{0 < X_s \leq \varepsilon\}} ds = o(\varepsilon). \tag{5.4.13}$$

Indeed,

$$\begin{aligned} \mathbb{E}_y \int_0^{\tau_{y,0}} 1_{\{0 < X_s \leq \varepsilon\}} ds &= \sum_{n \geq 1} \mathbb{E}_y \left(1_{\{S_n < \tau_{y,0}\}} 1_{\{X_{S_n} \leq \varepsilon\}} (I_{X_{S_n}}(\varepsilon) \wedge (S_{n+1} - S_n)) \right) \quad (5.4.14) \\ &\leq I_0(\varepsilon) \sum_{n \geq 1} \mathbb{E}_y \left(1_{\{S_n < \tau_{y,0}\}} 1_{\{X_{S_n} \leq \varepsilon\}} \right) = I_0(\varepsilon) \mathbb{E}_y \left(\sum_{n=1}^{\bar{\tau}_{y,0}-1} 1_{\{Z_n \leq \varepsilon\}} \right), \end{aligned}$$

where

$$\bar{\tau}_{y,0} = \inf\{n \geq 1 : Z_n = 0 \mid Z_0 = y\}$$

is the return time to 0 of the embedded chain $Z_n = X_{S_n}$.

Since $\pi(\beta) < \infty$, $(Z_n)_n$ is a positive Harris recurrent strong Feller chain (recall Proposition 5.1). Being strong Feller, every bounded measurable function f having compact support is a *special function* (see Revuz (2008), exercise 4.11, chapter 6, page 215). This means that for any function h such that $\pi^Z(h) > 0$, the function

$$x \mapsto \mathbb{E}_x \left(\sum_{n=1}^{\infty} (1 - h(Z_1)) \cdot \ldots \cdot (1 - h(Z_{n-1})) f(Z_n) \right)$$

is bounded. Taking $f = 1_{(0,1]}$ (which, being of compact support, is therefore a special function) and $h = 1_{\{0\}}$ (which satisfies $\pi^Z(h) > 0$ since Z_n is recurrent coming back to 0 infinitely often almost surely) we obtain that

$$x \mapsto \mathbb{E}_x \sum_{n=1}^{\bar{\tau}_{x,0}-1} 1_{\{Z_n \leq 1\}} \text{ is bounded,}$$

implying the assertion by dominated convergence.

Step 2. We now treat the remainder term $R(\varepsilon) = R_1(\varepsilon) + R_2(\varepsilon)$ appearing in (5.4.12), where

$$R_1(\varepsilon) = \mathbb{E}_0 \left(1_{\{T_0 > I_\varepsilon(0)\}} \int_{I_\varepsilon(0)}^{\tau_{0,0}} 1_{\{X_s \leq \varepsilon\}} ds \right) \text{ and } R_2(\varepsilon) = \mathbb{E}_0 \left(1_{\{T_0 \leq I_\varepsilon(0)\}} \int_0^{\tau_{0,0}} 1_{\{X_s \leq \varepsilon\}} ds \right).$$

Observe that

$$\begin{aligned} R_2(\varepsilon) &= \int_0^{I_0(\varepsilon)} \beta(x_t(0)) e^{-\int_0^t \beta(x_s(0)) ds} dt \\ &\qquad \left(t + \int_0^{x_t(0)} \overline{H}(x_t(0), dy) \mathbb{E}_y \int_0^{\tau_{y,0}} 1_{\{0 < X_u \leq \varepsilon\}} \right) \\ &\leq \int_0^\varepsilon \gamma(x) e^{-(\Gamma(x) - \Gamma(0))} \left(I_0(\varepsilon) + \int_0^x \overline{H}(x, dy) \mathbb{E}_y \int_0^{\tau_{y,0}} 1_{\{0 < X_u \leq \varepsilon\}} \right) dx = O(\varepsilon^2), \end{aligned}$$

since $t \leq I_0(\varepsilon)$.
Concerning the first remainder term, we first use that by the Markov property,

$$R_1(\varepsilon) = e^{-\Gamma(\varepsilon)}\mathbb{E}_\varepsilon \int_0^{\tau_{\varepsilon,0}} 1_{\{0<X_s\leq\varepsilon\}} ds.$$

We consider three different events.

We say that event E_1 is realized when the first jump of the process leads to an after-jump position $y \leq \varepsilon$ while the second jump of the process happens after the process has reached ε.

We say that event E_2 is realized when the first jump of the process leads to an after-jump position $y \leq \varepsilon$ and the second jump of the process happens before the process reaches ε again.

We say that event E_3 is realized when the first jump of the process leads to an after-jump position $y > \varepsilon$.

Clearly,

$$e^{-\Gamma(\varepsilon)}\mathbb{E}_\varepsilon 1_{E_1} \int_0^{\tau_{\varepsilon,0}} 1_{\{0<X_s\leq\varepsilon\}} ds = \int_\varepsilon^\infty \gamma(z)e^{-\Gamma(z)}dz \int_{0+}^{\varepsilon} \overline{H}(z,dy)e^{-(\Gamma(\varepsilon)-\Gamma(y))} \left(\int_y^\varepsilon \frac{1}{\alpha(t)}dt + E_\varepsilon \int_0^{\tau_{\varepsilon,0}} 1_{\{0<X_s\leq\varepsilon\}}\right) = o(\varepsilon)$$

under our hypotheses (where we have used (5.4.13)).

Similar arguments as those used to control $R_2(\varepsilon)$ show that

$$e^{-\Gamma(\varepsilon)}\mathbb{E}_\varepsilon 1_{E_2} \int_0^{\tau_{\varepsilon,0}} 1_{\{0<X_s\leq\varepsilon\}} ds \leq \int_\varepsilon^\infty \gamma(z)e^{-\Gamma(z)}dz \int_{0+}^{\varepsilon} \overline{H}(z,dy) \int_y^\varepsilon \gamma(z')e^{-(\Gamma(z')-\Gamma(y))}dz' \left(I_0(\varepsilon) + \int_{0+}^{z'} \overline{H}(z',du)\mathbb{E}_u \int_0^{\tau_{u,0}} 1_{\{0<X_s\leq\varepsilon\}} ds\right) = O(\varepsilon^2).$$

Moreover, using (5.4.14),

$$\begin{aligned} e^{-\Gamma(\varepsilon)}\mathbb{E}_\varepsilon 1_{E_3} \int_0^{\tau_{\varepsilon,0}} 1_{\{0<X_s\leq\varepsilon\}} ds &\leq I_0(\varepsilon)\mathbb{E}_\varepsilon \left(1_{\{Z_1\geq\varepsilon\}}\mathbb{E}_{Z_1}\Big(\sum_{n=1}^{\bar{\tau}_{Z_1,0}-1} 1_{\{Z_n\leq\varepsilon\}}\Big)\right) \\ &\leq I_0(\varepsilon)\mathbb{E}_\varepsilon \left(\sum_{n=1}^{\bar{\tau}_{Z_1,0}-1} 1_{\{Z_n\leq\varepsilon\}}\right) = I_0(\varepsilon)O(\varepsilon). \end{aligned}$$

All in all we have shown that $R(\varepsilon) = o(\varepsilon)$ which concludes the proof.

□

Theorem 5.3. *Grant the assumptions of Proposition 5.10 together with $\pi(\beta) < \infty$, and suppose that $H(x,y) = h(y)/h(x)$, where h is differentiable, non-decreasing, with $h(0) > 0$ and $\alpha(0) > 0$. We choose $\Gamma(0) = 0$.*
Then $u(x)$ is given by

$$\begin{aligned} u(x) &= u(0) + \int_0^x dy \frac{\gamma(y)\, e^{\Gamma(y)}}{h(y)} \int_y^\infty e^{-\Gamma(z)} \frac{h(z)}{\alpha(z)} dz - \int_0^x \frac{1}{\alpha(y)} dy \\ &= u(0) + s(x)\int_x^\infty \pi(y)dy + \int_0^x s(y)\pi(y)dy - \int_0^x \frac{1}{\alpha(y)}dy, \quad (5.4.15) \end{aligned}$$

where $u(0)$ is given by (5.4.11).

Proof. We come back to (5.4.9) and we put $\overline{H}(y) = h(y) - h(0)$. Using Fubini's theorem and the fact that for $y > 0, u(x) - u(y) = \int_y^x u'(z)dz$, since u differentiable on $(0, \infty)$,

$$\begin{aligned} u(x) - \int_{0+}^{x} \overline{H}(x, dy)\, u(y) &= H(x,0)u(x) + \int_{0+}^{x} (u(x) - u(y))\overline{H}(x, dy) \\ &= H(x,0)u(x) + \int_0^x \overline{H}(x,y)u'(y)dy \\ &= \frac{h(0)}{h(x)}u(x) + \frac{1}{h(x)}\int_0^x \overline{H}(y)u'(y)dy \\ &= \frac{h(0)}{h(x)}u(0) + \frac{1}{h(x)}\int_0^x h(y)u'(y)dy. \end{aligned}$$

Therefore, u solves

$$\alpha(x)u'(x) - \frac{\beta(x)}{h(x)}\int_0^x h(y)u'(y)dy - \frac{\beta(x)}{h(x)}h(0)u(0) = -1$$

on $(0, \infty)$.

Put $v(x) = \int_0^x h(y)u'(y)dy + h(0)u(0)$, for $x > 0$. Using integration by parts and the fact that $h'(y)u(y) \geq 0$, we obtain that $v(x) \leq h(x)u(x) < \infty$ for all x. Moreover, $v'(x) = h(x)u'(x)$ and $v(0) = h(0)u(0)$, and thus

$$v'(x) - \gamma(x)v(x) = -\frac{h(x)}{\alpha(x)}. \tag{5.4.16}$$

Putting $w(x) := e^{-\Gamma(x)}v(x)$, $w(x) < \infty$, since $v(x) < \infty$, we have

$$w'(x) = -e^{-\Gamma(x)}\frac{h(x)}{\alpha(x)} = -\frac{1}{C}\pi(x) < 0,$$

where π is the speed density given in (5.3.3).

By our assumptions, π, and hence w', is integrable on $\mathbb{R}_+$ implying that the explicit solution of the above equation is given by

$$w(x) = w(\infty) + \int_x^\infty e^{-\Gamma(y)}\frac{h(y)}{\alpha(y)}dy, \tag{5.4.17}$$

for some finite constant $w(\infty)$, so that

$$v(x) = e^{\Gamma(x)}\int_x^\infty e^{-\Gamma(y)}\frac{h(y)}{\alpha(y)}dy + e^{\Gamma(x)}w(\infty). \tag{5.4.18}$$

Since by (5.4.16)

$$\frac{v'(x)}{h(x)} = \gamma(x)\frac{v(x)}{h(x)} - \frac{1}{\alpha(x)} = u'(x),$$

this implies

$$u(x) = u(0) + \int_0^x u'(y)\,dy = u(0) + \int_0^x dy\frac{\gamma(y)\,e^{\Gamma(y)}}{h(y)}\left[\int_y^\infty e^{-\Gamma(z)}\frac{h(z)}{\alpha(z)}dz + w(\infty)\right] - \int_0^x \frac{1}{\alpha(y)}dy.$$

The value of $w(\infty)$ is deduced from the fact that on the one hand

$$w(0) = w(\infty) + \int_0^\infty e^{-\Gamma(y)} \frac{h(y)}{\alpha(y)} dy = w(\infty) + \frac{1}{C}$$

and on the other hand

$$w(0) = e^{-\Gamma(0)} v(0) = h(0)u(0).$$

Replacing $u(0)$ by its explicit value given in (5.4.11), we obtain from this that $w(0) = \frac{1}{C}$, whence $w(\infty) = 0$, which implies the assertion. □

Example 5.7. Let $h(x) = e^x$, $\alpha(x) = \alpha_1 x^a$, $a < 1$ (entailing 0 is reflecting), $\beta(x) = \beta_1 x^a$, $(b = a > a - 1)$. Assumptions 1 and 2 are satisfied. To ensure recurrence, we assume $\gamma(x) = \gamma_1 > 1$ and due to this, we obtain the expected first return time to 0 as

$$u(0) = \mathbb{E}(\tau_{0,0}) = \frac{1}{\alpha_1} \int_0^\infty y^{-a} e^{-(\gamma_1 - 1)y} = \frac{\Gamma(1-a)}{\alpha_1 (\gamma_1 - 1)^{1-a}} < \infty.$$

Note that, consistently, $u(0)$ diverges when $\gamma_1 \downarrow 1$ and also when $a \uparrow 1$. We also have

$$\begin{aligned} u(x) &= u(0) + \frac{\gamma_1}{(\gamma_1 - 1)\alpha_1} \int_0^x de^{(\gamma_1 - 1)y} \int_y^\infty \frac{e^{-(\gamma_1 - 1)z}}{z^a} dz - \frac{1}{\alpha_1 (1-a)} x^{1-a} \\ &\sim \frac{1}{(\gamma_1 - 1)\alpha_1 (1-a)} x^{1-a} \quad \text{as } x \to \infty, \end{aligned}$$

where, after integration by parts, we used a large x estimate of the integral Gamma function. The large x expected time to local extinction is algebraic. An exact expression (involving the integral Gamma function) of $u(x)$ for all x is available from the first expression of $u(x)$.

5.5 Some Simulations

We illustrate our results by some simulations involving a growth model with immigration. In our simulations we take $\alpha(x) = \alpha_0 + \alpha_1 x^a$ and $\beta(x) = x^b$ with $\alpha_0 = \alpha_1 = 1$, $a = 2$ and $b = \frac{3}{2}$. In this case, the state 0 is reflecting, and the process $x_t(x)$ reaches ∞ in finite time. Assumptions 1 and 2 are both satisfied. We work in the separable case $H(x, y) = \frac{h(y)}{h(x)}$.

The following simulations are done in discrete time by using the embedded chain $Z_n = X_{S_n}$ in the case where 0 is not absorbing. In this case, we have for all $x \geq 0$,

$$\mathbb{P}(Z_n \in dy \mid Z_{n-1} = x) = e^{\Gamma(x)} \int_x^\infty dz \gamma(z) e^{-\Gamma(z)} H(z, dy),$$

meaning that Z_n is a time-homogeneous discrete-time Markov chain on $[0, \infty]$.
We also have

$$\begin{aligned} \mathbb{P}(Z_n \leq y \mid Z_{n-1} = x) &= e^{\Gamma(x)} \int_x^\infty dz \gamma(z) e^{-\Gamma(z)} \int_0^y H(z, dy') \\ &= 1 - e^{-(\Gamma(x \vee y) - \Gamma(x))} + \int_{x \vee y}^\infty dz \gamma(z) e^{-(\Gamma(z) - \Gamma(x))} H(z, y). \quad (5.5.1) \end{aligned}$$

Indeed, since $H(z,y)=1$ for all $y \geq z$ and only whenever $y \geq x$, the second integral in the first equation has to be cut into two pieces corresponding to ($z > y$ and $x < z \leq y$).
To simulate the embedded chain, we have to decide first if, given $Z_{n-1} = x$, the forthcoming move is down or up.
- A move down occurs with probability

$$\mathbb{P}(Z_n \leq x \mid Z_{n-1} = x) = \int_x^{\infty} dz\gamma(z)\, e^{-(\Gamma(z)-\Gamma(x))} H(z,x).$$

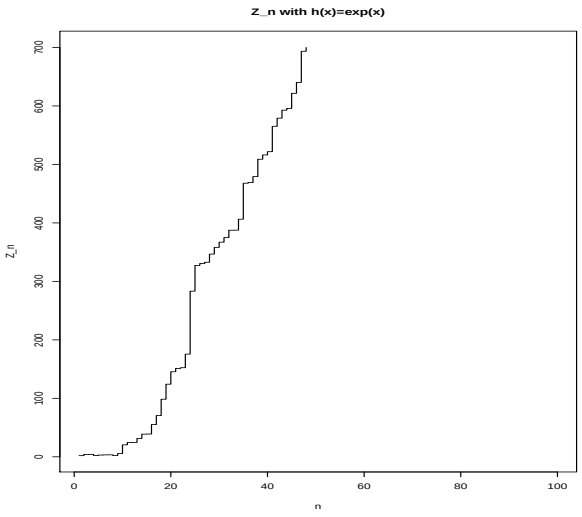
- A move up occurs with the complementary probability.
As soon as the type of move is fixed (down or up), to decide where the process goes precisely, we must use the inverse of the corresponding distribution function (5.5.1) (with $y \leq x$ or $y > x$), conditioned on the type of move.

Remark 5.3. *(i) If the jump kernel $H(z,y)$ is decreasing in z for each fixed y, then, from (5.5.1), the embedded chain is stochastically monotone, that is, for each fixed y, $\mathbb{P}(Z_n \leq y \mid Z_{n-1} = x)$ is decreasing in x. Note that*

$$\mathbb{P}(Z_n \in dy \mid Z_{n-1} = x) = e^{\Gamma(x)} \int_x^{\infty} dz\gamma(z)\, e^{-\Gamma(z)} H(z,dy) = \mathbb{E}H(G(x), dy).$$

(ii) If state 0 is absorbing, equation (5.5.1) is valid only when $x > 0$ and the boundary condition $\mathbb{P}(Z_n = 0 \mid Z_{n-1} = 0) = 1$ should be added.

The first simulation is done with the choice $h(x) = e^x$. Here, state $+\infty$ is an absorbing state.



We can remark the occurrence of many jumps for small values of the process and the scarcity of jumps for large values. In other words, the probability of disaster when the process is at position x tends to 0 when x tends to infinity. It is decreasing in x, i.e. the greater x is, the less is the probability of disaster at that point. In particular, $H(\infty, \{\infty\}) = 1$, that is, state $+\infty$ is absorbing.
By doing a simple calculation we notice that $s(\infty) < \infty$ and $I_\infty(x) < \infty$. Using the last criterion in section 4.4 we conclude that either the process X is transient (converges to $+\infty$ as $t \to \infty$) or hits $+\infty$ in finite time and then stays there forever.
In the next simulation we choose $h(x) = 1$ for all x (total disaster case). In this case $Z_n = 0$ for all $n \geq 1$. To obtain some information about the process, we have

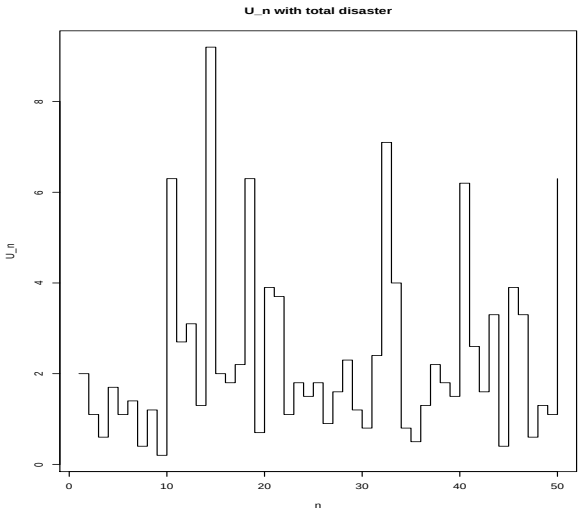
simulated $U_n = X_{S_n-}$. Since $s(\infty) = \infty$, the process X is recurrent and comes back to 0 infinitely often. We have

$$\begin{aligned}\mathbb{P}\left(U_n \in dy \mid U_{n-1} = x\right) &= \int_0^x H(x, dz) \int_0^\infty dt\beta\left(x_t\left(z\right)\right) e^{-\int_z^{x_t(z)} \gamma(u)du} \delta_{x_t(z)}(dy) \\ &= \int_0^x H(x, dz) e^{\Gamma(z)} \int_z^\infty du\gamma\left(u\right) e^{-\Gamma(u)} \delta_u(dy).\end{aligned}$$

In the particular case $h(x) = 1$, that is, $H(x, dz) = \delta_0(dz)$, this gives

$$\mathbb{P}\left(U_n \in dy \mid U_{n-1} = x\right) = \gamma(y) e^{-(\Gamma(y)-\Gamma(0))} dy, \tag{5.5.2}$$

that is, $(U_n)_{n\geq 1}$ is an i.i.d. sequence with common distribution given according to (5.5.2).



6

On decay-surge population models

Abstract. We consider continuous space-time decay-surge population models which are semi-stochastic processes for which deterministically declining populations, bound to fade away, are reinvigorated at random times by bursts or surges of random sizes. In a particular separable framework (in a sense made precise below) we provide explicit formulae for the scale (or harmonic) function and the speed measure of the process. The behavior of the scale function at infinity allows to formulate conditions under which such processes either explode or are transient at infinity, or Harris recurrent. A description of the structures of both the discrete-time embedded chain and extreme record chain of such continuous-time processes is supplied.

This chapter is contained in the submitted preprint, B.Goncalves, T. Huillet and E. Löcherbach. "*On decay-surge population models*" https://hal.archives-ouvertes.fr/hal-03035061.

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6.1 Introduction

This chapter deals with decay-surge population models where a deterministically declining evolution following some nonlinear flow is interrupted by bursts of random sizes occurring at random times. Decay-surge models are natural models of many physical and biological phenomena, including the evolution of aging and declining populations which are reinvigorated by immigration, the height of the membrane potential of a neuron decreasing in between successive spikes of its presynaptic neurons due to leakage effects and jumping upwards after the action potential emission of its presynaptic partners, work processes in single-server queueing systems as from Cohen (1982)... Our preferred physical image will be the one of aging populations subject to immigration.

Decay-surge models have been extensively studied in the literature, see among others Eliazar and Klafter (2006b), Harrison and Resnick (1976) and Harrison and Resnick (1978). Most studies concentrate however on non-Markovian models such as shot noise or Hawkes processes, where superpositions of overlapping decaying populations are considered, see Brémaud and Massoulié (1996), Eliazar and Klafter (2007) and Eliazar and Klafter (2009), Huillet (2021), Kella and Stadje (2001), Kella (2009) and Brockwell et al. (1982b).

Inspired by storage processes for dams, the chapters of Resnick and Tweedie, Kella (2009), Cairns and Pollett (2004), Asmussen and Kella (1996), Boxma et al. (2011), Boxma et al. (2006), Harrison and Resnick (1976), Brockwell et al. (1982b) are mostly concerned with growth-collapse models when growth is from *stochastic* additive inputs such as compound Poisson or Lévy processes or renewal processes. Here, the water level of a dam decreasing deterministically according to some fixed water release program is subject to sudden uprises due to rain or flood. Growth-collapse models are also very relevant in the Burridge-Knopoff stress-release model of earthquakes and continental drift, as from Carlson et al. (1994), and in stick-slip models of interfacial friction, as from Richetti et al. (2001). As we shall see, growth-collapse models are in some sense 'dual' to decay-surge models.

In contrast with these last papers, and as in the works of Eliazar and Klafter (2006), (2007) and (2009), we concentrate in the present work on a *deterministic* and continuous decay motion in between successive surges, described by a nonlinear flow, determining the decay rate of the population and given by

$$x_t(x) = x - \int_0^t \alpha(x_s(x))ds,\ t \geq 0,\ x_0(x) = x \geq 0.$$

In our process, upward jumps (surges) occur with state dependent rate $\beta(x)$, when the current state of the process is x. When a jump occurs, the present size of the

population x is replaced by a new random value $Y(x) > x$, distributed according to some transition kernel $K(x, dy), y \geq x$.

This leads to the study of a quite general family of continuous-time piecewise deterministic Markov processes (PDMP's) $X_t(x)$ representing the size of the population at time t when started from the initial value $x \geq 0$. See Davis (1984). The infinitesimal generator of this process is given for smooth test functions by

$$\mathcal{G}u(x) = -\alpha(x)u'(x) + \beta(x) \int_{(x,\infty)} K(x, dy)[u(y) - u(x)], \ x \geq 0,$$

under suitable conditions on the parameters α, β and $K(x, dy)$ of the process. In the sequel we focus on the study of separable kernels $K(x, dy)$ where for each $0 \leq x \leq y$,

$$\int_{(y,\infty)} K(x, dz) = \frac{k(y)}{k(x)}, \tag{6.1.1}$$

for some positive non-increasing function $k : [0, \infty) \to [0, \infty]$ which is continuous on $(0, \infty)$.

The present chapter proposes a precise characterization of the probabilistic properties of the above process in this separable frame. Supposing that $\alpha(x)$ and $\beta(x)$ are continuous and positive on $(0, \infty)$, the main ingredient of our study is the function

$$\Gamma(x) = \int_1^x \gamma(y)dy, \ \text{where } \gamma(y) = \beta(y)/\alpha(y), \ y, x \geq 0. \tag{6.1.2}$$

Supposing that $\Gamma(\cdot)$ is a **space transform,** that is, $\Gamma(0) = -\infty$ and $\Gamma(\infty) = \infty$, we show that

1. Starting from some strictly positive initial value $x > 0$, the process does not get extinct (does not hit 0) in finite time almost surely (Proposition 6.2). In particular, imposing additionally $k(0) < \infty$, we can study the process in restriction to the state space $(0, \infty)$. This is what we do in the sequel.
2. The function
$$s(x) = \int_1^x \gamma(y)e^{-\Gamma(y)}/k(y)dy, \ x \geq 0, \tag{6.1.3}$$
is a **scale function** of the process, that is, solves $\mathcal{G}s(x) = 0$ (Proposition 6.3). It is always strictly increasing and satisfies $s(0) = -\infty$ under our assumptions. But it might not be a space transform, that is, $s(\infty)$ can take finite values.
3. This scale function plays a key role in the understanding of the exit probabilities of the process and yields conditions under which the process either explodes in finite time or is transient at infinity. More precisely, if $s(\infty) < \infty$, we have the following explicit formula for the exit probabilities. For any $0 < a < x < b$,
$$\mathbb{P}(X_t \text{ enters } (0, a] \text{ before entering } [b, \infty) \ | X_0 = x) = \frac{s(x) - s(a)}{s(b) - s(a)}. \tag{6.1.4}$$
(Proposition 6.4).
Taking $b \to \infty$ in the above formula, we deduce from this that $s(\infty) < \infty$ implies either that the process explodes in finite time (possesses an infinite number of jumps within some finite time interval) or that it is transient at infinity.

Due to the asymmetric dynamic of the process (continuous motion downwards and jumps upwards such that entering the interval $[b, \infty)$ starting from $x < b$ always happens by a jump), (6.1.4) does not hold if $s(\infty) = \infty$.

4. Imposing additionally that $\beta(0) > 0$, **Harris recurrence (positive or null)** of the process is equivalent to the fact that s is a space transform, that is, $s(\infty) = \infty$ (Theorem 6.2).
   In this case, up to constant multiples, the unique invariant measure of the process possesses a Lebesgue density (speed density) given by

$$\pi(x) = \frac{k(x)e^{\Gamma(x)}}{\alpha(x)}, x > 0.$$

   More precisely, we show how the scale function can be used to obtain Foster-Lyapunov criteria in the spirit of Meyn and Tweedie (1993) implying the non-explosion of the process together with its recurrence under additional irreducibility properties. Additional conditions, making use of the speed measure, under which first moments of hitting times are finite, are also supplied in this setup.

Organization of the paper. In Section 2, we introduce our model and state some first results. Most importantly, we establish a simple relationship between decay-surge models and growth-collapse models as studied in Goncalves et al. (2022) that allows us to obtain explicit representations of the law of the first jump time and of the associated speed measure without any further study. Section 3 is devoted to the proof of the existence of the scale function (Proposition 6.3) together with the study of first moments of hitting times which are shown to be finite if the speed density is integrable at $+\infty$ (Proposition 6.6). Section 4 then collects our main results. If the scale function is a space transform, it can be naturally transformed into a Lyapunov function in the sense of Meyn and Tweedie (1993) such that the process does not explode in finite time and comes back to certain compact sets infinitely often (Proposition 6.7). Using the regularity produced by the jump heights according to the absolutely continuous transition kernel $K(x, dy)$, Theorem 6.1 establishes then a local Doeblin lower bound on the transition operator of the process - a key ingredient to prove Harris recurrence which is our main result Theorem 6.2. Several examples are supplied including one related to linear Hawkes processes and to shot-noise processes. In a last part of the work, we focus on the embedded chain of the process, sampled at the jump times, which, in addition to its fundamental relevance, is easily amenable to simulations. Following Adke (1993), we also draw the attention to the structure of the extreme record chain of $X_t(x)$, allowing in particular to derive the distribution of the first upper record time and overshoot value, as a level crossing time and value.

6.2 The model, some first results and a useful duality property

We study population decay models with random surges described by a Piecewise Deterministic Markov Process (PDMP) $X_t, t \geq 0$, starting from some initial value $x \geq 0$ at time 0 and taking values in $[0, \infty)$. The main ingredients of our model are

1. The drift function $\alpha(x)$. We suppose that $\alpha : [0,\infty) \to [0,\infty)$ is continuous, with $\alpha(x) > 0$ for all $x > 0$. In between successive jumps, the process follows the decaying dynamic

$$\dot{x}_t(x) = -\alpha\left(x_t(x)\right), x_0(x) = x \geq 0. \tag{6.2.1}$$

2. The jump rate function $\beta(x)$. We suppose that $\beta : (0,\infty) \to [0,\infty)$ is continuous and $\beta(x) > 0$ for all $x > 0$.
3. The jump kernel $K(x, dy)$. This is a transition kernel from $[0,\infty)$ to $[0,\infty)$ such that for any $x > 0$, $K(x,[x,\infty)) = 1$. Writing $K(x,y) = \int_{(y,\infty)} K(x,dz)$, we suppose that $K(x,y)$ is jointly continuous in x and y.

In between successive jumps, the population size follows the deterministic flow $x_t(x)$ given in (6.2.1). For any $0 \leq a < x$, the integral

$$t_a(x) := \int_a^x \frac{dy}{\alpha(y)} \tag{6.2.2}$$

is the time for the flow to hit a starting from x. In particular, starting from $x > 0$, the flow reaches 0 after some time $t_0(x) = \int_0^x \frac{dy}{\alpha(y)} \leq \infty$. We refer to Goncalves et al. (2020) for a variety of examples of such decaying flows that can hit zero in finite time or not.

Jumps occur at state dependent rate $\beta(x)$. At the jump times, the size of the population grows by a random amount $\Delta(X_{t-}) > 0$ of its current size X_{t-}. Writing $Y(X_{t-}) := X_{t-} + \Delta(X_{t-})$ for the position of the process right after its jump, $Y(X_{t-})$ is distributed according to $K(X_{t-}, dy)$.

Up to the next jump time, X_t then decays again, following the deterministic dynamics (6.2.1), started at the new value $Y(X_{t-}) := X_{t-} + \Delta(X_{t-})$.

We are thus led to consider the PDMP X_t with state-space $[0,\infty)$ solving

$$dX_t = -\alpha(X_t)\,dt + \Delta(X_{t-}) \int_0^\infty \mathbb{1}_{\{r \leq \beta(X_{t-}(x))\}} M(dt, dr), \; X_0 = x, \tag{6.2.3}$$

where $M(dt,dr)$ is a Poisson measure on $[0,\infty) \times [0,\infty)$. Taking $dt \ll 1$ be the system's scale, this dynamics means alternatively that we have transitions

$$\begin{aligned} X_{t-} &= x \to x - \alpha(x)\,dt \text{ with probability } 1 - \beta(x)\,dt \\ X_{t-} &= x \to x + \Delta(x) \text{ with probability } \beta(x)\,dt. \end{aligned}$$

It is a nonlinear version of the Langevin equation with jumps.

6.2.1 Discussion of the jump kernel

We have

$$\mathbb{P}\left(Y(x) > y \mid X_{t-} = x\right) = K(x,y) = \int_{(y,\infty)} K(x,dz).$$

Clearly $K(x,y)$ is a non-increasing function of y for all $y \geq x$ satisfying $K(x,y) = 1$ for all $y < x$. By continuity, this implies that $K(x,x) = 1$ such that the law of $\Delta(x)$ has no atom at 0.

In the sequel we concentrate on the separable case

$$K(x,y) = \frac{k(y)}{k(x)}, \tag{6.2.4}$$

where $k : [0,\infty) \to [0,\infty]$ is any positive non-increasing function. In what follows we suppose that k is continuous and finite on $(0,\infty)$.

Fix $z > 0$ and assume $y = x + z$. Then

$$\mathbb{P}(Y(x) > y) = \frac{k(x+z)}{k(x)}.$$

Depending on $k(x)$, this probability can be a decreasing or an increasing function of x for each z.

Example 6.1. Suppose $k(x) = e^{-x^\alpha}$, $\alpha > 0$, $x \geq 0$ (a Weibull distribution).

- If $\alpha < 1$, then $\partial_x K(x, x+z) > 0$, so that the larger x, the larger $\mathbb{P}(Y(x) > x+z)$. In other words, if the population stays high, the probability of a large number of immigrants will be enhanced. There is a positive feedback of x on $\Delta(x)$, translating a herd effect.

- If $\alpha = 1$, then $\partial_x K(x, x+z) = 0$ and there is no feedback of x on the number of immigrants, which is then exponentially distributed.

- If $\alpha > 1$, then $\partial_x K(x, x+z) < 0$ and the larger x, the smaller the probability $\mathbb{P}(Y(x) > x+z)$. In other words, if the population stays high, the probability of a large number of immigrants will be reduced. There is a negative feedback of x on $\Delta(x)$.

Example 6.2. **The case** $k(0) < \infty$. Without loss of generality, we may take $k(0) = 1$. Assume that $k(x) = \mathbb{P}(Z > x)$ for some proper random variable $Z > 0$ and that

$$Y(x) \overset{d}{=} Z \mid Z > x$$

so that $Y(x)$ is amenable to the truncation of Z above x. Thus

$$\mathbb{P}(Y(X_{t-}) > y \mid X_{t-} = x) = \frac{\mathbb{P}(Z > y, Z > x)}{\mathbb{P}(Z > x)} = \frac{k(y)}{k(x)}, \text{ for } y > x.$$

A particular (exponential) choice is

$$k(x) = e^{-\theta x}, \ \theta > 0,$$

with $\mathbb{P}(Y(X_{t-}) > y \mid X_{t-} = x) = e^{-\theta(y-x)}$ depending only on $y-x$. Another possible one is (Pareto) : $k(x) = (1+x)^{-c}$, $c > 0$.

Note $K(0,y) = k(y) > 0$ for all $y > 0$, and $k(y)$ turns out to be the cpdf of a jump above y, starting from 0 : state 0 is **reflecting**.

The case $k(0) = \infty$. Consider $k(x) = \int_x^\infty \mu(Z \in dy)$ for some positive Radon measure μ with infinite total mass. In this case,

$$\mathbb{P}(Y(X_{t-}) > y \mid X_{t-} = x) = \frac{\mu(Z > y, Z > x)}{\mu(Z > x)} = \frac{k(y)}{k(x)}, \text{ for } y > x.$$

Now, $K(0,y) = 0$ for all $y > 0$ and state 0 becomes **attracting**. An example is $k(x) = x^{-c}$, $c > 0$, which is not a cpdf.

The ratio $k(y)/k(x)$ is thus the conditional probability that a jump is greater than the level y given that it did occur and that it is greater than the level x, see Eq. (1) of Eliazar and Klafter (2009) for a similar choice.

Our motivation for choosing the separable form $K(x,y) = k(y)/k(x)$ is that it accounts for the possibility of having state 0 either absorbing or reflecting for upwards jumps launched from 0 and also that it can account for both a negative or positive feedback of the current population size on the number of incoming immigrants.

Example 6.3. One can think of many other important and natural choices of $K(x,y)$, not in the separable class, among which those for which

$$K(x,dy) = \delta_{Vx}(dy)$$

for some random variable $V > 1$. For this class of kernels, state 0 is **always** attracting. For example, choosing $V = 1 + E$ where :

1/ E is exponentially distributed with pdf $e^{-\theta x}$, $Y(x) = Vx$ yields

$$\mathbb{P}(Y(x) > y \mid X_- = x) = K(x,y) = \mathbb{P}((1+E)x > y) = e^{-\theta\left(\frac{y}{x}-1\right)},$$

2/ E is Pareto distributed with pdf $(1+x)^{-c}$, $c > 0$, $Y(x) = Vx$ yields

$$\mathbb{P}(Y(x) > y \mid X_- = x) = K(x,y) = \mathbb{P}((1+E)x > y) = (y/x)^{-c},$$

both with $K(0,y) = 0$. When
3/ : $V \sim \delta_v$, $v > 1$, then $K(x,y) = \mathbb{1}(y/x \leq v)$. The three kernels depend on y/x.

Note that in all three cases, $\partial_x K(x, x+z) > 0$, so that the larger x, the larger $\mathbb{P}(Y(x) > x+z)$. If the population stays high, the probability of a large number of immigrants will be enhanced. There is a positive feedback of x on $\Delta(x)$, translating a herd effect.

Remark 6.1. *A consequence of the separability condition of K is the following. Consider a Markov sequence of after-jump positions defined recursively by $Z_n = Y(Z_{n-1})$, $Z_0 = x_0$. With $x_m > x_{m-1}$, we have*

$$\mathbb{P}(Z_m > x_m \mid Z_{m-1} = x_{m-1}) = K(x_{m-1}, x_m),\ m = 1, ..., n,$$

so that, with $x_0 < x_1 < ... < x_n$, and under the separability condition on K, the product

$$\prod_{m=1}^{n} \mathbb{P}(Z_m > x_m \mid Z_{m-1} = x_{m-1}) = \prod_{m=1}^{n} K(x_{m-1}, x_m) = \prod_{m=1}^{n} \frac{k(x_m)}{k(x_{m-1})} = \frac{k(x_n)}{k(x_0)}$$

only depends on the initial and terminal states (x_0, x_n) and not on the full path $(x_0, ..., x_n)$.

6.2.2 The infinitesimal generator

In what follows we always work with separable kernels. Moreover, we write $X_t(x)$ for the process given in (6.2.3) to emphasize the dependence on the starting point x, that is, $X_t(x)$ designs the process with the above dynamics (6.2.3) and satisfying $X_0(x) = x$. If the value of the starting point x is not important, we shall also write X_t instead of $X_t(x)$.

Under the separability condition, the infinitesimal generator of X_t acting on bounded smooth test functions u takes the following simple form

$$(\mathcal{G}u)(x) = -\alpha(x)u'(x) + \frac{\beta(x)}{k(x)}\int_x^\infty k(y)u'(y)\,dy, x \geq 0. \tag{6.2.5}$$

Remark 6.2. *In Eliazar and Klafter (2006), a particular scale-free version of decay-surge models with $\alpha(x) \propto x^a$, $\beta(x) \propto x^b$ and $k(x) \propto x^{-c}$, $c > 0$, has been investigated.*

Remark 6.3. *If x_t goes extinct in finite time $t_0(x) < \infty$, since x_t is supposed to represent the size of some population, we need to impose $x_t = 0$ for $t \geq t_0(x)$, forcing state 0 to be absorbing. From this time on, X_t can re-enter the positive orthant if there is a positive probability to move from 0 to a positive state meaning $k(0) < \infty$ and $\beta(0) > 0$. In such a case, the first time X_t hits state 0 is only a first local extinction time the expected value of which needs to be estimated. The question of the time elapsed between consecutive local extinction times (excursions) also arises.*

On the contrary, for situations for which $k(0) = \infty$ or $\beta(0) = 0$, the first time X_t hits state 0 will be a global extinction time.

6.2.3 Relation between decay-surge and growth-collapse processes

In this subsection, we exhibit a natural relationship between decay-surge population models, as studied here, and growth-collapse models as developed in Boxma et al. (2006), Goncalves et al. (2022), Gripenberg (1983) and Hanson and Tuckwell (1978). Growth-collapse models describe deterministic population growth where at random jump times the size of the population undergoes a catastrophe reducing its current size to a random fraction of it. More precisely, the generator of a growth-collapse process, having parameters $(\tilde{\alpha}, \tilde{\beta}, \tilde{h})$, is given for all smooth test functions by

$$(\tilde{\mathcal{G}}u)(x) = \tilde{\alpha}(x)u'(x) - \tilde{\beta}(x)/\tilde{h}(x) \int_0^x u'(y)\tilde{h}(y)dy, x \geq 0. \tag{6.2.6}$$

In the above formula, $\tilde{\alpha}, \tilde{\beta}$ are continuous and positive functions on $(0, \infty)$, and $\tilde{h}$ is positive and non-decreasing on $(0, \infty)$.

In what follows, consider a decay-surge process X_t defined by the triple (α, β, k) and let $\widetilde{X}_t = 1/X_t$.

Proposition 6.1. *The process $\widetilde{X}_t$ is a growth-collapse process as studied in Goncalves et al. (2022) with triple $\left(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{h}\right)$ given by*

$$\widetilde{\alpha}(x) = x^2\alpha(1/x), \widetilde{\beta}(x) = \beta(1/x) \text{ and } \widetilde{h}(x) = k(1/x), \ x > 0.$$

Proof. Let u be any smooth test function and study $u(\tilde{X}_t) = u \circ g(X_t)$ with $g(x) = 1/x$. By Ito's formula for processes with jumps,

$$u(\tilde{X}_t) = u \circ g(X_t) = u(\tilde{X}_0) + \int_0^t \mathcal{G}(u \circ g)(X_{t'})dt' + M_t,$$

where M_t is a local martingale. We obtain

$$\begin{aligned} \mathcal{G}(u \circ g)(x) &= -\alpha(x)(u \circ g)'(x) + \frac{\beta(x)}{k(x)} \int_x^\infty (u \circ g)'(y)k(y)dy, \\ &= \frac{1}{x^2}\alpha(x)u'(\frac{1}{x}) + \frac{\beta(x)}{k(x)} \int_x^\infty u'(\frac{1}{y})k(y)\frac{-dy}{y^2} \end{aligned}$$

Using the change of variable $y = 1/x$, this last expression can be rewritten as

$$\frac{1}{x^2}\alpha(x)u'(\frac{1}{x}) - \frac{\beta(x)}{k(x)} \int_0^{1/x} u'(z)k(\frac{1}{z})dz = \widetilde{\alpha}(y)u'(y) - \frac{\widetilde{\beta}(y)}{\widetilde{h}(y)} \int_0^y u'(t)\widetilde{h}(t)dt$$

which is the generator of the process $\widetilde{X}_t$. □

In what follows we speak about the above relation between the decay-surge (DS) process X and the growth-collapse (GC) process $\tilde{X}$ as *DS-GC-duality*. Some simple properties of the process X follow directly from the above duality as we show next. Of course, the above duality does only hold up to the first time one of the two processes leaves the interior $(0, \infty)$ of its state space. Therefore a particular attention has to be paid to state 0 for X_t or equivalently to state $+\infty$ for $\tilde{X}_t$. Most of our results will only hold true under conditions ensuring that, starting from $x > 0$, the process X_t will not hit 0 in finite time.

Another important difference between the two processes is that the simple transformation $x \mapsto 1/x$ maps a priori unbounded sample paths X_t into bounded onces $\tilde{X}_t$ (starting from $\tilde{X}_0 = 1/x$, almost surely, $\tilde{X}_t \leq \tilde{x}_t(1/x)-$ a relation which does not hold for X).

6.2.4 First consequences of the DS-GC-duality

Given $X_0 = x > 0$, the first jump times both of the DS-process X_t starting from x, and of the GC-process $\tilde{X}_t$, starting from $1/x$, coincide and are given by

$$T_x = \inf\{t > 0 : X_t \neq X_{t-} | X_0 = x\} = \tilde{T}_{\frac{1}{x}} = \inf\{t > 0 : \tilde{X}_t \neq \tilde{X}_{t-} | \tilde{X}_0 = \frac{1}{x}\}.$$

Introducing

$$\Gamma(x) := \int_1^x \gamma(y)\, dy, \text{ where } \gamma(x) := \beta(x)/\alpha(x), x > 0,$$

and the corresponding quantity associated to the process $\tilde{X}_t$,

$$\widetilde{\Gamma}(x) = \int_1^x \tilde{\gamma}(y)\, dy, \ \tilde{\gamma}(x) = \tilde{\beta}(x)/\tilde{\alpha}(x), x > 0,$$

clearly, $\widetilde{\Gamma}(x) = -\Gamma(1/x)$ for all $x > 0$.

Arguing as in Sections 2.4 and 2.5 of Goncalves et al. (2022), a direct consequence of the above duality is the fact that for all $t < t_0(x)$,

$$\mathbb{P}(T_x > t) = e^{-\int_0^t \beta(x_s(x))ds} = e^{-[\Gamma(x)-\Gamma(x_t(x))]}. \tag{6.2.7}$$

To ensure that $\mathbb{P}(T_x < \infty) = 1$, in accordance with Assumption 1 of Goncalves et al. (2022) we will impose the condition

Assumption 4. $\Gamma(0) = -\infty$.

Proposition 6.2. *Under Assumption 4, the stochastic process $X_t(x), x > 0$, necessarily jumps before reaching 0. In particular, for any $x > 0$, $X_t(x)$ almost surely never reaches 0 in finite time.*

Proof. By duality, we have

$$\mathbb{P}(X \text{ jumps before reaching } 0 | X_0 = x) = \mathbb{P}(\tilde{X} \text{ jumps before reaching } +\infty | \tilde{X}_0 = \frac{1}{x}) = 1,$$

as has been shown in Section 2.5 of Goncalves et al. (2022), and this implies the assertion. □

In particular, the only situation where the question of the extinction of the process X makes sense (either local or total) is when $t_0(x) < \infty$ and $\Gamma(0) > -\infty$.

Example 6.4. We give an example where finite time extinction of the process is possible. Suppose $\alpha(x) = \alpha_1 x^a$ with $\alpha_1 > 0$ and $a < 1$. Then $x_t(x)$, started at x, hits 0 in finite time $t_0(x) = x^{1-a}/[\alpha_1(1-a)]$, with

$$x_t(x) = \left(x^{1-a} + \alpha_1(a-1)t\right)^{1/(1-a)}.$$

Suppose $\beta(x) = \beta_1 > 0$, constant. Then, with $\gamma_1 = \beta_1/\alpha_1 > 0$

$$\Gamma(x) = \int_1^x \gamma(y)\,dy = \frac{\gamma_1}{1-a}\left(x^{1-a} - 1\right)$$

with $\Gamma(0) = -\frac{\gamma_1}{1-a} > -\infty$. Assumption 4 is not fulfilled, so X can hit 0 in finite time and there is a positive probability that $T_x = \infty$. On this last event, the flow $x_t(x)$ has all the time necessary to first hit 0 and, if in addition the kernel k is chosen so as $k(0) = \infty$, to go extinct definitively. The time of extinction $\tau(x)$ of X itself can be deduced from the renewal equation in distribution

$$\tau(x) \overset{d}{=} t_0(x)\,\mathbb{1}_{\{T_x=\infty\}} + \tau'\left(Y\left(x_{T_x}(x)\right)\right)\mathbb{1}_{\{T_x<\infty\}}$$

where τ' is a copy of τ.

We conclude that for this family of models, X itself goes extinct in finite time. This is an interesting regime that we shall not investigate any further.

Let us come back to the discussion of Assumption 4. It follows immediately from Eq. (13) in Goncalves et al. (2022) that for $x > 0$, under Assumption 4 and supposing that $t_0(x) = \infty$ for all $x > 0$,

$$\mathbb{E}(T_x) = e^{-\Gamma(x)} \int_0^x \frac{dz}{\alpha(z)} e^{\Gamma(z)}.$$

Clearly, when $x \to 0$, $\mathbb{E}(T_x) \sim 1/\alpha(x)$.

Remark 6.4. *(i) If $\beta(0) > 0$ then Assumption 4 implies $t_0(x) = \infty$, such that 0 is not accessible.*

(ii) Notice also that $t_0(x) < \infty$ together with Assumption 4 implies that $\beta(0) = \infty$ such that the process $X_t(x)$ is prevented from hitting 0 even though $x_t(x)$ reaches it in finite time due to the fact that the jump rate $\beta(x)$ blows up as $x \to 0$.

Remark 6.5. *The DS-GC-duality makes it possible to translate known results on moments for growth-collapse models obtained e.g. in ?, ? or ?, to obtain analogous moment results for decay-surge models.*

6.2.5 Classification of state 0

Recall that for all $x > 0$,

$$t_0(x) = \int_0^x \frac{dy}{\alpha(y)}$$

represents the time required for x_t to move from $x > 0$ to 0. So :

$$\begin{aligned} &\text{If } t_0(x) < \infty \text{ and } \Gamma(0) > -\infty, && \text{state 0 is accessible.} \\ &\text{If } t_0(x) = \infty \text{ or } \Gamma(0) = -\infty, && \text{state 0 is inaccessible.} \end{aligned}$$

We therefore introduce the following conditions which apply in the separable case $K(x,y) = k(y)/k(x)$.

Condition (R) : $\beta(0) > 0$ and $K(0,y) = k(y)/k(0) > 0$ for some $y > 0$.

Condition (A) : $\frac{\beta(0)}{k(0)} k(y) = 0$ for all $y > 0$.

State 0 is reflecting if condition (R) is satisfied and it is absorbing if condition (A) is satisfied.

This leads to four possible combinations for the boundary state 0 :

1. **Condition (R)** is satisfied, and $t_0(x) < \infty$ and $\Gamma(0) > -\infty$: regular (reflecting and accessible).
2. **Condition (R)** is satisfied, and $t_0(x) = \infty$ or $\Gamma(0) = -\infty$: entrance (reflecting and inaccessible).
3. **Condition (A)** is satisfied, and $t_0(x) < \infty$ and $\Gamma(0) > -\infty$: exit (absorbing and accessible).
4. **Condition (A)** is satisfied, and $t_0(x) = \infty$ or $\Gamma(0) = -\infty$: natural (absorbing and inaccessible).

6.2.6 Speed measure.

Suppose now an invariant measure (or speed measure) $\pi(dy)$ exists. Since we supposed $\alpha(x) > 0$ for all $x > 0$, we necessarily have $x_\infty(x) = 0$ for all $x > 0$ and so the support of π is $[x_\infty(x) = 0, \infty)$. Thanks to our duality relation, by Eq. (19) of Goncalves et al. (2022) the explicit expression of the speed measure is given by $\pi(dy) = \pi(y)dy$ with

$$\pi(y) = C \frac{k(y)\, e^{\Gamma(y)}}{\alpha(y)}, \tag{6.2.8}$$

up to a multiplicative constant $C > 0$. If and only if this function is integrable at 0 and ∞, $\pi(y)$ can be tuned to a probability density function.

Remark 6.6. *(i) When $k(x) = e^{-\kappa_1 x}$, $\kappa_1 > 0$, $\alpha(x) = \alpha_1 x$ and $\beta(x) = \beta_1 > 0$ constant, $\Gamma(y) = \gamma_1 \log y$, $\gamma_1 = \beta_1/\alpha_1$ and*

$$\pi(y) = C y^{\gamma_1 - 1} e^{-\kappa_1 y}$$

a Gamma(γ_1, κ_1) distribution. This result is well-known, corresponding to the linear decay-surge model (a jump version of the damped Langevin equation) having an invariant (integrable) probability density, see Malrieu (2015). We shall show later that the corresponding process X is positive recurrent.

(ii) A less obvious power-law example is as follows : Assume $\alpha(x) = \alpha_1 x^a$ $(a > 1)$ and $\beta(x) = \beta_1 x^b$, $\alpha_1, \beta_1 > 0$ so that $\Gamma(y) = \frac{\gamma_1}{b-a+1} y^{b-a+1}$. We have $\Gamma(0) = -\infty$ if we assume $b - a + 1 = -\theta$ with $\theta > 0$, hence $\Gamma(y) = -\frac{\gamma_1}{\theta} y^{-\theta}$. Taking $k(y) = e^{-\kappa_1 y^\eta}$, $\kappa_1, \eta > 0$

$$\pi(y) = C y^{-a} e^{-\left(\kappa_1 y^\eta + \frac{\gamma_1}{\theta} y^{-\theta}\right)}$$

which is integrable both at $y = 0$ and $y = \infty$. As a special case, if $a = 2$ and $b = 0$ (constant jump rate $\beta(x)$), $\eta = 1$

$$\pi(y) = C y^{-2} e^{-\left(\kappa_1 y + \gamma_1 y^{-1}\right)},$$

an inverse Gaussian density.

(*iii*) *In Eliazar and Klafter (2007) and (2009), a special case of our model was introduced for which* $k(y) = \beta(y)$. *In such cases,*

$$\pi(y) = C\gamma(y)\, e^{\Gamma(y)}$$

so that

$$\int_0^x \pi(y)\, dy = C\left(e^{\Gamma(x)} - e^{\Gamma(0)}\right) = Ce^{\Gamma(x)},$$

under the Assumption $\Gamma(0) = -\infty$. *If in addition* $\Gamma(\infty) < \infty$, $\pi(y)$ *can be tuned to a probability density.*

6.3 Scale function and hitting times

In this section we start by studying the scale function of X_t, before switching to hitting times features that make use of it. A scale function $s(x)$ of the process is any function solving $(\mathcal{G}s)(x) = 0$. In other words, a scale function is a function that transforms the process into a local martingale. Of course, any constant function is solution. Notice that for the growth-collapse model considered in Goncalves et al. (2022), other scale functions than the constant ones do not exist.

In what follows we are interested in non-constant solutions and conditions ensuring the existence of those. To clarify ideas, we introduce the following condition

Assumption 5. *Let*

$$s(x) = \int_1^x \gamma(y)e^{-\Gamma(y)}/k(y)dy,\ x \geq 0, \tag{6.3.1}$$

and suppose that $s(\infty) = \infty$.

Notice that Assumption 5 implies that $k(\infty) = 0$ which is reasonable since it prevents jumps of the process X_t jumping from some finite position X_{t-} to an after jump position $X_t = X_{t-} + \Delta(X_{t-}) = +\infty$.

Proposition 6.3. *(1) Suppose* $\Gamma(\infty) = \infty$. *Then the function* s *introduced in* (6.3.1) *above is a strictly increasing version of the scale function of the process obeying* $s(1) = 0$.

(1.1) If additionally Assumptions 4 and 5 hold and if $k(0) < \infty$, *then* $s(0) = -\infty$ *and* $s(\infty) = \infty$, *such that* s *is a space transform* $[0, \infty) \to [-\infty, \infty)$.

(1.2) If Assumption 5 does not hold, then

$$s_1(x) = \int_x^\infty \gamma(y)e^{-\Gamma(y)}/k(y)dy = s(\infty) - s(x)$$

is a version of the scale function which is strictly decreasing, positive, such that $s_1(\infty) = 0$.

(2) Finally, suppose that $\Gamma(\infty) < \infty$. *Then the only scale functions belonging to* C^1 *are the constant ones.*

Remark 6.7. *1. We shall see later that – as in the case of one-dimensional diffusions, see e.g. Example 2 in Section 3.8 of Has'minskii (1980) – the fact that* s *is a space transform as in item (1.1) above is related to the Harris recurrence of the process.*

2. *The assumption $\Gamma(\infty) < \infty$ of item (3) above corresponds to Assumption 2 of Goncalves et al. (2022) where this was the only case that we considered. As a consequence, for the GC-model considered there we did not dispose of non-constant scale functions.*

Proof. To find a C^1-scale function s, it necessarily solves

$$(\mathcal{G}s)(x) = -\alpha(x)s'(x) + \beta(x)/k(x)\int_x^\infty k(y)s'(y)dy = 0$$

such that for all $x > 0$,

$$k(x)s'(x) - \gamma(x)\int_x^\infty k(y)s'(y)dy = 0. \tag{6.3.2}$$

Putting $u'(x) = k(x)s'(x)$, the above implies in particular that u' is integrable in a neighborhood of $+\infty$ such that $u(\infty)$ must be a finite number. We get $u'(x) = \gamma(x)(u(\infty) - u(x))$.

Case 1 : $u(\infty) = 0$ so that $u(x) = -c_1 e^{-\Gamma(x)}$ for some constant c_1, whence $\Gamma(\infty) = \infty$. We obtain

$$s'(x) = c_1 \frac{\gamma(x)}{k(x)} e^{-\Gamma(x)} \tag{6.3.3}$$

and thus

$$s(x) = c_2 + c_1 \int_1^x \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy \tag{6.3.4}$$

for some constants c_1, c_2. Taking $c_2 = 0$ and $c_1 = 1$ gives the formula (6.3.1), and both items (1.1) and (1.2) follow from this.

Case 2 : $u(\infty) \neq 0$ is a finite number. Putting $v(x) = e^{\Gamma(x)}u(x)$, v then solves

$$v'(x) = u(\infty)\gamma(x)e^{\Gamma(x)}$$

such that

$$v(x) = d_1 + u(\infty)e^{\Gamma(x)}$$

and thus

$$u(x) = e^{-\Gamma(x)}d_1 + u(\infty).$$

Letting $x \to \infty$, we see that the above is perfectly well-defined for any value of the constant d_1, if we suppose $\Gamma(\infty) = \infty$.

As a consequence, $u'(x) = -d_1\gamma(x)e^{-\Gamma(x)}$, leading us again to the explicit formula

$$s(x) = c_2 + c_1 \int_1^x \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy, \tag{6.3.5}$$

with $c_1 = -d_1$, implying items (1.1) and (1.2).

Finally, if $\Gamma(\infty) < \infty$, we see that we have to take $d_1 = 0$ implying that the only scale functions in this case are the constant ones. □

Example 6.5. In the linear case example with $\beta(x) = \beta_1 > 0$, $\alpha(x) = \alpha_1 x$, $\alpha_1 > 0$ and $k(y) = e^{-y}$, with $\gamma_1 = \beta_1/\alpha_1$, Assumption 5 is satisfied such that

$$s(x) = \gamma_1 \int_1^x y^{-(\gamma_1+1)} e^y dy$$

which is diverging both as $x \to 0$ and as $x \to +\infty$. Notice that 0 is inaccessible for this process, i.e., starting from a strictly positive position $x > 0$, X_t will never hit 0. Defining the process on the state space $(0, \infty)$, invariant under the dynamics, the process is recurrent and even positive recurrent as we know from the Gamma shape of its invariant speed density.

6.3.1 Hitting times

Fix $a < x < b$. In what follows we shall be interested in hitting times of the positions a and b, starting from x, under the condition $\Gamma(\infty) = \infty$. Due to the asymmetric structure of the process (continuous motion downwards and up-moves by jumps only), these times are given by

$$\tau_{x,b} = \inf\{t > 0 : X_t = b\} = \inf\{t > 0 : X_t \leq b, X_{t-} > b\}$$

and

$$\tau_{x,a} = \inf\{t > 0 : X_t = a\} = \inf\{t > 0 \ : \ X_t \leq a\}.$$

Obviously, $\tau_{a,a} = \tau_{b,b} = 0$.

Let $T = \tau_{x,a} \wedge \tau_{x,b}$. Contrarily to the study of processes with continuous trajectories, it is not clear that $T < \infty$ almost surely. Indeed, starting from x, the process could jump across the barrier of height b before hitting a and then never enter the interval $[0, b]$ again. So we suppose in the sequel that $T < \infty$ almost surely. Then

$$\mathbb{P}\left(\tau_{x,a} < \tau_{x,b}\right) + \mathbb{P}\left(\tau_{x,b} < \tau_{x,a}\right) = 1.$$

Proposition 6.4. *Suppose $\Gamma(\infty) = \infty$ and that Assumption 5 does not hold. Let $0 < a < x < b < \infty$ and suppose that $T = \tau_{x,a} \wedge \tau_{x,b} < \infty$ almost surely. Then*

$$\mathbb{P}\left(\tau_{x,a} < \tau_{x,b}\right) = \frac{\int_x^b \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy}{\int_a^b \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy}. \tag{6.3.6}$$

Proof. Under the above assumptions, $s_1(X_t)$ is a local martingale and the stopped martingale $M_t = s_1(X_{T \wedge t})$ is bounded, which follows from the fact that $X_{T \wedge t} \geq a$ together with the observation that s_1 is decreasing implying that $M_t \leq s_1(a)$. Therefore from the stopping theorem we have

$$s_1(x) = \mathbb{E}(M_0) = \mathbb{E}(s_1(X_0)) = \mathbb{E}(M_T).$$

Moreover,

$$\mathbb{E}(M_T) = s_1(a)\mathbb{P}(T = \tau_{x,a}) + s_1(b)\mathbb{P}(T = \tau_{x,b}).$$

But $\{T = \tau_{x,a}\} = \{\tau_{x,a} < \tau_{x,b}\}$ and $\{T = \tau_{x,b}\} = \{\tau_{x,b} < \tau_{x,a}\}$, such that

$$s_1(x) = s_1(a)\mathbb{P}(\tau_{x,a} < \tau_{x,b}) + s_1(b)\mathbb{P}(\tau_{x,b} < \tau_{x,a}).$$

□

Remark 6.8. - *See Kella and Stadje (2001) for similar arguments in a particular case of a constant flow and exponential jumps.*

- *We stress that it is not possible to deduce the above formula without imposing the existence of s_1 (that is, if Assumption 5 holds such that s_1 is not well-defined). Indeed, if we would want to consider the local martingale $s(X_t)$ instead, the stopped martingale $s(X_{t \wedge T})$ is not bounded since $X_{t \wedge T}$ might take arbitrary values in (a, ∞), such that it is not possible to apply the stopping rule.*

Remark 6.9. *Let* $\tau_{x,[b,\infty)} = \inf\{t > 0 : X_t \geq b\}$ *and* $\tau_{x,[0,a]} = \inf\{t > 0 : X_t \leq a\}$ *be the entrance times to the intervals* $[b, \infty)$ *and* $[0, a]$. *Observe that by the structure of the process, namely the continuity of the downward motion,*

$$\{\tau_{x,a} < \tau_{x,b}\} \subset \{\tau_{x,a} < \tau_{x,[b,\infty)}\} \subset \{\tau_{x,a} < \tau_{x,b}\}.$$

Indeed, the second inclusion is trivial since $\tau_{x,[b,\infty)} \leq \tau_{x,b}$. *The first inclusion follows from the fact that it is not possible to jump across* b *and then hit* a *without touching* b. *Therefore,* (6.3.6) *can be rewritten as*

$$\mathbb{P}\left(\tau_{x,[0,a]} < \tau_{x,[b,\infty)}\right) = \frac{s_1(x) - s_1(b)}{s_1(a) - s_1(b)}. \tag{6.3.7}$$

Now suppose that the process does not explode in finite time, that is, during each finite time interval, almost surely, only a finite number of jumps appear. In this case $\tau_{x,[b,\infty)} \to +\infty$ *as* $b \to \infty$. *Then, letting* $b \to \infty$ *in* (6.3.7), *we obtain for any* $a > 0$,

$$\mathbb{P}\left(\tau_{x,a} < \infty\right) = \frac{s_1(x)}{s_1(a)} = \frac{\int_x^\infty \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy}{\int_a^\infty \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy} < 1, \tag{6.3.8}$$

since $\int_a^x \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} dy > 0$ *by assumptions on* k *and* γ.

As a consequence, we obtain the following

Proposition 6.5. *Suppose that* $\Gamma(\infty) = \infty$ *and that Assumption 5 does not hold. Then either the processes explodes in finite time with positive probability or it is transient at* $+\infty$, *i.e. for all* $a < x$, $\tau_{x,a} = \infty$ *with positive probability.*

Proof. Let $a < x < b$ and $T = \tau_{x,a} \wedge \tau_{x,b}$ and suppose that almost surely, the process does not explode in finite time. We show that in this case, $\tau_{x,a} = \infty$ with positive probability.

Indeed, suppose that $\tau_{x,a} < \infty$ almost surely. Then (6.3.6) holds, and letting $b \to \infty$, we obtain (6.3.8) implying that $\tau_{x,a} = \infty$ with positive probability which is a contradiction. □

Example 6.6. Choose $\alpha(x) = x^2$, $\beta(x) = 1 + x^2$ so that $\gamma(x) = 1 + 1/x^2$and $\Gamma(x) = x - 1/x$ with $\Gamma(0) = -\infty$ and $\Gamma(\infty) = \infty$.

Choose also $k(x) = e^{-x/2}$. Then Assumption 5 is violated : the survival function of the big upward jumps decays too slowly in comparison to the decay of $e^{-\Gamma}$. The speed density is

$$\pi(x) = C\frac{k(x)\, e^{\Gamma(x)}}{\alpha(x)} = Cx^{-2} e^{x/2 - 1/x}$$

which is integrable at 0 but not at ∞. The process explodes (has an infinite number of upward jumps in finite time) with positive probability.

6.3.2 First moments of hitting times

Let $a > 0$. We are seeking for positive solutions of

$$(\mathcal{G}\phi_a)(x) = -1, x \geq a,$$

with boundary condition $\phi_a(a) = 0$. The above is equivalent to

$$(\mathcal{G}\phi_a)(x) = -\alpha(x)\phi_a'(x) + \frac{\beta(x)}{k(x)}\int_x^\infty k(y)\phi_a'(y)dy = -1.$$

This is also

$$-k(x)\phi_a'(x) + \gamma(x)\int_x^\infty k(y)\phi_a'(y)dy = -\frac{k(x)}{\alpha(x)}.$$

Putting $U(x) := \int_x^\infty k(y)\phi_a'(y)dy$, the latter integro-differential equation reads $U'(x) = -\gamma(x)U(x) - \frac{k(x)}{\alpha(x)}$. Supposing that $\int^{+\infty}\pi(y)dy < \infty$ (recall (6.2.8)), this leads to

$$\begin{aligned} U(x) &= e^{-\Gamma(x)}\int_x^\infty e^{\Gamma(y)}\frac{k(y)}{\alpha(y)}dy, \\ -U'(x) &= \gamma(x)e^{-\Gamma(x)}\int_x^\infty e^{\Gamma(y)}\frac{k(y)}{\alpha(y)}dy + \frac{k(x)}{\alpha(x)} = k(x)\phi_a'(x), \end{aligned}$$

such that

$$\begin{aligned} \phi_a(x) &= \int_a^x dy\frac{\gamma(y)}{k(y)}e^{-\Gamma(y)}\int_y^\infty e^{\Gamma(z)}\frac{k(z)}{\alpha(z)}dz + \int_a^x\frac{dy}{\alpha(y)} \\ &= \int_a^\infty dz\pi(z)\left[s_1(a) - s_1(x\wedge z)\right] + \int_a^x\frac{dy}{\alpha(y)}. \end{aligned} \tag{6.3.9}$$

Notice that $[a,\infty) \ni x \mapsto \phi_a(x)$ is non-decreasing and that $\phi_a(x) < \infty$ for all $x > a > 0$ under our assumptions. Dynkin's formula implies that for all $x > a$ and all $t \geq 0$,

$$\mathbb{E}_x(t\wedge\tau_{x,a}) = \phi_a(x) - \mathbb{E}_x(\phi_a(X_{t\wedge\tau_{x,a}})). \tag{6.3.10}$$

In particular, since $\phi_a(\cdot) \geq 0$,

$$\mathbb{E}_x(t\wedge\tau_{x,a}) \leq \phi_a(x) < \infty,$$

such that we may let $t \to \infty$ in the above inequality to obtain by monotone convergence that

$$\mathbb{E}_x(\tau_{x,a}) \leq \phi_a(x) < \infty.$$

In a second step, we obtain from (6.3.10), using Fatou's lemma, that

$$\begin{aligned} \mathbb{E}_x(\tau_{x,a}) = \lim_{t\to\infty}\mathbb{E}_x(t\wedge\tau_{x,a}) &= \phi_a(x) - \lim_{t\to\infty}\mathbb{E}_x(\phi_a(X_{t\wedge\tau_{x,a}})) \\ &\geq \phi_a(x) - \mathbb{E}_x(\liminf_{t\to\infty}\phi_a(X_{t\wedge\tau_{x,a}})) = \phi_a(x), \end{aligned}$$

where we have used that $\liminf_{t\to\infty}\phi_a(X_{t\wedge\tau_{x,a}}) = \phi_a(a) = 0$.

As a consequence we have just shown the following

Proposition 6.6. *Suppose that $\int^{+\infty}\pi(y)dy < \infty$. Then $\mathbb{E}_x(\tau_{x,a}) = \phi_a(x) < \infty$ for all $0 < a < x$, where ϕ_a is given as in (6.3.9).*

Remark 6.10. *The last term* $\int_a^x \frac{dy}{\alpha(y)}$ *in the RHS of the expression of* $\phi_a(x)$ *in* (6.3.9) *is the time needed for the deterministic flow to first hit* a *starting from* $x > a$*, which is a lower bound of* $\phi_a(x)$*. Considering the tail function of the speed density* $\pi(y)$*, namely* $\overline{\pi}(y) := \int_y^\infty e^{\Gamma(z)} \frac{k(z)}{\alpha(z)} dz$*, the first term in the RHS expression of* $\phi_a(x)$ *is*

$$\int_a^x -ds_1(y)\,\overline{\pi}(y) = -\left[s_1(y)\,\overline{\pi}(y)\right]_a^x - \int_a^x s_1(y)\,\pi(y)\,dy,$$

emphasizing the importance of the couple $(s_1(\cdot), \pi(\cdot))$ *in the evaluation of* $\phi_a(x)$*. If* a *is a small critical value below which the population can be considered in danger, this is the mean value of a 'quasi-extinction' event when the initial size of the population was* x*.*

Remark 6.11. *Notice that the above discussion is only possible for couples* $0 < a < x$*, since starting from* x*,* $X_{t\wedge\tau_{x,a}} \geq a$ *for all* t*. A similar argument does not hold true for* $x < b$ *and the study of* $\tau_{x,b}$*.*

6.3.3 Mean first hitting time of 0

Suppose that $\Gamma(0) > -\infty$. Then for flows $x_t(x)$ that go extinct in finite time $t_0(x)$, under the condition that $\int^{+\infty} \pi(y)dy < \infty$, one can let $a \to 0$ in the expression of $\phi_a(x)$ to obtain

$$\phi_0(x) = \int_0^x dy \frac{\gamma(y)}{k(y)} e^{-\Gamma(y)} \int_y^\infty e^{\Gamma(z)} \frac{k(z)}{\alpha(z)} dz + \int_0^x \frac{dy}{\alpha(y)},$$

which is the expected time to eventual extinction of X starting from x, that is, $\phi_0(x) = \mathbb{E}\tau_{x,0}$.

The last term $\int_0^x \frac{dy}{\alpha(y)} = t_0(x) < \infty$ in the RHS expression of $\phi_0(x)$ is the time needed for the deterministic flow to first hit 0 starting from $x > 0$, which is a lower bound of $\phi_0(x)$.

Notice that under the condition $t_0(x) < \infty$ and $k(0) < \infty$, $\int_0 \pi(y)dy < \infty$, such that π can be tuned into a probability. It is easy to see that $\Gamma(0) > -\infty$ implies then that $\phi_0(x) < \infty$.

Example 6.7. (Linear release at constant jump rate) : Suppose $\alpha(x) = \alpha_1 > 0$, $\beta(x) = \beta_1 > 0$, $\gamma(x) = \gamma_1 = \beta_1/\alpha_1$, $\Gamma(x) = \gamma_1 x$ with $\Gamma(0) = 0 > -\infty$. Choose $k(x) = e^{-x}$.

State 0 is reached in finite time $t_0(x) = x/\alpha_1$, and it turns out to be reflecting. We have $\int_0 \pi(x)\,dx < \infty$ and $\int^\infty \pi(x)\,dx < \infty$ if and only if $\gamma_1 < 1$. In such a case, the first integral term in the above expression of $\phi_0(x)$ is $\gamma_1/[\alpha_1(1-\gamma_1)]\,x$, so that $\phi_0(x) = x/[\alpha_1(1-\gamma_1)] < \infty$.

6.4 Non-explosion and Recurrence

In this section we come back to the scale function s introduced in (6.3.1) above. Despite the fact that we cannot use s to obtain explicit expressions for exit probabilities, we show how we might use it to obtain Foster-Lyapunov criteria in spirit of Meyn and Tweedie (1993) that imply the non-explosion of the process together with its recurrence under additional irreducibility properties.

Let $S_1 < S_2 < \ldots < S_n < \ldots$ be the successive jump times of the process and $S_\infty = \lim_{n\to\infty} S_n$. We start discussing how we can use the scale function s to obtain a general criterion for non-explosion of the process, that is, $S_\infty = +\infty$ almost surely.

Proposition 6.7. *Suppose $\Gamma(\infty) = \infty$ and suppose that Assumption 5 holds. Suppose also that β is continuous on $[0,\infty)$. Let V be any $\mathcal{C}^1-$function defined on $[0,\infty)$, such that $V(x) = 1 + s(x)$ on $[1,\infty)$ and such that $V(x) \geq 1/2$ for all x. Then V is a norm-like function in the sense of Meyn and Tweedie (1993), and we have*

1. *$\mathcal{G}V(x) = 0$, $\forall x \geq 1$.*
2. *$\sup_{x\in[0,1]} |\mathcal{G}V(x)| < \infty$.*

As a consequence, $S_\infty = \sup_n S_n = \infty$ almost surely, so that X is non-explosive.

Proof. We check that V satisfies the condition (CD0) of Meyn and Tweedie (1993). It is evident that V is norm-like since $\lim_{x\to\infty} V(x) = 1+\lim_{x\to\infty} s(x) = 1+s(\infty) = \infty$ since Assumption 5 holds. Moreover, since $K(x,y) = \frac{k(y)}{k(x)}$,

$$\mathcal{G}V(x) = -\alpha(x)V'(x) + \frac{\beta(x)}{k(x)}\int_x^\infty k(y)V'(y)dy.$$

Since for all $1 \leq x \leq y$, $V'(x) = s'(x)$ and $V'(y) = s'(y)$, we have $\mathcal{G}V = \mathcal{G}s = 0$ on $[1,\infty[$.

For the second point, for $x \in]0,1[$,

$$\begin{aligned}\mathcal{G}V(x) &= -\alpha(x)V'(x) + \frac{\beta(x)}{k(x)}\int_x^\infty k(y)V'(y)dy \\ &= -\alpha(x)V'(x) + \frac{\beta(x)}{k(x)}\int_1^\infty k(y)V'(y)dy + \beta(x)\int_x^1 K(x,y)V'(y)dy.\end{aligned}$$

$\alpha(x)V'(x)$ is continuous and thus bounded on $[0,1]$. Moreover, for all $y \geq x$, $K(x,y) \leq 1$ implying that $\beta(x)\int_x^1 K(x,y)V'(y)dy \leq \beta(x)\int_x^1 V'(y)dy < \infty$. We also have for all $y \geq 1$,

$$\int_1^\infty k(y)V'(y)dy = \int_1^\infty k(y)s'(y)dy.$$

Thus, using $\mathcal{G}V(x) = \mathcal{G}s(x) = 0$ on $]0,1[$ we have

$$\frac{\beta(x)}{k(x)}\int_1^\infty k(y)V'(y)dy = \frac{\beta(x)}{k(x)}k(1)V'(1)/\gamma(1) < \infty$$

because β is continuous on $[0,1]$ and k taking finite values on $(0,\infty)$. As a consequence, $\sup_{x\in[0,1]} |\mathcal{G}V(x)| < \infty$. □

We close this subsection with a stronger Foster-Lyapunov criterion implying the existence of finite hitting time moments.

Proposition 6.8. *Suppose there exist $x_* > 0$, $c > 0$ and a positive function V such that $\mathcal{G}V(x) \leq -c$ for all $x \geq x_*$. Then for all $x \geq a \geq x_*$,*

$$\mathbb{E}(\tau_{x,a}) \leq \frac{V(x)}{c}.$$

Proof. Using Dynkin's formula, we have for $x \geq a \geq x_*$,

$$\mathbb{E}(V(X_{t\wedge\tau_{x,a}})) = V(x) + \mathbb{E}(\int_0^{t\wedge\tau_{x,a}} \mathcal{G}V(X_s)ds) \leq V(x) - c\mathbb{E}(t \wedge \tau_{x,a}),$$

such that

$$\mathbb{E}(t \wedge \tau_{x,a}) \leq \frac{V(x)}{c},$$

which implies the assertion, letting $t \to \infty$. □

Example 6.8. Suppose $\alpha(x) = 1+x$, $\beta(x) = x$ and $k(x) = e^{-2x}$. We choose $V(x) = e^x$. Then

$$(\mathcal{G}V)(x) = -e^x \leq -1, \forall x \geq 0.$$

As a consequence $\mathbb{E}(\tau_{x,a}) < \infty$, for all $a > 0$.

6.4.1 Irreducibility and Harris recurrence

In this section we impose that $\Gamma(\infty) = \infty$, such that non-trivial scale functions do exist. We also assume that Assumption 5 holds since otherwise the process is either transient at ∞ or explodes in finite time. Then the function V introduced in Proposition 6.7 is a Lyapunov function. This is *almost* the Harris recurrence of the process, all we need to show is some irreducibility property that we are going to check now.

Theorem 6.1. *Suppose we are in the separable case, that $k \in C^1$ and that 0 is inaccessible, that is, $t_0(x) = \infty$ for all x. Then every compact set $C \subset]0, \infty[$ is 'petite' in the sense of Meyn and Tweedie. More precisely, there exist $t > 0$, $\alpha \in (0, 1)$ and a probability measure ν on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, such that*

$$P_t(x, dy) \geq \alpha \mathbb{1}_C(x)\nu(dy).$$

Proof. Suppose w.l.o.g. that $C = [a, b]$ with $0 < a < b$. Fix any $t > 0$. The idea of our construction is to impose that all processes $X_s(x), a \leq x \leq b$, have one single common jump during $[0, t]$. Indeed, notice that for each $x \in C$, the jump rate of $X_s(x)$ is given by $\beta(x_s(x))$ taking values in a compact set $[\beta_*, \beta^*]$ where $\beta_* = \min\{\beta(x_s(x)), 0 \leq s \leq t, x \in C\}$ and $\beta^* = \max\{\beta(x_s(x)), 0 \leq s \leq t, x \in C\}$. Notice that $0 < \beta_* < \beta^* < \infty$ since β is supposed to be positive on $(0, \infty)$. We then construct all processes $X_s(x), s \leq t, x \in C$, using the same underlying Poisson random measure M. It thus suffices to impose that E holds, where

$$E = \{M([0, t-\varepsilon] \times [0, \beta^*]) = 0, M([t-\varepsilon, t] \times [0, \beta_*]) = 1, M([t-\varepsilon, t] \times]\beta_*, \beta^*]) = 0\}.$$

Indeed, the above implies that up to time $t - \varepsilon$, none of the processes $X_s(x), x \in C$, jumps. The second and third assumption imply moreover that the unique jump time, call it S, of M within $[t - \varepsilon, t] \times [0, \beta^*]$ is a common jump of all processes. For each value of $x \in C$, the associated process $X(x)$ then chooses a new after-jump position y according to

$$\frac{1}{k(x_S(x))}|k'(y)|dy\mathbb{1}_{\{y \geq x_S(x)\}}. \tag{6.4.1}$$

Case 1. Suppose k is strictly decreasing, that is, $|k'|(y) > 0$ for all y. Fix then any open ball $B \subset [b, \infty[$ and notice that $\mathbb{1}_{\{y \geq x_S(x)\}} \geq \mathbb{1}_B(y)$, since $b \geq x_S(x)$. Moreover, since k is decreasing, $1/k(x_S(x)) \geq 1/k(x_t(a))$. Therefore, the transition density given in (6.4.1) can be lower-bounded, independently of x, by

$$\frac{1}{k(x_t(a))} \mathbb{1}_B(y)|k'(y)|dy = p\tilde{\nu}(dy),$$

where $\tilde{\nu}(dy) = c\mathbb{1}_B(y)|k'(y)|dy$, normalized to be a probability density, and where $p = \frac{1}{k(x_t(a))}/c$. In other words, on the event E, with probability p, all particles choose a new and common position $y \sim \tilde{\nu}(dy)$ and couple.

Case 2. $|k'|$ is different from 0 on a ball B (but not necessarily on the whole state space). We suppose w.l.o.g. that B has compact closure. Then it suffices to take t sufficiently large in the first step such that $x_{t-\varepsilon}(b) < \inf B$. Indeed, this implies once more that $\mathbb{1}_B(y) \leq \mathbb{1}_{\{y \geq x_s(x)\}}$ for all $x \in C$ and for s the unique common jump time.

Conclusion. In any of the above cases, let $\bar{b} := \sup\{x : x \in B\} < \infty$ and restrict the set E to

$$E' = E \cap \{M([t-\varepsilon]\times]\beta_*, \bar{b}] = 0\}.$$

Putting $\alpha := p \Pr(E')$ and

$$\nu(dy) = \int_{t_\varepsilon}^{t} \mathcal{L}(S|E')(ds) \int \tilde{\nu}(dz)\delta_{x_{t-s}(z)}(dy)$$

then allows to conclude. □

Remark 6.12. *If β is continuous on $[0, \infty)$ with $\beta(0) > 0$ and if moreover $k(0) < \infty$, the above construction can be extended to any compact set of the form $[0, b], b < \infty$ and to the case where $t_0(x) < \infty$.*

As a consequence of the above considerations we obtain the following theorem.

Theorem 6.2. *Suppose that $\beta(0) > 0, k(0) < \infty$, that $\Gamma(\infty) = \infty$ and moreover that Assumption 5 holds. Then the process is recurrent in the sense of Harris and its unique invariant measure is given by π.*

Proof. Condition (CD1) of Meyn and Tweedie (1993) holds with V given as in Proposition 6.7 and with compact set $C = [0, 1]$. By Theorem 6.1, all compact sets are 'petite'. Then Theorem 3.2 of Meyn and Tweedie (1993) allows to conclude. □

Example 6.9. A meaningful recurrent example consists of choosing $\beta(x) = \beta_1/x$, $\beta_1 > 0$ (the surge rate decreases like $1/x$), $\alpha(x) = \alpha_1/x$ (finite time extinction of x_t), $\alpha_1 > 0$, and $k(y) = e^{-y}$. In this case, all compact sets are 'petite'. Moreover, with $\gamma_1 = \beta_1/\alpha_1$, $\Gamma(x) = \gamma_1 x$ and

$$\pi(y) = \frac{y}{\alpha_1} e^{(\gamma_1 - 1)y}$$

which can be tuned into a probability density if $\gamma_1 < 1$. This also implies that Assumption 5 is satisfied such that s can be used to define a Lyapunov function. The associated process X is positive recurrent if $\gamma_1 < 1$, null-recurrent if $\gamma_1 = 1$.

6.5 The embedded chain

In this section, we illustrate some of the previously established theoretical results by simulations of the embedded chain that we are going to define now. Defining $T_1 = S_1, T_n = S_n - S_{n-1}, n \geq 2$, the successive inter-jump waiting times, we have

$$\begin{aligned}\mathbb{P}\left(T_n \in dt, X_{S_n} \in dy \mid X_{S_{n-1}} = x\right) &= dt\beta\left(x_t\left(x\right)\right) e^{-\int_0^t \beta(x_s(x))ds} K\left(x_t\left(x\right), dy\right) \\ &= dt\beta\left(x_t\left(x\right)\right) e^{-\int_{x_t(x)}^x \gamma(z)dz} K\left(x_t\left(x\right), dy\right).\end{aligned}$$

The embedded chain is then defined through $Z_n := X_{S_n}, n \geq 0$. If 0 is not absorbing, for all $x \geq 0$

$$\begin{aligned}\mathbb{P}\left(Z_n \in dy \mid Z_{n-1} = x\right) &= \int_0^\infty dt\beta\left(x_t\left(x\right)\right) e^{-\int_{x_t(x)}^x \gamma(z)dz} K\left(x_t\left(x\right), dy\right) \\ &= e^{-\Gamma(x)} \int_0^x dz\gamma\left(z\right) e^{\Gamma(z)} K\left(z, dy\right),\end{aligned}$$

where the last line is valid for $x > 0$ only, and only if $t_0(x) = \infty$. This implies that Z_n is a time-homogeneous discrete-time Markov chain on $[0, \infty)$.

Remark 6.13. *$(S_n, Z_n)_{n\geq 0}$ is also a discrete-time Markov chain on $\mathbb{R}_+^2$ with transition probabilities given by*

$$\mathbb{P}\left(S_n \in dt \mid Z_{n-1} = x, S_{n-1} = s\right) = dt\beta\left(x_{t-s}\left(x\right)\right) e^{-\int_0^{t-s} \beta(x_{s'}(x))ds'},\ t \geq s,$$

and

$$\mathbb{P}\left(Z_n \in dy \mid Z_{n-1} = x, S_{n-1} = s\right) = \int_0^\infty dt\beta\left(x_t\left(x\right)\right) e^{-\int_{x_t(x)}^x \gamma(z)dz} K\left(x_t\left(x\right), dy\right)$$

independent of s. Note

$$\mathbb{P}\left(T_n \in d\tau \mid Z_{n-1} = x\right) = d\tau\beta\left(x_\tau\left(x\right)\right) e^{-\int_0^\tau \beta(x_s(x))ds}, \tau \geq 0.$$

Coming back to the marginal Z_n and assuming $\Gamma\left(0\right) = -\infty$, the arguments of Sections 2.5 and 2.6 in Goncalves et al. (2022) imply that

$$\begin{aligned}\mathbb{P}\left(Z_n > y \mid Z_{n-1} = x\right) &= e^{-\Gamma(x)} \int_0^x dz\gamma\left(z\right) e^{\Gamma(z)} \int_y^\infty K\left(z, dy'\right) \\ &= 1 - e^{\Gamma(x\wedge y)-\Gamma(x)} + e^{-\Gamma(x)} \int_0^{x\wedge y} dz\gamma\left(z\right) e^{\Gamma(z)} K\left(z, y\right). \qquad (6.5.1)\end{aligned}$$

To obtain the last line, we have used that $K\left(z, y\right) = 1$ for all $y \leq z$ and, whenever $z < y$, we have split the second integral in the first line into the two parts corresponding to ($z < y \leq x$ and $z \leq x < y$).

To simulate the embedded chain, we have to decide first if, given $Z_{n-1} = x$, the forthcoming move is down or up.

- A move up occurs with probability given by

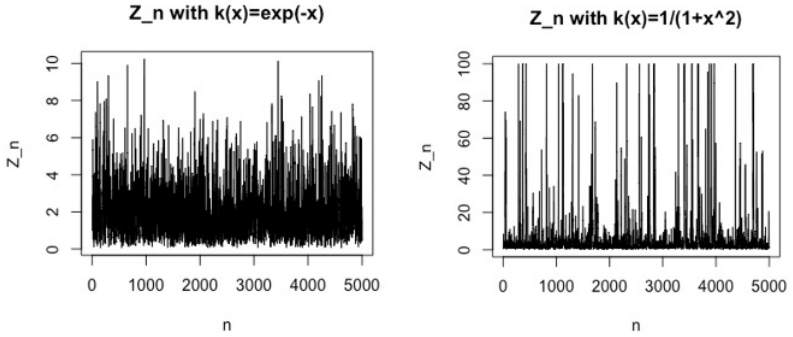
$$\mathbb{P}\left(Z_n > x \mid Z_{n-1} = x\right) = e^{-\Gamma(x)} \int_0^x dz\gamma\left(z\right) e^{\Gamma(z)} K\left(z, x\right).$$

- A move down occurs with complementary probability.

As soon as the type of move is fixed (down or up), to decide where the process goes precisely, we must use the inverse of the corresponding distribution function (6.5.1) (with $y \leq x$ or $y > x$), conditioned on the type of move.

Remark 6.14. *If state* 0 *is absorbing, Eq. (6.5.1) is valid only when* $x > 0$*, and the boundary condition* $\mathbb{P}(Z_n = 0 \mid Z_{n-1} = 0) = 1$ *should be added.*

In the following simulations, as before, we work in the separable case $K(x, y) = \frac{k(y)}{k(x)}$, where we choose $k(x) = e^{-x}$ in the first simulation and $k(x) = 1/(1 + x^2)$ in the second. Moreover, we take $\alpha(x) = \alpha_1 x^a$ and $\beta(x) = \beta_1 x^b$ with $\alpha_1 = 1$, $a = 2$, $\beta_1 = 1$ and $b = 1$. In these cases, there is no finite time extinction of the process $x_t(x)$; that is, in both cases, state 0 is not accessible.



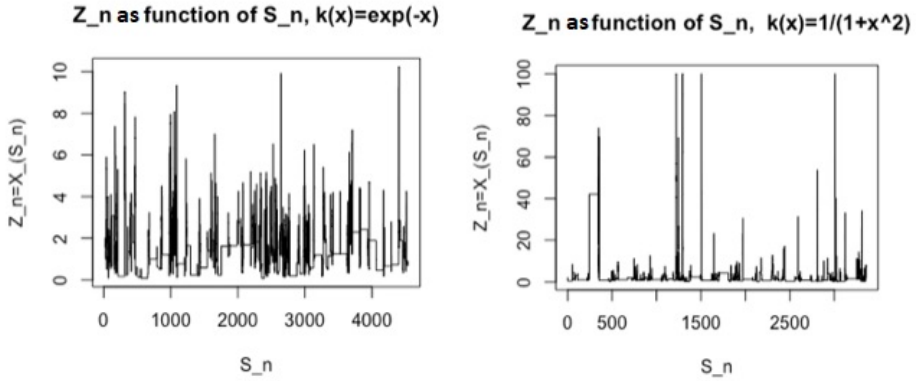
Notice that in accordance with the fact that $k(x) = 1/(1+x^2)$ has slower decaying tails than $k(x) = e^{-x}$, the process with jump distribution $k(x) = 1/(1 + x^2)$ has higher maxima than the process with $k(x) = e^{-x}$.

The graphs above do not provide any information about the jump times. In what follows we take this additional information into account and simulate the values Z_n of the embedded process as a function of the jump times S_n. To do so we must calculate the distribution $\mathbb{P}\left(S_n \leq t \mid X_{S_{n-1}} = x, S_{n-1} = s\right)$. Using Remark 6.13 we have

$$\mathbb{P}\left(S_n \leq t \mid X_{S_{n-1}} = x, S_{n-1} = s\right) = \int_s^t dt' \beta\left(x_{t'-s}(x)\right) e^{-\int_0^{t'-s} \beta(x_{s'}(x)) ds'}$$

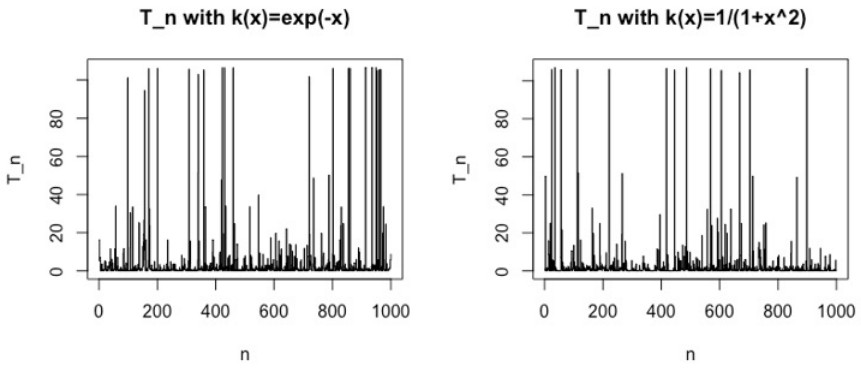
$$= \int_0^{t-s} du \beta\left(x_u(x)\right) e^{-\int_0^u \beta(x_{s'}(x)) ds'} = 1 - e^{-\int_0^{t-s} \beta(x_{s'}(x)) ds'} = 1 - e^{-[\Gamma(x) - \Gamma(x_{t-s}(x))]}.$$

The simulation of the jump times S_n then goes through a simple inversion of the conditional distribution function $\mathbb{P}\left(S_n \leq t \mid X_{S_{n-1}} = x, S_{n-1} = s\right)$. In the following simulations we use the same parameters as in the previous simulations.



The above graphs give the positions Z_n as a function of the jump times S_n. The waiting times between successive jumps are longer in the first process than in the

second one. Since we use the same jump rate function in both processes and since this rate is an increasing function of the positions, this is due to the fact that jumps lead to higher values in the second process than in the first such that jumps occur more frequently.



These graphs represent the sequence $T_n = S_n - S_{n-1}$ of the inter jump waiting times for the two processes showing once more that these waiting times are indeed longer in the first process than in the second.

6.6 The extremal record chain

Of interest are the upper record times and values sequences of Z_n, namely

$$\begin{aligned} R_n &= \inf\left(r \geq 1 : r > R_{n-1}, Z_r > Z_{R_{n-1}}\right) \\ Z_n^* &= Z_{R_n}. \end{aligned}$$

Unless X (and so Z_n) goes extinct, Z_n^* is a strictly increasing sequence tending to ∞.

Following Adke (1993), with $(R_0 = 0, Z_0^* = x)$, $(R_n, Z_n^*)_{n\geq 0}$ clearly is a Markov chain with transition probabilities for $y > x$

$$\begin{aligned} \overline{P}^*(k,x,y) &:= \mathbb{P}\left(R_n = r+k, Z_n^* > y \mid R_{n-1} = r, Z_{n-1}^* = x\right) \\ &= \overline{P}(x,y) \text{ if } k = 1 \\ &= \int_0^x \ldots \int_0^x \prod_{l=0}^{k-2} P(x_l, dx_{l+1})\, \overline{P}(x_{k-1}, y) \text{ if } k \geq 2, \end{aligned}$$

where $P(x,dy) = \mathbb{P}(Z_n \in dy \mid Z_{n-1} = x)$, $\overline{P}(x,y) = \mathbb{P}(Z_n > y \mid Z_{n-1} = x)$ and $x_0 = x$.

Clearly the marginal sequence $(Z_n^*)_{n\geq 0}$ is Markov with transition matrix

$$\overline{P}^*(x,y) := \mathbb{P}\left(Z_n^* > y \mid Z_{n-1}^* = x\right) = \sum_{k\geq 1} \overline{P}^*(k,x,y),$$

but the record times marginal sequence $(R_n)_{n\geq 0}$ is non-Markov. However

$$\mathbb{P}\left(R_n = r+k \mid R_{n-1} = r, Z_{n-1}^* = x\right) = \overline{P}^*(k,x,x),$$

showing that the law of $A_n := R_n - R_{n-1}$ (the age of the n-th record) is independent of R_{n-1} (although not of Z^*_{n-1}):

$$\mathbb{P}\left(A_n = k \mid Z^*_{n-1} = x\right) = \overline{P}^*(k, x, x), \ k \geq 1.$$

Of particular interest is $(R_1, Z^*_1 = Z_{R_1})$, the first upper record time and value because S_{R_1} is the first time $(X_t)_t$ exceeds the threshold x, and Z_{R_1} the corresponding overshoot at $y > x$. Its joint distribution is simply $(y > x)$

$$P^*(k, x, dy) = \mathbb{P}(R_1 = k, Z^*_1 \in dy \mid R_0 = 0, Z^*_0 = x)$$

$$\begin{aligned} &= P(x, dy) \text{ if } k = 1 \\ &= \int_0^x ... \int_0^x \prod_{l=0}^{k-2} P(x_l, dx_{l+1}) P(x_{k-1}, dy) \text{ if } k \geq 2. \end{aligned}$$

If $y_c > x$ is a critical threshold above which one wishes to evaluate the joint probability of $(R_1, Z^*_1 = Z_{R_1})$, then $P^*(k, x, dy)/\overline{P}^*(k, x, y_c)$ for $y > y_c$ is its right expression.

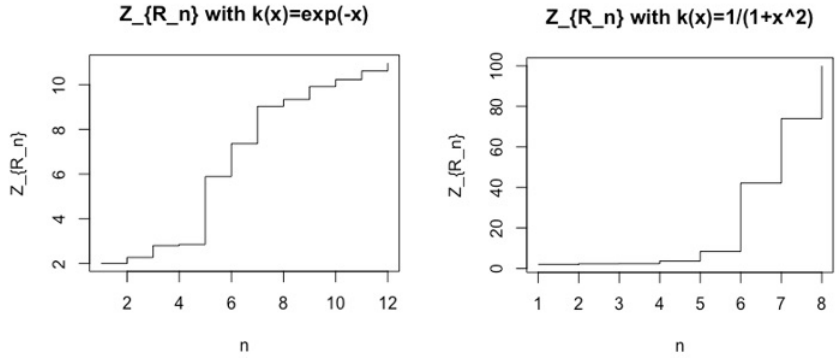
Note that $\mathbb{P}(R_1 = k \mid R_0 = 0, Z^*_0 = x) = \overline{P}^*(k, x, x)$ and also that

$$P^*(x, dy) := \mathbb{P}(Z^*_1 \in dy \mid Z^*_0 = x) = \sum_{k \geq 1} P^*(k, x, dy).$$

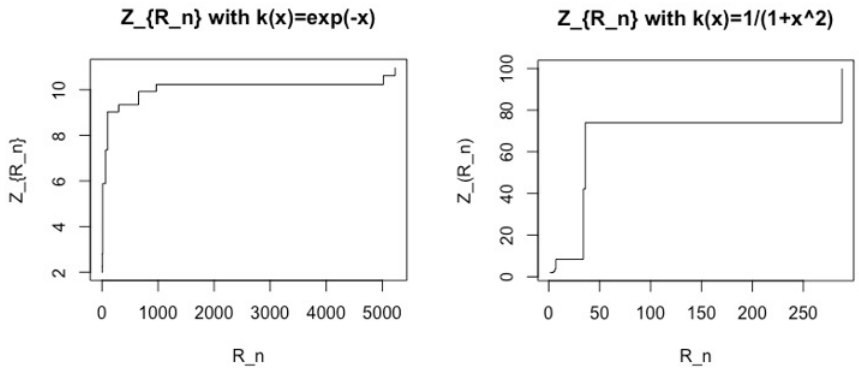
Of interest is also the number of records in the set $\{0, ..., N\}$:

$$\mathcal{R}_N := \#\{n \geq 0 : R_n \leq N\} = \sum_{n \geq 0} \mathbb{1}_{\{R_n \leq N\}}.$$

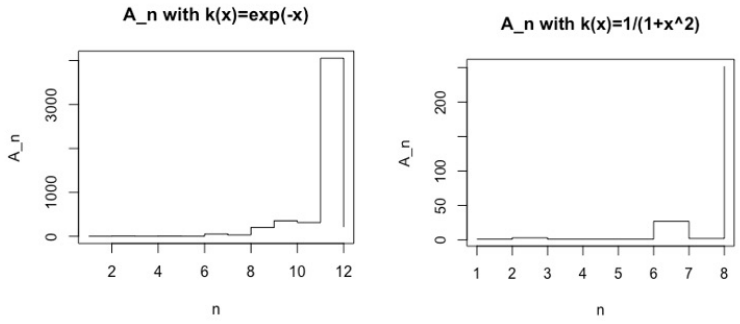
The following graphs represent the records of the two processes as a function of the ranks of the records, the one at the left with $k(x) = e^{-x}$ and the one at the right with $k(x) = 1/(1 + x^2)$. We can notice that there are more records in the first graph than in the second. Records occur more frequently in the first graph than in the second. On the other hand, the heights of the records are much lower in the first graph than in the second.



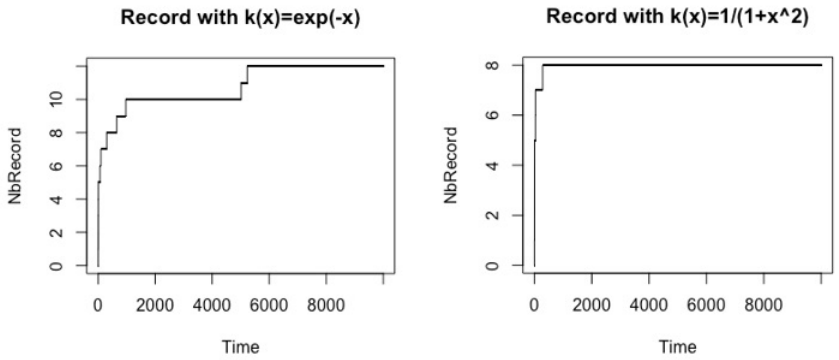
The following graphs give Z_{R_n} as a function of R_n. We remark that the gap between two consecutive records decreases over the time whereas the time between two consecutive records becomes longer. In other words, the higher is a record, the longer it takes to surpass it statistically.



The following graphs give $A_n = R_n - R_{n-1}$ as a function of n. The differences between two consecutive records are much greater in the first graph than in the second. It is also noted that the maximum time gap is reached between the penultimate record and the last record.



The following graphs give the obtained records from the simulation of the two processes, as a function of time. We can remark that the curve is slowly increasing with the time. In fact, to reach the 12th record, the first simulated process has needed 5500 units of time and analogously, the second simulated process has needed 1500 units of time to reach its 8th record.



6.7 Decay-Surge processes and the relation with Hawkes and Shot-noise processes

6.7.1 Hawkes processes

In the section we study the particular case $\beta(x) = \beta_1 x$, $\beta_1 > 0$ (the surge rate increases linearly with x), $\alpha(x) = \alpha_1 x$, $\alpha_1 > 0$ (exponentially declining population) and $k(y) = e^{-y}$. In this case, with $\gamma_1 = \beta_1/\alpha_1$, $\Gamma(x) = \gamma_1 x$.

In this case, we remark $\Gamma(0) = 0 > -\infty$. Therefore there is a strictly positive probability that the process will never jump (in which case it is attracted to 0). However we have $t_0(x) = \infty$, so the process never hits 0 in finite time. Finally, $\beta(0) = 0$ implies that state 0 is natural (absorbing and inaccessible).

Note that for this model,

$$\pi(y) = \frac{1}{\alpha_1 y} e^{(\gamma_1 - 1)y}$$

and we may take a version of the scale function given by

$$s(x) = \frac{\gamma_1}{1-\gamma_1}[e^{-(\gamma_1-1)y}]_0^x = \frac{\gamma_1}{1-\gamma_1}\left(e^{(1-\gamma_1)x} - 1\right).$$

Clearly, Assumption 5 is satisfied if and only if $\gamma_1 < 1$. We call the case $\gamma_1 < 1$ *subcritical*, the case $\gamma_1 > 1$ *supercritical* and the case $\gamma_1 = 1$ *critical*.

Supercritical case. It can be shown that the process does not explode almost surely, such that it is transient in this case (see Proposition 6.5). The speed density is neither integrable at 0 nor at ∞.

Critical and subcritical case. If $\gamma_1 < 1$, then π is integrable at $+\infty$, and we find

$$\phi_a(x) = -\left[s(y)\,\overline{\pi}(y)\right]_a^x - \int_a^x s(y)\,\pi(y)\,dy + \frac{1}{\alpha_1}\log\frac{x}{a},$$

where

$$\int_a^x s(y)\,\pi(y)\,dy = \frac{\gamma_1}{\alpha_1(1-\gamma_1)}\left[\ln\left(\frac{x}{a}\right) - Ei\left((\gamma_1-1)x\right) + Ei\left((\gamma_1-1)a\right)\right].$$

In the critical case $\gamma_1 = 1$ the hitting time of a will be finite without having finite expectation.

In both critical and subcritical cases, that is, when $\gamma_1 \leq 1$, the process X_t converges to 0 as $t \to \infty$ as we shall show now.

Due to the additive structure of the underlying deterministic flow and the exponential jump kernel, we have the explicit representation

$$X_t = e^{-\alpha_1 t}x + \sum_{n\geq 1 : S_n \leq t} e^{-\alpha_1(t-S_n)}Y_n, \tag{6.7.1}$$

where the $(Y_n)_{n\geq 1}$ are i.i.d. exponentially distributed random variables with mean 1, such that for all n, Y_n is independent of $S_k, k \leq n$, and of $Y_k, k < n$. Finally, in (6.7.1), the process X_t jumps at rate $\beta_1 X_{t-}$.

The above system is a *linear Hawkes process* without immigration, with kernel function $h(t) = e^{-\alpha_1 t}$ and with random jump heights $(Y_n)_{n\geq 1}$ (see Hawkes (1971), see also Brémaud and Massoulié (1996)). Such a Hawkes process can be interpreted as *inhomogeneous Poisson process with branching*. Indeed, the additive structure in (6.7.1) suggests the following construction.

— At time 0, we start with a Poisson process having time-dependent rate $\beta_1 e^{-\alpha_1 t} x$.

— At each jump time S of this process, a new (time inhomogeneous) Poisson process is born and added to the existing one. This new process has intensity $\beta_1 e^{-\alpha_1 (t-S)} Y$, where Y is exponentially distributed with parameter 1, independent of what has happened before. We call the jumps of this newborn Poisson process *jumps of generation* 1.

— At each jump time of generation 1, another time inhomogeneous Poisson process is born, of the same type, independently of anything else that has happened before. This gives rise to jumps of generation 2.

— The above procedure is iterated until it eventually stops since the remaining Poisson processes do not jump any more.

The total number of jumps of any of the offspring Poisson processes is given by

$$\beta_1 \mathbb{E}(Y) \int_S^\infty e^{-\alpha_1 (t-S)} dt = \gamma_1.$$

So we see that whenever $\gamma_1 \leq 1$, we are considering a subcritical or critical Galton-Watson process which goes extinct almost surely, after a finite number of reproduction events. This extinction event is equivalent to the fact that the total number of jumps in the system is finite almost surely, such that after the last jump, X_t just converges to 0 (without, however, ever reaching it). Notice that in the subcritical case $\gamma_1 < 1$, the speed density is integrable at ∞, while it is not at 0 corresponding to absorption in 0.

An interesting feature of this model is that it can exhibit a phase transition when γ_1 crosses the value 1.

Finally, in the case of a linear Hawkes process with immigration we have $\beta(x) = \mu + \beta_1 x$, with $\mu > 0$. In this case,

$$\pi(y) = \frac{1}{\alpha_1} e^{(\gamma_1 - 1)y} y^{\mu/\alpha_1 - 1}$$

which is always integrable in 0 and which can be tuned into a probability in the subcritical case $\gamma_1 < 1$ corresponding to positive recurrence.

Remark 6.15. *An interpretation of the decay-surge process in terms of Hawkes processes is only possible in case of affine jump rate functions β, additive drift α and exponential kernels k as considered above.*

6.7.2 Shot-noise processes

Let $h(t)$, $t \geq 0$, with $h(0) = 1$ be a causal non-negative non-increasing response function translating the way shocks will attenuate as time passes by in a shot-noise process. We assume $h(t) \to 0$ as $t \to \infty$ and

$$\int_0^\infty h(s)\, ds < \infty. \tag{6.7.2}$$

With $X_0 = x \geq 0$, consider then the linear shot-noise process

$$X_t = x + \int_0^t \int_{\mathbb{R}_+} y h(t-s)\, \mu(ds, dy), \tag{6.7.3}$$

where, with $(S_n; n \geq 1)$ the points of a homogeneous Poisson point process with intensity β, $\mu(ds, dy) = \sum_{n\geq 1} \delta_{S_n}(ds)\, \delta_{\Delta_n}(dy)$ (translating independence of the shots' heights Δ_n and occurrence times S_n). Note that, with $dN_s = \sum_{n\geq 1} \Delta_n \delta_{S_n}(ds)$, so with $N_t = \sum_{n\geq 1} \Delta_n \mathbb{1}_{\{S_n \leq t\}}$ representing a time-homogeneous compound Poisson process with jumps' amplitudes Δ,

$$X_t = x + \int_0^t h(t-s)\, dN_s \tag{6.7.4}$$

is a linearly filtered compound Poisson process. Under this form, it is clear that X_t cannot be Markov unless $h(t) = e^{-\alpha t}$, $\alpha > 0$. We define

$$\begin{aligned} \nu(dt, dy) &= \mathbb{P}(S_n \in dt, \Delta_n \in dy \text{ for some } n \geq 1) \\ &= \beta dt \cdot \mathbb{P}(\Delta \in dy). \end{aligned}$$

In the sequel, we shall assume without much loss of generality that $x = 0$.

The linear shot-noise process X_t has two alternative equivalent representations, emphasizing its superposition characteristics :

$$\begin{aligned} (1)\ X_t &= \sum_{n\geq 1} \Delta_n h(t - S_n)\, \mathbb{1}_{\{S_n \leq t\}} \\ (2)\ X_t &= \sum_{p=1}^{P_t} \Delta_p h(t - \mathcal{S}_p(t)), \end{aligned}$$

where $P_t = \sum_{n\geq 1} \mathbb{1}_{\{S_n \leq t\}}$.

Both show that X_t is the size at t of the whole decay-surge population, summing up all the declining contributions of the sub-families which appeared in the past at jump times (a shot-noise or filtered Poisson process model appearing also in Physics and Queuing theory, Snyder and Miller (2012), Parzen (1999)). The contributions $\Delta_p h(t - \mathcal{S}_p(t))$, $p = 1, ...P_t$, of the P_t families to X_t are stochastically ordered in decreasing sizes.

In the Markov case, $h(t) = e^{-\alpha t}$, $t \geq 0$, $\alpha > 0$, we have

$$X_t = e^{-\alpha t} \int_0^t e^{\alpha s} dN_s, \tag{6.7.5}$$

so that

$$dX_t = -\alpha X_t dt + dN_t,$$

showing that X_t is a time-homogeneous Markov process driven by N_t, known as the classical linear shot-noise. This is clearly the only choice of the response function that makes X_t Markov. In that case, by Campbell's formula (see Parzen (1999)),

$$\begin{aligned} \Phi_t^X(q) &: = \mathbb{E}e^{-qX_t} = e^{-\beta \int_0^t (1 - \phi_\Delta(qe^{-\alpha(t-s)}))ds} \\ &= e^{-\frac{\beta}{\alpha}\int_{e^{-\alpha t}}^1 \frac{1-\phi_\Delta(qu)}{u} du} \text{ where } e^{-\alpha s} = u, \end{aligned}$$

with

$$\Phi_t^X(q) \to \Phi_\infty^X(q) = e^{-\frac{\beta}{\alpha}\int_0^1 \frac{1-\phi_\Delta(qu)}{u} du} \text{ as } t \to \infty.$$

The simplest explicit case is when $\phi_\Delta(q) = 1/(1+q/\theta)$ (else $\Delta \sim \text{Exp}(\theta)$) so that, with $\gamma = \beta/\alpha$,

$$\Phi_\infty^X(q) = (1+q/\theta)^{-\gamma}$$

the Laplace-Stieltjes-Transform of a Gamma(γ, θ) distributed random variable X_∞, with density

$$\frac{\theta^\gamma}{\Gamma(\gamma)} x^{\gamma-1} e^{-\theta x},\ x > 0.$$

This time-homogeneous linear shot-noise with exponential attenuation function and exponentially distributed jumps is a decay-surge Markov process with triple

$$\left(\alpha(x) = -\alpha x;\ \beta(x) = \beta;\ k(x) = e^{-\theta x}\right).$$

Shot-noise processes being generically non-Markov, there is no systematic relationship of decay-surge Markov processes with shot-noise processes. In Eliazar and Klafter (2009), it is pointed out that decay-surge Markov processes could be related to the maximal process of nonlinear shot noise ; see Eliazar and Klafter (2007, 2009).

6.7.3 Appendix

Remark 6.16. *Suppose β is non-decreasing and $K(x,y) = k(y)/k(x)$. Let Z_t be the Markov jump process having non-decreasing trajectories, which is solution of*

$$dZ_t = \Delta(Z_{t-}) \int_0^\infty \mathbb{1}_{\{r \le \beta(Z_{t-}(x))\}} M(dt, dr),\ Z_0(x) = x.$$

In other words, Z jumps at rate $\beta(x)$, when being in position x, to a new position chosen according to $K(x, dy)$. Since β is non-decreasing, we may couple $X_t(x)$ and $Z_t(x)$, using the same underlying Poisson random measure M, in such a way that for all $t \ge 0$,

$$Z_t(x) \ge X_t(x).$$

Indeed, let τ be the first jump time of one of the two processes. Since β is non-decreasing, τ is necessarily a jump time of Z. Let $X_{\tau-} = x_1$ and $Z_\tau = x_2$ be the positions of the two processes right before the jump. Clearly, $x_1 \le x_2$.

In case that both processes jump, write $Y(x_1)$ and $Y(x_2)$ for the associated positions right after the jump. Since for all $y \ge x_2$,

$$k(y)/k(x_1) = \mathbb{P}(Y(x_1) \ge y) \le \mathbb{P}(Y(x_2) \ge y) = k(y)/k(x_2),$$

$Y(x_1)$ and $Y(x_2)$ are stochastically ordered, implying that there exists a coupling of them such that $Y(x_1) \le Y(x_2)$ almost surely, preserving the order of Z and X.

In case that only Z jumps, clearly $Z_\tau \ge Z_{\tau-} \ge X_{\tau-} = X_\tau$, such that the order is still preserved.

Proposition 6.9. *Suppose that X_t is Harris recurrent having invariant probability measure π such that $\pi(\beta) \in (0, \infty)$. Let $S_n, n \ge 1$, be the successive jump times of the process. Then Z_n is Harris recurrent with invariant measure π^Z given by*

$$\pi^Z(g) = \frac{1}{\pi(\beta)} \pi(\beta K g),$$

for any $g : \mathbb{R}^N \to \mathbb{R}$ measurable and bounded, where

$$\beta K g(x) = \beta(x) \int_x^\infty K(x, dy) g(y).$$

After an integration by parts, it follows from the latter that for separable kernels $K(x,y) = k(y)/k(x)$,

$$\text{if } k\left(x\right) e^{-\Gamma(x)} \to 0 \text{ as } x \to 0 \text{ and } x \to \infty, \text{ then } \pi^{Z}(dx) = \frac{e^{\Gamma(x)} dk\left(x\right)}{\int_0^\infty e^{\Gamma(x)} dk\left(x\right)}$$

is the explicit representation of the invariant measure of Z.

7

An interacting neuronal network with inhibition : theoretical analysis and perfect simulation

Abstract. We study a purely inhibitory neural network model where neurons are represented by their state of inhibition. The study we present here is partially based on the work of Cottrell (1992) and Fricker et al. (1993). The spiking rate of a neuron depends only on its state of inhibition. When a neuron spikes, its state is replaced by a random new state, independently of anything else and the inhibition state of the other neurons increase by a positive value. Using the Perron-Frobenius theorem, we show the existence of a Lyapunov function for the process. Furthermore, we prove a local Doeblin condition which implies the existence of an invariant measure for the process. Finally, we extend our model to the case where the neurons are indexed by $\mathbb{Z}$. We construct a perfect simulation algorithm to show the recurrence of the process under certain conditions. To do this, we rely on the classical contour technique used in the study of contact processes, and assuming that the spiking rate lies on the interval $[\beta_*, \beta^*]$, we show that there is a critical threshold for the ratio $\delta = \frac{\beta_*}{\beta^* - \beta_*}$ over which the process is ergodic.

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7.1 Introduction

For the operation of a neural network, neurons excite and/or inhibit each other. Here, we study a model of a purely inhibitory neural network where neurons are represented by their inhibitory state. The study we present is partially based on the work of Cottrell (1992). Her model consists of considering N interacting neurons described their state of inhibition. In her work, a neuron spikes when its state touches the value 0. When a neuron spikes, the state of inhibition of the other neurons increase by a non-negative deterministic constant θ. The spiking neuron immediately receives a random inhibition independently of anything else. In Cottrell's work the state of inhibition is just the waiting time until the next spike. In the present work we generalize Cottrell's model in several natural ways. Actually, in Cottrell's model, the next spiking time in the neural net is deterministic and we will lift this assumption. A random spiking time is more realistic than a deterministic one since stochasticity is present all over in the brain functioning. Secondly, to allow formal general models we allow the state of inhibition to decrease at a general rate in between the successive spikes of the network while in Cottrell's work the drift of the flow is equal to -1.

In the first part of this paper, we consider systems of N interacting neurons, in which any neuron can spike at any time. The spiking neuron takes a new random state of inhibition, and the others increase their inhibitory state by a deterministic quantity that we will call the inhibition weight, which depends on the distance between the spiking neuron and the "receiving" neuron, so that a neuron located far away of the spiking neuron is less impacted by the spike. The model thus presented obviously extends Cottrell (1992) and Fricker et al. (1993) in two ways : the spiking time is no more deterministic but it is random ; the dynamic of the process is no more constant.

Firstly, we show the existence of a Lyapunov function that allows us to formulate a sufficient condition of non-evanescence of the process in the sense of Meyn and Tweedie (1993), i.e. a condition ensuring that the process does not escape at infinity. To do so, we introduce a reproduction matrix H and we suppose the spectral radius of H is lower than 1. The eigenvector associated with the spectral radius of H allows us to find a Lyapunov function for the process.

Secondly, we study the recurrence of the process relying on Doeblin conditions which we establish for the embedded chain sampled at the jump times. We show the existence of an invariant probability measure for the process. We do this in the case when the distribution of the new states has an absolutely continuous density and the jump rate is bounded.

In a second part, we consider the case where we have an infinite number of neurons indexed by $\mathbb{Z}$ (Comets et al. (2002), Galves and Löcherbach (2013), Galves et al.

(2010) and André (2019)). The mean field behavior of such models has been studied by Cormier et al. (2020) and Robert and Touboul (2016) who were also interested in the stationary distributions of these processes. In the work of Ferrari et al. (2018), considering an infinite system of interacting point processes with memory of variable length, the authors investigated the conditions for the existence of a phase transition using the classical contour technique, based on the classical work of Griffeath (1981) on a contact process. Morgan André in this work André (2019), proves that the model described in Ferrari et al. (2018) presents a metastable behavior while relying on the contour technique used in Griffeath (1981). Following the idea of Ferrari et al. (2018), Galves et al. (2013) and Griffeath (1981), we construct a perfect simulation algorithm that allows us to show the recurrence of the process. Assuming that the spiking rate takes values in the interval $[\beta_*, \beta^*]$, we show that there is a critical threshold for the ratio $\delta = \frac{\beta_*}{\beta^* - \beta_*}$ over which the process is ergodic.

This paper is organized as follows. In Section 2 we describe the model and study the law of the first jump time of the process. The Foster-Lyapunov and Doeblin conditions are discussed to find non-evanescence criteria and to show the existence of a unique invariant probability measure of the process in Section 3 which is our first main result. Finally, in Section 4, we present a perfect simulation algorithm and we simulate the law of the state of inhibition of a given neuron in its invariant regime.

7.2 The model

7.2.1 Description of the model

In our paper, let us consider we have N neurons that are related to each other. For all $i \in \{1, \cdots, N\}$, $X_t^{i,N}$ describes the state of inhibition of neuron i at time t. When the neuron $i \in \{1, \cdots, N\}$ spikes,

- The current state of inhibition of neuron i is replaced by a new value Y^i independently of anything else with distribution F^i. Y^i is the new position of the jumping particle right after the jump.
- The state of inhibition of any neuron $j \neq i$ is increased by a positive value $W_{i \to j}$ at time t.

In between successive jumps of the system, each neuron i follows the deterministic dynamic

$$\dot{x}_t^i(x^i) = -\alpha_i \left(x_t^i(x^i) \right), \; x_0^i = x^i, \tag{7.2.1}$$

where α_i is positive on $(0, \infty)$, locally Lipschitz on $[0, \infty)$, that is, there exists a positive constant ζ such that, for all $x, x' \in [0, \infty)$,

$$|\alpha(x) - \alpha(x')| \leq \zeta |x - x'|$$

and $\alpha_i(0) = 0$ such that the process can not enter the negative values.

Let β_i be a continuous positive and decreasing rate function on $[0, \infty)$. We have taken β_i to be decreasing so that the larger x_t^i is, the lower its probability of spiking and the smaller x_t^i is, the higher its probability of spiking. $x_t^i(x^i)$ designs the solution of the equation (7.2.1) at time t starting from x^i at time 0.

We are thus led to consider the piecewise deterministic Markov process (PDMP) $X_t^N = (X_t^{1,N}, \cdots, X_t^{N,N}) \in \mathbb{R}_+^N$. For $i \in \{1, \cdots, N\}$, the state of inhibition of neuron i at time t, $X_t^{i,N}$ is given by :

$$X_t^{i,N} = X_0^{i,N} - \int_0^t \alpha_i\left(X_{s-}^{i,N}\right) ds + \int_0^t \int_0^\infty \int_0^\infty (y^i - X_{s-}^{i,N}) \mathbb{1}_{\left\{r \le \beta_i\left(X_{s-}^{i,N}\right)\right\}} M^i\left(ds, dr, dy^i\right) + \sum_{j \neq i} W_{j \to i} \int_0^t \int_0^\infty \int_0^\infty \mathbb{1}_{\left\{r \le \beta_j\left(X_{s-}^{j,N}\right)\right\}} M^j\left(ds, dr, dy^j\right), \quad (7.2.2)$$

where M^i is a random Poisson measure with intensity $dtdrF^i(dy)$ and for all i, the M^i are all independent. This model extends that of Goncalves et al. Goncalves et al. (2022) in the multidimensional case.

Remark 7.1. *For all $i \in \{1, \cdots N\}$, $X_t^{i,N}$ can be interpreted as the inhibition state of the neuron i at time t and $W_{j \to i}$ as the inhibition weight of the neuron j on the neuron i. When $W_{i \to j} \le 0$, we say that the neuron i is excitatory for the neuron j and when $W_{i \to j} \ge 0$, we say that the neuron i is inhibitory for the neuron j. In our paper we are interested in the case where neuron i is inhibitory for neuron j i.e., $W_{i \to j} \ge 0$.*

Remark 7.2. *The formula (7.2.2) is well-posed in the sense that there is non explosion of the process . Since $\beta_i(X_s^{i,N}) \le \beta_i(0)$ for all i we deduce that $\int_0^t \beta_i(X_s^{i,N}) ds < \infty$ whence the non explosion, that is, almost surely, the process has only a finite number of jumps within each finite time interval.*

The infinitesimal generator associated with this model is given by :

$$G^N V(x) = -\sum_{i=1}^N \alpha_i(x^i) \frac{\partial}{\partial x^i} V(x) + \sum_{i=1}^N \beta_i(x^i) \int_0^\infty F^i(dy^i)[V(x + e^i y^i - e^i x^i + \sum_{j \neq i} e^j W_{i \to j}) - V(x)] \quad (7.2.3)$$

where V is a smooth function and e^i is the $i-th$ unit vector.
In other words, at each jump of the process, only one neuron spikes. If it is neuron i then its state is replaced by Y^i and all other neurons receive the inhibition weight $W_{i \to j} \ge 0$ for any $j \neq i$.

7.2.2 First jump time

Let N_t^i be the counting process of successive jumps of neuron i, that is,

$$N_t^i = \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbb{1}_{\{r \le \beta_i(X_{s-}^{i,N}(x^i))\}} M^i(ds, dr, dy^i)$$

and S_1^i the first spike time of neuron i, so we have

$$S_1^i = \inf\{t > 0, N_t^i = 1\}.$$

Let S_1 be the first jump time of the state process (X_t), that is, $S_1 = \min_i S_1^i$. Let for all i,

$$t_0(x^i) := \int_0^{x^i} \frac{dy}{\alpha_i(y)}$$

the time for the neuron i to hit 0 starting from x^i.

Proposition 7.1. *For $t < \min_i t_0(x^i)$,*

$$\mathbb{P}(S_1 > t) = \prod_{i=1}^{N} e^{-[\Gamma_i(x^i) - \Gamma_i(x_t^i(x^i))]}, \tag{7.2.4}$$

with $\Gamma_i(x^i) := \int^{x^i} \gamma_i(y)dy$ and $\gamma_i(x^i) = \beta_i(x^i)/\alpha_i(x^i)$.

Proof. For all $t > 0$,

$$\mathbb{P}(S_1 > t) = \mathbb{P}(\min_i S_1^i > t) = \prod_{i=1}^{N} e^{-\int_0^t \beta_i(x_s^i(x^i))ds}. \tag{7.2.5}$$

Moreover, if $t < \min_i t_0(x^i)$ by making a change of variables $z = x_s^i(x^i)$ and using the fact that $dz = \dot{x}_s^i(x^i)ds = -\alpha_i(z)ds$ we have

$$\begin{aligned}\mathbb{P}(S_1 > t) &= \prod_{i=1}^{N} e^{-\int_0^t \beta_i(x_s^i(x^i))ds} = \prod_{i=1}^{N} e^{-\int_{x^i}^{x_t^i(x^i)} \beta_i(z)(-dz/\alpha_i(z))} \\ &= \prod_{i=1}^{N} e^{-[\Gamma_i(x^i) - \Gamma_i(x_t^i(x^i))]}.\end{aligned}$$

□

Assumption 6. *There exists $1 \le i \le N$ suchthat$\Gamma_i(0) = -\infty$.*

Proposition 7.2. *1. Suppose Assumption 6 holds. Then $S_1 < \infty$ almost surely.*

2. Suppose Assumption 6 does not hold.
- If there exists $i \in [|1, N|]$, such that $t_0(x^i) < \infty$ then $S_1 < \infty$ almost surely if and only if $\beta_i(0) > 0$.
- If for all $i \in [|1, N|]$, $t_0(x^i) = \infty$ then $\mathbb{P}(S_1 = \infty) > 0$ i.e. with a positive probability the first jump time is infinite.

Proof. 1. Let N be fixed and suppose Assumption 6 holds.
- If for all $i \in [|1, N|]$, $t_0(x^i) = \infty$ and letting t tend to ∞ in (7.2.4) we have

$$\mathbb{P}(S_1 = \infty) = \prod_{i=1}^{N} e^{-[\Gamma_i(x^i) - \Gamma_i(x_\infty^i(x^i))]} = \prod_{i=1}^{N} e^{-[\Gamma_i(x^i) - \Gamma_i(0)]},$$

since $x_\infty^i(x^i) = 0$ for all $i \in [|1, N|]$. Then by Assumption 6, $\mathbb{P}(S_1 = \infty) = 0$ that is $S_1 < \infty$ almost surely.
- If for some $i \in [|1, N|]$, $t_0(x^i) < \infty$ and letting $t \uparrow \min_i t_0(x^i)$ in (7.2.4), we obtain

$$\mathbb{P}(S_1 \ge \min_i t_0(x^i)) = \lim_{t \uparrow \min_i t_0(x^i)} \mathbb{P}(S_1 > t) = \prod_{i=1}^{N} e^{-[\Gamma_i(x^i) - \Gamma_i(0)]} = 0$$

by Assumption 6, implying that $S_1 < \infty$ almost surely.

2. Suppose Assumption 6 does not hold.
- If there exists $i \in [|1, N|]$, such that i.e. the time for neuron i to hit 0 starting from x^i is finite then it is obvious (by definition of $t_0(x^i)$) to see that it is sufficient for $\beta_i(0) > 0$ to have $S_1 < \infty$ almost surely.
- If Assumption 6 does not hold and for all $i \in [|1, N|]$, $t_0(x^i) = \infty$ then by making $t \to \infty$ in (7.2.4) we have $\mathbb{P}(S_1 = \infty) > 0$ that is $S_1 = \infty$ with a positive probability.

□

We finish this section with a simulation of the process starting from some fixed initial configuration $(x_0^1, \cdots, x_0^N)$. For this, we assume that for all i the jump rate $\beta_i(x^i)$ is bounded and lower bounded, that is, $\beta_i(x^i) \in [\beta_*, \beta^*]$ for all $x^i > 0$, where $0 < \beta_* < \beta^* < \infty$.
The following variables will be used to write our simulation algorithm.

- $T = (T_1, T_2, \cdots)$ where $T_1 < T_2 < \cdots$ are the times of successive proposals of jumps for the total system, to be accepted or rejected
- L is the label associated with T. It will be $\{sure\}$ or $\{possible\}$
- $P = (P^1, \cdots, P^N)$ is the vector of positions of the N neurons at a fixed instant
- I is the vector which represents the number of the neuron which spikes.

Algorithm 7.1. *1. We set* $T_1 \sim \exp(\beta^* N)$
- $L_1 = \{possible\}$ *with probability* $\frac{\beta^* - \beta_*}{\beta^*}$
- $L_1 = \{sure\}$ *with probability* $\frac{\beta_*}{\beta^*}$

2. We initialize the vector P *with the values* $(x_0^1, \cdots, x_0^N)$

3. We choose $I_1 = k$ *with probability* $\frac{1}{N}$
- *If* $L_1 = \{sure\}$,

$$(a) \begin{cases} P^k \sim F^k, \\ P^i \leftarrow x^i_{T_1}(P^i) + W_{k \to i} \end{cases}$$

-*If* $L_1 = \{possible\}$ *we accept the jump with probability*

$$p = \frac{\beta_k(x^k_{T_1}(P^k)) - \beta_*}{\beta^* - \beta_*},$$

and we apply (a).
-*Else* $P^j \leftarrow x^j_{T_1}(P^j), \quad \forall j \in \{1, \cdots, N\}$.

4. We update the vector P *and start the procedure again from* (1).
We stop the procedure after a fixed finite number n *of iterations.*

We plot in the following figure a typical trajectory of $X_t^{i,N}$ with $N = 2$ neurons.
In both figures $N = 2$ neurons, $n = 50$ iterations and $F^i \sim \exp(1)$ for all $1 \leq i \leq 2$. It can be seen that in Figure 7.1 there are more occurrences of jumps than in Figure 7.2. In both, Neuron $i = 1$ is plotted in blue and neuron $i = 2$ in red.

7.3 Foster-Lyapunov and Doeblin conditions

In this section, we want to find conditions of non-evanescence of the process and show the existence of an invariant probability measure of the process.

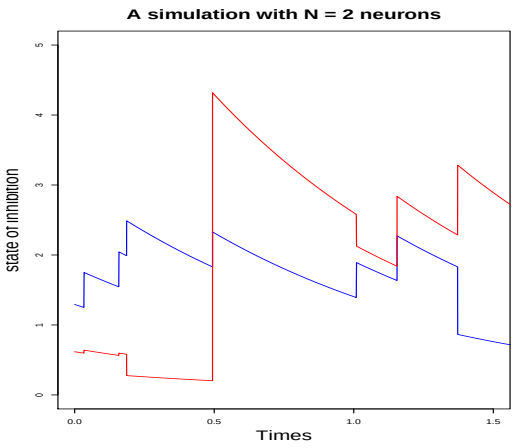


Figure 7.1 – A simulation of the neuron system with $N = 2$ neurons, $\alpha(x) = x$, $\beta(x) = 3 + \mathbf{1}(x \leq 2)$, $W_{2\to1} = 1/2$ and $W_{1\to2} = 1$. In this case, the process is ergodic.

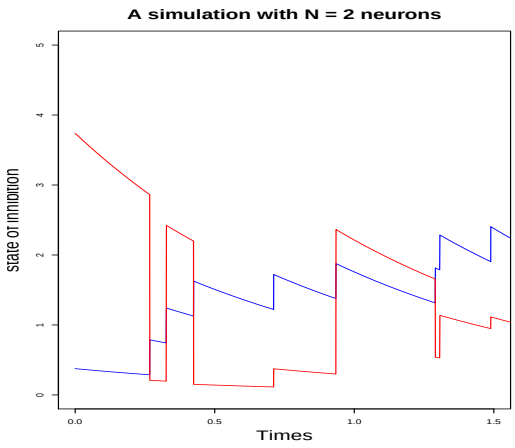


Figure 7.2 – A simulation of the neuron system with $N = 2$ neurons, $\alpha(x) = x/(1+x)$, $\beta(x) = 1/(1+x)$ and $W_{1\to2} = W_{2\to1} = 1/2$. In this case, the process is transient.

7.3.1 Foster-Lyapunov condition

In what follows, K is a fixed compact set, and we suppose that $\gamma_i(x) := \beta_i(x)/\alpha_i(x)$ is such that

$$\|\gamma_i\|_{\infty,K^c} := \sup_{x\in K^c} |\gamma_i(x)| < \infty.$$

We define W the matrix of inhibition weight by $W_{ij} := W_{j\to i},\ i \neq j$ and $W_{ii} = 0$. It is further assumed that the matrix W is irreducible in the sense that there exists an integer $p > 0$ such that W^p has only positive coefficients. We introduce the reproduction matrix

$$H_{ij} = W_{j\to i}\|\gamma_i\|_{\infty,K^c},\ i \neq j,\ H_{ii} = \|\gamma_i\|_{\infty,K^c} \int_0^\infty y^i F^i(dy^i)$$

which is also irreducible.

Suppose that

$$\rho(H) < 1$$

where $\rho(H)$ is the largest eigenvalue of H, that is, the spectral radius of H. By the Perron Frobenius theorem, there exists a left eigenvector κ associated to this eigenvalue ρ, that is, for all i,

$$\sum_j \kappa_j H_{ji} = \rho\kappa_i.$$

On the other hand, put $m_i = \kappa_i\|\gamma_i\|_{\infty,K^c}$.

Finally, let $V : \mathbb{R}_+^N \to \mathbb{R}_+$ such that

$$V(x) = \sum_{i=1}^N m_i x^i \tag{7.3.1}$$

and recall that the infinitesimal generator is given by :

$$G^N V(x) = -\sum_{i=1}^N \alpha_i(x^i)\frac{\partial}{\partial x^i}V(x) + \sum_{i=1}^N \beta_i(x^i)\int_0^\infty F^i(dy^i)[V(x + e^i y^i - e^i x^i + \sum_{j\neq i} e^j W_{i\to j}) - V(x)]$$

So by replacing V by its expression in the infinitesimal generator $G^N V(x)$ we have for all $x \in K^c$:

$$G^N V(x) = -\sum_{i=1}^N \alpha_i(x^i) m_i + \sum_{i=1}^N \beta_i(x^i) \int_0^\infty dF^i(y^i)[\sum_{j=1, j\neq i}^N (W_{i\to j} + x^j) m_j + y^i m_i - \sum_{j=1}^N x^j m_j]$$

$$= -\sum_{i=1}^N \alpha_i(x^i) m_i + \sum_{i=1}^N \beta_i(x^i)(m_i \int_0^\infty y^i F^i(dy^i) + \sum_{j\neq i} W_{i\to j} m_j) - \sum_{i=1}^N \beta_i(x^i) x^i m_i.$$

Then, since $-\beta_i(x^i)x^i \leq 0$,

$$G^N V(x) \leq -\sum_i \alpha_i(x^i) m_i + \sum_i \beta_i(x^i)(m_i \int_0^\infty y^i F^i(dy^i) + \sum_{j\neq i} W_{i\to j} m_j)$$

$$= -\sum_i \alpha_i(x^i) \left(m_i - \gamma_i(x^i) \left[\frac{m_i}{\|\gamma_i\|_{\infty, K^c}} H_{ii} + \sum_{j\neq i} \frac{1}{\|\gamma_j\|_{\infty, K^c}} H_{ji} m_j \right] \right)$$

$$= -\sum_i \alpha_i(x^i) \left(m_i - \gamma_i(x^i) \left[\kappa_i H_{ii} + \sum_{j\neq i} \kappa_j H_{ji} \right] \right)$$

$$= -\sum_i \alpha_i(x^i) \left(\|\gamma_i\|_{\infty, K^c} \kappa_i - \gamma_i(x^i) \rho \kappa_i \right)$$

$$= -\sum_i \alpha_i(x^i) \|\gamma_i\|_{\infty, K^c} \kappa_i \left(1 - \frac{\gamma_i(x^i)}{\|\gamma_i\|_{\infty, K^c}} \rho \right).$$

This calculus leads us to introduce the following

Assumption 7. *Let $\alpha > 0$. For all i, there exists r_i, such that $\forall x^i \geq r_i$, $\alpha_i(x^i) \geq \alpha x^i$.*

Corollary 7.1. *Under assumption 7 we have for all $x \in K^c$ such that $x^i \geq r_i$ for all i and such $\alpha_i(x^i) \geq \alpha x^i$:*

$$G^N V(x) \leq -\alpha \sum_i x^i m_i \left(1 - \frac{\gamma_i(x^i)}{\|\gamma_i\|_{\infty, K^c}} \rho \right) \leq -cV(x)$$

where c is a positive constant.

Definition 7.1. *We call the process non evanescent if there exists a compact K such that for all x, $\mathbb{P}_x-$ almost surely, $\limsup_t 1_K(X_t) = 1$.*

Proposition 7.3. *If $\rho < 1$, then the process is non-evanescent.*

Démonstration. $V(x)$ defined in (7.3.1) above is a norm-like function because the eigenvector κ is positive. Indeed, we call $V : \mathbb{R}_+^N \to \mathbb{R}$ a norm-like function if V is a positive, measurable function and $V(x) \to \infty$ when $x \to \infty$. The condition (CD1) of theorem 3.1 of Meyn and Tweedie (1993) implies the result. □

Remark 7.3. *We can refine the above conditions as follows. Let K be a compact set. Then we replace the above definition of H_{ii} by*

$$H_{ii} = \|\gamma_i\|_\infty \sup\{(\int y^i F^i(dy^i) - x^i)_+ : x \in K^c, \beta_i(x^i) > 0\}$$

and suppose that $\rho(H) < 1$. As a consequence, we obtain, with the same calculus, that on K^c,

$$G^N V(x) < 0.$$

Example 7.1. **(Mean-field interaction)** Suppose we have N neurons. We suppose also the function γ_i such that $\|\gamma_i\|_{\infty,K^c} < \infty$ and $F^i = F$, $W_{j\to i} = \theta$ for all i. In this case the reproduction matrix is

$$H_{ij} = \theta\|\gamma_i\|_{\infty,K^c},\ i \neq j,\ H_{ii} = \mathbb{E}(Y)\|\gamma_i\|_{\infty,K^c}$$

for some fixed compact set K. Suppose $\rho(H)$ is the spectral radius of H. Then, $\rho(H) = \sup_{x\in K^c}\gamma_i(x)$
$(\mathbb{E}(Y) + (N-1)\theta)$ and its associated eigenvector is $\kappa = (1, \cdots, 1)$. The condition $\rho(H) < 1$ is therefore equivalent to $\sup_{x\in K^c}\gamma_i(x) < 1/(\mathbb{E}(Y) + (N-1)\theta)$.

Example 7.2. **(Torus)** Suppose we have $N \geq 3$ neurons such that each neuron interacts with its two nearest neighbors (its left and right neighbors). Neuron 1 interacts with neuron 2 and neuron N . Neuron N interacts with neuron $N-1$ and neuron 1, so we have a torus.
We suppose also γ_i such that $\|\gamma_i\|_{\infty,K^c} < \infty$ and $F^i = F$, $W_{j\to i} = \theta$ for all $j \in \{i+1, i-1\}$ and $W_{j\to i} = 0$ if $j \neq \{i+1, i-1\}$. In this case the reproduction matrix is

$$H_{ij} = \begin{cases} \theta\|\gamma_i\|_{\infty,K^c} & \text{, if } i \neq j,\ j \in \{i+1, i-1\} \\ 0 & \text{, if } i \neq j,\ j \notin \{i+1, i-1\} \\ \mathbb{E}(Y)\|\gamma_i\|_{\infty,K^c} & \text{, if } i = j. \end{cases}$$

If $\rho(H)$ is the spectral radius of H then $\rho(H) = \|\gamma_i\|_{\infty,K^c}(\mathbb{E}(Y) + 2\theta)$ and its associated eigenvector is $\kappa = (1, \cdots, 1)$. The condition $\rho(H) < 1$ is equivalent to $\|\gamma_i\|_{\infty,K^c}(\mathbb{E}(Y) + 2\theta) < 1$.

7.3.2 Doeblin condition

Let $S_0 < S_1 < \cdots < S_n < \cdots$ be the instants of successive jumps of the N neurons. It is obvious that the embedded chain $Z_n := X_{S_n}$ is a Markov chain. Let I_n be the index of the neuron which jumps at time S_n.

Proposition 7.4. *Suppose that the assumptions of proposition 7.2 hold. Then, (Z_n, I_n) is a Markov chain and its transition $Q(x, dy)$ is given by :*

$$\mathbb{P}(Z_n \in dy, I_n = j | Z_{n-1} = x, I_{n-1} = i) = \int_0^\infty ds e^{-\int_0^s dl \sum_{i=1}^N \beta_i(x_l^i(x^i))}\beta_j(x_s^j(x^j))$$
$$\times \int F^j(du)\delta_{(x_s^1(x^1)+W_{j\to 1},\cdots,x_s^{j-1}(x^{j-1})+W_{j\to j-1},u,\, x_s^{j+1}(x^{j+1})+W_{j\to j+1},\cdots,x_s^N(x^N)+W_{j\to N})}(dy). \tag{7.3.2}$$

Theorem 7.1. *Suppose for all $1 \leq i \leq N$, $\alpha_i \in \mathcal{C}^1$ and there exists a compact set $K \subset (0,\infty)^N$ such that for all $x \in K$, for all $1 \leq i \leq N$, $\beta_i(x^i + \sum_{j=1}^{i-1} W_{j\to i}) > 0$. Moreover we suppose that $F^i(dy)$ is absolutely continuous and $\|\beta_i\|_\infty < \infty$ for all i. Then there exist $d \in (0,1)$ and a probability measure ν on $(\mathbb{R}_+^N, \mathcal{B}(\mathbb{R}_+^N))$, such that*

$$Q^N(x, dy) \geq d1_K(x)\nu(dy) \tag{7.3.3}$$

where Q is the transition operator of embedded chain $Z_n = X_{S_n}$ and Q^N is its $N-th$ iterate.

To prove this theorem we need the following proposition. First of all :
Fix any deterministic sequence $s_1 < \cdots < s_N$. In the sequel we shall work on the event $S_1 = s_1, \cdots, S_N = s_N, I_1 = 1, \cdots, I_N = N$ and $Y_1 = y_1, \cdots, Y_N = y_N$. Assume that the jumps are ordered such that neuron 1 jumps before neuron 2 and etc. Let $y = (y^1, \cdots, y^N)$ where y^i is the new state of inhibition of neuron i after the spike.

Let $t_k = s_k - s_{k-1}$ for all $1 \leq k \leq N$ the inter jump times of the N neurons which implies that $s_k = t_1 + \cdots + t_k$.
Conditionally on this event, let Ψ_{s_N} be the vecteur of positions of the process at time s_N. We can define Ψ_{s_N} as a function of the positions $y_1, \cdots, y_N$ such that $\Psi_{s_N} : \mathbb{R}^N \to \mathbb{R}^N$ is given by :

$$\Psi^k_{s_N}(y) = \begin{cases} \psi^{k,N}_{t_N} \circ \cdots \circ \psi^{k,k+1}_{t_{k+1}}(y^k) & \text{, if } 1 \leq k < N \\ y^N & \text{, if } k = N \end{cases}$$

where for all $l \neq k$,

$$\psi^{k,l}_s(u) = x^k_s(u) + W_{l \to k} \tag{7.3.4}$$

and $x^k_s(u)$ means the solution of the deterministic dynamic $\dot{x}^k_s = -\alpha_i(x^k_s),\ x^k_0 = u$.

Remark 7.4. *In the definition of $\Psi^k_{s_N}(y)$, we note that it depends only on y^k. There we have for all $i \neq j$ we have*

$$\frac{\partial \Psi^i_{s_N}}{\partial y^j} = 0.$$

Proposition 7.5. *For all $1 \leq k \leq N$ let α_k be globally Lipschitz function. For all $y \in \mathbb{R}^N_+$, there exists an open neighborhood $\mathcal{B}$ of y such that $\Psi_{s_N} : \mathcal{B} \to \mathbb{R}^N$ is a local diffeomorphism.*

Proof. Let $J_{\Psi_{s_N}(y)}$ be the Jacobian matrix of $\Psi_{s_N}(y)$. Using the remark 7.4 we have :

$$\det(J_{\Psi_{s_N}(y)}) = \det \begin{pmatrix} \frac{\partial \Psi^1_{s_N}(y)}{\partial y^1} & \cdots & \frac{\partial \Psi^1_{s_N}(y)}{\partial y^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Psi^N_{s_N}(y)}{\partial y^1} & \cdots & \frac{\partial \Psi^N_{s_N}(y)}{\partial y^N} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial \Psi^1_{s_N}(y)}{\partial y^1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\partial \Psi^N_{s_N}(y)}{\partial y^N} \end{pmatrix}.$$

We obtain $\det(J_{\Psi_{s_N}(y)}) \neq 0$ if and only if $\prod_{j=1}^N \frac{\partial \Psi^j_{s_N}(y)}{\partial y^j} \neq 0$ that is $\frac{\partial \Psi^j_{s_N}(y)}{\partial y^j} \neq 0, \forall 1 \leq j \leq N$. It is obvious to see that $\frac{\partial \Psi^N_{s_N}(y)}{\partial y^N} = 1$.
For all $1 \leq j \leq N-1$, we have :

$$\frac{\partial \Psi^j_{s_N}}{\partial y^j}(y) = \prod_{i=1}^{N-j} \exp\left(- \int_{s_{N-i}}^{s_{N-(i-1)}} \alpha'_j \left(x^j_s \left(\Psi^j_{s_{N-i}}(y) \right) \right) ds \right) \neq 0. \tag{7.3.5}$$

It means that $|\det(J_{\Psi_{s_N}(y)})| \neq 0$ then $\Psi_{s_N}(y)$ is a local diffeomorphism.

Localizing, we may therefore conclude that for each y there exists $\mathcal{B}$ such that $\Psi_{s_N} : \mathcal{B} \to \Psi_{S_N}(\mathcal{B})$ is a diffeomorphism. □

Proof of theorem 7.1. Let $\varepsilon > 0$ fixed. We will work on the event

$$E = \{S_1 \leq \varepsilon, \cdots, S_{n+1} - S_n \leq \varepsilon, \forall n < N : (I_1, \cdots, I_N) = (1, \cdots, N)\}.$$

In particular, on E, the index I_n of the $n-$th neuron is equal to n for all $n \in \{1, \cdots, N\}$.

Knowing that the first jump takes place at time $S_1 = s_1$, the probability that the index I_1 of the first jump is equal to 1 is given by :

$$\mathbb{P}(I_1 = 1 | S_1 = s_1) = \mathbb{P}(S_1^1 < S_1^j, \forall j \neq 1) = \frac{\beta_1(x_{s_1}^1(x^1))}{\sum_{j=1}^N \beta_j(x_{s_1}^j(x^j))}.$$

We want to compute, $\mathbb{P}(I_1 = 1, \cdots, I_N = N | S_1 = s_1, S_2 = s_2, \cdots, S_N = s_N)$. To obtain a compact formula, using formula (7.3.4) we define

$$\phi_j^k(x^k, y^k, s_1, \cdots, s_N) = \begin{cases} \psi_{t_j}^{k,j} \circ \cdots \circ \psi_{t_{k+1}}^{k,k+1}(y^k), & \text{if } 1 \leq k \leq j-1 \\ \psi_{t_j}^{k,j} \circ \cdots \circ \psi_{t_1}^{k,1}(x^k), & \text{if } j \leq k \leq N \end{cases}$$

giving the states of neuron k at time S_j depending on whether neuron k jumped before or after time S_j.

Let

$$x_j^k = x_{t_j}^k(\phi_{j-1}^k(x^k, y^k, s_1, \cdots, s_N)$$

be the state of neuron k before the $j-th$ jump. We know that as long as neuron k has not yet jumped, it receives each time a quantity $W_{j\to k}$, $\forall j \neq k$ from the other neurons that jumped before it. So knowing all the jump times where other neurons jumped, we have :

$$\mathbb{P}(I_1 = 1, \cdots, I_N = N | S_1 = s_1, S_2 = s_2, \cdots, S_N = s_N) = \\ \frac{\beta_1(x_{s_1}^1(x^1))}{\sum_{i=1}^N \beta_i(x_{s_1}^i(x^i))} \int_{\mathbb{R}_+^{N-1}} \frac{\prod_{i=2}^N \beta_i(x_i^i)}{\prod_{i=2}^N (\sum_{k=1}^N \beta_k(x_i^k))} \prod_{k=1}^{N-1} \mathbb{P}(Y^k \in dy^k).$$

For any Borel subset B of $\mathbb{R}^N$ we have

$$Q^N(x, B) \geq \mathbb{P}_x(Z_N \in B, E) = \int_{[0,\varepsilon]^N} dt_1 \cdots dt_N \int_{\mathbb{R}^N} F^1(dy^1) \cdots F^N(dy^N) \\ \times (\prod_{k=1}^N \beta_k(x_s^k)) e^{-\int_0^{s_N} \sum \beta_k(\Psi_t(y))dt} \mathbb{1}_B(\Psi_{s_N}(y)).$$

Remark that on the event E, $x_s^k \leq x^k + \sum_{j=1}^{k-1} W_{j\to k}$. Recall β_k is decreasing function and let $\mu_k = \inf_k\{\beta_k(x^k + \sum_{j=1}^{k-1} W_{j\to k}) : x \in K\}$ the lowerbound on K of $\beta_k(x^k + \sum_{j=1}^{k-1} W_{j\to k})$.
Using the fact that $\|\beta_i\|_\infty < \infty$ for all i, let $c = (\prod_{k=1}^N \mu_k) e^{-N\|\beta_i\|_\infty N\varepsilon}$. Then we have

$$Q^N(x, B) \geq c \int_{[0,\varepsilon]^N} dt_1 \cdots dt_N \int_{\mathbb{R}^N} F^1(dy^1) \cdots F^N(dy^N) \mathbb{1}_B(\Psi_{s_N}(y)). \tag{7.3.6}$$

Following the arguments of Benaïm et al. (2015), for any $t^* \leq N\varepsilon$, there exists a ball $B_r(t^*)$ of radius r, of center t^* and an open set $I \subset \mathbb{R}^N$ such that we can find for all $s_N \in B_r(t^*)$, an open set $W_{s_N} \subset \mathbb{R}^N$:

$$\tilde{\Psi}_{s_N} : \begin{cases} W_{s_N} \to I \\ y \mapsto \Psi_{s_N}(y) \end{cases}$$

is a diffeomorphism (see Benaïm et al. (2015), Lemma 6.2). In the above formula, $\tilde{\Psi}_{s_N}$ denotes the restriction of Ψ_{s_N} to W_{s_N}.This allows us to apply the theorem of a change of variables in the inequality (7.3.6).

$\alpha_j'\left(x_s^j\left(\Psi^j_{s_{N-i}}(y)\right)\right)$ is upper bounded since α_j is a global Lipschitz function. Then, for all $1 \leq j \leq N-1$ we obtain :

$$\frac{\partial \Psi^j_{s_N}}{\partial y^j}(y) \leq \exp(N-j)\varepsilon \|\alpha_j'\|_\infty.$$

Then, $\forall y \in W_{S_N}, c\,|\det(J_{\Psi_{s_N}(y)})|^{-1} \geq c' > 0$ and the inequality (7.3.6) becomes :

$$\begin{aligned} Q^N(x,B) &\geq c\int_{[0,\varepsilon]^N} dt_1 \cdots dt_N \int_{\mathcal{B}} F^1(dy^1)\cdots F^N(dy^N)\mathbb{1}_B(\Psi_{s_N}(y)) \\ &\geq c'\int_{B_r(t^*)} dt_1 \cdots dt_N \int_{W_{s_N}\cap\mathcal{B}} dy \mathbb{1}_B(\tilde{\Psi}_{s_N}(y))|\det(J_{\tilde{\Psi}_{s_N}(y)})| \\ &\geq c'\lambda(B_r(t^*))\int_I \mathbb{1}_B(x)dx \;=\; d\mathbb{1}_B(x)\nu(I) \end{aligned}$$

where $d = c'\lambda(B_r(t^*))$ with $\lambda(B_r(t^*))$ the Lebesgue measure of the ball $B_r(t^*)$ and $\nu(I)$ the uniform measure of I.

□

Corollary 7.2. *If for all $k \leq N$, β_k is strictly lower bounded and bounded, then the process is recurrent.*

Remark 7.5. *When β_k is strictly lower-bounded and bounded, we can notice that the lower bound obtained in theorem 7.1 holds on the whole state space $\mathbb{R}_+$, that is, without 1_K. This allows us to have the global lower bound $Q^N(x,dy) \geq d\nu(dy)$ and thus the uniform ergodicity of the process.*

7.4 Perfect simulation

In this section, we consider a framework with an infinity of neurons indexed by $\mathbb{Z}$. We want to build a perfect simulation algorithm to show in another way the recurrence of our process under certain conditions. Let $\mathcal{V}_{\cdot\to i} = \{j : W_{j\to i} \neq 0\}$ and $\mathcal{V}_{i\to\cdot} = \{j : W_{i\to j} \neq 0\}$ be the incoming and out-coming neighbourhoods of the neuron i (see Comets et al. (2002) and Galves and Löcherbach (2013)).

We consider the case where each neuron has a finite number of neighbours.
We assume throughout this section that for all i the jump rate $\beta_i(x^i)$ is bounded, that is, $\beta_i(x^i) \in [\beta_*, \beta^*]$ for all $x^i > 0$, where $0 < \beta_* < \beta^* < \infty$.

The following variables will be used to write the perfect simulation algorithm :

- T is the time vector
- P is the matrix of positions where each row of this matrix represents the different positions of the N neurons at a fixed instant
- I is the vector which represents the index of the neuron which spikes.

We fix a neuron $i \in \mathbb{Z}$ and in what follows we are interested in finding the state of i at time 0 in the stationary regime, that is, assuming that the process starts from $-\infty$. To do so we explore the past in order to determine all sets of indices and times which affect the value of neuron i at time 0.
To explain what we mean by this, let us consider the following example where the interactions are given in the case of nearest neighbors. In the following example, the red dots represent possible jumps and the blue stars represent sure jumps. The sure and possible jumps are the same as in Algorithm 7.1.

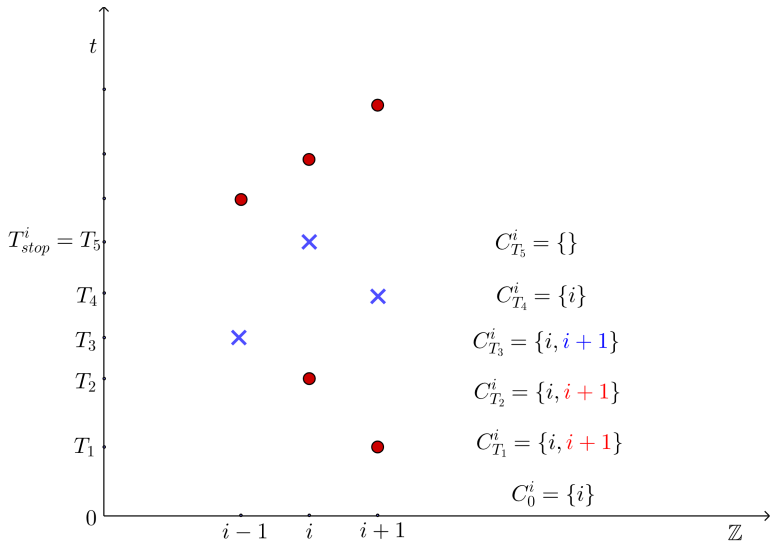


Figure 7.3 – Clan of ancestors if neuron i

In this example, we have fixed a neuron i in $\mathbb{Z}$ at time 0 and we say that the clan of ancestors of neuron i is reduced to neuron i itself. It is assumed that the space of neurons is reduced to $i-1, i, i+1$. Then, at time T_1, neuron $i+1$ makes a possible jump. We record the time T_1 and we add the neuron $i+1$ to the clan of the ancestors of the neuron i. At time T_2 neuron i makes a possible jump. As neuron i is already in the ancestor clan then the clan remains unchanged and we download the time T_2. At time T_3 neuron $i-1$ makes a sure jump. We register the time T_3 and the neuron $i-1$ but the clan remains unchanged. At time T_4 the neuron $i+1$ makes a sure jump and as the neuron $i+1$ is already in the clan, we remove from the clan and only the neuron i remains in the clan. At time T_5 neuron i makes a sure jump and as neuron i is already in the clan, we remove from the clan and the clan becomes empty. Our algorithm stops the first time the clan becomes empty. In the following algorithm we will work in a general case.
The set of neurons thus constructed will be called the ancestor clan of neuron i. (see Galves and Löcherbach (2013), Galves et al. (2010)). The clan of ancestors is a process that evolves in time by successive jumps. We start with $C^i_0 = \{i\}$ and in the following we will define the updates of this process at the times of the jumps. More precisely we do the following :

Algorithm 7.2 (Backward procedure). *1. We simulate , $\forall\, l \in \mathbb{Z}$, $N_t^{l,s}$ and $N_t^{l,p}$ two Poisson processes with respective intensities β_* and $\beta^* - \beta_*$. The jump times of $N_t^{l,s}$ and $N_t^{l,p}$ are respectively $T_n^{l,s}$ and $T_n^{l,p}$ for the neuron l after n jumps.*

2. Let $i \in \mathbb{Z}$ be fixed and $T_1 = \inf\{T_1^{j,r} > 0 :\ j \in \mathcal{V}_{\cdot\to i},\ T_1^{i,r} > 0\}$ where $r \in \{p, s\}$ and $\mathcal{V}_{\cdot\to i}$ is the incoming neighborhood of i.
- If $T_1 = T_1^{j,p}$, we set $C_{T_1}^i = \{i, j\}$ and we set $I_1 = j$.

- If $T_1 = T_1^{j,s}$, we set $C_{T_1}^i = \{i\}$ and $I_1 = j$.

- If $T_1 = T_1^{i,p}$, we set $C_{T_1}^i = \{i\}$ and we set $I_1 = i$.

- If $T_1 = T_1^{i,s}$, we set $C_{T_1}^i = \emptyset$ and we stop the algorithm. In this case we set $I_1 = i$.

3. Suppose T_n is the $n-th$ jump time of $C_{T_n}^i$. Then,

$$T_{n+1} = \inf\{T_m^{j,s}, T_m^{j,p} > T_n : \exists l \in C_{T_n}^i,\ j \in \mathcal{V}_{\cdot\to l},\ T_m^{k,p},\ T_m^{k,s} > T_n,\ k \in C_{T_n}^i\}.$$

- If $T_{n+1} = T_m^{j,s}$ we set $I_{n+1} = j$ and then $C_{T_{n+1}}^i = C_{T_n}^i$.

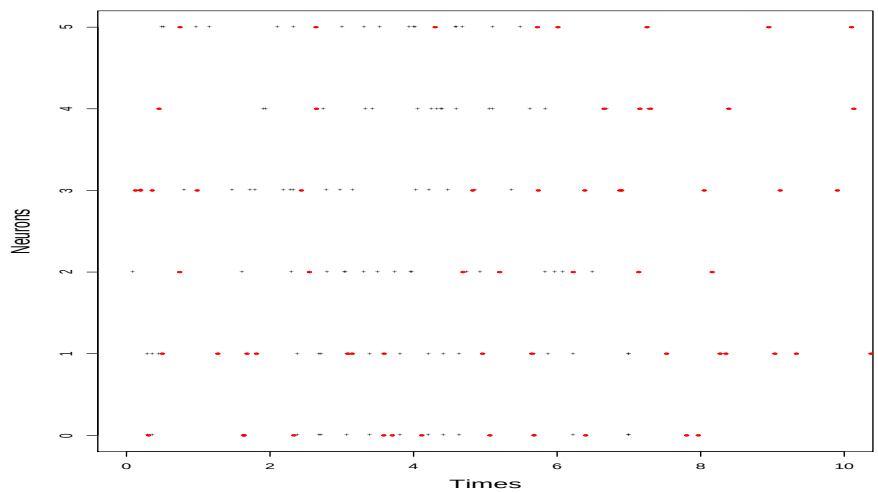
- If $T_{n+1} = T_m^{j,p}$ we set $I_{n+1} = j$ and then $C_{T_{n+1}}^i = C_{T_n}^i \cup \{j\}$.

- If $T_{n+1} = T_m^{k,p}$ we set $I_{n+1} = k$ and then $C_{T_{n+1}}^i = C_{T_n}^i$.

- If $T_{n+1} = T_m^{k,s}$ we set $I_{n+1} = k$ and then $C_{T_{n+1}}^i = C_{T_n}^i \setminus \{k\}$ where $k \in C_{T_n}^i$.
We stop the procedure at time $T_{stop}^i = \inf\{t : C_t^i = \emptyset\}$.

To ensure that the algorithm stops it will be necessary to find a criterion so that $T_{stop}^i < \infty$. This will be done in Theorem 7.2 below. The above algorithm is called the backward procedure.

The following graph is an example for $N = 6$ neurons of simulations of successive jump times (certain or possible). The certain jumps are made according to the Poisson process $N_t^{l,s}$ and the possible jumps according to the Poisson process $N_t^{l,p}$ with respective intensities β_* and $\beta^* - \beta_*$.



The process $N_t^{l,s}$ is represented in red and the process $N_t^{l,p}$ in blue. In this example, the number of iteration is $n = 50$, the intensities are $\beta_* = 3$ and $\beta^* - \beta_* = 1$.

In the following we will write a forward procedure of the process in case where each neuron has a finite number of neighbors and in case $T^i_{stop} < \infty$.
For this we define :

$$N^i_{stop} = \inf\{n > 0 : C^i_{T_n} = \emptyset\},\ \bar{\mathcal{C}}^i = \cup_{n=0}^{N^i_{stop}} C^i_{T_n} \text{ and } \partial_{ext}(C^i_t) = \{j \notin C^i_t : \exists k \in C^i_t, W_{j\to k} > 0\}$$

where N^i_{stop} is the number of steps of the backward procedure, $\bar{\mathcal{C}}^i$ is the union of all clans of ancestors up to N^i_{stop} and $\partial_{ext}(C^i_t)$ is the set of neurons not belonging to the clan of ancestor of neuron i but having an interaction with at least one neuron in the ancestor clan of neuron i.

In this algorithm, we will rely on the a priori realizations of the processes $N_t^{i,s}$ and $N_t^{i,p}$.

Algorithm 7.3 (Forward procedure)**.** *1. We initialize the set of sites for which the decision to accept can be made by*

$$\mathcal{S}^i = \{(I_m, T_m) \in \bar{\mathcal{C}}^i \times \mathbb{R}_+, C^{I_m}_{T_m} = \emptyset\}$$

For $n = N^i_{stop}$ we have $P_n^{I_n} \sim F^{I_n}$. Starting from $n \to n-1$:

2. If $(I_{n-1}, T_{n-1}) \in \mathcal{S}^i$ then $P_{n-1}^{I_{n-1}} \sim F^{I_{n-1}}$.
- If for $j \in \mathcal{V}_{I_{n-1}\to.}$, we have $j \in C^i_{T_{n-1}}$ then

$$P^j_{n-1} = x^j_{T_n - T_{n-1}}(P^j_n) + W_{I_{n-1}\to j}$$

- If for $j \notin \mathcal{V}_{I_{n-1}\to.}$, we have $j \in C^i_{T_{n-1}}$ then

$$P^j_{n-1} = x^j_{T_n - T_{n-1}}(P^j_n)$$

3. If $k := I_{n-1} \in \partial_{ext}(C^i_{T_{n-1}})$, we have

$$P^l_{n-1} = x^l_{T_n - T_{n-1}}(P^l_n) + W_{k\to l}$$

where there exists l such that $k \to l \in C^i_{T_{n-1}}$

4. If $(I_{n-1}, T_{n-1}) \in (\bar{\mathcal{C}}^i \times \mathbb{R}_+) \setminus \mathcal{S}^i$ then :
- We decide according to the probabilities

$$p = \frac{\beta_{I_{n-1}}(x^{I_{n-1}}_{T_n - T_{n-1}}(P^{I_{n-1}}_n)) - \beta_*}{\beta^* - \beta_*}$$

to accept the presence of a spike of neuron I_{n-1}.
We update

$$\mathcal{S}^i \leftarrow \mathcal{S}^i \cup \{(I_m, T_m) \in \bar{\mathcal{C}}^i \times \mathbb{R}_+, C^{I_m}_{T_m} \subset \mathcal{S}^i\}$$

and go back to step 2.
- Else with the probabilities $1 - p$ we reject the presence of a spike of neuron I_{n-1} and $P^{I_{n-1}}_{n-1} = x^{I_{n-1}}_{T_n - T_{n-1}}(P^{I_{n-1}}_n)$.

We consider all the elements of $\mathcal{S}^i$ and we always start with the last element to get out of the clan. The update of $\mathcal{S}^i$ allows us to start the procedure again.

We stop the procedure when all the elements of $\bar{\mathcal{C}}^i$ are filled.

Remark 7.6. - *The output of the above algorithm 7.3 is a sample of the process in its stationary state. For more on this, see page 21 of Galves et al. (2013).*

- For any site $(i,t) \in \mathbb{Z} \times \mathbb{R}_+$, C_t^i is a Markov jump process taking values in the finite subset of $\mathbb{Z}$ (see Galves et al. (2010)) and its infinitesimal generator is given by

$$A^{clan}g(C) = \sum_{j \in C} \beta_*[g(C \setminus \{j\}) - g(C)] + \sum_{j \in \partial_{ext}(C)} (\beta^* - \beta_*)[g(C \cup \{j\}) - g(C)]$$

where g is a test function.

Proposition 7.6. *Let $d_j = \min_{x^j} \beta_j(x^j)$, $d^j = \max_{x^j} \beta_j(x^j)$ and $b_j = \sum_{k \to j}(d^k - d_k)$. If $b_j < d_j$ for all j then T^i_{stop} is finite almost surely.*

Proof. We shall construct a process $Z = (Z_n)_n$ such that for all n, $|C^j_{T_n}| \leq Z_n$ and such that Z_n evolves as follows : with probability $\frac{\sum_{j \in Z_n} d_j}{\sum_{j \in Z_n}(b_j + d_j)}$ we have $Z_{n+1} = Z_n - 1$ and with probability $\frac{\sum_{j \in Z_n} b_j}{\sum_{j \in Z_n}(b_j + d_j)}$ we have $Z_{n+1} = Z_n + 1$. If $b_j < d_j$ for all j then almost surely $\lim_{n \to \infty} Z_n = 0$. Which implies that $\lim_{n \to \infty} |C^j_{T_n}| = 0$ this implies T^j_{stop} is finite almost surely.

□

In this general case where a neuron has a number of neighbours (more than two neighbours) with which it interacts, we can say no more than proposition 7.6. Thus, in the following, we put ourselves in the case where each neuron i has exactly two neighbours so that the neuron i interacts only with the neurons $i+1$ and $i-1$. In other words, the incoming neighbourhood of i is $\mathcal{V}_{\cdot \to i} = \{i+1, i-1\}$.

Algorithm 7.4 (Backward procedure).

1. *We simulate , $\forall\, l \in \mathbb{Z}$, $N_t^{l,s}$ and $N_t^{l,p}$ two Poisson processes with respective intensities β_* and $\beta^* - \beta_*$. The jump times of $N_t^{l,s}$ and $N_t^{l,p}$ are respectively $T_n^{l,s}$ and $T_n^{l,p}$ for the neuron l after n jumps. The jump times $T_n^{l,s}$ will be considered as times of sure jumps (counted by the process $N_t^{l,s}$) and the jump times $T_n^{l,p}$ will be considered as times of possible jumps (counted by the process $N_t^{l,p}$)*

2. *Let $i \in \mathbb{Z}$ fix and $T_1 = \inf\{T_1^{i\pm1,s}, T_1^{i\pm1,p}, T_1^{i,s}, T_1^{i,p}\}$.*
   - If $T_1 = T_1^{i\pm1,p}$, we set $C^i_{T_1} = \{i, i \pm 1\}$. We put $I_1 = i \pm 1$.

   - If $T_1 = T_1^{i\pm1,s}$, we set $C^i_{T_1} = i$ and we put $I_1 = i \pm 1$.

   - If $T_1 = T_1^{i,p}$, we set $C^i_{T_1} = i$ and we put $I_1 = i$.

   - If $T_1 = T_1^{i,s}$, we set $C^i_{T_1} = \emptyset$ and we stop the algorithm. We put $I_1 = i$.

3. *Suppose T_n is the $n-$th jump time of $C^i_{T_n}$. We have :*

$$T_{n+1} = \inf\{T_m^{j,s}, T_m^{j,p} > T_n : |j - C^i_{T_n}| \leq 1,\ T_m^{k,p},\ T_m^{k,s} > T_n,\ k \in C^i_{T_n}\}.$$

- *If* $T_{n+1} = T_m^{j,p}$ *we set :*

$$\begin{cases} \text{If } j \in C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \\ \text{If } j \notin C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \cup \{j\} \end{cases}$$

- *If* $T_{n+1} = T_m^{k,s}$ *we set :*

$$\begin{cases} \text{If } k \in C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \setminus \{k\} \\ \text{If } k \notin C^i_{T_n}, & C^i_{T_{n+1}} = C^i_{T_n} \end{cases}$$

We update C^i_t *and start the procedure again. We stop the procedure at time* T^i_{stop} *where* $T^i_{stop} = \inf\{t : C^i_t = \emptyset\}$.

In this second algorithm, we can realize the acceptance/rejection procedure of the elements in the clans of ancestors already determined in the first algorithm.
We start with the positions for which the jumps are certains. Then, we keep in mind all the jumps that we accept regardless of everything. During the algorithm, we will gradually update all the certains positions of the jumps.

Algorithm 7.5 (Forward procedure)**.** *1. We initialize the set of sites for which the decision to accept can be made by*

$$\mathcal{S}^i = \{(I_m, T_m) \in \bar{\mathcal{C}}^i \times \mathbb{R}_+, C^{I_m}_{T_m} = \emptyset\}$$

The sites in this set are those that are kept in memory independently of everything.

For $n = N^i_{stop}$ *we have* $P_n^{I_n} \sim F^{I_n}$,. *Starting from* $n \to n-1$:

2. *If* $(I_{n-1}, T_{n-1}) \in \mathcal{S}^i$ *then* $P_{n-1}^{I_{n-1}} \sim F^{I_{n-1}}$.

- *If* $j \in \mathcal{V}_{I_{n-1}\to.}$, *and* $j \in C^i_{T_{n-1}}$ *then*

$$P^j_{n-1} = \begin{cases} x^j_{T_n - T_{n-1}}(P^j_n) + W_{I_{n-1}\to j}, & \text{if } j = I_{n-1} \pm 1 \\ x^j_{T_n - T_{n-1}}(P^j_n), & \text{if } j \neq \{I_{n-1}, I_{n-1} \pm 1\} \end{cases}$$

3. *If* $k := I_{n-1} \in \partial_{ext}(C^i_{T_{n-1}})$ *we have*

$$P^{k\pm1}_{n-1} = x^{k\pm1}_{T_n - T_{n-1}}(P^{\pm1}_n) + W_{k\to\pm1}$$

4. *If* $(I_{n-1}, T_{n-1}) \in (\bar{\mathcal{C}}^i \times \mathbb{R}_+) \setminus \mathcal{S}^i$ *then we decide according to the probabilities*

$$p = \frac{\beta(x^{I_{n-1}}_{T_n - T_{n-1}}(P^{I_{n-1}}_n)) - \beta_*}{\beta^* - \beta_*}$$

to accept the presence of a spike of neuron I_{n-1}.

5. *If we accept the jump, we update*

$$\mathcal{S}^i \leftarrow \mathcal{S}^i \cup \{(I_m, T_m) \in \bar{\mathcal{C}}^i \times \mathbb{R}_+, C^i_t \subset \mathcal{S}^i\}$$

and we return to (2). *Else we continue the algorithm.*
Once all the elements of $\bar{\mathcal{C}}^i$ *are filled, our perfect simulation algorithm stops.*

Indeed, the whole procedure makes sense only if $T^i_{stop} < \infty$ almost surely. The following theorem gives conditions on the finitude of the extinction time.

Theorem 7.2. *We set* $\delta = \frac{\beta_*}{\beta^* - \beta_*}$. *There exists a critical value* $0 < \delta_c < \infty$ *such that :*

- *if* $\delta > \delta_c$, *then the extinction time is finite almost surely that is,* $\mathbb{P}(\forall i,\ T^i_{stop} < \infty) = 1$
- *if* $\delta < \delta_c$, *then the extinction time is infinite with a positive probability that is,* $\mathbb{P}(\forall i,\ T^i_{stop} = \infty) > 0$.

Proof. We first show that $T^i_{stop} < +\infty$ almost surely for sufficiently large δ. We observe that we can upper bound $|C^i_t|$ by Z_t almost surely for all $t \geq 0$ where $Z_0 = 1$ and $(Z_t)_{t\geq 0}$ is a branching process. With a rate $n(\beta^* - \beta_*)$ the transition from Z_t is from n to $n+1$ and with a rate $n\beta_*$ this transition is from n to $n-1$.
We can therefore define for any bounded test function f, the associated infinitesimal generator of $(Z_t)_{t\geq 0}$ as follows :

$$Af(n) = n[(\beta^* - \beta_*)\,(f(n+1) - f(n)) + \beta_*\,(f(n-1) - f(n))].$$

Take $f(n) = n$, we obtain :

$$Af = f[(\beta^* - \beta_*) - \beta_*] = f(\beta^* - \beta_*)(1 - \delta).$$

Then, for $\delta > 1$, we have $Af(n) = -cf(n)$ where $-c = (\beta^* - \beta_*)(1-\delta)$. Assuming $x_t = \mathbb{E}(f(Z_t))$ and using the Itô formula, we have :

$$x_t = x_0 + \mathbb{E}\int_0^t Af(Z_s)ds = x_0 - c\int_0^t x_s ds = x_0 e^{-ct}.$$

Therefore, when $t \to \infty$, we have $x_t \to 0$. Which implies that if $\delta > 1$, $\mathbb{P}(T^i_{stop} < \infty) \geq \mathbb{P}(\lim_{t\to\infty} Z_t = 0) = 1$ thus ensuring that $\delta_c \leq 1$.

We now show that for all $\delta < \delta_c$, $T^i_{stop} = +\infty$ with positive probability.
For this proof, we will use the classical graphical construction of C^i_t. We work within the space-time diagram $\mathbb{Z} \times [0, \infty[$. For each $i \in \mathbb{Z}$, we consider $N^{i,s}_t$ and $N^{i,p}_t$ two independent Poisson processes with respective intensities β_* and $\beta^* - \beta_*$. The jump times of $N^{i,s}_t$ and $N^{i,p}_t$ are respectively $T^{i,s}_n$ and $T^{i,p}_n$ for the neuron i after n jumps. For each $i \in \mathbb{Z}$, we draw graphical sequences as follows. First draw arrows pointing from $(i-1, T^{i,p}_n)$ to $(i, T^{i,p}_n)$ and from $(i+1, T^{i,p}_n)$ to $(i, T^{i,p}_n)$ for all $n \geq 1$, $i \in \mathbb{Z}$. Second, δ's at all $(i, T^{i,s}_n)$, for all $n \geq 1$, $i \in \mathbb{Z}$. We also suppose that time is going up which implies that we thus obtain a random graph $\mathcal{P}$. Let us say that there is a chain of vertical upward and horizontal directed edges in the random graph that leads from $(i, 0)$ to (j, t) (with $j \in \{i+1, i-1\}$) without passing through a δ. Notice that C^i_t is the set of the clan of ancestors of site (i, t), that is

$$C^i_t = \{j :\ \text{there is a path from } (i,0) \text{ to } (j,t) \text{ for } j = i \pm 1\}.$$

It is obvious to notice that $T^i_{stop} < \infty$ if and only if $\bar{\mathcal{C}}^i = \cup_{t\geq 0} C^i_t$ is a finite set. We will therefore show that $\mathbb{P}(T^i_{stop} < \infty) = \mathbb{P}(|\bar{\mathcal{C}}^i| < \infty) < 1$ for sufficiently small values of δ using classical contour techniques. (see Griffeath (1981).)

For this, on $|\bar{\mathcal{C}}^i| < \infty$, we draw the contour of $\bar{\mathcal{C}}^i$ as follow.
Starting from $(i - \frac{1}{2}, 0)$. Let Γ be a possible path of the graph $\mathcal{P}$. Γ consists of $4n$ alternating vertical and horizontal edges for some $n \geq 1$ which we encode as a succession of direction vectors $(D_1, \cdots, D_{2n})$. Each of the D_i can be one the seven triplets

$$dld, drd, dru, ulu, uru, urd, dlu,$$

where d, u, l and r stand for down, up, left and right, respectively. Note that uld cannot occur in a possible path Γ because the direction of uld is counter-clockwise.

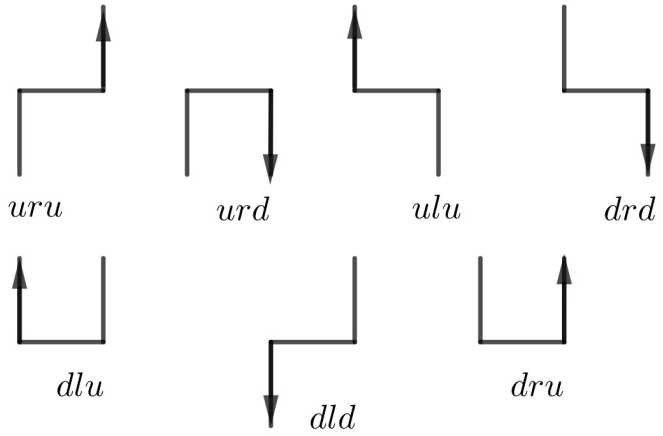


Figure 7.4 – Possible directions

We start at $(i - \frac{1}{2}, 0)$ and move clockwise around the curve.
The two figures below show examples of possible paths for $n = 3$ and $n = 4$. Figure.1 shows a possible path with $n = 3$ and in this case we have

$$\Gamma : ulu, ulu, urd, drd, drd, dlu.$$

For $n = 4$, Figure.2 gives

$$\Gamma : ulu, ulu, urd, drd, dru, urd, dld, dlu.$$

Writing $N(dld), N(drd), \cdots$ for the number of appearances of the different direction vectors, we have that $N(dlu) = 1$ (dlu is the last triplet of which appears exactly one single time) and

$$N(dru) = N(urd) - 1 \leq n/2, \;\; N(drd) + N(dru) + N(uru) + N(urd) = n.$$

(for more details, see Ferrari et al. (2018)).

We first observe that the occurrence of either uru, urd, or drd can be upper bounded by δ. This is due the fact that the probability associated with uru or drd is $\frac{\delta}{1+2\delta}$ and that of urd is $\frac{\delta}{2+\delta}$. In the same way, we observe that the occurrence of either dld, ulu or dlu can be upper bounded by 1. Indeed, the associated probability with

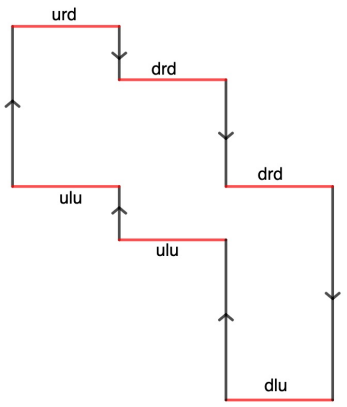


Figure 1

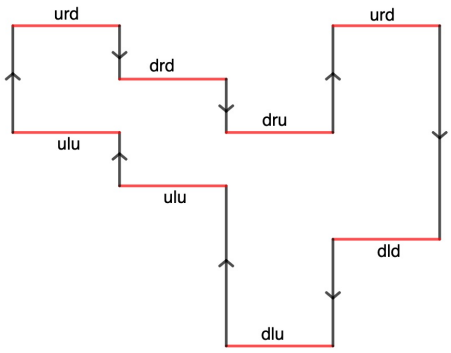


Figure 2

Figure 7.5 – Contour of clan of ancestors in the case where $n = 3$ and $n = 4$ respectively.

its directions is $\frac{1}{1+\delta}$. Therefore we obtain the following list of upper bounds

$$
\begin{aligned}
uru &\text{ occurs with probability at most } \delta \\
urd &\text{ occurs with probability at most } \delta \\
drd &\text{ occurs with probability at most } \delta \\
dru &\text{ occurs with probability at most } 1 \\
dld &\text{ occurs with probability at most } 1 \\
ulu &\text{ occurs with probability at most } 1 \\
dlu &\text{ occurs with probability at most } 1.
\end{aligned}
$$

In the above list, we have upper bounded the probability associated with dru which is given by $\frac{\delta}{3\delta} = \frac{1}{3}$, by 1.
For a given contour having $4n$ edges, with $n \geq 2$, its probability is therefore upper bounded by

$$\delta^{N(drd)+N(uru)+N(urd)} = \delta^{n-N(dru)} \leq \delta^{n/2}.$$

Indeed, for each triplet we have 4 possible choices. The first entry of a given triplet is always fixed by the previous triplet in the sequence, and for the first triplet D_1 the first entry is always u.
Then, for $n = 1$, the probability of appearance of a contour of length 4 is equal to $\mathbb{P}(D_1 = urd) = \frac{\delta}{2+\delta} \leq \delta$.
We also have, for $n = 2$, the probability of appearance of a contour of length 8 is equal to

$$
\begin{aligned}
&\mathbb{P}(D_1 = ulu,\ D_2 = urd,\ D_3 = drd) + \mathbb{P}(D_1 = ulu,\ D_2 = uru,\ D_3 = urd) + \\
&\mathbb{P}(D_1 = uru,\ D_2 = urd,\ D_3 = dld) + \mathbb{P}(D_1 = urd,\ D_2 = drd,\ D_3 = dld) \leq 4\delta^2.
\end{aligned}
$$

Remark 7.7. *In the above probabilities, we have not put the direction $D_4 = dlu$ because it is a certain direction. It is common to all possible paths and its probability of occurrence is 1.*

Therefore, a very approximate upper bound on the total number of possible triplets $(D_1, \cdots, D_{2n})$ is given by $4^{2n} = 16^n$. We get for all $\delta < \frac{1}{(16)^2}$,

$$\mathbb{P}(T^i_{stop} < \infty) \leq \delta + 4\delta^2 + \sum_{n\geq 3}(16)^n \delta^{n/2} = \delta + 4\delta^2 + \frac{(16\sqrt{\delta})^3}{1-16\sqrt{\delta}}.$$

We set $\phi : \delta \mapsto \phi(\delta) = \delta + 4\delta^2 + \frac{(16\sqrt{\delta})^3}{1-16\sqrt{\delta}}$. Then, $\mathbb{P}(T^i_{stop} < \infty) \leq \phi(\delta)$.
As $\delta \to 0$, $\phi(\delta) \to 0$ which implies that there exists δ_c such that $\phi(\delta_c) = 1$. As a consequence, $\mathbb{P}(T^i_{stop} < \infty) < 1$, $\forall\, 0 < \delta < \delta_c$.
We therefore conclude that δ_c exists and $0 < \delta_c < 1$.

□

7.4.1 Some simulations

We simulate the state $X_0(i)$ in the stationary regime for a fixed neuron $i \in \mathbb{Z}$ at time 0 and estimate its density. The main purpose of this simulation is to have an idea about the theoretical distribution of $X_0(i)$ and whether this distribution is impacted by the specification of F^i.

We denote by $\mathcal{D}$ the set of neurons which belong to a clan of ancestors of neuron i at a time t or to its neighborhood.
To do this, we apply the following algorithm :

Algorithm 7.6. *1. Initialize the family $\mathcal{V}_{.\to i}$ of non empty neighborhoods of the neuron i*

2. Initialize $C^i_0 = i$ the clan of ancestors of neuron i at time $t = 0$.

3. For all time $t > 0$ we let C^i_t the clan of ancestors of neuron i at time t

4. While $|C^i_t| > 0$ (where $|C^i_t|$ denotes the cardinality of C^i_t) do
- Determine the next jump time $t_{next} > t$ in the clan of ancestors of neuron i at time t_{next} and in $\partial_{ext}(clan)$, the correspondant neuron j and the nature of jump
- If neuron $j \in C^i_t$ and the jump is sure, then $C^i_{t_{next}} = C^i_t \setminus \{i\}$
- If $j \in C^i_t$ and the jump is possible $C^i_{t_{next}} = C^i_t$
- If $j \in V(C^i_t)$ (where $V(C^i_t) := \cup_{j\in C^i_t} \mathcal{V}_{.\to j}$) and the jump is sure, then $C^i_{t_{next}} = C^i_t$
- If $j \in V(C^i_t)$ and the jump is possible $C^i_{t_{next}} = C^i_t \cup \{j\}$
- We update $t \leftarrow t_{next}$
end While.

5. We determine the chronological list of the different jump times from 0 to the last time which makes the clan empty.
- For each of these jump times, we indicate the associated neuron and the nature of the jump.
- If the jump is sure, we simulate a random state following a distribution F^i at the neuron associated with this jump time.

6. *We set $m = \infty$. While $m > 0$ do*

   - Let m be the rank of the last possible jump time T_m of $\mathcal{D}$ in the chronology of jump times. Let k be the neuron associated with this jump.

7. *We determine the rank r of the last certain jump time $T_r > T_m$ of k in the chronology of jump times. The state of k is determined recursively from its state at time T_r to its state at time T_m as follows :*

   - For $s \in \{1, \cdots, r-m-1\}$ let $x =$ state of k at time T_{r-s+1} .

   - Let $dt = T_{r-s+1} - T_{r-s}$ and j the neuron associated with the jump time T_{r-s}. The state of k at time T_{r-s} is $x^k_{-dt}(x) + W_{j\to k} * 1_{\{\text{sure jump of } j \text{ at } T_{r-s}\}}$ with $W_{j\to k}$ the inhibition weight of j on k.

   - We determine rather the occurence is effective or not of the jump of k at time T_m thanks to its state at time T_{m+1}. [*]

   - If the jump is effective, we simulate a random state for k at time T_m following a distribution F^i. Otherwise, we determine the state of k at time T_m as $x_{T_m - T_{m+1}}(x)$ where $x =$ state of k at time T_{m+1}. Let m be the new rank of the last possible jump time of $\mathcal{D}$ and repeat the procedure.

   end While.

   Remark 7.8. *After this step, we know the exact nature of all jumps.*

8. *Determine for neuron i its first safe jump time T_n where n is the rank of this time in the chronology of jump times.*

9. *The state of neuron i is determined recursively from its state at time T_n to T_0 as follows :*

   - For $s \in \{1, \cdots, n-1\}$ let $x =$ state of neuron i at time T_{n-s+1}.

   - Let $dt = T_{n-s+1} - T_{n-s}$ and j the neuron associated with the jump time T_{n-s}. The state of neuron i at time T_{n-s} is $x^i_{-dt}(x) + W_{j\to i} * 1_{\{\text{sure jump of } j \text{ at } T_{r-i}\}}$ with $W_{j\to i}$ the inhibition weight of j on i.

   Remark 7.9. *The last value determined is the initial state of neuron i.*

Remark 7.10. *This algorithm is inspired by Galves et al. (2013) (page 20-21) which shows that if we find the state of a fixed neuron $i \in \mathbb{Z}$ at time 0, it is necessarily the state of the neuron i in its stationary regime. The algorithm is not a proof in itself, but allows to have an idea of the theoretical distribution of the value of neuron i at time 0 in its stationary regime.*

In the three following examples we consider $\alpha_i(x) = x$, $\beta_i(x) = 3 + \mathbb{1}_{\{x \leq 2\}}$, $W_{i\to j} = 1$. To verify if the distribution of inhibition state depends on the distribution F^i, we consider three different distributions for F^i that are $\mathcal{E}(1)$, $\mathcal{E}(10)$ and $0.5\delta_1 + 0.5\delta_2$. We simulate, with the algorithm described above $N = 1000$ values for the inhibition state . We then estimate non-parametrically the distribution of the inhibition state in these three cases of distribution F^i and we compare them.

The stationary distribution of the process in the three following cases seems to be continuous. We do not provide a proof here, this is outside the scope of this paper.

∗. The jump occurs with a Bernoulli distribution with parameter $(\beta(x^k_{-dt}(x)) - \beta_*)/(\beta^* - \beta_*)$

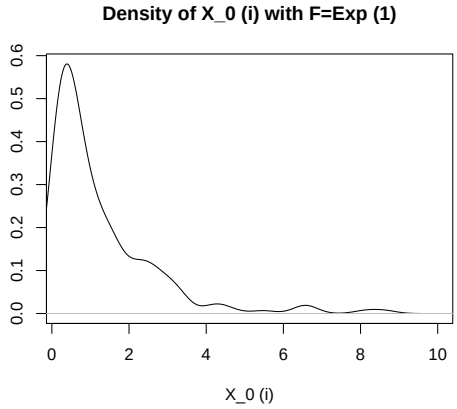


Figure 7.6 – Density of $X_0(i)$ with $F = \exp(1)$

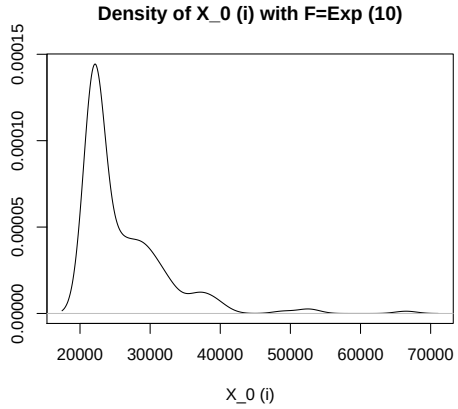


Figure 7.7 – Density of $X_0(i)$ with $F = \exp(10)$

We can remark that the distribution of state of inhibition $X_0(i)$ in stationary regime is concentrated in the interval $(0, 4)$ when $F^i = \mathcal{E}(1)$ whereas this distribution is rather concentrated on the interval $(20000, 40000)$ when $F^i = \mathcal{E}(10)$. This shows that these two distributions of state $X_0(i)$ are different.

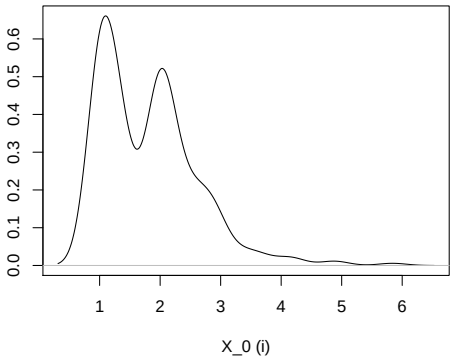


Figure 7.8 – Density of $X_0(i)$ with $F = 0.5\delta_1 + 0.5\delta_2$

In this example, the distribution of the state of inhibition $X_0(i)$ in stationary regime seems to be continuous although F^i is discrete. We do not provide a proof here, this is outside the scope of this paper. We observe two local extrema at 1 and 2 which are linked to the jumps because of the Dirac. These extrema suggest that jumps are very frequent in this process.

8

Discussions et perspectives

Ce travail de thèse se focalise sur l'étude théorique de processus aléatoires (processus avec catastrophes ou processus avec résurgence). Les résultats démontrés dans ce manuscrit trouvent leur place dans le cadre de la dynamique de population et de la modélisation de neurones.

La première partie de cette thèse est basée sur l'étude de chaînes de Markov à temps discret impliquant des catastrophes. Plus particulièrement le chapitre 2 permet la modélisation d'une chaîne de Markov avec catastrophes totales dans le sens où chaque fois qu'il y a une catastrophe dans la population alors toute la population est éteinte à ce moment. On s'est naturellement penché sur la question du premier temps d'extinction de la population et le temps moyen qu'il faudra pour que la population revienne à l'origine après sa première extinction.
Dans le chapitre 3 on prend une population de croissance (par exemple une population d'arbres dans une forêt) où lors d'une catastrophe chaque individu de la population a une probabilité de survie indépendante l'une de l'autre. On s'est intéressé dans ce cas au premier temps d'extinction de la population. On a pour cela utilisé le noyau de Green qui nous a permis de trouver une équation fonctionnelle pour le premier temps d'extinction.
Des modèles de croissance interrompue par des catastrophes, conduisant à une diminution modérée ou massive des individus ont été envisagés dans le chapitre 4. On s'intéresse principalement au cas où la probabilité d'occurrence d'une catastrophe dépend de la taille actuelle de la population. Ces genres de population sont confrontés à deux menaces majeures qui sont l'extinction et la surpopulation. Pour la menace d'extinction, la probabilité d'une extinction éventuelle et la loi du temps jusqu'à l'extinction éventuelle ont été expliquées. Pour la menace de surpopulation, le temps nécessaire pour atteindre un seuil élevé a été analysé en utilisant la théorie des valeurs extrêmes pour les chaînes de Markov.

Une perspective d'étude possible :

— mettre en réseau ces processus, c.-à-d. au lieu d'étudier une population, on pourra étudier un nombre fini de populations et mettre une dépendance (une interaction) au sein du réseau.

Dans la deuxième partie de ce manuscrit, nous avons étudié dans les chapitres 5 et 6

une classe particulière de processus de Markov déterministes par morceaux (PDMP) qui sont des versions catastrophes semi-stochastiques de modèles de croissance et de décroissance démographique déterministes. Nous avons discuté des conditions sous lesquelles ces processus sont récurrents ou transients et de la finitude du temps d'extinction ou du temps d'explosion de ces processus. Nous avons également prêté attention à la loi de hauteur des excursions. Cette étude a nécessité une hypothèse forte sur la séparabilité du noyau de saut du processus. On a montré explicitement la relation entre les modèles de type croissance et ceux de type décroissance de populations. Dans le modèle de croissance de populations, on a constaté qu'une fonction d'échelle n'existait pas ce qui était dû à l'asymétrie en 0 du processus. À la question portant sur l'existence d'une fonction d'échelle dans le cas du modèle de décroissance de populations, on a pu montrer que sous certaines hypothèses, il existe une fonction d'échelle pour ce modèle.

Deux perspectives d'études possibles :

— se débarrasser de l'hypothèse de la séparabilité du noyau de saut,

— s'intéresser aux comportements en temps long.

Dans le cas du chapitre 7, on a mis en réseau un certain nombre (fini ou infini) de neurones. On s'est intéressé à la modélisation de l'état d'inhibition d'un neurone du réseau à un temps donné. On a trouvé des conditions sur la récurrence du système fini en utilisant une condition de Lyapunov et des conditions de Doeblin locales. Dans le cas du système infini, on s'est beaucoup basé sur les articles de Galves et al. (2010), Ferrari et al. (2018), Comets et al. (2002) et Griffeath (1981) pour trouver des critères de récurrence du système. On a pour cela construit graphiquement le processus à l'aide d'un algorithme de simulation parfaite. Cet algorithme a été scindé en deux procédures. La procédure directe qui est basée sur l'algorithme d'acceptation rejet et la procédure inverse qui est basée sur un retournement du temps. Cela a permis de montrer qu'il existait une valeur critique telle qu'en deçà de cette valeur le temps d'extinction du processus est fini presque sûrement.

Deux angles de travail possibles :

— l'approximation de la valeur critique,

— faire l'étude dans le cas de la modélisation de l'état d'excitation d'un neurone.

Bibliographie

J. Abate and W. Whitt. An operational calculus for probability distributions via laplace transforms. *Advances in Applied Probability*, 28(1) :75–113, 1996.

S. Adke. Records generated by Markov sequences. *Statistics and probability letters*, 18(4) :257–263, 1993.

W. J. Anderson. *Continuous-time Markov chains : An applications-oriented approach*. Springer Science & Business Media, 2012.

M. André. A result of metastability for an infinite system of spiking neurons. *Journal of Statistical Physics*, 177(5) :984–1008, 2019.

J. R. Artalejo, A. Economou, and M. Lopez-Herrero. Evaluating growth measures in an immigration process subject to binomial and geometric catastrophes. *Mathematical Biosciences & Engineering*, 4(4) :573, 2007.

S. Asmussen and O. Kella. Rate modulation in dams and ruin problems. *Journal of Applied Probability*, 33(2) :523–535, 1996.

J. Azéma, M. Duflo, and D. Revuz. Mesure invariante des processus de Markov récurrents. *Séminaire de probabilités de Strasbourg*, 3 :24–33, 1969. URL http://www.numdam.org/item/SPS_1969__3__24_0/.

H. Baumann and W. Sandmann. Steady state analysis of level dependent quasi-birth-and-death processes with catastrophes. *Computers & operations research*, 39(2) :413–423, 2012.

I. Ben-Ari, A. Roitershtein, and R. B. Schinazi. A random walk with catastrophes. *Electronic journal of probability*, 24, 2019.

M. Benaïm, S. Le Borgne, F. Malrieu, and P.-A. Zitt. Qualitative properties of certain piecewise deterministic Markov processes. *Annales de l'I.H.P. Probabilités et statistiques*, 51(3) :1040–1075, 2015. doi : 10.1214/14-AIHP619. URL http://www.numdam.org/articles/10.1214/14-AIHP619/.

J. Bertoin. On overshoots and hitting times for random walks. *Journal of applied probability*, 36(2) :593–600, 1999.

N. Bingham. Random walk and fluctuation theory. *Handbook of statistics*, 19 : 171–213, 2001.

L. Bondesson. *Generalized gamma convolutions and related classes of distributions and densities*, volume 76. Springer Science & Business Media, 2012.

L. Bondesson. A class of probability distributions that is closed with respect to addition as well as multiplication of independent random variables. *Journal of Theoretical Probability*, 28(3) :1063–1081, 2015.

O. Boudali and A. Economou. The effect of catastrophes on the strategic customer behavior in queueing systems. *Naval Research Logistics (NRL)*, 60(7) :571–587, 2013.

N. Bouzar, S. Satheesh, et al. Comments on a-decomposability. *Metron-International Journal of Statistics*, 66(2) :243–252, 2008.

O. Boxma, D. Perry, W. Stadje, and S. Zacks. A Markovian growth-collapse model. *Advances in Applied Probability*, 38(1) :221–243, 2006. doi : 10.1239/aap/1143936148.

O. Boxma, O. Kella, and D. Perry. On some tractable growth-collapse processes with renewal collapse epochs. *Journal of Applied Probability*, 48(A) :217–234, 2011.

P. Brémaud and L. Massoulié. Stability of nonlinear hawkes processes. *The Annals of Probability*, pages 1563–1588, 1996.

P. J. Brockwell, J. Gani, and S. I. Resnick. Birth, immigration and catastrophe processes. *Advances in Applied Probability*, 14(4) :709–731, 1982a.

P. J. Brockwell, S. I. Resnick, and R. L. Tweedie. Storage processes with general release rule and additive inputs. *Advances in Applied Probability*, 14(2) :392–433, 1982b. doi : 10.2307/1426528.

P. J. Brockwell, J. Gani, and S. I. Resnick. Catastrophe processes with continuous state-space 1. *Australian Journal of Statistics*, 25(2) :208–226, 1983.

B. Cairns and P. Pollett. Extinction times for a general birth, death and catastrophe process. *Journal of applied probability*, 41(4) :1211–1218, 2004.

J. M. Carlson, J. S. Langer, and B. E. Shaw. Dynamics of earthquake faults. *Reviews of Modern Physics*, 66(2) :657, 1994.

R. Chacon and D. Ornstein. A general ergodic theorem. *Illinois, J. Math*, 2(4) : 153–160, 1960.

G. Christoph and K. Schreiber. Scaled Sibuya distribution and discrete self-decomposability. *Statistics & probability letters*, 48(2) :181–187, 2000.

P. Collet, S. Martınez, and J. San Martın. Quasi-stationary distributions. probability and its applications, 2013.

F. Comets, R. Fernández, and P. A. Ferrari. Processes with long memory : regenerative construction and perfect simulation. *The Annals of Applied Probability*, 12 (3) :921–943, 2002.

L. Comtet. *Analyse combinatoire*, volume 2. Presses universitaires de France, 1970.

Q. Cormier, E. Tanré, and R. Veltz. Long time behavior of a mean-field model of interacting neurons. *Stochastic Processes and their Applications*, 130(5) :2553–2595, 2020.

M. Cottrell. Mathematical analysis of a neural network with inhibitory coupling. *Stochastic Processes and their Applications*, 40(1) :103–126, 1992.

C. Dabrowski. Catastrophic event phenomena in communication networks : A survey. *Computer Science Review*, 18 :10–45, 2015.

M. H. Davis. Piecewise-deterministic Markov processes : A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society : Series B (Methodological)*, 46(3) :353–376, 1984.

H. Dette. On the generating functions of a random walk on the non-negative integers. *Journal of applied probability*, 33(4) :1033–1052, 1996.

A. Economou. The compound Poisson immigration process subject to binomial catastrophes. *Journal of applied probability*, 41(2) :508–523, 2004.

A. Economou and D. Fakinos. Alternative approaches for the transient analysis of Markov chains with catastrophes. *Journal of Statistical Theory and Practice*, 2 (2) :183–197, 2008.

I. Eliazar and J. Klafter. Non-linear shot noise : Lévy, noah, & joseph. *Physica A : Statistical Mechanics and its Applications*, 360(2) :227–260, 2006a.

I. Eliazar and J. Klafter. Growth-collapse and decay-surge evolutions, and geometric langevin equations. *Physica A : Statistical Mechanics and its Applications*, 367 : 106–128, 2006b.

I. Eliazar and J. Klafter. Nonlinear shot noise : From aggregate dynamics to maximal dynamics. *EPL (Europhysics Letters)*, 78(4) :40001, 2007.

I. Eliazar and J. Klafter. The maximal process of nonlinear shot noise. *Physica A : Statistical Mechanics and its Applications*, 388(9) :1755–1779, 2009.

W. Feller. *An introduction to probability theory and its applications, vol 2*. John Wiley & Sons, 2008.

R. Fernández, P. Ferrari, and A. Galves. Coupling, renewal and perfect simulation of chains of infinite order. *Lecture Notes for the vth Brazilian school of Probability, Ubatuba*, 2001, 2001.

P. A. Ferrari and A. Galves. Construction of stochastic processes, coupling and regeneration. *XIII Escuela Venezolana de Matemática*, 2000.

P. A. Ferrari, A. Galves, I. Grigorescu, and E. Löcherbach. Phase transition for infinite systems of spiking neurons. *Journal of Statistical Physics*, 172(6) :1564–1575, 2018.

C. Fricker, P. Robert, E. Saada, and D. Tibi. Analysis of a network model. In *Cellular Automata and Cooperative Systems*, pages 159–172. Springer, 1993.

A. Galves and E. Löcherbach. Infinite systems of interacting chains with memory of variable length—a stochastic model for biological neural nets. *Journal of Statistical Physics*, 151(5) :896–921, 2013.

A. Galves, E. Löcherbach, and E. Orlandi. Perfect simulation of infinite range Gibbs measures and coupling with their finite range approximations. *Journal of Statistical Physics*, 138(1) :476–495, 2010.

A. Galves, N. L. Garcia, E. Löcherbach, and E. Orlandi. Kalikow-type decomposition for multicolor infinite range particle systems. *The Annals of Applied Probability*, 23(4) :1629–1659, 2013.

P. W. Glynn and P. Y. Desai. A probabilistic proof of the Perron-Frobenius theorem. *arXiv preprint arXiv:1808.04964*, 2018.

B. Goncalves. An interacting neuronal network with inhibition : theoretical analysis and perfect simulation. *arXiv preprint arXiv:2110.06714*, 2021.

B. Goncalves and T. Huillet. Scaling features of two special Markov chains involving total disasters. *Journal of Statistical Physics*, 178(2) :499–531, 2020. URL https://doi.org/10.1007/s10955-019-02439-5.

B. Goncalves and T. Huillet. A generating function approach to Markov chains undergoing binomial catastrophes. *Journal of Statistical Mechanics : Theory and Experiment*, 2021(3) :033402, 2021a. URL https://doi.org/10.1088/1742-5468/abdfcb.

B. Goncalves and T. E. Huillet. Keeping random walks safe from extinction and overpopulation in the presence of life-taking disasters. *Mathematical Population Studies*, pages 1–30, 2021b. URL https://doi.org/10.1080/08898480.2021.1976476.

B. Goncalves, T. Huillet, and E. Löcherbach. On decay-surge population models. *arXiv preprint arXiv:2012.00716*, 2020.

B. Goncalves, T. Huillet, and E. Löcherbach. On population growth with catastrophes. *Stochastic Models*, 0(0) :1–36, 2022. URL https://doi.org/10.1080/15326349.2021.2020660.

F. Gosselin and J.-D. Lebreton. Asymptotic properties of infinite leslie matrices. *Journal of theoretical biology*, 256(2) :157–163, 2009.

M. Greenwood. On the statistical measure of infectiousness. *Epidemiology & Infection*, 31(3) :336–351, 1931.

D. Griffeath. The basic contact processes. *Stochastic Processes and their Applications*, 11(2) :151–185, 1981.

G. Gripenberg. A stationary distribution for the growth of a population subject to random catastrophes. *Journal of mathematical biology*, 17(3) :371–379, 1983.

G. Gripenberg. Extinction in a model for the growth of a population subject to catastrophes. *Stochastics*, 14(2) :149–163, 1985. doi : 10.1080/17442508508833336. URL https://doi.org/10.1080/17442508508833336.

C. Halgreen. Self-decomposability of the generalized inverse Gaussian and hyperbolic distributions. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 47(1) :13–17, 1979.

F. B. Hanson and H. C. Tuckwell. Persistence times of populations with large random fluctuations. *Theoretical Population Biology*, 14(1) :46–61, 1978. ISSN 0040-5809. doi : https://doi.org/10.1016/0040-5809(78)90003-5. URL https://www.sciencedirect.com/science/article/pii/0040580978900035.

F. B. Hanson and H. C. Tuckwell. Logistic growth with random density independent disasters. *Theoretical Population Biology*, 19(1) :1–18, 1981. ISSN 0040-5809. doi : https://doi.org/10.1016/0040-5809(81)90032-0. URL https://www.sciencedirect.com/science/article/pii/0040580981900320.

T. E. Harris et al. The theory of branching processes. 6, 1963.

J. M. Harrison and S. I. Resnick. The stationary distribution and first exit probabilities of a storage process with general release rule. *Mathematics of Operations Research*, 1(4) :347–358, 1976.

J. M. Harrison and S. I. Resnick. The recurrence classification of risk and storage processes. *Mathematics of Operations Research*, 3(1) :57–66, 1978.

A. G. Hawkes. Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58(1) :83–90, 1971.

R. Horn and C. Johnson. Matrix analysis second edition, 2013.

T. Huillet. A shot-noise approach to decay/surge collective population models . working paper or preprint, Feb. 2021. URL https://hal.archives-ouvertes.fr/hal-03138215.

T. E. Huillet. On a Markov chain model for population growth subject to rare catastrophic events. *Physica A : Statistical Mechanics and its Applications*, 390 (23-24) :4073–4086, 2011.

S. Janson. Tail bounds for sums of geometric and exponential variables. *Statistics & Probability Letters*, 135 :1–6, 2018.

M. Kac. Random walk and the theory of brownian motion. *The American Mathematical Monthly*, 54(7P1) :369–391, 1947.

S. Kapodistria, T. Phung-Duc, and J. Resing. Linear birth/immigration-death process with binomial catastrophes. *Probability in the Engineering and Informational Sciences*, 30(1) :79–111, 2016.

O. Kella. On growth-collapse processes with stationary structure and their shot-noise counterparts. *Journal of Applied Probability*, 46(2) :363–371, 2009.

O. Kella and W. Stadje. On hitting times for compound Poisson dams with exponential jumps and linear release rate. *Journal of applied probability*, 38(3) : 781–786, 2001.

G. Kersting and F. C. Klebaner. Sharp conditions for nonexplosions and explosions in Markov jump processes. *The Annals of Probability*, 23(1) :268 – 272, 1995. doi : 10.1214/aop/1176988386. URL https://doi.org/10.1214/aop/1176988386.

R. Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer Science & Business Media, 1980.

T. J. Kozubowski and K. Podgórski. A generalized Sibuya distribution. *Annals of the Institute of Statistical Mathematics*, 70(4) :855–887, 2018.

E. Löcherbach, D. Loukianova, and O. Loukianov. Polynomial bounds in the ergodic theorem for one-dimensional diffusions and integrability of hitting times. In *Annales de l'IHP Probabilités et statistiques*, volume 47, pages 425–449, 2011.

E. McKenzie. Ch. 16. Discrete variate time series. *Handbook of statistics*, 21 : 573–606, 2003.

S. P. Meyn and R. L. Tweedie. Stability of Markovian processes iii : Foster–lyapunov criteria for continuous-time processes. *Advances in Applied Probability*, 25(3) : 518–548, 1993.

M. F. Neuts. An interesting random walk on the non-negative integers. *Journal of applied probability*, 31(1) :48–58, 1994.

J. Neveu. Chaînes de Markov et théorie du potentiel. *Annales scientifiques de l'Université de Clermont. Mathématiques*, 24(3) :37–89, 1964.

J. R. Norris. *Markov chains*. Number 2. Cambridge university press, 1998.

A. Pakes, A. Trajstman, and P. Brockwell. A stochastic model for a replicating population subjected to mass emigration due to population pressure. *Mathematical Biosciences*, 45(1-2) :137–157, 1979.

E. Parzen. Stochastic processes : Classics in applied mathematics. *Society for Industrial and Applied Mathematics*, 1999.

D. Revuz. *Markov chains*. Elsevier, 2008.

P. Richetti, C. Drummond, J. Israelachvili, M. In, and R. Zana. Inverted stick-slip friction. *EPL (Europhysics Letters)*, 55(5) :653, 2001.

P. Robert and J. Touboul. On the dynamics of random neuronal networks. *Journal of Statistical Physics*, 165(3) :545–584, 2016.

T. Sapatinas. Characterizations of probability distributions based on discrete p-monotonicity. *Statistics & probability letters*, 24(4) :339–344, 1995.

K.-I. Sato. Potential operators for Markov processes. In *Contributions to Probability Theory*, pages 193–212. University of California Press, 2020.

K. Schreiber. *Discrete self-decomposable distributions. Dr. rer. nat.* PhD thesis, Thesis dissertation. Ottovon-Guericke-Universität Magdeburg, 1999.

E. Seneta. Finite approximations to infinite non-negative matrices, ii : refinements and applications. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 64, pages 465–470. Cambridge University Press, 1968.

H. A. Simon. On a class of skew distribution functions. *Biometrika*, 42(3/4) : 425–440, 1955.

D. L. Snyder and M. I. Miller. *Random point processes in time and space*. Springer Science & Business Media, 2012.

F. Steutel and K. van Harn. *Infinite divisibility of probability distributions on the real line*. Number 259 in Pure and Applied Mathematics. Marcel Dekker, 2004. StHa04.

F. W. Steutel and K. van Harn. Discrete analogues of self-decomposability and stability. *The Annals of Probability*, pages 893–899, 1979.

R. J. Swift. Transient probabilities for a simple birth-death-immigration process under the influence of total catastrophes. *International Journal of Mathematics and Mathematical Sciences*, 25(10) :689–692, 2001.

M. Sibuya. Generalized hypergeometric, digamma and trigamma distributions. *Annals of the Institute of Statistical Mathematics*, 31(3) :373–390, 1979.

A. Trajstman. A bounded growth population subjected to emigrations due to population pressure. *Journal of Applied Probability*, 18(3) :571–582, 1981.

K. Van Harn, F. Steutel, and W. Vervaat. Self-decomposable discrete distributions and branching processes. In *Analytical Methods in Probability Theory*, pages 60–64. Springer, 1981.

D. Vere-Jones. Ergodic properties of nonnegative matrices. i. *Pacific Journal of Mathematics*, 22(2) :361–386, 1967.

C. Wilcox and B. Elderd. The effect of density-dependent catastrophes on population persistence time. *Journal of Applied Ecology*, pages 859–871, 2003.

W. Woess. Denumerable Markov chains. *Math. Eur. Math. Soc., Zürich xviii*, 2009.

A. M. Yaglom. Certain limit theorems of the theory of branching random processes. In *Doklady Akad. Nauk SSSR (NS)*, volume 56, page 3, 1947.

M. Yajima and T. Phung-Duc. A central limit theorem for a Markov-modulated infinite-server queue with batch Poisson arrivals and binomial catastrophes. *Performance Evaluation*, 129 :2–14, 2019.

G. U. Yule. II. A mathematical theory of evolution, based on the conclusions of Dr. JC Willis, FR S. *Philosophical transactions of the Royal Society of London. Series B, containing papers of a biological character*, 213(402-410) :21–87, 1925.