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ABSTRACT. This paper addresses the ubiquity of remarkable measures on graphs, and their applications. In many queueing systems, it is necessary to take into account the compatibility constraints between users, or between supplies and demands, and so on. The stability region of such systems can then be seen as a set of measures on graphs, where the measures under consideration represent the arrival flows to the various classes of users, supplies, demands, etc., and the graph represents the compatibilities between those classes. In this paper, we show that these 'stabilizing' measures can always be easily constructed as a simple function of a family of weights on the edges of the graph. Second, we show that the latter measures always coincide with invariant measures of random walks on the graph under consideration. Some arguments in the proofs rely on the so-called matching rates of specific stochastic matching models. As a by-product of these arguments we show that, in several cases, the matching rates are independent of the matching policy, that is, the rule for choosing a match between various compatible elements.

1. INTRODUCTION

Queueing models with compatibilities have recently gained an increasing interest, both from a theoretical and from an applied standpoint. In the original skill-based routing problems, customers and servers are bound by compatibility constraints, and it is the role of the routing algorithm to determine which customer should be matched to which server upon arrival. See the classical references [32], and then [33, 2, 20] or the recent survey [15].

However, in many applications, customers and servers may play symmetric roles, such as in peer-to-peer applications, job search, dating websites, housing programs, organ transplant allocations, supplies and demands, and so on. In all such applications, the system is basically just an interface used by the elements, to be matched, and to depart the system by pairs.

To account for this variety of applications, in [14] (see also [1]), a variant of skill-based systems was introduced, commonly referred to as 'Bipartite matching models' (BM): Couples (customer, server) enter the system at each time point (or are investigated one by one, from left to right in an infinite sequence), and servers and customers play symmetric roles: As soon as they find a match, they leave the system right away by couples (customer, server). A (bipartite) graph, representing the class of users, supplies, demands, etc., determines the compatibility between customers and servers. Each time a couple (customer, server) is formed, the couple leaves the system right away. The matching policy addressed in the seminal works [14, 1] is 'First Come, First Served' (FCFS). Interestingly, the stationary state in such cases is shown to enjoy a remarkable product form (see [3], and the subsequent performance analysis in [17]), that can be generalized to a wider class of systems with compatibilities, such as matching queues implementing the so-called FCFS-ALIS (Assign the Longest Idling Server) service discipline - see e.g. [2], [4] and [5] for an overview of various extensions of BM models or redundancy models. Let us also mention other approaches to such bipartite models, such as fluid and diffusion limits for systems with compatibility probabilities [11], applications to taxi hubs [10], or ride-sharing. The original settings of [14, 1] have then been generalized to the case of non-independent arrivals and other matching policies (*extended* bipartite matching (EBM) model - see [12, 28]).

To take into account applications such as assemble-to-order systems, dating websites, car-sharing and cross-kidney transplants, a variant of this model has been more recently introduced, in which arrivals are made one-by-one, and the compatibility graph is general, i.e. not necessarily bipartite: This is the so-called General stochastic matching model (GM), introduced in [26]. Reference [27] shows that the GM model under the First Come, First Matched matching policy also enjoys a product form in steady state. Then, the GM model was studied along various angles, among which, fluid limits of a continuous-time variant [29], optimization [30] or optimal control [22, 13], matching models on hypergraphs [31], on graphs with self-loops [8], or models with reneging, see [9, 24, 6]. Recently, GM models have been shown to share remarkable similarities with order-independent loss queues, see [16].

For all BM, EBM and GM models studied in the existing literature, the stability region is expressed as a set of measures on nodes representing the arrival rates to each class of users, such that the system is positive recurrent. (These measures are probability measures for discrete-time models, and arrival intensities for continuous-time models.) As will be specified below, the conditions on these measures typically take the form of a constraint on the arrival rates to all independent sets of nodes, to be less than the arrival rates to the subset of the neighboring nodes - a condition that is reminiscent of the Hall condition for the existence of a perfect matching on bipartite a graph, see [23]. This condition was shown to be necessary for the stability of EBM (respectively, GM) models, and sufficient for EBM (resp., GM) models under the 'Match the Longest' (ML) matching policy (see respectively [12] and [26]). This sufficiency result has then been extended to all policies of the 'Max-weight' type (generalizing the ML policy - see a precise definition below), for GM models with possible reneging in [24]. The above condition is also sufficient for the stability of designated GM models implementing priority policies, see [26]. However, [29] shows that, for a wide class of compatibility graphs, there always exist priority and random policies such that the latter condition is *not* sufficient for stability. In [3] (respectively, [27]), it is also proven that the same condition is sufficient for the stability of BM (resp., GM) models under the 'First Come, First Matched' (FCFM) matching policy, and that under this condition the stationary distribution can be expressed in a product form (but the sufficiency for EBM models remains an open problem). Similar results are obtained for matching queues under the FCFS-ALIS discipline and for redundancy models, see e.g. [2], [4] and the recent survey [5].

In all these models, the stability regions are thus expressed as sets of measures on the set of nodes of the compatibility graph. However, the aforementioned references do not provide general explicit construction procedures of such sets, that is, of the arrival rates that stabilize the corresponding system. This is a crucial problem, for instance, for designing access control procedures into these systems.

In this work, we characterize exactly these sets of measures. While in such systems, the typical procedure is to construct an optimal control (a Markovian matching policy, for instance) that is able to optimize a given criterion *given the arrival rates* (that are thus seen as a constraint to the problem), hereafter we somehow reverse this procedure: Having fixed a control that is able to achieve stability (along the cases, a 'Max-weight' policy, or the 'First Come First Matched' policy, for instance), we make explicit the construction procedure of a set of arrival rates rendering the system stable. This aim is natural for practical purposes: The explicit construction of a set of arrival rates rendering the considered system stable, provides the feasible admission controls for a given graph topology, and a given Markovian control of the system.

We obtain two remarkable results: (i) We show that, in the wide range of systems ruled by compatibility constraints that are introduced in Section 3, the 'stabilizing' arrival rates of the considered system can *always* be constructed as a simple weighted measure on the edges of the graph, and (ii) we deduce from this, that a stabilizing set of arrival rates always coincides with an invariant measure of a random walk on the graph. As a by-product of (i), for some graph topologies we are able to determine uniquely the matching rates of the various edges, defined as the frequency of matchings executed on the various edges over time, *independently* of the matching policy.

Beyond its practical interest for admission control, this ubiquity of the stabilizing measures of matching models establishes an insightful connection between stochastic matching models and random walks on graphs which, we believe, opens the way for an interesting avenue of research.

This paper is organized as follows. After some preliminaries in Section 2, in Section 3 we present the stability regions of a wide class of service systems with compatibilities, including various stochastic matching models and skill-based queues. Then, in Section 4 we present our main results, establishing various representations of these stability regions in terms of sets of *weighted measures* - a notion that will be properly defined below. Proofs for our two main results are provided respectively in Sections 5 and 8. In Sections 6 and 7 we study the matching rates of general and bipartite stochastic matching models, and connect these objects with the previous discussion. Last, in Section 9 we show how our main results can be applied, to show the insensitivity of the matching rates to the matching policy of the considered matching model.

2. Preliminaries

2.1. **General notation.** We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}_+$ the set of non-negative real numbers and by $\mathbb{R}_+^*$ the subset of positive real numbers. Likewise, we denote by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{N}_*$ the subset of positive integers. For a and b in $\mathbb{N}$, we denote by $[\![a;b]\!]$ the set $[a;b]\cap\mathbb{N}$. For any finite set A, we denote by $|A|$ the cardinality of A and by $\mathcal{P}(A)$ the power set of A, namely, the set of all the subsets of A. For any set E, we denote by $\mathbf{1}_E$ the indicator function of E.

The set of finite positive measures having full support on A is denoted by $\mathcal{M}(A)$, whereas the set of probability measures having full support on A is denoted by $\overline{\mathcal{M}}(A)$. For any $\mu \in \mathcal{M}(A)$, we denote by $\bar{\mu} \in \overline{\mathcal{M}}(A)$, its normalized counterpart, namely $\bar{\mu}(.) = \frac{\mu(.)}{\mu(A)}$. Let us also denote by $\mathcal{M}^{\geq 0}(A)$, the set of non-negative measures on A having support included in A.

For a real matrix M, we let ${}^tM$ denote the transpose of M.

2.2. **Multigraphs.** Hereafter, a *multigraph* is given by a couple $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the (finite) set of nodes and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of edges. All multigraphs considered hereafter are undirected, that is, $(u,v) \in \mathcal{E} \implies (v,u) \in \mathcal{E}$, for all $u,v \in \mathcal{V}$. We write $u{-}v$ or $v{-}u$ for $(u,v)\in\mathcal{E}$, and $u\not{-}v$ (or $v\not{-}u$) else. Elements of the form $(v,v)\in\mathcal{E}$, for $v\in\mathcal{V}$, are called *self-loops*. A multigraph having no self-loop is simply a graph. With respect to the classical notion of multigraphs, we assume hereafter that all the edges are simple.

For any multigraph $G = (\mathcal{V}, \mathcal{E})$ and any $U \subset \mathcal{V}$, we denote

$$\mathcal{E}(U) \triangleq \{v \in \mathcal{V} : \exists u \in U,\ u - v\}$$

the neighborhood of U, and for $u \in \mathcal{V}$, we write for short $\mathcal{E}(u) \triangleq \mathcal{E}(\{u\})$.

An *independent set* of G is a non-empty subset $\mathcal{I} \subset \mathcal{V}$ which does not include any pair of neighbors, i.e.: $\forall (i,j) \in \mathcal{I}^2,\ i \not- j$. We let $\mathbb{I}(G)$ be the set of independent sets of G. For any subset $\mathcal{V}_1$ of $\mathcal{V}$, we denote

$$\mathbb{I}(\mathcal{V}_1) = \{\mathcal{I} \cap \mathcal{V}_1 \,:\, \mathcal{I} \in \mathbb{I}(G)\}\,. \tag{1}$$

A graph $G = (\mathcal{V}, \mathcal{E})$ is said *bipartite* if the set of its vertices $\mathcal{V}$ can be partitioned into two independent sets $\mathcal{V}_1$ and $\mathcal{V}_2$, namely, each edge connects an element of $\mathcal{V}_1$ to an element of $\mathcal{V}_2$.

Throughout this paper, all considered (multi)graphs have at least two nodes and are connected, that is, for any $u, v \in \mathcal{V}$, there exists a subset $\{v_0 \triangleq u, v_1, v_2, \dots, v_p \triangleq v\} \subset \mathcal{V}$ such that $v_i - v_{i+1}$, for any $i \in [\![0; p-1]\!]$.

2.3. **Weighted measures on a multigraph.** For any multigraph $G = (\mathcal{V}, \mathcal{E})$, we say that a family $\alpha = (\alpha_{i,j})_{(i,j)\in\mathcal{E}}$ of real numbers is a family of *weights* on $\mathcal{E}$ if for all $(i,j) \in \mathcal{E},\ \alpha_{i,j} = \alpha_{j,i} > 0$. Hereafter, a family of weights α on $\mathcal{E}$ will be seen as an element of the set $\mathcal{M}(\mathcal{E})$, and for any such α and any edge $e = (i,j) \in \mathcal{E}$, we will write indifferently α_e, $\alpha_{i,j}$ or $\alpha_{j,i}$.

For any family of weights $\alpha \in \mathcal{M}(\mathcal{E})$, we define the associated positive measure on nodes $\mu^\alpha \in \mathcal{M}(\mathcal{V})$, by

$$\mu^\alpha(i) \triangleq \sum_{j\in\mathcal{E}(i)} \alpha_{i,j}, \quad i \in \mathcal{V}, \tag{2}$$

and the associated probability distribution $\bar{\mu}^\alpha \triangleq \overline{\mu^\alpha} \in \overline{\mathcal{M}}(\mathcal{V})$, by

$$\bar{\mu}^\alpha(i) = \frac{\sum\limits_{j\in\mathcal{E}(i)} \alpha_{i,j}}{\sum\limits_{l\in\mathcal{V}} \sum\limits_{j\in\mathcal{E}(l)} \alpha_{l,j}}, \quad i \in \mathcal{V}. \tag{3}$$

The measures μ^α and $\bar{\mu}^\alpha$, $\alpha \in \mathcal{M}(\mathcal{E})$, will be called hereafter *weighted measures,* and we let

$$\mathcal{W}(G) \triangleq \{\mu^\alpha : \alpha \in \mathcal{M}(\mathcal{E})\}\,; \tag{4}$$

$$\overline{\mathcal{W}}(G) \triangleq \{\bar{\mu}^\alpha : \alpha \in \mathcal{M}(\mathcal{E})\}\,, \tag{5}$$

be respectively the set of weighted measures on the multigraph G, and the sets of their associated probability measures.

3. Stability regions of various stochastic models on (multi-)graphs

In this section we introduce several stochastic systems studied in the recent literature, and defined on a multi-graph. These are service systems that all depend on compatibilities constraints between items (users, requests, customers, supply, demands, etc.), that are precisely represented by a multi-graph. As will be recalled below, the stability regions of these systems, defined as the sets of arrival rates of items such that the considered systems reach a steady state, all have a similar form. The main aim of this paper is to characterize these sets, and to construct explicitly the corresponding arrival rates, so as to achieve stability.

3.1. **General stochastic matching model.** We first consider the matching model introduced in [26] (and generalized in [8] for multigraphs): Take a connected multigraph $G = (\mathcal{V}, \mathcal{E})$, where elements of $\mathcal{V}$ represent classes of items. We say that two items of respective classes i and $j \in \mathcal{V}$ are *compatible* if $(i,j) \in \mathcal{E}$, that is, if i and j are neighbors in G. Items enter the system one by one in discrete-time, and, for all $n \in \mathbb{N}_*$, we denote by $V_n \in \mathcal{V}$ the class of the item entering the system at time n. The sequence $\{V_n\}_{n\in\mathbb{N}_*}$ is supposed to be IID, of common distribution $\bar{\mu} \in \overline{\mathcal{M}}(\mathcal{V})$.

In other words, at any time point, for any $i \in \mathcal{V}$, $\bar{\mu}(i)$ is the probability that the incoming item is of class i.

Then, upon the arrival of an element, say, of class i, we investigate whether there is in line a compatible element with the incoming item (and possibly an item of the same class, if the corresponding node has a self-loop). If this is the case, then the incoming i-item is matched with one of these elements (if there are more than one), following a fixed criterion called *matching policy*, and the two matched elements leave the system right away. The matching policy is denoted by Φ, and depends only on the arrival dates of the stored items, their classes, and possibly on a random draw that is independent of everything else - we then say that the policy is *admissible*. For instance:

- For the *First Come, First Matched* policy $\Phi = \textsc{fcfm}$, the oldest compatible item in line is chosen as the match of i.
- In a Max-weight policy $\Phi = \textsc{mw}$, the i-item choses its match among the stored items of class j, for

$$j = \text{Argmax}\left\{X(k) + w_{ki} \,:\, k \in \mathcal{E}(i)\right\},$$

  where we denote for all k as above, by $X(k)$ the number of k-items in line upon the arrival of the i-item, and where the weights w_{ki}, $(k,i) \in \mathcal{E}$, are fixed non-negative real numbers, and possible ties are broken by a uniform draw that is independent of everything else. As a particular case, if $w_{ki} = 0$ for all (k,i), the policy is $\Phi = \textsc{ml}$ for *Match the Longest*, that is, j is simply chosen as a member of the class having the largest amount of stored items in line.

See [26, 8] for formal definitions, and other examples of admissible matching policies. The system is then fully characterized by the triple $(G, \Phi, \bar{\mu})$.

For any fixed connected multigraph G, any fixed admissible matching policy Φ and any fixed probability measure $\bar{\mu}$, the system can be represented by the Markov chain $\left\{W_n^{\Phi,\bar{\mu}}\right\}_{n\in\mathbb{N}}$, valued in the space $\mathbb{W}$ of words on the alphabet $\mathcal{V}$, where for any n and any word $w = w_1 w_2 \cdots w_p$ of length p in $\mathbb{W}$, we set $W_n^{\Phi,\bar{\mu}} = w$ if, just before time n, the oldest item in line is of class w_1, the second oldest is of class w_2, and so on, and w_p is the class of the most recent item in line (again, see [8]).

The primary question of stability for such models is thus translated in the following terms: For a given compatibility graph and a given control of the matches, represented by the matching policy Φ, what is the set of measures rendering the latter Markov chain positive recurrent? For any connected multigraph $G = (\mathcal{V}, \mathcal{E})$ and any admissible matching policy Φ, we thus define the *stability region* associated to G and Φ as the set of measures

$$\overline{\textsc{stab}}(G,\Phi) \triangleq \left\{\bar{\mu} \in \overline{\mathcal{M}}\left(\mathcal{V}\right) \,:\, \left\{W_n^{\Phi,\bar{\mu}}\right\}_{n\in\mathbb{N}} \text{ is positive recurrent}\right\},$$

and one aims at a precise characterization of the set $\overline{\textsc{stab}}(G,\Phi)$. Following results established for (simple) graphs in [26], it was shown, first, in Proposition 4.3 of [8], that for any admissible matching policy Φ,

$$\overline{\textsc{stab}}(G,\Phi) \subset \overline{\mathcal{N}}(G),$$

where the set of probability measures $\overline{\mathcal{N}}(G)$ is defined by

$$\overline{\mathcal{N}}(G) \triangleq \left\{\bar{\mu} \in \overline{\mathcal{M}}\left(\mathcal{V}\right) : \forall \mathcal{I} \in \mathbb{I}(G),\ \bar{\mu}\left(\mathcal{I}\right) < \bar{\mu}\left(\mathcal{E}\left(\mathcal{I}\right)\right)\right\}, \tag{6}$$

and moreover, that the two sets *coincide*, for $\Phi = \textsc{fcfm}$ (Theorem 1 in [27] for graphs and Theorem 4.5 in [8] for multigraphs), and $\Phi = \textsc{mw}$ (Theorem 4.7 in [8]).

3.2. **Continuous-time matching model.** A continuous-time matching model associated to the triple (G,Φ,μ), for $\mu \in \mathcal{M}(\mathcal{V})$, is defined exactly as its discrete-time analogue, except that arrivals to each node i are given by a Poisson process of intensity $\mu(i)$, independently of everything else - see e.g. [29] for details. Then, Section 2.3 in [29] shows that for any matching policy Φ, the continuous-time model associated to (G,Φ,μ) is stable if and only if the discrete-time model associated to (G,Φ,μ) is itself stable. In other words, a measure $\mu \in \mathcal{M}(\mathcal{V})$ belongs to the stability region $\textsc{stab}(G,\Phi)$ of the continuous-time model if the normalized probability measure $\bar\mu$ is an element of $\overline{\textsc{stab}}(G,\Phi)$, and $\bar\mu \in \overline{\textsc{stab}}(G,\Phi)$ if any measure $\mu = K\bar\mu$, for $K>0$ is an element of $\textsc{stab}(G,\Phi)$. Then, the uniformization arguments showing this statement remain valid whenever G is a multigraph. So the results of Section 3.1 remain valid in this context, replacing $\overline{\mathcal{N}}(G)$ by $\mathcal{N}(G)$, where

$$\mathcal{N}(G) \triangleq \left\{\mu \in \mathcal{M}\left(\mathcal{V}\right) : \forall \mathcal{I} \in \mathbb{I}(G),\ \mu\left(\mathcal{I}\right) < \mu\left(\mathcal{E}\left(\mathcal{I}\right)\right)\right\}. \tag{7}$$

3.3. **(Extended) Bipartite stochastic matching models.** In the so-called *extended bipartite matching model*, as introduced in [12], we are given a connected bipartite graph $G=(\mathcal{V}_1\cup\mathcal{V}_2,\mathcal{E})$. As above, elements of $\mathcal{V}_1$ and $\mathcal{V}_2$ represent classes of items, and we say that two items $i\in\mathcal{V}_1$ and $j\in\mathcal{V}_2$ are compatible if there is an edge between i and j. In an extended bipartite matching model, exactly one element of class in $\mathcal{V}_1$, and one element of class in $\mathcal{V}_2$, enter the system at each time point, in other words the arrivals are done *two-by-two.* The sequence $\{V_n\}_{n\in\mathbb{N}_*}$ represents the couples of classes of incoming items; namely, for any $n\in\mathbb{N}_*$, $V_n=(i,j)$ for $i\in\mathcal{V}_1$ and $j\in\mathcal{V}_2$, means that an element of class i and an element of class j enter the system together at time n. The sequence $\{V_n\}_{n\in\mathbb{N}_*}$ is supposed to be IID, of common distribution $\tilde\mu \in \overline{\mathcal{M}}(\mathcal{V}_1\times\mathcal{V}_2)$, with marginals $\tilde\mu_1$ and $\tilde\mu_2$ respectively. (With some abuse, we then denote $\tilde\mu=(\tilde\mu_1,\tilde\mu_2)$.) Then, upon the arrival of each couple, say of a couple (i,j), we first investigate whether there is in line an element of class in $\mathcal{V}_1$ that is compatible with the incoming j-item. If this is the case, then the incoming j-item is matched with one of these elements (if there are more than one), that is chosen following a given matching policy Φ, and the two elements leave the system right away. Then we apply the exact same procedure to the incoming i-item. If one of the two incoming items did not find a match while the other did, then the element is stored in the buffer. If none of the two did find a match in the buffer, then, either the two incoming items are compatible and then they are matched and leave the system right away, or they are both stored in the buffer. Again, the matching policy Φ is supposed to be admissible, i.e. to depend only on the arrival dates of the stored items, their classes, and possibly on a random draw independent of everything else. For instance, for $\Phi=\textsc{fcfm}$, the oldest compatible item in line is chosen or, for $\Phi=\textsc{ml}$ ('Match the Longest'), an item of the compatible class having the largest number of elements in line is chosen, ties being broken uniformly at random. See again [12] for formal definitions, and other examples of matching policies. Altogether, the system is again fully characterized by the triple $(G,\Phi,\tilde\mu)$.

For any fixed connected bipartite graph $G=(\mathcal{V}_1\cup\mathcal{V}_2,\mathcal{E})$, any fixed admissible matching policy Φ and any fixed probability measure $\tilde\mu$, the bipartite matching model associated to $(G,\Phi,\tilde\mu)$ can be fully represented by the Markov chain $\left\{Y_n^{\Phi,\tilde\mu}\right\}_{n\in\mathbb{N}}$ valued in the space $\mathbb{Y}$ gathering all couples of the form (u,v), where u and v are two words, respectively over the alphabets $\mathcal{V}_1$ and $\mathcal{V}_2$. For any n and any such couple (u,v), we denote $Y_n^{\Phi,\tilde\mu}=(u,v)$ if, just before time n, the oldest item in line having class in $\mathcal{V}_1$ is of class u_1, the second oldest having class in $\mathcal{V}_1$ is of class u_2, and so on, and the oldest item in line having class in $\mathcal{V}_2$ is of class v_1, the second oldest having class in $\mathcal{V}_2$ is of class v_2, and so on. See again [12].

Similarly to the discussion above, the stability problem amounts to determining, for any $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ and any admissible Φ, the stability region associated to G, defined as

$$\overline{\text{STAB}}_{\text{EB}}(G,\Phi) \triangleq \left\{\tilde{\mu} \in \overline{\mathcal{M}}\left(\mathcal{V}_1 \times \mathcal{V}_2\right) : \left\{Y_n^{\Phi,\tilde{\mu}}\right\}_{n\in\mathbb{N}} \text{ is positive recurrent}\right\}.$$

It was shown in Lemma 3.2 of [12] that for all Φ, $\overline{\text{STAB}}_{\text{EB}}(G,\Phi)$ is included in the set

$$(8)\quad \overline{\mathcal{N}}_{\text{EB}}(G) \triangleq \left\{\tilde{\mu} = (\tilde{\mu}_1,\tilde{\mu}_2) \in \overline{\mathcal{M}}\left(\mathcal{V}_1 \times \mathcal{V}_2\right) : \left\{\begin{array}{l} \forall \mathcal{I}_1 \in \mathbb{I}(\mathcal{V}_1)\backslash\{\mathcal{V}_1\},\ \tilde{\mu}_1\left(\mathcal{I}_1\right) < \tilde{\mu}_2\left(\mathcal{E}\left(\mathcal{I}_1\right)\right) \\ \forall \mathcal{I}_2 \in \mathbb{I}(\mathcal{V}_2)\backslash\{\mathcal{V}_2\},\ \tilde{\mu}_2\left(\mathcal{I}_2\right) < \tilde{\mu}_1\left(\mathcal{E}\left(\mathcal{I}_2\right)\right) \end{array}\right\}\right\}.$$

Moreover, it can be shown that the two sets $\overline{\text{STAB}}_{\text{EB}}(G,\Phi)$ and $\overline{\mathcal{N}}_{\text{EB}}(G)$ actually coincide for $\Phi = \text{ML}$ (see Theorem 7.1. in [12]).

The *bipartite stochastic matching model* introduced in [14], is defined similarly to the extended bipartite model, except that the measure $\tilde{\mu}$ now satisfies $\tilde{\mu} = \tilde{\mu}_1 \otimes \tilde{\mu}_2$, that is, the classes of the two elements of each entering couple are independent. Similarly to the above discussion, we define the corresponding stability region

$$\overline{\text{STAB}}_{\text{B}}(G,\Phi) \triangleq \left\{\tilde{\mu} = \tilde{\mu}_1 \otimes \tilde{\mu}_2 \in \overline{\mathcal{M}}\left(\mathcal{V}_1 \times \mathcal{V}_2\right) : \left\{Y_n^{\Phi,\tilde{\mu}}\right\}_{n\in\mathbb{N}} \text{ is positive recurrent}\right\}.$$

Then, it is shown in [1] that the set $\overline{\text{STAB}}_{\text{B}}(G,\Phi)$ coincides with the set

$$(9)\quad \overline{\mathcal{N}}_{\text{B}}(G) \triangleq \left\{\tilde{\mu} = \tilde{\mu}_1 \otimes \tilde{\mu}_2 \in \overline{\mathcal{M}}\left(\mathcal{V}_1 \times \mathcal{V}_2\right) : \left\{\begin{array}{l} \forall \mathcal{I}_1 \in \mathbb{I}(\mathcal{V}_1)\backslash\{\mathcal{V}_1\},\ \tilde{\mu}_1\left(\mathcal{I}_1\right) < \tilde{\mu}_2\left(\mathcal{E}\left(\mathcal{I}_1\right)\right) \\ \forall \mathcal{I}_2 \in \mathbb{I}(\mathcal{V}_2)\backslash\{\mathcal{V}_2\},\ \tilde{\mu}_2\left(\mathcal{I}_2\right) < \tilde{\mu}_1\left(\mathcal{E}\left(\mathcal{I}_2\right)\right) \end{array}\right\}\right\},$$

and Theorem 1 in [3] shows that this condition is also necessary and sufficient for the existence of a unique matching of two bi-infinite streams of elements of $\mathcal{V}_1$ and of $\mathcal{V}_2$.

3.4. **General matching model with reneging.** We now consider a continuous-time general stochastic matching model with reneging, as defined in [24]: The model is exactly the one in Section 3.2, except that $G = (\mathcal{V},\mathcal{E})$ is a graph, and for a given subset $\mathcal{V}_1$ of $\mathcal{V}$, elements of $\mathcal{V} \setminus \mathcal{V}_1$ have a random and integrable finite lifetime in the system, after which they are immediately discarded, if they did not find a match before that. This assumption naturally accounts for various applications of matching models in which the elements abandon, if they waited too long before finding a match: This abandonment may represent the death of patients waiting for a transplant, the expiration of organs that were not transplanted quickly enough, the balking of candidates looking for a job, and so on. Recalling (1), it is shown in Theorem 1 of [24] that for all G and Φ, the stability region of this model is included in the set of measures

$$(10)\qquad \mathcal{N}_{\mathcal{V}_1}(G) \triangleq \left\{\mu \in \mathcal{M}\left(\mathcal{V}\right) : \forall \mathcal{I} \in \mathbb{I}(\mathcal{V}_1),\ \mu\left(\mathcal{I}\right) < \mu\left(\mathcal{E}\left(\mathcal{I}\right)\right)\right\},$$

and that the two sets coincide for $\Phi = \text{MW}$. The region $\mathcal{N}_{\mathcal{V}_1}(G)$ is thus a superset of $\mathcal{N}(G)$, defined as a restriction of the condition in (7) to the classes of 'non-reneging' nodes. Using uniformization as above, one can easily show that the same results hold for the corresponding discrete-time matching model with reneging, replacing $\mathcal{N}_{\mathcal{V}_1}(G)$ by the set of probability measures

$$(11)\qquad \overline{\mathcal{N}}_{\mathcal{V}_1}(G) \triangleq \left\{\bar{\mu} \in \overline{\mathcal{M}}\left(\mathcal{V}\right) : \forall \mathcal{I} \in \mathbb{I}(\mathcal{V}_1),\ \bar{\mu}\left(\mathcal{I}\right) < \bar{\mu}\left(\mathcal{E}\left(\mathcal{I}\right)\right)\right\}.$$

3.5. **Skill-based queues.** Various manufacturing systems are modeled by the so-called *multi-type jobs and multi-type servers*, as introduced in [33]: Consider a bipartite graph $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$. There are $|\mathcal{V}_1|$ independent Poisson processes of respective intensities $\mu(i)$, $i \in \mathcal{V}_1$, and the i-th process represents the arrival of jobs of type i. The jobs request i.i.d. service times of exponential $\varepsilon(1)$ distribution. There are $|\mathcal{V}_2|$ servers, and server $j \in \mathcal{V}_2$ can only serve jobs of types in $\mathcal{E}(j)$, at rate $\gamma(j)$ (whatever the class of the job in $\mathcal{E}(j)$). Similarly, jobs of type i can be served only by a server of a class in $\mathcal{E}(i)$. Customers are put in line in a single queue, and are chosen (by compatible servers) in a First Come, First Served order. If a given job finds several idle servers in $\mathcal{E}(i)$, it choses a server amongst them, at random. We view $\mu = (\mu(1), ..., \mu(|\mathcal{V}_1|))$ and $\gamma = (\gamma(1), ..., \gamma(|\mathcal{V}_2|))$ as measures of $\mathcal{M}(\mathcal{V}_1)$ and $\mathcal{M}(\mathcal{V}_2)$, respectively. It is then shown in Theorem 2 of [33] that the model can be stable only if μ belongs to the set

$$\mathcal{N}_{\mathcal{V}_1,\gamma}(G) \triangleq \{\mu \in \mathcal{M}(\mathcal{V}_1) : \forall \mathcal{I}_1 \in \mathbb{I}(\mathcal{V}_1)\ \mu(\mathcal{I}_1) < \gamma(\mathcal{E}(\mathcal{I}_1))\}, \tag{12}$$

and that this condition is also sufficient if the random assignment of jobs follows a given *assignment condition*, see section 3.6 of [33]. The set $\mathcal{N}_{\mathcal{V}_1,\gamma}(G)$ can thus be seen as the subset of $\mathcal{N}_{\mathcal{V}_1}(G)$ (for a bipartite G) containing only measures whose restriction to $\mathcal{V}_2$ is γ. In [2], the stability region is also shown to coincide with $\mathcal{N}_{\mathcal{V}_1,\gamma}(G)$ in the case of a queueing system having the same structure, and ruled by the so-called FCFS-ALIS policy, namely, customers are queued in a single line and chosen by compatible server in First Come, First Served order, but customers finding several idle servers upon arrival are assigned to the server that has been idling for the longest period of time (*Assign Longest Idle Server*). Surprisingly, it is shown that both models then have the same stationary distribution, expressed in a product-form.

The so-called *multi-class* and *multi-pool* queue (see e.g. the survey [15]) is a generalization of this class of models, in which:

- For all $j \in \mathcal{V}_2$, there are $s_j \in \mathbb{N}_*$ indistinguishable servers of class j;
- Whenever they are served by a server of class $j \in \mathcal{V}_2$, the service times of customers of class $i \in \mathcal{V}_1$ are exponentially distributed with a parameter $\gamma((i,j)) > 0$ that might depend on both i and j.

It is shown in [32] that for any measure $\gamma \in \mathcal{M}(\mathcal{E})$ and any $(s_j)_{j\in\mathcal{V}_2} \in (\mathbb{N}_*)^{\mathcal{V}_2}$, the stability region of the model is included in the set of measures

$$\mathcal{N}_{\mathcal{V}_1,\gamma,s}(G) \triangleq \left\{ \mu \in \mathcal{M}(\mathcal{V}_1) : \exists \pi \in \mathcal{M}(\mathcal{E}) \text{ s.t.} \begin{array}{l} -\ \forall i \in \mathcal{V}_1,\ \mu(i) = \sum_{j\in\mathcal{V}_2} \gamma((i,j)) s_j \pi((i,j)) \\ -\ \forall j \in \mathcal{V}_2,\ \sum_{i\in\mathcal{V}_1} \pi((i,j)) < 1 \end{array} \right\}, \tag{13}$$

within a set of admission controls (deciding the classes of servers the incoming customers are assigned to) and service policies (determining the classes of customers the available servers choose to serve) termed SBR (for *Skill-based routing*) policies. Moreover, it is also shown in [32] that the stability regions is precisely $\mathcal{N}_{\mathcal{V}_1,\gamma,s}(G)$ whenever the so-called *Maximum Pressure Policy* MPP is implemented - such a policy is then termed *throughput optimal.*

3.6. **Redundancy models.** In [21], a skill-based service system similar to the one introduced in Section 3.5 is considered, in which *redundancy* is allowed: Each server has its own queue, and jobs may be replicated in several copies, in the queues of several compatible servers, and served contemporarily by several servers. It is then shown in Theorem 1 of [21] that this system again has the same stability region $\mathcal{N}_{\mathcal{V}_1,\gamma}(G)$ (and then the same product form stationary distribution) as the one in Section 3.5. We refer to [20] and [5], for a survey of an even wider class of skill-based queueing systems having the same stability region $\mathcal{N}_{\mathcal{V}_1,\gamma}(G)$. This class also includes e.g. the so called order independent (loss) queues (see [7, 25], and insightful connections with matching systems in [16]).

3.7. **Towards exact characterizations of the stability regions.** In most of the stochastic matching and queueing models presented above, the stability regions are included in, and sometimes coincides with, sets of measures that roughly have the same form, and are reminiscent of Hall's condition. They are respectively given by (6), (7), (8), (9), (10), (11) and (12). In all cases, it must then be checked that for any independent set (possibly in a given subset of nodes), the measure of the set must be less than the measure of the set of its neighboring nodes. Guaranteeing stability for given arrival rates, and a given matching policy, amounts to checking that this condition holds for all independent sets, a task that becomes more and more cumbersome as the size of the graph G grows large. Specifically, it is indicated in Proposition 1 in [26] (resp., Proposition 3.5 in [12]), that checking that a given measure μ belongs to $\overline{\mathcal{N}}(G)$ defined by (6) (resp., to $\overline{\mathcal{N}}_{\text{EB}}(G)$ defined by (8)) can be done in the order of $O(|\mathcal{V}|^3)$ operations. Hence the need to precisely characterize, and possibly to explicity *construct* these measures. Corollaries 1 and 2 below, show that these sets *coincide* with sets of weighted measures on the considered multi-graphs. In other words, *any* measure that might stabilize the considered system can be obtained by just assigning weights to the edges of G - and any such procedure leads to a 'stabilizing' measure.

4. Results

In this section we present our main results, which characterize the stability regions of the systems presented in Section 3. As we will show in Section 4.2, these regions can be identified with sets of weighted measures. But, as we first demonstrate in Section 4.1, the latter sets coincide in turn with sets of invariant measures of weighted random walks on multi-graphs. Aside from their interest for the admission control of the systems introduced in Section 3, these results establish a deep and insightful connection between these two, apparently disjoint, classes of models.

4.1. **Weighted random walk on a multi-graph.** Given a multigraph $G = (\mathcal{V},\mathcal{E})$, a *random walk* on G is a discrete time Markov chain on the vertices of G such that for $i, j \in \mathcal{V}$, a transition is allowed from i to j only if there is an edge relying i and j. Formally, we say that a stochastic matrix $P = (P(i,j))_{(i,j)\in\mathcal{V}^2}$ defines a *random walk* on the edges of the multigraph $G = (\mathcal{V},\mathcal{E})$ if $P(i,j) = 0$ as soon as $(i,j) \notin \mathcal{E}$. We say furthermore that the random walk is *reversible* if the associated Markov chain admits a reversible invariant measure. Recalling (4), we easily obtain the following result.

Proposition 1. *Let $G = (\mathcal{V},\mathcal{E})$ be a connected multigraph, and $\mu \in \mathcal{M}(\mathcal{V})$. Then, μ is invariant for a reversible random walk on G if, and only if $\mu \in \mathcal{W}(G)$, that is, μ is a weighted measure. In that case, $\bar{\mu}$ is the unique stationary distribution of the chain.*

Proof. Let $\alpha \in \mathcal{M}(\mathcal{E})$. We define the corresponding *weighted random walk*, as the Markov chain on $\mathcal{V}$ of transition P^α, defined by

$$P^\alpha(i,j) = \frac{\alpha_{i,j}}{\sum_{l\in\mathcal{E}(i)} \alpha_{i,l}}, \quad i \in \mathcal{V}, \quad j \in \mathcal{E}(i).$$

We show that the measure μ^α is a reversible invariant measure for that weighted random walk. Let us indeed consider two vertices $i,j \in \mathcal{V}$. If $(i,j) \notin \mathcal{E}$, it is immediate that $\mu^\alpha(i)P^\alpha(i,j) = 0 = \mu^\alpha(j)P^\alpha(j,i)$. And if $(i,j) \in \mathcal{E}$, using $\alpha_{i,j} = \alpha_{j,i}$, we have

$$\begin{aligned}\mu^\alpha(i)P^\alpha(i,j) &= \Big(\sum_{l\in\mathcal{E}(i)} \alpha_{i,l}\Big)\, \frac{\alpha_{i,j}}{\sum_{l\in\mathcal{E}(i)}\alpha_{i,l}}\\ &= \Big(\sum_{l\in\mathcal{E}(j)} \alpha_{j,l}\Big)\, \frac{\alpha_{j,i}}{\sum_{l\in\mathcal{E}(j)}\alpha_{j,l}} = \mu^\alpha(j)P^\alpha(j,i).\end{aligned}$$

Conversely, if μ is a reversible invariant measure for the random walk of matrix P, then we can set

$$\alpha_{i,j} = \mu(i)P(i,j) = \mu(j)P(j,i)$$

and check that μ is the weighted measure associated to the family of weights $(\alpha_{i,j})_{(i,j)\in\mathcal{E}}$. In such case, as the Markov chain is clearly irreducible on $\mathcal{V}$, $\bar{\mu}^\alpha$ is the unique stationary probability of the random walk. □

4.2. **Main results.** Our main result is the following,

Theorem 1. *Let $G = (\mathcal{V},\mathcal{E})$ be a connected multigraph. Then, the set of weighted measures $\mathcal{W}(G)$ defined by* (4) *coincides with:*

- *The set $\mathcal{N}(G)$ defined by* (7)*, if G is not a bipartite graph;*
- *The set*

$$\mathcal{N}_2(G) \triangleq \left\{\mu \in \mathcal{M}(\mathcal{V}) : \left\{\begin{array}{l} \forall \mathcal{I} \in \mathbb{I}(G)\setminus\{\mathcal{V}_1,\mathcal{V}_2\},\ \mu(\mathcal{I}) < \mu(\mathcal{E}(\mathcal{I})) \\ \mu(\mathcal{V}_1) = \mu(\mathcal{V}_2)\end{array}\right.\right\}, \tag{14}$$

if G is a bipartite graph of bipartition $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$.

Theorem 1 is proven in Section 5. In the statement above, the set $\mathcal{N}_2(G)$ appears as a counterpart of $\mathcal{N}(G)$, in the case where G is a bipartite graph. As a matter of fact, the set $\mathcal{N}(G)$ is empty whenever G is bipartite, of bipartition $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$. Indeed, for any μ in the latter set, we would have $\mu(\mathcal{V}_1) = \mu(\mathcal{E}(\mathcal{V}_2)) > \mu(\mathcal{V}_2)$, and symmetrically $\mu(\mathcal{V}_2) > \mu(\mathcal{V}_1)$. Specifically, from Proposition 4.2 in [8] we obtain that the set $\mathcal{N}(G)$ is non-empty if and only if G is not a bipartite graph.

Theorem 1 has the following primary consequence,

Corollary 1. *Let $G = (\mathcal{V},\mathcal{E})$ be a connected multigraph that is not a bipartite graph, and let $\mu \in \mathcal{M}(\mathcal{V})$. Then the following properties of the measure μ are equivalent:*

(i) It is invariant for a reversible random walk on the edges of the multigraph G.
(ii) It is a weighted measure, that is, there exist a family of weights $\alpha = (\alpha_{i,j})_{(i,j)\in\mathcal{E}}$ on $\mathcal{E}$ such that $\mu = \mu^\alpha$.
(iii) The general stochastic matching models associated to $(G, \textsc{fcfm}, \bar{\mu})$ and $(G, \textsc{mw}, \bar{\mu})$ are stable.
(iv) The continuous-time general stochastic matching models associated to $(G, \textsc{fcfm}, \mu)$ and $(G, \textsc{mw}, \mu)$ are stable.

Proof. We know that $(i) \iff (ii)$ from Proposition 1. Then, it follows from Theorems 4.5 and 4.7 in [8] (respectively for FCFM and MW) that assertion (iii) is equivalent to $\bar{\mu} \in \overline{\mathcal{N}}(G)$, where $\overline{\mathcal{N}}(G)$ is defined by (6). This is in turn, clearly equivalent to writing that $\mu \in \mathcal{N}(G)$ which, from Theorem 1, is equivalent to assertion (ii). Last, the equivalence $(iii) \iff (iv)$ was shown in Section 2.3 in [29], following a simple uniformization argument. □

Example 1. *Consider the graph of Figure 1. The set of independent sets of G is*

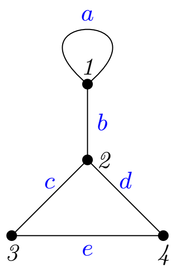

FIGURE 1. A multigraph with a self-loop

$\{\{2\},\{3\},\{4\}\}$, *and so the set* $\mathcal{N}(G)$ *reads*

$$\mathcal{N}(G)=\left\{\mu\in\mathcal{M}(\mathcal{V})\,:\,\mu(2)<\frac{1}{2},\,\mu(3)<\mu(2)+\mu(4)\text{ and }\mu(4)<\mu(2)+\mu(3)\right\}.$$

Let $\alpha=(a,b,c,d,e)$ *represent a family of weights on the edges of* $\mathcal{E}$. *The corresponding weighted measure* μ^α *reads*

$$\begin{cases}\mu^\alpha(1) &= a+b\\ \mu^\alpha(2) &= b+c+d\\ \mu^\alpha(3) &= c+e\\ \mu^\alpha(2) &= d+e.\end{cases}$$

As assertion (i) of Theorem 1 demonstrates, any such measure μ^α *is an element of* $\mathcal{N}(G)$. *Moreover, according to Corollary 1, for any* α *the continuous-time matching models* $(G,\textsc{mw},\mu^\alpha)$ *and* $(G,\textsc{fcfm},\mu^\alpha)$, *and the discrete-time matching models* $(G,\textsc{mw},\bar\mu^\alpha)$ *and* $(G,\textsc{fcfm},\bar\mu^\alpha)$ *are stable. Reciprocally, any measure rendering these models stable, can be constructed in such a way.*

On another hand, for any α *the measure* μ^α *is invariant for the reversible Markov chain on* $\{1,2,3,4\}$ *having transitions*

$$\begin{cases}P(1,1) &= \frac{a}{a+b};\ P(1,2)=\frac{b}{a+b};\\ P(2,1) &= \frac{b}{b+c+d};\ P(2,3)=\frac{c}{b+c+d};\ P(2,4)=\frac{d}{b+c+d}\\ P(3,2) &= \frac{c}{c+e};\ P(3,4)=\frac{e}{c+e};\\ P(4,2) &= \frac{d}{d+e};\ P(4,3)=\frac{e}{d+e}.\end{cases}$$

Moreover, any invariant measure of a reversible random walk on G *can be constructed in such a way.*

Corollary 1 has an immediate practical interest for admission control, in the wide class of systems described above: It allows to determine the exact set of arrival rates (or probabilities) for which the corresponding system can be stabilizable, and provides a simple way to construct such arrival rates, and so to calibrate the admission control of the corresponding arrivals into the system. For this, it is necessary and sufficient to set any family α of weights on the edges of G, and then to set μ (resp. $\bar\mu$) in function of α according to (2) (resp. (3)). Moreover, an explicit representation of the family of weights α such that $\mu=\mu^\alpha$ is provided in the proof of Theorem 1, as a simple function of the matching rates of the related general stochastic matching model, see (19), (20) and Corollary 4.

Hereafter, for a bipartite graph $G=(\mathcal{V}_1\cup\mathcal{V}_2,\mathcal{E})$ and a measure $\mu\in\mathcal{M}(\mathcal{V})$, let us define the measures $\tilde\mu_1\in\overline{\mathcal{M}}(\mathcal{V}_1)$ and $\tilde\mu_2\in\overline{\mathcal{M}}(\mathcal{V}_2)$, by

$$\tilde\mu_1(i)=\frac{\mu(i)}{\mu(\mathcal{V}_1)},\ i\in\mathcal{V}_1\quad\text{and}\quad\tilde\mu_2(j)=\frac{\mu(j)}{\mu(\mathcal{V}_2)},\ j\in\mathcal{V}_2,\tag{15}$$

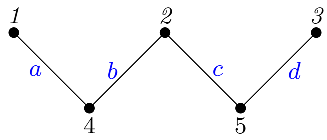

FIGURE 2. A bipartite graph

and recall that we denote by $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2)$, a probability measure of $\overline{\mathcal{M}}(\mathcal{V}_1 \times \mathcal{V}_2)$ having first (resp. second) marginal $\tilde{\mu}_1$ (resp. $\tilde{\mu}_2$). We have the following corollary to Theorem 1,

Corollary 2. *Let $G = (\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ be a bipartite graph, and let $\mu \in \mathcal{M}(\mathcal{V})$ be such that $\mu(\mathcal{V}_1) = \mu(\mathcal{V}_2)$. Then the following properties of μ are equivalent:*

(i) It is invariant for a reversible random walk on the edges of G.
(ii) It is a weighted measure, that is, there exists a family of weights α on $\mathcal{E}$ such that $\mu = \mu^\alpha$.
(iii) The extended bipartite matching model associated to $(G, \text{ML}, \tilde{\mu})$ is stable.
(iv) The bipartite matching model associated to $(G, \text{FCFM}, \tilde{\mu}_1 \otimes \tilde{\mu}_2)$ is stable.

Proof of Corollary 2. First observe that the equivalence $(i) \iff (ii)$ proven in Theorem 1 is not peculiar to the non-bipartite case, and remains valid in the present context. Then, we show that

$$\mathcal{N}_2(G) = \left\{ \mu \in \mathcal{M}\left(\mathcal{V}\right) : \left\{ \begin{array}{l} \forall \mathcal{I}_1 \in \mathbb{I}(\mathcal{V}_1) \setminus \{\mathcal{V}_1\},\ \mu\left(\mathcal{I}_1\right) < \mu\left(\mathcal{E}\left(\mathcal{I}_1\right)\right) \\ \forall \mathcal{I}_2 \in \mathbb{I}(\mathcal{V}_2) \setminus \{\mathcal{V}_2\},\ \mu\left(\mathcal{I}_2\right) < \mu\left(\mathcal{E}\left(\mathcal{I}_2\right)\right) \\ \mu(\mathcal{V}_1) = \mu(\mathcal{V}_2) \end{array} \right. \right\}. \tag{16}$$

The left inclusion being trivial, let us focus on the converse. Let μ be an element of the right-hand set, and let $\mathcal{I}$ be an independent set of G that is different from $\mathcal{V}_1$ and $\mathcal{V}_2$. Then, as $\mathcal{V}_1$ and $\mathcal{V}_2$ form a bipartition of G we have that

$$\begin{aligned} \mu(\mathcal{I}) = \mu\left(\mathcal{I} \cap \mathcal{V}_1\right) + \mu\left(\mathcal{I} \cap \mathcal{V}_2\right) &< \mu\left(\mathcal{E}\left(\mathcal{I} \cap \mathcal{V}_1\right)\right) + \mu\left(\mathcal{E}\left(\mathcal{I} \cap \mathcal{V}_2\right)\right) \\ &= \mu\left(\mathcal{E}\left(\mathcal{I}\right) \cap \mathcal{V}_2\right) + \mu\left(\mathcal{E}\left(\mathcal{I}\right) \cap \mathcal{V}_1\right) = \mu\left(\mathcal{E}\left(\mathcal{I}\right)\right). \end{aligned}$$

Hence μ is an element of $\mathcal{N}_2(G)$, which completes the proof of (16).

Now, from Theorem 7.1 of [12], the set $\text{STAB}_{\text{EB}}(G, \text{ML})$ coincides with the set $\overline{\mathcal{N}}_{\text{EB}}(G)$ defined by (8). Second, Theorem 1 in [3] shows that $\text{STAB}_{\text{B}}(G, \text{FCFM})$ coincides with the set $\overline{\mathcal{N}}_{\text{B}}(G)$ defined by (9). We conclude by observing that, for any $\mu \in \mathcal{M}(\mathcal{V})$ such that $\mu(\mathcal{V}_1) = \mu(\mathcal{V}_2)$, $\mu \in \mathcal{N}_2(G)$ is equivalent to $(\tilde{\mu}_1, \tilde{\mu}_2) \in \overline{\mathcal{N}}_{\text{EB}}(G)$ and to $\tilde{\mu}_1 \otimes \tilde{\mu}_2 \in \overline{\mathcal{N}}_{\text{B}}(G)$, from (16).

□

Example 2. *Consider the "W" graph of Figure 2. The set $\mathcal{N}_2(G)$ then reads*

$$\begin{aligned} \mathcal{N}_2(G) = \Big\{ \mu \in \mathcal{M}(\mathcal{V})\ &: \mu(1) + \mu(2) + \mu(3) = \mu(4) + \mu(5);\ \mu(1) < \mu(4), \\ &\mu(3) < \mu(5),\ \mu(1) + \mu(2) < \mu(4) + \mu(5),\ \mu(2) + \mu(3) < \mu(4) + \mu(5), \\ &\qquad \mu(4) < \mu(1) + \mu(2),\ \mu(5) < \mu(2) + \mu(3) \Big\}. \end{aligned}$$

Let $\alpha = (a, b, c, d)$ represent a family of weights on the edges of $\mathcal{E}$. The corresponding weighted measure μ^α reads

$$\begin{cases} \mu^\alpha(1) &= a \\ \mu^\alpha(2) &= b + c \\ \mu^\alpha(3) &= d \\ \mu^\alpha(4) &= a + b \\ \mu^\alpha(5) &= c + d. \end{cases}$$

From assertion (ii) of Theorem 1, any such measure μ^α is an element of $\mathcal{N}_2(G)$. Second, recalling (15) *we define the measures $\tilde{\mu}_1^\alpha$ on $\{1, 2, 3\}$ and $\tilde{\mu}_2^\alpha$ on $\{4, 5\}$, as*

$$\begin{cases} \tilde{\mu}_1^\alpha(1) &= \frac{a}{a+b+c+d}; \\ \tilde{\mu}_1^\alpha(2) &= \frac{b+c}{a+b+c+d}; \\ \tilde{\mu}_1^\alpha(3) &= \frac{d}{a+b+c+d}; \end{cases} \qquad \begin{cases} \tilde{\mu}_1^\alpha(4) &= \frac{a+b}{a+b+c+d}; \\ \tilde{\mu}_1^\alpha(5) &= \frac{c+d}{a+b+c+d}, \end{cases}$$

and set $\tilde{\mu}^\alpha = (\tilde{\mu}_1^\alpha, \tilde{\mu}_2^\alpha)$. Then, in view of Corollary 2 the extended bipartite matching model $(G, \text{ML}, \tilde{\mu}^\alpha)$ and the bipartite matching model $(G, \text{FCFM}, \tilde{\mu}_1^\alpha \otimes \tilde{\mu}_2^\alpha)$ are stable, and any measure rendering these models stable can be constructed in such a way.

On another hand, the measure μ^α is invariant for the reversible random walk having transitions

$$\begin{cases} P(1,4) &= 1; \\ P(2,4) &= \frac{b}{b+c};\ P(2,5) = \frac{c}{b+c}; \\ P(3,5) &= 1; \\ P(4,1) &= \frac{a}{a+b};\ P(4,2) = \frac{b}{a+b}; \\ P(5,2) &= \frac{c}{c+d};\ P(5,3) = \frac{d}{c+d}, \end{cases}$$

and any invariant measure of a reversible random walk on G can be constructed in such a way.

Corollary 2 can, again, be easily exploited for the admission control of bipartite stochastic matching models, and matching queues: It is necessary and sufficient to set any family α of weights on the edges of G, to define μ^α in function of α according to (2), and then to deduce $\tilde{\mu}^\alpha$ from μ^α as in (15) to obtain an extended bipartite matching model that is stabilizable by the policy ML, or a bipartite matching model that is stabilizable by FCFM. Moreover, as above, an explicit representation of the family of weights α is provided as a simple function of the matching rates of the related extended bipartite matching model, see (23) and Corollary 5.

We now turn to our second main result,

Theorem 2. *For any connected graph $G = (\mathcal{V}, \mathcal{E})$ and any non-trivial partition $\mathcal{V}_1 \cup \mathcal{V}_2$ of $\mathcal{V}$, the set $\mathcal{N}_{\mathcal{V}_1}(G)$ defined by* (10) *coincides with*

$$\mathcal{W}_{\prec,\mathcal{V}_2}(G) \triangleq \left\{ \mu \in \mathcal{M}(\mathcal{V}) : \exists \alpha \in \mathcal{M}(\mathcal{E}) \text{ s.t. } \left\{ \begin{array}{l} \forall i \in \mathcal{V}_1,\ \mu^\alpha(i) = \mu(i) \\ \forall j \in \mathcal{V}_2,\ \mu^\alpha(j) < \mu(j) \end{array} \right\} \right\}.$$

In Section 8, two different proofs of Theorem 2 are given. One using Theorem 1 whereas the other is self-contained, and involves flow network theory. Theorem 2 readily implies that the set of probability measures $\overline{\mathcal{N}}_{\mathcal{V}_1}(G)$ defined by (11), coincides with

$$\left\{ \bar{\mu} \in \overline{\mathcal{M}}(\mathcal{V}) : \exists \alpha \in \mathcal{M}(\mathcal{E}) \text{ s.t. } \left\{ \begin{array}{l} \forall i \in \mathcal{V}_1,\ \bar{\mu}^\alpha(i) = \bar{\mu}(i) \\ \forall j \in \mathcal{V}_2,\ \bar{\mu}^\alpha(j) < \bar{\mu}(j) \end{array} \right\} \right\}.$$

Therefore, recalling the discussion of Section 3.4, the stability region of continuous-time and discrete-time matching models with reneging are fully characterized, and easy to construct, using weighted measures.

We now focus again on the case where the graph $G = (\mathcal{V}, \mathcal{E})$ is bipartite, of bipartition $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$. As a matter of fact, the stability region $\mathcal{N}_{\mathcal{V}_1,\gamma}(G)$ of skill-based service systems (sections 3.5 and 3.6), defined by (12), appears as a particular case of $\mathcal{N}_{\mathcal{V}_1,\gamma}(G)$, whenever the restriction of the measure μ to the subset $\mathcal{V}_2$ is fixed, and set to γ. The following result then immediately follows from Theorem 2,

Corollary 3. *For any connected bipartite graph $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$, for any $\gamma \in \mathcal{M}(\mathcal{V}_2)$, $\mathcal{N}_{\mathcal{V}_1,\gamma}(G)$ coincide with the set*

$$\left\{ \mu \in \mathcal{M}(\mathcal{V}_1) : \exists \alpha \in \mathcal{M}(\mathcal{E}) \text{ s.t. } \left\{ \begin{array}{l} \forall i \in \mathcal{V}_1,\ \mu^\alpha(i) = \mu(i) \\ \forall j \in \mathcal{V}_2,\ \mu^\alpha(j) < \gamma(j) \end{array} \right. \right\}.$$

Example 3 (Example 2, continued)**.** *Consider again the "W" graph of Figure 2, and fix $\gamma(4)$ and $\gamma(5)$, two positive numbers. The stability region $\mathcal{N}_{\mathcal{V}_1,\gamma}(G)$ of the corresponding skill-based queues then reads*

$$\mathcal{N}_{\mathcal{V}_1,\gamma}(G) = \Big\{ \mu \in \mathcal{M}(\mathcal{V}) : \mu(1) < \gamma(4),\ \mu(3) < \gamma(5),$$
$$\mu(1) + \mu(2) < \gamma(4) + \gamma(5),\ \mu(2) + \mu(3) < \mu(4) + \mu(5) \Big\}.$$

From Corollary 3 (setting $\mathcal{V}_1 = \{1, 2, 3\}$ and $\mathcal{V}_2 = \{4, 5\}$), the above set can be fully characterized as the set of measures μ such that there exists a family of weights $\alpha = (a, b, c, d)$ on $\mathcal{E}$ satisfying

$$\left\{ \begin{array}{lll} \mu(1) = & \mu^\alpha(1) & = a \\ \mu(2) = & \mu^\alpha(2) & = b + c \\ \mu(3) = & \mu^\alpha(3) & = d \\ & \mu^\alpha(4) & = a + b < \gamma(4) \\ & \mu^\alpha(5) & = c + d < \gamma(5). \end{array} \right.$$

Now, observe the following immediate characterization of the maximal stability region of multi-class and multi-pool queues, as a set of weighted measures on $\mathcal{V}_1$,

Proposition 2. *Let $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ be a connected bipartite graph. Then, for any measure $\gamma \in \mathcal{M}(\mathcal{E})$ and any $(s_j)_{j \in \mathcal{V}_2} \in (\mathbb{N}_*)^{\mathcal{V}_2}$ the set $\mathcal{N}_{\mathcal{V}_1,\gamma,s}(G)$ defined by (13) equals*

$$\left\{ \mu \in \mathcal{M}(\mathcal{V}_1) : \exists \alpha \in \mathcal{M}(\mathcal{E}) \text{ s.t. } \left\{ \begin{array}{l} \forall i \in \mathcal{V}_1,\ \mu^\alpha(i) = \mu(i) \\ \forall j \in \mathcal{V}_2,\ \sum\limits_{i \in \mathcal{E}(j)} \frac{\alpha_{i,j}}{\gamma((i,j))} < s_j \end{array} \right. \right\}.$$

Proof. For the left inclusion, it suffices to set $\alpha_{i,j} = \gamma((i,j)) s_j \pi((i,j))$ for all $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$. As for the right inclusion, just set

$$\pi((i,j)) = \frac{\alpha_{i,j}}{\gamma((i,j)) s_j} \mathbf{1}_{\{i - j\}}, \quad i \in \mathcal{V}_1,\ j \in \mathcal{V}_2.$$

□

In the particular case where service times depend only on the classes of servers and not on the classes of customers they serve (i.e., $\gamma((i,j)) = \gamma(j)$ for all $(i,j) \in \mathcal{E}$), by gathering Corollary 3 and Proposition 2 we immediately obtain the following characterization of the maximal stability region,

Proposition 3. *Let $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ be a connected bipartite graph, $\gamma \in \mathcal{M}(\mathcal{V}_2)$ and $(s_j)_{j \in \mathcal{V}_2} \in (\mathbb{N}_*)^{\mathcal{V}_2}$. Let $\hat{\gamma} \in \mathcal{M}(\mathcal{V}_2)$ and $\tilde{\gamma} \in \mathcal{M}(\mathcal{E})$ be respectively defined by*

$\hat{\gamma}(j) \triangleq s_j\gamma(j)$ for all $j \in \mathcal{V}_2$, and by $\tilde{\gamma}((i,j)) \triangleq \gamma(j)$ for all $(i,j) \in \mathcal{E}$. Then, the sets $\mathcal{N}_{\mathcal{V}_1,\tilde{\gamma},s}(G)$, $\mathcal{N}_{\mathcal{V}_1,\hat{\gamma}}(G)$ and

$$\left\{\mu \in \mathcal{M}(\mathcal{V}_1) : \exists \alpha \in \mathcal{M}(\mathcal{E}) \ s.t. \left\{ \begin{array}{l} \forall i \in \mathcal{V}_1,\ \mu^\alpha(i) = \mu(i) \\ \forall j \in \mathcal{V}_2,\ \mu^\alpha(j) < s_j\gamma(j) \end{array} \right. \right\}$$

coincide.

5. Proof of Theorem 1

We now prove our main result. Let G be a connected multigraph.

Case 1: G is not a bipartite graph. We first show that $\mathcal{W}(G)$ is included in $\mathcal{N}(G)$, an inclusion that was shown in Lemma 11 in [16] whenever G is a graph. We can easily extend that proof to multigraphs, as follows: For any subset $\mathcal{X} \subset \mathcal{V}$, denote by $\mathcal{A}_\mathcal{X} \subset \mathcal{E}$ the set of edges of the graph having at least one extremity in $\mathcal{X}$ (including self-loops (i,i) for $i \in \mathcal{X}$). Let $\alpha \in \mathcal{M}(\mathcal{E})$ be a family of weights, and fix an independent set $\mathcal{I} \in \mathbb{I}(G)$. Then, first observe that $\mathcal{A}_\mathcal{I} \subsetneq \mathcal{A}_{\mathcal{E}(\mathcal{I})}$. Indeed, for any $(a_1,a_2) \in \mathcal{A}_\mathcal{I}$, we have either $a_1 \in \mathcal{I}$ and thus $a_2 \in \mathcal{E}(\mathcal{I})$, or the other way around, and in both cases, $(a_1,a_2) \in \mathcal{A}_{\mathcal{E}(\mathcal{I})}$. Therefore, we have that $\mathcal{A}_\mathcal{I} \subset \mathcal{A}_{\mathcal{E}(\mathcal{I})}$. On another hand, as G is connected and non-bipartite, there exists an edge e connecting an element of $\mathcal{E}(\mathcal{I})$ to an element of $\mathcal{I}^c$, otherwise we would have $\mathcal{E}(\mathcal{E}(\mathcal{I})) = \mathcal{I}$ and, since $\mathcal{I} \in \mathbb{I}(G)$, $(\mathcal{I}, \mathcal{E}(\mathcal{I}))$ would form a bipartition of G. In particular, as $\mathcal{I}$ is an independent set, we have that $e \in \mathcal{A}_{\mathcal{E}(\mathcal{I})} \cap (\mathcal{A}_\mathcal{I})^c$, entailing that $\mathcal{A}_\mathcal{I} \subsetneq \mathcal{A}_{\mathcal{E}(\mathcal{I})}$. We obtain that

$$\mu^\alpha(\mathcal{I}) = \sum_{i\in\mathcal{I}} \sum_{e\in\mathcal{A}_i} \alpha_e = \sum_{e\in\mathcal{A}_\mathcal{I}} \alpha_e < \sum_{e\in\mathcal{A}_{\mathcal{E}(\mathcal{I})}} \alpha_e \le \sum_{i\in\mathcal{E}(\mathcal{I})} \sum_{e\in\mathcal{A}_i} \alpha_e = \mu^\alpha(\mathcal{E}(\mathcal{I})), \tag{17}$$

where the second equality holds due to the fact that the set $\mathcal{I}$ is independent. This shows that $\mu^\alpha \in \mathcal{N}(G)$.

We now show the converse inclusion. We will show that $\mathcal{N}(G)$ is included in the set of weighted measures with non-negative weights, that is,

$$\left\{\mu \in \mathcal{M}(\mathcal{V}) : \forall \mathcal{I} \in \ \mathbb{I}(G),\ \mu(\mathcal{I}) < \mu(\mathcal{E}(\mathcal{I}))\right\} \subset \left\{\mu \in \mathcal{M}(\mathcal{V}) : \exists (\alpha_{i,j})_{(i,j)\in\mathcal{E}} \in \mathbb{R}_+^\mathcal{E},\ \forall i \in \mathcal{V},\ \mu(i) = \sum_{j\in\mathcal{E}(i)} \alpha_{i,j}\right\}.$$

By taking the interior on each side of the inclusion, we will obtain the desired inclusion. Indeed, the set $\mathcal{N}(G)$ on the left of the inclusion is open so that it is equal to its interior, and it can be seen that the interior of the polyhedral cone defining the set of right-hand side above coincides with the set of weighted measures with positive weights.

Let $\mu \in \mathcal{M}(\mathcal{V})$ be a measure. We will reason by contraposition and show that if μ is not a weighted measure with positive or null weights, then there exists an independent set $\mathcal{I}$ of G such that $\mu(\mathcal{I}) \ge \mu(\mathcal{E}(\mathcal{I}))$. Let A be the matrix indexed by $\mathcal{V} \times \mathcal{E}$ such that for $(v,e) \in \mathcal{V} \times \mathcal{E}$, $A_{v,e} = 1$ if the vertex v is an extremity of the edge e, and $A_{v,e} = 0$ otherwise. Introduce the vector $b = (\mu(v))_{v\in\mathcal{V}}$, and, for a given family of weights $\alpha \in \mathcal{M}^{\ge 0}(\mathcal{E})$, introduce the vector $x_\alpha = (\alpha_e)_{e\in\mathcal{E}}$. With these notations, the measure μ is the weighted measure associated to the family of weights α if and only if $Ax_\alpha = b$. The rest of the proof will rely on Farkas' lemma (see [18]), that asserts that one and only one of the following linear systems has a solution:

(1) the system $Ax = b$, for x indexed by $\mathcal{E}$ satisfying $x \ge 0$;
(2) the system ${}^tAy \ge 0$, for y indexed by $\mathcal{V}$ satisfying ${}^tby < 0$.

Let us consider the matrix ${}^tA$, indexed by $\mathcal{E} \times \mathcal{V}$. On the line indexed by the edge e,

- if e is not a self-loop, there are exactly two occurrences of the value 1, at the positions corresponding to the two extremities of e,
- if e is a self-loop, there is exactly one occurrence of the value 1, at the position corresponding to the vertex having the self-loop e.

It follows that the equation ${}^tAy \geq 0$ is equivalent to:

- for any edge $e = (i, j)$ which is not a self-loop, $y(i) + y(j) \geq 0$,
- for any self-loop $e = (i, i)$, $y(i) \geq 0$.

Let us assume that μ is not a weighted measure, that is, that the system $Ax = b,\ x \geq 0$ has no solution. Then, by Farkas' lemma, the system ${}^tAy \geq 0,\ {}^tby < 0$ has at least one solution, which means that there exists a vector y satisfying ${}^tAy \geq 0$, and such that $\sum_{i\in\mathcal{V}} \mu(i)y(i) < 0$. Let us choose a solution y of the system such that the $|y(i)|, i \in \mathcal{V}$, take as few different values as possible.

- If the $|y(i)|, i \in \mathcal{V}$, take a single value $c > 0$, we let

  $$\mathcal{V}_+ \triangleq \{i \in \mathcal{V} : y(i) = c\} \quad \text{and} \quad \mathcal{V}_- \triangleq \{i \in \mathcal{V} : y(i) = -c\}.$$

  In view of the constraints that y must satisfy on the edges, one can check that $\mathcal{V}_-$ is an independent set, and that $\mathcal{E}(\mathcal{V}_-) \subset \mathcal{V}_+$. Furthermore, the condition $\sum_{i\in\mathcal{V}} \mu(i)y(i) < 0$ implies that $\mu(\mathcal{V}_+) < \mu(\mathcal{V}_-)$, so that $\mu(\mathcal{E}(\mathcal{V}_-)) < \mu(\mathcal{V}_-)$. Thus, the measure μ is not an element of $\mathcal{N}(G)$.
- Otherwise, let p be the largest absolute value of a coordinate of y, and let q be the second largest absolute value. We introduce the sets

  $$\mathcal{V}_+^p \triangleq \{i \in \mathcal{V} : y(i) = p\} \quad \text{and} \quad \mathcal{V}_-^p \triangleq \{i \in \mathcal{V} : y(i) = -p\}.$$

  Again, the constraints on the edges imply that $\mathcal{V}_-^p$ is an independent set, and we have $\mathcal{E}(\mathcal{V}_-^p) \subset \mathcal{V}_+^p$, so that $\mu(\mathcal{E}(\mathcal{V}_-^p)) \leq \mu(\mathcal{V}_+^p)$.
  - Let us first assume that $\mu(\mathcal{V}_+^p) \leq \mu(\mathcal{V}_-^p)$. Then, $\mathcal{V}_-^p$ is an independent set such that $\mu(\mathcal{E}(\mathcal{V}_-^p)) \leq \mu(\mathcal{V}_-^p)$, so that $\mu \notin \mathcal{N}(G)$.
  - Let us now assume that $\mu(\mathcal{V}_+^p) > \mu(\mathcal{V}_-^p)$. Then, let us modify y by assigning the value q to all the vertices of $\mathcal{V}_+^p$, and the value $-q$ to all the vertices of $\mathcal{V}_-^p$. To begin with, let us show that the new vector y' then defined still satisfies the constraints on the edges and self-loops. First, if an edge (i, j) has no extremity in $\mathcal{V}_-^p \cup \mathcal{V}_+^p$, then $y'(i)+y'(j) = y(i) + y(j) \geq 0$. Second, if $i \in \mathcal{V}_-^p$, then any adjacent vertex j of i belongs to $\mathcal{V}_+^p$, so that $y'(i)+y'(j) = -q+q = 0$. Third, if $i \in \mathcal{V}_+^p$, then for any adjacent vertex j we have that $y'(i) + y'(j) = q + y'(j) \geq 0$, since q is the largest value $|y'(j)|$ can possibly take. Last, for any self-loop $e = (i, i)$, we have $y(i) \geq 0$, and since $y'(i)$ has the same sign as $y(i)$ we also have that $y'(i) \geq 0$. Furthermore, observe that

    $$\sum_{i\in\mathcal{V}} \mu(i)y'(i) = \sum_{i\in\mathcal{V}} \mu(i)y(i) - (p - q)(\mu(\mathcal{V}_+^p) - \mu(\mathcal{V}_-^p)) < 0.$$

    This shows that y' is also a solution to the system ${}^tAy \geq 0$ and ${}^tby < 0$. We have thus a contradiction with the fact that the $|y(i)|, i \in \mathcal{V}$, takes as few different values as possible.

Case 2: G is a bipartite graph. Let us now assume that $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ is a connected bipartite graph. Fix $\alpha \in \mathcal{M}(\mathcal{E})$. First, it is immediate that

$$\mu^\alpha(\mathcal{V}_1) = \sum_{e\in\mathcal{E}} \alpha_e = \mu^\alpha(\mathcal{V}_2).$$

Then, we can apply the same arguments as above, showing that $\mathcal{W}(G) \subset \mathcal{N}(G)$ if G would not be a bipartite graph, by replacing the generic independent set $\mathcal{I}$ by an independent set $\mathcal{I}_1 \in \mathbb{I}(\mathcal{V}_1)\backslash\{\mathcal{V}_1\}$. This leads to the conclusion that $\mu^\alpha(\mathcal{I}_1) < \mu^\alpha(\mathcal{E}(\mathcal{I}_1))$. By a symmetric argument, we also have that $\mu^\alpha(\mathcal{I}_2) < \mu^\alpha(\mathcal{E}(\mathcal{I}_2))$, for any $\mathcal{I}_2 \in \mathbb{I}(\mathcal{V}_2)\backslash\{\mathcal{V}_2\}$. Using (16), this shows that $\mu^\alpha \in \mathcal{N}_2(G)$.

We now prove the converse inclusion. By a similar approach to the case where G is not a bipartite graph, based on Farkas' lemma, we will prove that if a measure μ belongs to $\mathcal{N}_2(G)$, then it is a weighted measure with non-negative weights. Again, by taking the interior of these sets of measures (within the affine hyperplane of measures satisfying $\mu(\mathcal{V}_1) = \mu(\mathcal{V}_2)$), the result also follows for positive weights.

We keep the same notation as above, and let $\mu \in \mathcal{N}_2(G)$. We reason similarly: If μ is not a weighted measure, then, by Farkas' lemma, the system ${}^tAy \geq 0$, ${}^tby < 0$ has at least one solution, which means that there exists a vector y satisfying ${}^tAy \geq 0$ and such that $\sum_{i\in\mathcal{V}} \mu(i)y(i) < 0$. Then, by choosing again a solution y whose components $|y(i)|, i \in \mathcal{V}$, take as few different values as possible, we have the following alternative:

- If the components $|y(i)|, i \in \mathcal{V}$, take a single value c, then the exact same argument as above lead to the construction of an independent set $\mathcal{V}_-$ of G that is such that $\mu(\mathcal{E}(\mathcal{V}_-)) < \mu(\mathcal{V}_-)$, a contradiction.
- Otherwise, we let again p, q, $\mathcal{V}_+^p$ and $\mathcal{V}_-^p$ be defined as above. Then, again, $\mathcal{V}_-^p$ is an independent set that is such that $\mu(\mathcal{E}(\mathcal{V}_-^p)) \leq \mu(\mathcal{V}_+^p)$.
  - If $\mu(\mathcal{V}_+^p) \leq \mu(\mathcal{V}_-^p)$, then, $\mathcal{V}_-^p$ is an independent set such that $\mu(\mathcal{E}(\mathcal{V}_-^p)) \leq \mu(\mathcal{V}_-^p)$, so as μ is an element of $\mathcal{N}_2(G)$, we can assume without loss of generality that $\mathcal{V}_-^p = \mathcal{V}_1$. This implies that
  $$\mathcal{E}(\mathcal{V}_-^p) = \mathcal{E}(\mathcal{V}_1) = \mathcal{V}_2 \subset \mathcal{V}_+^p,$$
  and hence $|y(i)|$ equals p for all $i \in \mathcal{V}$, a contradiction.
  - If now $\mu(\mathcal{V}_+^p) > \mu(\mathcal{V}_-^p)$, then we can apply the exact same argument as in the non-bipartite case, leading to the same contradiction.

6. On the matching rates of general stochastic models

We now provide an alternative proof of the implication $(iii) \implies (ii)$ in Corollary 1, relying on the *matching rates* of a related stochastic matching model, defined as the long-run frequency of matchings executed on each given edge of the considered graph. Specifically, as will be shown in Corollary 4, for any probability measure $\bar{\mu}$ in the stability region of such a model, we can explicitly define the rates α such that $\bar{\mu} = \bar{\mu}^\alpha$, in terms of the asymptotic matching rates of the considered model, see (22). Moreover, from this characterization we can deduce interesting properties of the matching models at stake, among which, invariance properties of the matching rates with respect to the matching policy, as will be shown in Section 9.

6.1. Asymptotic matching rates.

Definition 1. *For a general matching model associated to any fixed* $(G, \Phi, \bar{\mu})$*, for any time* $n \geq 1$ *and any* $i, j \in \mathcal{V}$*, we set*

$$M_n[i,j] \triangleq \text{Number of matchings } (i,j) \text{ performed up to and including time } n,$$

where the above is set as null if $(i,j) \notin \mathcal{E}$*. Then, the* matching rate *of* i *with* j *(or equivalently, of* j *with* i*) up to* n *is defined as the proportion* $M_n[i,j]/n$*.*

In the sequel, for any fixed couple (G, Φ), any $i \in \mathcal{V}$ and any $j \in \mathcal{E}(i)$, we set

$$A_{i\to j} \triangleq \{\text{states of } \mathbb{W} \text{ s.t. an incoming } i\text{-item gets matched with a } j\text{-item}\}. \tag{18}$$

Lemma 1. *For any stable general matching model associated to* $(G, \Phi, \bar{\mu})$ *and any* $(i,j) \in \mathcal{E}$*, the asymptotic matching rates satisfy*

$$\frac{M_n[i,j]}{n} \xrightarrow[n\to+\infty]{a.s.} \Theta^{\Phi,\bar{\mu}}[i,j],$$

where

$$\Theta^{\Phi,\bar{\mu}}[i,j] \triangleq \bar{\mu}(i)\Pi_W^{\Phi,\bar{\mu}}\left(A_{i\to j}\right) + \bar{\mu}(j)\Pi_W^{\Phi,\bar{\mu}}\left(A_{j\to i}\right)\mathbf{1}_{\{i\neq j\}}, \tag{19}$$

and $\Pi_W^{\Phi,\bar{\mu}}$ *represents the unique stationary distribution of the chain* $\left\{W_n^{\Phi,\bar{\mu}}\right\}_{n\in\mathbb{N}}$.

Proof. Fix an edge $(i,j) \in \mathcal{E}$ with $i \neq j$. Then, we have a.s.

$$\begin{aligned}\frac{M_n[i,j]}{n} &= \frac{1}{n}\sum_{k=1}^{n}\mathbf{1}_{\{\text{a match } (i,j) \text{ is performed at time } k\}}\\ &= \frac{1}{n}\sum_{k=1}^{n}\left[\mathbf{1}_{\{V_k=i\}}\mathbf{1}_{\left\{W_{k-1}^{\Phi,\bar{\mu}}\in A_{i\to j}\right\}} + \mathbf{1}_{\{V_k=j\}}\mathbf{1}_{\left\{W_{k-1}^{\Phi,\bar{\mu}}\in A_{j\to i}\right\}}\right]\\ &=: \frac{1}{n}\sum_{k=0}^{n-1} f_{i,j}\left(W_k^{\Phi,\bar{\mu}}, V_{k+1}\right).\end{aligned}$$

Moreover, for all $k \in \mathbb{N}$, $W_k^{\Phi,\bar{\mu}}$ and V_{k+1} are independent, and $\left\{W_k^{\Phi,\bar{\mu}}\right\}_{k\in\mathbb{N}}$ is an ergodic Markov chain of stationary distribution $\Pi_W^{\Phi,\bar{\mu}}$. Thus, $\left\{\left(W_k^{\Phi,\bar{\mu}}, V_{k+1}\right)\right\}_{k\in\mathbb{N}}$ is an ergodic Markov chain on $\mathbb{W}\times\mathcal{V}$, whose unique stationary distribution is given by $\Pi^{\Phi,\bar{\mu}} \triangleq \Pi_W^{\Phi,\bar{\mu}} \otimes \bar{\mu}$. As $f_{i,j}$ is bounded by 1, it is integrable with respect to $\Pi^{\Phi,\bar{\mu}}$ and the ergodic theorem for Markov chains gives that a.s.,

$$\begin{aligned}\lim_{n\to+\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} f_{i,j}\left(W_k^{\Phi,\bar{\mu}}, V_{k+1}\right)\right) &= \sum_{w\in\mathbb{W}}\sum_{v\in\mathcal{V}} f_{i,j}(w,v)\Pi^{\Phi,\bar{\mu}}(w,v)\\ &= \sum_{w\in\mathbb{W}}\mathbf{1}_{\{w\in A_{i\to j}\}}\Pi_W^{\Phi,\bar{\mu}}(w)\sum_{v\in\mathcal{V}}\mathbf{1}_{\{v=i\}}\bar{\mu}(v)\\ &\quad + \sum_{w\in\mathbb{W}}\mathbf{1}_{\{w\in A_{j\to i}\}}\Pi_W^{\Phi,\bar{\mu}}(w)\sum_{v\in\mathcal{V}}\mathbf{1}_{\{v=j\}}\bar{\mu}(v)\\ &= \bar{\mu}(i)\Pi_W^{\Phi,\bar{\mu}}\left(A_{i\to j}\right) + \bar{\mu}(j)\Pi_W^{\Phi,\bar{\mu}}\left(A_{j\to i}\right),\end{aligned}$$

and the computation is the same without the second term, in the case $i = j$. □

By combining the result above and the strong law of large numbers, we get the following result, which formalizes the intuitive fact that in steady state, the probability of arrival to each node must equate the sum of the asymptotic matching rates of all the adjacent edges to that node. Specifically,

Lemma 2. *For any stable general matching model associated to* $(G, \Phi, \bar{\mu})$*, we have*

$$\bar{\mu}(i) = \sum_{j\in\mathcal{V}}\left(1+\mathbf{1}_{\{j=i\}}\right)\Theta^{\Phi,\bar{\mu}}[i,j], \quad \textit{for all } i \in \mathcal{V}. \tag{20}$$

Proof. Consider an item $i \in \mathcal{V}$ and denote, for all $n \in \mathbb{N}_*$, by $A_n(i)$ and $D_n(i)$ the number of arrivals and the number of departures of i-items up to and including time n, respectively. By the strong law of large numbers, we have

$$\frac{A_n(i)}{n} = \frac{1}{n}\sum_{k=1}^{n} 1_{\{V_k=i\}} \xrightarrow[n\to+\infty]{\text{a.s.}} \bar{\mu}(i).$$

Moreover, Lemma 1 entails that

$$\frac{D_n(i)}{n} = \frac{2M_n[i,i]}{n} + \sum_{j\in\mathcal{V}\setminus\{i\}} \frac{M_n[i,j]}{n}$$
$$\xrightarrow[n\to+\infty]{\text{a.s.}} 2\Theta^{\Phi,\bar\mu}[i,i] + \sum_{j\in\mathcal{V}\setminus\{i\}} \Theta^{\Phi,\bar\mu}[i,j] = \Theta^{\Phi,\bar\mu}[i,i] + \sum_{j\in\mathcal{V}} \Theta^{\Phi,\bar\mu}[i,j].$$

Comparing the two limits above, it remains to prove that

$$\frac{A_n(i)-D_n(i)}{n} \xrightarrow[n\to+\infty]{\text{a.s.}} 0.$$

For this, observe that we have a.s., for all $n \geq 1$,

$$\left|W_n^{\Phi,\bar\mu}\right|_i = A_n(i) - D_n(i) + \left|W_0^{\Phi,\bar\mu}\right|_i, \tag{21}$$

where $|w|_i$ denotes the number of occurrences of i in the word w. Let us suppose that, on some event $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') > 0$, we have

$$\lim_{n\to+\infty} \frac{A_n(i)-D_n(i)}{n} \neq 0.$$

Fix a realization $\omega \in \Omega'$. First, if

$$\lim_{n\to+\infty} \frac{A_n(i)(\omega)-D_n(i)(\omega)}{n} > 0,$$

from (21), there would exist a rank $M(\omega)$ such that $\left|W_n^{\Phi,\bar\mu}(\omega)\right|_i > 0$ for all $n \geq M(\omega)$. Likewise, if we have that

$$\lim_{n\to+\infty} \frac{A_n(i)(\omega)-D_n(i)(\omega)}{n} < 0,$$

then in view of (21), for some $M(\omega)$ we would have that $\left|W_n^{\Phi,\bar\mu}(\omega)\right|_i < 0$ for all $n \geq M(\omega)$. All in all, we get that $\Omega' \subset \liminf_{n\to\infty} \left\{\omega \in \Omega : \left|W_n^{\Phi,\bar\mu}(\omega)\right|_i \neq 0\right\}$, an absurdity in view of the recurrence of $\left\{W_n^{\Phi,\bar\mu}\right\}_{n\in\mathbb{N}}$. This concludes the proof. □

6.2. **Alternative proof of** $(iii) \implies (ii)$ **in Corollary 1.** We deduce from Lemma 2, the following result, from which the implication $(iii) \implies (ii)$ of Corollary 1 directly follows,

Corollary 4. *For any connected multigraph $G = (\mathcal{V}, \mathcal{E})$ and any admissible matching policy Φ, recalling (5) we have*

$$\overline{\text{STAB}}(G,\Phi) \subset \overline{\mathcal{W}}(G).$$

Specifically, we have that

$$\bar\mu \in \overline{\text{STAB}}(G,\Phi) \Longrightarrow \bar\mu = \bar\mu^{\alpha^{\Phi,\bar\mu}},$$

where

$$\alpha^{\Phi,\bar\mu} \;:\; \begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathbb{R}_+^* \\ (i,j) & \longmapsto & \left(1+\mathbf{1}_{\{i=j\}}\right)\Theta^{\Phi,\bar\mu}[i,j], \end{array} \tag{22}$$

for $\Theta^{\Phi,\bar\mu}$ defined by (19).

Proof. Fix a connected multigraph $G = (\mathcal{V}, \mathcal{E})$ and a matching policy Φ. Let $\bar\mu \in \overline{\text{STAB}}(G,\Phi)$ (if the set is empty, the result is trivial), in a way that the general stochastic matching model associated to $(G, \Phi, \bar\mu)$ is stable. Then, for the family of weights $\alpha^{\Phi,\bar{\mu}} \in \mathcal{M}(\mathcal{E})$ defined by (22), recalling (3) and in view of Lemma 2, we have that for all $i \in \mathcal{V}$,

$$\bar{\mu}^{\alpha^{\Phi,\bar{\mu}}}(i) = \frac{\Theta^{\Phi,\bar{\mu}}[i,i] + \sum_{j\in\mathcal{V}} \Theta^{\Phi,\bar{\mu}}[i,j]}{\sum_{l\in\mathcal{V}} \left(\Theta^{\Phi,\bar{\mu}}[l,l] + \sum_{j\in\mathcal{V}} \Theta^{\Phi,\bar{\mu}}[l,j] \right)} = \frac{\bar{\mu}(i)}{\sum_{l\in\mathcal{V}} \bar{\mu}(l)} = \bar{\mu}(i),$$

which concludes the proof. □

7. Matching rates of extended bipartite matching models

As will be shown below, the arguments of Section 6 can be adapted to bipartite matching models. Specifically, we show hereafter that any measure in the stability region of a given extended bipartite model can be represented as a weighted measure, associated to a family of weights that are explicitly defined in function of the matching rates of the corresponding model - thereby providing an alternative proof of implication $(iii) \Longrightarrow (ii)$ in Corollary 2.

We first define the matching rates in extended bipartite stochastic matching models as follows,

Definition 2. *For any extended bipartite matching model associated to a triple $(G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}), \Phi, \tilde{\mu})$, for any time $n \geq 1$ and any couple $(i,j) \in \mathcal{V}_1 \times \mathcal{V}_2$, we set*

$$M_n^{\text{EB}}[i,j] \triangleq \text{Number of matchings } (i,j) \text{ performed up to time } n \text{ included},$$

where the above is set as null if $(i,j) \notin \mathcal{E}$. Then, the matching rate of i with j up to n is defined as the proportion $M_n^{\text{EB}}[i,j]/n$.

As above, for any fixed bipartite graph $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ and any admissible Φ, for any $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$, we define the following sets:

$$A_{i\to j}^{\text{EB}} \triangleq \{\text{buffers s.t. an incoming } i\text{-item gets matched with a stored } j\text{-item}\},$$
$$A_{i\leftrightarrow j}^{\text{EB}} \triangleq \{\text{buffers s.t. an incoming } (i,j)\text{-couple gets matched together}\}.$$

(Observe that the latter sets are non-empty only if $(i,j) \in \mathcal{E}$, and that the second one does not depend on Φ.) The following result, whose proof is analogous to that of Lemma 1, makes precise the asymptotic matching rates in a stable bipartite matching model,

Proposition 4. *Consider an extended bipartite matching model associated to the triple $(G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}), \Phi, \tilde{\mu})$ such that $\tilde{\mu} \in \text{STAB}_{\text{EB}}(G, \Phi)$. Then, for any $(i,j) \in \mathcal{E}$, we have that*

$$\begin{aligned}\frac{M_n^{\text{EB}}[i,j]}{n} \xrightarrow[n\to+\infty]{a.s.} \; & \Theta_{\text{EB}}^{\Phi,\tilde{\mu}}[i,j] \\ \triangleq \; & \tilde{\mu}((i,\mathcal{V}_2 \setminus \{j\}))\tilde{\Pi}_W^{\Phi,\tilde{\mu}}\left(A_{i\to j}^{\text{EB}}\right) + \tilde{\mu}((\mathcal{V}_1 \setminus \{i\}, j))\tilde{\Pi}_W^{\Phi,\tilde{\mu}}\left(A_{j\to i}^{\text{EB}}\right) \\ & + \tilde{\mu}((i,j))\tilde{\Pi}_W^{\Phi,\mu}\left(A_{i\leftrightarrow j}^{\text{EB}}\right),\end{aligned}$$

where $\tilde{\Pi}_W^{\Phi,\tilde{\mu}}$ is the unique stationary probability distribution of the positive recurrent Markov chain $\left\{Y_n^{\Phi,\tilde{\mu}}\right\}_{n\in\mathbb{N}}$.

We can then equate the arrival rates to the cumulative matching rates of the nodes. The following result can be proven similarly to Lemma 2,

Lemma 3. *For an extended bipartite matching model associated to any* $(G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}), \Phi, \tilde{\mu})$*, such that* $\tilde{\mu} \in \text{STAB}_{\text{EB}}(G, \Phi)$ *of marginals* $\tilde{\mu}_1$ *and* $\tilde{\mu}_2$*, we have that*

$$\begin{cases} \tilde{\mu}_1(i) & = \sum_{j \in \mathcal{E}(i)} \Theta^{\Phi,\tilde{\mu}}_{\text{EB}}[i,j], \text{ for all } i \in \mathcal{V}_1; \\ \tilde{\mu}_2(i) & = \sum_{j \in \mathcal{E}(i)} \Theta^{\Phi,\tilde{\mu}}_{\text{EB}}[i,j], \text{ for all } i \in \mathcal{V}_2. \end{cases} \tag{23}$$

Let us introduce the two following probability distributions,

Definition 3. *For any fixed connected bipartite graph* $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ *and any fixed family of weights* $\alpha \in \mathcal{M}(\mathcal{E})$*, we define the conditional probability measures* $\tilde{\mu}_1^\alpha \in \overline{\mathcal{M}}(\mathcal{V}_1)$ *and* $\tilde{\mu}_2^\alpha \in \overline{\mathcal{M}}(\mathcal{V}_2)$ *as follows: for any* $i \in \mathcal{V}$*,*

$$\tilde{\mu}_1^\alpha(i) \triangleq \frac{\sum_{j \in \mathcal{E}(i)} \alpha_{i,j}}{\sum_{l \in \mathcal{V}_1} \sum_{j \in \mathcal{E}(l)} \alpha_{l,j}}, \, i \in \mathcal{V}_1 \quad \text{and} \quad \tilde{\mu}_2^\alpha(i) \triangleq \frac{\sum_{j \in \mathcal{E}(i)} \alpha_{i,j}}{\sum_{l \in \mathcal{V}_2} \sum_{j \in \mathcal{E}(l)} \alpha_{l,j}}, i \in \mathcal{V}_2.$$

As $(\mathcal{V}_1, \mathcal{V}_2)$ realizes a bipartition of $\mathcal{V}$, we can deduce that

$$\forall i \in \mathcal{V}_1, \; \bar{\mu}^\alpha(i) = \frac{\tilde{\mu}_1^\alpha(i)}{2} \quad \text{and} \quad \forall j \in \mathcal{V}_2, \; \bar{\mu}^\alpha(j) = \frac{\tilde{\mu}_2^\alpha(j)}{2}, \tag{24}$$

and thus

$$\bar{\mu}^\alpha(\mathcal{V}_1) = \bar{\mu}^\alpha(\mathcal{V}_2) = \frac{1}{2} \quad \text{and} \quad \mu^\alpha(\mathcal{V}_1) = \mu^\alpha(\mathcal{V}_2) = \frac{\mu^\alpha(\mathcal{V})}{2}. \tag{25}$$

Corollary 5. *Let* $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ *be a connected bipartite graph. Then, for any admissible matching policy* Φ*, we get that*

$$\overline{\text{STAB}}_{\text{EB}}(G, \Phi) \subset \left\{ \tilde{\mu} \in \overline{\mathcal{M}}(\mathcal{V}_1 \times \mathcal{V}_2) \text{ of marginals } \tilde{\mu}_1^\alpha \text{ and } \tilde{\mu}_2^\alpha : \alpha \in \mathcal{M}(\mathcal{E}) \right\}.$$

Proof. Fix a connected bipartite graph $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$, an admissible policy Φ, and let $\tilde{\mu}$ be a measure of $\overline{\text{STAB}}_{\text{EB}}(G, \Phi)$ of marginals $\tilde{\mu}_1$ and $\tilde{\mu}_2$, in a way that the bipartite matching model associated to $(G, \Phi, \tilde{\mu})$ is stable. Let us also define the family of weights $\alpha^{\Phi,\tilde{\mu}}_{\text{EB}} \in \mathcal{M}(\mathcal{E})$, by

$$\begin{array}{rccc} \alpha^{\Phi,\tilde{\mu}}_{\text{EB}} & : & \mathcal{E} & \longrightarrow & \mathbb{R}_+^* \\ & & (i,j) & \longmapsto & \Theta^{\Phi,\tilde{\mu}}_{\text{EB}}[i,j]. \end{array} \tag{26}$$

From Lemma 3, for all $i \in \mathcal{V}_1$, we have that

$$\tilde{\mu}_1^{\alpha^{\Phi,\tilde{\mu}}_{\text{EB}}}(i) = \frac{\sum_{j \in \mathcal{E}(i)} \Theta^{\Phi,\tilde{\mu}}_{\text{EB}}[i,j]}{\sum_{l \in \mathcal{V}_1} \sum_{j \in \mathcal{E}(l)} \Theta^{\Phi,\tilde{\mu}}_{\text{EB}}[l,j]} = \frac{\tilde{\mu}_1(i)}{\sum_{l \in \mathcal{V}_1} \tilde{\mu}_1(l)} = \tilde{\mu}_1(i).$$

So we have $\tilde{\mu}_1 = \tilde{\mu}_1^{\alpha^{\Phi,\tilde{\mu}}_{\text{EB}}}$, and likewise $\tilde{\mu}_2 = \mu_2^{\alpha^{\Phi,\tilde{\mu}}_{\text{EB}}}$, which concludes the proof. □

Remark 1. *Observe that these results can be easily adapted to a bipartite matching model, as defined in Section 3.3, namely, whenever we assume that* $\tilde{\mu} = \tilde{\mu}_1 \otimes \tilde{\mu}_2$*. In particular, a similar result to Corollary 5 then establishes a characterization of the weights of any stable measure for a bipartite model, associated for example to the matching policy* FCFM*. This provides an alternative proof of implication* (iv) $\Longrightarrow$ (ii) *in Corollary 2.*

8. Proof of Theorem 2

8.1. **Proof using Theorem 1.** We first prove Theorem 2, by using Theorem 1. For this, in Proposition 5 we first show that for any graph $G = (\mathcal{V}, \mathcal{E})$ and any partition $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, the set $\mathcal{N}_{\mathcal{V}_1}(G)$ coincides with the set of weighted measures on an augmented multigraph of G. The latter set is then shown to coincide with $\mathcal{W}_{\prec,\mathcal{V}_2}(G)$.

Proposition 5. *For any connected graph $G = (\mathcal{V}, \mathcal{E})$ and any non-trivial partition $\mathcal{V}_1 \cup \mathcal{V}_2$ of $\mathcal{V}$, we have*

$$\mathcal{N}_{\mathcal{V}_1}(G) = \mathcal{W}(\hat{G}),$$

where $\hat{G} = (\mathcal{V}, \hat{\mathcal{E}})$ is the multigraph obtained from G by adding a self-loop at any element of $\mathcal{V}_2$.

Proof of Proposition 5. From Theorem 1, as $\hat{G}$ is not a bipartite graph by construction, we have that $\mathcal{W}(\hat{G}) = \mathcal{N}(\hat{G})$. So it suffices to show that

$$\mathcal{N}_{\mathcal{V}_1}(G) = \mathcal{N}(\hat{G}). \tag{27}$$

The right inclusion in (27) is immediate: If we let $\mu \in \mathcal{N}(\hat{G})$, and fix an independent set $\mathcal{I}_1 \in \mathbb{I}(\mathcal{V}_1)$, then as $\mathcal{I}_1 \subsetneq \mathcal{V}$ we readily obtain that

$$\mu(\mathcal{I}_1) < \mu(\hat{\mathcal{E}}(\mathcal{I}_1)) = \mu(\mathcal{E}(\mathcal{I}_1)),$$

where the second equality follows from the definition of $\hat{\mathcal{E}}$. Hence, $\mu \in \mathcal{N}_{\mathcal{V}_1}(G)$.

We now turn to the left inclusion. Let $\mu \in \mathcal{N}_{\mathcal{V}_1}(G)$ and fix an independent set $\mathcal{I} \in \mathbb{I}(G)$. Denote in the remainder of this proof, $\mathcal{I}_1 \triangleq \mathcal{I} \cap \mathcal{V}_1$ and $\mathcal{I}_2 \triangleq \mathcal{I} \cap \mathcal{V}_2$. We get that

$$\mu(\hat{\mathcal{E}}(\mathcal{I})) - \mu(\mathcal{I}) = \mu(\hat{\mathcal{E}}(\mathcal{I}) \cap \mathcal{I}_2) + \mu(\hat{\mathcal{E}}(\mathcal{I}) \cap \mathcal{I}_1) + \mu(\hat{\mathcal{E}}(\mathcal{I}) \cap \mathcal{I}^c) - \mu(\mathcal{I}_2) - \mu(\mathcal{I}_1).$$

But by the very definition of $\hat{G}$, we get that $\hat{\mathcal{E}}(\mathcal{I}) \cap \mathcal{I}_2 = \mathcal{I}_2$, $\hat{\mathcal{E}}(\mathcal{I}) \cap \mathcal{I}_1 = \emptyset$, and that $\hat{\mathcal{E}}(\mathcal{I}) \cap \mathcal{I}^c$ is the disjoint union of $\mathcal{E}(\mathcal{I}_1)$ and $\mathcal{E}(\mathcal{I}_2) \cap (\mathcal{E}(\mathcal{I}_1))^c$. Thus we obtain

$$\mu(\hat{\mathcal{E}}(\mathcal{I})) - \mu(\mathcal{I}) = \mu(\hat{\mathcal{E}}(\mathcal{I}) \cap \mathcal{I}^c) - \mu(\mathcal{I}_1) \geq \mu(\mathcal{E}(\mathcal{I}_1)) - \mu(\mathcal{I}_1) > 0,$$

which concludes the proof. □

For any finite set A, for two measures μ, μ' of $\mathcal{M}(A)$, and for a subset $B \subset A$, we denote $\mu \prec_B \mu'$ whenever $\mu(i) < \mu'(i)$ for any $i \in B$, and $\mu(i) = \mu'(i)$ for any $i \in A \setminus B$. For any $\mathcal{V}' \subset \mathcal{V}$, we define the set

$$\mathcal{W}_{\prec,\mathcal{V}'}(G) \triangleq \{\mu \in \mathcal{M}(\mathcal{V}) \,:\, \mu^\alpha \prec_{\mathcal{V}'} \mu \text{ for some } \alpha \in \mathcal{M}(\mathcal{E})\}.$$

We can now prove Theorem 2.

Proof of Theorem 2. From Proposition 5, it is enough to show that

$$\mathcal{W}(\hat{G}) = \mathcal{W}_{\prec,\mathcal{V}_2}(G).$$

Regarding the left inclusion, for any $\mu \in \mathcal{W}(\hat{G})$, letting $\hat{\alpha} \in \mathcal{M}(\hat{\mathcal{E}})$ be such that $\mu = \mu^{\hat{\alpha}}$, and $\alpha \in \mathcal{M}(\mathcal{E})$ be the restriction of $\hat{\alpha}$ to $\mathcal{E}$ we get that $\mu(i) = \mu^{\hat{\alpha}}(i) = \mu^\alpha(i)$ for all $i \in \mathcal{V}_1$, while for any $i \in \mathcal{V}_2$ we get

$$\mu(i) = \mu^{\hat{\alpha}}(i) = \hat{\alpha}_{i,i} + \sum_{j \neq i:\, j \in \mathcal{E}(i)} \hat{\alpha}_{i,j} > \sum_{j \neq i:\, j \in \mathcal{E}(i)} \alpha_{i,j} = \mu^\alpha(i),$$

hence $\mu \in \mathcal{W}_{\prec,\mathcal{V}_2}(G)$. As for the converse, let $\mu \in \mathcal{W}_{\prec,\mathcal{V}_2}(G)$, and $\alpha \in \mathcal{M}(\mathcal{E})$ be such that $\mu^\alpha \prec_{\mathcal{V}_2} \mu$. Then, setting $\hat{\alpha}_{i,j} = \alpha_{i,j}$ for all $i \neq j$ and $\hat{\alpha}_{i,i} = \mu(i) - \mu^\alpha(i)$ for all $i \in \mathcal{V}_2$, we easily retrieve that $\mu = \mu^{\hat{\alpha}}$. Hence $\mu \in \mathcal{W}(\hat{G})$, which completes the proof. □

8.2. **Bipartite case: proof using flow theory.** Interestingly enough, in the particular case where G is bipartite of bipartition $\mathcal{V}_1 \cup \mathcal{V}_2$, Theorem 2 is in fact reminiscent of a simple flow argument. Hereafter, we give an independent proof of this result in the bipartite case, using flow network theory, see [19]. We start with the following result,

Proposition 6. *For any connected bipartite graph* $G = (\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ *and any measure* $\mu \in \mathcal{N}_{\mathcal{V}_1}(G)$*, there exists a* $s-t$ *flow graph* $\vec{G} = \left(\vec{\mathcal{V}}, \vec{\mathcal{E}}\right)$ *associated to* G*, and a* $s-t$ *flow* $\overline{f}$ *of* $\vec{G}$*, such that*

$$\sum_{i \in \mathcal{V}_1} \overline{f}(s,i) = \mu(\mathcal{V}_1). \tag{28}$$

Proof of Proposition 6. Fix $G = (\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$, and a measure $\mu \in \mathcal{N}_{\mathcal{V}_1}(G)$. We can then easily define a measure $\delta \in \mathcal{M}(\mathcal{V}_2)$ such that

$$\begin{cases} \forall \mathcal{I}_1 \in \mathbb{I}(\mathcal{V}_1),\ \mu(\mathcal{I}_1) < \delta(\mathcal{E}(\mathcal{I}_1)); \\ \forall j \in \mathcal{V}_2,\ \delta(j) < \mu(j). \end{cases} \tag{29}$$

We then construct a $s-t$ flow graph $\vec{G} = \left(\vec{\mathcal{V}}, \vec{\mathcal{E}}\right)$ associated to G, as follows:

- We set a source s, and add the edges between s and each vertex of $\mathcal{V}_1$;
- We set a well t, and add the edges between each vertex of $\mathcal{V}_2$ and t;
- For all $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$, we orient the edges from s to i, from j to t and from i to j, for all edge $(i,j) \in \mathcal{E}$;
- We associate a capacity c to each edge as follows: For all $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$, we set $c(s,i) \triangleq \mu(i)$, $c(j,t) \triangleq \delta(j)$ and $c(i,j) \triangleq +\infty$, for all edge $(i,j) \in \mathcal{E}$, see Figure 3.

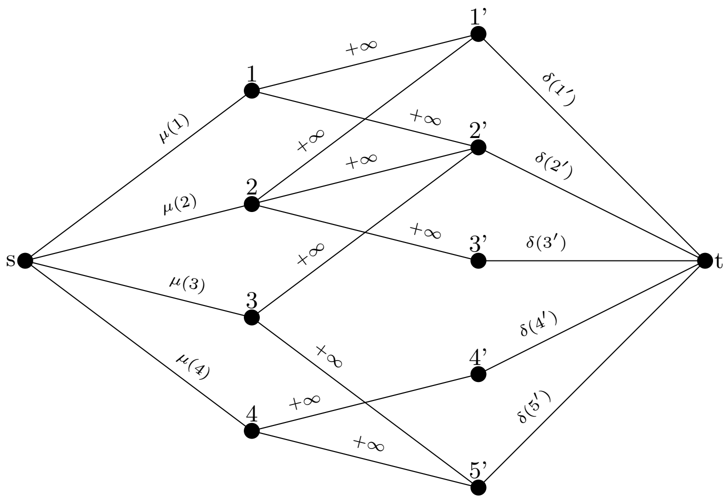

FIGURE 3. A flow graph $\vec{G}$ associated to a connected bipartite $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$, with $\mathcal{V}_1 = \{1,2,3,4\}$ and $\mathcal{V}_2 = \{1',2',3',4',5'\}$.

It is then easily checked by hand that we have

$$\min_{(\mathscr{S},\mathscr{T}) \in \mathfrak{C}_{s,t}\left(\vec{G}\right)} c(\mathscr{S},\mathscr{T}) = c\left(\{s\}\,,\, \vec{\mathcal{V}} \backslash \{s\}\right) = \mu(\mathcal{V}_1), \tag{30}$$

where $\mathfrak{C}_{s,t}\left(\vec{G}\right)$ is the set of all $s-t$ cuts of $\vec{G}$. The max-flow/min-cut theorem (see [19]) concludes the proof. $\square$

We are then able to retrieve the proof of Theorem 2 in the bipartite case.

Proof of Theorem 2 for G bipartite. Fix $G = (\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$. We will show the equality between the sets $\mathcal{N}_{\mathcal{V}_1}(G)$ and

$$\mathcal{W}^{\geq 0}_{\prec,\mathcal{V}_2}(G) \triangleq \left\{ \mu \in \mathcal{M}(\mathcal{V}) : \exists \alpha \in \mathcal{M}^{\geq 0}(\mathcal{E}) \text{ s.t. } \left\{ \begin{array}{l} \forall i \in \mathcal{V}_1,\ \mu^\alpha(i) = \mu(i) \\ \forall j \in \mathcal{V}_2,\ \mu^\alpha(j) < \mu(j) \end{array} \right. \right\},$$

which implies in turn the equality between $\mathcal{N}_{\mathcal{V}_1}(G)$ and $\mathcal{W}_{\prec,\mathcal{V}_2}(G)$, as $\mathcal{W}^{\geq 0}_{\prec,\mathcal{V}_2}(G)$ is the interior of $\mathcal{W}_{\prec,\mathcal{V}_2}(G)$ and $\mathcal{N}_{\mathcal{V}_1}(G)$ is an open set.

In order to prove the first inclusion, let us fix a measure $\mu \in \mathcal{N}_{\mathcal{V}_1}(G)$ and define the non-negative measure $\alpha^{\overline{f}} \in \mathcal{M}^{\geq 0}(\mathcal{E})$, by

$$\forall (i,j) \in \mathcal{E},\ \alpha^{\overline{f}}_{i,j} \triangleq \overline{f}(i,j),$$

where $\overline{f}$ is the flow of $\vec{G}$, given by Proposition 6. On the one hand, by the flow conservation condition on $\vec{G}$ and from (29), we have that

$$\forall j \in \mathcal{V}_2,\ \mu^{\alpha^{\overline{f}}}(j) = \sum_{i \in \mathcal{V}_1} \alpha^{\overline{f}}_{i,j} = \sum_{i \in \mathcal{V}_1} \overline{f}(i,j) = \delta(j) < \mu(j).$$

On the other hand, the maximality of the flow $\overline{f}$ in Proposition 6 and the capacity constraint implies that

$$\forall i \in \mathcal{V}_1,\ \overline{f}(s,i) = \mu(i),$$

and it follows from the flow conservation condition, that

$$\forall i \in \mathcal{V}_1, \mu^{\alpha^{\overline{f}}}(i) = \sum_{j \in \mathcal{V}_2} \alpha^{\overline{f}}_{i,j} = \sum_{j \in \mathcal{V}_2} \overline{f}(i,j) = \overline{f}(s,i) = \mu(i).$$

implying that $\mu \in \mathcal{W}^{\geq 0}_{\prec,\mathcal{V}_2}(G)$.

Now, to prove the converse inclusion we consider a measure $\mu \in \mathcal{W}^{\geq 0}_{\prec,\mathcal{V}_2}(G)$. Then there exists a non-negative measure $\alpha \in \mathcal{M}^{\geq 0}(\mathcal{E})$ such that

$$\left\{ \begin{array}{l} \forall i \in \mathcal{V}_1,\ \mu^\alpha(i) = \mu(i) \\ \forall j \in \mathcal{V}_2,\ \mu^\alpha(j) < \mu(j) \end{array} \right. . \tag{31}$$

Let us consider an independent set $\mathcal{I}_1 \in \mathbb{I}(\mathcal{V}_1)$. Then, by (31) and since $\alpha \in \mathcal{M}^{\geq 0}(\mathcal{E})$, we have that

$$\mu(\mathcal{I}_1) = \mu^\alpha(\mathcal{I}_1) = \sum_{j \in \mathcal{E}(\mathcal{I}_1)} \sum_{i \in \mathcal{I}_1} \alpha_{i,j} \leq \sum_{j \in \mathcal{E}(\mathcal{I}_1)} \sum_{i \in \mathcal{V}_1} \alpha_{i,j} = \mu^\alpha(\mathcal{E}(\mathcal{I}_1)) < \mu(\mathcal{E}(\mathcal{I}_1)),$$

which concludes the proof. □

9. About uniqueness of the matching rates

9.1. **Insensitivity of the matching rates.** In Lemma 2 (resp. Lemma 3), we have proven that the arrival rates of stable general (resp. bipartite) stochastic matching models are given as explicit functions of the matching rates of the various edges - see (20) (resp. (23)). Specifically, if in the bipartite case we define for any $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2) \in \overline{\mathcal{M}}(\mathcal{V}_1 \times \mathcal{V}_2)$, the probability measure $\bar{\mu}$ on $\overline{\mathcal{M}}(\mathcal{V})$, by

$$\bar{\mu}(i) = \frac{1}{2}\tilde{\mu}_1(i)\mathbf{1}_{\{i \in \mathcal{V}_1\}} + \frac{1}{2}\tilde{\mu}_2(i)\mathbf{1}_{\{i \in \mathcal{V}_2\}}, \quad i \in \mathcal{V},$$

then both systems of equations (20) and (23) can be rewritten under the generic form

$$\bar{\mu}(i) = \sum_{j \in \mathcal{E}(i)} \alpha_{i,j}, \quad i \in \mathcal{V}, \tag{S}$$

where the weights $\alpha_{i,j}$, $(i,j) \in \mathcal{E}$ represent the matching rates in the various cases, or twice the matching rate for $i=j$. These weights depend on the matching policy, and are not unique in general. However, for various (multi)graph geometries, they are uniquely defined and thereby, *independent of the matching policy*, as we will show hereafter.

In what follows, for a rooted tree $G=(\mathcal{V},\mathcal{E},r)$ of root r having at least two nodes, we denote by $d \triangleq d(G)$, the depth of G, that is, the maximal distance from the root. For any $\ell \in [\![1;d]\!]$, we let n_ℓ be the number of elements of generation ℓ, and label by $\ell_1,...,\ell_{n_\ell}$, the nodes of generation ℓ in an arbitrary manner. In that way, nodes $d_1,...,d_{n_d}$ are the leaves of G that are at maximal distance from the root. For any $\ell \in [\![1;d]\!]$ and $i \in [\![1;n_\ell]\!]$, we denote by $f(\ell_i)$, the father of node ℓ_i. Notice that by construction, we have $f(1_i)=r$ for all $i \in [\![1;n_1]\!]$. Last, for any $\ell \in [\![1;d]\!]$ and $i \in [\![1;n_\ell]\!]$, we denote by $\mathscr{S}(\ell_i)$ and $\mathscr{D}(\ell_i)$, the sets of sons and of descendants of ℓ_i, respectively. Observe that the above sets are empty if and only if ℓ_i is a leaf.

Lemma 4. *Let $G=(\mathcal{V},\mathcal{E},r)$ be a rooted tree having at least two nodes, and let $\bar{\mu} \in \overline{\mathcal{M}}(\mathcal{V})$. Then, the system of equations*

$$\bar{\mu}(i) = \sum_{j \in \mathcal{E}(i)} \alpha_{i,j}, \quad i \in \mathcal{V} \setminus \{r\} \tag{S*}$$

admits the following unique solution in $\overline{\mathcal{W}}(G)$ defined by (5),

$$\alpha_{\ell_i,f(\ell_i)} = \bar{\mu}(\ell_i) + \sum_{k_j \in \mathscr{D}(\ell_i)} (-1)^{k-\ell} \bar{\mu}(k_j), \quad \ell \in [\![1;d]\!], \, i \in [\![1;n_\ell]\!], \tag{32}$$

where sums over empty sets are understood as null.

Proof. If ℓ_i is a leave of G, it is immediate that we have

$$\alpha_{\ell_i,f(\ell_i)} = \bar{\mu}(\ell_i).$$

The result then follows by induction, observing that (S*) is equivalent, for a graph G that is a tree, to

$$\alpha_{\ell_i,f(\ell_i)} = \bar{\mu}(\ell_i) - \sum_{(\ell+1)_j \in \mathscr{S}(\ell_i)} \alpha_{\ell_i,(\ell+1)_j}, \quad \ell \in [\![1;d]\!], \, i \in [\![1;n_\ell]\!].$$

□

From this, we first deduce the following result,

Proposition 7. *Let $G=(\mathcal{V},\mathcal{E})$ be a tree having at least two nodes, and let $\bar{\mu} \in \overline{\mathcal{M}}(\mathcal{V})$. Then, the system (S) admits a unique solution in $\overline{\mathcal{W}}(G)$ if, and only if $\bar{\mu}(\mathcal{V}_1) = \bar{\mu}(\mathcal{V}_2) = \frac{1}{2}$, where $\mathcal{V}_1 \cup \mathcal{V}_2$ is the bipartition of $\mathcal{V}$. Otherwise there is no solution to (S).*

Proof. Fix a root r, and let $\mathcal{V}_1$ and $\mathcal{V}_2$ be respectively given as the sets of nodes at even and odd generations, respectively. For such G, the system (S) is equivalent to (S*) together with the relation

$$\bar{\mu}(r) = \sum_{i=1}^{n_1} \alpha_{r,1_i}.$$

By the connectedness assumption, all nodes of G are descendants of the sons of r. Therefore, from (32), we have that

$$\begin{aligned}\bar{\mu}(r) &= \sum_{i=1}^{n_1} \left(\bar{\mu}(1_i) + \sum_{k_j \in \mathscr{D}(1_i)} (-1)^{k-1} \bar{\mu}(k_j) \right) \\ &= \sum_{\substack{k=1;\\ k \text{ odd}}}^{d} \sum_{i=1}^{n_k} \bar{\mu}(k_i) - \sum_{\substack{k=1;\\ k \text{ even}}}^{d} \sum_{i=1}^{n_k} \bar{\mu}(k_i), \end{aligned} \tag{33}$$

which amounts to $\bar{\mu}(\mathcal{V}_1) = \bar{\mu}(\mathcal{V}_2)$. Therefore, the system (S) admits the unique solution given by (32) if $\bar{\mu}(\mathcal{V}_1) = \bar{\mu}(\mathcal{V}_2)$, and no solution otherwise. □

Proposition 8. *Let $G = (\mathcal{V}, \mathcal{E})$ be a cycle, and let $\bar{\mu} \in \overline{\mathcal{M}}(\mathcal{V})$. Then,*

1. *If G is of odd length, there exists a unique solution to* (S)*.*
2. *If G is of even length, and if we denote by $\mathcal{V}_1 \cup \mathcal{V}_2$ the bipartition of $\mathcal{V}$, then there are infinitely many solutions if $\bar{\mu}(\mathcal{V}_1) = \bar{\mu}(\mathcal{V}_2) = \frac{1}{2}$, and no solution otherwise.*

Proof. Let n be the size of the cycle, and index the nodes of G by $1, ..., n$. The linear system (S) of unknown α, identified with the column matrix ${}^t\big(\alpha_{1,2}\, \alpha_{2,3}\, \cdots\, \alpha_{n-1,n}\, \alpha_{n,1}\big)$, can be written under the form $A\alpha = \bar{\mu}$, where $\bar{\mu}$ is identified with the column vector ${}^t\big(\bar{\mu}(1)\, \bar{\mu}(2)\, \cdots\, \bar{\mu}(n)\big)$ and

$$A = \begin{pmatrix} 1 & 0 & \cdots & & & 0 & 1 \\ 1 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 \\ & & & & & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ 0 & 0 & \dots & & 0 & 1 & 1 \end{pmatrix}.$$

An immediate computation shows that $\det A = 1 + (-1)^{n+1}$. Thus the system (S) admits the unique solution $A^{-1}\bar{\mu}$ if n is odd. If n is even, then it is easily seen that A has rank $n-1$, and that the system is compatible if and only if

$$\sum_{\substack{i=1;\\ i \text{ odd}}}^{n} \bar{\mu}(i) = \sum_{\substack{i=1;\\ i \text{ even}}}^{n} \bar{\mu}(i) = \frac{1}{2}.$$

In that case, there are infinitely many solutions. □

We deduce the following result:

Proposition 9. *Let $G = (\mathcal{V}, \mathcal{E})$ be a graph consisting of a tree having at least two nodes and of an additional edge, and let $\bar{\mu} \in \overline{\mathcal{M}}(\mathcal{V})$. Then,*

1. *If G is a non-bipartite graph (i.e. the additional edge forms a cycle of odd length), then the system (S) admits a unique solution.*
2. *If G is a bipartite graph (i.e. the additional edge forms a cycle of even length), the system (S) admits infinitely many solutions in $\overline{\mathcal{W}}(G)$ if and only if $\bar{\mu}(\mathcal{V}_1) = \bar{\mu}(\mathcal{V}_2) = \frac{1}{2}$. Otherwise, there is no solution to (S).*
3. *If the additional edge is a self-loop, then the system (S) admits a unique solution.*

Proof. Adding an edge that is not a self-loop to the tree generates a cycle $\mathscr{C}$. If the cycle is of even size, then the resulting graph is bipartite, while if the cycle is of odd size we obtain a non-bipartite graph.

1. First suppose that the resulting graph G is non-bipartite, and is not reduced to an odd cycle (otherwise, assertion 1. of Proposition 8 applies). Then G can be seen as the odd cycle $\mathscr{C}$, to which are appended one or several disconnected rooted trees. More precisely, denote by N the (odd) number of nodes of $\mathscr{C}$, by $r^1, ..., r^p$ the nodes of $\mathscr{C}$ having a degree strictly larger than two, and by $s^1, ..., s^{N-p}$ the nodes of $\mathscr{C}$ of degree 2, if any. Then, for any $l \in [\![1;p]\!]$, to the node r^l is appended a rooted tree $\mathscr{T}^l$ of root r^l. Lemma 4 yields that the weights associated to all the edges of $\mathscr{T}^l$ are unique. In particular, denoting by a^l and b^l the two neighbors of r^l in $\mathscr{C}$, (32) implies that

$$\bar{\mu}(r^l) = \alpha_{r^l,a^l} + \alpha_{r^l,b^l} + \sum_{\substack{k=1;\\ k \text{ odd}}}^{d^l} \sum_{i=1}^{n_k^l} \bar{\mu}(k_i^l) - \sum_{\substack{k=1;\\ k \text{ even}}}^{d^l} \sum_{i=1}^{n_k^l} \bar{\mu}(k_i^l), \tag{34}$$

using the same notations as in (33) for $\mathscr{T}^l$. This is true for any $l \in [\![1;p]\!]$, so the restriction to $\mathscr{C}$ of any solution α to (S) solves in particular the system

$$\breve{\mu}(\ell) = \sum_{j \in \mathcal{E}(\ell) \cap \mathscr{C}} \alpha_{\ell,j}, \quad \ell \in \mathscr{C}, \tag{S**}$$

where

$$\begin{cases} \breve{\mu}(r^l) & = \bar{\mu}(r^l) - \sum\limits_{\substack{k=1;\\ k \text{ odd}}}^{d^l} \sum\limits_{i=1}^{n_k^l} \bar{\mu}(k_i^l) + \sum\limits_{\substack{k=1;\\ k \text{ even}}}^{d^l} \sum\limits_{i=1}^{n_k^l} \bar{\mu}(k_i^l), \quad l \in [\![1,p]\!], \\ \breve{\mu}(s^l) & = \bar{\mu}(s^l), \quad l \in [\![1; N-p]\!]. \end{cases}$$

From 1. of Proposition 8, after normalizing $\breve{\mu}$, the solution to (S**) is unique, and so is the solution to (S) in view of the uniqueness of the solution to (S*) on each tree $\mathscr{T}^l$, $l \in [\![1;p]\!]$, from Lemma 4.
2. Suppose that the resulting graph G is bipartite and is not reduced to an even cycle (otherwise, assertion 2. of Proposition 8 applies). Applying the same construction as in case 1., we obtain again (34) for all $l \in [\![1;p]\!]$. Assertion 2. of Proposition 8 shows that the resulting system (S**) on $\mathscr{C}$ admits no solution unless $\breve{\mu}(\mathcal{V}_1) = \breve{\mu}(\mathcal{V}_2)$. But this is equivalent to $\bar{\mu}(\mathcal{V}_1) = \bar{\mu}(\mathcal{V}_2)$, as in each rooted tree $\mathscr{T}^l$ of G, the nodes of the even generations belong to the same subset of the bipartition as r^l. So, under that condition, (S**) and thereby (S), admit infinitely many solutions.
3. If the additional edge is a self-loop on node r, then G can be seen as a rooted tree of root r. Then, it is immediate in view of Lemma 4 that (S) admits a unique solution, that from (32), is given by

$$\begin{cases} \alpha_{\ell_i,f(\ell_i)} & = \bar{\mu}(\ell_i) + \sum\limits_{k_j \in \mathscr{D}(\ell_i)} (-1)^{k-\ell} \bar{\mu}(k_j), \quad \ell \in [\![1;d]\!], i \in [\![1;n_\ell]\!], \\ \alpha_{r,r} & = \bar{\mu}(r) - \sum\limits_{\substack{k=1;\\ k \text{ odd}}}^{d} \sum\limits_{i=1}^{n_k} \bar{\mu}(k_i) + \sum\limits_{\substack{k=1;\\ k \text{ even}}}^{d} \sum\limits_{i=1}^{n_k} \bar{\mu}(k_i), \end{cases}$$

which completes the proof.

□

Let us now come back to the representation of solutions of (S) as asymptotic matching rates for corresponding matching models.

Definition 4. *Let $G = (\mathcal{V}, \mathcal{E})$ be a multigraph (resp. a bipartite graph). We say that the matching rates are* policy-invariant *for G, if for any $\mu \in \mathcal{N}(G)$ (resp.*

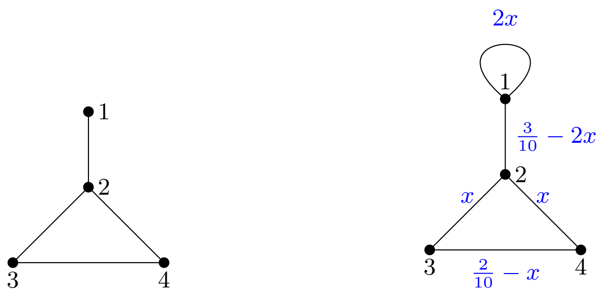

FIGURE 4. Graphs G_1 *(left)* and G_2 *(right)*.

any $\mu \in \mathcal{N}_2(G)$*), the asymptotic matching rates* $\Theta^{\Phi,\bar{\mu}}$ *(resp.* $\Theta^{\Phi,\tilde{\mu}}_{\text{EB}}$*) in the matching model associated to* $(G, \Phi, \bar{\mu})$ *(resp.* $(G, \Phi, \tilde{\mu})$*) do not depend on* Φ *as long as* $\bar{\mu} \in \overline{\text{STAB}}(G, \Phi)$ *(resp.* $\tilde{\mu} \in \overline{\text{STAB}}_{\text{EB}}(G, \Phi)$*).*

Recalling equations (20) and (23), and gathering Propositions 7, 8 and 9, we readily obtain the following result.

Theorem 3. *Matching rates are policy-invariant for* G *in the following cases:*

(1) G *is an odd cycle,*
(2) G *is a tree,*
(3) G *is a tree with an additional edge forming an odd cycle,*
(4) G *is a tree with an additional self-loop.*

Furthermore, the matching rates are given by the unique solution of the system (S).

Observe that in case *(2)* above, since we consider a bipartite matching model, we assume implicitly that the bipartition satisfies $\bar{\mu}(\mathcal{V}_1) = \bar{\mu}(\mathcal{V}_2) = \frac{1}{2}$ (otherwise, there are obviously no stable policies, so that regardless of the policy chosen, the matching rates are not well-defined).

9.2. **Examples.**

9.2.1. *Case with unicity.* The graph $G_1 = (\mathcal{V}, \mathcal{E})$ represented on the left of Figure 4 falls within case (3) above. For any $\bar{\mu} \in \overline{\mathcal{N}}(G_1)$ and any Φ such that the matching model $(G_1, \Phi, \bar{\mu})$ is stable, the matching rates on the various edges are given by $\alpha = A^{-1}\bar{\mu}$, for

$$\bar{\mu} = \begin{pmatrix} \bar{\mu}(1) \\ \bar{\mu}(2) \\ \bar{\mu}(3) \\ \bar{\mu}(4) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \alpha := \begin{pmatrix} \alpha_{1,2} \\ \alpha_{2,3} \\ \alpha_{2,4} \\ \alpha_{3,4} \end{pmatrix}.$$

9.2.2. *Case without unicity.* The graph $G_2 = (\mathcal{V}, \mathcal{E})$ given on the right of Figure 4 does not satisfy the assumptions of Theorem 3, and the corresponding system (S) does not have a unique solution. It is immediate that the measure

$$\bar{\mu} = {}^t(3/10 \ 3/10 \ 2/10 \ 2/10)$$

is an element of $\overline{\mathcal{N}}(G_2)$, however we get that $A\alpha^x = \bar{\mu}$, for

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \alpha^x := \begin{pmatrix} \alpha^x_{1,1} \\ \alpha^x_{1,2} \\ \alpha^x_{2,3} \\ \alpha^x_{2,4} \\ \alpha^x_{3,4} \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{3}{10} - 2x \\ x \\ x \\ \frac{2}{10} - x \end{pmatrix},$$

for any $x \in \left(0\,;\frac{3}{20}\right)$ (see Figure 4). For instance, in our simulations the FCFM policy yields approximate matching rates of $\alpha^{0.098}$.

9.3. **About fairness of the matching rates.** One possible way to define a notion of *fairness* of a stable measure $\bar{\mu}$, with respect to a given policy Φ, is to say that $\bar{\mu}$ is *fair* (for the matching rates) if $\Theta^{\Phi,\bar{\mu}}$ is constant on $\mathcal{E}$, i.e., $\Theta^{\Phi,\bar{\mu}}[i,j]$ takes the same value for all edges $(i,j) \in \mathcal{E}$. In other words, it means that asymptotically, all couples of compatible items leave the system together as often as each other.

Let us assume that $\bar{\mu} \in \overline{\mathcal{M}}(\mathcal{V})$ is fair with respect to a given policy, with $\Theta^{\Phi,\bar{\mu}}$ equal to a constant c. Then, as a consequence of Lemma 2, for any $i \in \mathcal{E}$, we have

$$\bar{\mu}(i) = \sum_{j\in\mathcal{V}} (1 + \mathbf{1}_{\{i=j\}})\; c = \Big(|\mathcal{E}(i)| + \mathbf{1}_{\{(i,i)\in\mathcal{E}\}}\Big)\, c.$$

If follows that $\bar{\mu}$ is equal to the weighted measure $\bar{\mu}^{\alpha^*}$ defined by the family of weights

$$\begin{array}{cccc} \alpha^* & : & \mathcal{E} & \to & \mathbb{R}_+^* \\ & & (i,j) & \mapsto & 1 + \mathbf{1}_{\{i=j\}}. \end{array}$$

However, the measure $\bar{\mu}^{\alpha^*}$ may not be fair (in which case there exists no fair measure). Indeed, if $\bar{\mu}^{\alpha^*}$ also coincides with $\bar{\mu}^{\alpha}$, for some α non-proportional to α^*, the matching rates might as well be given by the family α, or by yet another family of weights. We refer to the above discussion for the graph G_2 of Figure 4: The family α^* corresponds to $\alpha^{1/10}$, but there are other families of weights generating the same measure, so that the values of the matching rates cannot be directly predicted.

Nevertheless, observe that if α^* is the only family of weights (up to a multiplicative constant) that generates $\bar{\mu}^{\alpha^*}$, then $\bar{\mu}^{\alpha^*}$ is indeed fair for any policy Φ such that $\bar{\mu}^{\alpha^*} \in \overline{\text{STAB}}(G,\Phi)$. Thanks to Theorem 3, we deduce that the weighted measure $\bar{\mu}^{\alpha^*}$ is the one and only one to be fair in the different contexts specified in Theorem 3, for any policy Φ such that $\bar{\mu}^{\alpha^*} \in \overline{\text{STAB}}(G,\Phi)$.

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