Exact computation of an error bound for the balanced linear complementarity problem with unique solution

Jean-Pierre Dussault, Jean Charles Gilbert

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Jean-Pierre Dussault [†] and Jean Charles Gilbert [‡]

This paper considers the balanced form of the standard linear complementarity problem with unique solution and provides a more precise expression of an upper error bound discovered by Chen and Xiang and published in 2006. This expression has at least two advantages. It makes possible the exact computation of the error bound factor and it provides a satisfactory upper estimate of that factor in terms of the data bitlength when the data is formed of rational numbers. Along the way, we show that, when any rowwise convex combination of two square matrices is nonsingular, the ℓ_∞ norm of the inverse of these rowwise convex combinations is maximized by an extreme diagonal matrix.

Keywords: Balanced linear complementarity problem, complexity, data bitlength, error bound, extreme diagonal matrix, matrix inverse norm, P-matrix, rowwise convex combination of matrices, separable function, strong duality.

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1 Introduction

Error bounds play a prominent role in the analysis of mathematical problems and the algorithms to solve them, in particular in numerical optimization [28]. This paper focuses on error bounds discovered by Chen and Xiang [8; 2006] for the linear complementarity problem with a **P**-matrix and simplifies the expression of its upper factor. The paper also deduces some consequences of this new expression.

In its standard form [11], the *linear complementarity problem* (LCP) reads

$$0 \leqslant x \perp (Mx+q) \geqslant 0, \tag{1.1}$$

where the unknown is $x \in \mathbb{R}^n$ (the set of real vectors with n components), while $M \in \mathbb{R}^{n\times n}$ (the set of real matrices of order n) and $q \in \mathbb{R}^n$ are data. Inequalities on vectors must be understood componentwise (for example $x \geqslant 0$ in (1.1) means $x_i \geqslant 0$ for all $i \in [1:n]$, the set of the first n integers). The compact writing of the problem in (1.1) means that one has to find a vector x in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geqslant 0\}$ such that $Mx+q \geqslant 0$ and $x^{\mathsf{T}}(Mx+q) = 0$ (the superscript "$^{\mathsf{T}}$" is used to denote vector or matrix transposition).

A matrix $M \in \mathbb{R}^{n\times n}$ is said to be a **P**-*matrix* if all its principal minors are positive (i.e., the determinant $\det M_{II} > 0$, for all $I \subseteq [1:n]$; by convention $\det M_{\varnothing\varnothing} = 1$). One denotes by **P** the class of **P**-matrices. It is known that problem (1.1) has a unique solution, whatever q is, if and only if $M \in \mathbf{P}$ [32; 1958]. There are many other characterizations of the **P**-matricity [11], including algorithmic ones [2, 3].

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For the sake of generality and for taking advantage of its symmetric formulation, which allows us to shorten some proofs, this paper considers an LCP in a slightly more general form than (1.1), namely

$$0 \leqslant (Ax + a) \perp (Bx + b) \geqslant 0, \tag{1.2}$$

where A, $B \in \mathbb{R}^{n\times n}$ and a, $b \in \mathbb{R}^n$ are the data. By comparison with problem (1.1), we name this problem the *balanced LCP*. This is a special case of the so-called (*extended*) *vertical LCP*, which uses more than two matrices and vectors in its formulation (see [10, 33]). Throughout this work, we assume that problem (1.2) has a unique solution $\bar{x}$.

An error bound associated with a set S is an estimate of the distance to S by quantities that are easier to evaluate than this distance, usually those that are used to define the set. The set considered in this paper is the solution set of the LCP (1.2), which has been said to be reduced to the singleton $\{\bar{x}\}$, while the quantity used to estimate the distance to $\bar{x}$ is defined as follows.

Let $\|\cdot\|$ denote an arbitrary norm on $\mathbb{R}^n$. The *natural residual* [22, 23] associated with the linear complementarity problem (1.2) is the function $r : \mathbb{R}^n \to \mathbb{R}^n$ whose value at $x \in \mathbb{R}^n$ is given by

$$r(x) := \min(Ax + a, Bx + b), \tag{1.3}$$

where the minimum operator "min" acts componentwise (for two vectors u, $v \in \mathbb{R}^n$ and $i \in [1:n]$: $[\min(u, v)]_i = \min(u_i, v_i)$). It is clear that x solves (1.2) if and only if $r(x) = 0$, since $\min(Ax+a, Bx+b) = 0$ if and only if, for all $i \in [1:n]$, $(Ax+a)_i \geqslant 0$, $(Bx+b)_i \geqslant 0$ and either $(Ax + a)_i$ or $(Bx + b)_i$ vanishes. Therefore $\|r(x)\|$ is a possible measure of the proximity of x to $\bar{x}$. In this paper, we consider error bounds of the form

$$\forall\, x \in \mathbb{R}^n : \quad \check{\beta}\, \|r(x)\| \leqslant \|x - \bar{x}\| \leqslant \beta\, \|r(x)\|,$$

where $\check{\beta}$ and β are positive constants (independent of x), that we call the *lower* and *upper error bound factors*, respectively. Error bounds for the LCP have been the subject of many contributions, see [30, 26, 24, 22, 23, 21, 8, 9, 12, 20, 7] for entry points.

For P and $Q \in \mathbb{R}^{n\times n}$, we define

$$[P, Q] := \{X \in \mathbb{R}^{n\times n} : P \leqslant X \leqslant Q\},$$

where the inequalities act again componentwise (i.e., $P \leqslant X \leqslant Q$ means $P_{ij} \leqslant X_{ij} \leqslant Q_{ij}$ for all i, $j \in [1:n]$). Hence, for the identity matrix I, $[0, I]$ is a compact notation for the set of diagonal matrices with diagonal elements in the interval $[0, 1]$. Note also that the set of *extreme points* of $[0, I]$, denoted by $\operatorname{ext}[0, I]$, is the set of diagonal matrices with diagonal elements in $\{0, 1\}$ (see [31; p. 162] for the definition of an extreme point of a convex set; one can use [14; proposition 2.12] for a meticulous proof of this claim).

For $D \in [0, I]$, we denote by

$$C_D := (I - D)A + DB \tag{1.4}$$

the *rowwise convex combination* of the matrices A and $B \in \mathbb{R}^{n\times n}$. The LCP (1.2) has a unique solution whatever the vectors a and b are, if and only if [1, 27, 34, 35]

$$\forall\, D \in [0, I] : \quad C_D \text{ is nonsingular.} \tag{1.5}$$

Then, the following lower and upper error bounds hold [8, 35]:

$$\forall\, x \in \mathbb{R}^n : \quad \left(\max_{D\in[0,I]} \|C_D\|\right)^{-1} \|r(x)\| \leqslant \|x-\bar{x}\| \leqslant \left(\max_{D\in[0,I]} \|C_D^{-1}\|\right) \|r(x)\|, \tag{1.6}$$

where $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$ and the induced matrix norm.

In this paper, we are interested in giving more precision on the way the lower and upper error bound factors appearing in (1.6) can be computed, when the ℓ_∞ norm is used. If the lower bound factor is easy to evaluate (see section 3.1), the upper bound factor

$$\beta := \max_{D\in[0,I]} \|C_D^{-1}\|_\infty \tag{1.7}$$

raises more difficulty. This concern makes perfect sense because, as far as we know, this upper error bound factor is the best one obtained so far for the LCP (1.1) with $M \in \mathbf{P}$; in particular, it is smaller, hence better, than the one of Mathias and Pang [24] (see [8; theorem 2.3]). We shall show that the evaluation of β can be simplified since one has

$$\beta = \max_{D\in\operatorname{ext}[0,I]} \|C_D^{-1}\|_\infty. \tag{1.8}$$

This extends to higher dimensions the simple observation that, when $n = 1$, the map $D \in [0,1] \mapsto \|C_D^{-1}\|_\infty$ is monotone, so that it attains its maximum on $[0,1]$ at a point in $\{0,1\}$. For $n > 1$, however, $D_{kk} \in [0,1] \mapsto \|C_D^{-1}\|_\infty$ may be nonmonotone, so that an analysis along this line is troublesome. Furthermore, this map can be neither convex nor concave (see [13]). For these reasons, we shall present a specific, rather long and indirect, proof of (1.8). The simplification (1.8) of (1.7) may look minor at first glance, but it may be interesting for reasons that are discussed in section 4: it provides a way of computing β exactly, it simplifies its computation for small n and it may be crucial for giving an upper estimate of β in terms of the data bitlength in some complexity analysis.

The paper is organized as follows. The next section presents two results that will play an important role in getting the expression (1.8) of β: the first one deals with the norm of a matrix inverse and the second deals with min-max duality in optimization. Section 3 is dedicated to the proof of (1.8). We conclude by some thoughts on complexity issues.

This paper is an abridged version of the more detailed report [13].

Notation

The unit closed ball associated with a norm $\|\cdot\|$ is denoted by $\bar{\mathfrak{B}} := \{x : \|x\| \leqslant 1\}$ and the unit sphere by $\partial\mathfrak{B} := \{x : \|x\| = 1\}$.

2 Preliminaries

This section presents two results that will play a major part in our strategy to get the desired result in section 3. The first one (lemma 2.1) gives an expression of $\|\mathcal{A}^{-1}\|$, for a nonsingular matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$, in terms of an optimization problem. Consequences of this expression for the ℓ_∞ norm are given in corollary 2.2 and in the technical lemma 2.3. The second result (lemma 2.4) highlights conditions to have strong duality on a product space $X \times [1\!:\!p]$ for a pairing function that has a separable property.

2.1 Norm of a matrix inverse

For a given nonsingular matrix function $z \in \mathbb{R}^p \mapsto \mathcal{A}(z) \in \mathbb{R}^{n\times n}$, analyzing the map $\|[\mathcal{A}(\cdot)]^{-1}\|$ is often more difficult than analyzing $\|[\mathcal{A}(\cdot)\|$. It is possible, however, to toggle from one map to the other thanks to the following lemma (formula (2.1a) can be found in [18; problem 5.6.P46]; a proof of the lemma is given in [13]).

Lemma 2.1 (norm of a matrix inverse) *If $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a nonsingular square matrix and if $\|\cdot\|$ denotes a vector norm and its induced matrix norm, then*

$$\min_{\|v\|=1} \|\mathcal{A}v\| = \|\mathcal{A}^{-1}\|^{-1}. \tag{2.1a}$$

In addition, $\bar{v}$ solves the problem in the left-hand side of (2.1a) if and only if $\bar{w} := \|\mathcal{A}^{-1}\|\,\mathcal{A}\bar{v}$ solves the problem in the left-hand side of

$$\max_{\|w\|=1} \|\mathcal{A}^{-1}w\| = \|\mathcal{A}^{-1}\|. \tag{2.1b}$$

In the sequel, the infinity vector and its induced matrix norms, both denoted by $\|\cdot\|_\infty$, are used. For this reason, we consider this case in corollary 2.2 below and bring some precision. We denote by e^i the ith basis vector of $\mathbb{R}^n$ and set $e := \sum_{i\in[1:n]} e^i$, which is the vector of all ones. By definition and computation [17; § 5.6.5] (see also (2.7a)-(2.7d) in the proof of corollary 2.2 below), for a matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$, one has

$$\|\mathcal{A}\|_\infty := \max_{\|w\|_\infty=1} \|\mathcal{A}w\|_\infty = \max_{i\in[1:n]} \|\mathcal{A}_{i:}\|_1, \tag{2.2}$$

where $\mathcal{A}_{i:} := (e^i)^{\mathsf{T}}\mathcal{A}$ denotes the ith row of $\mathcal{A}$ and $\|v\|_1 := \sum_{i\in[1:n]} |v_i|$ denotes the ℓ_1-norm of $v \in \mathbb{R}^n$. We also denote by "sign" the maximal monotone multifunction $\mathbb{R} \multimap \mathbb{R}$ that is the subdifferential of the absolute value function: it associates with $t \in \mathbb{R}$ the following set of $\mathbb{R}$:

$$\operatorname{sign} t := \begin{cases} \{-1\} & \text{if } t < 0 \\ [-1, 1] & \text{if } t = 0 \\ \{1\} & \text{if } t > 0. \end{cases} \tag{2.3}$$

One finds other definitions of sign(0), in particular to make the map "sign" a single-valued function, but our choice of definition is important for the sequel, like in the formulas (2.4b) below. Recall that $\|\cdot\|_1$ is the dual norm of $\|\cdot\|_\infty$ with respect to the Euclidean scalar product, which means that

$$\|v\|_1 = \max_{\|w\|_\infty=1} v^{\mathsf{T}}w = \max_{\|w\|_\infty=1} |v^{\mathsf{T}}w|. \tag{2.4a}$$

The solution sets of these maximum problems are

$$\operatorname*{Arg\,max}_{\|w\|_\infty=1} v^{\mathsf{T}}w = (\operatorname{sign} v) \cap \partial\mathfrak{B}_\infty \quad\text{and}\quad \operatorname*{Arg\,max}_{\|w\|_\infty=1} |v^{\mathsf{T}}w| = (\pm\operatorname{sign} v) \cap \partial\mathfrak{B}_\infty, \tag{2.4b}$$

where, for a vector $v \in \mathbb{R}^n$, $\operatorname{sign} v := (\operatorname{sign} v_1) \times \cdots \times (\operatorname{sign} v_n) \subseteq \mathbb{R}^n$ (hence $\operatorname{sign} 0_{\mathbb{R}^n} = \mathfrak{B}_\infty$), $\pm \operatorname{sign} v := (\operatorname{sign} v) \cup (-\operatorname{sign} v)$ and the boundary $\partial\mathfrak{B}_\infty$ of $\mathfrak{B}_\infty$ is present only to deal with the case where $v = 0$. For a nonsingular square matrix $\mathcal{A}$, we adopt the following notation

$$\mathcal{W}_\infty(\mathcal{A}) := \operatorname*{Arg\,max}_{\|w\|_\infty=1} \|\mathcal{A}^{-1}w\|_\infty = \{w \in \partial\mathfrak{B}_\infty : \|\mathcal{A}^{-1}w\|_\infty = \|\mathcal{A}^{-1}\|_\infty\}, \tag{2.5a}$$

$$\mathcal{V}_\infty(\mathcal{A}) := \operatorname*{Arg\,min}_{\|v\|_\infty=1} \|\mathcal{A}v\|_\infty = \{v \in \partial\mathfrak{B}_\infty : \|\mathcal{A}v\|_\infty = \|\mathcal{A}^{-1}\|_\infty^{-1}\}. \tag{2.5b}$$

The second equality in (2.5a) comes from the definition of the induced matrix norm $\|\cdot\|_\infty$ in (2.2), while the second equality in (2.5b) is deduced from the identity (2.1a). The next corollary gives other expressions of these sets.

Corollary 2.2 (ℓ_∞-norm of a matrix inverse) *Suppose that $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. Set $\beta := \|\mathcal{A}^{-1}\|_\infty$ and $\alpha := 1/\beta$. Then,*

$$\mathcal{W}_\infty(\mathcal{A}) = \bigcup \left\{\pm \operatorname{sign}(\mathcal{A}^{-\mathsf{T}}e^i) : \|(\mathcal{A}^{-1})_{i:}\|_1 = \beta\right\}, \tag{2.6a}$$

$$\mathcal{V}_\infty(\mathcal{A}) = \alpha\, \mathcal{A}^{-1}\Big(\mathcal{W}_\infty(\mathcal{A})\Big). \tag{2.6b}$$

Proof. [(2.6a)] Observe first that

$$\begin{aligned}
\beta &= \max_{\|w\|_\infty=1} \|\mathcal{A}^{-1}w\|_\infty \qquad \text{[definition of the matrix norm } \|\cdot\|_\infty] & (2.7a)\\
&= \max_{\|w\|_\infty=1} \max_{i\in[1:n]} |(e^i)^\mathsf{T}\mathcal{A}^{-1}w| \qquad \text{[definition of the vector norm } \|\cdot\|_\infty] & (2.7b)\\
&= \max_{i\in[1:n]} \max_{\|w\|_\infty=1} |(e^i)^\mathsf{T}\mathcal{A}^{-1}w| \qquad \text{[the max's commute]} & (2.7c)\\
&= \max_{i\in[1:n]} \|\mathcal{A}^{-\mathsf{T}}e^i\|_1 \qquad [(2.4a)]. & (2.7d)
\end{aligned}$$

We can now establish the identity (2.6a).

[$\subseteq$] If $\bar{w} \in \mathcal{W}_\infty(\mathcal{A})$, $\bar{w}$ solves the problem in (2.7a)-(2.7b), by definition. Let $\bar{\imath} \in [1:n]$ be a solution to the inner problem $\max\{|(e^i)^\mathsf{T}\mathcal{A}^{-1}\bar{w}| : i \in [1:n]\}$ appearing in (2.7b). Then, the pair $(\bar{w}, \bar{\imath})$ maximizes the map $(w, i) \in \partial\mathfrak{B}_\infty \times [1:n] \mapsto |(e^i)^\mathsf{T}\mathcal{A}^{-1}w|$. It follows that $\bar{\imath}$ solves to the problems in (2.7c)-(2.7d) and $\bar{w}$ is a solution to the inner problem $\max\{|(e^{\bar{\imath}})^\mathsf{T}\mathcal{A}^{-1}w| : \|w\|_\infty = 1\}$ appearing in (2.7c). Hence, by (2.4b), $\bar{w} \in \pm \operatorname{sign}(\mathcal{A}^{-\mathsf{T}}e^{\bar{\imath}})$ and, by (2.7d), $\beta = \|\mathcal{A}^{-\mathsf{T}}e^{\bar{\imath}}\|_1 = \|(\mathcal{A}^{-1})_{\bar{\imath}:}\|_1$.

[$\supseteq$] Suppose now that $\bar{w} \in \pm \operatorname{sign}(\mathcal{A}^{-\mathsf{T}}e^{\bar{\imath}})$ for some $\bar{\imath} \in [1:n]$ satisfying $\|\mathcal{A}^{-\mathsf{T}}e^{\bar{\imath}}\|_1 = \beta$.

- By this last identity, $\bar{\imath}$ solves the problem in (2.7d), hence the problem in (2.7c).
- By the nonsingularity of $\mathcal{A}^{-\mathsf{T}}$, one component of $\mathcal{A}^{-\mathsf{T}}e^{\bar{\imath}}$ does not vanish, so that $\bar{w} \in \pm \operatorname{sign}(\mathcal{A}^{-\mathsf{T}}e^{\bar{\imath}}) \cap \partial\mathfrak{B}_\infty$. By (2.4b), this implies that $\bar{w}$ solves the problem $\max\{|(e^{\bar{\imath}})^\mathsf{T}\mathcal{A}^{-1}w| : \|w\|_\infty = 1\}$.

It results from these last two observations and (2.7a)-(2.7c), that $\bar{w}$ solves the problem in (2.7a)-(2.7b). We have shown that $\bar{w} \in \mathcal{W}_\infty(\mathcal{A})$.

[(2.6b)] This is a consequence of the last claim in lemma 2.1, according to which $\bar{v} \in \mathcal{V}_\infty(\mathcal{A})$ if and only if $\bar{v} = \alpha\, \mathcal{A}^{-1}\bar{w}$ with $\bar{w} \in \mathcal{W}_\infty(\mathcal{A})$. □

We conclude this section by synthesizing in the following lemma a mechanism that, despite its innocuous appearance, plays a major part in the proof of proposition 3.1 below. As shown in the lemma's proof, this mechanism is only operational when some element of $\mathcal{A}^{-1}$ vanishes, but this fact is revealed indirectly, through a property of a vector $v \in \mathcal{V}_\infty(\mathcal{A})$.

Lemma 2.3 (technical) *Suppose that $\mathcal{A} \in \mathbb{R}^{n\times n}$ is nonsingular, that $\alpha := \|\mathcal{A}^{-1}\|_\infty^{-1}$ and that $v \in \mathcal{V}_\infty(\mathcal{A})$ has the property that $|(\mathcal{A}v)_k| < \alpha$ for some $k \in [1:n]$. Then, there exists a $v' \in \mathcal{V}_\infty(\mathcal{A})$ such that $(\mathcal{A}v')_k = 0$.*

PROOF. Let $\beta := 1/\alpha$. Since $v \in \mathcal{V}_\infty(\mathcal{A})$, the vector defined by $w := \beta\mathcal{A}v$ is in $\mathcal{W}_\infty(\mathcal{A})$, by (2.6b). By assumption, $(\mathcal{A}v)_k \in (-\alpha, \alpha)$, so that $w_k \in (-1, 1)$. These two facts on w and (2.6a) imply that there must be some index i such that

$$w \in \pm\,\mathrm{sign}(\mathcal{A}^{-\mathsf{T}}e^i), \qquad \|(\mathcal{A}^{-1})_{i:}\|_1 = \beta \qquad \text{and} \qquad (\mathcal{A}^{-1})_{ik} = 0.$$

Define the vector $w' \in \mathbb{R}^n$ by vanishing the kth component of w:

$$w'_i := \begin{cases} w_i & \text{if } i \neq k \\ 0 & \text{otherwise.} \end{cases}$$

Then, we also have $w' \in \pm\,\mathrm{sign}(\mathcal{A}^{-\mathsf{T}}e^i)$, implying that $w' \in \mathcal{W}_\infty(\mathcal{A})$. The sought vector is $v' := \alpha\mathcal{A}^{-1}w'$. Indeed, on the one hand, $v' \in \mathcal{V}_\infty(\mathcal{A})$ by (2.6b). On the other hand, $\mathcal{A}v' = \alpha w'$ implying that $(\mathcal{A}v')_k = 0$, as desired. □

2.2 Strong duality for separable functions

Let be given a set X and p functions $\varphi_i : X \to \bar{\mathbb{R}}$. Usually, equality does not hold in the weak duality inequality [16, 15, 5, 14]

$$\inf_{x\in X}\ \max_{i\in[1:p]}\ \varphi_i(x) \geqslant \max_{i\in[1:p]}\ \inf_{x\in X}\ \varphi_i(x). \tag{2.8}$$

Take for example, $X = \mathbb{R}$, $p = 2$, $\varphi_1(x) = (x+1)^2$ and $\varphi_2(x) = (x-1)^2$, in which case the left-hand side value is 1, while the right-hand side value is 0 (see [15; lemma 4.5] for a way of modifying (2.8) that ensures equality). The situation is very different, more elementary and more favorable, when X is a Cartesian product $X = X_1 \times \cdots \times X_p$ of sets X_i and each function φ_i only depends on the ith component $x_i \in X_i$ of $x = (x_1, \ldots, x_p) \in X$; then equality holds above with some other interesting properties. This particular situation, which occurs below, is analyzed in the next lemma. In this one, the problems

$$\inf_{x\in X}\ \max_{i\in[1:p]}\ \varphi_i(x_i) \qquad \text{and} \qquad \max_{i\in[1:p]}\ \inf_{x_i\in X_i}\ \varphi_i(x_i).$$

are called the *primal* and *dual* problems, respectively. A *primal* (resp. *dual*) *solution* is a solution to this primal (resp. dual) problem.

The next lemma not only shows the lack of duality gap for the separable case, but also describes the sets of primal and dual solutions. It also shows how to construct a primal solution from the dual solutions, as well as a dual solution from the primal solutions.

Lemma 2.4 (strong duality for separable functions) *Let $X := X_1 \times \cdots \times X_p$ be the Cartesian product of nonempty sets X_i and let $\varphi_i : X_i \to \mathbb{R}$, $i \in [1:p]$, be arbitrary functions. An $x \in X$ is written $x = (x_1, \ldots, x_p)$, with $x_i \in X_i$ for $i \in [1:p]$.*

1) (No duality gap) *The following identity holds*

$$\inf_{x\in X} \max_{i\in[1:p]} \varphi_i(x_i) = \max_{i\in[1:p]} \inf_{x_i\in X_i} \varphi_i(x_i). \tag{2.9a}$$

Denote by $\bar{v}$ the common value of the two sides of this identity.

2) (Set of primal solutions) *The set of primal solutions is the possibly empty set $\bar{X} := \bar{X}_1 \times \cdots \times \bar{X}_p$, where*

$$\bar{X}_i := \{x_i \in X_i : \varphi_i(x_i) \leqslant \bar{v}\}. \tag{2.9b}$$

3) (Set of dual solutions) *The set of dual solutions is the nonempty set*

$$\bar{I} := \{i \in [1:p] : \varphi_i(x_i) \geqslant \bar{v} \textit{ for all } x_i \in X_i\}. \tag{2.9c}$$

4) (Saddle-point property) *The following properties are equivalent:*

   (i) $(\bar{x}, \bar{\imath}) \in \bar{X} \times \bar{I}$,

   (ii) $(\bar{x}, \bar{\imath})$ *is a saddle-point of the map* $(x, i) \in X \times [1:p] \mapsto \varphi_i(x_i)$, *meaning that*

$$\forall\, (x, i) \in X \times [1:p] : \qquad \varphi_i(\bar{x}_i) \leqslant \varphi_{\bar{\imath}}(\bar{x}_{\bar{\imath}}) \leqslant \varphi_{\bar{\imath}}(x_{\bar{\imath}}), \tag{2.9d}$$

   (iii) $\bar{x}_{\bar{\imath}}$ *minimizes* $\varphi_{\bar{\imath}}$ *on* $X_{\bar{\imath}}$ *and* $\bar{\imath}$ *maximizes* $\varphi_i(\bar{x}_i)$ *on* $[1:p]$.

5) (Deducing a primal solution from the dual solutions) *Suppose that, for any dual solution $\bar{\imath} \in \bar{I}$, the problem $\inf\{\varphi_{\bar{\imath}}(x_{\bar{\imath}}) : x_{\bar{\imath}} \in X_{\bar{\imath}}\}$ has a solution $\hat{x}_{\bar{\imath}}$, then the primal problem has a solution $\bar{x} \in X$ satisfying*

$$\bar{x}_{\bar{I}} = \hat{x}_{\bar{I}} \qquad \textit{and} \qquad \bar{I} = \operatorname*{Arg\,max}_{i\in[1:p]} \varphi_i(\bar{x}). \tag{2.9e}$$

6) (Deducing a dual solution from the primal solutions) *Suppose that $\bar{X} \neq \varnothing$. Then, $\bar{\imath} \in \bar{I}$ if and only if, for all $\bar{x} \in \bar{X}$, $\bar{\imath}$ maximizes $i \in [1:p] \mapsto \varphi_i(\bar{x}_i)$,*

Proof. We only prove points 1 and 2, which are those intervening below. For a proof of the other points, see [13].

1) By the weak duality property (2.8) and the fact that φ_i only depends on the ith component of x, the inequality "$\geqslant$" holds in (2.9a). Let us prove the reverse inequality. Let $\varepsilon > 0$. For any $i \in [1:p]$, there is an $x_i^\varepsilon \in X_i$ such that

$$\varphi_i(x_i^\varepsilon) \leqslant \inf_{x_i\in X_i} \varphi_i(x_i) + \varepsilon.$$

Therefore,

$$\max_{i\in[1:p]} \varphi_i(x_i^\varepsilon) \leqslant \max_{i\in[1:p]} \inf_{x_i\in X_i} \varphi_i(x_i) + \varepsilon. \tag{2.10}$$

It is here that the separability assumption intervenes. Since the left-hand side of (2.10) is the value at $x^\varepsilon := (x_1^\varepsilon, \dots, x_p^\varepsilon)$ of the function $x = (x_1, \dots, x_p) \in X \mapsto \max_{i\in[1:p]} \varphi_i(x_i)$, the following inequality certainly holds

$$\inf_{x\in X} \max_{i\in[1:p]} \varphi_i(x_i) \leqslant \max_{i\in[1:p]} \varphi_i(x_i^\varepsilon).$$

Combining with (2.10), we get

$$\inf_{x\in X} \max_{i\in[1:p]} \varphi_i(x_i) \leqslant \max_{i\in[1:p]} \inf_{x_i\in X_i} \varphi_i(x_i) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the inequality "$\leqslant$" holds in (2.9a).

2) The point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p)$ is a primal solution if and only if

$$\max_{i\in[1:p]} \varphi_i(\bar{x}_i) \leqslant \inf_{x\in X} \max_{i\in[1:p]} \varphi_i(x_i) = \bar{v} \qquad \text{or} \qquad \forall\, i \in [1:p]:\ \varphi_i(\bar{x}_i) \leqslant \bar{v}.$$

This fact reads $\bar{x} \in \bar{X}$, for the given $\bar{X}$. □

3 Finitely computable error bounds for the LCP

This section focusses on the exact computation of the lower and upper error bound factors in (1.6). More specifically, its main result, proposition 3.1, shows that a solution to (1.7) can always be found in $\operatorname{ext}[0, I]$, the set of extreme points of $[0, I]$.

3.1 Computation of the lower error bound factor

Before focusing in section 3.2 on the main objective of this paper, which is the simplification of the upper error bound factor in (1.6), let us mention that the lower error bound factor in (1.6), with the ℓ_∞ norm, namely

$$\left(\max_{D\in[0,I]} \|C_D\|_\infty \right)^{-1}, \tag{3.1}$$

can be easily computed.

Observe first that, for two vectors u and v and a vector norm $\|\cdot\|$, one has

$$\max_{t\in[0,1]} \|(1-t)u + tv\|) = \max(\|u\|, \|v\|) = \max_{t\in\{0,1\}} \|(1-t)u + tv\|). \tag{3.2}$$

Next,

$$\begin{aligned}
\max_{D\in[0,I]} \|C_D\|_\infty &= \max_{D\in[0,I]} \max_{i\in[1:n]} \|(1-D_{ii})A_{i:} + D_{ii}B_{i:}\|_1 \qquad [(1.4) \text{ and } (2.2)] \\
&= \max_{i\in[1:n]} \max_{D_{ii}\in[0,1]} \|(1-D_{ii})A_{i:} + D_{ii}B_{i:}\|_1 \qquad [\text{the max's commute}] \\
&= \max_{i\in[1:n]} \max(\|A_{i:}\|_1, \|B_{i:}\|_1) \qquad [(3.2)] \\
&= \max(\|A\|_\infty, \|B\|_\infty) \qquad [\text{the max's commute}].
\end{aligned} \tag{3.3}$$

This shows that (3.1) can be easily computed. Note that, in view of (3.3), the maximum in (3.1) is obtained for $D \in \{0, I\}$.

3.2 Computation of the upper error bound factor

By the compactness of $[0, I]$ and the continuity of $D \in [0, I] \mapsto \|C_D^{-1}\|_\infty$, the maximization problem (1.7), recalled below

$$\beta := \max_{D \in [0,I]} \|C_D^{-1}\|_\infty, \tag{3.4}$$

has a solution, say $\bar{D}$. Since $C_{\bar{D}}$ is nonsingular, β given by (3.4) is finite and positive. Then, one can also define the positive number

$$\alpha := \beta^{-1}. \tag{3.5}$$

The goal of this section is to show that the value β can be obtained by restricting the feasible set of problem (3.4) to $\text{ext}[0, I]$, the set of extreme diagonal matrices of $[0, I]$:

$$\beta = \max_{D \in \text{ext}[0,I]} \|C_D^{-1}\|_\infty. \tag{3.6}$$

Proposition 3.1 (validity of (3.6)) *Suppose that A and $B \in \mathbb{R}^{n \times n}$ satisfy (1.5) and that $\bar{D}$ solves the optimization problem in (3.4). Then, if $\bar{D}_{kk} \in (0, 1)$ for some $k \in [1:n]$, $\bar{D}$ remains optimal if $\bar{D}_{kk}$ is changed to any value in $[0, 1]$. In particular, the value of β defined by (3.4) is also given by (3.6).*

Before starting the analysis, let us observe that the objective of problem (3.4) is made of the composition of the nonlinear smooth function $D \mapsto C_D^{-1}$ and the convex function $\|\cdot\|_\infty$, but this objective is maximized, not minimized, so that the theory developed for the class of composite problems [6, 29, 4] does not apply. For this reason, we provide a specific proof of proposition 3.1. This one is postponed to page 11, after the following preliminary considerations on problem (3.4).

Part of the analysis is based on the following rewriting of β, defined by (3.4) (some more justifications are given after (3.7e), C_D is defined by (1.4)):

$$\max_{D \in [0,I]} \|C_D^{-1}\|_\infty = \max_{D \in [0,I]} \left(\min_{\|v\|_\infty = 1} \|C_D v\|_\infty \right)^{-1} \qquad [(2.1a)] \tag{3.7a}$$

$$= \left(\min_{D \in [0,I]} \min_{\|v\|_\infty = 1} \|C_D v\|_\infty \right)^{-1} \tag{3.7b}$$

$$= \left(\min_{\|v\|_\infty = 1} \min_{D \in [0,I]} \|C_D v\|_\infty \right)^{-1} \qquad \text{[the min's commute]} \tag{3.7c}$$

$$= \left(\min_{\|v\|_\infty = 1} \min_{D \in [0,I]} \max_{i \in [1:n]} |(C_D v)_i| \right)^{-1} \qquad [\text{definition of } \|\cdot\|_\infty] \tag{3.7d}$$

$$= \left(\min_{\|v\|_\infty = 1} \max_{i \in [1:n]} \min_{D_{ii} \in [0,1]} |(1 - D_{ii})(Av)_i + D_{ii}(Bv)_i| \right)^{-1}, \tag{3.7e}$$

where we have been able to switch $\min_D$ and $\max_i$ from (3.7d) to (3.7e), without duality gap, thanks to point 1 of lemma 2.4 and the fact that $[0, I] = [0, 1] \times \cdots \times [0, 1]$ (n times) is a Cartesian product and that $|(C_D v)_i| = |(1 - D_{ii})(Av)_i + D_{ii}(Bv)_i|$ only depends on D_{ii}.

In (3.7c), we have a minimum in v (i.e., the infimum is attained), since by (3.7b) the function $(D, v) \mapsto \|C_D v\|_\infty$ has a minimizer $(\bar{D}, \bar{v})$ on $[0, I] \times \partial\mathfrak{B}_\infty$, which implies that $\bar{v}$ solves the problem in (3.7c) (this property of nested optimization problems is discussed around [14; corollary 1.10]). Let us deduce some consequences of the identities in (3.7).

According to (3.4), the value of the left-hand side in (3.7a) is $\beta > 0$ and, according to (3.5), the optimal values of the optimization problems inside the parentheses in (3.7b)-(3.7c) is $\alpha > 0$, so that

$$\alpha = \min_{D\in[0,I]} \min_{\|v\|_\infty=1} \|C_D v\|_\infty, \tag{3.8a}$$

$$= \min_{\|v\|_\infty=1} \min_{D\in[0,I]} \|C_D v\|_\infty. \tag{3.8b}$$

Therefore, one can write

$$\boxed{\bar{D} \text{ solves (3.4)} \quad \Longleftrightarrow \quad \exists\, \bar{v} \text{ such that } (\bar{D}, \bar{v}) \text{ solves problems (3.8).}} \tag{3.9}$$

We also have

$$\boxed{\left.\begin{array}{r} \bar{D} \text{ solves (3.4)} \\ \bar{v} \in \mathcal{V}_\infty(C_{\bar{D}}) \end{array}\right\} \quad \Longleftrightarrow \quad (\bar{D}, \bar{v}) \text{ solves problems (3.8).}} \tag{3.10}$$

This is because, when $\bar{D}$ solves (3.4) and $\bar{v} \in \mathcal{V}_\infty(C_{\bar{D}})$ (i.e., $\bar{v}$ minimizes $\|C_{\bar{D}} v\|_\infty$ on $\partial\mathfrak{B}_\infty$ by (2.5b)), $(\bar{D}, \bar{v})$ solves the problems in (3.8). Reciprocally, when $(\bar{D}, \bar{v})$ solves the problems in (3.8), then $\bar{D}$ solves (3.4) by (3.9) and $\bar{v}$ minimizes $\|C_{\bar{D}} v\|_\infty$ on $\partial\mathfrak{B}_\infty$, which also reads $\bar{v} \in \mathcal{V}_\infty(C_{\bar{D}})$ by (2.5b).

Pursuing along the vein that exploits (3.7), we see that the optimal value of the optimization problems inside the parentheses in (3.7d)-(3.7e) is also $\alpha > 0$, so that, for a $\bar{v}$ such that $(\bar{D}, \bar{v})$ solves the problems in (3.8) for some $\bar{D} \in [0, I]$, one has

$$\alpha = \min_{D\in[0,I]} \max_{i\in[1:n]} |(C_D \bar{v})_i|, \tag{3.11a}$$

$$= \max_{i\in[1:n]} \min_{D_{ii}\in[0,1]} |(1 - D_{ii})(A\bar{v})_i + D_{ii}(B\bar{v})_i|. \tag{3.11b}$$

We shall also use the following implication:

$$\boxed{\left.\begin{array}{r} (\bar{D}, \bar{v}) \text{ solves problems (3.8)} \\ \bar{D}' \text{ solves (3.11a)} \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} (\bar{D}', \bar{v}) \text{ solves problems (3.8)} \\ \bar{D}' \text{ solve (3.4).} \end{array}\right.} \tag{3.12}$$

Indeed, by the left-hand side of the implication, $(\bar{D}, \bar{v})$ minimizes $\|C_D v\|_\infty$ on $[0, I] \times \partial\mathfrak{B}_\infty$ and $\bar{D}'$ minimizes $\|C_D \bar{v}\|_\infty$ on $[0, I]$. Then, $(\bar{D}', \bar{v})$ minimizes $\|C_D v\|_\infty$ on $[0, I] \times \partial\mathfrak{B}_\infty$ or, equivalently, $(\bar{D}', \bar{v})$ solves problems (3.8). Next, $\bar{D}'$ solves (3.4), by (3.9).

We conclude this preliminary discussion with an elementary lemma [13].

Lemma 3.2 (elementary) *Suppose that ν and $\mu \in \mathbb{R}$, that $\alpha > 0$ and that*

$$\min_{\delta\in[0,1]} |(1-\delta)\nu + \delta\mu| = \alpha. \tag{3.13}$$

Then, the solution set of the optimization problem in (3.13) is $\{0\}$, $\{1\}$ or $[0,1]$.

Proof of proposition 3.1. Suppose that A and $B \in \mathbb{R}^{n\times n}$ satisfy (1.5) and that $\bar{D}$ solves the optimization problem in (3.4). Since the last claim of the proposition is clear, we only focus on the first part of it, assuming that $\bar{D}_{kk} \in (0,1)$ for some $k \in [1\!:\!n]$. By (3.9)-(3.10), there is a $\bar{v} \in \mathcal{V}_\infty(C_{\bar{D}})$ such that $(\bar{D}, \bar{v})$ solves the problems in (3.8). The goal of the proof is now to show that one can replace $\bar{D}_{kk}$ by any value in $[0,1]$, to form a diagonal matrix $\bar{D}'$ that is still a solution to (3.4). Sometimes (case 1 below), this goal will be reached with the chosen initial $\bar{v}$; other times (case 2 below), it will be necessary to change the optimal $\bar{D}$ and $\bar{v} \in \mathcal{V}_\infty(C_{\bar{D}})$ several times (infinitely often is not excluded, with a limit argument) to reach the goal. Before introducing these cases, we highlight the principal argument that is used in the proof.

Principal argument. Recall that the optimal value of (3.4) is denoted by $\beta := \|C_{\bar{D}}^{-1}\|_\infty$, which is positive, and that $\alpha := 1/\beta = \|C_{\bar{D}}\bar{v}\|_\infty$ for the chosen $\bar{v} \in \mathcal{V}_\infty(C_{\bar{D}})$. Now, we want to determine the other values that the kth diagonal element $\bar{D}_{kk}$ of the optimal $\bar{D}$ can take, if any, while keeping the optimality of the resulting diagonal matrix. Here is the mechanism that allows us to change $\bar{D}_{kk}$. By point 2 of lemma 2.4, for the current $\bar{v}$ and for any value $\bar{D}'_{kk}$ taken in the interval

$$[a_k, b_k] := \{D_{kk} \in [0,1] : |(1-D_{kk})(A\bar{v})_k + D_{kk}(B\bar{v})_k| \leqslant \alpha\}, \tag{3.14}$$

the diagonal matrix $\bar{D}'$ defined by

$$\bar{D}'_{ii} := \begin{cases} \bar{D}'_{kk} & \text{if } i = k \\ \bar{D}_{ii} & \text{otherwise,} \end{cases}$$

is a solution to problem (3.11a). By (3.12), we get that $\bar{D}'$ is a solution to problem (3.4). In conclusion, for any $\bar{v} \in \mathcal{V}_\infty(C_{\tilde{D}})$ with $\tilde{D}$ solving (3.4), the interval $[a_k, b_k]$ defined by (3.14) is a set of optimal values for $\bar{D}_{kk}$. These intervals depend on $\bar{v}$. Our objective is to show that the union of these intervals $[a_k, b_k]$ for some well chosen $\bar{v} \in \mathcal{V}_\infty(C_{\tilde{D}})$ and solutions $\tilde{D}$ to (3.4) is $[0,1]$ (the reasoning only holds when $\bar{D}_{kk} \in (0,1)$). In case 1 below, one has $[a_k, b_k] = [0,1]$, immediately. In case 2 below, the objective is realized by changing $\bar{v}$ and $\tilde{D}$, alternatively, possibly infinitely often.

By optimality of $\bar{D}$ for (3.11a), we have $|(C_{\bar{D}}\bar{v})_k| \leqslant \alpha$ (recall that $(C_{\bar{D}}\bar{v})_k = (1-\bar{D}_{kk})(A\bar{v})_k + \bar{D}_{kk}(B\bar{v})_k$ only depends on $\bar{D}_{kk}$). Therefore, either $\min\{|(C_D\bar{v})_k| : D_{kk} \in [0,1]\} = \alpha$ or $\min\{|(C_D\bar{v})_k| : D_{kk} \in [0,1]\} < \alpha$. We now examine these two complementary cases.

1) *Case where*

$$\min_{D_{kk}\in[0,1]} |(1-D_{kk})(A\bar{v})_k + D_{kk}(B\bar{v})_k| = \alpha. \tag{3.15}$$

By lemma 3.2, with $\nu = (A\bar{v})_k$ and $\mu = (B\bar{v})_k$, the solution set of problem (3.15) is either $\{0\}$, $\{1\}$ or $[0,1]$. From (3.14) and (3.15), this solution set is also the interval $[a_k, b_k]$.

Therefore, by the *principal argument* described above, for the considered vector $\bar{v}$ solving (3.8b), the kth element of the optimal $\bar{D}$ can be $\{0\}$, $\{1\}$ or any value in $[0, 1]$. Since $\bar{D}_{kk} \in (0, 1)$, by asumption, one has $[a_k, b_k] = [0, 1]$, which concludes the proof in this case.

2) *Case where*

$$\min_{D_{kk}\in[0,1]} |(1 - D_{kk})(A\bar{v})_k + D_{kk}(B\bar{v})_k| < \alpha. \tag{3.16}$$

In that case, the interval $[a_k, b_k]$ defined by (3.14) is not guaranteed to contain 0 or 1. By modifying the vector $\bar{v}$, however, we show that one can find intervals of substitutes for $\bar{D}_{kk}$, maintaining the optimality of the diagonal matrix, that cover all the interval $[0, 1]$; this is the desired result.

It suffices to extend the interval $[a_k, b_k]$ of optimal values for $\bar{D}_{kk}$ to the left so that it contains 0, because, by symmetry, the interval $[a_k, b_k]$ can then also be extended to the right so that it contains 1 (switch A and B and replace D by $I - D$).

One can assume that $a_k > 0$, since otherwise there is nothing to prove. This implies that $\alpha < \|A\|_\infty$ (because, by optimality of $\bar{D}$, one has $\alpha = \|C_{\bar{D}}\bar{v}\|_\infty \leqslant \|C_0\bar{v}\|_\infty = \|A\bar{v}\|_\infty \leqslant \|A\|_\infty$ and $\alpha \neq \|A\|_\infty$ since otherwise $|(A\bar{v})_k| \leqslant \|A\bar{v}\|_\infty = \alpha$ and $a_k = 0$ by (3.14)).

We do this extension by an iterative procedure whose iterates, indexed by $j \in \mathbb{N}$, are pairs $(\bar{D}^j, \bar{v}^j)$ verifying

$$(\bar{D}^j, \bar{v}^j) \text{ solves the problems in (3.8)}, \tag{3.17a}$$

$$\bar{D}^j_{ii} = \bar{D}_{ii} \text{ for } i \neq k, \tag{3.17b}$$

$$(1 - \bar{D}^j_{kk})(A\bar{v}^j)_k + \bar{D}^j_{kk}(B\bar{v}^j)_k = 0, \tag{3.17c}$$

$$0 < \bar{D}^{j+1}_{kk} \leqslant (1 - \alpha/(2\|A\|_\infty))\bar{D}^j_{kk}. \tag{3.17d}$$

The iterative process is interrupted as soon as 0 is in the interval

$$[a^j_k, b^j_k] := \{D_{kk} \in [0, 1] : |(1 - D_{kk})(A\bar{v}^j)_k + D_{kk}(B\bar{v}^j)_k| \leqslant \alpha\}, \tag{3.18}$$

that is, as soon as $a^j_k = 0$. It will be clear from the construction of these intervals that their union will be formed of solutions for $\bar{D}_{kk}$. Actually, the reasoning below does not control directly a^j_k but it controls $\bar{D}^j_{kk} \in [a^j_k, b^j_k]$, which tends to zero by (3.17d).

- Let us determine $(\bar{D}^0, \bar{v}^0)$ and verify (3.17a)-(3.17c) for $j = 0$ ((3.17d) for $j = 0$ will be verified when $\bar{D}^1_{kk}$ will be determined, in the next point).

  When (3.16) holds, point 2 of lemma 2.4 ensures that changing $\bar{D}_{kk}$ in order to have $|(C_{\bar{D}}\bar{v})_k| < \alpha$ will not change the optimality of $\bar{D}$, so that we can actually assume that $|(C_{\bar{D}}\bar{v})_k| < \alpha$. Then, lemma 2.3 with $\mathcal{A} = C_{\bar{D}}$ and $v = \bar{v} \in \mathcal{V}_\infty(C_{\bar{D}})$ tells us that one can find a $\bar{v}^0 \in \mathcal{V}_\infty(C_{\bar{D}})$ such that $(C_{\bar{D}}\bar{v}^0)_k = 0$, which reads $(1 - \bar{D}_{kk})(A\bar{v}^0)_k + \bar{D}_{kk}(B\bar{v}^0)_k = 0$. Therefore, setting $\bar{D}^0 := \bar{D}$, we see that (3.17b) and (3.17c) hold. Furthermore, (3.17a) also holds since, by (3.10), the fact that $\bar{D}^0$ solves (3.4) and that $\bar{v}^0 \in \mathcal{V}_\infty(C_{\bar{D}^0})$ implies that $(\bar{D}^0, \bar{v}^0)$ solves the problems in (3.8).

- Let us now show how to construct $(\bar{D}^{j+1}, \bar{v}^{j+1})$ from $(\bar{D}^j, \bar{v}^j)$, if this is necessary.

Assume that $a_k^j > 0$ (otherwise, there is no reason to pursue the iterative process). Then, $0 \neq (A\bar{v}^j)_k \neq (B\bar{v}^j)_k$ (otherwise $a_k^j = 0$) and by definition of a_k^j in (3.18):

$$(1 - a_k^j)(A\bar{v}^j)_k + a_k^j(B\bar{v}^j)_k = \alpha \operatorname{sign}((A\bar{v}^j)_k). \tag{3.19}$$

Indeed, if $(1-a_k^j)(A\bar{v}^j)_k + a_k^j(B\bar{v}^j)_k = \alpha$, one has, by definition of a_k^j, $(1-t)(A\bar{v}^j)_k + t(B\bar{v}^j)_k > \alpha$ for all $t \in [0, a_k^j)$, in particular $(1-0)(A\bar{v}^j)_k + 0(B\bar{v}^j)_k > \alpha$, so that $(A\bar{v}^j)_k > \alpha > 0$. Similarly, $(1-a_k^j)(A\bar{v}^j)_k + a_k^j(B\bar{v}^j)_k = -\alpha$ implies that $(A\bar{v}^j)_k < 0$.

Now, define the diagonal matrix $\bar{D}^{j+1} \in [0, I]$ by

$$\bar{D}_{ii}^{j+1} \in \begin{cases} (a_k^j + \bar{D}_{kk}^j)/2 & \text{if } i = k \\ \bar{D}_{ii} & \text{otherwise,} \end{cases} \tag{3.20}$$

so that (3.17b) is verified with $j+1$ instead of j. Adding side by side (3.17c) and (3.19), and using the definition (3.20) of $\bar{D}^{j+1}$, we get

$$(1 - \bar{D}_{kk}^{j+1})(A\bar{v}^j)_k + \bar{D}_{kk}^{j+1}(B\bar{v}^j)_k = \frac{1}{2}\,\alpha \operatorname{sign}((A\bar{v}^j)_k). \tag{3.21}$$

Subtracting side by side (3.17c) from (3.21), using $(A\bar{v}^j)_k \neq (B\bar{v}^j)_k$, $(A\bar{v}^j)_k - (B\bar{v}^j)_k = (A\bar{v}^j)_k/\bar{D}_{kk}^j$ by (3.17c) again and finally $|(A\bar{v}^j)_k| \leqslant \|A\|_\infty$ yields

$$\bar{D}_{kk}^j - \bar{D}_{kk}^{j+1} = \frac{(\alpha/2)\operatorname{sign}(A\bar{v}^j)_k}{(A\bar{v}^j)_k - (B\bar{v}^j)_k} = \frac{\alpha/2}{|(A\bar{v}^j)_k|}\,\bar{D}_{kk}^j \geqslant \frac{\alpha}{2\|A\|_\infty}\bar{D}_{kk}^j,$$

which is (3.17d).

We still have to determine $\bar{v}^{j+1}$ and to verify (3.17a) and (3.17c) with $j+1$ instead of j. By (3.20) and (3.17c), $\bar{D}_{kk}^{j+1} \in [a_k^j, \bar{D}_{kk}^j] \subseteq [a_k^j, b_k^j]$. This implies that, like $\bar{D}^j$, $\bar{D}^{j+1}$ solves problem (3.11a) with $\bar{v} = \bar{v}^j$ (point 2 of lemma 2.4) and, by (3.17a) and (3.12), $(\bar{D}^{j+1}, \bar{v}^j)$ solves the problems in (3.8). Now, by (3.21),

$$|(1 - \bar{D}_{kk}^{j+1})(A\bar{v}^j)_k + \bar{D}_{kk}^{j+1}(B\bar{v}^j)_k| < \alpha \quad \text{or} \quad |(C_{\bar{D}^{j+1}}\bar{v}^j)_k| < \alpha.$$

Then, lemma 2.3 with

$$A = C_{\bar{D}^{j+1}} \qquad \text{and} \qquad v = \bar{v}^j \in \mathcal{V}_\infty(C_{\bar{D}^{j+1}})$$

(the last membership comes from the fact that $(\bar{D}^{j+1}, \bar{v}^j)$ solves the problems in (3.8) and the implication "$\Leftarrow$" in (3.10)) tells us that one can find a

$$\bar{v}^{j+1} \in \mathcal{V}_\infty(C_{\bar{D}^{j+1}}) \text{ such that } (C_{\bar{D}^{j+1}}\bar{v}^{j+1})_k = 0.$$

The first membership implies (3.17a) with $j+1$ replacing j, by the implication "$\Rightarrow$" of (3.10) (note that $\bar{D}^{j+1}$ solves (3.4) by the implication "$\Leftarrow$" of (3.10)). The second identity reads (3.17c) with $j+1$ replacing j.

By the two previous points, the iterative procedure defining $(\bar{D}^j, \bar{v}^j)$, for $j \in \mathbb{N}$, is well defined, unless it is interrupted by the fact that $a_k^j = 0$ for some $j \in \mathbb{N}$, which is a desirable property since then $\bar{D}$ is solution to (3.4) with any $\bar{D}_{kk} \in [0, b_k]$.

If the procedure does not terminate, one has $\bar{D}_{kk}^j \to 0$ by (3.17d) and $\bar{D}$ is optimal for any $\bar{D}_{kk} \in [\bar{D}_{kk}^j, b_k]$. Since the set of solutions to problem (3.4) is closed, we get that $\bar{D}$ is optimal for any $\bar{D}_{kk} \in [0, b_k]$. □

4 Discussion

The simplification (1.8) of the error bound factor β given by (1.7) allows us to compute it by evaluating the map $D \in [0, I] \mapsto \|C_D^{-1}\|_\infty$ at the 2^n extreme points of $[0, I]$, which are the diagonal matrices D with diagonal entries in $\{0, 1\}$. This is an improvement. Nevertheless, for large n, this exponential number of evaluations can make this exact exhaustive computation approach very time consuming. Now, it is not unlikely that, for special classes of matrices, the simplification (1.8) can yield an efficient way of computing the error bound factor. Finally, we are also exploring the possibility to simplify this comprehensive evaluation by a specific algorithm based on the developments made in this paper.

Another interest of the simplified formula (1.8) of β deals with the complexity analysis of some algorithms for solving the balanced linear complementarity problem (1.2) with matrices A and B verifying (1.5) (equivalent to the **P**-matricity of M if $(A, B) = (I, M)$) and integer (or rational) data. When the complexity is expressed in terms of the data bitlength, which is the number of bits required to represent the problem data in the computer memory, and when the error bound (1.6) intervenes, the question may arise to know whether the upper error bound factor can be bounded above by a formula using the data bitlength or the bitlength of the matrix A and B, denoted $\mathfrak{L}(A, B)$ say, since the data bitlength is certainly larger than $\mathfrak{L}(A, B)$. It is known from [25; paragraph straddling pages 209-210] (probably also implicit in [19]), that, for an arbitrary nonsingular matrix $M \in \mathbb{R}^{n \times n}$,

$$\|M^{-1}\|_\infty \leqslant n\, 2^{\mathfrak{L}(M)+1}.$$

Thanks to the formula (1.8) of β, the upper error bound factor is equal to $\|C_{\bar{D}}^{-1}\|_\infty$, for some $\bar{D} \in \operatorname{ext}[0, I]$. Therefore, the rows of $C_{\bar{D}}$ defined by (1.4) are those of A or B. As a result, one certainly has

$$\mathfrak{L}(C_{\bar{D}}) \leqslant \mathfrak{L}(A, B). \tag{4.1}$$

As a result, with $\bar{D} \in \operatorname{ext}[0, I]$ solving the optimization problem in (1.8), one has

$$\max_{D \in [0,I]} \|C_D^{-1}\|_\infty = \|C_{\bar{D}}^{-1}\|_\infty \leqslant n\, 2^{\mathfrak{L}(C_{\bar{D}})+1} \leqslant n\, 2^{\mathfrak{L}(A,B)+1}.$$

Without (1.8), formula (4.1) could fail to hold and the upper bound of $\max\{\|C_D^{-1}\|_\infty : D \in [0, I]\}$ could be in terms of $\mathfrak{L}(C_{\bar{D}})$, which could be infinite since the optimal diagonal matrix $\bar{D}$ could then have irrational numbers in some entries. Therefore, thanks to (1.8), for the balanced linear complementarity problem (1.2), with matrices A and B verifying (1.5), one has the error bound

$$\forall\, x \in \mathbb{R}^n : \quad \|x - \bar{x}\|_\infty \leqslant n\, 2^{\mathfrak{L}(A,B)+1}\, \|\min(Ax + a, Bx + b)\|_\infty, \tag{4.2}$$

where $\bar{x}$ is the unique solution to the balanced LCP.

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