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▶ To cite this version:

Tayssir Touili, Xin Ye. LTL Model Checking of Self Modifying Code. Formal Methods in System Design, 2022, 60 (2). hal-03863361

HAL Id: hal-03863361
https://hal.science/hal-03863361

Submitted on 21 Nov 2022

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LTL Model Checking of Self Modifying Code

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Abstract

Self modifying code is code that can modify its own instructions during the execution of the program. It is extensively used by malware writers to obfuscate their malicious code. Thus, analysing self modifying code is nowadays a big challenge. In this paper, we consider the LTL model-checking problem of self modifying code. We model such programs using self-modifying pushdown systems (SM-PDS), an extension of pushdown systems that can modify its own set of transitions during execution. We reduce the LTL model-checking problem to the emptiness problem of self-modifying Büchi pushdown systems (SM-BPDS). We implemented our techniques in a tool that we successfully applied for the detection of several self-modifying malware. Our tool was also able to detect several malwares that well-known antiviruses such as BitDefender, Kinsoft, Avira, eScan, Kaspersky, Qihoo-360, Baidu, Avast, and Symantec failed to detect.

Keywords: malware detection, model checking, automata

1. Introduction

Binary code presents several complex aspects that cannot be encountred in source code. One of these aspects is self-modifying code, i.e., code that can modify its own instructions during the execution of the program. Self-modifying code makes reverse code engineering harder. Thus, it is extensively used to protect software intellectual property. It is also heavily used by malware writers in order to make their malwares hard to analyse and detect by static analysers and anti-viruses. Thus, it is crucial to be able to analyse self-modifying code.

There are several kinds of self-modifying code. In this work, we consider self-modifying code caused by **self-modifying instructions**. These kind of instructions treat code as data. This allows them to read and write into code, leading to **self-modifying instructions**. These self-modifying instructions are usually **mov** instructions, since **mov** allows to access memory and read and write into it.

Let us consider the example shown in Figure 1. For simplicity, the addresses' length is assumed to be 1 byte. In the right box, we give, respectively, the binary code, the addresses of the different instructions, and the corresponding assembly code, obtained by translating syntactically the binary code at each address. For example, `0c` is the binary code of the jump `jmp`. Thus, `0c 02` is translated to `jmp 0x2` (jump to address 0x2). The second line is translated to `push 0x9`, since `ff` is the binary code of the instruction `push`. The third instruction `mov 0x2 0xc` will replace the first byte at address `0x2` by `0xc`. Thus, at address `0x2`, ff 09 is replaced by 0c 09. This means the instruction push 0x9 is replaced by the jump instruction jmp 0x9 (jump to address 0x9), etc. Therefore, this code is self-modifying: the **mov** instruction was able to modify the instructions of the program via its ability to read and write the memory. If we study this code without looking at the semantics of the self-modifying instructions, we will extract from it the Control Flow Graph CFG a that is in the left of the figure, and we will reach the conclusion that the call to the API function CopyFileA at address 0x9 cannot be made. However, you can see that the correct CFG is the one on the right hand side CFG b, where the call to the API function CopyFileA at address 0x9 can be reached. Thus, it is very important to be able to take into account the semantics of the self-modifying instructions in binary code.

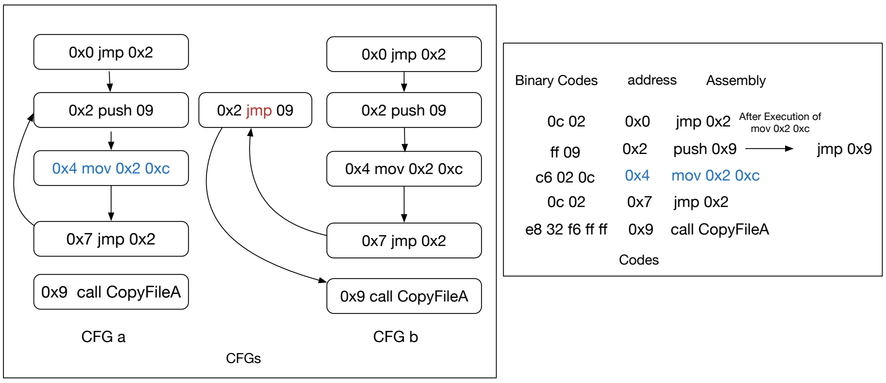


Figure 1: An Example of a Self-modifying Code

In this paper, we consider the LTL model-checking problem of self-modifying code. To this aim, we use Self-Modifying Pushdown Systems (SM-PDSs) [1] to model self-modifying code. Indeed, SM-PDSs were shown in [1] to be an adequate model for self-modifying code since they allow to mimic the program's stack while taking into account the self-modifying semantics of the transitions. This is very important for binary code analysis and malware detection, since malwares are based on calls to API functions of the operating system. Thus, antiviruses check the API calls to determine whether a program is malicious or not. Therefore, to evade from these antiviruses, malware writers try to hide the API calls they make by replacing calls by push and jump instructions. Thus, to be able to analyse such malwares, it is crucial to be able to analyse the program's stack. Hence the need to a model like pushdown systems and self-modifying pushdown systems for this purpose, since they allow to mimic the program's stack.

Intuitively, a SM-PDS is a pushdown system (PDS) with self-modifying rules, i.e., with rules that allow to modify the current set of transitions during execution. This model was introduced in [1] in order to represent self-modifying code. In [1], the authors have proposed algrithms to compute finite automata that accept the forward and backward reachability sets of SM-PDSs. In this work, we tackle the problem of LTL model-checking of SM-PDSs. Since SM-PDSs are equivalent to PDSs [1], one possible approach for LTL model checking of SM-PDS is to translate the SM-PDS to a standard PDS and then run the LTL model checking algorithm on the equivalent PDS [2, 3]. But translation from a SM-PDS to a standard PDS is exponential. Thus, performing the LTL model checking on the equivalent PDS is not efficient.

To overcome this limitation, we propose a *direct* LTL model checking algorithm for SM-PDSs. Our algorithm is based on reducing the LTL model checking problem to the emptiness problem of Self Modifying Büchi Pushdown Systems (SM-BPDS). Intuitively, we obtain this SM-BPDS by taking the product of the SM-PDS with a Büchi automaton accepting an LTL formula φ. Then, we solve the emptiness problem of an SM-BPDS by computing its repeating heads. This computation is based on computing labelled pre^* configurations by applying a saturation procedure on labelled finite automata.

We implemented our algorithm in a tool. Our experiments show that our *direct* techniques are much more efficient than translating the SM-PDS to an equivalent PDS and then applying the standard LTL model checking for PDSs [2, 3]. Moreover, we successfully applied our tool to the analysis of 892 self-modifying malwares. Our tool was also able to detect several self-modifying malwares that well-known antiviruses like BitDefender, Kinsoft, Avira, eScan, Kaspersky, Qihoo-360, Baidu, Avast, and Symantec were not able to detect.

This paper is an expanded version of the conference paper [4]. Compared to [4], this journal version includes the proofs of all our results (no proof is provided in [4]).

Related Work. Model checking and static analysis approaches have been widely used to analyze binary programs, for instance, in [5, 6, 7, 8, 9]. Temporal Logics were chosen to describe malicious behaviors in [10, 8, 9, 11, 12]. However, these works cannot deal with self-modifying code.

POMMADE [9, 11] is a malware detector based on LTL and CTL model-checking of PDSs. STAMAD [13, 14, 15] is a malware detector based on PDSs and machine learning. However, POMMADE and STAMAD cannot deal with self-modifying code.

Cai et al. [16] use local reasoning and separation logic to describe self-modifying code and treat program code uniformly as regular data structure. However, [16] requires programs to be manually annotated with invariants. In [17], the authors propose a formal semantics for self-modifying codes, and use that to represent self-unpacking code. This work only deals with packing and unpacking behaviours. Bonfante et al. [18] provide an operational semantics for self-modifying programs and show that they can be constructively rewritten to a non-modifying program. However, all these specifications [18, 16, 17] are too abstract to be used in practice.

In [19], the authors propose a new representation of self-modifying code named State Enhanced-Control Flow Graph (SE-CFG). SE-CFG extends standard control flow graphs with a new data structure, keeping track of the possible states programs can reach, and with edges that can be conditional on the state of the target memory location. It is not easy to analyse a binary program only using its SE-CFG, especially that this representation does not allow to take into account the stack of the program.

The authors in [20] propose abstract interpretation techniques to compute an over-approximation of the set of reachable states of a self-modifying program, where for each control point of the program, an over-approximation of the memory state at this control point is provided. Static and dynamic analysis techniques are combined to analyse self-modifying programs in [21] . Unlike our approach, these techniques [20, 21] cannot handle the program's stack.

Unpacking binary code is also considered in [22, 23, 24, 17]. These works do not consider self-modifying **mov** instructions.

Outline. The rest of the paper is structured as follows: Section 2 recalls the definition of Self Modifying pushdown systems. LTL model checking and SM-BPDSs are defined in Section 3. Section 4 solves the emptiness problem of SM-BPDS. Finally, the experiments are reported in Section 5.

2. Self-Modifying Pushdown Systems

2.1. Definition

We recall in this section the definition of Self-modifying Pushdown Systems [1].

Definition 1. *A Self-modifying Pushdown System (SM-PDS) is a tuple* $\mathcal{P} = (P, \Gamma, \Delta, \Delta_c)$*, where* P *is a finite set of control points,* Γ *is a finite set of stack symbols,* $\Delta \subseteq (P \times \Gamma) \times (P \times \Gamma^*)$ *is a finite set of transition rules, and* $\Delta_c \subseteq P \times (\Delta \cup \Delta_c) \times (\Delta \cup \Delta_c) \times P$ *is a finite set of modifying transition rules. If* $((p, \gamma), (p', w)) \in \Delta$*, we also write* $\langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle \in \Delta$*. If* $(p, r_1, r_2, p') \in \Delta_c$*, we also write* $p \xhookrightarrow{(r_1,r_2)} p' \in \Delta_c$*. A Pushdown System (PDS) is a SM-PDS where* $\Delta_c = \emptyset$*.*

Intuitively, a Self-modifying Pushdown System is a Pushdown System that can dynamically modify its set of rules during the execution time: rules Δ are standard PDS transition rules, while rules Δ_c modify the current set of transition rules: $\langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle \in \Delta$ expresses that if the SM-PDS is in control point p and has γ on top of its stack, then it can move to control point p', pop γ and push w onto the stack, while $p \xhookrightarrow{(r_1,r_2)} p' \in \Delta_c$ expresses that when the PDS is in control point p, then it can move to control point p', remove the rule r_1 from its current set of transition rules, and add the rule r_2.

Formally, a configuration of a SM-PDS is a tuple $c = (\langle p, w \rangle, \theta)$ where $p \in P$ is the control point, $w \in \Gamma^*$ is the stack content, and $\theta \subseteq \Delta \cup \Delta_c$ is the current set of transition rules of the SM-PDS. θ is called the current *phase* of the SM-PDS. When the SM-PDS is a PDS, i.e., when $\Delta_c = \emptyset$, a configuration is a tuple $c = (\langle p, w \rangle, \Delta)$, since there is no changing rule, so there is only one possible phase. In this case, we can also write $c = \langle p, w \rangle$. Let $\mathcal{C}$ be the set of configurations of a SM-PDS. A SM-PDS defines a transition relation $\Rightarrow_{\mathcal{P}}$ between configurations as follows: Let $c = (\langle p, w \rangle, \theta)$ be a configuration, and let r be a rule in θ, then:

1. if $r \in \Delta_c$ is of the form $r = p \xhookrightarrow{(r_1,r_2)} p'$, such that $r_1 \in \theta$, then $(\langle p, w \rangle, \theta) \Rightarrow_{\mathcal{P}} (\langle p', w \rangle, \theta')$, where $\theta' = (\theta \setminus \{r_1\}) \cup \{r_2\}$. In other words, the transition rule r updates the current set of transition rules θ by removing r_1 from it and adding r_2 to it.
2. if $r \in \Delta$ is of the form $r = \langle p, \gamma \rangle \hookrightarrow \langle p', w' \rangle \in \Delta$, then $(\langle p, \gamma w \rangle, \theta) \Rightarrow_{\mathcal{P}} (\langle p', w' w \rangle, \theta)$. In other words, the transition rule r moves the control point from p to p', pops γ from the stack and pushes w' onto the stack. This transition keeps the current set of transition rules θ unchanged.

Let $\Rightarrow_{\mathcal{P}}^*$ be the transitive, reflexive closure of $\Rightarrow_{\mathcal{P}}$ and $\Rightarrow_{\mathcal{P}}^+$ be its transitive closure. An execution (a run) of $\mathcal{P}$ is a sequence of configurations $\pi = c_0 c_1 ...$ s.t. $c_i \Rightarrow_{\mathcal{P}} c_{i+1}$ for every $i \geq 0$. Given a configuration c, the set of immediate predecessors (resp. successors) of c is $pre_{\mathcal{P}}(c) = \{c' \in \mathcal{C} : c' \Rightarrow_{\mathcal{P}} c\}$ (resp. $post_{\mathcal{P}}(c) = \{c' \in \mathcal{C} : c \Rightarrow_{\mathcal{P}} c'\}$). These notations can be generalized straightforwardly to sets of configurations. Let $pre_{\mathcal{P}}^*$ (resp. $post_{\mathcal{P}}^*$) denote the reflexive-transitive closure of $pre_{\mathcal{P}}$ (resp. $post_{\mathcal{P}}$). We remove the subscript $\mathcal{P}$ when it is clear from the context.

We suppose w.l.o.g. that rules in Δ are of the form $\langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle$ such that $|w| \leq 2$, and that the self-modifying rules $r = p \xhookrightarrow{(r_1, r_2)} p'$ in Δ_c are such that $r \neq r_1$. Note that this is not a restriction, since for a given SM-PDS, one can compute an equivalent SM-PDS that satisfies these conditions [1] .

Example 1. *Let $\mathcal{P} = (P, \Gamma, \Delta, \Delta_c)$ be a SM-PDS where $P = \{p_1, p_2, p_3, p_4\}$, $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$, $\Delta = \{r_1 : \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_2, \gamma_2\gamma_1 \rangle, r_2 : \langle p_2, \gamma_2 \rangle \hookrightarrow \langle p_3, \epsilon \rangle, r_3 : \langle p_4, \gamma_1 \rangle \hookrightarrow \langle p_2, \gamma_2\gamma_3 \rangle\}$, $\Delta_c = \{r' : p_3 \xhookrightarrow{(r_1, r_3)} p_4\}$. Let $c_0 = (\langle p_1, \gamma_1\gamma_1 \rangle, \theta_0)$ where $\theta_0 = \{r_1, r_2, r'\}$. Applying rule r_1, we get $(\langle p_1, \gamma_1\gamma_1 \rangle, \theta_0) \Rightarrow_{\mathcal{P}} (\langle p_2, \gamma_2\gamma_1\gamma_1 \rangle, \theta_0)$. Then, applying rule r_2, we get $(\langle p_2, \gamma_2\gamma_1\gamma_1 \rangle, \theta_0) \Rightarrow_{\mathcal{P}} (\langle p_3, \gamma_1\gamma_1 \rangle, \theta_0)$. Then, applying rule r', we get $(\langle p_3, \gamma_1\gamma_1 \rangle, \theta_0) \Rightarrow_{\mathcal{P}} (\langle p_4, \gamma_1\gamma_1 \rangle, \theta_1)$ where r' is self-modifying, thus, it leads the SM-PDS from phase $\theta_0 = \{r_1, r_2, r'\}$ to phase $\theta_1 = \theta_0 \setminus \{r_1\} \cup \{r_3\} = \{r_2, r_3, r'\}$. Then, applying rule r_3, we get $(\langle p_4, \gamma_1\gamma_1 \rangle, \theta_1) \Rightarrow_{\mathcal{P}} (\langle p_2, \gamma_2\gamma_3\gamma_1 \rangle, \theta_1)$. Then, applying rule r_2 again, we get $(\langle p_2, \gamma_2\gamma_3\gamma_1 \rangle, \theta_1) \Rightarrow_{\mathcal{P}} (\langle p_3, \gamma_3\gamma_1 \rangle, \theta_1)$.*

2.2. SM-PDS vs. PDS

Let $\mathcal{P} = (P, \Gamma, \Delta, \Delta_c)$ be a SM-PDS. It was shown in [1] that:

1. $\mathcal{P}$ can be described by an equivalent pushdown system (PDS). Indeed, since the number of phases is finite, we can encode phases in the control point of the PDS. However, this translation is not efficient since the number of control points of the equivalent PDS is $|P| \cdot 2^{\mathcal{O}(|\Delta| + |\Delta_c|)}$.
2. $\mathcal{P}$ can also be described by an equivalent Symbolic pushdown system [25], where each SM-PDS rule is represented by a *single, symbolic* transition, where the different values of the phases are encoded in a symbolic way using relations between phases. This translation is not efficient neither since the size of the relations used in the symbolic transitions is $2^{\mathcal{O}(|\Delta| + |\Delta_c|)}$.

2.3. Modeling self-modifying code with SM-PDSs

2.3.1. Self-modifying instructions

There are different techniques to implement self-modifying code. We consider in this work code that uses self-modifying instructions. These are instructions that can access the memory locations and write onto them, thus changing the instructions that are in these memory locations. In assembly, the only instructions that can do this are the **mov** instructions. In this case, the self-modifying instructions are of the form `mov` l v, where l is a location of the program that stores executable data and v is a value. This instruction replaces the value at location l (in the binary code) with the value v. This means if at location l there is a binary value v' that is involved in an assembly instruction i_1, and if by replacing v' by v, we obtain a new assembly instruction i_2, then the instruction i_1 is replaced by i_2. E.g., `ff` is the binary code of `push`, `40` is the binary code of `inc`, `0c` is the binary code of `jmp`, `c6` is the binary code of `mov`, etc. Thus, if we have `mov` l `ff`, and if at location l there was initially the value 40 01 (which corresponds to the assembly instruction inc %edx), then 40 is replaced by `ff`, which means the instruction `inc %edx` is replaced by `push 01`. If at location l there was initially the value c6 01 02 (which corresponds to the assembly instruction `mov edx 0x2`), then c6 is replaced by `ff`, which means the instruction `mov edx 0x2` is replaced by `push 02`.

Note that if the instructions i_1 and i_2 do not have the same number of operands, then `mov` l v will, in addition to replacing i_1 by i_2, change several other instructions that follow i_1. Currently, we cannot handle this case, thus we assume that i_1 and i_2 have the same number of operands.

Note also that $mov\ l\ v$ is self-modifying only if l is a location of the program that stores executable data, otherwise, it is not; e.g., $mov\ eax\ v$ does not change the instructions of the program, it just writes the value v to the register eax. Thus, from now on, by self-modifying instruction, we mean an instruction of the form $mov\ l\ v$, where l is a location of the program that stores executable data. Moreover, to ensure that only one instruction is modified, we assume that the corresponding instructions i_1 and i_2 have the same number of operands.

2.3.2. From self-modifying code to SM-PDS

We show in what follows how to build a SM-PDS from a binary program. We suppose we are given an oracle $\mathcal{O}$ that extracts from the binary code a corresponding assembly program, together with informations about the values of the registers and the memory locations at each control point of the program. In our implementation, we use Jakstab [26] to get this oracle. We translate the assembly program into a self-modifying pushdown system where the control locations store the control points of the binary program and the stack memics the program's stack. The non self-modifying instructions of the program define the rules Δ of the SM-PDS (which are standard PDS rules), and can be obtained following the translation of [9] that models non self-modifying instructions of the program by a PDS.

As for the self-modifying instructions of the program, they define the set of changing rules Δ_c. As explained above, these are instructions of the form $mov\ l\ v$, where l is a location of the program that stores executable data. This instruction replaces the value at location l (in the binary code) with the value v. Let i_1 be the initial instruction involving the location l, and let i_2 be the new instruction involving the location l, after applying the $mov\ l\ v$ instruction. As mentioned previously, we assume that i_1 and i_2 have the same number of operands (to ensure that only one instruction is modified). Let r_1 (resp. r_2) be the SM-PDS rule corresponding to the instruction i_1 (resp. i_2). Suppose from control point n to n', we have this $mov\ l\ v$ instruction, then we add $n \xhookrightarrow{(r_1,r_2)} n'$ to Δ_c. This is the SM-PDS rule corresponding to the instruction $mov\ l\ v$ at control point n.

3. LTL Model-Checking of SM-PDSs

3.1. The linear-time temporal logic LTL

Let At be a finite set of atomic propositions. LTL formulas are defined as follows (where $A \in At$):

$$\varphi := \ A \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid X\varphi \mid \varphi_1 U \varphi_2$$

Formulae are interpreted on infinite words over 2^{At}. Let $\omega = \omega^0\omega^1...$ be an infinite word over 2^{At}. We write ω_i for the suffix of ω starting at ω^i. We denote $\omega \models \varphi$ to express that ω satisfies a formula φ:

$\omega \models A \iff A \in \omega^0$
$\omega \models \neg\varphi \iff \omega \not\models \varphi$
$\omega \models \varphi_1 \vee \varphi_2 \iff \omega \models \varphi_1$ or $\omega \models \varphi_2$
$\omega \models X\varphi \iff \omega_1 \models \varphi$
$\omega \models \varphi_1 U \varphi_2 \iff \exists i \geq 0, \omega_i \models \varphi_2$ and $\forall 0 \leq j < i, \omega_j \models \varphi_1$

The temporal operators G (globally) and F (eventually) are defined as follows: $F\varphi = (A \vee \neg A)U\varphi$ and $G\varphi = \neg F \neg \varphi$. Let $W(\varphi)$ be the set of infinite words that satisfy an LTL formula φ. It is well known that $W(\varphi)$ can be accepted by Büchi automata:

Definition 2. *A Büchi automaton $\mathcal{B}$ is a quintuple $(Q, \Gamma, \eta, q_0, F)$ where Q is a finite set of states, Γ is a finite input alphabet, $\eta \subseteq (Q \times \Gamma \times Q)$ is a set of transitions, $q_0 \in Q$ is the initial state and $F \subseteq Q$ is the set of accepting states. A run of $\mathcal{B}$ on a word $\gamma_0\gamma_1... \in \Gamma^\omega$ is a sequence of states $q_0q_1q_2...$ s.t. $\forall i \geq 0, (q_i, \gamma_i, q_{i+1}) \in \eta$. An infinite word ω is accepted by $\mathcal{B}$ if $\mathcal{B}$ has a run on ω that starts at q_0 and visits accepting states from F infinitely often.*

Theorem. *[27] Given an LTL formula φ, one can effectively construct a Büchi automaton $\mathcal{B}_\varphi$ which accepts $W(\varphi)$.*

3.2. Self Modifying Büchi Pushdown Systems

Definition 3. *A Self Modifying Büchi Pushdown Systems (SM-BPDS) is a tuple $\mathcal{BP} = (P, \Gamma, \Delta, \Delta_c, G)$ where P is a set of control locations, $G \subseteq P$ is a set of accepting control locations, $\Delta \subseteq (P \times \Gamma) \times (P \times \Gamma^*)$ is a finite set of transition rules, and $\Delta_c \subseteq P \times 2^{\Delta \cup \Delta_c} \times 2^{\Delta \cup \Delta_c} \times P$ is a finite set of modifying transition rules of the form $p \xhookrightarrow{(\sigma,\sigma')} p'$ where $\sigma, \sigma' \subseteq \Delta \cup \Delta_c$.*

Let $\Rightarrow_{\mathcal{BP}}$ be the transition relation between configurations as follows: Let $\theta \subseteq \Delta \cup \Delta_c, \gamma \in \Gamma, w \in \Gamma^*$, and $p \in P$, then

1. *If $r : \langle p, \gamma \rangle \hookrightarrow \langle p', w' \rangle \in \Delta$ and $r \in \theta$, then $(\langle p, \gamma w \rangle, \theta) \Rightarrow_{\mathcal{BP}} (\langle p', w'w \rangle, \theta)$.*
2. *If $r : p \xhookrightarrow{(\sigma,\sigma')} p' \in \Delta_c$, $\sigma \cap \theta \neq \emptyset$ and $r \in \theta$, then $(\langle p, \gamma w \rangle, \theta) \Rightarrow_{\mathcal{BP}} (\langle p', \gamma w \rangle, \theta')$ where $\theta' = \theta \backslash \sigma \cup \sigma'$.*

A run π of $\mathcal{BP}$ is a sequence of configurations $\pi = c_0c_1...$ s.t. $c_i \Rightarrow_{\mathcal{BP}} c_{i+1}$ for every $i \geq 0$. π is accepting iff it infinitely often visits configurations having control locations in G.

Let c and c' be two configurations of the SM-BPDS $\mathcal{BP}$. The relation $\Rightarrow^r_{\mathcal{BP}}$ is defined as follows: $c \Rightarrow^r_{\mathcal{BP}} c'$ iff there exists a configuration $(\langle g, u \rangle, \theta)$, $g \in G$ s.t. $c \Rightarrow^*_{\mathcal{BP}} (\langle g, u \rangle, \theta) \Rightarrow^+_{\mathcal{BP}} c'$. We remove the subscript $\mathcal{BP}$ when it is clear *from the context. We define* $\overset{i}{\Rightarrow}$ *as follows:* $c \overset{i}{\Rightarrow} c'$ *iff there exists a sequence of configurations* $c_0 \Rightarrow_{\mathcal{BP}} c_1 \Rightarrow_{\mathcal{BP}} ... \Rightarrow_{\mathcal{BP}} c_i$ *s.t.* $c_0 = c$ *and* $c_i = c'$.

A head of SM-BPDS is a tuple $(\langle p, \gamma\rangle, \theta)$ *where* $p \in P$, $\gamma \in \Gamma$ *and* $\theta \subseteq \Delta \cup \Delta_c$. *A head* $((p, \gamma), \theta)$ *is repeating if there exists* $v \in \Gamma^*$ *such that* $(\langle p, \gamma\rangle, \theta) \Rightarrow^r_{\mathcal{BP}} (\langle p, \gamma v\rangle, \theta)$. *The set of repeating heads of SM-BPDS is called* $Rep_{\mathcal{BP}}$.

We assume w.l.o.g. that for every rule in Δ_c of the form $r : p \xhookrightarrow{(\sigma,\sigma')} p'$, $r \notin \sigma$. Note that this is not a restriction since a rule of the form $r = p \xhookrightarrow{(\sigma,\sigma')} p'$ where $r \in \sigma$ can be simulated by a set of rules that satisfy the above condition.

3.3. From LTL Model-Checking of SM-PDSs to the emptiness problem of SM-BPDSs

Let $\mathcal{P} = (P, \Gamma, \Delta, \Delta_c)$ be a self modifying pushdown system. Let At be a set of atomic propositions. Let $\nu : P \to 2^{At}$ be a labelling function. Let $\pi = (\langle p_0, w_0\rangle, \theta_0)(\langle p_1, w_1\rangle, \theta_1)...$ be an execution of the SM-PDS $\mathcal{P}$. Let φ be an LTL formula over the set of atomic propositions At. We say that

$$\pi \models_\nu \varphi \text{ iff } \nu(p_0)\nu(p_1)\cdots \models \varphi$$

Let $(\langle p, w\rangle, \theta)$ be a configuration of $\mathcal{P}$. We say that $(\langle p, w\rangle, \theta) \models_\nu \varphi$ iff $\mathcal{P}$ has an execution π starting at $(\langle p, w\rangle, \theta)$ such that $\pi \models_\nu \varphi$.

Our goal in this paper is to perform LTL model-checking for self-modifying pushdown systems. Since SM-PDSs can be translated to standard (symbolic) pushdown systems, one way to solve this LTL model-checking problem is to compute the (symbolic) pushdown system that is equivalent to the SM-PDS (see section 2.2), and then apply the standard LTL model-checking algorithms on standard PDSs [25]. However, this approach is not efficient (as will be witnessed later in the experiments). Thus, we need a *direct* approach that performs LTL model-checking on the SM-PDS, without translating it to an equivalent PDS. Let $\mathcal{B}_\varphi = (Q, 2^{At}, \eta, q_0, F)$ be a Büchi automaton that accepts $W(\varphi)$. We compute the SM-BPDS $\mathcal{BP}_\varphi = (P \times Q, \Gamma, \Delta', \Delta'_c, G)$ by performing a kind of product between the SM-PDS $\mathcal{P}$ and the Büchi automaton $\mathcal{B}_\varphi$ as follows:

1. if $r = \langle p, \gamma\rangle \hookrightarrow \langle p', w\rangle \in \Delta$ and $(q, \nu(p), q') \in \eta$, then $\langle (p, q), \gamma\rangle \hookrightarrow \langle (p', q'), w\rangle \in \Delta'$. Let $prod(r)$ be the set of rules of Δ' obtained from the rule r, i.e., rules of Δ' of the form $\langle (p, q), \gamma\rangle \hookrightarrow \langle (p', q'), w\rangle$.
2. if a rule $r = p \xhookrightarrow{(r_1,r_2)} p' \in \Delta_c$ and $(q, \nu(p), q') \in \eta$, then $(p, q) \xhookrightarrow{(\sigma,\sigma')} (p', q') \in \Delta'_c$ where $\sigma = prod(r_1), \sigma' = prod(r_2)$. Let $prod(r)$ be the set of rules of Δ' obtained from the rule r, i.e., rules of Δ'_c of the form $(p, q) \xhookrightarrow{(\sigma,\sigma')} (p', q')$.
3. $G = P \times F$.

Remark. Note that a rule $r = p \xhookrightarrow{(r_1,r_2)} p' \in \Delta_c$ generates rules of the form $(p, q) \xhookrightarrow{(\sigma,\sigma')} (p', q') \in \Delta'_c$, where $\sigma = prod(r_1)$ and $\sigma' = prod(r_2)$ are sets of rules. This is why we require that a Self Modifying Büchi Pushdown System has modifying transition rules of the form $p \xhookrightarrow{(\sigma,\sigma')} p'$ where $\sigma, \sigma' \subseteq \Delta \cup \Delta_c$ are sets of rules.

We can show that:

Theorem 3.1. *Let $(\langle p, w\rangle, \theta)$ be a configuration of the SM-PDS $\mathcal{P}$. $(\langle p, w\rangle, \theta) \models_\nu \varphi$ iff $\mathcal{BP}_\varphi$ has an accepting run from $(\langle (p, q_0), w\rangle, prod(\theta))$ where $prod(\theta)$ is the set of rules of $\Delta \cup \Delta_c$ obtained from the rules of θ as described above.*

Thus, LTL model-checking for SM-PDSs can be reduced to checking whether a SM-BPDS has an accepting run. The rest of the paper is devoted to this problem.

4. The Emptiness Problem of SM-BPDSs

From now on, we fix a SM-BPDS $\mathcal{BP} = (P, \Gamma, \Delta, \Delta_c, G)$. Following [3], we can show that $\mathcal{BP}$ has an accepting run starting from a configuration c if and only if from c, it can reach a configuration with a repeating head:

Proposition 1. *A SM-BPDS $\mathcal{BP}$ has an accepting run starting from a configuration c if and only if there exists a repeating head $((p, \gamma), \theta)$ such that $c \Rightarrow^*_{\mathcal{BP}} (\langle p, \gamma w\rangle, \theta)$ for some $w \in \Gamma^*$.*

Proof: " $\Rightarrow$ ": Let $\sigma = c_0 c_1 ...$ be an accepting run starting at configuration c where $c_0 = c$ and $c_i = (\langle p_i, w_i\rangle, \theta_i)$. We construct an increasing sequence of indices $i_0, i_1 ...$ with a property that once any of the configurations c_{i_k} is reached, the rest of the run never changes the bottom $|w_{i_k}| - 1$ elements of the stack anymore. This property can be written as follows:

$$|w_{i_0}| = \min\{|w_j| \mid j \geq 0\}$$

$$|w_{i_k}| = \min\{|w_j| \mid j > i_{k-1}\}, k \geq 1$$

Because $\mathcal{BP}$ has only finitely many different heads, there must be a head $(\langle p, \gamma\rangle, \theta)$ which occurs infinitely often as a head in the sequence $c_{i_0} c_{i_1} ....$ Moreover, as some $g \in G$ becomes a control location infinitely often, we can find a subsequence of indices $i_{j_0}, i_{j_1}, ...$ with the following property: for every $k \geq 1$, there exist $v, w \in \Gamma^*$

$$c_{i_{j_k}} = (\langle p, \gamma w\rangle, \theta) \Rightarrow^r (\langle p, \gamma v w\rangle, \theta) = c_{i_{j_{k+1}}}$$

Because w is never looked at or changed in this path, we can have $(\langle p, \gamma\rangle, \theta) \Rightarrow^r (\langle p, \gamma v\rangle, \theta)$. This proves this direction of the proposition.

" $\Leftarrow$ ": Because $(\langle p, \gamma\rangle, \theta)$ is a repeating head, we can construct the following run for some $u, v, w \in \Gamma^*$, $\theta' \subseteq (\Delta \cup \Delta_c)$ and $g \in G$:

$$c \Rightarrow^* (\langle p, \gamma w\rangle, \theta) \Rightarrow^* (\langle g, u w\rangle, \theta') \Rightarrow^+ (\langle p, \gamma v w\rangle, \theta) \Rightarrow^* (\langle g, u v w\rangle, \theta') \Rightarrow^+ (\langle p, \gamma v v w\rangle, \theta) \Rightarrow^* ...$$

Since g occurs infinitely often, the run is accepting. $\square$

Thus, since there exists an efficient algorithm to compute the pre^* of SM-PDSs [1], the emptiness problem of a SM-BPDS can be reduced to computing its repeating heads.

4.1. The Head Reachability Graph $\mathcal{G}$

Our goal is to compute the set of repeating heads $Rep_{\mathcal{BP}}$, i.e., the set of heads $(\langle p,\gamma\rangle,\theta)$ such that there exists $v \in \Gamma^*$, $(\langle p,\gamma\rangle,\theta) \Rightarrow^r (\langle p,\gamma v\rangle,\theta)$. I.e., $(\langle p,\gamma\rangle,\theta) \Rightarrow^* (\langle p,\gamma v\rangle,\theta)$ s.t. this path goes through an accepting location in G. To this aim, we will compute a finite graph $\mathcal{G}$ whose nodes are the heads of $\mathcal{BP}$ of the form $((p,\gamma),\theta)$, where $p \in P$, $\gamma \in \Gamma$ and $\theta \subseteq \Delta \cup \Delta_c$; and whose edges encode the reachability relation between these heads. More precisely, given two heads $((p,\gamma),\theta)$ and $((p',\gamma'),\theta')$, $((p,\gamma),\theta) \xrightarrow{b} ((p',\gamma'),\theta')$ is an edge of the graph $\mathcal{G}$ means that the configuration $(\langle p,\gamma\rangle,\theta)$ can reach a configuration having $(\langle p',\gamma'\rangle,\theta')$ as head, i.e., it means that there exists $v \in \Gamma^*$ s.t. $(\langle p,\gamma\rangle,\theta) \Rightarrow^* (\langle p',\gamma' v\rangle,\theta')$. Moreover, we need to keep the information whether this path visits an accepting location in G or not. This information is recorded in the label of the edge b: $b = 1$ means that the path visits an accepting location in G, i.e. that $(\langle p,\gamma\rangle,\theta) \Rightarrow^r (\langle p',\gamma' v\rangle,\theta')$. Otherwise, $b = 0$. Therefore, if the graph $\mathcal{G}$ contains a loop from a head $((p,\gamma),\theta)$ to itself such that this loop goes through an edge labelled by 1, then $((p,\gamma),\theta)$ is a repeating head. Thus, computing $Rep_{\mathcal{BP}}$ can be reduced to computing the graph $\mathcal{G}$ and finding 1-labelled loops in this graph.

More precisely, we define the head reachability graph $\mathcal{G}$ as follows:

Definition 4. *The head reachability graph $\mathcal{G}$ is a tuple $(P \times \Gamma \times 2^{\Delta\cup\Delta_c}, \{0,1\}, \delta)$ such that $((p,\gamma),\theta) \xrightarrow{b} ((p',\gamma'),\theta')$ is an edge of δ iff:*

1. *there exists a transition $r_c : p \xhookrightarrow{(\sigma,\sigma')} p' \in \theta \cap \Delta_c$, $\gamma = \gamma'$, $\theta' = \theta \setminus \sigma \cup \sigma'$, and $b = 1$ iff $p \in G$;*
2. *there exists a transition $\langle p,\gamma\rangle \hookrightarrow \langle p',\gamma'\rangle \in \theta \cap \Delta, \theta = \theta'$ and $b = 1$ iff $p \in G$;*
3. *there exists a transition $\langle p,\gamma\rangle \hookrightarrow \langle p'',\gamma_1\gamma'\rangle \in \theta \cap \Delta$, for $\gamma_1 \in \Gamma$, $p'' \in P$, s.t. $(\langle p'',\gamma_1\rangle,\theta) \Rightarrow^*_{\mathcal{BP}} (\langle p',\epsilon\rangle,\theta')$, and $b = 1$ iff $p \in G$ or $(\langle p'',\gamma_1\rangle,\theta) \Rightarrow^r_{\mathcal{BP}} (\langle p',\epsilon\rangle,\theta')$*

Let $\mathcal{G}$ be the head reachability graph. We define $\underset{i}{\rightarrow}$ as follows: let $((p,\gamma),\theta)$ and $((p',\gamma'),\theta')$ be two heads of $\mathcal{BP}$. We write $((p,\gamma),\theta) \underset{i}{\rightarrow} ((p',\gamma'),\theta')$ iff $\exists$ booleans $b_1, b_2 ... b_i \in \{0,1\}$, $\exists$ heads $((p_j,\gamma_j),\theta_j), 0 \leq j \leq i$ s.t. $\mathcal{G}$ contains the path $((p_0,\gamma_0),\theta_0) \xrightarrow{b_1} ((p_1,\gamma_1),\theta_1) \xrightarrow{b_2} ... \xrightarrow{b_i} ((p_i,\gamma_i),\theta_i)$ where $((p_0,\gamma_0),\theta_0) = ((p,\gamma),\theta)$ and $((p_i,\gamma_i),\theta_i) = ((p',\gamma'),\theta')$.

Let $\rightarrow^*$ be the reflexive transitive closure of the graph relation $\xrightarrow{b}$, and let $\rightarrow^r$ be defined as follows: Given two heads $((p,\gamma),\theta)$ and $((p',\gamma'),\theta')$, $((p,\gamma),\theta) \rightarrow^r ((p',\gamma'),\theta')$ iff there is in $\mathcal{G}$ a path between $((p,\gamma),\theta)$ and $((p',\gamma'),\theta')$ that goes through a 1-labelled edge, i.e., iff there exist heads $((p_1,\gamma_1),\theta_1)$ and $((p_2,\gamma_2),\theta_2)$ s.t. $((p,\gamma),\theta) \rightarrow^* ((p_1,\gamma_1),\theta_1) \xrightarrow{1} ((p_2,\gamma_2),\theta_2) \rightarrow^* ((p',\gamma'),\theta')$.

We can show that:

Theorem 4.1. *Let $\mathcal{BP} = (P,\Gamma,\Delta,\Delta_c,G)$ be a self-modifying Büchi pushdown system, and let $\mathcal{G}$ be its corresponding head reachability graph. A head $((p,\gamma),\theta)$ of $\mathcal{BP}$ is repeating iff $\mathcal{G}$ has a loop on the node $((p,\gamma),\theta)$ that goes through a 1-labeled edge.*

To prove this theorem, we first need to prove the following lemma:

Lemma 1. *The relations $\rightarrow^*$ and $\rightarrow^r$ have the following properties: For any heads $((p,\gamma),\theta_1)$ and $((p',\gamma'),\theta_2)$:*

(a) $((p,\gamma),\theta_1) \rightarrow^* ((p',\gamma'),\theta_2)$ *iff* $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^* (\langle p',\gamma' v\rangle,\theta_2)$ *for some* $v \in \Gamma^*$.

(b) $((p,\gamma),\theta_1) \rightarrow^r ((p',\gamma'),\theta_2)$ *iff* $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^r (\langle p',\gamma' v\rangle,\theta_2)$ *for some* $v \in \Gamma^*$.

Proof: "$\Rightarrow$": Assume $((p,\gamma),\theta_1) \underset{i}{\rightarrow} ((p',\gamma'),\theta_2)$. We proceed by induction on i.

(a) **Basis.** $i = 0$. In this case, $((p,\gamma),\theta_1) = ((p',\gamma'),\theta_2)$, then we can get $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^* (\langle p,\gamma\rangle,\theta_1) = (\langle p',\gamma'\rangle,\theta_2)$

Step. $i > 0$. Then there exist $p_1 \in P, \gamma'' \in \Gamma^*$ and $\theta' \subseteq \Delta \cup \Delta_c$ such that $((p,\gamma),\theta_1) \underset{1}{\rightarrow} ((p_1,\gamma''),\theta') \xrightarrow[i-1]{} ((p',\gamma'),\theta_2)$. From the induction hypothesis, there exists $u \in \Gamma^*$ such that $(\langle p_1,\gamma''\rangle,\theta') \Rightarrow^* (\langle p',\gamma' u\rangle,\theta_2)$
Since $((p,\gamma),\theta_1) \rightarrow ((p_1,\gamma''),\theta')$, we have $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^* (\langle p_1,\gamma'' w\rangle,\theta')$ for $w \in \Gamma^*$, hence $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^* (\langle p',\gamma' uw\rangle,\theta_2)$.
The property holds.

(b) $((p,\gamma),\theta_1) \rightarrow^r ((p,\gamma),\theta_1)$ cannot hold for the case $i = 0$.

Basis. $i = 1$. In this case, $((p,\gamma),\theta_1) \rightarrow^r ((p',\gamma'),\theta_2)$, then we can get $p \in G$ and $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^r (\langle p',\gamma'\rangle,\theta_2)$. The property holds.

Step. $i > 0$. As done in the proof of part (a) of this lemma, there exists $p_1, \gamma'' \in \Gamma, \theta'' \subseteq \Delta \cup \Delta_c$ s.t. $((p,\gamma),\theta_1) \underset{1}{\rightarrow} ((p_1,\gamma''),\theta') \xrightarrow[i-1]{} ((p',\gamma'),\theta_2)$. Then if $((p,\gamma),\theta_1) \rightarrow^r ((p',\gamma'),\theta_2)$, either $((p_1,\gamma''),\theta') \rightarrow^r ((p',\gamma'),\theta_2)$ or $((p,\gamma),\theta_1) \xrightarrow{1} ((p_1,\gamma''),\theta')$ holds. In the first case i.e. $((p_1,\gamma''),\theta') \rightarrow^r ((p',\gamma'),\theta_2)$, by the induction hypothesis, we can have $(\langle p_1,\gamma''\rangle,\theta') \Rightarrow^r (\langle p',\gamma' u\rangle,\theta_2)$, hence, $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^r (\langle p',\gamma' u\rangle,\theta_2)$ holds

The second case depends on the rule applied to get $((p,\gamma),\theta_1) \xrightarrow{1} ((p_1,\gamma''),\theta')$ according to Definition 4.

- If this edge corresponds to a transition $r_c : p \xhookrightarrow{(\sigma,\sigma')} p_1 \in \theta_1$, then $\gamma = \gamma'', \theta' = \theta_1 \backslash \sigma \cup \sigma'$ and $p \in G$. Since we can obtain $(\langle p,\gamma\rangle,\theta_1) \Rightarrow_{\mathcal{BP}} (\langle p_1,\gamma\rangle,\theta') \Rightarrow^* (\langle p',\gamma' uw\rangle,\theta_2)$ from part (a) and $p \in G$, then $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^r (\langle p_1,\gamma\rangle,\theta') \Rightarrow^* (\langle p',\gamma' uw\rangle,\theta_2)$. This implies that $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^r (\langle p',\gamma' v\rangle,\theta_2)$ for some $v \in \Gamma^*$.

- If this edge corresponds to a transition $r : \langle p,\gamma\rangle \hookrightarrow \langle p_1,\gamma''\rangle \in \theta_1 \cap \Delta$, then $\theta' = \theta_1$ and $p \in G$. Since we can obtain $(\langle p,\gamma\rangle,\theta_1) \Rightarrow_{\mathcal{BP}} (\langle p_1,\gamma''\rangle,\theta_1) \Rightarrow^* (\langle p',\gamma' uw\rangle,\theta_2)$ from part (a) and $p \in G$, then $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^r (\langle p_1,\gamma''\rangle,\theta_1) \Rightarrow^* (\langle p',\gamma' uw\rangle,\theta_2)$. This implies that $(\langle p,\gamma\rangle,\theta_1) \Rightarrow^r (\langle p',\gamma' v\rangle,\theta_2)$ for some $v \in \Gamma^*$.

- If this edge corresponds to a transition $r : \langle p, \gamma\rangle \hookrightarrow \langle p'', \gamma_1\gamma''\rangle \in \theta_1$, then either $p \in G$ or $(\langle p'', \gamma_1\rangle, \theta_1) \Rightarrow^r (\langle p_1, \epsilon\rangle, \theta')$ holds. If $p \in G$, then we have $(\langle p, \gamma\rangle, \theta_1) \Rightarrow^r (\langle p'', \gamma_1\gamma''\rangle, \theta_1)$. Otherwise, $(\langle p'', v_1\gamma''w\rangle, \theta_1) \Rightarrow^r (\langle p_1, \gamma''w\rangle, \theta')$. Since we can obtain $(\langle p_1, \gamma''\rangle, \theta') \Rightarrow^* (\langle p', \gamma'u\rangle, \theta_2)$ from part (a). Therefore, $(\langle p, \gamma\rangle, \theta_1) \Rightarrow^r (\langle p_1, \gamma''\rangle, \theta') \Rightarrow^* (\langle p', \gamma'u\rangle, \theta_2)$. This implies that $(\langle p, \gamma\rangle, \theta_1) \Rightarrow^r (\langle p', \gamma'v\rangle, \theta_2)$ for some $v \in \Gamma^*$.

'$\Leftarrow$": Assume $(\langle p, \gamma\rangle, \theta_1) \overset{i}{\Rightarrow} (\langle p', \gamma'v\rangle, \theta_2)$. We proceed by induction on i.

(a) **Basis.** $i = 0$. In this case, $v = \epsilon$ and $(\langle p, \gamma\rangle, \theta_1) = (\langle p', \gamma'\rangle, \theta_2)$, then $((p, \gamma), \theta_1) \rightarrow^* ((p', \gamma'), \theta_2)$ holds.

Step. $i > 0$. Then there exist $p_1 \in P, u \in \Gamma^*$ and $\theta' \subseteq \Delta \cup \Delta_c$ such that $(\langle p, \gamma\rangle, \theta_1) \overset{1}{\Rightarrow} (\langle p_1, u\rangle, \theta') \overset{i-1}{\Rightarrow} (\langle p', \gamma'v\rangle, \theta_2)$. There are 2 cases:

1. Case $\theta' = \theta_1$: There must exist a rule $r : \langle p, \gamma\rangle \hookrightarrow \langle p_1, u\rangle \in \Delta$ such that $r \in \theta'$ and $|u| \geq 1$. Let l denote the minimal length of the stack on the path from $(\langle p_1, u\rangle, \theta_1)$ to $(\langle p', \gamma'v\rangle, \theta_2)$. Then u can be written as $u''\gamma_1u'$ where $|u'| = l - 1$ (that means u' will remain on the stack for the path). Furthermore, there exists p''' such that $(\langle p_1, u''\rangle, \theta_1) \Rightarrow^* (\langle p''', \epsilon\rangle, \theta'')$ for some $\theta'' \subseteq (\Delta_c \cup \Delta)$. We have $(\langle p, \gamma\rangle, \theta_1) \overset{k}{\Rightarrow} (\langle p''', \gamma_1u'\rangle, \theta'')$ for $k < i$. By the induction on i, we have $((p, \gamma), \theta_1) \rightarrow^* ((p''', \gamma_1), \theta'')$. Because u' has to remain on the stack for the rest of the path, v is of the form $v'u'$ for some $v' \in \Gamma^*$. That means $(\langle p''', \gamma_1\rangle, \theta'') \overset{j}{\Rightarrow} (\langle p', \gamma'v'\rangle, \theta_2)$ for $j < i$. By the induction hypothesis, $((p''', \gamma_1), \theta'') \rightarrow^* ((p', \gamma'), \theta_2)$ holds. Moreover, we have $((p, \gamma), \theta_1) \rightarrow^* ((p''', \gamma_1), \theta'')$, hence $((p, \gamma), \theta_1) \rightarrow^* ((p', \gamma'), \theta_2)$.

2. Case $\theta' \neq \theta_1$: There must be a rule $r_c : p \xhookrightarrow{(\sigma,\sigma')} p_1 \in \Delta_c$ such that $r_c \in \theta_1$ and $\sigma \cap \theta_1 \neq \emptyset$, then $\theta' = \theta_1 \setminus \sigma \cup \sigma'$. After the execution of r_c, the content of the stack will remain the same, thus, $u = \gamma$. Then $(\langle p, \gamma\rangle, \theta_1) \overset{1}{\Rightarrow} (\langle p_1, \gamma\rangle, \theta') \overset{i-1}{\Rightarrow} (\langle p', \gamma'v\rangle, \theta_2)$. By the induction hypothesis to $(\langle p_1, \gamma\rangle, \theta') \overset{i-1}{\Rightarrow} (\langle p', \gamma'v\rangle, \theta_2)$, we can obtain that $((p_1, \gamma), \theta') \rightarrow^* ((p', \gamma'), \theta_2)$. Since $(\langle p, \gamma\rangle, \theta_1) \overset{1}{\Rightarrow} (\langle p_1, \gamma\rangle, \theta')$, then we can have a path $((p, \gamma), \theta_1) \rightarrow ((p_1, \gamma), \theta') \rightarrow^* ((p', \gamma'), \theta_2)$ that implies $((p, \gamma), \theta_1) \rightarrow^* ((p', \gamma'), \theta_2)$. The property holds.

(b) $(\langle p, \gamma\rangle, \theta_1) \Rightarrow^r (\langle p, \gamma'v\rangle, \theta_1)$ is impossible in 0 steps.

Basis. $i = 1$. $(\langle p, \gamma\rangle, \theta_1) \Rightarrow^r (\langle p, \gamma\rangle, \theta_1)$, then $p \in G$. Thus, $((p, \gamma), \theta_1) \rightarrow^r ((p, \gamma), \theta_1)$ holds.

Step. $i > 1$. $(\langle p, \gamma\rangle, \theta_1) \Rightarrow^r (\langle p', \gamma'v\rangle, \theta_2)$ holds, then there exist $p_1 \in P, u \in \Gamma^*$ and $\theta' \subseteq \Delta \cup \Delta_c$ such that $(\langle p, \gamma\rangle, \theta_1) \overset{1}{\Rightarrow} (\langle p_1, u\rangle, \theta') \overset{i-1}{\Rightarrow} (\langle p', \gamma'v\rangle, \theta_2)$. Thus, either $(\langle p, \gamma\rangle, \theta_1) \Rightarrow^r (\langle p_1, u\rangle, \theta')$ or $(\langle p_1, u\rangle, \theta') \Rightarrow^r (\langle p', \gamma'v\rangle, \theta_2)$ holds.

The first case implies $p \in G$. There are 2 cases:

1. Case $\theta' = \theta_1$: then as in the previous proof of part (a), we can have a path $((p,\gamma),\theta_1) \to^* ((p''',\gamma_1),\theta'') \to^* ((p',\gamma'),\theta_2)$. Since $p \in G$, we get by Definition 4 $((p,\gamma),\theta_1) \to^* ((p''',\gamma_1),\theta'') \to^* ((p',\gamma'),\theta_2)$. Thus, we have that $((p,\gamma),\theta_1) \to^r ((p',\gamma'),\theta_2)$. The property holds.
2. Case $\theta' \neq \theta_1$: then as in the previous proof of part (a), we can have a path $((p,\gamma),\theta_1) \to ((p_1,\gamma),\theta') \to^* ((p',\gamma'),\theta_2)$. Since $p \in G$, we get $((p,\gamma),\theta_1) \xrightarrow{1} ((p_1,\gamma),\theta') \to^* ((p',\gamma'),\theta_2)$. Thus, we have that $((p,\gamma),\theta_1) \to^r ((p',\gamma'),\theta_2)$. The property holds.

In the second case, $(\langle p_1,u\rangle,\theta') \Rightarrow^r (\langle p',\gamma' v\rangle,\theta_2)$ holds. As previously, there are 2 cases:

1. Case $\theta' = \theta_1$: then as in case (a) we have $(\langle p_1,u\rangle,\theta_1) \Rightarrow^* (\langle p''',\gamma_1 u'\rangle,\theta'')$ and $(\langle p''',\gamma_1\rangle,\theta'') \Rightarrow^* (\langle p',\gamma' v'\rangle,\theta_2)$. If $(\langle p_1,u\rangle,\theta_1) \Rightarrow^r (\langle p',\gamma' v\rangle,\theta_2)$, then either $(\langle p_1,u\rangle,\theta_1) \Rightarrow^r (\langle p''',\gamma_1 u'\rangle,\theta'')$ or $(\langle p''',\gamma_1\rangle,\theta'') \Rightarrow^r (\langle p',\gamma' v'\rangle,\theta_2)$.
   - If $(\langle p_1,u\rangle,\theta_1) \Rightarrow^r (\langle p''',\gamma_1 u'\rangle,\theta'')$, let $u'' \in \Gamma^*$ s.t. $u = u''\gamma_1 u'$ and $(\langle p_1,u''\rangle,\theta_1) \Rightarrow^r (\langle p''',\epsilon\rangle,\theta'')$, then, we have $((p,\gamma),\theta_1) \to^r ((p''',\gamma_1),\theta'')$. We have $(\langle p,\gamma\rangle,\theta_1) \overset{k}{\Rightarrow} (\langle p''',\gamma_1 u'\rangle,\theta'')$ for $k < i$. By the induction on i, we have $((p,\gamma),\theta_1) \to^* ((p''',\gamma_1),\theta'')$. Because u' has to remain on the stack for the rest of the path, v is of the form $v'u'$ for some $v' \in \Gamma^*$. That means $(\langle p''',\gamma_1\rangle,\theta'') \overset{j}{\Rightarrow} (\langle p',\gamma' v'\rangle,\theta_2)$ for $j < i$. By the induction hypothesis, $((p''',\gamma_1),\theta'') \to^* ((p',\gamma'),\theta_2)$ holds. Moreover, we have $((p,\gamma),\theta_1) \to^* ((p''',\gamma_1),\theta'')$, hence $((p,\gamma),\theta_1) \to^* ((p',\gamma'),\theta_2)$. So we can have a path $((p,\gamma),\theta_1) \to^* ((p''',\gamma_1),\theta'') \to^* ((p',\gamma'),\theta_2)$, thus we have that $((p,\gamma),\theta_1) \to^r ((p',\gamma'),\theta_2)$;
   - If $(\langle p''',\gamma_1\rangle,\theta'') \Rightarrow^r (\langle p',\gamma' v'\rangle,\theta_2)$, then by the induction hypothesis we have $((p''',\gamma_1),\theta'') \to^r ((p',\gamma'),\theta_2)$. Thus, we can have a path $((p,\gamma),\theta_1) \to^* ((p''',\gamma_1),\theta'') \to^* ((p',\gamma'),\theta_2)$, then we have that $((p,\gamma),\theta_1) \to^r ((p',\gamma'),\theta_2)$;
2. Case $\theta' \neq \theta_1$: then $(\langle p_1,\gamma\rangle,\theta') \Rightarrow^r (\langle p',\gamma' v\rangle,\theta_2)$. By the induction hypothesis we have $((p_1,\gamma),\theta') \to^r ((p',\gamma'),\theta_2)$. Since $(\langle p,\gamma\rangle,\theta_1) \overset{1}{\Rightarrow} (\langle p_1,\gamma\rangle,\theta') \overset{i-1}{\Rightarrow} (\langle p',\gamma' v\rangle,\theta_2)$.
   By the induction hypothesis to $(\langle p_1,\gamma\rangle,\theta') \overset{i-1}{\Rightarrow} (\langle p',\gamma' v\rangle,\theta_2)$, we can obtain that $((p_1,\gamma),\theta') \to^* ((p',\gamma'),\theta_2)$. Since $(\langle p,\gamma\rangle,\theta_1) \overset{1}{\Rightarrow} (\langle p_1,\gamma\rangle,\theta')$, then we can have a path $((p,\gamma),\theta_1) \to ((p_1,\gamma),\theta') \to^* ((p',\gamma'),\theta_2)$. Thus, we have that $((p,\gamma),\theta_1) \to^r ((p',\gamma'),\theta_2)$;

Thus, the property holds.

□

Proof of Theorem 4.1

We can now prove Theorem 4.1.

Proof: Let $((p,\gamma),\theta)$ be a repeating head, then there exists some $v \in \Gamma^*, \theta \subseteq \Delta_c \cup \Delta$ such that $(\langle p,\gamma\rangle,\theta) \Rightarrow^r (\langle p,\gamma v\rangle,\theta)$. By Lemma 1, this is the case if and only if $((p,\gamma),\theta) \to^r ((p,\gamma),\theta)$. From the definition of $\to^r$, that means that there exist heads $((p_1,\gamma_1),\theta')$ and $((p_2,\gamma_2),\theta'')$ such that $((p,\gamma),\theta) \to^* ((p_1,\gamma_1),\theta') \xrightarrow{1} ((p_2,\gamma_2),\theta'') \to^* ((p,\gamma),\theta)$. Then $((p,\gamma),\theta), ((p_1,\gamma_1),\theta')$ and $((p_2,\gamma_2),\theta'')$ are all in the same loop with a 1-labelled edge. Conversely, whenever $((p,\gamma),\theta)$ is in a component with such an edge, $((p,\gamma),\theta) \to^r ((p,\gamma),\theta)$ holds, then Lemma 1 implies that $(\langle p, \gamma\rangle, \theta) \Rightarrow^r (\langle p, \gamma v\rangle, \theta)$ which means that $((p, \gamma), \theta)$ is a repeating head.

□

4.2. Labelled configurations and labelled $\mathcal{BP}$-automata

To compute $\mathcal{G}$, we need to be able to compute predecessors of configurations of the form $(\langle p', \epsilon\rangle, \theta')$, and to determine whether these predecessors were backward-reachable using some control points in G (item 3 in Definition 4). To solve this question, we will label configurations $(\langle p'', w\rangle, \theta)$ s.t. $(\langle p'', w\rangle, \theta) \Rightarrow^* (\langle p', \epsilon\rangle, \theta')$ by 1 if this path went through an accepting location in G, i.e., if $(\langle p'', w\rangle, \theta) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$, and by 0 if not. To this aim, we define a labelled configuration as a tuple $[(\langle p, w\rangle, \theta), b]$, s.t. $(\langle p, w\rangle, \theta)$ is a configuration and $b \in \{0, 1\}$.

Multi-automata were introduced in [2, 3] to finitely represent regular infinite sets of configurations of a PDS. Since a labelled configuration $c = [(\langle p, w\rangle, \theta), b]$ of a SM-PDS involves a PDS configuration $\langle p, w\rangle$, together with the current set of transition rules (phase) θ, and a boolean b, in order to take into account the phases θ, and these new 0/1-labels in configurations, we extend multi-automata to labelled $\mathcal{BP}$-automata as follows:

Definition 5. *Let $\mathcal{BP} = (P, \Gamma, \Delta, \Delta_c, G)$ be a SM-BPDS. A labelled $\mathcal{BP}$-automaton is a tuple $\mathcal{A} = (Q, \Gamma, T, I, F)$ where Γ is the automaton alphabet, Q is a finite set of states, $I \subseteq P \times 2^{\Delta \cup \Delta_c} \subseteq Q$ is the set of initial states, $T \subset Q \times \big((\Gamma \cup \{\epsilon\}) \times \{0, 1\}\big) \times Q$ is the set of transitions, $F \subseteq Q$ is the set of final states.*

If $\big(q, [\gamma, b], q'\big) \in T$, we write $q \xrightarrow{[\gamma,b]}_T q'$. We extend this notation in the obvious way to sequences of symbols: (1) $\forall q \in Q, q \xrightarrow{[\epsilon,0]}_T q$, and (2) $\forall q, q' \in Q, \forall b \in \{0, 1\}, \forall w \in \Gamma^*$ for $w = \gamma_0 ... \gamma_{n+1}, q \xrightarrow{[w,b]}_T q'$ iff $\exists q_0, ..., q_n \in Q, b_0, ..., b_{n+1} \in \{0, 1\}, b = b_0 \vee b_1 \vee ... \vee b_{n+1}$ and $q \xrightarrow{[\gamma_0,b_0]}_T q_0 \xrightarrow{[\gamma_1,b_1]}_T q_1 \cdots q_n \xrightarrow{[\gamma_{n+1},b_{n+1}]}_T q'$. If $q \xrightarrow{[w,b]}_T q'$ holds, we say that $q \xrightarrow{[w,b]}_T q'$ and $q \xrightarrow{[\gamma_0,b_0]}_T q_0 \xrightarrow{[\gamma_1,b_1]}_T q_1 \cdots q_n \xrightarrow{[\gamma_{n+1},b_{n+1}]}_T q'$ is a path of $\mathcal{A}$.

A labelled configuration $[(\langle p, w\rangle, \theta), b]$ is accepted by the automaton $\mathcal{A}$ iff there exists a path $(p, \theta) \xrightarrow{[\gamma_0,b_0]}_T q_1 \xrightarrow{[\gamma_1,b_1]}_T q_2 \cdots q_n \xrightarrow{[\gamma_n,b_n]}_T q_{n+1}$ in $\mathcal{A}$ such that $w = \gamma_0\gamma_1 \cdots \gamma_n$, $b = b_0 \vee b_1 \vee ... \vee b_n$, $(p, \theta) \in I$, and $q_{n+1} \in F$. Let $L(\mathcal{A})$ be the set of labelled configurations accepted by $\mathcal{A}$.

4.3. Computing $pre^*\big((\langle p', \epsilon\rangle, \theta')\big)$

Given a configuration of the form $(\langle p', \epsilon\rangle, \theta')$, our goal is to compute a labelled $\mathcal{BP}$-automaton $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$ that accepts labelled configurations of the form $[c, b]$ where c is a configuration and $b \in \{0, 1\}$ such that $c \Rightarrow^* (\langle p', \epsilon\rangle, \theta')$ (i.e., $c \in pre^*\big((\langle p', \epsilon\rangle, \theta')\big)$) and $b = 1$ iff this path went through final control points, i.e., $c \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$. Otherwise, $b = 0$.
Let $p \in P$, we define $B(p) = 1$ if $p \in G$ and $B(p) = 0$ otherwise. $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big) = (Q, \Gamma, T, I, F)$ is computed as follows: Initially, $Q = I = F = \{(p', \theta')\}$ and $T = \emptyset$. We add to T transitions as follows:

α_1: If $r = \langle p, \gamma \rangle \hookrightarrow \langle p_1, w \rangle \in \Delta$. If there exists in T a path $(p_1, \theta) \xrightarrow{[w,b]}_T q$ (in case $|w| = 0$, we have $w = \epsilon$) with $r \in \theta$. Then, add (p, θ) to I, and $\big((p, \theta), [\gamma, B(p) \vee b], q\big)$ to T.

α_2: if $r = p \xhookrightarrow{(\sigma,\sigma')} p_1 \in \Delta_c$ and there exists in T a transition $(p_1, \theta) \xrightarrow{[\gamma,b]}_T q$ with $r \in \theta$, where $\gamma \in \Gamma$. Then add (p, θ') to I, and $\big((p, \theta'), [\gamma, B(p) \vee b], q\big)$ to T, for θ' such that $\theta = \theta' \setminus \sigma \cup \sigma'$.

The procedure above terminates since there is a finite number of states and phases. Note that by construction, $F = \{(p', \theta')\}$, and, since initially $Q = \{(p', \theta')\}$, states of $\mathcal{A}_{pre^*}\big((\langle p', \epsilon \rangle, \theta')\big)$ are all of the form (p, θ) for $p \in P$ and $\theta \subseteq \Delta \cup \Delta_c$.
Let us explain the intuition behind rule (α_1). Let $r = \langle p, \gamma \rangle \hookrightarrow \langle p_1, w \rangle \in \Delta$. Let $c = (\langle p_1, ww' \rangle, \theta)$ and $c' = (\langle p, \gamma w' \rangle, \theta)$. Then, if $c \Rightarrow^* (\langle p', \epsilon \rangle, \theta')$, then necessarily, $c' \Rightarrow^* (\langle p', \epsilon \rangle, \theta')$. Moreover, $c' \Rightarrow^r (\langle p', \epsilon \rangle, \theta')$ iff either $c \Rightarrow^r (\langle p', \epsilon \rangle, \theta')$ or $p \in G$ (i.e. $B(p) = 1$). Thus, we would like that if the automaton $\mathcal{A}_{pre^*}\big((\langle p', \epsilon \rangle, \theta')\big)$ accepts the labelled configuration $[c, b]$ (where $b = 1$ means $c \Rightarrow^r (\langle p', \epsilon \rangle, \theta')$), then it should also accept the labelled configuration $[c', b \vee B(p)]$ ($b \vee B(p) = 1$ means $c' \Rightarrow^r (\langle p', \epsilon \rangle, \theta')$). Thus, if the automaton $\mathcal{A}_{pre^*}\big((\langle p', \epsilon \rangle, \theta')\big)$ contains a path of the form $\pi = (p_1, \theta) \xrightarrow{[w,b_1]}_T q \xrightarrow{[w',b_2]}_T q_f$ where $q_f \in F$ that accepts the labelled configuration $[c, b]$, then the automaton should also accept the labelled configuration $[c', b \vee B(p)]$. This configuration is accepted by the run $(p, \theta) \xrightarrow{[\gamma, B(p) \vee b_1]}_T q \xrightarrow{[w',b_2]}_T q_f$ added by rule (α_1).

Rule (α_2) deals with modifying rules: Let $r = p \xhookrightarrow{(r_1,r_2)} p_1 \in \Delta_c$. Let $c = (\langle p_1, \gamma w' \rangle, \theta)$ and $c' = (\langle p, \gamma w' \rangle, \theta'')$ s.t. $\theta = \theta'' \backslash \{r_1\} \cup \{r_2\}$. Then, if $c \Rightarrow^* (\langle p', \epsilon \rangle, \theta')$, then necessarily, $c' \Rightarrow^* (\langle p', \epsilon \rangle, \theta')$. Moreover, $c' \Rightarrow^r (\langle p', \epsilon \rangle, \theta')$ iff either $c \Rightarrow^r (\langle p', \epsilon \rangle, \theta')$ or $p \in G$ (i.e. $B(p) = 1$). Thus, we need to impose that if the automaton $\mathcal{A}_{pre^*}\big((\langle p', \epsilon \rangle, \theta')\big)$ contains a path of the form $(p_1, \theta) \xrightarrow{[\gamma,b_1]}_T q \xrightarrow{[w',b_2]}_T q_f$ (where $q_f \in F$) that accepts the labelled configuration $[c, b], b = b_1 \vee b_2$ ($b = 1$ means $c \Rightarrow^r (\langle p', \epsilon \rangle, \theta')$), then necessarily, the automaton $\mathcal{A}_{pre^*}\big((\langle p', \epsilon \rangle, \theta')\big)$ should also accept the labelled configuration $[c', b \vee B(p)]$. This configuration is accepted by the run $(p, \theta'') \xrightarrow{[\gamma, B(p) \vee b_1]}_T q \xrightarrow{[w',b_2]}_T q_f$ added by rule (α_2).

4.3.1. Example

Let us illustrate the procedure by an example. Consider the SM-BPDS $\mathcal{BP} = (P, \Gamma, \Delta, \Delta_c, G)$ shown in the left (i.e. part a) of Fig.2 where $P = \{p_1, p_2, p_3, p'\}, \Delta = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}, \Delta_c = \{r'\}$ and $G = \{p_2\}$. We show how to compute a $\mathcal{BP}$-automaton $\mathcal{A}_{pre^*}((\langle p', \epsilon \rangle, \theta'))$. Let $\mathcal{A}$ be the automaton that accepts the set $\{(\langle p', \epsilon \rangle, \theta')\}$ with $Q = I = F = \{(p', \theta')\}$. Initially, T is empty. The result is obtained through the following steps:

1. First we note that $(p', \theta') \xrightarrow{[\epsilon,b]}_T (p', \theta'), b = 0$ holds. Since $\langle p', \epsilon \rangle$ occurs on the right hand side of rule $r_1 \in \theta'$ and $r_3 \in \theta'$, moreover, $p_1 \notin G$ i.e. $B(p_1) = 0$ and $p' \notin G$ i.e. $B(p') = 0$, then Rule (α_1) adds the transition $(p_1, \theta') \xrightarrow{[\gamma_1,b_1]}_T (p', \theta')$ with $b_1 = B(p_1) \vee b = 0$ and $(p', \theta') \xrightarrow{[\gamma',b_2]}_T (p', \theta')$ with $b_2 = B(p') \vee b = 0$.

Δ :

$r_1 : \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p', \epsilon \rangle \quad r_2 : \langle p_2, \gamma_2 \rangle \hookrightarrow \langle p_1, \gamma_1 \gamma' \rangle$

$r_3 : \langle p', \gamma' \rangle \hookrightarrow \langle p', \epsilon \rangle \quad r_4 : \langle p_2, \gamma' \rangle \hookrightarrow \langle p_3, \gamma_2 \rangle$

$r_5 : \langle p', \gamma' \rangle \hookrightarrow \langle p_2, \gamma' \rangle \quad r_6 : \langle p_2, \gamma' \rangle \hookrightarrow \langle p_1, \gamma_1 \rangle$

$r_7 : \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_2, \gamma_2 \rangle$

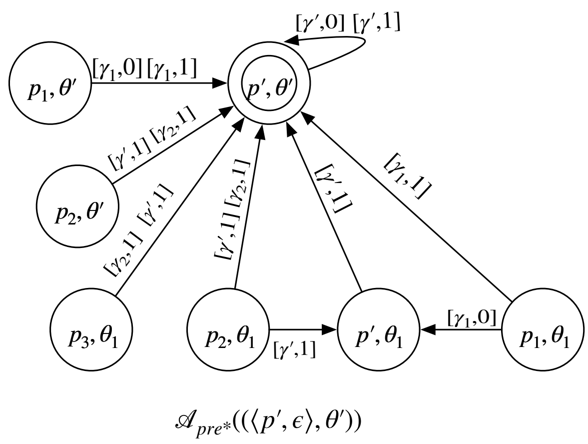
Δ_c :

$r' : p_3 \xrightarrow{(\{r_4\},\{r_3\})} p_2$

$\theta' = \{r_1, r_2, r_3, r_5, r_6, r_7, r'\}$, $\theta_1 = \{r_1, r_2, r_4, r_5, r_6, r_7, r'\}$

$G = \{p_2\}$, $\Gamma = \{\gamma_1, \gamma_2, \gamma'\}$

a



b

Figure 2: $\mathcal{BP}$-automaton $\mathcal{A}_{pre^*}((\langle p', \epsilon \rangle, \theta'))$

2. Now that we have the path $(p_1, \theta') \xrightarrow{[\gamma_1, b_1]}_T (p', \theta') \xrightarrow{[\gamma', b_2]}_T (p', \theta')$, $b_1 = 0$ and $b_2 = 0$, since $r_2 \in \theta'$, moreover, $p_2 \in G$ i.e. $B(p_2) = 1$ then Rule (α_1) adds $(p_2, \theta') \xrightarrow{[\gamma_2, b]}_T (p', \theta')$ with $b = B(p_2) \vee b_1 \vee b_2 = 1$.
3. Since we have $(p_1, \theta') \xrightarrow{[\gamma_1, b_1]}_T (p', \theta')$, $b_1 = 0$ and $r_6 \in \theta'$, moreover, $p_2 \in G$ i.e. $B(p_2) = 1$, then Rule (α_1) adds the transition $(p_2, \theta') \xrightarrow{[\gamma', b]}_T (p', \theta')$ with $b = B(p_2) \vee b_1 = 1$.
4. Now we have $(p_2, \theta') \xrightarrow{[\gamma_2, b_1]}_T (p', \theta')$, $b_1 = 1$ and $r_7 \in \theta'$, moreover, $p_2 \in G$ i.e. $B(p_2) = 1$, then Rule (α_1) adds the transition $(p_1, \theta') \xrightarrow{[\gamma_1, b]}_T (p', \theta')$ with $b = B(p_2) \vee b_1 = 1$.
5. Now we have $(p_2, \theta') \xrightarrow{[\gamma', b_1]}_T (p', \theta')$, $b_1 = 1$ and $r_5 \in \theta'$, moreover, $p' \notin G$ i.e. $B(p') = 0$, then Rule (α_1) adds the transition $(p', \theta') \xrightarrow{[\gamma', b]}_T (p', \theta')$ with $b = B(p') \vee b_1 = 1$.
6. Since we have $(p_2, \theta') \xrightarrow{[\gamma', b_1]}_T (p', \theta')$ and $(p_2, \theta') \xrightarrow{[\gamma_2, b_2]}_T (p', \theta')$, $b_1 = 1, b_2 = 1$, the self-modifying rule $r' \in \theta'$ can be applied. Moreover, $p_3 \notin G$ i.e. $B(p_3) = 0$ Thus, Rule (α_2) adds $(p_3, \theta_1) \xrightarrow{[\gamma_2, b]}_T (p', \theta')$ and $(p_3, \theta_1) \xrightarrow{[\gamma', b']}_T (p', \theta')$ where $\theta_1 = (\theta' \setminus \{r_3\}) \cup \{r_4\}$ with $b = B(p_3) \vee b_1 = 1, b' = B(p_3) \vee b_2 = 1$.
7. Now we have $(p_3, \theta_1) \xrightarrow{[\gamma_2, b_1]}_T (p', \theta')$, $b_1 = 1$ and $r_4 \in \theta_1$, moreover, $p_2 \in G$ i.e. $B(p_2) = 1$, then Rule (α_1) adds the transition $(p_2, \theta_1) \xrightarrow{[\gamma', b]}_T (p', \theta')$ with $b = B(p_2) \vee b_1 = 1$.
8. Since $(p_2, \theta_1) \xrightarrow{[\gamma', b_1]}_T (p', \theta')$, $b_1 = 1$ and $r_5 \in \theta_1$, moreover, $p_2 \in G$ i.e. $B(p_2) = 1$, then Rule (α_1) adds the transition $(p', \theta_1) \xrightarrow{[\gamma', b]}_T (p', \theta')$ with $b = B(p_2) \vee b_1 = 1$.

9. We note that $(p', \theta_1) \xrightarrow{[\epsilon, b_1]}_T (p', \theta_1)$, $b_1 = 0$ holds. Since $\langle p', \epsilon\rangle$ occurs on the right hand side of rule $r_1 \in \theta_1$, moreover, $p_1 \notin G$ i.e. $B(p_1) = 0$ and $p' \notin G$ i.e. $B(p') = 0$, then Rule (α_1) adds the transition $(p_1, \theta_1) \xrightarrow{[\gamma_1, b]}_T (p', \theta')$ with $b = B(p_1) \vee b_1 = 0$.
10. Now we have $(p_1, \theta_1) \xrightarrow{[\gamma_1, b_1]}_T (p', \theta_1)$, $b_1 = 0$ and $r_6 \in \theta_1$, moreover, $p_2 \in G$ i.e. $B(p_2) = 1$, then Rule (α_1) adds the transition $(p_2, \theta_1) \xrightarrow{[\gamma', b]}_T (p', \theta_1)$ with $b = B(p_2) \vee b_1 = 1$.
11. Now that we have the path $(p_1, \theta_1) \xrightarrow{[\gamma_1, b_1]}_T (p', \theta_1) \xrightarrow{[\gamma', b_2]}_T (p', \theta')$, since $r_2 \in \theta_1$, $b_1 = 0, b_2 = 1$, moreover, $p_2 \in G$ i.e. $B(p_2) = 1$ then Rule (α_1) adds $(p_2, \theta_1) \xrightarrow{[\gamma_2, b]}_T (p', \theta')$ with $b = B(p_2) \vee b_1 \vee b_2 = 1$.
12. Since we have $(p_2, \theta_1) \xrightarrow{[\gamma_2, b_1]}_T (p', \theta')$, $b_1 = 1$ and $r_7 \in \theta_1$, Rule (α_1) adds the transition $(p_1, \theta_1) \xrightarrow{[\gamma_1, b]}_T (p', \theta')$ with $b = B(p_1) \vee b_1 = 1$.
13. No further additions are possible. Thus, the procedure terminates.

The result is depicted in the right side of Fig.2

4.3.2. Proof

Before proving that our construction is correct, we introduce the following definition:

Definition 6. *Let $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big) = (Q, \Gamma, T, P, F)$ be the labelled $\mathcal{P}$-automaton computed by the saturation procedure above. In this section, we use $\underset{i}{\rightarrow}_T$ to denote the transition relation of $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$ obtained after adding i transitions using the saturation procedure above. Let us notice that due to the fact that initially $Q = \{(p', \theta')\}$ and due to rules (α_1) and (α_2) that at step i add only transitions of the form $(p, \theta) \xrightarrow{\gamma}_T q$ for a state q that is already in the automaton at step $i-1$, then, states of $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$ are all of the form (p, θ) for $p \in P$ and $\theta \subseteq \Delta \cup \Delta_c$.*

We can show that:

Lemma 2. *Let $p, p'' \in P$ and $\theta, \theta'' \subseteq \Delta \cup \Delta_c$. Let $w \in \Gamma^*$ and $b \in \{0, 1\}$. If a path $(p, \theta) \xrightarrow{[w,b]}_T (p'', \theta'')$ is in $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$, then $(\langle p, w\rangle, \theta) \Rightarrow^* (\langle p'', \epsilon\rangle, \theta'')$. Moreover, if $b = 1$, then $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'')$.*

Proof: Initially, the automaton contains no transitions. Let i be an index such that $(p, \theta) \underset{i}{\xrightarrow{[w,b]}}_T (p'', \theta'')$ holds. We proceed by induction on i.

Basis. $i = 0$, then $(p'', \theta'') \underset{0}{\xrightarrow{[\epsilon,0]}}_T (p'', \theta'')$. This means $p'' = p'$, $\theta'' = \theta'$. Since initially $Q = \{(p', \theta')\}$, then $(\langle p'', \epsilon\rangle, \theta'') \Rightarrow^* (\langle p'', \epsilon\rangle, \theta'')$ always holds.

Step. $i > 0$. Let $t = \big((p_1, \theta_1), [\gamma, b_1], (p_0, \theta_0)\big)$ be the i-th transition added to $\mathcal{A}_{pre^*}$ and j be the number of times that t is used in the path $(p, \theta) \underset{i}{\xrightarrow{[w,b]}}_T (p'', \theta'')$. The proof is by induction on j. If $j = 0$, then we have $(p, \theta) \underset{i-1}{\xrightarrow{[w,b]}}_T (p'', \theta'')$ in the automaton, and we apply the induction hypothesis (induction on i) then we obtain $(\langle p, w\rangle, \theta) \Rightarrow^* (\langle p'', \epsilon\rangle, \theta'')$. So assume that $j > 0$. Then, there exist $u, v \in \Gamma^*$, $b', b'' \in \{0,1\}$ such that $w = u\gamma v$, $b = b' \vee b_1 \vee b''$ and

$$(p,\theta) \xrightarrow[i-1]{[u,b']}_T (p_1,\theta_1) \xrightarrow[i]{[\gamma,b_1]}_T (p_0,\theta_0) \xrightarrow[i]{[v,b'']}_T (p'',\theta'') \tag{1}$$

The application of the induction hypothesis (induction on i) to $(p,\theta) \xrightarrow[i-1]{[u,b']}_T (p_1,\theta_1)$ gives that

$$(\langle p, u\rangle, \theta) \Rightarrow^* (\langle p_1, \epsilon\rangle, \theta_1), \text{ moreover, if } b' = 1, (\langle p, u\rangle, \theta) \Rightarrow^r (\langle p_1, \epsilon\rangle, \theta_1) \tag{2}$$

There are 2 cases depending on whether transition t was added by saturation rule α_1 or α_2.

1. Case t was added by rule α_1: There exist $p_2 \in P$ and $w_2 \in \Gamma^*$ such that

$$r = \langle p_1, \gamma\rangle \hookrightarrow \langle p_2, w_2\rangle \in \Delta \cap \theta_1 \tag{3}$$

and $\mathcal{A}_{pre^*}$ contains the following path:

$$\pi' = (p_2,\theta_1) \xrightarrow[i-1]{[w_2,b_2]}_T (p_0,\theta_0) \xrightarrow[i]{[v,b'']}_T (p'',\theta''), \quad b_1 = b_2 \vee B(p_1) \tag{4}$$

Applying the transition rule r, we get that

$$(\langle p_1, \gamma v\rangle, \theta_1) \Rightarrow (\langle p_2, w_2 v\rangle, \theta_1) \tag{5}$$

By induction on j (since transition t is used $j-1$ times in π'), we get from (4) that

$$(\langle p_2, w_2 v\rangle, \theta_1) \Rightarrow^* (\langle p'', \epsilon\rangle, \theta'') \text{ moreover, if } b_2 \vee b'' = 1, (\langle p_2, w_2 v\rangle, \theta_1) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'') \tag{6}$$

Putting (2), (5) and (6) together, we can obtain that

$$(\langle p, w\rangle, \theta) = (\langle p, u\gamma v\rangle, \theta) \Rightarrow^* (\langle p_1, \gamma v\rangle, \theta_1) \Rightarrow (\langle p_2, w_2 v\rangle, \theta_1) \Rightarrow^* (\langle p'', \epsilon\rangle, \theta'')$$

Furthermore, if $b = b' \vee b_1 \vee b'' = 1$, then $b' = 1$ or $b_1 \vee b'' = 1$.

For the first case, $b' = 1$, then we can have $(\langle p, u\rangle, \theta) \Rightarrow^r (\langle p_1, \epsilon\rangle, \theta_1)$ from (2). Thus, we can obtain that $(\langle p, u\gamma v\rangle, \theta) \Rightarrow^r (\langle p_1, \gamma v\rangle, \theta_1) \Rightarrow^* (\langle p'', \epsilon\rangle, \theta'')$ i.e. $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'')$.

The second case $b_1 \vee b'' = 1$ i.e. $B(p_1) \vee b_2 \vee b'' = 1$ implies that $B(p_1) = 1$ (that means $p_1 \in G$ and $(\langle p_1, \gamma v\rangle, \theta_1) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'')$) or $b_2 \vee b'' = 1$ (that implies $(\langle p_2, w_2 v\rangle, \theta_1) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'')$ from (6)). Therefore, $(\langle p, w\rangle, \theta_1) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'')$.

2. Case t was added by rule α_2 : there exist $p_2 \in P$ and $\theta_2 \subseteq \Delta \cup \Delta_c$ such that

$$r = p_1 \xhookrightarrow{(\sigma,\sigma')} p_2 \in \Delta_c \cap \theta_2, \theta_2 = (\theta_1 \backslash \sigma) \cup \sigma' \tag{7}$$

and the following path in the current automaton (self-modifying rule won't change the stack) with $r \in \theta_2$:

$$(p_2,\theta_2) \xrightarrow[i-1]{[\gamma,b_1']}_T (p_0,\theta_0) \xrightarrow[i]{[v,b'']}_T (p'',\theta''), \quad b_1 = B(p_1) \vee b_1' \tag{8}$$

Applying the transition rule, we can get from (7) that

$$(\langle p_1, \gamma v\rangle, \theta_1) \Rightarrow (\langle p_2, \gamma v\rangle, \theta_2) \tag{9}$$

We can apply the induction hypothesis (on j) to (8), and obtain

$$(\langle p_2, \gamma v\rangle, \theta_2) \Rightarrow^* (\langle p'', \epsilon\rangle, \theta''), \text{ moreover, if } b'_1 \vee b'' = 1, (\langle p_2, \gamma v\rangle, \theta_2) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'') \tag{10}$$

From (2),(9) and (10), we get

$$(\langle p, w\rangle, \theta) = (\langle p, u\gamma v\rangle, \theta) \Rightarrow^* (\langle p_1, \gamma v\rangle, \theta_1) \Rightarrow (\langle p_2, \gamma v\rangle, \theta_2) \Rightarrow^* (\langle p'', \epsilon\rangle, \theta'')$$

Furthermore, if $b = b' \vee b_1 \vee b'' = 1$, then $b' = 1$ or $b_1 \vee b'' = 1$.

For the first case, $b' = 1$, then we can have $(\langle p, u\rangle, \theta) \Rightarrow^r (\langle p_1, \epsilon\rangle, \theta_1)$ from (2). Thus, we can obtain that $(\langle p, u\gamma v\rangle, \theta) \Rightarrow^r (\langle p_1, \gamma v\rangle, \theta_1) \Rightarrow^* (\langle p'', \epsilon\rangle, \theta'')$ i.e. $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'')$. The second case $b_1 \vee b'' = 1$ i.e. $B(p_1) \vee b'_1 \vee b'' = 1$ implies that $B(p_1) = 1$ (that means $p_1 \in G$ and $(\langle p_1, \gamma v\rangle, \theta_1) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$) or $b'_1 \vee b'' = 1$ (that implies $(\langle p_2, \gamma v\rangle, \theta_2) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'')$ from (10)) i.e. $(\langle p, w\rangle, \theta_1) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$. Therefore, we can get that if $b = 1$, then $(\langle p, w\rangle, \theta_1) \Rightarrow^r (\langle p'', \epsilon\rangle, \theta'')$.

□

Lemma 3. *If there is a labelled configuration $[(\langle p, w\rangle, \theta), b]$ such that $(\langle p, w\rangle, \theta) \Rightarrow^* (\langle p', \epsilon\rangle, \theta')$, then there is a path $(p, \theta) \xrightarrow{[w,b]}_T (p', \theta')$ in $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$. Moreover, if $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$, then $b = 1$.*

Proof: Assume $(\langle p, w\rangle, \theta) \overset{i}{\Rightarrow} (\langle p', \epsilon\rangle, \theta')$. We proceed by induction on i.

Basis. $i = 0$. Then $\theta = \theta', p' = p$ and $w = \epsilon$. Initially, we have that $Q = \{(p', \theta')\}$, therefore, by the definition of $\rightarrow_T$, we have $(p', \theta') \xrightarrow{\epsilon}_T (p', \theta')$. We cannot have $(\langle p', \epsilon\rangle, \theta') \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$ in 0-step.

Step. $i > 0$. Then there exists a configuration $(\langle p'', u\rangle, \theta'')$ such that

$$(\langle p, w\rangle, \theta) \Rightarrow (\langle p'', u\rangle, \theta'') \overset{i-1}{\Rightarrow} (\langle p', \epsilon\rangle, \theta')$$

We apply the induction hypothesis to $(\langle p'', u\rangle, \theta'') \overset{i-1}{\Rightarrow} (\langle p', \epsilon\rangle, \theta')$, and obtain that there exists in $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$ a path $(p'', \theta'') \xrightarrow{[u,b'']}_T (p', \theta')$. If $(\langle p'', u\rangle, \theta'') \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$, $b'' = 1$.

Let (p_0, θ_0) be a state of $\mathcal{A}_{pre^*}$. Let $w_1, u_1 \in \Gamma^*, \gamma \in \Gamma, b''_0, b''_1 \in \{0, 1\}$ be such that $w = \gamma w_1$, $u = u_1 w_1$, $b'' = b''_0 \vee b''_1$ and

$$(p'', \theta'') \xrightarrow{[u_1, b''_0]}_T (p_0, \theta_0) \xrightarrow{[w_1, b''_1]}_T (p', \theta') \tag{1}$$

There are two cases depending on which rule is applied to get $(\langle p, w\rangle, \theta) \Rightarrow (\langle p'', u\rangle, \theta'')$.

1. Case $(\langle p, w\rangle, \theta) \Rightarrow (\langle p'', u\rangle, \theta'')$ is obtained by a rule of the form: $\langle p, \gamma\rangle \hookrightarrow \langle p'', u_1\rangle \in \Delta$. In this case, $\theta'' = \theta$. By the saturation rule α_1, we have

$$(p, \theta'') \xrightarrow{[\gamma, b_0]}_T (p_0, \theta_0),\ b_0 = B(p) \vee b_0'' \tag{2}$$

Putting (1) and (2) together, we can obtain that

$$\pi = (p, \theta'') \xrightarrow{[\gamma, b_0]}_T (p_0, \theta_0) \xrightarrow{[w_1, b_1'']}_T (p', \theta') \tag{3}$$

Thus, $(p, \theta'') \xrightarrow{[\gamma w_1, b_0 \vee b_1'']}_T (p', \theta')$ i.e. $(p, \theta) \xrightarrow{[w, b]}_T (p', \theta')$ where $b = b_0 \vee b_1''$.

2. Case $(\langle p, w\rangle, \theta) \Rightarrow (\langle p'', u\rangle, \theta'')$ is obtained by a rule of the form $p \xhookrightarrow{(\sigma, \sigma')} p'' \in \Delta_c$ i.e $\theta'' \neq \theta$. In this case, $u_1 = \gamma$. By the saturation rule β_2, we obtain that

$$(p, \theta) \xrightarrow{[\gamma, b_0]}_T (p_0, \theta_0) \text{ where } \theta'' = \theta \backslash \{r_1\} \cup \{r_2\}, b_0 = B(p) \vee b_0''. \tag{4}$$

Putting (1) and (4) together, we have the following path

$$(p, \theta) \xrightarrow{[\gamma, b_0]}_T (p_0, \theta_0) \xrightarrow{[w_1, b_1'']}_T (p', \theta') \text{ i.e. } (p, \theta) \xrightarrow{[w, b]}_T (p', \theta') \text{ where } b = b_0 \vee b_1'' \tag{5}$$

Furthermore, if $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$, then $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p'', u\rangle, \theta'')$ or $(\langle p'', u\rangle, \theta'') \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$.

For the first case, $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p'', u\rangle, \theta'')$, then $p \in G$ i.e. $B(p) = 1$. For the second case, $(\langle p'', u\rangle, \theta'') \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$, we can get $b'' = 1$ (from induction hypothesis). Thus, $b = b_0 \vee b_1'' = B(p) \vee b_0'' \vee b_1'' = B(p) \vee b'' = 1$. Therefore, if $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$, then we can obtain $b = 1$.

□

From these two lemmas, we get:

Theorem 4.2. *Let $[c, b]$ be a labelled configuration. Then $[c, b]$ is in $L(\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$ iff $c \in pre^*\big((\langle p', \epsilon\rangle, \theta')\big)$. Moreover, $c \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$ iff $b = 1$.*

Proof: Let $[(\langle p, w\rangle, \theta), b]$ be a labelled configuration of $pre^*\big((\langle p', \epsilon\rangle, \theta')\big)$. Then $(\langle p, w\rangle, \theta) \Rightarrow^* (\langle p', \epsilon\rangle, \theta')$. By Lemma 2, we can obtain that there exists a path $(p, \theta) \xrightarrow{[w, b]}_T (p', \theta')$ in $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$. So $[(\langle p, w\rangle, \theta), b]$ is in $L(\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big))$. Moreover, if $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$, then $b = 1$.

Conversely, let $[(\langle p, w\rangle, \theta), b]$ be a labelled configuration accepted by $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$ i.e. there exists a path $(p, \theta) \xrightarrow{[w, b]}_T (p', \theta')$ in $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$. By Lemma 3, $(\langle p, w\rangle, \theta) \Rightarrow^* (\langle p', \epsilon\rangle, \theta')$ i.e. $(\langle p, w\rangle, \theta) \in pre^*(L(A))$. Moreover, if $b = 1$, $(\langle p, w\rangle, \theta) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$.

□

4.4. Computing the Head Reachability Graph $\mathcal{G}$

Based on the definition of the Head Reachability Graph $\mathcal{G}$, and on Theorem 4.2, we can compute $\mathcal{G}$ as follows. Initially, $\mathcal{G}$ has no edges.

α'_1: if $r_c : p \xhookrightarrow{(\sigma,\sigma')} p' \in \Delta_c$, then for every phase θ such that $r_c \in \theta$ and every $\gamma \in \Gamma$, we add the edge $((p,\gamma),\theta) \xrightarrow{B(p)} ((p',\gamma),\theta_0)$ to the graph $\mathcal{G}$, where $\theta_0 = \theta \setminus \sigma \cup \sigma'$.

α'_2: if $r : \langle p,\gamma\rangle \hookrightarrow \langle p_0,\gamma_0\rangle \in \Delta$, then for every phase θ such that $r \in \theta$, we add the edge $((p,\gamma),\theta) \xrightarrow{B(p)} ((p_0,\gamma_0),\theta)$ to the graph $\mathcal{G}$.

α'_3: if $r : \langle p,\gamma\rangle \hookrightarrow \langle p_0,\gamma_0\gamma'\rangle \in \Delta$, then for every phase θ such that $r \in \theta$, we add to the graph $\mathcal{G}$ the edge $((p,\gamma),\theta) \xrightarrow{B(p)} ((p_0,\gamma_0),\theta)$. Moreover, for every control point $p' \in P$ and phase θ' such that $\mathcal{A}_{pre^*}\big((\langle p',\epsilon\rangle,\theta')\big)$ contains a transition of the form $t = (p_0,\theta) \xrightarrow{[\gamma_0,b]}_T (p',\theta')$, we add to the graph $\mathcal{G}$ the edge $((p,\gamma),\theta) \xrightarrow{b \vee B(p)} ((p',\gamma'),\theta')$.

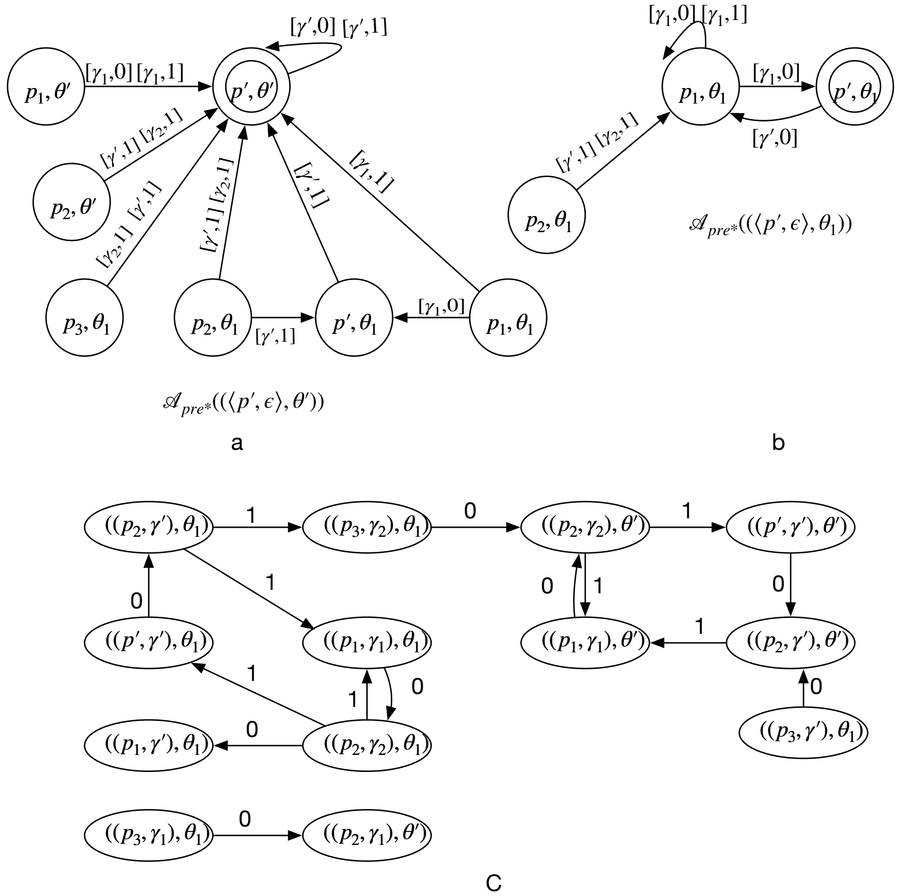


Figure 3: An Example of the SM-BPDS and the graph $\mathcal{G}$

Items α'_1 and α'_2 are obvious. They respectively correspond to item 1 and item 2 of Definition 4 (since $B(p) = 1$ iff $p \in G$). Item α'_3 is based on Lemma 1 and on item 3 of Definition 4. Indeed, it follows from Lemma 1 that $\mathcal{A}_{pre^*}\big((\langle p', \epsilon\rangle, \theta')\big)$ contains a transition of the form $(p_0, \theta) \xrightarrow{[\gamma_0, b]}_T (p', \theta')$ implies that $(\langle p_0, \gamma_0\rangle, \theta) \Rightarrow^* (\langle p', \epsilon\rangle, \theta')$, and if $b = 1$, then $(\langle p_0, \gamma_0\rangle, \theta) \Rightarrow^r (\langle p', \epsilon\rangle, \theta')$. Thus, in this case, the edge $((p, \gamma), \theta) \xrightarrow{b \vee B(p)} ((p', \gamma'), \theta')$ is added to $\mathcal{G}$ (item 3 of Definition 4) since $\langle p, \gamma\rangle \hookrightarrow \langle p_0, \gamma_0\gamma'\rangle \in \Delta$.

Example: Let us illustrate the procedure by an example. Consider the previous example shown in Fig.2. $\mathcal{A}_{pre*((\langle p',\epsilon\rangle,\theta'))}$ is shown in Fig. 3 (a) and $\mathcal{A}_{pre*((\langle p',\epsilon\rangle,\theta_1))}$ is shown in Fig. 3 (b). The result $\mathcal{G}$ shown in Fig. 3 (c) is obtained as follows:

1. Since $r_5 \in \theta', r_5 \in \theta_1$ and $p' \notin G$ i.e. $B(p') = 0$, Rule α'_2 adds edges $((p', \gamma'), \theta') \xrightarrow{b} ((p_2, \gamma'), \theta')$ and $((p', \gamma'), \theta_1) \xrightarrow{b} ((p_2, \gamma'), \theta_1)$, $b = B(p') = 0$ to $\mathcal{G}$.
2. Because $r_6 \in \theta', r_6 \in \theta_1$ and $p_2 \in G$ i.e. $B(p_2) = 1$, Rule α'_2 adds edges $((p_2, \gamma'), \theta') \xrightarrow{b} ((p_1, \gamma_1), \theta')$ and $((p_2, \gamma'), \theta_1) \xrightarrow{b} ((p_1, \gamma_1), \theta_1)$, $b = B(p_2) = 1$ to $\mathcal{G}$.
3. Because $r_7 \in \theta', r_7 \in \theta_1$ and $p_1 \notin G$ i.e. $B(p_1) = 0$, Rule α'_2 adds edges $((p_1, \gamma_1), \theta') \xrightarrow{b} ((p_2, \gamma_2), \theta')$ and $((p_1, \gamma_1), \theta_1) \xrightarrow{b} ((p_2, \gamma_2), \theta_1)$, $b = B(p_1) = 0$ to $\mathcal{G}$.
4. Because $r_4 \in \theta_1$ and $p_2 \in G$ i.e. $B(p_2) = 1$, Rule α'_2 adds the edge $((p_2, \gamma'), \theta_1) \xrightarrow{b} ((p_3, \gamma_2), \theta_1)$, $b = B(p_2) = 1$ to $\mathcal{G}$.
5. Now we have $r' \in \theta_1$ and $r_4 \in \theta_1$, for every $\gamma \in \Gamma$, Rule α'_1 adds edges $((p_3, \gamma_1), \theta_1) \xrightarrow{b} ((p_2, \gamma_1), \theta')$, $((p_3, \gamma_2), \theta_1) \xrightarrow{b} ((p_2, \gamma_2), \theta')$ and $((p_3, \gamma'), \theta_1) \xrightarrow{b} ((p_2, \gamma'), \theta')$, $b = B(p_3) = 0$ to $\mathcal{G}$.
6. Since $r_2 \in \theta'$ and $p_2 \in G$ i.e. $B(p_1) = 1$, Rule α'_3 first adds to the graph $\mathcal{G}$ the edge $((p_2, \gamma_2), \theta') \xrightarrow{b} ((p_1, \gamma_1), \theta')$, $b = B(p_2) = 1$. Then only $\mathcal{A}_{pre*((\langle p',\epsilon\rangle,\theta'))}$ contains transitions of the form $(p_1, \theta') \xrightarrow{[\gamma_1, b']}_T (p', \theta')$ i.e. transition $(p_1, \theta') \xrightarrow{[\gamma_1, b'_1]}_T (p', \theta')$,$b'_1 = 0$ and $(p_1, \theta') \xrightarrow{[\gamma_1, b'_2]}_T (p', \theta')$, $b'_2 = 1$. Then Rule α'_3 adds edges $((p_2, \gamma_2), \theta') \xrightarrow{b_1} ((p', \gamma'), \theta')$ with $b_1 = B(p_2) \vee b'_1 = 1$ and $((p_2, \gamma_2), \theta') \xrightarrow{b_2} ((p', \gamma'), \theta')$ with $b_2 = B(p_2) \vee b'_2 = 1$ to $\mathcal{G}$.
7. Since $r_2 \in \theta_1$ and $p_2 \in G$ i.e. $B(p_1) = 1$, Rule α'_3 first adds to the graph $\mathcal{G}$ the edge $((p_2, \gamma_2), \theta_1) \xrightarrow{b} ((p_1, \gamma_1), \theta_1)$, $b = B(p_1) = 1$. Then only $\mathcal{A}_{pre*((\langle p',\epsilon\rangle,\theta'))}$ contain transitions of the form $(p_1, \theta') \xrightarrow{[\gamma_1, b']}_T (p', \theta')$ and $\mathcal{A}_{pre*((\langle p',\epsilon\rangle,\theta_1))}$ contain transitions of the form $(p_1, \theta') \xrightarrow{[\gamma_1, b']}_T (p', \theta_1)$ i.e. transition $(p_1, \theta_1) \xrightarrow{[\gamma_1, b_1]}_T (p', \theta_1)$ in $\mathcal{A}_{pre*((\langle p',\epsilon\rangle,\theta'))}$ and $(p_1, \theta_1) \xrightarrow{[\gamma_1, b_1]}_T (p', \theta_1)$ in $\mathcal{A}_{pre*((\langle p',\epsilon\rangle,\theta_1))}$, $b_1 = 0$. Then Rule α'_3 adds the edge $((p_2, \gamma_2), \theta_1) \xrightarrow{b_2} ((p', \gamma'), \theta_1)$, $b_2 = b \vee b_1 = 1$ to $\mathcal{G}$.
8. No further additions are possible. Thus, the procedure terminates.

The result is depicted in Fig. 3 (c). By finding 1-labelled loops in $\mathcal{G}$, the repeating heads are

$$\{((p_2, \gamma_2), \theta_1), ((p', \gamma'), \theta_1), ((p_2, \gamma'), \theta_1), ((p_1, \gamma_1), \theta_1)\}$$

and

$$\{((p_2, \gamma_2), \theta'), ((p', \gamma'), \theta'), ((p_2, \gamma'), \theta'), ((p_1, \gamma_1), \theta')\}.$$

5. Experiments

5.1. Our approach vs. standard LTL for PDSs

We implemented our approach in a tool[1] and we compared its performance against the approaches that consist in translating the SM-PDS to an equivalent standard (or symbolic) PDS, and then applying the standard LTL model checking algorithms implemented in the PDS model-checker tool Moped [25]. All our experiments were run on Ubuntu 16.04 with a 2.7 GHz CPU, 2GB of memory. To perform the comparison, we randomly generate several SM-PDSs and LTL formulas of different sizes. For this, we use the function `int rand(void)` several times to randomly generate states and transitions. The results (CPU Execution time) are shown in Table 1. **Column** *Size* is the size of SM-PDS (S_1 for non-modifying transitions Δ and S_2 for modifying transitions Δ_c). **Column** *LTL* gives the size of the transitions of the Büchi automaton generated from the LTL formula (using the tool LTL2BA[28]). **Column** *SM-PDS* gives the cost of our direct algorithm presented in this paper. **Column** *PDS* shows the cost it takes to get the equivalent PDS from the SM-PDS. **Column** *Result* reports the cost it takes to run the LTL PDS model-checker Moped [25] for the PDS we got. **Column** *Total* is the total cost it takes to translate the SM-PDS into a PDS and then apply the standard LTL model checking algorithm of Moped (Total=PDS+Result). **Column** *Symbolic PDS* reports the cost it takes to get the equivalent Symbolic PDS from the SM-PDS. **Column** $Result_1$ is the cost to run the Symbolic PDS LTL model-checker Moped. **Column** $Total_1$ is the total cost it takes to translate the SM-PDS into a symbolic PDS and then apply the standard LTL model checking algorithm of Moped. You can see that our direct algorithm (**Column** *SM-PDS*) is much more efficient than translating the SM-PDS to an equivalent (symbolic) PDS, and then run the standard LTL model-checker Moped. **Translating the SM-PDS to a standard PDS may take more than 20 days, whereas our direct algorithm takes only a few seconds.** Moreover, since the obtained standard (symbolic) PDS is huge, Moped failed to handle several cases (the time limit that we set for Moped is 20 minutes), whereas our tool was able to deal with all the cases in only a few seconds.

5.2. Malicious Behavior Detection on Self-Modifying Code

5.2.1. Specifying Malicious Behaviors using LTL.

As described in [11], several malicious behaviors can be described by LTL formulas. We give in what follows four examples of such malicious behaviors and show how they can be described by LTL formulas:

Registry Key Injecting: In order to get started at boot time, many malwares add themselves into the registry key listing. This behavior is typically implemented by first calling the API function GetModuleFileNameA to retrieve

[1] https://lipn.univ-paris13.fr/∼touili/smodic/

Size	LTL	SM-PDS	PDS	Result	Total	Symbolic PDS	$Result_1$	$Total_1$
$S_1 : 5, S_2 : 2$	$\|\delta\|$:15	**0.07s**	0.09s	0.01s	0 .10s	0.08s	0.00s	0.08s
$S_1 : 5, S_2 : 3$	$\|\delta\|$:8	**0.06s**	0.08s	0.01s	0.09s	0.09s	0.00s	0.09s
$S_1 : 11, S_2 : 4$	$\|\delta\|$:8	**0.16s**	0.13s	0.05s	0.18s	0.10s	0.00s	0.10s
$S_1 : 5, S_2 : 3$	$\|\delta\|$:10	**0.06s**	0.15s	0.01s	0.16s	0.09s	0.00s	0.09s
$S_1 : 110, S_2 : 4$	$\|\delta\|$:8	**0.34s**	186.10s	0.79s	186.99s	0.35s	0.00s	0.35s
$S_1 : 255, S_2 : 8$	$\|\delta\|$:8	**0.39s**	281.02s	0.94s	281.96s	4.82s	0.05s	4.87s
$S_1 : 255, S_2 : 8$	$\|\delta\|$:10	**0.42s**	281.02s	0.97s	281.99s	4.82s	0.06s	4.88s
$S_1 : 110, S_2 : 4$	$\|\delta\|$:15	**0.28s**	186.10s	1.05s	187.15s	0.35s	0.06s	0.41s
$S_1 : 255, S_2 : 8$	$\|\delta\|$:15	**0.46s**	281.02s	1.92s	282.94s	4.82s	0.08s	4.90s
$S_1 : 110, S_2 : 4$	$\|\delta\|$:20	**0.37s**	186.10s	1.05s	187.15s	0.35s	0.06s	0.41s
$S_1 : 255, S_2 : 8$	$\|\delta\|$:20	**0.55s**	281.02s	1.97s	282.99s	4.82s	0.17s	4.99s
$S_1 : 255, S_2 : 8$	$\|\delta\|$:25	**0.59s**	281.02s	1.23s	282.99s	4.82s	0.24s	5.36s
$S_1 : 2059, S_2 : 7$	$\|\delta\|$:8	**0.86s**	19525.01s	20.71s	19545.72s	20.70s	error	-
$S_1 : 2059, S_2 : 9$	$\|\delta\|$:8	**1.49s**	19784.7s	79.12s	19863.32	128.12s	error	-
$S_1 : 2059, S_2 : 11$	$\|\delta\|$:8	**3.73s**	30011.67s	168.15s	30179.82s	261.07s	error	-
$S_1 : 2059, S_2 : 11$	$\|\delta\|$:28	**6.88s**	30011.67s	169.55s	30180.22s	261.07s	error	-
$S_1 : 3050, S_2 : 10$	$\|\delta\|$:8	**5.21s**	39101.57s	killed	-	438.27s	error	-
$S_1 : 3090, S_2 : 10$	$\|\delta\|$:8	**5.86s**	40083.07s	killed	-	438.69s	error	-
$S_1 : 3050, S_2 : 10$	$\|\delta\|$:20	**7.24s**	39101.57s	killed	-	438.27s	error	-
$S_1 : 3090, S_2 : 10$	$\|\delta\|$:30	**8.38s**	40083.07s	killed	-	438.69s	error	-
$S_1 : 3090, S_2 : 10$	$\|\delta\|$:25	**8.89s**	40083.07s	killed	-	438.69s	error	-
$S_1 : 4050, S_2 : 10$	$\|\delta\|$:8	**9.21s**	81408.91s	killed	-	699.19s	error	-
$S_1 : 4050, S_2 : 10$	$\|\delta\|$:28	**11.64s**	81408.91s	killed	-	699.19s	error	-
$S_1 : 4058, S_2 : 11$	$\|\delta\|$:8	**9.83s**	93843.37s	killed	-	802.07s	error	-
$S_1 : 4058, S_2 : 11$	$\|\delta\|$:25	**13.59s**	93843.37s	killed	-	802.07s	error	-
$S_1 : 5050, S_2 : 11$	$\|\delta\|$:8	**10.34s**	173943.37s	killed	-	921.16s	error	-
$S_1 : 5090, S_2 : 11$	$\|\delta\|$:8	**10.52s**	179993.54s	killed	-	929.32s	error	-
$S_1 : 5090, S_2 : 11$	$\|\delta\|$:10	**12.89s**	179993.54s	killed	-	929.32s	error	-
$S_1 : 6090, S_2 : 11$	$\|\delta\|$:8	**13.49s**	190293.64s	killed	-	1002.73s	error	-
$S_1 : 6090, S_2 : 11$	$\|\delta\|$:10	**15.81s**	190293.64s	killed	-	1002.73s	error	-
$S_1 : 6090, S_2 : 11$	$\|\delta\|$:40	**32.39s**	190293.64s	killed	-	1002.73s	error	-
$S_1 : 7090, S_2 : 11$	$\|\delta\|$:25	**39.86s**	198932.32s	killed	-	1092.28s	error	-
$S_1 : 7090, S_2 : 11$	$\|\delta\|$:30	**43.24s**	198932.32s	killed	-	1092.28s	error	-
$S_1 : 9090, S_2 : 11$	$\|\delta\|$:8	**29.98s**	199987.98s	killed	-	1128.19s	error	-
$S_1 : 9090, S_2 : 11$	$\|\delta\|$:20	**45.29s**	199987.98s	killed	-	1128.19s	error	-
$S_1 : 10050, S_2 : 12$	$\|\delta\|$:8	**48.53s**	2134587.14s	killed	-	1469.28s	error	-
$S_1 : 10050, S_2 : 12$	$\|\delta\|$:25	**59.69s**	2134587.14s	killed	-	1469.28s	error	-
$S_1 : 10050, S_2 : 12$	$\|\delta\|$:30	**61.42s**	2134587.14s	kille d	-	1469.28s	error	-
$S_1 : 10150, S_2 : 12$	$\|\delta\|$:35	**64.17s**	2134633.28s	killed	-	1469.28s	error	-
$S_1 : 10150, S_2 : 14$	$\|\delta\|$:8	**58.34s**	2181975.64s	killed	-	2849.96s	error	-
$S_1 : 10150, S_2 : 14$	$\|\delta\|$:40	**82.72s**	2181975.64s	killed	-	2849.96s	error	-
$S_1 : 10150, S_2 : 12$	$\|\delta\|$:40	**76.61s**	2134633.28s	killed	-	1469.28s	error	-
$S_1 : 10150, S_2 : 16$	$\|\delta\|$:45	**89.83s**	2211008.82s	killed	-	3665.59s	error	-
$S_1 : 10150, S_2 : 12$	$\|\delta\|$:60	**97.56s**	2134633.28s	killed	-	1469.28s	error	-
$S_1 : 10150, S_2 : 12$	$\|\delta\|$:65	**105.89s**	2134633.28s	killed	-	1469.28s	error	-
$S_1 : 10150, S_2 : 16$	$\|\delta\|$:65	**134.45s**	2211008.82s	killed	-	3665.59s	error	-
$S_1 : 10180, S_2 : 16$	$\|\delta\|$:65	**175.29s**	2134643.52s	killed	-	3689.83s	error	-
$S_1 : 10180, S_2 : 16$	$\|\delta\|$:78	**214.36s**	2134643.52s	killed	-	3689.83s	error	-

Table 1: Our approach vs standard LTL for PDSs

the path of the malware's executable file. Then, the API function RegSetValueExA is called to add the file path into the registry key listing. This malicious behavior can be described in LTL as follows:

$\phi_{rk} = \mathbf{F}\big(call\ GetModuleFileNameA \wedge \mathbf{F}(\ call\ RegSetValueExA)\big)$

This formula expresses that if a call to the API function GetModuleFileNameA is followed by a call to the API function RegSetValueExA, then probably a malware is trying to add itself into the registry key listing.

Data-Stealing: Stealing data from the host is a popular malicious behavior that intend to steal any valuable information including passwords, software codes, bank information, etc. To do this, the malware needs to scan the disk to find the interesting file that he wants to steal. After finding the file, the malware needs to locate it. To this aim, the malware first calls the API function GetModuleHandleA to get a base address to search for a location of the file. Then the malware starts looking for the interesting file by calling the API function FindFirstFileA. Then the API functions CreateFileMappingA and MapViewOfFile are called to access the file. Finally, the specific file can be copied by calling the API function CopyFileA. Thus, this data-stealing malicious behavior can be described by the following LTL formula as follows:

$\phi_{ds} = \mathbf{F}(call\ GetModuleHandleA\ \wedge \mathbf{F}(call\ FindFirstFileA \wedge \mathbf{F}\ (call\ CreateFileMappingA$
$\wedge\ \mathbf{F}\ (call\ MapViewofFile \wedge \mathbf{F}\ call\ CopyFileA))))$

Spy-Worm: A spy worm is a malware that can record data and send it using the Socket API functions. For example, Keylogger is a spy worm that can record the keyboard states by calling the API functions GetAsyKeyState and GetKeyState and send that to the specific server by calling the socket function sendto. Another spy worm can also spy on the I/O device rather than the keyboard. For this, it can use the API function GetRawInputData to obtain input from the specified device, and then send this input by calling the socket functions send or sendto. Thus, this malicious behavior can be described by the following LTL formula:

$\phi_{sw} = \mathbf{F}\big((call\ GetAsyncKeyState \vee call\ GetRawInputData) \wedge \mathbf{F}(call\ sendto \vee call\ send)\big)$

Appending virus: An appending virus is a virus that inserts a copy of its code at the end of the target file. To achieve this, since the real OFFSET of the virus' variables depends on the size of the infected file, the virus has to first compute its real absolute address in the memory. To perform this, the virus has to call the sequence of instructions: l_1: call f; l_2: ....; f: pop eax;. The instruction call f will push the return address l_2 onto the stack. Then, the pop instruction in f will put the value of this address into the register eax. Thus, the virus can get its real absolute address from the register eax. This malicious behavior can be described by the following LTL formula:

$\phi_{av} = \bigvee \mathbf{F}\Big(call \wedge \mathbf{X}(\text{top-of-stack} = a) \wedge \mathbf{G}\neg\big(ret \wedge (\text{top-of-stack} = a)\big)\Big)$

where the $\bigvee$ is taken over all possible return addresses a, and top-of-stack$=a$ is a predicate that indicates that the top of the stack is a. The subformula $call \wedge \mathbf{X}(\text{top-of-stack} = a)$ means that there exists a procedure call having a as return address. Indeed, when a procedure call is made, the program pushes its corresponding return address a to the stack. Thus, at the next step, a will be on the top of the stack. Therefore, the formula above expresses that there exists a procedure call having a as return address, such that there is no *ret* instruction which will return to a.

Note that this formula uses predicates that indicate that the top of the stack is a. Our techniques work for this case as well: it suffices to encode the top of the stack in the control points of the SM-PDS. Our implementation works for this case as well and can handle appending viruses.

5.2.2. Applying our tool for malware detection.

We applied our tool to detect several malwares. We use the unpack tool unpacker [29] to handle packers like UPX, and we use Jakstab [26] as disassembler. We consider 160 malwares from the malware library VirusShare [30], 184 malwares from the malware library MalShare [31], 288 email-worms from VX heaven [32] and 260 new malwares generated by NGVCK, one of the best malware generators. We also choose 200 benign samples from Windows programs. We consider self-modifying versions of these programs[2]. In these versions, the malicious behaviors are unreachable if the semantics of the self-modifying instructions are not taken into account, i.e., if the self-modifying instructions are considered as "standard" instructions that do not modify the code, then the malicious behaviors cannot be reached. To check this, we model such programs in two ways:

1. First, we take into account the self-modifying instructions and model these programs using SM-PDSs as described in Section 2.3. Then, we check whether these SM-PDSs satisfy at least one of the malicious LTL formulas presented above. If yes, the program is declared as malicious, if not, it is declared as benign. Our tool was able to detect all the 892 self-modifying malwares as malicious, and to determine that benign programs are benign. We report in Table 3 some partial results of our experiments. **Column** *Size* is the number of control locations, **Column** *Result* gives the result of our algorithm: **Yes** means malicious and **No** means benign; and **Column** *cost* gives the cost to apply our LTL model-checker to check one of the LTL properties (**Column** *Formula*) described above. For every program, we consider all the formulas mentionned above. A program is declared malicious if it satisfes at least one of the formulas.
2. Second, we abstract away the self-modifying instructions and proceed as if these instructions were not self-modifying. In this case, we translate the binary codes to standard pushdown systems as described in [9]. By using PDSs as models, none of the malwares that we consider was detected as malicious, whereas, as reported in Table 3, using self-modifying PDSs as models, and applying our LTL model-checking algorithm allowed to detect all the 892 malwares that we considered.

Remark. Note that checking the formulas ϕ_{rk}, ϕ_{ds}, and ϕ_{sw} could be done using multiple pre^* queries on SM-PDSs using the pre^* algorithm of [1]. However, this would be less efficient than performing our direct LTL model-checking algorithm, as shown in Table 2, where **Column** *Size* gives the number of control locations, **Column** *LTL* gives the time of applying our LTL model-checking algorithm; and **Column** *Multiple* pre^* gives the cost of applying multiple pre^* on

[2] Self-modifying instructions are embedded into these programs.

Example	Size	LTL	Multiple pre^*	Example	Size	LTL	Multiple pre^*
Tanatos.b	12315	16.261s	46.635s	Netsky.c	45	0.002s	0.092s
Win32.Happy	23	0.042s	0.075s	MyDoom-N	16980	30.231s	98.418s
Kelino.g	470	0.672s	3.446s	Netsky.b	45	0.057s	0.183s
Netsky.a	45	0.047s	0.085s	Mydoom.c	155	0.014s	0.206s
klez.c	30	0.039s	0.088s	Mydoom.v	5965	3.971s	83.988s
Netsky.d	45	0.083s	0.123s	Ardurk.d	1913	0.482s	3.212s
klez.f	27	0.054s	4.518s	Magistr.a.poly	36989	49.863s	159.195s
klez.e	27	0.094s	0.482s	Magistr.b	4670	3.987s	53.235s
Adon.1703	37	0.358s	0.884s	Adon.1559	37	0.255s	4.088s
Spam.Tedroo.AB	487	0.924s	4.894s	Alaul.c	355	0.109s	5.757s
Akez	273	0.136s	1.863s	Alcaul.d	845	0.165s	0.392s
Haharin.A	210	1.462s	4.318s	fsAutoB.F026	245	1.698s	4.503s
Haharin.dr	235	1.558s	4.312s	LdPinch.Win32.5558	2015	6.907s	8.981s
LdPinch.bx.dll	2010	6.965s	8.128s	LdPinch.fmye	1845	6.194s	9.232s
LdPinch-15	580	1.008s	3.957s	LdPinch.e	578	1.185s	3.392s
Win32/Toga!rfn	590	2.023s	3.978s	klez-N	6281	3.252s	78.419s
Mydoom.y	26902	12.462s	102.559s	Mydoom.j	22355	11.262s	111.617s
Plage.b	395	0.291s	3.138s	Urbe.a	123	0.376s	2.981s
Mydoom-EG	230	0.242s	6.172s	Email.W32!c	220	0.249s	5.946s
W32.Mydoom.L	235	0.288s	6.452s	Mydoom.DN.worm	220	0.299s	8.928s
Mydoom.5	228	0.307s	8.163s	Mydoom.cjdz	225	0.392s	9.968s
Mydoom.R	230	0.322s	9.086s	Win32.Mydoom	235	0.296s	7.985s
Mydoom.o@MM!zip	235	0.403s	10.323s	Win32.Mydoom.288	248	0.410s	2.983s
Sramota.avf	240	0.383s	2.691s	Mydoom	238	0.278	2.749s
Win32.Runouce	51678	92.692s	248.146s	Win32.Chur.A	51895	98.161s	298.047s
Win32.CNHacker	51095	94.952s	245.452s	Netsky.ah@MM	4480	6.991s	16.018s
Win32.Skybag	4180	6.891s	13.739s	Skybag.A	4310	6.205s	15.452s
LdPinch.by	970	4.092s	11.327s	Generic.20269	433	2.402s	9.614s
LdPinch.arr	1250	1.848s	9.986s	LdPnch-Fam	195	1.440s	4.097s
Troj.LdPinch.er	205	2.529s	6.154s	LdPinch.Gen.3	210	1.482s	4.973s

Table 2: Multiple pre^* v.s. our direct LTL model-checking algorithm

Example	Size	Formula	Result	*cost*	Example	Size	Formula	Result	*cost*	Example	Size	Formula	Result	*cost*
klez-N	6281	ϕ_{ds}	Yes	3.252s	klez.c	30	ϕ_{ds}	Yes	0.039s	klez.f	27	ϕ_{ds}	Yes	0.054s
klez.d	31	ϕ_{ds}	Yes	0.085s	Alcaul.d	845	ϕ_{av}	Yes	0.165s	Alaul.c	355	ϕ_{av}	Yes	0.109s
Akez	273	ϕ_{av}	Yes	0.136s	Akez.Win32.1	455	ϕ_{av}	Yes	4.008s	Mydoom.y	26902	ϕ_{ds}	Yes	12.462s
Mydoom.j	22355	ϕ_{ds}	Yes	11.262s	Mydoom.c	155	ϕ_{ds}	Yes	0.014s	MyDoom-N	16980	ϕ_{ds}	Yes	30.231s
Mydoom.v	5965	ϕ_{ds}	Yes	3.971s	Mydoom.M	5965	ϕ_{ds}	Yes	5.633s	Netsky.a	45	ϕ_{ds}	Yes	0.047s
Netsky.d	45	ϕ_{ds}	Yes	0.083s	Worm.Skybag-1	4820	ϕ_{rk}	Yes	7.119s	Win32.Agent.R	4490	ϕ_{rk}	Yes	7.898s
LdPinch.BX	2010	ϕ_{rk}	Yes	6.965s	LdPinch.fmye	1845	ϕ_{rk}	Yes	6.194s	LdPinch.Win32	2015	ϕ_{rk}	Yes	6.907s
Lydra.a	3450	ϕ_{rk}	Yes	8.289s	Trojan.StartPage	2985	ϕ_{rk}	Yes	5.982s	PSWTroj.au	2985	ϕ_{rk}	Yes	6.198s
LdPinch-21	3180	ϕ_{rk}	Yes	6.917s	LdPinch-R	3025	ϕ_{rk}	Yes	7.005s	LdPinch.Gen	2990	ϕ_{rk}	Yes	6.992s
Repah.b	221	ϕ_{rk}	Yes	2.428s	Gibe.b	5358	ϕ_{rk}	Yes	4.229s	Magistr.b	4670	ϕ_{rk}	Yes	3.699s
Win32.Anar.a	215	ϕ_{ds}	Yes	1.631s	Anar.24576	240	ϕ_{ds}	Yes	2.738s	Anar.S	155	ϕ_{ds}	Yes	2.093s
Kelino.l	495	ϕ_{rk}	Yes	0.326s	Kipis.t	20378	ϕ_{rk}	Yes	25.345s	W32.HfsAutoB.	3398	ϕ_{rk}	Yes	5.092s
Worm.Anarxy	210	ϕ_{ds}	Yes	1.913s	Plage.b	395	ϕ_{rk}	Yes	0.291s	Urbe.a	123	ϕ_{rk}	Yes	0.376s
calculation.exe	9952	ϕ_{ds}	No	31.176s	cisvc.exe	4105	ϕ_{ds}	No	9.114s	simple.exe	52	ϕ_{ds}	No	0.053s
calculation.exe	9952	ϕ_{rk}	No	14.932s	cisvc.exe	4105	ϕ_{rk}	No	3.454s	simple.exe	52	ϕ_{rk}	No	0.001s
calculation.exe	9952	ϕ_{av}	No	40.118s	cisvc.exe	4105	ϕ_{av}	No	5.092s	simple.exe	52	ϕ_{av}	No	0.197s
calculation.exe	9952	ϕ_{sw}	No	19.539s	cisvc.exe	4105	ϕ_{sw}	No	2.864s	simple.exe	52	ϕ_{sw}	No	0.003s
shutdown.exe	2529	ϕ_{ds}	No	13.228s	loop.exe	529	ϕ_{ds}	No	4.373s	cmd.exe	1324	ϕ_{ds}	No	13.466s
shutdown.exe	2529	ϕ_{rk}	No	9.152s	loop.exe	529	ϕ_{rk}	No	8.029s	cmd.exe	1324	ϕ_{rk}	No	6.233s
shutdown.exe	2529	ϕ_{av}	No	19.031s	loop.exe	529	ϕ_{av}	No	13.478s	cmd.exe	1324	ϕ_{av}	No	9.620s
shutdown.exe	2529	ϕ_{sw}	No	0.397s	loop.exe	529	ϕ_{sw}	No	9.249s	cmd.exe	1324	ϕ_{sw}	No	9.268s
notepad.exe	10529	ϕ_{rk}	No	24.583s	java.exe	800	ϕ_{rk}	No	15.852s	eclipse.exe	21324	ϕ_{rk}	No	42.373s
notepad.exe	10529	ϕ_{av}	No	89.131s	java.exe	800	ϕ_{av}	No	18.079s	eclipse.exe	21324	ϕ_{av}	No	63.447s
notepad.exe	10529	ϕ_{sw}	No	22.830s	java.exe	800	ϕ_{sw}	No	13.472s	eclipse.exe	21324	ϕ_{sw}	No	51.693s
notepad.exe	10529	ϕ_{ds}	No	36.119s	java.exe	800	ϕ_{ds}	No	22.357s	java.exe	21324	ϕ_{ds}	No	69.683s
sort.exe	8529	ϕ_{rk}	No	29.789s	bibDesk.exe	32800	ϕ_{rk}	No	50.279s	interface.exe	1005	ϕ_{rk}	No	8.462s
sort.exe	8529	ϕ_{sw}	No	34.427s	bibDesk.exe	32800	ϕ_{sw}	No	197.628s	interface.exe	1005	ϕ_{sw}	No	11.309s
sort.exe	8529	ϕ_{av}	No	69.140s	bibDesk.exe	32800	ϕ_{av}	No	408.925s	interface.exe	1005	ϕ_{av}	No	32.193s
ipv4.exe	968	ϕ_{rk}	No	4.186s	TextWrangler.exe	14675	ϕ_{rk}	No	45.221s	sogou.exe	45219	ϕ_{rk}	No	55.259s

Example	Size	Formula	Result	*cost*	Example	Size	Formula	Result	*cost*	Example	Size	Formula	Result	*cost*
SdBot.zk	3430	ϕ_{rk}	Yes	23.242s	Virus.Gen	661	ϕ_{av}	Yes	9.437s	AutoRun	240	ϕ_{rk}	Yes	4.181s
Spam.AB	487	ϕ_{rk}	Yes	0.924s	Haharin.A	210	ϕ_{rk}	Yes	1.462s	Alaul.c	355	ϕ_{ds}	Yes	0.109s
Virus.klk	5235	ϕ_{rk}	Yes	15.863s	Virus.Agent	5340	ϕ_{rk}	Yes	15.968s	Hoax.Gen	5455	ϕ_{rk}	Yes	13.569s
eHeur.Virus02	420	ϕ_{ds}	Yes	4.985s	Akez.11255	440	ϕ_{rk}	Yes	3.985s	Akez.5	490	ϕ_{rk}	Yes	3.958s
Weird.c	430	ϕ_{rk}	Yes	3.929s	PEAKEZ.A	450	ϕ_{rk}	Yes	2.998s	Weird.d	473	ϕ_{ds}	Yes	3.302s
Win32.Runouce	51678	ϕ_{rk}	Yes	92.692s	Chur.A	51895	ϕ_{rk}	Yes	98.161s	WCNHacker.C	51095	ϕ_{rk}	Yes	94.952s
Agent.xpro	533	ϕ_{rk}	Yes	0.352s	Vilsel.lhb	15036	ϕ_{rk}	Yes	4.972s	Generic.20269	433	ϕ_{av}	Yes	3.489s
NewApt!generic	4815	ϕ_{sw}	Yes	9.002s	NewApt.A@mm	4485	ϕ_{sw}	Yes	8.159s	Newapt.1	4155	ϕ_{sw}	Yes	7.885s
NGVCK1	329	ϕ_{rk}	Yes	0.933s	NGVCK2	455	ϕ_{sw}	Yes	1.109s	NGVCK3	2300	ϕ_{rk}	Yes	1.388s
NGVCK4	550	ϕ_{ds}	Yes	1.149s	NGVCK5	1555	ϕ_{rk}	Yes	1.825s	NGVCK6	1698	ϕ_{rk}	Yes	1.689s
NGVCK7	6902	ϕ_{av}	Yes	14.524s	NGVCK8	2355	ϕ_{rk}	Yes	4.254s	NGVCK9	281	ϕ_{sw}	Yes	13.301s
NGVCK10	2980	ϕ_{rk}	Yes	9.262s	NGVCK11	5965	ϕ_{ds}	Yes	11.456s	NGVCK12	4529	ϕ_{ds}	Yes	10.094s
NGVCK13	2210	ϕ_{rk}	Yes	8.902s	NGVCK14	5358	ϕ_{ds}	Yes	10.294s	NGVCK15	970	ϕ_{ds}	Yes	1.912s
NGVCK16	658	ϕ_{rk}	Yes	0.935s	NGVCK17	913	ϕ_{rk}	Yes	1.392s	NGVCK18	90	ϕ_{rk}	Yes	0.094s
NGVCK19	1295	ϕ_{ds}	Yes	6.958s	NGVCK20	4378	ϕ_{ds}	Yes	15.449s	NGVCK21	31	ϕ_{rk}	Yes	0.097s
NGVCK22	370	ϕ_{ds}	Yes	0.898s	NGVCK23	3955	ϕ_{ds}	Yes	9.498s	NGVCK24	6924	ϕ_{ds}	Yes	11.983s
NGVCK25	8127	ϕ_{ds}	Yes	15.018s	NGVCK26	4970	ϕ_{ds}	Yes	9.982s	NGVCK27	7989	ϕ_{ds}	Yes	13.197s
NGVCK28	227	ϕ_{rk}	Yes	0.098s	NGVCK29	960	ϕ_{rk}	Yes	0.692s	NGVCK30	89	ϕ_{rk}	Yes	0.088s
NGVCK31	550	ϕ_{rk}	Yes	0.875s	NGVCK32	60	ϕ_{rk}	Yes	0.059s	NGVCK33	65	ϕ_{rk}	Yes	0.069s
NGVCK34	5990	ϕ_{ds}	Yes	9.848s	NGVCK35	4590	ϕ_{ds}	Yes	10.178s	NGVCK36	825	ϕ_{ds}	Yes	2.934s
NGVCK37	80	ϕ_{rk}	Yes	0.998s	NGVCK38	150	ϕ_{rk}	Yes	1.093s	NGVCK39	395	ϕ_{rk}	Yes	1.048s
mfc.dll	110	ϕ_{rk}	No	2.014s	Uedit32	98	ϕ_{rk}	No	0.926s	wechat.exe	12252	ϕ_{rk}	No	45.147s
mfc.dll	110	ϕ_{sw}	No	24.571s	Uedit32	98	ϕ_{sw}	No	2.572s	wechat.exe	12252	ϕ_{sw}	No	68.327s
mfc.dll	110	ϕ_{av}	No	19.132s	Uedit32	98	ϕ_{av}	No	36.176s	wechat.exe	12252	ϕ_{av}	No	57.129s
mfc.dll	110	ϕ_{ds}	No	7.746s	Uedit32	98	ϕ_{ds}	No	6.529s	wechat.exe	12252	ϕ_{ds}	No	54.373s
game.exe	34325	ϕ_{rk}	No	82.424s	cycle.exe	9014	ϕ_{rk}	No	42.555s	calender.exe	892	ϕ_{rk}	No	35.039s
game.exe	34325	ϕ_{ds}	No	60.119s	cycle.exe	9014	ϕ_{ds}	No	73.306s	calender.exe	892	ϕ_{ds}	No	42.148s
game.exe	34325	ϕ_{av}	No	126.037s	cycle.exe	9014	ϕ_{av}	No	110.191s	calender.exe	892	ϕ_{av}	No	65.983s
game.exe	34325	ϕ_{sw}	No	61.254s	cycle.exe	9014	ϕ_{sw}	No	51.026s	calender.exe	892	ϕ_{sw}	No	27.105s

Table 3: Experimental results

our tool	McAfee	Norman	BitDefender	Kinsoft	Avira	eScan	Kaspersky	Qihoo360	Baidu	Avast	Symantec
100%	24.8%	19.5%	31.2%	9.7%	34.1%	21.9%	53.1%	51.7%	1.4%	68.3%	82.4%

Table 4: Detection rate: Our tool vs. well known antiviruses

SM-PDSs to check the properties ϕ_{rk}, ϕ_{ds}, and ϕ_{sw}. It can be seen that applying our *direct* LTL model checking algortihm is more efficient. Furthermore, the appending virus formula ϕ_{av} cannot be solved using multiple pre^* queries. Our direct LTL model-checking algorithm is needed in this case. Note that some of the malwares we considered in our experiments are appending viruses. Thus, our algorithm and our implementation are crucial to be able to detect these malwares.

5.2.3. Comparison with well-known antiviruses.

We compare our tool against well-known and widely used antiviruses. Since known antiviruses update their signature database as soon as a new malware is known, in order to have a fair comparision with these antiviruses, we need to consider new malwares. We use the sophisticated malware generator NGVCK available at VX Heavens [32] to generate 205 malwares. We obfuscate these malwares with self-modifying code, and we fed them to our tool and to well known antiviruses such as BitDefender, Kinsoft, Avira, eScan, Kaspersky, Qihoo-360, Baidu, Avast, and Symantec. Our tool was able to detect all these programs as malicious, whereas none of the well-known antiviruses was able to detect all these malwares. Table 4 reports the detection rates of our tool and the well-known anti-viruses.

6. Conclusion and discussion

In this paper, we propose a **direct** LTL model checking algorithm for SM-PDSs. Our algorithm is based on reducing the LTL model checking problem to the emptiness problem of Self Modifying Büchi Pushdown Systems (SM-BPDSs). Intuitively, we obtain this SM-BPDS by taking the product of the SM-PDS with a Büchi automaton accepting an LTL formula φ. Then, we solve the emptiness problem of an SM-BPDS by computing its repeating heads. This computation is based on computing labelled pre^* configurations by applying a saturation procedure on labelled finite automata.

We implemented our techniques in a tool for self-modifying code analysis. We successfully used our tool to model-check more than 900 self-modifying binary codes. In particular, we applied our tool for malware detection, since malwares usually use self-modifying instructions, and since malicious behaviors can be described by LTL formulas. In our experiments, our tool was able to detect 895 malwares and to prove that 200 benign programs were benign. It was also able to detect several malwares that well-known antiviruses such as Bit-Defender, Kinsoft, Avira, eScan, Kaspersky, Avast, and Symantec failed to detect.

Malware detection is nowadays a big challenge. This work brings just a stone to the building, and helps dealing with self-modifying code. However, a lot of work remains to be done in order to have a robust tool for malware detection. Indeed, as mentionned in Section 2.3.1, this work assumes that if instruction i_1

is replaced by i_2, then i_1 and i_2 must have the same number of operands. This of course does not hold for all malicious programs. Moreover, to disassemble programs, we use Jakstab [26]. This tool has a lot of limitations and offers sometimes rough translations, as discussed in [26]. Another limitation of our tool is that currently it considers only the four malicious behaviors described in Section 5.2.1. Several other malicious behaviors can be found in malwares. Thus, we need to study these behaviors and specify them as LTL formulas. To this aim, we plan to apply machine learning techniques in order to extract the maximum number of malicious behaviors from malwares.

References

[1] T.Touili, X.Ye, Reachability analysis of self modifying code, in: 2017 22nd International Conference on Engineering of Complex Computer Systems (ICECCS), IEEE, 2017, pp. 120–127.

[2] A. Bouajjani, J. Esparza, O. Maler, Reachability Analysis of Pushdown Automata: Application to Model Checking, in: International Conference on Concurrency Theory (CONCUR), Springer, 1997, pp. 135–150.

[3] J.Esparza, D.Hansel, P.Rossmanith, S.Schwoon, Efficient algorithms for model checking pushdown systems, in: International Conference on Computer Aided Verification (CAV), Springer, 2000, pp. 232–247.

[4] T.Touili, X.Ye, Ltl model checking of self modifying code, in: 2019 24th International Conference on Engineering of Complex Computer Systems (ICECCS), IEEE, 2019, pp. 1–10.

[5] J.Bergeron, M.Debbabi, et al., Static detection of malicious code in executable programs, Int. J. of Req. Eng 2001 (184-189) (2001) 79.

[6] G.Balakrishnan, T. Reps, N.Kidd, A.Lal, J.Lim, et al., Model checking x86 executables with codesurfer/x86 and WPDS++, in: International Conference on Computer Aided Verification (CAV), Springer, 2005, pp. 158–163.

[7] P.Singh, A.Lakhotia, Static verification of worm and virus behavior in binary executables using model checking, in: IEEE Systems, Man and Cybernetics SocietyInformation Assurance Workshop, 2003., IEEE, 2003, pp. 298–300.

[8] J.Kinder, S.Katzenbeisser, C.Schallhart, H.Veith, Detecting malicious code by model checking, in: International Conference on Detection of Intrusions and Malware, and Vulnerability Assessment, Springer, 2005, pp. 174–187.

[9] F.Song, T.Touili, Efficient malware detection using model-checking, in: International Symposium on Formal Methods, Springer, 2012, pp. 418–433.

[10] P. Beaucamps, I. Gnaedig, J. Marion, Behavior abstraction in malware analysis, in: International Conference on Runtime Verification, Springer, 2010, pp. 168–182.

[11] F.Song, T.Touili, Ltl model-checking for malware detection, in: International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS), Springer, 2013, pp. 416–431.

[12] H.Nguyen, T.Touili, CARET model checking for malware detection, in: Proceedings of the 24th ACM SIGSOFT International SPIN Symposium on Model Checking of Software, 2017.

[13] K.Dam, T.Touili, Learning malware using generalized graph kernels, in: Proceedings of the 13th International Conference on Availability, Reliability and Security, 2018, pp. 1–6.

[14] K.Dam, T.Touili, Precise extraction of malicious behaviors, in: 42nd Annual Computer Software and Applications Conference (COMPSAC), Vol. 1, IEEE, 2018, pp. 229–234.

[15] K.Dam, T.Touili, Malware detection based on graph classification, in: ICISSP, 2017.

[16] H.Cai, Z.Shao, A.Vaynberg, Certified self-modifying code, ACM SIGPLAN Notices 42 (6).

[17] S.Debray, K.Coogan, G.Townsend, On the semantics of self-unpacking malware code, Tech. rep. University of Arizona, Computer Science.

[18] G.Bonfante, J. Marion, D.Reynaud-Plantey, A computability perspective on self-modifying programs, in: International Conference on Software Engineering and Formal Methods (SEFM), IEEE, 2009, pp. 231–239.

[19] A.Bertrand, M.Matias, D.Koen, A model for self-modifying code, in: International Workshop on Information Hiding, Springer, 2006, pp. 232–248.

[20] S.Blazy, V.Laporte, D.Pichardie, Verified abstract interpretation techniques for disassembling low-level self-modifying code, Journal of Automated Reasoning 56 (3) (2016) 283–308.

[21] K.Roundy, B.Miller, Hybrid analysis and control of malware, in: International Workshop on Recent Advances in Intrusion Detection, Springer, 2010, pp. 317–338.

[22] K.Coogan, S.Debray, T.Kaochar, G.Townsend, Automatic static unpacking of malware binaries, in: 2009 16th Working Conference on Reverse Engineering, IEEE, 2009, pp. 167–176.

[23] K.Gyung, et al., Renovo: A hidden code extractor for packed executables, in: Proceedings of the 2007 ACM workshop on Recurring malcode, 2007, pp. 46–53.

[24] P.Royal, M.Halpin, et al., Polyunpack: Automating the hidden-code extraction of unpack-executing malware, in: 22nd Annual Computer Security Applications Conference (ACSAC'06), IEEE, 2006, pp. 289–300.

[25] S.Schwoon, Model-checking pushdown systems, Ph.D. thesis, Technische Universität München, Universitätsbibliothek (2002).

[26] H. J.Kinder, Jakstab: A static analysis platform for binaries, in: International Conference on Computer Aided Verification (CAV), Springer, 2008.

[27] M.Vardi, P.Wolper, Reasoning about infinite computations, Inf. Comput. 115 (1).

[28] P.Gastin, D.Oddoux, Fast ltl to büchi automata translation, in: International Conference on Computer Aided Verification (CAV), Springer, 2001.

[29] U. Tool, Automated unpacking: A behaviour based approach, `https://github.com/malwaremusings/unpacker`.

[30] VirusShare, vxshare, `https://virusshare.com`.

[31] S.Cutler, malshare, `https://malshare.com`.

[32] V.Heaven, V.heavens, `http://vxer.org/lib/`.