Approximation rate in Wasserstein distance of probability measures on the real line by deterministic empirical measures

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Abstract

We are interested in the approximation in Wasserstein distance with index $\rho \geq 1$ of a probability measure μ on the real line with finite moment of order ρ by the empirical measure of N deterministic points. The minimal error converges to 0 as $N \to +\infty$ and we try to characterize the order associated with this convergence. In [17], Xu and Berger show that, apart when μ is a Dirac mass and the error vanishes, the order is not larger than 1 and give a sufficient condition for the order to be equal to this threshold 1 in terms of the density of the absolutely continuous with respect to the Lebesgue measure part of μ. They also prove that the order is not smaller than $1/\rho$ when the support of μ is bounded and not larger when the support is not an interval. We complement these results by checking that for the order to lie in the interval $(1/\rho, 1)$, the support has to be bounded and by stating a necessary and sufficient condition in terms of the tails of μ for the order to be equal to some given value in the interval $(0, 1/\rho)$, thus precising the sufficient condition in terms of moments given in [17]. We also give a necessary condition for the order to be equal to the boundary value $1/\rho$. In view of practical application, we emphasize that in the proof of each result about the order of convergence of the minimal error, we exhibit a choice of points explicit in terms of the quantile function of μ which exhibits the same order of convergence.

Keywords: deterministic empirical measures, Wasserstein distance, rate of convergence.

AMS Subject Classification (2010): *49Q22, 60-08*

Introduction

Let $\rho \geq 1$ and μ be a probability measure on the real line with finite moment of order ρ. We are interested in the rate of convergence to 0 in terms of $N \in \mathbb{N}^*$ of

$$e_N(\mu,\rho) := \inf\left\{\mathcal{W}_\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \mu\right) : -\infty < x_1 \leq x_2 \leq \cdots \leq x_N < +\infty\right\}, \tag{0.1}$$

where $\mathcal{W}_\rho$ denotes the Wasserstein distance with index ρ. The Hoeffding-Fréchet or comonotone coupling between two probability measures on the real line is optimal for $\mathcal{W}_\rho$ so that when $-\infty < x_1 \leq x_2 \leq \cdots \leq x_N < +\infty$,

$$\mathcal{W}_\rho^\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \mu\right) = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|x_i - F^{-1}(u)\right|^\rho du, \tag{0.2}$$

where F^{-1} denotes the quantile function of μ defined by $F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}$ for $u \in (0,1)$ with $F(x) = \mu((-\infty, x])$ for $x \in \mathbb{R}$. The motivation for our study is the approximation of the probability measure μ by finitely supported probability measures. Examples of application are provided by the optimal initialization

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of systems of particles with mean-field interaction [14, 2], where, to preserve the mean-field feature, it is important to get N points with equal weight $1/N$ (of course, nothing prevents several of these points to be equal) and by the numerical analysis of restricted Monte Carlo methods which may only use random bits instead of random numbers [11]. The optimal approximation in the quadratic case $\rho = 2$ has been shown by Baker [1] to preserve the convex order. This is not the case for the optimal quantization [12] which is obtained by also optimizing over the weights and considering :

$$\inf \left\{ \mathcal{W}_\rho \left(\sum_{i=1}^N p_i \delta_{x_i}, \mu \right) : -\infty < x_1 \leq x_2 \leq \cdots \leq x_N < +\infty,\ (p_1, \cdots, p_N) \in [0,1]^N,\ \sum_{i=1}^N p_i = 1 \right\}.$$

The optimal quantization was introduced in signal processing [4] but has turned out since to be useful in scientific computing [16]. Both the minimization problem with respect to the locations $x_1 \leq x_2 \leq \cdots \leq x_N$ under prescribed but non necessarily uniform weights $(p_1, \cdots, p_N)$ and with respect to the weights $(p_1, \cdots, p_N)$ under prescribed locations $x_1 \leq x_2 \leq \cdots \leq x_N$ have been studied by Xu and Berger [17]. The case when the weights are not prescribed but only satisfy a size constraint has been studied in [5] while [15] addresses the minimization with respect to the locations under prescribed uniform weights when the Wasserstein distance is replaced by the energy distance.

According to Corollary 5.12 [17], the fact that $\int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty$ ensures that $e_N(\mu, \rho)$ goes to 0 as $N \to \infty$. The main purpose of the paper is to study the rate at which this convergence occurs. In particular, we would like to give necessary conditions on μ, which, when possible, are also sufficient, to ensure convergence at a rate $N^{-\alpha}$ with $\alpha > 0$ called the order of convergence.

One of course has

$$e_N(\mu, \rho) \leq \mathcal{W}_\rho(\mu_N, \mu) \text{ and } e_N(\mu, \rho) \leq \mathbb{E}^{1/\rho}\left[\mathcal{W}_\rho^\rho(\mu_N, \mu)\right] \text{ where } \mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$$

is the usual empirical measure of random variables $(X_i)_{i \geq 1}$ i.i.d. according to μ. In the one-dimensional setting of the present paper, the convergence rate of $\mathcal{W}_\rho(\mu_N, \mu)$ has been studied in [8] for $\rho = 1$ and in [9] in the quadratic case $\rho = 2$, the one of $\mathbb{E}^{1/\rho}\left[\mathcal{W}_\rho^\rho(\mu_N, \mu)\right]$ for $\rho \geq 1$ in the book [6] by Bobkov and Ledoux. In general dimension, estimations of $\mathbb{E}^{1/\rho}\left[\mathcal{W}_\rho^\rho(\mu_N, \mu)\right]$ and concentration inequalities for $\mathcal{W}_\rho^\rho(\mu_N, \mu)$ are given in [10]. In the random case, the largest possible order of convergence (apart from the case when μ is a Dirac mass and the error vanishes) is $\alpha = 1/2$, which matches the rate of convergence in the standard strong law of large numbers given by the central limit theorem under square integrability.

The rate of convergence of $e_N(\mu, \rho)$ has already been addressed by Xu and Berger [17] in the one-dimensional setting of the present paper, by Chevallier [7] in higher finite dimension and by [11] along the subsequence $(N = 2^n)_{n \in \mathbb{N}}$ in the quadratic case $\rho = 2$ when μ is a Gaussian measure on an Hilbert space or the law of the solution to a scalar autonomous stochastic differential equation. In particular, according to Theorem 5.21 (ii) [17], when the support of μ is bounded, then $\sup_{N \geq 1} N^{\frac{1}{\rho}} e_N(\mu, \rho) < +\infty$, while Remark 5.22 (ii) [17] ensures that $\limsup_{N \to +\infty} N^{\frac{1}{\rho}} e_N(\mu, \rho) > 0$ when F^{-1} is discontinuous. According to Theorem 5.20 [17], apart when μ is a Dirac mass and $e_N(\mu, \rho)$ vanishes for all $N, \rho \geq 1$, $\limsup_{N \to \infty} N e_N(\mu, \rho) > 0$ so that the order of convergence cannot exceed 1 . It is equal to 1 when the density f of the absolutely continuous with respect to the Lebesgue measure part of μ is dx a.e. positive on $\{x \in \mathbb{R} : 0 < F(x) < 1\}$ (or equivalently F^{-1} is absolutely continuous), since Theorem 5.15 [17] then ensures that

$$\lim_{N \to +\infty} N e_N(\mu, \rho) = \frac{1}{2(\rho+1)^{1/\rho}} \left(\int_{\mathbb{R}} \frac{\mathbf{1}_{\{0 < F(x) < 1\}}}{f^{\rho-1}(x)}\, dx \right)^{1/\rho},$$

where, by Theorem 2.4 [3], the right-hand side is not smaller than $\liminf_{N \to +\infty} N e_N(\mu, \rho)$ even without the positivity assumption on the density. In [7], Chevallier addresses the multidimensional setting and proves in Theorem 3 that for a probability measure μ on $\mathbb{R}^d$ with support bounded by r, there exist points $x_1, \ldots, x_N \in \mathbb{R}^d$ such that $\frac{1}{4r}\mathcal{W}_\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \mu\right) \leq f_{\rho,d}(N)$ where $f_{\rho,d}(N)$ is respectively equal to $\left(\frac{d}{d-\rho}\right)^{\frac{1}{\rho}} N^{-\frac{1}{d}}$, $\left(\frac{1+\ln N}{N}\right)^{\frac{1}{d}}$, and $\zeta(p/d) N^{-\frac{1}{\rho}}$ with ζ denoting the zeta Riemann function when $\rho < d$, $\rho = d$ and $\rho > d$.

The case when the support of μ is not bounded is also considered by Xu and Berger [17] in the one-dimensional setting of the present paper and by [7] in the multidimensional setting. In Corollary 1 [7], Chevallier proves that $\lim_{N\to\infty}(f_{\rho,d}(N))^{-\alpha\rho}\mathcal{W}_\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i},\mu\right)=0$ when $\int_{\mathbb{R}^d}|x|^{\frac{\rho}{1-\alpha\rho}}\mu(dx)<+\infty$ for some $\alpha\in(0,1/\rho)$. This generalizes the one-dimensional statement of Theorem 5.21 (i) [17] : under the same moment condition, $\lim_{N\to\infty}N^\alpha e_N(\mu,\rho)=0$.

In the main contribution of the present paper, Theorem 2.2, we refine this result by stating the following necessary and sufficient condition

$$\forall\alpha\in(0,1/\rho)\,,\;\lim_{x\to+\infty}x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)=0\Leftrightarrow\lim_{N\to+\infty}N^\alpha e_N(\mu,\rho)=0.$$

We also check that

$$\forall\alpha\in(0,1/\rho)\,,\quad\sup_{x\geq 0}x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)<+\infty\Leftrightarrow\sup_{N\geq 1}N^\alpha\,e_N(\mu,\rho)<+\infty,$$

a condition under which, the order of convergence α of the minimal error $e_N(\mu,\rho)$ is preserved by choosing $x_1=F^{-1}\left(\frac{1}{N}\right)\wedge(-N^{\frac{1}{\rho}-\alpha})$, $x_N=F^{-1}\left(\frac{N-1}{N}\right)\vee N^{\frac{1}{\rho}-\alpha}$ and any $x_i\in[F^{-1}(\frac{i-1}{N}+),F^{-1}(\frac{i}{N})]$ for $2\leq i\leq N-1$. We also show that for $(N^\alpha e_N(\mu,\rho))_{N\geq 1}$ to be bounded for $\alpha>1/\rho$, the support of μ has to be bounded. Then we address the boundary case $\alpha=1/\rho$: among Weibull distributions, we exhibit probability measures μ with unbounded support such that, for $\rho>1$, $\lim_{N\to+\infty}N^{1/\rho}\,e_N(\mu,\rho)=0$ and give a necessary condition for $\left(N^{1/\rho}e_N(\mu,\rho)\right)_{N\geq 1}$ to be bounded, which unfortunately is not sufficient but ensures the boundedness of $\left(\frac{N^{1/\rho}}{1+\ln N}e_N(\mu,\rho)\right)_{N\geq 1}$. These results are summarized together with the ones obtained by Xu and Berger [17] in Table 1. They are stated and proved in the second section of the paper.

The first section is devoted to preliminary results. We first recall that, when $\int_{\mathbb{R}}|x|^\rho\mu(dx)<\infty$, then the infimum in (0.1) is attained:

$$e_N\left(\mu,\rho\right)=\mathcal{W}_\rho\left(\frac{1}{N}\sum_{i=1}^N\delta_{x_i^N},\mu\right)=\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left|x_i^N-F^{-1}(u)\right|^\rho du \tag{0.3}$$

for some points $x_i^N\in[F^{-1}(\frac{i-1}{N}+),F^{-1}(\frac{i}{N})]\cap\mathbb{R}$ which are unique as long as $\rho>1$ and explicit when $\rho\in\{1,2\}$. To circumvent the otherwise lack of explicit formula for the optimal point x_i^N when studying the order of convergence of $e_N(\mu,\rho)$ to 0, we also derive bounds where x_i^N does not appear for the i-th contribution in the right-hand side of (0.3). We then give an alternative expression of each contribution in terms of the cumulative distribution function F in place of the quantile function F^{-1}, before taking advantage of the induced alternative formula for $e_N(\mu,\rho)$ to recover that the error goes to 0 as $N\to\infty$ when $\int_{\mathbb{R}}|x|^\rho\mu(dx)<\infty$. We also precise Theorem 5.21 (ii) [17] which states that the boundedness of the support of μ implies that of the sequence $\left(N^{1/\rho}e_N(\mu,\rho)\right)_{N\geq 1}$, by remarking that when F^{-1} is moreover continuous, then the sequence goes to 0 as $N\to\infty$ for $\rho>1$, with an order of convergence arbitrarily low as examplified by the beta distributions. Last, when μ is not a Dirac mass, we complement in a non-asymptotic way the positivity of $\limsup_{N\to\infty}Ne_N(\mu,\rho)$ proved by Xu and Berger in Theorem 5.20 [17].

The proofs of two technical lemmas are given in the appendix.

Notation :

- We set $F(x)=\mu\left((-\infty,x]\right)$ for $x\in\mathbb{R}$ and denote $F^{-1}(u)=\inf\{x\in\mathbb{R}:F(x)\geq u\}$ for $u\in(0,1)$. We have $u\leq F(x)\Leftrightarrow F^{-1}(u)\leq x$. The quantile function F^{-1} is left-continuous and non-decreasing and we denote by $F^{-1}(u+)$ its right-hand limit at $u\in[0,1)$ (in particular $F^{-1}(0+)=\lim\limits_{u\to 0+}F^{-1}(u)\in[-\infty,+\infty)$) and set $F^{-1}(1)=\lim\limits_{u\to 1-}F^{-1}(u)\in(-\infty,+\infty]$.
- We respectively denote by $x\wedge y$ and $x\vee y$ the minimum and the maximum of two real numbers x and y.
- We denote by $\lfloor x\rfloor$ (resp. $\lceil x\rceil$) the integer j such that $j\leq x<j+1$ (resp. $j-1<x\leq j$) and by $\{x\}=x-\lfloor x\rfloor$ the fractional part of $x\in\mathbb{R}$.

α	Necessary condition	Sufficient condition
$\alpha = 1$	$\int_{\mathbb{R}} \frac{\mathbf{1}_{\{f(x)>0\}}}{f^{\rho-1}(x)}\,dx < +\infty$ (Thm. 2.4 [3])	$f(x) > 0$ dx a.e. on $\{x \in \mathbb{R} : 0 < F(x) < 1\}$ and $\int_{\mathbb{R}} \frac{\mathbf{1}_{\{f(x)>0\}}}{f^{\rho-1}(x)}\,dx < +\infty$ (Thm. 5.15 [17])
$\alpha \in \left(\frac{1}{\rho}, 1\right)$ when $\rho > 1$	F^{-1} continuous (Remark 5.22 (ii) [17]) and μ with bounded support (Prop. 2.1)	related to the modulus of continuity of F^{-1} (Example 1.8)
$\alpha = \frac{1}{\rho}$	$\exists \lambda > 0, \forall x \geq 0, F(-x) + 1 - F(x) \leq \frac{e^{-\lambda x}}{\lambda}$ (Prop. 2.7)	μ with bounded support (Thm. 5.21 (ii) [17]) For $\rho > 1$, Example 2.6 with unbounded supp.
$\alpha \in \left(0, \frac{1}{\rho}\right)$	$\sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} \Big(F(-x) + 1 - F(x)\Big) < +\infty$ (Thm. 2.2)	$\sup_{x \geq 0} x^{\frac{\rho}{1-\alpha\rho}} \Big(F(-x) + 1 - F(x)\Big) < +\infty$ (Thm. 2.2)

Table 1: Conditions for the convergence of $e_N(\mu, \rho)$ with order α : $\sup_{N \geq 1} N^\alpha e_N(\mu, \rho) < +\infty$.

- For two sequences $(a_N)_{N\geq 1}$ and $(b_N)_{N\geq 1}$ of real numbers with $b_N > 0$ for $N \geq 2$ we denote $a_N \asymp b_N$ when $0 < \inf_{N\geq 2} \left(\frac{a_N}{b_N}\right)$ and $\sup_{N\geq 2} \left(\frac{a_N}{b_N}\right) < +\infty$.

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1 Preliminary results

In the present section, we state preliminary results which will prove useful for the study of the order of convergence of $e_N(\mu, \rho)$ to 0 when $N \to 0$ in the next section devoted to the case when the support of μ is not bounded. We first recall that the infimum over $(x_1, \ldots, x_N)$ is attained in (0.1) and bound the first (resp. last) coordinate of the minimizer from below (resp. above). Next, we give estimations from below and above not involving x_i^N of each contribution in the sum in the right-hand side of (0.3). We also rewrite these contributions in terms of the cumulative distribution function F in place of the quantile function F^{-1}. We recall that the finiteness of the ρ-th order moment of μ is a necessary and sufficient condition for the error $e_N(\mu, \rho)$ to go to 0 as $N \to \infty$ before checking that when μ has a bounded support and a continuous quantile function, then $\lim_{N\to\infty} N^{1/\rho} e_N(\mu, \rho) = 0$ with an order of convergence arbitrarily low as examplified by the beta distributions. We finally derive a non-asymptotic lower bound for $Ne_N(\mu, \rho) + (N + 1)e_{N+1}(\mu, \rho)$.

1.1 Infimum is attained in (0.1)

When $\rho = 1$ (resp. $\rho = 2$), $\mathbb{R} \ni y \mapsto N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|y - F^{-1}(u)\right|^\rho du$ is minimal for y belonging to the set $\left[F^{-1}\left(\frac{2i-1}{2N}\right), F^{-1}\left(\frac{2i-1}{2N}+\right)\right]$ of medians (resp. equal to the mean $N \int_{\frac{i-1}{N}}^{\frac{i}{N}} F^{-1}(u)\,du$) of the image of the uniform law on $\left[\frac{i-1}{N}, \frac{i}{N}\right]$ by F^{-1}.

For general $\rho > 1$, the function $\mathbb{R} \ni y \mapsto \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|y - F^{-1}(u)\right|^\rho du$ is strictly convex and continuously differentiable with derivative

$$\rho \int_{\frac{i-1}{N}}^{\frac{i}{N}} \Big(\mathbf{1}_{\{y \geq F^{-1}(u)\}} \left(y - F^{-1}(u)\right)^{\rho-1} - \mathbf{1}_{\{y < F^{-1}(u)\}} \left(F^{-1}(u) - y\right)^{\rho-1}\Big)\,du \tag{1.1}$$

non-positive for $y = F^{-1}\left(\frac{i-1}{N}+\right)$ when either $i = 1$ and $F^{-1}(0+) > -\infty$ or $i \geq 2$ and non-negative for $y = F^{-1}\left(\frac{i}{N}\right)$ when either $i \leq N - 1$ or $i = N$ and $F^{-1}(1) < +\infty$. Since the derivative has a positive limit as

$y \to +\infty$ and a negative limit as $y \to -\infty$, we deduce that $\mathbb{R} \ni y \mapsto \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|y - F^{-1}(u)\right|^\rho du$ admits a unique minimizer $x_i^N \in \left[F^{-1}\left(\frac{i-1}{N}+\right), F^{-1}\left(\frac{i}{N}\right)\right] \cap \mathbb{R}$, as already stated in Corollary 4.4 [17] (to keep notations simple, we do not explicit the dependence of x_i^N on ρ). Therefore

(1.2)

$$e_N^\rho(\mu,\rho) = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|x_i^N - F^{-1}(u)\right|^\rho du \text{ with } \left[F^{-1}\left(\frac{i-1}{N}+\right), F^{-1}\left(\frac{i}{N}\right)\right] \ni x_i^N = \begin{cases} F^{-1}\left(\frac{2i-1}{2N}\right) & \text{if } \rho = 1, \\ N\int_{\frac{i-1}{N}}^{\frac{i}{N}} F^{-1}(u)\,du & \text{if } \rho = 2, \\ \text{not explicit otherwise.} \end{cases}$$

We will see that when the support of μ is unbounded, the contributions with $i \in \{1, N\}$ in the right-hand side of (0.3) are dominant. To prove our main result, Theorem 2.2 below which characterizes the convergence of $e_N(\mu,\rho)$ to 0 with order $\alpha \in (0, \frac{1}{\rho})$, we will need the following estimates on x_1^N and x_N^N.

Lemma 1.1. *Let $\rho \geq 1$ and $\alpha \in \left(0, \frac{1}{\rho}\right)$. There is a finite constant C only depending on ρ and α such that the two extremal points in the optimal sequence $(x_i^N)_{1\leq i\leq N}$ for $e_N(\mu,\rho)$ satisfy*

$$\forall N \geq 1,\ x_1^N \geq CN^{\frac{1}{\rho}-\alpha} \inf_{u\in(0,\frac{1}{N})} u^{\frac{1}{\rho}-\alpha}F^{-1}(u) \text{ and } x_N^N \leq CN^{\frac{1}{\rho}-\alpha} \sup_{u\in(0,\frac{1}{N})} u^{\frac{1}{\rho}-\alpha}F^{-1}(1-u).$$

If $\sup_{u\in(0,1/2]} u^{\frac{1}{\rho}-\alpha}\left(F^{-1}(1-u) - F^{-1}(u)\right) < +\infty$, then $\sup_{N\geq 1} N^{\alpha-\frac{1}{\rho}}\left(x_N^N \vee \left(-x_1^N\right)\right) < +\infty$.

The proof which relies on the expression (1.1) of the derivative when $\rho \notin \{1,2\}$ is postponed to the appendix.

1.2 Bounds on $e_N(\mu,\rho)$

To circumvent the lack of explicit formula for the optimal point x_i^N (unless $\rho \in \{1,2\}$) when studying the order of convergence of $e_N(\mu,\rho)$ to 0, we are now going to derive bounds where x_i^N does not appear for the i-th contribution in the right-hand side of (0.3).

When needing to bound $e_N(\mu,\rho)$ from above (see in particular the derivation of the order of convergence of $e_N\left(\mathbf{1}_{\{x>0\}}\beta x^{\beta-1}\exp\left(-x^\beta\right)dx, \rho\right)$ for $\beta > 0$ in Example 2.6 below), we may replace the optimal point x_i^N by $F^{-1}\left(\frac{2i-1}{2N}\right)$:

$$\forall i \in \{1,\dots,N\},\quad \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|F^{-1}(u) - x_i^N\right|^\rho du \leq \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|F^{-1}(u) - F^{-1}\left(\frac{2i-1}{2N}\right)\right|^\rho du, \tag{1.3}$$

a simple choice particularly appropriate when linearization is possible since $\left[\frac{i-1}{N}, \frac{i}{N}\right] \ni v \mapsto \int_{\frac{i-1}{N}}^{\frac{i}{N}} |u-v|^\rho du$ is minimal for $v = \frac{2i-1}{2N}$.

To bound $e_N(\mu,\rho)$ from below, we can use that, by Jensen's inequality and the minimality of $F^{-1}\left(\frac{2i-1}{2N}\right)$ for $\rho = 1$,

$$\begin{aligned}\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|F^{-1}(u) - x_i^N\right|^\rho du &\geq N^{\rho-1}\left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|F^{-1}(u) - x_i^N\right| du\right)^\rho \geq N^{\rho-1}\left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|F^{-1}(u) - F^{-1}\left(\frac{2i-1}{2N}\right)\right| du\right)^\rho \\ &\geq N^{\rho-1}\left(\frac{1}{4N}\left(F^{-1}\left(\frac{2i-1}{2N}\right) - F^{-1}\left(\frac{4i-3}{4N}\right) + F^{-1}\left(\frac{4i-1}{4N}\right) - F^{-1}\left(\frac{2i-1}{2N}\right)\right)\right)^\rho \\ &\geq \frac{1}{4^\rho N}\left(F^{-1}\left(\frac{4i-1}{4N}\right) - F^{-1}\left(\frac{4i-3}{4N}\right)\right)^\rho. \end{aligned} \tag{1.4}$$

This estimate will be used in the proof that the order of convergence of $e_N(\mu,\rho)$ cannot exceed $1/\rho$ when the support of μ is not bounded (Proposition 2.1 below), in the derivation of the order of convergence of $e_N(\beta\mathbf{1}_{[0,1]}(x)x^{\beta-1}dx, \rho)$ and $e_N\left(\mathbf{1}_{\{x>0\}}\beta x^{\beta-1}\exp\left(-x^\beta\right)dx, \rho\right)$ for $\beta > 0$ in Examples 1.8 and 2.6 below and in the derivation of the necessary condition for convergence with boundary order $1/\rho$ (Proposition 2.7 below).

1.3 Alternative formula in terms of the cumulative distribution function

In the proof of our main result (Theorem 2.2 below which characterizes the convergence of $e_N(\mu,\rho)$ to 0 with order $\alpha \in (0, \frac{1}{\rho}))$ and in the derivation of our necessary condition for the convergence of $e_N(\mu,\rho)$ with boundary order $1/\rho$ (Proposition 2.7 below), we will need to rewrite in terms of the cumulative distribution function F in place of the quantile function F^{-1} contributions in the decomposition over $i \in \{1,\dots,N\}$ in the right-hand side of (0.3). This is possible thanks to the next lemma.

Lemma 1.2. *Assume that* $\int_{\mathbb{R}} |x|^\rho \mu(dx) < +\infty$ *with* $\rho \geq 1$*. For* $i \in \{1,\dots,N\}$ *and* $x \in \left[F^{-1}\left(\frac{i-1}{N}\right), F^{-1}\left(\frac{i}{N}\right)\right] \cap \mathbb{R}$ *(with convention* $F^{-1}(0) = F^{-1}(0+)$*), we have:*

$$\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|x - F^{-1}(u)\right|^\rho du = \rho \int_{F^{-1}\left(\frac{i-1}{N}\right)}^{x} (x-y)^{\rho-1} \left(F(y) - \frac{i-1}{N}\right) dy + \rho \int_{x}^{F^{-1}\left(\frac{i}{N}\right)} (y-x)^{\rho-1} \left(\frac{i}{N} - F(y)\right) dy,$$

and the right-hand side is minimal for $x = x_i^N$*.*

Remark 1.3. *Under the convention* $F^{-1}(0) = -\infty$*, when, for some* $i \in \{1,\dots,N\}$*,* $F^{-1}\left(\frac{i-1}{N}+\right) > F^{-1}\left(\frac{i-1}{N}\right)$*, then* $F(y) = \frac{i-1}{N}$ *for* $y \in \left[F^{-1}\left(\frac{i-1}{N}\right), F^{-1}\left(\frac{i-1}{N}+\right)\right)$ *and* $\int_{F^{-1}\left(\frac{i-1}{N}\right)}^{F^{-1}\left(\frac{i-1}{N}+\right)} (x-y)^{\rho-1}\left(F(y) - \frac{i-1}{N}\right) dy = 0$ *so that the lower integration limit in the first integral in the right-hand side of* (1.5) *may be replaced by* $F^{-1}\left(\frac{i-1}{N}+\right)$*. In a similar way, the upper integration limit in the second integral may be replaced by* $F^{-1}\left(\frac{i}{N}+\right)$ *under the convention* $F^{-1}(1+) = +\infty$*.*

Plugging this equality written for $x = x_i^N$ in the right-hand side of (0.3), we immediately deduce the following alternative formulation of $e_N(\mu,\rho)$ in terms of the cumulative distribution function F:

Proposition 1.4.

(1.5)
$$e_N^\rho(\mu,\rho) = \rho \sum_{i=1}^{N} \left(\int_{F^{-1}\left(\frac{i-1}{N}+\right)}^{x_i^N} \left(x_i^N - y\right)^{\rho-1} \left(F(y) - \frac{i-1}{N}\right) dy + \int_{x_i^N}^{F^{-1}\left(\frac{i}{N}\right)} \left(y - x_i^N\right)^{\rho-1} \left(\frac{i}{N} - F(y)\right) dy \right).$$

Remark 1.5. *When* $\rho = 1$*, the equality* (1.5) *follows from the interpretation of* $\mathcal{W}_1(\nu,\eta)$ *as the integral of the absolute difference between the cumulative distribution functions of* ν *and* η *(equal, as seen with a rotation with angle* $\frac{\pi}{2}$*, to the integral of the absolute difference between their quantile functions) and the integral simplifies into:*

(1.6)
$$e_N(\mu,1) = \sum_{i=1}^{N} \left(\int_{F^{-1}\left(\frac{i-1}{N}+\right)}^{F^{-1}\left(\frac{2i-1}{N}\right)} \left(F(y) - \frac{i-1}{N}\right) dy + \int_{F^{-1}\left(\frac{2i-1}{N}\right)}^{F^{-1}\left(\frac{i}{N}\right)} \left(\frac{i}{N} - F(y)\right) dy \right) = \frac{1}{N} \int_{\mathbb{R}} \min_{j\in\mathbb{N}} |NF(y) - j|\, dy.$$

For $\rho > 1$*, this equality can be deduced from the general formula for* $\mathcal{W}_\rho^\rho(\nu,\eta)$ *in terms of the cumulative distribution functions of* μ *and* η *(see for instance Lemma B.3 [13]).*

Proof of Lemma 1.2. Let $i \in \{1,\dots,N\}$ and $x \in \left[F^{-1}\left(\frac{i-1}{N}\right), F^{-1}\left(\frac{i}{N}\right)\right] \cap \mathbb{R}$. We have $\frac{i-1}{N} \leq F(x)$ and $F(x-) \leq \frac{i}{N}$. Since $F^{-1}(u) \leq x \Leftrightarrow u \leq F(x)$ and $F^{-1}(u) = x$ for $u \in (F(x-), F(x)]$, we have:

$$\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|x - F^{-1}(u)\right|^\rho du = \int_{\frac{i-1}{N}}^{F(x)} \left(x - F^{-1}(u)\right)^\rho du + \int_{F(x)}^{\frac{i}{N}} \left(F^{-1}(u) - x\right)^\rho du.$$

Using the well-known fact that the image of $\mathbf{1}_{[0,1]}(v)\, dv\mu(dz)$ by $(v,z) \mapsto F(z-) + v\mu(\{z\})$ is the Lebesgue measure on $[0,1]$ and that $\mathbf{1}_{[0,1]}(v)\, dv\mu(dz)$ a.e., $F^{-1}\left(F(z-) + v\mu(\{z\})\right) = z$, we obtain that:

$$\begin{aligned}
\int_{\frac{i-1}{N}}^{F(x)} \left(x - F^{-1}(u)\right)^\rho du &= \int_{v=0}^{1} \int_{z\in\mathbb{R}} \mathbf{1}_{\left\{\frac{i-1}{N} \leq F(z-) + v\mu(\{z\}) \leq F(x)\right\}} (x-z)^\rho \mu(dz)\, dv \\
&= \int_{v=0}^{1} \int_{z\in\mathbb{R}} \mathbf{1}_{\left\{\frac{i-1}{N} \leq F(z-) + v\mu(\{z\}) \leq F(x)\right\}} \int \rho (x-y)^{\rho-1} \mathbf{1}_{\{z \leq y \leq x\}}\, dy \mu(dz)\, dv \\
&= \rho \int_{y=-\infty}^{x} (x-y)^{\rho-1} \int_{v=0}^{1} \int_{z\in\mathbb{R}} \mathbf{1}_{\left\{\frac{i-1}{N} \leq F(z-) + v\mu(\{z\})\right\}} \mathbf{1}_{\{z\leq y\}} \mu(dz)\, dv\, dy. \qquad (1.7)
\end{aligned}$$

For $v>0$, $\{z\in\mathbb{R}:F(z-)+v\mu(\{z\})\le F(y)\}=(-\infty,y]\cup\{z\in\mathbb{R}:z>y \text{ and } F(z)=F(y)\}$ with $\mu\left(\{z\in\mathbb{R}:z>y \text{ and } F(z)=F(y)\}\right)=0$ and therefore

$$\int_{z\in\mathbb{R}}\mathbf{1}_{\left\{\frac{i-1}{N}\le F(z-)+v\mu(\{z\})\right\}}\mathbf{1}_{\{z\le y\}}\mu(dz)=\int_{z\in\mathbb{R}}\mathbf{1}_{\left\{\frac{i-1}{N}\le F(z-)+v\mu(\{z\})\le F(y)\right\}}\mu(dz).$$

Plugging this equality in (1.7), using again the image of $\mathbf{1}_{[0,1]}(v)\,dv\mu(dz)$ by $(v,z)\mapsto F(z-)+v\mu(\{z\})$ and the equivalence $\frac{i-1}{N}\le F(y)\Leftrightarrow F^{-1}\left(\frac{i-1}{N}\right)\le y$, we deduce that:

$$\int_{\frac{i-1}{N}}^{F(x)}\left(x-F^{-1}(u)\right)^\rho\,du=\rho\int_{y=-\infty}^{x}(x-y)^{\rho-1}\int_{u=0}^{1}\mathbf{1}_{\left\{\frac{i-1}{N}\le u\le F(y)\right\}}\,du\,dy=\rho\int_{F^{-1}\left(\frac{i-1}{N}\right)}^{x}(x-y)^{\rho-1}\left(F(y)-\frac{i-1}{N}\right)dy.$$

In a similar way, we check that:

$$\int_{F(x)}^{\frac{i}{N}}\left(F^{-1}(u)-x\right)^\rho\,du=\rho\int_{x}^{F^{-1}\left(\frac{i}{N}\right)}(y-x)^{\rho-1}\left(\frac{i}{N}-F(y)\right)dy,$$

which concludes the proof. □

1.4 Convergence of the error to 0

According to Corollary 5.12 [17], the finiteness of the moment $\int_{\mathbb{R}}|x|^\rho\mu(dx)$ with order ρ implies that the error $e_N(\mu,\rho)$ goes to 0 as $N\to\infty$. Since, by the inverse transform sampling, the moment clearly has to be finite for $e_N(\mu,\rho)$ to be finite for some $N\ge 1$, the following equivalence holds.

Proposition 1.6. *For each $\rho\ge 1$, we have* $\int_{\mathbb{R}}|x|^\rho\mu(dx)<+\infty\Leftrightarrow\lim_{N\to+\infty}e_N(\mu,\rho)=0.$

The direct implication can also be deduced from the inequality $e_N(\mu,\rho)\le\mathcal{W}_\rho\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_i},\mu\right)$ and the almost sure convergence to 0 of $\mathcal{W}_\rho\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_i},\mu\right)$ for $(X_i)_{i\ge1}$ i.i.d. according to μ deduced from the strong law of large numbers and stated for instance in Theorem 2.13 [6]. For the sake of completeness, we give an alternative simple argument based on (1.5).

Proof. According to the introduction, the finiteness of $e_N(\mu,\rho)$ for some $N\ge1$ implies that $\int_{\mathbb{R}}|x|^\rho\mu(dx)<+\infty$. So it is enough to check the zero limit property under the finite moment condition.

When respectively $F^{-1}\left(\frac{i}{N}\right)\le 0$, $F^{-1}\left(\frac{i-1}{N}+\right)<0<F^{-1}\left(\frac{i}{N}\right)$ or $F^{-1}\left(\frac{i-1}{N}+\right)\ge 0$, then, by Lemma 1.2, the term with index i in (1.5) is respectively bounded from above by

$$\int_{F^{-1}\left(\frac{i-1}{N}+\right)}^{F^{-1}\left(\frac{i}{N}\right)}\left(F^{-1}\left(\frac{i}{N}\right)-y\right)^{\rho-1}\left(F(y)-\frac{i-1}{N}\right)dy\le\int_{F^{-1}\left(\frac{i-1}{N}+\right)}^{F^{-1}\left(\frac{i}{N}\right)}(-y)^{\rho-1}\left(\frac{1}{N}\wedge F(y)\right)dy,$$

$$\int_{F^{-1}\left(\frac{i-1}{N}+\right)}^{0}(-y)^{\rho-1}\left(\frac{1}{N}\wedge F(y)\right)dy+\int_{0}^{F^{-1}\left(\frac{i}{N}\right)}y^{\rho-1}\left(\frac{1}{N}\wedge(1-F(y))\right)dy,$$

$$\int_{F^{-1}\left(\frac{i-1}{N}+\right)}^{F^{-1}\left(\frac{i}{N}\right)}\left(y-F^{-1}\left(\frac{i-1}{N}+\right)\right)^{\rho-1}\left(\frac{i}{N}-F(y)\right)dy\le\int_{F^{-1}\left(\frac{i-1}{N}+\right)}^{F^{-1}\left(\frac{i}{N}\right)}y^{\rho-1}\left(\frac{1}{N}\wedge(1-F(y))\right)dy.$$

After summation, we deduce that:

$$e_N^\rho(\mu,\rho)\le\rho\int_{-\infty}^{0}(-y)^{\rho-1}\left(\frac{1}{N}\wedge F(y)\right)dy+\rho\int_{0}^{+\infty}y^{\rho-1}\left(\frac{1}{N}\wedge(1-F(y))\right)dy.$$

Since, by Fubini's theorem, $\rho\int_{-\infty}^{0}(-y)^{\rho-1}F(y)\,dy+\rho\int_{0}^{+\infty}y^{\rho-1}(1-F(y))\,dy=\int_{\mathbb{R}}|x|^\rho\mu(dx)<+\infty$, Lebesgue's theorem ensures that the right-hand side and therefore $e_N(\mu,\rho)$ go to 0 as $N\to+\infty$. □

According to Theorem 5.21 (ii) [17], when the support of μ is bounded, then $\sup_{N\geq 1} N^{\frac{1}{\rho}} e_N(\mu,\rho) < +\infty$, while Remark 5.22 (ii) [17] ensures that $\limsup_{N\to+\infty} N^{\frac{1}{\rho}} e_N(\mu,\rho) > 0$ when F^{-1} is discontinuous. We complement these results by the following lemma.

Lemma 1.7. *If F^{-1} is continuous and the support of μ bounded, then for each $\rho > 1$, $\lim\limits_{N\to+\infty} N^{1/\rho} e_N(\mu,\rho) = 0$.*

Proof. By (1.3),

$$e_N^\rho(\mu,\rho) \leq \frac{1}{2N}\sum_{i=1}^N \left\{\left(F^{-1}\left(\frac{2i-1}{2N}\right) - F^{-1}\left(\frac{i-1}{N}+\right)\right)^\rho + \left(F^{-1}\left(\frac{i}{N}\right) - F^{-1}\left(\frac{2i-1}{2N}\right)\right)^\rho\right\}$$
$$\leq \frac{1}{2N}\left(F^{-1}(1) - F^{-1}(0+)\right) \max_{1\leq j\leq 2N}\left(F^{-1}\left(\frac{j}{2N}\right) - F^{-1}\left(\frac{j-1}{2N}\right)\right)^{\rho-1},$$

where we use the convention $F^{-1}(0) = F^{-1}(0+)$ in $\max\limits_{1\leq j\leq 2N}\left(F^{-1}\left(\frac{j}{2N}\right) - F^{-1}\left(\frac{j-1}{2N}\right)\right)^{\rho-1}$. When the support of μ is bounded then $F^{-1}(1) - F^{-1}(0+) < \infty$ and when moreover F^{-1} is continuous, then this function is uniformly continuous on $(0,1)$ and the conclusion follows. ☐

The next example shows that when μ is compactly supported with F^{-1} continuous then, for each $\rho > 1$, the rate of convergence of $N^{1/\rho} e_N(\mu,\rho)$ to 0 as $N \to +\infty$ may be arbitrarily small.

Example 1.8. *Let $\mu_\beta(dx) = \beta \mathbf{1}_{[0,1]}(x) x^{\beta-1}\,dx$ with $\beta > 0$. Then $F^{-1}(u) = u^{1/\beta}$. Let us suppose that $\rho > 1$ and $\beta \geq \frac{\rho}{\rho-1}$. Using (1.4) with $i = 1$ for the second inequality, we obtain that*

$$e_N^\rho(\mu_\beta,\rho) \geq \int_0^{\frac{1}{N}} \left|F^{-1}(u) - x_1^N\right|^\rho du \geq \frac{1}{4^\rho N^{1+\frac{\rho}{\beta}}}\left(\left(\frac{3}{4}\right)^{\frac{1}{\beta}} - \left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)^\rho.$$

On the other hand, under the convention $F^{-1}(0) = 0$:

$$e_N^\rho(\mu_\beta,\rho) \leq \mathcal{W}_\rho^\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{F^{-1}\left(\frac{i-1}{N}\right)}, \mu_\beta\right) = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(u^{\frac{1}{\beta}} - \left(\frac{i-1}{N}\right)^{\frac{1}{\beta}}\right)^\rho du$$
$$\leq \int_0^{\frac{1}{N}} u^{\frac{\rho}{\beta}}\,du + \frac{1}{N^{1+\frac{\rho}{\beta}}}\sum_{i=2}^N \left(i^{\frac{1}{\beta}} - (i-1)^{\frac{1}{\beta}}\right)^\rho$$
$$\leq \frac{\beta}{\beta+\rho} \times \frac{1}{N^{1+\frac{\rho}{\beta}}} + \frac{1}{\beta^\rho N^{1+\frac{\rho}{\beta}}}\sum_{i=2}^N (i-1)^{\frac{\rho}{\beta}-\rho}. \tag{1.8}$$

When $\beta > \frac{\rho}{\rho-1}$, the last sum is smaller than $\sum\limits_{j\in\mathbb{N}^*} j^{-(\rho-\frac{\rho}{\beta})}$ which is finite since $\rho - \frac{\rho}{\beta} > 1$ and $e_N(\mu_\beta,\rho) \asymp N^{-\frac{1}{\rho}-\frac{1}{\beta}}$. Notice that according Theorem 5.15 [17],

$$\forall \beta > 0,\ \forall \rho \geq 1, \quad \lim_{N\to+\infty} N e_N(\mu_\beta,\rho) = \frac{1}{2\beta(\rho+1)^{1/\rho}}\left(\int_0^1 u^{\frac{\rho}{\beta}-\rho} du\right)^{1/\rho},$$

with the right-hand side finite if and only if $\rho = 1$ or $\rho > 1$ and $\beta < \frac{\rho}{\rho-1}$ and then equal to $\frac{1}{2\beta(\rho+1)^{1/\rho}}\left(\frac{\beta}{\rho+\beta-\rho\beta}\right)^{1/\rho}$. When $\rho > 1$, for the limiting value $\beta = \frac{\rho}{\rho-1}$, one has $\frac{1}{\rho}+\frac{1}{\beta} = 1$ and, by (1.8), $e_N^\rho(\mu_\beta,\rho) \leq \frac{1}{\rho} + \frac{1}{\beta^\rho N^\rho}\sum_{i=2}^N \frac{1}{i-1} \sim \frac{\ln N}{\beta^\rho N^\rho}$ as $N \to +\infty$. On the other hand, according to (1.4),

$$e_N^\rho(\mu_\beta,\rho) \geq \frac{1}{4^{2\rho-1}N^\rho}\sum_{i=1}^N \left((4i-1)^{\frac{1}{\beta}} - (4i-3)^{\frac{1}{\beta}}\right)^\rho \geq \frac{2^\rho}{4^{2\rho-1}\beta^\rho N^\rho}\sum_{i=1}^N \frac{1}{4i-1} \geq \frac{2^\rho}{4^{2\rho}\beta^\rho N^\rho}\sum_{i=1}^N \frac{1}{i} \sim \frac{2^\rho \ln N}{4^{2\rho}\beta^\rho N^\rho}$$

so that $e_N(\mu_\beta,\rho) \asymp N^{-1}(\ln N)^{\frac{1}{\rho}}$.

According to Corollary 6.15 [6], for $(X_i)_{i\geq 1}$ *i.i.d. according to* μ_β, $\mathbb{E}^{1/\rho}\left[\mathcal{W}_\rho^\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_i},\mu_\beta\right)\right] \asymp N^{-\frac{1}{\rho}-\frac{1}{\beta}}$ *if* $\rho > 2$ *and* $\beta > \frac{2\rho}{\rho-2}$ *and* $\mathbb{E}^{1/\rho}\left[\mathcal{W}_\rho^\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_i},\mu_\beta\right)\right] \asymp N^{-1/2}$ *if* $\rho \leq 2$ *and* $\beta \geq 1$ *or* $\rho > 2$ *and* $\beta \in [1,\frac{2\rho}{\rho-2})$.

Note that apart from the restriction $\beta \geq 1$ *made in [6] to ensure that the distribution is log-concave, the results concerning the optimal deterministic choice and the random choice share the same structure with different maximal orders of convergence* 1 *and* 1/2. *When* $\rho > 2$ *and* $\beta > \frac{2\rho}{\rho-2}$, *the deterministic and random orders of convergence are both equal to* $\frac{1}{\rho}+\frac{1}{\beta}$.

1.5 Non-asymptotic lower bound for the error

According to Theorem 5.20 [17] and its proof, $\limsup_{N\to\infty} Ne_N(\mu,\rho) \geq \frac{1}{2}\int_0^{\frac{1}{2}}(F^{-1}(u+\frac{1}{2})-F^{-1}(u))du$ with the right-hand side positive apart when μ is a Dirac mass and $e_N(\mu,\rho)$ vanishes for all $N,\rho\geq 1$. This result may be complemented by the following non-asymptotic bound.

Lemma 1.9. $\forall \rho \geq 1$, $\forall N \geq 1$, $\quad Ne_N(\mu,\rho)+(N+1)e_{N+1}(\mu,\rho) \geq \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} F(x)\wedge(1-F(x))\,dx.$

Remark 1.10.

- *One has*

$$\int_{\mathbb{R}} F(x)\wedge(1-F(x))\,dx = \int_{-\infty}^{F^{-1}(\frac{1}{2})} F(x)dx + \int_{F^{-1}(\frac{1}{2})}^{+\infty}\left(\frac{1}{2}-\left(F(x)-\frac{1}{2}\right)\right)dx = \int_0^{\frac{1}{2}}\left(F^{-1}\left(u+\frac{1}{2}\right)-F^{-1}(u)\right)du$$

  as easily seen since the sum and the last integral correspond to the area of the points at the right to $(F(x))_{-\infty\leq x\leq F^{-1}(\frac{1}{2})}$, *at the left to* $(F(x)-\frac{1}{2})_{F^{-1}(\frac{1}{2})<x<+\infty}$, *above* 0 *and below* $\frac{1}{2}$ *respectively computed by integration with respect to the abscissa and to the ordinate.*

- *The analalogous result in the random case is stated in Theorem 3.1 [6] : when* $(X_i)_{i\geq 1}$ *are i.i.d. according to* μ, $\mathbb{E}\left[\mathcal{W}_1\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_i},\mu\right)\right] \geq \frac{1}{2\sqrt{2N}}\mathbb{E}\left[|X_1-F^{-1}(1/2)|\right]$. *In other words, unless* μ *is a Dirac mass, the random rate cannot be quicker than the usual Monte Carlo rate* $\frac{1}{\sqrt{N}}$.

Proof. Note that when $\rho \geq \tilde{\rho} \geq 1$ and $\int_{\mathbb{R}}|x|^\rho\mu(dx) < +\infty$, with $(x_i^N)_{1\leq i\leq N}$ denoting the optimal points for $\rho \geq 1$,

$$e_N(\mu,\rho) = \mathcal{W}_\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i^N},\mu\right) \geq \mathcal{W}_{\tilde{\rho}}\left(\frac{1}{N}\sum_{i=1}^N \delta_{x_i^N},\mu\right) \geq e_N(\mu,\tilde{\rho}). \tag{1.9}$$

Hence $\rho \mapsto e_N(\mu,\rho)$ is non-decreasing and it is enough to check the statement for $\rho = 1$. It follows by plugging into (1.6) the following inequality written for $v = F(x)$:

$$\forall v \in [0,1], \forall N \geq 1, \quad \min_{j\in\mathbb{N}}|Nv-j| \vee \min_{j\in\mathbb{N}}|(N+1)v-j| \geq \frac{v\wedge(1-v)}{2},$$

To prove this inequality, we remark that for $v \in (0,1)$, there are two possibilities

- Either $\lfloor Nv\rfloor \leq Nv < (N+1)v \leq \lfloor Nv\rfloor+1$, which implies that $(Nv-\lfloor Nv\rfloor)\vee(\lfloor Nv\rfloor+1-(N+1)v) \geq \frac{1-v}{2}$ while $\lfloor Nv\rfloor+1-Nv = \lfloor Nv\rfloor+1-(N+1)v+v \geq v$ and $(N+1)v-\lfloor Nv\rfloor = Nv-\lfloor Nv\rfloor+v \geq v$ so that

$$\min_{j\in\mathbb{N}}|Nv-j| \vee \min_{j\in\mathbb{N}}|(N+1)v-j| \geq v\wedge\frac{1-v}{2}.$$

- Or $\lfloor Nv\rfloor \leq Nv < \lfloor Nv\rfloor+1 \leq (N+1)v$, which implies that $(\lfloor Nv\rfloor+1-Nv)\vee((N+1)v-(\lfloor Nv\rfloor+1)) \geq \frac{v}{2}$ while $Nv-\lfloor Nv\rfloor = (N+1)v-(\lfloor Nv\rfloor+1)+1-v \geq 1-v$ and $\lfloor Nv\rfloor+2-(N+1)v = \lfloor Nv\rfloor+1-Nv+1-v > 1-v$ so that

$$\min_{j\in\mathbb{N}}|Nv-j| \vee \min_{j\in\mathbb{N}}|(N+1)v-j| \geq \frac{v}{2}\wedge(1-v).$$

Synthetising the two cases and remarking that the inequality still holds for $v \in \{0,1\}$, we conclude. □

2 The case when the support of μ is unbounded

According to Theorem 5.21 (ii) [17], when the support of μ is bounded, $\sup_{N\geq 1} N^{1/\rho}e_N(\mu,\rho)<+\infty$ and, when moreover F^{-1} is continuous, then $\lim_{N\to\infty} N^{1/\rho}e_N(\mu,\rho)=0$ for $\rho>1$ by Lemma 1.7. In this section, we first show that in the unbounded support case, the order of convergence cannot exceed the minimal order $1/\rho$ in the bounded support case. Next we state our main result : a necessary and sufficient condition for convergence with order $\alpha\in(0,\frac{1}{\rho})$ which precises the implication stated in Theorem 5.21 (i) [17] :

$$\int_{\mathbb{R}^d}|x|^{\frac{\rho}{1-\alpha\rho}}\mu(dx)<+\infty\Rightarrow\lim_{N\to\infty}N^{\alpha}e_N(\mu,\rho)=0.$$

We finally address the boundary case $\alpha=1/\rho$ where we only obtain a necessary condition and illustrate the possibility that, when $\rho>1$, $\lim_{N\to\infty}N^{1/\rho}e_N(\mu,\rho)=0$ for some probability measure μ with unbounded support.

2.1 The order of convergence cannot exceed $1/\rho$

According to the next result, the order of convergence of $e_N(\mu,\rho)$ cannot exceed $\frac{1}{\rho}$ when the support of μ is not bounded.

Proposition 2.1. *Let $\rho>1$. Then $\exists\,\alpha>\frac{1}{\rho}$, $\sup_{N\geq1}N^{\alpha}e_N(\mu,\rho)<+\infty \implies F^{-1}(1)-F^{-1}(0+)<+\infty$.*

Proof. Let $\rho>1$ and $\alpha>\frac{1}{\rho}$ be such that $\sup_{N\geq1}N^{\alpha}e_N(\mu,\rho)<+\infty$ so that, by (1.2),

$$\sup_{N\geq1}N^{\alpha\rho}\left(\int_0^{\frac{1}{N}}\left|F^{-1}(u)-x_1^N\right|^{\rho}du+\int_{\frac{N-1}{N}}^1\left|F^{-1}(u)-x_N^N\right|^{\rho}du\right)<+\infty.$$

By (1.4) for $i=1$ and $N\geq1$, we have:

$$\int_0^{\frac{1}{N}}\left|F^{-1}(u)-x_1^N\right|^{\rho}du\geq\frac{1}{4^{\rho}N}\left(F^{-1}\left(\frac{1}{2N}\right)-F^{-1}\left(\frac{1}{4N}\right)\right)^{\rho}.$$

Therefore $C:=\sup_{N\geq1}(2N)^{\alpha-\frac{1}{\rho}}\left(F^{-1}\left(\frac{1}{2N}\right)-F^{-1}\left(\frac{1}{4N}\right)\right)<+\infty$. For $k\in\mathbb{N}^*$, we deduce that $F^{-1}\left(2^{-(k+1)}\right)-F^{-1}\left(2^{-k}\right)\geq-C2^{\frac{1-\alpha\rho}{\rho}k}$, and after summation that:

$$\forall k\in\mathbb{N}^*,F^{-1}\left(2^{-k}\right)\geq F^{-1}\left(1/2\right)-\frac{C}{2^{\alpha-\frac{1}{\rho}}-1}\left(1-2^{\frac{1-\alpha\rho}{\rho}(k-1)}\right). \tag{2.1}$$

When $k\to+\infty$, the right-hand side goes to $\left(F^{-1}\left(\frac{1}{2}\right)-\frac{C}{2^{\alpha-\frac{1}{\rho}}-1}\right)>-\infty$ so that $F^{-1}(0+)>-\infty$. In a symmetric way, we check that $F^{-1}(1)<+\infty$ so that μ is compactly supported. □

2.2 Necessary and sufficient condition for convergence with order $\alpha\in(0,\frac{1}{\rho})$

Our main result is the following necessary and sufficient condition for $e_N(\mu,\rho)$ to go to 0 with order $\alpha\in\left(0,\frac{1}{\rho}\right)$.

Theorem 2.2. *Let $\rho\geq1$ and $\alpha\in\left(0,\frac{1}{\rho}\right)$. We have*

$$\sup_{x\geq0}x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)<+\infty\Leftrightarrow\sup_{N\geq1}N^{\alpha}\,e_N(\mu,\rho)<+\infty\Leftrightarrow\sup_{N\geq2}\sup_{x_{2:N-1}}N^{\alpha}\mathcal{W}_{\rho}(\mu_N(x_{2:N-1}),\mu)<+\infty$$

where $\mu_N(x_{2:N-1})=\frac{1}{N}\left(\delta_{F^{-1}\left(\frac{1}{N}\right)\wedge(-N^{\frac{1}{\rho}-\alpha})}+\sum_{i=2}^{N-1}\delta_{x_i}+\delta_{F^{-1}\left(\frac{N-1}{N}\right)\vee N^{\frac{1}{\rho}-\alpha}}\right)$ and $\sup_{x_{2:N-1}}$ means the supremum over the choice of $x_i\in[F^{-1}(\frac{i-1}{N}+),F^{-1}(\frac{i}{N})]$ for $2\leq i\leq N-1$. Moreover,

$$\lim_{x\to+\infty}x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)=0\Leftrightarrow\lim_{N\to+\infty}N^{\alpha}e_N(\mu,\rho)=0.$$

Remark 2.3. • *Let us relate the order of convergence to the maximal integrable power* $\hat\beta := \sup\{\beta \geq 0 : \int_{\mathbb{R}} |x|^\beta \mu(dx) < \infty\}$ *of* μ. *For* $\beta \in (0,\hat\beta)$, $\int_{\mathbb{R}} |x|^\beta \mu(dx) < \infty$ *while when* $\hat\beta < \infty$, *for* $\beta > \hat\beta$, $\int_{\mathbb{R}} |x|^{\frac{\hat\beta+\beta}{2}} \mu(dx) = +\infty$, *so that, by Lemma 2.4 below,* $\sup_{x\geq 0} x^\beta\Big(F(-x)+1-F(x)\Big) = +\infty$. *Let* $\rho \geq 1$. *If* $\hat\beta > \rho$, *we deduce from Theorem 5.21 (i) [17] that for each* $\alpha \in (0, \frac{1}{\rho} - \frac{1}{\hat\beta})$, $\lim_{N\to\infty} N^\alpha e_N(\mu,\rho) = 0$ *and, when* $\hat\beta < +\infty$, *Theorem 2.2 ensures that for each* $\alpha > \frac{1}{\rho} - \frac{1}{\hat\beta}$, $\sup_{N\geq 1} N^\alpha e_N(\mu,\rho) = +\infty$ *since* $\frac{\rho}{1-\alpha\rho} > \hat\beta$. *In this sense, when* $\rho < \hat\beta < +\infty$ *the order of convergence of* $e_N(\mu,\rho)$ *to 0 is* $\frac{1}{\rho} - \frac{1}{\hat\beta}$. *Moreover, the boundedness and the vanishing limit at infinity for the sequence* $(N^{\frac{1}{\rho}-\frac{1}{\hat\beta}} e_N(\mu,\rho))_{N\geq 1}$ *are respectively equivalent to the same property for the function* $\mathbb{R}_+ \ni x \mapsto x^{\hat\beta}\Big(F(-x)+1-F(x)\Big)$. *Note that* $\limsup_{x\to+\infty} x^{\hat\beta}\Big(F(-x)+1-F(x)\Big)$ *can be* $+\infty$, *in which case,* $\sup_{N\geq 1} N^{\frac{1}{\rho}-\frac{1}{\hat\beta}} e_N(\mu,\rho) = +\infty$.

- *In the proof (see (2.4) below), we estimate* $\sup_{N\geq 2}\sup_{x_{2:N-1}} N^{\alpha\rho}\mathcal{W}_\rho^\rho(\mu_N(x_{2:N-1}),\mu)$ *in terms of* $C = \sup\limits_{x\geq 0} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)$.

- *According to Theorem 7.16 [6], for* $(X_i)_{i\geq 1}$ *i.i.d. according to* μ,

$$\sup_{N\geq 1} N^{\frac{1}{2\rho}} \mathbb{E}^{1/\rho}\left[\mathcal{W}_\rho^\rho\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_i},\mu\right)\right] \leq \left(\rho 2^{\rho-1}\int_{\mathbb{R}} |x|^{\rho-1}\sqrt{F(x)(1-F(x))}\,dx\right)^{1/\rho}$$

with $\exists\, \varepsilon > 0$, $\int_{\mathbb{R}} |x|^{2\rho+\varepsilon}\mu(dx) < +\infty \Rightarrow \int_{\mathbb{R}} |x|^{\rho-1}\sqrt{F(x)(1-F(x))}\,dx < +\infty \Rightarrow \int_{\mathbb{R}} |x|^{2\rho}\mu(dx) < +\infty$ *(the reverse implications fail) by the discussions just after this theorem and after Theorem 3.2 [6]. The condition* $\sup_{x\geq 0} x^{2\rho}\Big(F(-x)+1-F(x)\Big) < +\infty$ *equivalent to* $\sup_{N\geq 1} N^{\frac{1}{2\rho}} e_N(\mu,\rho) < +\infty$ *is slightly weaker than* $\int_{\mathbb{R}} |x|^{2\rho}\mu(dx) < +\infty$, *according to Lemma 2.4 just below. Moreover, we address similarly any order of convergence* α *with* $\alpha \in \Big(0,\frac{1}{\rho}\Big)$ *for* $e_N(\mu,\rho)$, *while the order* $\frac{1}{2\rho}$ *seems to play a special role for* $\mathbb{E}^{1/\rho}\left[\mathcal{W}_\rho^\rho\left(\frac{1}{N}\sum\limits_{i=1}^N \delta_{X_i},\mu\right)\right]$ *in the random case. When* $\rho = 1$, *the order of convergence* α *for* $\alpha \in (0,1/2)$ *is addressed in the random case in Theorem 2.2 [8] where the finiteness of* $\sup_{x\geq 0} x^{\frac{1}{1-\alpha}}\Big(F(-x)+1-F(x)\Big)$ *is stated to be equivalent to the stochastic boundedness of the sequence* $\left(N^\alpha \mathcal{W}_1\left(\frac{1}{N}\sum\limits_{i=1}^N \delta_{X_i},\mu\right)\right)_{N\geq 1}$. *When* $\alpha = 1/2$, *the stochastic boundedness property is, according to Theorem 2.1 (b) [8], equivalent to* $\int_{\mathbb{R}} \sqrt{F(x)(1-F(x))}\,dx < +\infty$.

The proof of Theorem 2.2 relies on the next lemma, the proof of which is postponed to the appendix.

Lemma 2.4. *For* $\beta > 0$, *we have*

$$\int_{\mathbb{R}} |y|^\beta \mu(dy) < +\infty \implies \lim_{x\to+\infty} x^\beta\Big(F(-x)+1-F(x)\Big) = 0$$
$$\implies \sup_{x\geq 0} x^\beta\Big(F(-x)+1-F(x)\Big) < +\infty \implies \forall \varepsilon \in (0,\beta]\,,\ \int_{\mathbb{R}} |y|^{\beta-\varepsilon}\mu(dy) < +\infty$$

and $\sup_{x\geq 0} x^\beta\Big(F(-x)+1-F(x)\Big) < +\infty \Leftrightarrow \sup_{u\in(0,1/2]} u^{\frac{1}{\beta}}\left(F^{-1}(1-u)-F^{-1}(u)\right) < +\infty$ *with*

$$\sup_{u\in(0,1/2]} u^{\frac{1}{\beta}}\left(F^{-1}(1-u)-F^{-1}(u)\right) \leq \left(\sup_{x\geq 0} x^\beta F(-x)\right)^{\frac{1}{\beta}} + \left(\sup_{x\geq 0} x^\beta (1-F(x))\right)^{\frac{1}{\beta}}. \tag{2.2}$$

Last, $\lim_{x\to+\infty} x^\beta\Big(F(-x)+1-F(x)\Big) = 0 \Leftrightarrow \lim_{u\to 0+} u^{\frac{1}{\beta}}\left(F^{-1}(1-u)-F^{-1}(u)\right) = 0$.

Proof of Theorem 2.2. Since by Lemma 2.4,

$$\sup_{u\in(0,1/2]} u^{\frac{1}{\rho}-\alpha}\left(F^{-1}(1-u)-F^{-1}(u)\right) < +\infty \implies \sup_{x\geq 0} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big) < +\infty$$

and, by (1.2),

$$e_N^\rho(\mu,\rho) \geq \int_0^{\frac{1}{N}} \left|F^{-1}(u) - x_1^N\right|^\rho du + \int_{\frac{N-1}{N}}^1 \left|F^{-1}(u) - x_N^N\right|^\rho du$$

to prove the equivalence, it is enough to check that

$$\sup_{x\geq 0} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big) < +\infty \implies \sup_{N\geq 1} N^{\alpha\rho}\, e_N^\rho(\mu,\rho) < +\infty \text{ and that}$$

$$\sup_{N\geq 1} N^{\alpha\rho}\left(\int_0^{\frac{1}{N}} \left|F^{-1}(u) - x_1^N\right|^\rho du + \int_{\frac{N-1}{N}}^1 \left|F^{-1}(u) - x_N^N\right|^\rho du\right) < +\infty$$

$$\implies \sup_{u\in(0,1/2]} u^{\frac{1}{\rho}-\alpha}\left(F^{-1}(1-u) - F^{-1}(u)\right) < +\infty.$$

We are now going to do so and thus prove that the four suprema in the two last implications are simultaneously finite or infinite.

Let us first suppose that $C := \sup_{x\geq 0} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big) < +\infty$ and set $N \geq 2$. Let $x_i \in [F^{-1}(\frac{i-1}{N}+), F^{-1}(\frac{i}{N})]$ for $2 \leq i \leq N-1$. We have

$$e_N^\rho(\mu,\rho) \leq \mathcal{W}_\rho^\rho\left(\mu_N(x_{2:N-1}),\mu\right) = L_N + M_N + U_N$$

with $L_N = \int_0^{\frac{1}{N}} \left|F^{-1}(u) - F^{-1}\left(\frac{1}{N}\right) \wedge (-N^{\frac{1}{\rho}-\alpha})\right|^\rho du$, $U_N = \int_{\frac{N-1}{N}}^1 \left|F^{-1}(u) - F^{-1}\left(\frac{N-1}{N}\right) \vee N^{\frac{1}{\rho}-\alpha}\right|^\rho du$ and

$$\begin{aligned}
M_N &= \sum_{i=2}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left|F^{-1}(u) - x_i\right|^\rho du \leq \sum_{i=2}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(F^{-1}\left(\frac{i}{N}\right) - F^{-1}\left(\frac{i-1}{N}\right)\right)^\rho du\\
&\leq \frac{1}{N}\sum_{i=1}^{N}\left(F^{-1}\left(\frac{N-1}{N}\right) - F^{-1}\left(\frac{1}{N}\right)\right)^{\rho-1}\left(F^{-1}\left(\frac{i}{N}\right) - F^{-1}\left(\frac{i-1}{N}\right)\right)\\
&= \frac{1}{N}\left(F^{-1}\left(\frac{N-1}{N}\right) - F^{-1}\left(\frac{1}{N}\right)\right)^\rho \qquad (2.3)\\
&\leq 2^\rho C^{1-\alpha\rho} N^{-\alpha\rho},
\end{aligned}$$

where we used (2.2) applied with $\beta = \frac{\rho}{1-\alpha\rho}$ for the last inequality. Let $x_+ = 0 \vee x$ denote the positive part of any real number x. Applying Lemma 1.2 with $x = F^{-1}\left(\frac{1}{N}\right) \wedge \left(-N^{\frac{1}{\rho}-\alpha}\right)$, we obtain that

$$\begin{aligned}
L_N &= \rho\int_{-\infty}^{F^{-1}(\frac{1}{N})\wedge\left(-N^{\frac{1}{\rho}-\alpha}\right)} \left(F^{-1}\left(\frac{1}{N}\right) \wedge \left(-N^{\frac{1}{\rho}-\alpha}\right) - y\right)^{\rho-1} F(y)\,dy\\
&\quad + \rho\int_{F^{-1}(\frac{1}{N})\wedge\left(-N^{\frac{1}{\rho}-\alpha}\right)}^{F^{-1}(\frac{1}{N})} \left(y - F^{-1}\left(\frac{1}{N}\right) \wedge \left(-N^{\frac{1}{\rho}-\alpha}\right)\right)^{\rho-1}\left(\frac{1}{N} - F(y)\right) dy\\
&\leq \rho\int_{N^{\frac{1}{\rho}-\alpha}}^{+\infty} y^{\rho-1}F(-y)\,dy + \frac{1}{N}\left(N^{\frac{1}{\rho}-\alpha} + F^{-1}\left(\frac{1}{N}\right)\right)_+^\rho.
\end{aligned}$$

In a symmetric way, we check that $U_N \leq \rho\int_{N^{\frac{1}{\rho}-\alpha}}^{+\infty} y^{\rho-1}(1-F(y))\,dy + \frac{1}{N}\left(N^{\frac{1}{\rho}-\alpha} - F^{-1}\left(\frac{N-1}{N}\right)\right)_+^\rho$ so that

$$\begin{aligned}
L_N + U_N &\leq \rho C\int_{N^{\frac{1}{\rho}-\alpha}}^{+\infty} y^{-1-\frac{\alpha\rho^2}{1-\alpha\rho}}\,dy + \frac{1}{N}\left(\left(N^{\frac{1}{\rho}-\alpha} + F^{-1}(1/2)\right)_+^\rho + \left(N^{\frac{1}{\rho}-\alpha} - F^{-1}(1/2)\right)_+^\rho\right)\\
&\leq \frac{1-\alpha\rho}{\alpha\rho}CN^{-\alpha\rho} + \left(1+2^{\rho-1}\right)N^{-\alpha\rho} + 2^{\rho-1}\left|F^{-1}(1/2)\right|^\rho N^{-1}.
\end{aligned}$$

Since $N^{-1} \leq 2^{\alpha\rho-1} N^{-\alpha\rho}$, we conclude that with $\sup_{x_{2:N-1}}$ denoting the supremum over $x_i \in [F^{-1}(\frac{i-1}{N}+), F^{-1}(\frac{i}{N})]$ for $2 \leq i \leq N-1$,

(2.4)
$$\sup_{N\geq 2} N^{\alpha\rho} e_N^\rho(\mu,\rho) \leq \sup_{N\geq 2}\sup_{x_{2:N-1}} N^{\alpha\rho}\mathcal{W}_\rho^\rho(\mu_N(x_{2:N-1}),\mu) \leq 2^\rho C^{1-\alpha\rho} + \frac{1-\alpha\rho}{\alpha\rho}C + 1 + 2^{\rho-1} + 2^{\rho+\alpha\rho-2}\left|F^{-1}(1/2)\right|^\rho.$$

We may replace $\sup_{N\geq 2} N^{\alpha\rho} e_N^\rho(\mu,\rho)$ by $\sup_{N\geq 1} N^{\alpha\rho} e_N^\rho(\mu,\rho)$ in the left-hand side, since, applying Lemma 1.2 with $x=0$, then using that for $y \geq 0$, $F(-y)+1-F(y) = \mu((-\infty,-y]\cup(y,+\infty)) \leq 1$, we obtain that

$$e_1^\rho(\mu,\rho) \leq \rho\int_0^{+\infty} y^{\rho-1}\Big(F(-y)+1-F(y)\Big)\,dy \leq \rho\int_0^1 y^{\rho-1}\,dy + \rho C\int_1^{+\infty} y^{-1-\frac{\alpha\rho^2}{1-\alpha\rho}}\,dy = 1 + \frac{1-\alpha\rho}{\alpha\rho}C.$$

Let us next suppose that $\sup_{N\geq 1} N^{\alpha\rho}\left(\int_0^{\frac{1}{N}}\left|F^{-1}(u)-x_1^N\right|^\rho du + \int_{\frac{N-1}{N}}^1 \left|F^{-1}(u)-x_N^N\right|^\rho du\right) < +\infty$. Like in the proof of Proposition 2.1, we deduce (2.1). With the monotonicity of F^{-1}, this inequality implies that

$$\exists C < +\infty,\ \forall u \in (0,1/2],\quad F^{-1}(u) \geq F^{-1}(1/2) - \frac{C}{1-2^{\alpha-\frac{1}{\rho}}}\left(u^{\alpha-\frac{1}{\rho}}-1\right),$$

and therefore that $\inf_{u\in(0,1/2]}\left(u^{\frac{1}{\rho}-\alpha}F^{-1}(u)\right) > -\infty$. With a symmetric reasoning, we conclude that

$$\sup_{u\in(0,1/2]} u^{\frac{1}{\rho}-\alpha}\Big(F^{-1}(1-u)-F^{-1}(u)\Big) < +\infty.$$

Let us now assume that $\limsup_{x\to+\infty} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big) \in (0,+\infty)$, which, in particular implies that $\sup_{x\geq 0} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big) < +\infty$ and check that $\limsup_{N\to\infty} N^\alpha e_N(\mu,\rho) > 0$. For $x > 0$, we have, on the one hand

$$\begin{aligned} x^{\frac{\alpha\rho^2}{1-\alpha\rho}}\int_x^{+\infty} y^{\rho-1}\Big(F(-y)+1-F(y)\Big)\,dy &\geq x^{\frac{\alpha\rho^2}{1-\alpha\rho}}\int_x^{2x} x^{\rho-1}\Big(F(-2x)+1-F(2x)\Big)\,dy \\ &= x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-2x)+1-F(2x)\Big). \end{aligned}$$

On the other hand, still for $x > 0$,

$$\begin{aligned} x^{\frac{\alpha\rho^2}{1-\alpha\rho}}\int_x^{+\infty} y^{\rho-1}\Big(F(-y)+1-F(y)\Big)\,dy &\leq x^{\frac{\alpha\rho^2}{1-\alpha\rho}}\sup_{y\geq x} y^{\frac{\rho}{1-\alpha\rho}}\Big(F(-y)+1-F(y)\Big)\int_x^{+\infty} y^{-\frac{\alpha\rho^2}{1-\alpha\rho}-1}\,dy \\ &= \frac{1-\alpha\rho}{\alpha\rho^2}\sup_{y\geq x} y^{\frac{\rho}{1-\alpha\rho}}\Big(F(-y)+1-F(y)\Big) \end{aligned} \tag{2.5}$$

Therefore $\limsup_{x\to+\infty} x^{\frac{\alpha\rho^2}{1-\alpha\rho}}\int_x^{+\infty} y^{\rho-1}\Big(F(-y)+1-F(y)\Big)\,dy \in (0,+\infty)$ and, by monotonicity of the integral,

$$\limsup_{N\to+\infty} y_N^{\frac{\alpha\rho^2}{1-\alpha\rho}}\int_{y_N}^{+\infty} y^{\rho-1}\Big(F(-y)+1-F(y)\Big)\,dy \in (0,+\infty) \tag{2.6}$$

along any sequence $(y_N)_{N\in\mathbb{N}}$ of positive numbers increasing to $+\infty$ and such that $\limsup_{N\to+\infty}\frac{y_{N+1}}{y_N} < +\infty$. By Lemmas 2.4 and 1.1, we have $\kappa := \sup_{N\geq 1} N^{\alpha-\frac{1}{\rho}}\left(x_N^N \vee (-x_1^N)\right) < +\infty$ (notice that since $x_1^N \leq x_N^N$, $\kappa \geq 0$). With (1.5), we deduce that:

$$\begin{aligned} \frac{e_N^\rho(\mu,\rho)}{\rho} &\geq \int_{-\infty}^{x_1^N}\left(x_1^N-y\right)^{\rho-1}F(y)\,dy + \int_{x_N^N}^{+\infty}\left(y-x_N^N\right)^{\rho-1}(1-F(y))\,dy \\ &\geq \int_{-\infty}^{-\kappa N^{\frac{1}{\rho}-\alpha}}\left(-\kappa N^{\frac{1}{\rho}-\alpha}-y\right)^{\rho-1}F(y)\,dy + \int_{\kappa N^{\frac{1}{\rho}-\alpha}}^{+\infty}\left(y-\kappa N^{\frac{1}{\rho}-\alpha}\right)^{\rho-1}(1-F(y))\,dy \\ &\geq 2^{1-\rho}\int_{2\kappa N^{\frac{1}{\rho}-\alpha}}^{+\infty} y^{\rho-1}\Big(F(-y)+1-F(y)\Big)\,dy. \end{aligned}$$

Applying (2.6) with $y_N=2\kappa N^{\frac{1}{\rho}-\alpha}$, we conclude that $\limsup_{N\to+\infty} N^{\alpha\rho}e_N^\rho(\mu,\rho)>0$. If $x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)$ does not go to 0 as $x\to+\infty$ then either $\sup_{x\geq 0} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)=+\infty=\sup_{N\geq 1}N^\alpha\, e_N(\mu,\rho)$ or $\limsup_{x\to+\infty} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)\in(0,+\infty)$ and $\limsup_{N\to+\infty} N^\alpha e_N(\mu,\rho)\in(0,+\infty)$ so that, synthesizing the two cases, $N^\alpha e_N(\mu,\rho)$ does not go to 0 as $N\to+\infty$. Therefore, to conclude the proof of the second statement, it is enough to suppose $\lim_{x\to+\infty} x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)=0$ and deduce $\lim_{N\to+\infty} N^\alpha e_N(\mu,\rho)=0$, which we now do. By Lemma 2.4, $\lim_{N\to\infty} N^{\alpha-\frac{1}{\rho}}\left(F^{-1}\left(\frac{N-1}{N}\right)-F^{-1}\left(\frac{1}{N}\right)\right)=0$. Since, reasoning like in the above derivation of (2.3), we have $\sum_{i=2}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}|F^{-1}(u)-x_i^N|^\rho du\leq\frac{1}{N}\left(F^{-1}\left(\frac{N-1}{N}\right)-F^{-1}\left(\frac{1}{N}\right)\right)^\rho$ for $N\geq 3$, we deduce that $\lim_{N\to\infty} N^{\alpha\rho}\sum_{i=2}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}|F^{-1}(u)-x_i^N|^\rho du=0$. Let

$$S_N=\sup_{x\geq N^{\frac{1}{2\rho}-\frac{\alpha}{2}}}\left(x^{\frac{\rho}{1-\alpha\rho}}\Big(F(-x)+1-F(x)\Big)\right)^{\frac{1-\alpha\rho}{2\alpha\rho^2}}\text{ and } y_N=F^{-1}\left(\frac{N-1}{N}\right)\vee N^{\frac{1}{2\rho}-\frac{\alpha}{2}}\vee(S_NN^{\frac{1}{\rho}-\alpha}).$$

Using Lemma 1.2 for the first inequality, (2.5) for the second one, then the definition of y_N for the third, we obtain that

$$\begin{aligned}
&N^{\alpha\rho}\int_{\frac{N-1}{N}}^1|F^{-1}(u)-x_N^N|^\rho du\\
&\leq\rho N^{\alpha\rho}\int_{F^{-1}\left(\frac{N-1}{N}\right)}^{y_N}(y_N-y)^{\rho-1}\left(F(y)-\frac{N-1}{N}\right)dy+\rho N^{\alpha\rho}\int_{y_N}^{+\infty}(y-y_N)^{\rho-1}(1-F(y))dy\\
&\leq N^{\alpha\rho-1}\int_{F^{-1}\left(\frac{N-1}{N}\right)}^{y_N}\rho(y_N-y)^{\rho-1}dy+\frac{1-\alpha\rho}{\alpha\rho^2}N^{\alpha\rho}y_N^{-\frac{\alpha\rho^2}{1-\alpha\rho}}\sup_{y\geq y_N}y^{\frac{\rho}{1-\alpha\rho}}\Big(F(-y)+1-F(y)\Big)\\
&\leq N^{\alpha\rho-1}\left(N^{\frac{1}{2\rho}-\frac{\alpha}{2}}\vee(S_NN^{\frac{1}{\rho}-\alpha})-F^{-1}\left(\frac{N-1}{N}\right)\right)_+^\rho\\
&+\frac{1-\alpha\rho}{\alpha\rho^2}N^{\alpha\rho}\left(S_NN^{\frac{1}{\rho}-\alpha}\right)^{-\frac{\alpha\rho^2}{1-\alpha\rho}}\sup_{y\geq N^{\frac{1}{2\rho}-\frac{\alpha}{2}}}y^{\frac{\rho}{1-\alpha\rho}}\Big(F(-y)+1-F(y)\Big)\\
&\leq\left(N^{\frac{\alpha}{2}-\frac{1}{2\rho}}\vee S_N-N^{\alpha-\frac{1}{\rho}}F^{-1}\left(\frac{N-1}{N}\right)\right)_+^\rho+\frac{1-\alpha\rho}{\alpha\rho^2}S_N^{\frac{\alpha\rho^2}{1-\alpha\rho}}.
\end{aligned}$$

Since $\lim_{N\to\infty} N^{\alpha-\frac{1}{\rho}}F^{-1}\left(\frac{N-1}{N}\right)=0=\lim_{N\to\infty} N^{\frac{\alpha}{2}-\frac{1}{2\rho}}=\lim_{N\to+\infty} S_N$, we deduce that $\lim_{N\to\infty} N^{\alpha\rho}\int_{\frac{N-1}{N}}^1|F^{-1}(u)-x_N^N|^\rho du=0$. Dealing in a symmetric way with $N^{\alpha\rho}\int_0^{\frac{1}{N}}|F^{-1}(u)-x_N^N|^\rho du$, we conclude that $\lim_{N\to\infty} N^{\alpha\rho}e_N^\rho(\mu,\rho)=0$.

□

Example 2.5. *let $\mu_\beta(dx)=f(x)\,dx$ with $f(x)=\beta\frac{\mathbf{1}_{\{x\geq1\}}}{x^{\beta+1}}$ be the Pareto distribution with parameter $\beta>0$. Then $F(x)=\mathbf{1}_{\{x\geq1\}}\left(1-x^{-\beta}\right)$ and $F^{-1}(u)=(1-u)^{-\frac{1}{\beta}}$. To ensure that $\int_{\mathbb{R}}|x|^\rho\mu(dx)<+\infty$, we suppose that $\beta>\rho$. Since $\frac{\rho}{1-\rho\left(\frac{1}{\rho}-\frac{1}{\beta}\right)}=\beta$ we have $\lim_{x\to+\infty} x^{\frac{\rho}{1-\rho\left(\frac{1}{\rho}-\frac{1}{\beta}\right)}}(F(-x)+1-F(x))=1$. Replacing* lim sup *by* lim inf *in the last step of the proof of Theorem 2.2, we check that $\liminf_{N\to+\infty} N^{\frac{1}{\rho}-\frac{1}{\beta}}e_N(\mu_\beta,\rho)>0$ and deduce with the statement of this theorem that $e_N(\mu_\beta,\rho)\asymp N^{-\frac{1}{\rho}+\frac{1}{\beta}}\asymp\sup_{x_{2:N-1}}\mathcal{W}_\rho(\mu_N(x_{2:N-1}),\mu_\beta)$.*

2.3 The boundary case $\alpha=1/\rho$

Before giving a necessary condition for convergence with the boundary order $1/\rho$, we show in the next example that for $\beta>0$, $e_N(\mathbf{1}_{\{x>0\}}\beta x^{\beta-1}\exp\left(-x^\beta\right)dx,\rho)$ converges to 0 with this order up to some logarithmic factor. In particular, the case $\beta>1$ illustrates the possibility that, when $\rho>1$, $\lim_{N\to+\infty} N^{1/\rho}e_N(\mu,\rho)=0$ for some probability measures μ with unbounded support. Of course, F^{-1} is then continuous on $(0,1)$, since, by Remark 5.22 (ii) [17], $\limsup_{N\to+\infty} N^{1/\rho}e_N(\mu,\rho)>0$ otherwise.

Example 2.6. *For the Weibull distribution* $\mu_\beta(dx) = f(x)\,dx$ *with* $f(x) = \mathbf{1}_{\{x>0\}}\beta x^{\beta-1}\exp\left(-x^\beta\right)$ *with* $\beta > 0$ *(the exponential distribution case* $\beta = 1$*, was addressed in Example 5.17 and Remark 5.22 (i) [17]), we have that* $F(x) = \mathbf{1}_{\{x>0\}}\left(1-\exp(-x^\beta)\right)$*,* $F^{-1}(u) = (-\ln(1-u))^{\frac{1}{\beta}}$ *and* $f\left(F^{-1}(u)\right) = \beta(1-u)(-\ln(1-u))^{1-\frac{1}{\beta}}$*. The density* f *is decreasing on* $[x_\beta, +\infty)$ *where* $x_\beta = \left(\frac{(\beta-1)\vee 0}{\beta}\right)^{\frac{1}{\beta}}$*. Using (1.4), the equality* $F^{-1}(w)-F^{-1}(u) = \int_u^w \frac{dv}{f(F^{-1}(v))}$ *valid for* $u, w \in (0,1)$ *and the monotonicity of the density, we obtain that for* N *large enough so that* $\lceil NF(x_\beta)\rceil \le N-1$*,*

$$e_N^\rho(\mu_\beta,\rho) \ge \frac{1}{4^\rho N}\sum_{i=\lceil NF(x_\beta)\rceil+1}^{N}\left(\int_{\frac{4i-3}{4N}}^{\frac{4i-1}{4N}}\frac{du}{f(F^{-1}(u))}\right)^\rho \ge \frac{1}{8^\rho N^{\rho+1}}\sum_{i=\lceil NF(x_\beta)\rceil+1}^{N}\frac{1}{f^\rho\left(F^{-1}\left(\frac{4i-3}{4N}\right)\right)}$$

$$\ge \frac{1}{(8N)^\rho}\sum_{i=\lceil NF(x_\beta)\rceil+2}^{N}\int_{\frac{i-2}{N}}^{\frac{i-1}{N}}\frac{du}{f^\rho\left(F^{-1}(u)\right)} = \frac{1}{(8N)^\rho}\int_{\frac{\lceil NF(x_\beta)\rceil}{N}}^{\frac{N-1}{N}}\frac{du}{f^\rho\left(F^{-1}(u)\right)}. \tag{2.7}$$

Using (1.3) for the first inequality, Hölder's inequality for the second inequality, then Fubini's theorem for the third, we obtain that

$$e_N^\rho(\mu_\beta,\rho) - \int_{\frac{N-1}{N}}^1 \left|x_N^N - F^{-1}(u)\right|^\rho du \le \sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left|\int_{\frac{2i-1}{2N}}^{u}\frac{dv}{f(F^{-1}(v))}\right|^\rho du$$

$$\le \sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left|u-\frac{2i-1}{2N}\right|^{\rho-1}\left|\int_{\frac{2i-1}{2N}}^{u}\frac{dv}{f^\rho\left(F^{-1}(v)\right)}\right|du \le \frac{1}{(2N)^\rho\rho}\int_0^{\frac{N-1}{N}}\frac{dv}{f^\rho\left(F^{-1}(v)\right)}.$$

We have $F(x_\beta) < 1$ *and, when* $\beta > 1$*,* $F(x_\beta) > 0$*. By integration by parts, for* $\rho > 1$*,*

$$(\rho-1)\int_{F(x_\beta)}^{\frac{N-1}{N}}\frac{\beta^\rho\,du}{f^\rho\left(F^{-1}(u)\right)} = \int_{F(x_\beta)}^{\frac{N-1}{N}}(\rho-1)(1-u)^{-\rho}(-\ln(1-u))^{\frac{\rho}{\beta}-\rho}\,du$$

$$= \left[(1-u)^{1-\rho}(-\ln(1-u))^{\frac{\rho}{\beta}-\rho}\right]_{F(x_\beta)}^{\frac{N-1}{N}} + \left(\frac{\rho}{\beta}-\rho\right)\int_{F(x_\beta)}^{\frac{N-1}{N}}(1-u)^{-\rho}(-\ln(1-u))^{\frac{\rho}{\beta}-\rho-1}\,du$$

$$= N^{\rho-1}(\ln N)^{\frac{\rho}{\beta}-\rho} + o\left(\int_{F(x_\beta)}^{\frac{N-1}{N}}(1-u)^{-\rho}(-\ln(1-u))^{\frac{\rho}{\beta}-\rho}\,du\right) \sim N^{\rho-1}(\ln N)^{\frac{\rho}{\beta}-\rho},$$

as $N \to +\infty$*. We obtain the same equivalent when replacing the lower integration limit* $F(x_\beta)$ *in the left-hand side by* $\frac{\lceil NF(x_\beta)\rceil}{N}$ *or* 0 *since* $\lim_{N\to+\infty}\int_{F(x_\beta)}^{\frac{\lceil NF(x_\beta)\rceil}{N}}\frac{du}{f^\rho(F^{-1}(u))} = 0$ *and* $\int_0^{F(x_\beta)}\frac{du}{f^\rho(F^{-1}(u))} < +\infty$*. On the other hand,*

$$\int_{\frac{N-1}{N}}^1\left|x_N^N - F^{-1}(u)\right|^\rho du \le \int_{\frac{N-1}{N}}^1\left((-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}}\right)^\rho du.$$

When $\beta < 1$*, for* $u \in \left[\frac{N-1}{N}, 1\right]$*,* $(-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}} \le \frac{1}{\beta}(-\ln(1-u))^{\frac{1}{\beta}-1}(-\ln(1-u)-\ln N)$ *so that*

$$\int_{\frac{N-1}{N}}^1\left((-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}}\right)^\rho du \le \frac{1}{\beta^\rho}\int_{\frac{N-1}{N}}^1(-\ln(1-u))^{\frac{\rho}{\beta}-\rho}(-\ln(N(1-u)))^\rho\,du$$

$$= \frac{1}{\beta^\rho N}\int_0^1(\ln N - \ln v)^{\frac{\rho}{\beta}-\rho}(-\ln(v))^\rho\,dv$$

$$\le \frac{2^{(\frac{\rho}{\beta}-\rho-1)\vee 0}}{\beta^\rho N}\left((\ln N)^{\frac{\rho}{\beta}-\rho}\int_0^1(-\ln(v))^\rho\,dv + \int_0^1(-\ln(v))^{\frac{\rho}{\beta}}\,dv\right). \tag{2.8}$$

When $\beta \ge 1$*, for* $N \ge 2$ *and* $u \in \left[\frac{N-1}{N}, 1\right]$*,* $(-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}} \le \frac{1}{\beta}(\ln N)^{\frac{1}{\beta}-1}(-\ln(1-u)-\ln N)$ *so that*

$$\int_{\frac{N-1}{N}}^1\left((-\ln(1-u))^{\frac{1}{\beta}} - (\ln N)^{\frac{1}{\beta}}\right)^\rho du \le \frac{(\ln N)^{\frac{\rho}{\beta}-\rho}}{\beta^\rho N}\int_0^1(-\ln(v))^{\frac{\rho}{\beta}}\,dv. \tag{2.9}$$

We conclude that for $\rho>1$ *and* $\beta>0$, $e_N(\mu_\beta,\rho)\asymp N^{-\frac{1}{\rho}}(\ln N)^{\frac{1}{\beta}-1}\asymp \mathcal{W}_\rho(\frac{1}{N}(\sum_{i=1}^{N-1}\delta_{F^{-1}\left(\frac{2i-1}{2N}\right)}+\delta_{F^{-1}\left(\frac{N-1}{N}\right)}),\mu_\beta)$. *In view of Theorem 5.20 [17], this rate of convergence does not extend continuously to* $e_N(\mu_\beta,1)$*, at least for* $\beta>1$. *In fact, by Remark 2.2 [14], for* $\beta>0$, $e_N(\mu_\beta,1)\asymp N^{-1}(\ln N)^{\frac{1}{\beta}}$*, which in view of* (2.8) *and* (2.9)*, implies that* $\sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left|x_i^N-F^{-1}(u)\right|^\rho du\asymp N^{-1}(\ln N)^{\frac{1}{\beta}}$.

In the Gaussian tail case $\beta=2$, $e_N(\mu_2,\rho)\asymp N^{-\frac{1}{\rho}}(\ln N)^{-\frac{1}{2}+\mathbf{1}_{\{\rho=1\}}}$ *for* $\rho\geq 1$*, like for the true Gaussian distribution, according to Example 5.18 [17]. This matches the rate obtained when* $\rho>2$ *in Corollary 6.14 [6] for* $\mathbb{E}^{1/\rho}\left[\mathcal{W}_\rho^\rho\left(\frac{1}{N}\sum_{i=1}^N\delta_{X_i},\mu\right)\right]$ *where* $(X_i)_{i\geq1}$ *are i.i.d. with respect to some Gaussian distribution* μ *with positive variance. When* $\rho=2$*, still according to this corollary the random rate is* $N^{-1/2}(\ln\ln N)^{1/2}$ *(of course worse than the standard Monte Carlo rate* $N^{-1/2}$*).*

The next proposition gives a necessary condition for $e_N(\mu,\rho)$ to go to 0 with order $\alpha=\frac{1}{\rho}$.

Proposition 2.7. *For* $\rho\geq1$,

$$\begin{aligned}
&\sup_{N\geq1}N^{1/\rho}e_N(\mu,\rho)<+\infty\\
\Rightarrow&\sup_{N\geq1}N\left(\int_0^{\frac{1}{N}}\left|F^{-1}(u)-x_1^N\right|^\rho du+\int_{\frac{N-1}{N}}^1\left|F^{-1}(u)-x_N^N\right|^\rho du\right)<+\infty\\
\Leftrightarrow&\sup_{u\in(0,1/2]}\left(F^{-1}(1-u/2)-F^{-1}(1-u)+F^{-1}(u)-F^{-1}(u/2)\right)<+\infty\\
\Rightarrow&\sup_{u\in(0,1/2]}\frac{F^{-1}(1-u)-F^{-1}(u)}{\ln(1/u)}<+\infty\\
\Leftrightarrow&\exists\lambda\in(0,+\infty),\ \forall x\geq0,\ \Big(F(-x)+1-F(x)\Big)\leq e^{-\lambda x}/\lambda\\
\Rightarrow&\sup_{N\geq2}\sup_{x_{2:N-1}}\frac{N^{1/\rho}}{1+\ln N}\mathcal{W}_\rho(\mu_{N,\lambda}(x_{2:N-1}),\mu)<+\infty\\
\Rightarrow&\sup_{N\geq1}\frac{N^{1/\rho}}{1+\ln N}e_N(\mu,\rho)<+\infty,
\end{aligned}$$

where $\mu_{N,\lambda}(x_{2:N-1})=\frac{1}{N}\left(\delta_{F^{-1}\left(\frac{1}{N}\right)\wedge\left(-\frac{\ln N}{\lambda}\right)}+\sum_{i=2}^{N-1}\delta_{x_i}+\delta_{F^{-1}\left(\frac{N-1}{N}\right)\vee\frac{\ln N}{\lambda}}\right)$ *and* $\sup_{x_{2:N-1}}$ *means the supremum over the choice of* $x_i\in[F^{-1}(\frac{i-1}{N}+),F^{-1}(\frac{i}{N})]$ *for* $2\leq i\leq N-1$.

Remark 2.8.

- *The first implication is not an equivalence for* $\rho=1$*. Indeed, in Example 2.6, for* $\beta\geq1$, $\lim_{N\to+\infty}Ne_N(\mu,1)=+\infty$ *while* $\sup_{N\geq1}N\left(\int_0^{\frac{1}{N}}\left|F^{-1}(u)-x_1^N\right|du+\int_{\frac{N-1}{N}}^1\left|F^{-1}(u)-x_N^N\right|du\right)<+\infty$.

- *The second implication is not an equivalence as examplified by* $F^{-1}(u)=g(u)\ln(u)$ *where* $g(u)=\sum_{k\in\mathbb{N}}\frac{k}{k+1}\mathbf{1}_{(2^{-(k+1)^3},2^{-k^3}]}(u)$*. The function* $(0,1)\ni u\mapsto g(u)\ln(u)$ *is a quantile function since it is left-continuous and non-decreasing as the product of the left-continuous, non-increasing and non-negative function* g *by the continuous, non-decreasing and non-positive logarithm function. Moreover, since* g *is bounded by* 1*, one easily checks that* $\sup_{u\in(0,1/2]}\frac{F^{-1}(1-u)-F^{-1}(u)}{\ln(1/u)}<+\infty$*. On the other hand,*

$$\begin{aligned}
F^{-1}\left(2^{1-(k+1)^3}\right)-F^{-1}\left(2^{-(k+1)^3}\right)&=\frac{k}{k+1}(1-(k+1)^3)\ln2+\frac{k+1}{k+2}(k+1)^3\ln2\\
&=\frac{k}{k+1}\ln2+\frac{(k+1)^2}{k+2}\ln2
\end{aligned}$$

  goes to ∞ *with* k.

- *The exponential tail condition* $\exists\lambda\in(0,+\infty),\ \forall x\geq0,\ \Big(F(-x)+1-F(x)\Big)\leq e^{-\lambda x}/\lambda$ *is not equivalent to* $\sup_{N\geq1}\frac{N^{1/\rho}}{1+\ln N}e_N(\mu,\rho)<+\infty$ *when* $\rho>1$ *since in Example 2.6, for* $\beta\in[1/2,1)$, $\sup_{N\geq1}\frac{N^{1/\rho}}{1+\ln N}e_N(\mu,\rho)<+\infty$ *while the exponential tail condition fails.*

Proof. The first implication is an immediate consequence of (1.2).
To prove the first equivalence, we first suppose that:

$$\sup_{N\geq 1} N^{\frac{1}{\rho}}\left(\int_0^{\frac{1}{N}}\left|F^{-1}(u)-x_1^N\right|^\rho du+\int_{\frac{N-1}{N}}^1\left|F^{-1}(u)-x_N^N\right|^\rho du\right)^{\frac{1}{\rho}}<+\infty. \tag{2.10}$$

and denote by C the finite supremum in this equation. By (1.4) for $i=1$, $\forall N\geq 1$, $F^{-1}\left(\frac{1}{2N}\right)-F^{-1}\left(\frac{1}{4N}\right)\leq 4C$. For $u\in(0,1/2]$, there exists $N\in\mathbb{N}^*$ such that $u\in\left[\frac{1}{2(N+1)},\frac{1}{2N}\right]$ and, by monotonicity of F^{-1} and since $4N\geq 2(N+1)$, we get

$$\begin{aligned}F^{-1}(u)-F^{-1}(u/2)&\leq F^{-1}\left(\frac{1}{2N}\right)-F^{-1}\left(\frac{1}{4(N+1)}\right)\\&\leq F^{-1}\left(\frac{1}{2N}\right)-F^{-1}\left(\frac{1}{4N}\right)+F^{-1}\left(\frac{1}{2(N+1)}\right)-F^{-1}\left(\frac{1}{4(N+1)}\right)\leq 8C.\end{aligned}$$

Dealing in a symmetric way with $F^{-1}(1-u/2)-F^{-1}(1-u)$, we obtain that

$$\sup_{u\in(0,1/2]}\Big(F^{-1}(1-u/2)-F^{-1}(1-u)+F^{-1}(u)-F^{-1}(u/2)\Big)\leq 16C.$$

On the other hand, for $N\geq 2$, by Lemma 1.2 applied with $x=F^{-1}\left(\frac{1}{N}\right)$,

$$\begin{aligned}\frac{1}{\rho}\int_0^{\frac{1}{N}}\left|F^{-1}(u)-x_1^N\right|^\rho du&\leq\sum_{k\in\mathbb{N}}\int_{F^{-1}\left(\frac{1}{2^{k+1}N}\right)}^{F^{-1}\left(\frac{1}{2^kN}\right)}\left(F^{-1}\left(\frac{1}{N}\right)-y\right)^{\rho-1}F(y)\,dy\\&\leq\sum_{k\in\mathbb{N}}\frac{F^{-1}\left(\frac{1}{2^kN}\right)-F^{-1}\left(\frac{1}{2^{k+1}N}\right)}{2^kN}\left(\sum_{j=0}^k\left(F^{-1}\left(\frac{1}{2^jN}\right)-F^{-1}\left(\frac{1}{2^{j+1}N}\right)\right)\right)^{\rho-1}\\&\leq\frac{1}{N}\left(\sup_{u\in(0,1/2]}\left(F^{-1}(u)-F^{-1}(u/2)\right)\right)^\rho\sum_{k\in\mathbb{N}}\frac{(k+1)^{\rho-1}}{2^k},\end{aligned}$$

where the last sum is finite. Dealing in a symmetric way with $\int_{\frac{N-1}{N}}^1\left|F^{-1}(u)-x_N^N\right|^\rho du$, we conclude that (2.10) is equivalent to the finiteness of $\sup_{u\in(0,1/2]}\Big(F^{-1}(1-u/2)-F^{-1}(1-u)+F^{-1}(u)-F^{-1}(u/2)\Big)$. Under (2.10) with C denoting the finite supremum, for $k\in\mathbb{N}^*$, $F^{-1}\left(2^{-(k+1)}\right)-F^{-1}\left(2^{-k}\right)\geq -4C$ and, after summation,

$$F^{-1}(2^{-k})\geq F^{-1}(1/2)-4C(k-1).$$

With the monotonicity of F^{-1}, we deduce that:

$$\forall u\in(0,1/2],\ F^{-1}(u)\geq F^{-1}(1/2)+\frac{4C}{\ln 2}\ln u$$

and therefore that $\sup_{u\in(0,1/2]}\frac{-F^{-1}(u)}{\ln(1/u)}<+\infty$. With the inequality $F^{-1}(F(x))\leq x$ valid for $x\in\mathbb{R}$, this implies that $\sup_{\{x\in\mathbb{R}:0<F(x)\leq 1/2\}}\frac{-x}{\ln(1/F(x))}<+\infty$ and therefore that $\exists\,\lambda\in(0,+\infty)$, $\forall x\leq 0$, $F(x)\leq e^{\lambda x}/\lambda$. Under the latter condition, since $u\leq F(F^{-1}(u))$ and $F^{-1}(u)\leq 0$ for $u\in(0,F(0)]$, we have $\sup_{u\in(0,F(0)]}\frac{-F^{-1}(u)}{\ln(1/u)}<\infty$ and even $\sup_{u\in(0,1/2]}\frac{-F^{-1}(u)}{\ln(1/u)}<\infty$ since when $F(0)<\frac{1}{2}$, $\sup_{u\in(F(0),1/2]}\frac{-F^{-1}(u)}{\ln(1/u)}\leq 0$. By a symmetric reasoning, we obtain the two equivalent tail properties $\sup_{u\in(0,1/2]}\frac{F^{-1}(1-u)-F^{-1}(u)}{\ln(1/u)}<+\infty$ and $\exists\,\lambda\in(0,+\infty)$, $\forall x\geq 0$, $\Big(F(-x)+1-F(x)\Big)\leq e^{-\lambda x}/\lambda$.

Let us finally suppose these two tail properties and deduce that $\sup_{N\geq 2}\sup_{x_{2:N-1}}\frac{N^{1/\rho}}{1+\ln N}\mathcal{W}_\rho(\mu_{N,\lambda}(x_{2:N-1}),\mu)<+\infty$. We use the decomposition $\mathcal{W}_\rho^\rho(\mu_{N,\lambda}(x_{2:N-1}),\mu)=L_N+M_N+U_N$ introduced in the proof of Theorem

2.2 but with $F^{-1}\left(\frac{1}{N}\right)\wedge\left(-\frac{\ln N}{\lambda}\right)$ and $F^{-1}\left(\frac{N-1}{N}\right)\vee\left(\frac{\ln N}{\lambda}\right)$ respectively replacing $F^{-1}\left(\frac{1}{N}\right)\wedge\left(-N^{\frac{1}{\rho}-\alpha}\right)$ and $F^{-1}\left(\frac{N-1}{N}\right)\vee\left(N^{\frac{1}{\rho}-\alpha}\right)$ in L_N and U_N. By (2.3), we get:

$$\forall N\geq 3,\ M_N\leq\frac{1}{N}\left(F^{-1}\left(\frac{N-1}{N}\right)-F^{-1}\left(\frac{1}{N}\right)\right)^{\rho}\leq\left(\sup_{u\in(0,1/2]}\frac{F^{-1}(1-u)-F^{-1}(u)}{\ln(1/u)}\right)^{\rho}\frac{(\ln N)^{\rho}}{N}.$$

Applying Lemma 1.2 with $x=F^{-1}\left(\frac{1}{N}\right)\wedge\left(-\frac{\ln N}{\lambda}\right)$ then the estimation of the cumulative distribution function, we obtain that for $N\geq 2$,

$$\begin{aligned}
L_N &\leq \rho\int_{-\infty}^{F^{-1}\left(\frac{1}{N}\right)\wedge\left(-\frac{\ln N}{\lambda}\right)}\left(F^{-1}\left(\frac{1}{N}\right)\wedge\left(-\frac{\ln N}{\lambda}\right)-y\right)^{\rho-1}F(y)\,dy\\
&+\rho\int_{F^{-1}\left(\frac{1}{N}\right)\wedge\left(-\frac{\ln N}{\lambda}\right)}^{F^{-1}\left(\frac{1}{N}\right)}\left(y-F^{-1}\left(\frac{1}{N}\right)\wedge\left(-\frac{\ln N}{\lambda}\right)\right)^{\rho-1}\left(\frac{1}{N}-F(y)\right)dy\\
&\leq\frac{\rho}{\lambda}\int_{-\infty}^{-\frac{\ln N}{\lambda}}(-y)^{\rho-1}e^{\lambda y}\,dy+\frac{1}{N}\left(\frac{\ln N}{\lambda}+F^{-1}\left(\frac{1}{N}\right)\right)_+^{\rho}\\
&\leq\frac{\rho}{\lambda}\sum_{k\geq 1}\int_{k\frac{\ln N}{\lambda}}^{(k+1)\frac{\ln N}{\lambda}}\left((k+1)\frac{\ln N}{\lambda}\right)^{\rho-1}e^{-\lambda y}\,dy+\frac{1}{N}\left(\frac{\ln N}{\lambda}+F^{-1}\left(\frac{1}{2}\right)\right)_+^{\rho}\\
&\leq\frac{\rho(\ln N)^{\rho}}{\lambda^{\rho+1}N}\sum_{k\geq 1}\frac{(k+1)^{\rho-1}}{2^{k-1}}+\frac{1}{N}\left(\frac{1}{\lambda}\ln N+F^{-1}\left(\frac{1}{2}\right)\right)_+^{\rho},
\end{aligned}$$

where we used that $N^k\geq N2^{k-1}$ for the last inequality. Dealing in a symmetric way with U_N, we conclude that $\sup_{N\geq 2}\sup_{x_{2:N-1}}\frac{N\mathcal{W}_\rho^\rho(\mu_{N,\lambda}(x_{2:N-1}),\mu)}{1+(\ln N)^{\rho}}<+\infty$. □

Conclusion

In the present paper, we have characterized the convergence of $e_N(\mu,\rho)$ to 0 with order $\alpha\in(0,\frac{1}{\rho})$ and also studied the convergence with boundary order $\alpha=1/\rho$ between the unbounded support case and the bounded support case. In view of Example 2.6, it would be nice to investigate whether the leading factor remains $N^{-1/\rho}$ for μ with unbounded support and superpolynomial but subexponential tails. Characterizing the order of convergence when the support of μ is bounded and its quantile function F^{-1} is continuous is another open question.

Of course, generalizing our results to higher dimension would be of great interest. This appears to be challenging since our approach heavily relies on one-dimensional tools like the cumulative distribution function and the quantile function.

Appendix

Proof of Lemma 1.1. Since the finiteness of $\sup_{u\in(0,1/2]}u^{\frac{1}{\rho}-\alpha}\left(F^{-1}(1-u)-F^{-1}(u)\right)$ implies the finiteness of both $\sup_{u\in(0,1)}u^{\frac{1}{\rho}-\alpha}F^{-1}(1-u)$ and $\inf_{u\in(0,1)}u^{\frac{1}{\rho}-\alpha}F^{-1}(u)$, the second statement is a consequence of the first one, that we are now going to prove. When $\rho=1$ (resp. $\rho=2$), then the conclusion easily follows from the explicit form $x_1^N=F^{-1}\left(\frac{1}{2N}\right)$ and $x_N^N=F^{-1}\left(\frac{2N-1}{2N}\right)$ (resp. $x_1^N=N\int_0^{\frac{1}{N}}F^{-1}(u)\,du$ and $x_N^N=N\int_{\frac{N-1}{N}}^1F^{-1}(u)\,du$). In the general case $\rho>1$, we are going to take advantage of the expression

$$f(y)=\rho\int_0^{\frac{1}{N}}\left(\mathbf{1}_{\{y\geq F^{-1}(1-u)\}}\left(y-F^{-1}(1-u)\right)^{\rho-1}-\mathbf{1}_{\{y<F^{-1}(1-u)\}}\left(F^{-1}(1-u)-y\right)^{\rho-1}\right)du$$

of the derivative of the function $\mathbb{R}\ni y\mapsto\displaystyle\int_{\frac{N-1}{N}}^1\left|y-F^{-1}(u)\right|^{\rho}du$ minimized by x_N^N. Since this function is strictly convex $x_N^N=\inf\{y\in\mathbb{R}:f(y)\geq 0\}$. Let us first suppose that $S_N:=\sup_{u\in(0,\frac{1}{N})}u^{\frac{1}{\rho}-\alpha}F^{-1}(1-u)\in$

$(0,+\infty)$. Since for fixed $y\in\mathbb{R}$, $\mathbb{R}\ni x\mapsto \left(\mathbf{1}_{\{y\geq x\}}(y-x)^{\rho-1}-\mathbf{1}_{\{y<x\}}(x-y)^{\rho-1}\right)$ is non-increasing, we deduce that $\forall y\in\mathbb{R}$, $f(y)\geq \rho S_N^{\rho-1}g(\frac{y}{S_N})$ where

$$g(z)=\int_0^{\frac{1}{N}}\left(\mathbf{1}_{\left\{z\geq u^{\alpha-\frac{1}{\rho}}\right\}}\left(z-u^{\alpha-\frac{1}{\rho}}\right)^{\rho-1}-\mathbf{1}_{\left\{z<u^{\alpha-\frac{1}{\rho}}\right\}}\left(u^{\alpha-\frac{1}{\rho}}-z\right)^{\rho-1}\right)du.$$

For $z\geq (4N)^{\frac{1}{\rho}-\alpha}$, we have $z^{\frac{\rho}{\alpha\rho-1}}\leq\frac{1}{4N}$ and $z-(2N)^{\frac{1}{\rho}-\alpha}\geq\left(1-2^{\alpha-\frac{1}{\rho}}\right)z$ so that

$$\begin{aligned}g(z)&=\int_{z^{\frac{\rho}{\alpha\rho-1}}}^{\frac{1}{N}}\left(z-u^{\alpha-\frac{1}{\rho}}\right)^{\rho-1}du-\int_0^{z^{\frac{\rho}{\alpha\rho-1}}}\left(u^{\alpha-\frac{1}{\rho}}-z\right)^{\rho-1}du\geq\int_{\frac{1}{2N}}^{\frac{1}{N}}\left(z-(2N)^{\frac{1}{\rho}-\alpha}\right)^{\rho-1}du-\int_0^{z^{\frac{\rho}{\alpha\rho-1}}}u^{(\rho-1)\frac{\alpha\rho-1}{\rho}}du\\&\geq\left(1-2^{\alpha-\frac{1}{\rho}}\right)^{\rho-1}z^{\rho-1}\int_{\frac{1}{2N}}^{\frac{1}{N}}du-\frac{\rho z^{\frac{\rho}{\alpha\rho-1}+\rho-1}}{1+(\rho-1)\alpha\rho}=z^{\rho-1}\left(\frac{\left(1-2^{\alpha-\frac{1}{\rho}}\right)^{\rho-1}}{2N}-\frac{\rho z^{\frac{\rho}{\alpha\rho-1}}}{1+(\rho-1)\alpha\rho}\right).\end{aligned}$$

The right-hand side is positive for $z>(\kappa N)^{\frac{1}{\rho}-\alpha}$ with $\kappa:=\frac{2\rho}{\left(1-2^{\alpha-\frac{1}{\rho}}\right)^{\rho-1}(1+(\rho-1)\alpha\rho)}$. Hence for $z>((\kappa\vee 4)\,N)^{\frac{1}{\rho}-\alpha}$, $g(z)>0$ so that for $y>((\kappa\vee 4)\,N)^{\frac{1}{\rho}-\alpha}S_N$, $f(y)>0$ and therefore

$$x_N^N\leq((\kappa\vee 4)\,N)^{\frac{1}{\rho}-\alpha}S_N.$$

Clearly, this inequality remains valid when $S_N=+\infty$. It holds in full generality since $S_N\geq 0$ and $S_N=0\Leftrightarrow F^{-1}(1)\leq 0$ a condition under which $x_N^N\leq 0$ since $x_N^N\in\left[F^{-1}\left(\frac{N-1}{N}+\right),F^{-1}(1)\right]\cap\mathbb{R}$. By a symmetric reasoning, we check that $x_1^N\geq((\kappa\vee 4)\,N)^{\frac{1}{\rho}-\alpha}\inf_{u\in(0,\frac{1}{N})}u^{\frac{1}{\rho}-\alpha}F^{-1}(u)$. □

Proof of Lemma 2.4. Let $\beta>0$. For $x>0$, using the monotonicity of F for the first inequality then that for $y\in[\frac{x}{2},x]$, $y^{\beta-1}\geq\left(\frac{x}{2}\right)^{\beta-1}\wedge x^{\beta-1}=\frac{x^{\beta-1}}{2^{(\beta-1)\vee 0}}$, we obtain that

$$F(-x)+1-F(x)\leq\frac{2}{x}\int_{x/2}^{x}\Big(F(-y)+1-F(y)\Big)dy\leq\frac{2^{\beta\vee 1}}{x^\beta}\int_{x/2}^{+\infty}y^{\beta-1}\Big(F(-y)+1-F(y)\Big)dy.$$

Since $\int_0^{+\infty}y^{\beta-1}\Big(F(-y)+1-F(y)\Big)dy=\frac{1}{\beta}\int_{\mathbb{R}}|y|^\beta\mu(dy)$, the finiteness of $\int_{\mathbb{R}}|y|^\beta\mu(dy)$ implies by Lebesgue's theorem that $\lim_{x\to\infty}x^\beta\left(F(-x)+1-F(x)\right)=0$. Since $x\mapsto x^\beta\left(F(-x)+1-F(x)\right)$ is right-continuous with left-hand limits on $[0,+\infty)$,

$$\sup_{x\geq 0}x^\beta\Big(F(-x)+1-F(x)\Big)<+\infty\Leftrightarrow\limsup_{x\to\infty}x^\beta\Big(F(-x)+1-F(x)\Big)<+\infty,$$

with the latter property clearly implied by $\lim_{x\to\infty}x^\beta\left(F(-x)+1-F(x)\right)=0$.

For $\varepsilon\in(0,\beta)$, using that for $y\geq 0$, $F(-y)+1-F(y)=\mu((-\infty,-y]\cup(y,+\infty))\leq 1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}}|x|^{\beta-\varepsilon}\mu(dx)&=(\beta-\varepsilon)\int_0^{+\infty}y^{\beta-\varepsilon-1}(F(-y)+1-F(y))\,dy\\&\leq(\beta-\varepsilon)\int_0^1 y^{\beta-\varepsilon-1}dy+(\beta-\varepsilon)\sup_{x\geq 0}x^\beta\Big(F(-x)+1-F(x)\Big)\int_1^{+\infty}y^{-\varepsilon-1}\,dy\\&=1+\frac{\beta-\varepsilon}{\varepsilon}\sup_{x\geq 0}x^\beta\Big(F(-x)+1-F(x)\Big).\end{aligned}$$

Therefore $\sup_{x\geq 0}x^\beta\Big(F(-x)+1-F(x)\Big)<+\infty\implies\forall\varepsilon\in(0,\beta),\ \int_{\mathbb{R}}|x|^{\beta-\varepsilon}\mu(dx)<+\infty.$

Let us next check that

$$\sup_{x\geq 0}x^\beta\Big(F(-x)+(1-F(x))\Big)<+\infty\Leftrightarrow\sup_{u\in(0,1/2]}u^{\frac{1}{\beta}}(F^{-1}(1-u)-F^{-1}(u))<+\infty.\tag{2.11}$$

For the necessary condition, we set $u \in (0, 1/2]$. Either $F^{-1}(u) \geq 0$ or, since for all $v \in (0,1)$, $F(F^{-1}(v)) \geq v$, we have $\left(-F^{-1}(u)\right)^{\beta} u \leq \sup_{x \geq -F^{-1}(u)} x^{\beta} F(-x)$ and therefore $F^{-1}(u) \geq -\left(\sup_{x\geq 0} x^{\beta} F(-x)\right)^{\frac{1}{\beta}} u^{-\frac{1}{\beta}}$. Either $F^{-1}(1-u) \leq 0$ or, since for all $v \in (0,1)$, $F(F^{-1}(v)-) \leq v$, we have $(F^{-1}(1-u))^{\beta} u \leq \sup_{x \geq F^{-1}(1-u)} x^{\beta}(1 - F(x-))$ and therefore $F^{-1}(1-u) \leq \left(\sup_{x\geq 0} x^{\beta}(1-F(x))\right)^{\frac{1}{\beta}} u^{-\frac{1}{\beta}}$. Hence (2.2) holds.

For the sufficient condition, we remark that the finiteness of $\sup_{u\in(0,1/2]} u^{\frac{1}{\beta}} (F^{-1}(1-u) - F^{-1}(u))$ and the inequality $F^{-1}(1-u) - F^{-1}(u) \geq \left(F^{-1}(1/2) - F^{-1}(u)\right) \vee \left(F^{-1}(1-u) - F^{-1}(1/2)\right)$ valid for $u \in (0,1/2]$ imply that $\inf_{u\in(0,1/2]} u^{\frac{1}{\beta}} F^{-1}(u) > -\infty$ and $\sup_{u\in(0,1/2]} u^{\frac{1}{\beta}} F^{-1}(1-u) < +\infty$. With the inequality $x \geq F^{-1}(F(x))$ valid for $x \in \mathbb{R}$ such that $0 < F(x) < 1$, this implies that $\inf_{x\in\mathbb{R}:F(x)\leq 1/2} (F(x))^{\frac{1}{\beta}} x > -\infty$ and therefore that $\sup_{x\geq 0} x^{\beta} F(-x) < +\infty$. With the inequality $x \leq F^{-1}(F(x)+)$ valid for $x \in \mathbb{R}$ such that $0 < F(x) < 1$, we obtain, in a symmetric way $\sup_{x\geq 0} x^{\beta}(1 - F(x)) < +\infty$.

Let us finally check that $\lim_{x\to+\infty} x^{\beta} \Big(F(-x) + 1 - F(x)\Big) = 0 \Leftrightarrow \lim_{u\to 0+} u^{\frac{1}{\beta}} \left(F^{-1}(1-u) - F^{-1}(u)\right) = 0$. For the necessary condition, we remark that either $F^{-1}(1) < +\infty$ and $\lim_{u\to 0+} u^{\frac{1}{\beta}} F^{-1}(1-u) = 0$ or $F^{-1}(1-u)$ goes to $+\infty$ as $u \to 0+$. For u small enough so that $F^{-1}(1-u) > 0$, we have $(F^{-1}(1-u))^{\beta} u \leq \sup_{x\geq F^{-1}(1-u)} x^{\beta}(1 - F(x-))$, from which we deduce that $\lim_{u\to 0+} u(F^{-1}(1-u))^{\beta} = 0$. The fact that $\lim_{u\to 0+} u^{\frac{1}{\beta}} F^{-1}(u) = 0$ is deduced by a symmetric reasoning.

For the sufficient condition, we use that $x(1-F(x))^{\frac{1}{\beta}} \leq \sup_{u\leq 1-F(x)} u^{\frac{1}{\beta}} F^{-1}((1-u)+)$ and $x\,(F(x))^{\frac{1}{\beta}} \geq \inf_{u\leq F(x)} u^{\frac{1}{\beta}} F^{-1}(u)$ for $x \in \mathbb{R}$ such that $0 < F(x) < 1$.

□

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