A new model of technical change and an application to the Solow model

Mehdi Senouci, Hugo Mauron

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Abstract

We present an alternative form of technical change within the traditional two-input framework. The aggregate production function is the convex hull of an increasing, finite number of Leontief production functions. At each date, each of these local production functions mutates into two Leontief production functions: one featuring exogenously increased labor-augmenting productivity, the other featuring exogenously increased capital-augmenting productivity. We embed this model of technical change into an otherwise standard, discrete-time Solow model. We do not specify technical change as purely labor-augmenting; still, it comes out that this modified Solow model has a globally stable balanced growth path. Along this path, technical change jointly determines the growth rate, capital-output ratio, and marginal productivity of capital and the competitive factor shares.

Keywords: Directed technical change, Uzawa theorem, Balanced growth path, Solow model.
JEL codes: O33, O40, O41.

Introduction

Production is the process by which inputs are converted into output, and is usually represented mathematically by a production function. By definition, technical change transforms the production process and consequently shows up mathematically as an alteration of the production function.

Economists have mainly investigated and used so-called "factor-augmenting" forms of technical change. Under the assumption of factor-augmenting technical change, the production function evolves only through the increase of some productivity indexes. Thus, if the inputs are $(X_1,\ldots,X_n)$, the factor-augmenting assumption amounts to assuming that the production function at any date t can be written $F_t(X_1,\ldots,X_n) = F(A_{1t}X_1,\ldots,A_{nt}X_n)$ where the terms A_{jt} are the productivity indexes and F is some unchanging production function, often dubbed the 'core' production function.

Pioneers of economic growth theory have primarily searched for forms of technical change that generate macroeconomic dynamics consistent with the first five Kaldor facts.[1] With this regards, the literature has reached a central

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[1] Kaldor (1961).

result with the Robinson-Uzawa theorem.[2] In its most general form –which is due to Jones and Scrimgeour (2008) – the theorem states that if all input and output quantities grow at a constant rate within a neoclassical dynamic model, then technical change necessarily appears as purely labor-augmenting along this balanced growth path. It is noteworthy that this result does not require technical change to be factor-augmenting. The influence of the Robinson-Uzawa theorem on growth theory is overwhelming. Most theoreticians expect growth models to exhibit balanced growth paths only if technical change is specified purely as labor-augmenting.

On the other hand, models featuring labor-augmenting technical change often have the prominent characteristic that relative effective inputs are constant along any balanced growth path. In turn, this property implies that variables such as the marginal product of capital or the factor shares are determined by the derivative of the core production function, which is assumed to be unaffected by technical change.

On the empirical side, this feature of the workhorse neoclassical growth model has notably shown its limits when some scholars attempted to use it to explain the recent worldwide decline of the labor share. For instance, Karabarbounis and Neiman (2014) argue that the decline of the labor share can be pictured as the consequence of technical change in the investment-goods sector with gross capital-labor substitutability at the macroeconomic level, a view widely dismissed as most empirical estimates of the elasticity of substitution fall significantly below 1.[3]

On the theoretical side, the assumption of the existence of a fixed core production function – only affected by technological change through variations of some productivity indexes – entails a problematic dichotomy. Why wouldn't technical change alter the core production function itself?

In response to these limitations, several scholars have developed alternative views of technical change; from the Atkinson-Stiglitz informal argument of localized technical change[4] to quite sophisticated theories such as the task approach to technical change.[5]

This paper aims to prove that orthodox neoclassical models might yield predictions about the influence of technical change over the marginal product of capital and competitive factor shares that are not heavily dependent on the shape of the core production function. We incorporate a new form of technical change in the Solow (1956) model in discrete time and analyze the resulting growth path. Our specification of technical change obeys to the following logic. We stick to the two-input (capital and labor) framework. At the initial date, there exists one, exogenously given Leontief production function f_0. At date one, f_0 gives rise to two new Leontief production functions. The first one (f_λ) results from labor-augmenting technical change (LATC), the second one (f_μ) from capital-augmenting technical change (KATC). Neither of these two production functions dominates the other, and the global production available at date one is the convex hull of f_λ and f_μ. Then, at date two, each of the production functions f_λ and f_μ gives rise to two Leontief production functions through the same process. We assume that the rates of LATC and KATC are independent of the production function to which they apply. The same process goes on at all subsequent dates. Since it is equivalent to apply LATC and then KATC, or KATC and then LATC, at each date t there exists $t+1$ available Leontief production functions – instead of 2^t. The global production function at any date t is thus the convex envelope of these $t+1$ local production functions.

We then investigate the properties of the Solow model fueled by our specification of technical change. It comes out that this model possesses a globally stable growth path that reproduces all the first five Kaldor facts. Along the balanced growth path, the common steady-state growth rate of the capital-labor and output-labor ratios is simply equal to the rate of LATC experienced by the local production functions. The steady-state marginal product of capital and competitive capital share depend on both the rates of LATC and KATC experienced by the local production functions.

In light of the prevailing interpretation of the Robinson-Uzawa theorem, we believe this result to be unexpected.

[2] Robinson (1938), Uzawa (1961). The result is also known as the Uzawa's theorem, the Steady-State Growth theorem or the Balanced Growth theorem.

[3] See Chirinko (2008) for a literature review on the estimation of the elasticity of substitution.

[4] Atkinson and Stiglitz (1969).

[5] Acemoglu and Zilibotti (2001), Autor and Acemoglu (2011), Acemoglu and Restrepo (2018). See also Blanchard (1997, p. 114), Zuleta (2008), Zuleta and Young (2013) and Peretto and Seater (2013) on models where technical change manifests itself as a change in the exponents of some Cobb-Douglas production function.

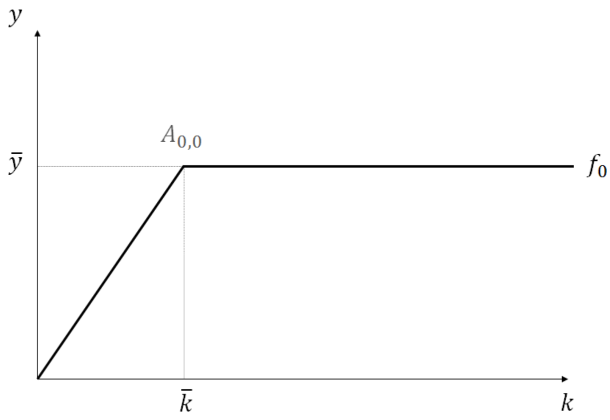


Figure 1: The production function available at $t = 0$.

Indeed, the global production function does not undergo factor-augmenting technical change – only the local production functions do – let alone pure LATC. Still, for all the parameters' values, the modified Solow model that we present does have a balanced growth path. This seeming contradiction is quickly resolved below.[6] The appropriate prime conclusion from our results is that technical change does not need to be specified as purely labor-augmenting to yield a balanced growth path. Secondly, our approach suggests that non-factor augmenting forms of technical change are promising ways to endogenize the whole production function.

This paper belongs to the theoretical literature on the determination of the global production function from a set of local production functions (aka a 'technology menu'). In contrast with Jones (2005) and Growiec (2008, 2018), we assume a finite technology menu. Under this assumption, efficiency implies that two local production functions are used jointly for most values of the capital-labor ratio. The balanced growth path that we highlight has the new feature that two technologies are used in steady state, and these two technologies jointly determine the characteristics of the global production function.

The rest of the paper is organized as follows. Section 1 presents our model of technical change and the shape of the aggregate production function. Section 2 embeds this production function in a discrete-time Solow model. Section 3 concludes.

1 A new type of technical change

In this section, we present the model of technical change. There are two inputs, capital (K) and labor (L) and time is discrete. For the sake of tractability, we proceed step by step by first presenting the evolution of the production function between dates $t = 0$ and $t = 1$, then between dates $t = 1$ and $t = 2$, before deriving the production function available at any date $t \geq 0$.

1.1 Dates $t = 0$ and $t = 1$

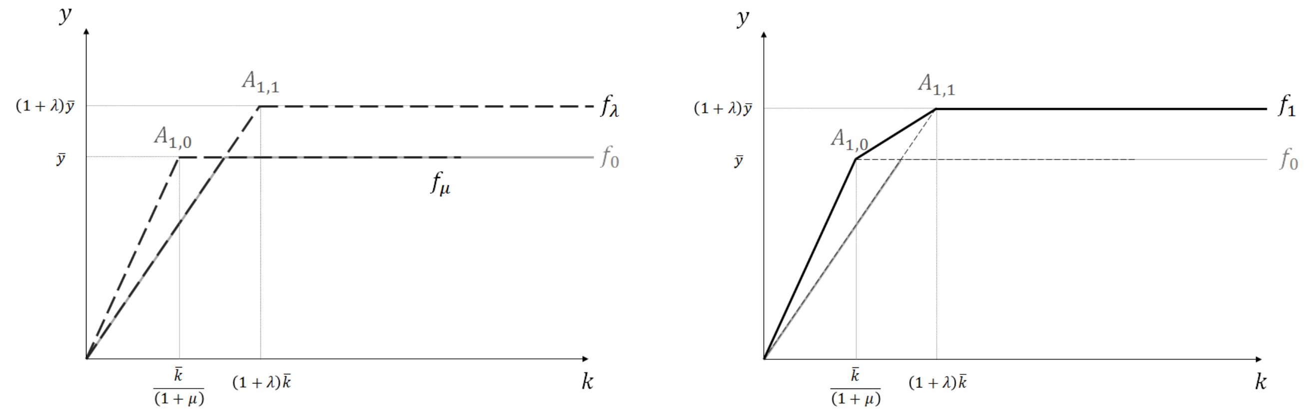
At date $t = 0$, the production function (in intensive terms) is some Leontief production function f_0 defined by:

$$\forall k \geq 0, \quad f_0(k) = \overline{y} \min\left(\frac{k}{\overline{k}}, 1\right)$$

where $\overline{y}$ and $\overline{k}$ are strictly positive parameters.[7] Figure 1 represents f_0 in the (k, y) plane. We call $A_{0,0}$ the characteristic point of the Leontief production function whose coordinates are $(\overline{k}, \overline{y})$.

[6] See subsection 2.2.

[7] The corresponding extensive production function is F_0, defined by: $\forall K, L > 0,\ F_0(K, L) = L f_0\left(\frac{K}{L}\right) = \overline{y} \min\left(\frac{K}{\overline{k}}, L\right)$



(a) The two production functions available at date $t=1$: f_λ and f_μ.

(b) The global production function f_1 as the convex hull of f_λ and f_μ.

Figure 2: Technical change between dates $t=0$ and $t=1$.

At date $t=1$, two production functions emerge from f_0. We call them respectively f_λ and f_μ, with:

$$\forall k \geq 0, \quad \begin{cases} f_\lambda(k) & = & (1+\lambda)\overline{y}\min\left(\frac{k}{(1+\lambda)\overline{k}}, 1\right) \\ f_\mu(k) & = & \overline{y}\min\left(\frac{(1+\mu)k}{\overline{k}}, 1\right) \end{cases}$$

where λ and μ are strictly positive constants. λ and μ respectively denote the rates of labor-augmenting technical change and the rate of capital-augmenting technical change ('LATC' and 'KATC'). This be seen in the extensive forms of f_λ and f_μ, respectively defined by formulas:

$$\begin{aligned} F_\lambda(K,L) &= Lf_\lambda\left(\tfrac{K}{L}\right) &= \overline{y}\min\left(\tfrac{K}{\overline{k}}, (1+\lambda)L\right) \\ F_\mu(K,L) &= Lf_\mu\left(\tfrac{K}{L}\right) &= \overline{y}\min\left((1+\mu)\tfrac{K}{\overline{k}}, L\right) \end{aligned}$$

Figure 2a represents f_λ and f_μ. We denote respectively by $A_{1,1} = \left((1+\lambda)\overline{k}, (1+\lambda)\overline{y}\right)$ and $A_{1,0} = \left(\overline{k}/(1+\mu), \overline{y}\right)$ their characteristic points.

At date $t=1$, the two production functions f_λ and f_μ can be used in conjunction. We assume that capital and labor inputs K and L are allocated efficiently to production functions f_λ and f_μ:

- If the capital-labor ratio $k = K/L$ is less than $\overline{k}_{1,0} = \overline{k}/(1+\mu)$, all inputs are allocated to F_μ;
- If the capital-labor ratio k is greater than $\overline{k}_{1,1} = (1+\lambda)\overline{k}$, all inputs are allocated to F_λ;
- In the intermediate range, i.e. if $k \in \left(\overline{k}_{1,0}, \overline{k}_{1,1}\right)$, both production functions F_λ and F_μ are used.

In the latter case, by allocating capital and labor inputs to production functions F_λ and F_μ according to

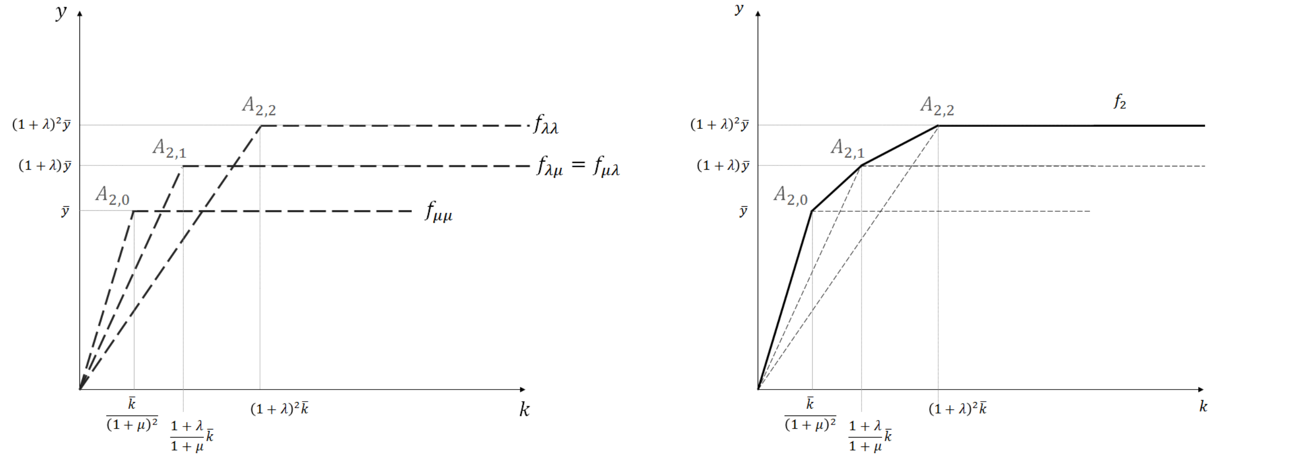
$$\begin{cases} L_\lambda & = & L\frac{k-\overline{k}_{1,0}}{\overline{k}_{1,1}-\overline{k}_{1,0}} \\ L_\mu & = & L\frac{\overline{k}_{1,1}-k}{\overline{k}_{1,1}-\overline{k}_{1,0}} \end{cases}, \quad \begin{cases} K_\lambda & = & \overline{k}_{1,1}L_\lambda \\ K_\mu & = & \overline{k}_{1,0}L_\mu \end{cases}$$

realized production is:

$$F_\lambda(K_\lambda, L_\lambda) + L_\mu F\left(K_\mu, L_\mu\right) = L_\lambda f_\lambda\left(\frac{K_\lambda}{L_\lambda}\right) + L_\mu f_\mu\left(\frac{K_\mu}{L_\mu}\right) = L\left(\frac{k-\overline{k}_{1,0}}{\overline{k}_{1,1}-\overline{k}_{1,0}}(1+\lambda)\overline{y} + \frac{\overline{k}_{1,1}-k}{\overline{k}_{1,1}-\overline{k}_{1,0}}\overline{y}\right)$$

For any $k \in \left(\overline{k}_{1,0}, \overline{k}_{1,1}\right)$, the quantity $\frac{k-\overline{k}_{1,0}}{\overline{k}_{1,1}-\overline{k}_{1,0}}(1+\lambda)\overline{y} + \frac{\overline{k}_{1,1}-k}{\overline{k}_{1,1}-\overline{k}_{1,0}}\overline{y}$ thus stands on the $\left[A_{1,0}A_{1,1}\right]$ line segment.

The graph of the global production function at date $t=1$ is then the broken line $\left[OA_{1,0}A_{1,1}\right]$ supplemented by the half-line $\left[A_{1,1}\left(+\infty, (1+\lambda)\overline{y}\right)\right)$. For the ease of notation, we'll denote the graph of f_1 by $\left[OA_{1,0}A_{1,1}\infty\right)$. f_1 is depicted on figure 2b. Notice that f_1 is the convex hull of f_λ and f_μ.



(a) The three production functions available at date $t = 2$: $f_{\lambda\lambda}$, $f_{\lambda\mu}$ and $f_{\mu\mu}$.

(b) The global production function f_2 as the convex hull of $f_{\lambda\lambda}$, $f_{\lambda\mu}$ and $f_{\mu\mu}$.

Figure 3: Technical change between dates $t = 1$ and $t = 2$.

1.2 Date $t = 2$

Between dates $t = 1$ and $t = 2$, functions f_λ and f_μ each give rise to two production functions through the same process than the one that led to f_λ and f_μ from f_0. More specifically, f_λ gives rise to functions $f_{\lambda\lambda}$ and $f_{\lambda\mu}$ defined by:

$$\forall k \geq 0, \quad \begin{cases} f_{\lambda\lambda}(k) &= (1+\lambda)^2 \overline{y} \min\left(\frac{k}{(1+\lambda)^2 \overline{k}}, 1\right) \\ f_{\lambda\mu}(k) &= (1+\lambda)\overline{y} \min\left(\frac{(1+\mu)k}{(1+\lambda)\overline{k}}, 1\right) \end{cases}$$

while function f_μ gives rise to functions $f_{\mu\lambda}$ and $f_{\mu\mu}$ defined by:

$$\forall k \geq 0, \quad \begin{cases} f_{\mu\lambda}(k) &= (1+\lambda)\overline{y} \min\left(\frac{(1+\mu)k}{(1+\lambda)\overline{k}}, 1\right) \\ f_{\mu\mu}(k) &= \overline{y} \min\left(\frac{(1+\mu)^2 k}{\overline{k}}, 1\right) \end{cases}$$

One can see that $f_{\lambda\mu} = f_{\mu\lambda}$. As we assume that the rates of LATC (λ) and KATC (μ) are constant, regardless of the production function to which they apply, applying LATC and then KATC to f_0 is equivalent to applying KATC and then LATC to f_0. The three production functions $f_{\mu\mu}$, $f_{\lambda\mu}$ and $f_{\lambda\lambda}$ are represented on figure 3a.

The global production function f_2 is the convex hull of the three production functions available $f_{\mu\mu}$, $f_{\lambda\mu}$ and $f_{\lambda\lambda}$ as depicted on figure 3b. Their characteristic points are respectively $A_{2,0} = \left(\overline{k}/(1+\mu)^2, \overline{y}\right)$, $A_{2,1} = \left((1+\lambda)\overline{k}/(1+\mu), (1+\lambda)\overline{y}\right)$ and $A_{2,2} = \left((1+\lambda)^2\overline{k}, (1+\lambda)^2\overline{y}\right)$. With the notations introduced in subsection 1.1, the graph of f_2 is $\left[OA_{2,0}A_{2,1}A_{2,2}\infty\right)$.

1.3 Date $t \geq 0$

The way our specification of technical change alters the global production function over time should now be clear. At each date $t \geq 0$, a family of $t+1$ production functions $\left(f_{t,i}\right)_{i\in\{0,\dots,t\}}$ exists, where the analytical formula for $f_{t,i}$ is:

$$\forall k \geq 0, \quad f_{t,i}(k) = (1+\lambda)^i \overline{y} \min\left(\frac{(1+\mu)^{t-i}}{(1+\lambda)^i} \frac{k}{\overline{k}}, 1\right)$$

For any $t \geq 0$ and for any $i \in \{0,\dots,t\}$, the characteristic point of function $f_{t,i}$ is $A_{t,i}$ of coordinates $\left(\overline{k}_{t,i}, \overline{y}_i\right) = \left((1+\lambda)^i \overline{k}/(1+\mu)^{t-i}, (1+\lambda)^i \overline{y}\right)$. The global production function at date t, f_t, is the convex hull of the $t+1$ functions $\left(f_{t,i}\right)_{i\in\{0,\dots,t\}}$. The graph of f_t is the broken line $\left[OA_{t,0}A_{t,1}\dots A_{t,t}\infty\right)$.

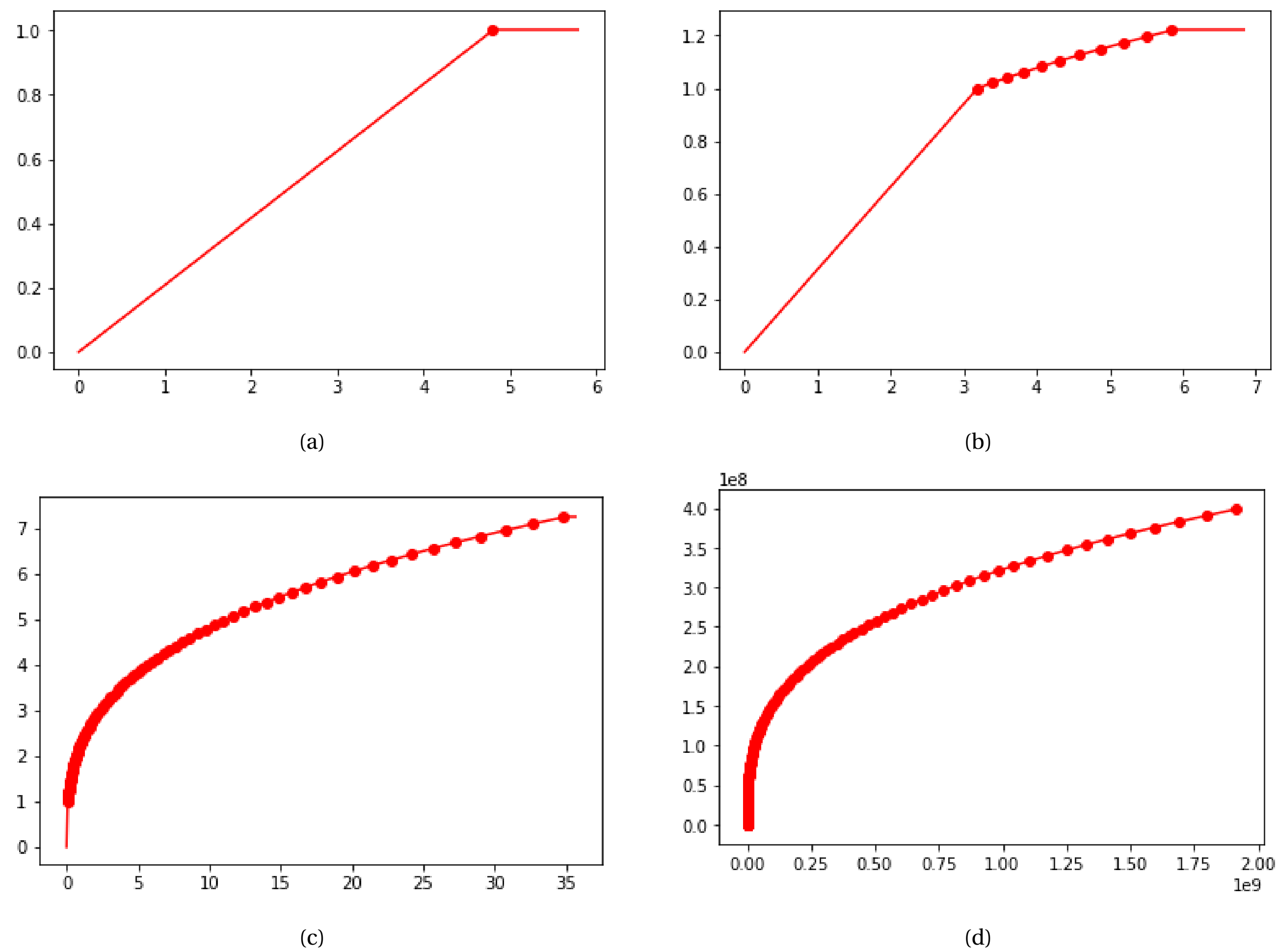


Figure 4: Evolution of the global production function for $\overline{k}=5$, $\overline{y}=1$, $\lambda=0.02$ and $\mu=0.04$. (4a), (4b), (4c) and (4d) respectively represent f_0, f_{10}, f_{100} and f_{1000}. Red dots are the $A_{t,i}$ points. Curves obtained by the Python code displayed in Appendix A.

Figure 4 represents the global production function f_t for $t=0$; $t=10$; $t=100$; $t=1{,}000$ for chosen parameters $\overline{k}$, $\overline{y}$, λ and μ.

An analytical formula for function f_t is given by:

$$\forall k \geq 0, \quad f_t(k)=\begin{cases} (1+\mu)^t \frac{\overline{y}}{\overline{k}} \times k & \text{if} \quad k \leq \overline{k}_{t,0} \\ (1+\lambda)^i \overline{y}+\frac{\lambda(1+\mu)^{t-i}}{((1+\lambda)(1+\mu)-1)} \frac{\overline{y}}{\overline{k}} \times \left(k-\overline{k}_{t,i}\right) & \text{if} \quad \overline{k}_{t,i} \leq k \leq \overline{k}_{t,i+1} \text{ for some } i \in \{0,\ldots,t-1\} \\ (1+\lambda)^t \overline{y} & \text{if} \quad \overline{k}_{t,t} \leq k \end{cases} \tag{1}$$

1.4 Marginal returns of capital and the capital share in production function f_t

Marginal returns to capital are constant over each part of this broken line. For $k \in \left(0, \overline{k}_{t,0}\right)$, marginal returns to capital are at $f_t'(k)=\overline{y}_0/\overline{k}_{t,0}=(1+\mu)^t \overline{y}/\overline{k}$. If $t \geq 1$, and $0 \leq i \leq t-1$, for $k \in \left(\overline{k}_{t,i}, \overline{k}_{t,i+1}\right)$ marginal returns to capital are at $f_t'(k)=(\overline{y}_{i+1}-\overline{y}_i)/\left(\overline{k}_{t,i+1}-\overline{k}_{t,i}\right)=\lambda(1+\mu)^{t-i}/\left((1+\lambda)(1+\mu)-1\right)*\overline{y}/\overline{k}$. For $k>\overline{k}_{t,t}$, $f_t'(k)=0$.

The competitive capital share $\alpha_t(k)=k f_t'(k)/f_t(k)$ is equal to 1 for $k \in \left(0, \overline{k}_{t,0}\right)$, and to 0 for $k>\overline{k}_{t,t}$. If $t \geq 1$, and $0 \leq i \leq t-1$, for $k \in \left(\overline{k}_{t,i}, \overline{k}_{t,i+1}\right)$ the capital share is at $\alpha_t(k)=\lambda(1+\mu)^{t-i}/\left((1+\lambda)(1+\mu)-1\right)*\overline{y}/\overline{k}*k/f_t(k)$. This quantity is strictly increasing in k over the $\left(\overline{k}_{t,i}, \overline{k}_{t,i+1}\right)$ interval. The right-hand limit of $\alpha_t(k)$ near $\overline{k}_{t,i}$ is:

$$\lim_{k \to \overline{k}_{t,i}^+} \alpha_t(k)=\frac{\lambda(1+\mu)^{t-i}}{(1+\lambda)(1+\mu)-1} \frac{\overline{y}}{\overline{k}} \frac{\overline{k}_{t,i}}{\overline{y}_i}=\frac{\lambda}{(1+\lambda)(1+\mu)-1}$$

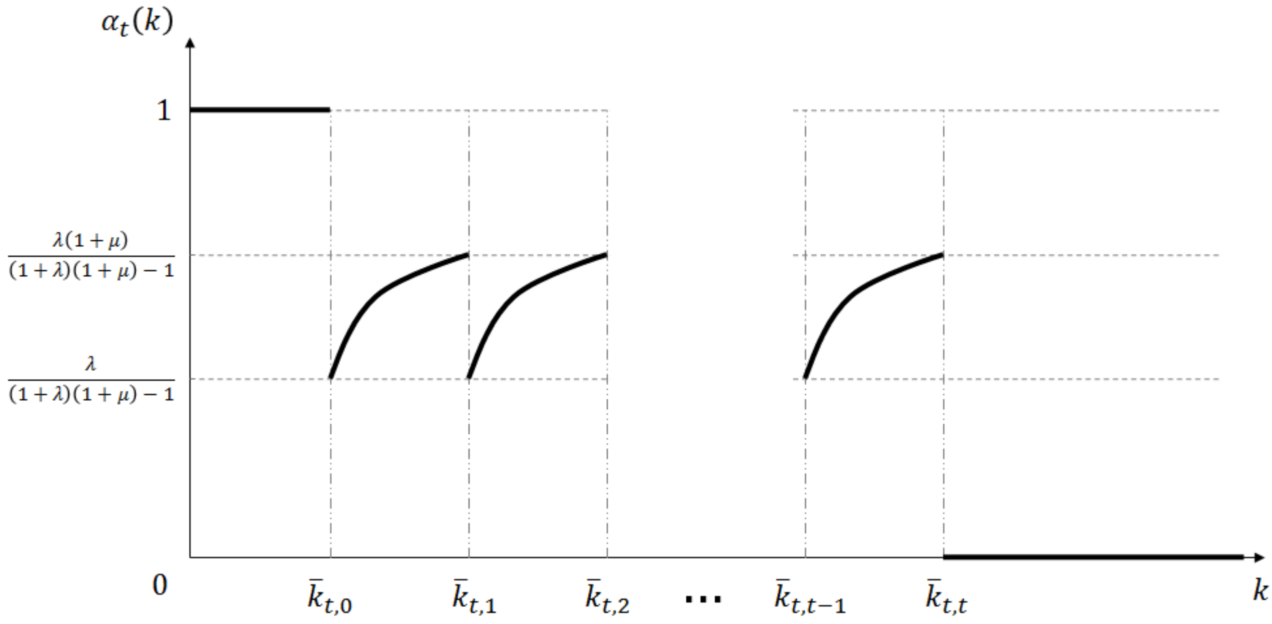


Figure 5: The shape of the date-t capital share function $\alpha_t(\cdot)$ for $t \geq 1$.

while the left-hand limit of $\alpha_t(k)$ near $\overline{k}_{t,i+1}$ is:

$$\lim_{k \to \overline{k}_{t,i+1}^-} \alpha_t(k) = \frac{\lambda(1+\mu)^{t-i}}{(1+\lambda)(1+\mu)-1} \frac{\overline{y}}{\overline{k}} \frac{\overline{k}_{t,i+1}}{\overline{y}_{i+1}} = \frac{\lambda(1+\mu)}{(1+\lambda)(1+\mu)-1}$$

The graph of function $\alpha_t(\cdot)$ is represented on figure 5.

1.5 The relationship between f_t and Cobb-Douglas production functions

Figure 4 suggests that f_t looks like a Cobb-Douglas production function as time tends to infinity. Indeed, this intuition is partly misleading. Indeed, it is true that for all $t \geq 0$, all the points $A_{t,0}A_{t,1}\dots A_{t,t}$ lie the graph of some identified Cobb-Douglas production function, as the following result proves.

Proposition 1. *Let $\alpha \in (0,1)$ be defined as:*

$$\alpha = \frac{\ln(1+\lambda)}{\ln(1+\lambda)+\ln(1+\mu)}$$

For all $t \geq 0$, let f_t^{CD} be the Cobb-Douglas production function defined by:

$$\forall k \geq 0, \;\; f_t^{CD}(k) = (1+\lambda)^{(1-\alpha)t} \left(\frac{k}{\overline{k}}\right)^{\alpha} \overline{y}$$

Then, for all $t \geq 0$ and for all $i \in \{0,\dots,t\}$:

$$f_t^{CD}\left(\overline{k}_{t,i}\right) = f_t\left(\overline{k}_{t,i}\right)$$

Proof. Let $t \geq 0$ and $i \in \{0,\dots,t\}$. Then:

$$\begin{aligned}
f_t^{\mathrm{CD}}\left(\overline{k}_{t,i}\right) &= (1+\lambda)^{(1-\alpha)t}\left(\tfrac{(1+\lambda)^i}{(1+\mu)^{t-i}}\right)^{\alpha}\overline{y} \\
&= \exp\left[(1-\alpha)t\ln(1+\lambda)+\alpha i\ln(1+\lambda)-\alpha(t-i)\ln(1+\mu)\right]\overline{y} \\
&= \exp\left[\tfrac{\ln(1+\mu)}{\ln(1+\lambda)+\ln(1+\mu)}t\ln(1+\lambda)+\tfrac{\ln(1+\lambda)}{\ln(1+\lambda)+\ln(1+\mu)}i\ln(1+\lambda)-\tfrac{\ln(1+\lambda)}{\ln(1+\lambda)+\ln(1+\mu)}(t-i)\ln(1+\mu)\right]\overline{y} \\
&= \exp\left[i\ln(1+\lambda)\right]\overline{y} \\
&= (1+\lambda)^i\overline{y} \\
&= f_t\left(\overline{k}_{t,i}\right)
\end{aligned}$$

□

However, the sequence of functions $(f_t)_{t\geq 0}$ is *not* equivalent in any sense to the sequence of functions $\left(f_t^{\mathrm{CD}}\right)_{t\geq 0}$. Indeed, for all $k > 0$, the quantity $f_t^{\mathrm{CD}}(k)/f_t(k)$ does not tend to 1 as time tends to infinity.

2 The Solow model under our model of technical change

We are interested in the following dynamic system for all $t \in \{0, 1, \ldots\}$:

$$y_t = f_t(k_t) \tag{2}$$

$$k_{t+1} = s y_t + (1-\delta) k_t \tag{3}$$

where $s \in (0,1]$ is the saving rate and $\delta \in (0,1]$ is the depreciation rate of capital. The initial capital-labor ratio $k_0 > 0$ is given by initial conditions. For simplicity, we assume that population is constant at $L > 0$.[8] Clearly, the dynamic system is well-defined and yields nonnegative sequences for the capital-labor ratio k_t and the output per worker y_t for all dates $t \geq 0$ for any value of the parameters $\left(s, \delta, \lambda, \mu, \overline{y}, \overline{k}\right)$. A Python code simulating the model is available in Appendix B.

In line with the theoretical literature on economic growth, we are mostly interested in matching the first five Kaldor facts. Consequently, we first investigate the existence of a stable, balanced growth path in this section.

Definition 1. *Let $(k_t)_{t\geq 0}$ and $(y_t)_{t \geq 0}$ denote, respectively, the sequences of capital-labor ratios and levels of output per worker generated by the model for some initial condition k_0. We say that the model economy is:*

- *On a Balanced Growth Path (BGP) if and only if k_t and y_t grow at constant rates at all dates $t \geq 0$;*
- *On a Balanced Growth Path in Finite Time (BGPFT) if and only if k_t and y_t grow at constant rates at all dates $t \geq T$ for some $T \geq 0$;*
- *On an asymptotic balanced growth path (ABGP) if and only if the growth rates of k_t and y_t tend to constants as t tends to infinity.*

Clearly, any BGP is also a BGPFT, and any BGPFT is an ABGP.

For convenience, we introduce a variable $i(t)$ that labels the interval of the production function that is used at any date $t \geq 0$.

Definition 2. *Let $(k_t)_{t\geq 0}$ and $(y_t)_{t \geq 0}$ denote, respectively, the sequences of capital-labor ratios and levels of output per worker generated by the model for some initial condition k_0.*

- *We define $i(0)$ as equal to -1 if $k_0 < \overline{k}$ and equal to 0 if $k_0 > \overline{k}$.*
- *For all $t \geq 1$, we define $i(t)$ as:*

$$i(t) = \begin{cases} -1 & \text{if } k_t < \overline{k}_{t,0} \\ i & \text{if there exists } i \in \{0, \ldots, t-1\} \text{ such that } \overline{k}_{t,i} < k_t < \overline{k}_{t,i+1} \\ t & \text{if } k_t > \overline{k}_{t,t} \end{cases}$$

2.1 A necessary condition on any potential BGPFT

In this subsection, we assume that some BGPFT exists, and we derive some necessary conditions on the resulting growth path.

As usual in the treatment of Solow models, the equation of accumulation of capital (3) implies:

$$\frac{k_{t+1}}{k_t} = s\frac{y_t}{k_t} + 1 - \delta$$

which entails that the constant growth rates of k_t and y_t must be equal on a BGPFT. Let g be this common growth rate. Then, the output-capital ratio y_t/k_t is constant for at $(g+\delta)/s$.

[8] The model can easily accomodate some constant population growth rate n, the result would be that n would add to δ in all formulas.

Proposition 2. *Assume that the model economy is on a BGPFT starting at some date $T \geq 0$. Call g the common growth rate of k_t and y_t for $t \geq T$. Then:*

$$g = \lambda$$

Proof. Let $(k_t, y_t)_{t\geq 0}$ be a BGPFT, starting at some date $T \geq 0$, and featuring a growth rate g. For all $t \geq T$, the output-capital ratio is then constant at $(g+\delta)/s$. Let $\mathscr{D}$ be the line of equation $y = ((g+\delta)/s) * k$ in the (k, y) plane. Notice that for all $t \geq 0$, $(OA_{t,t})$ all have the same slope which is equal to $\overline{y}/\overline{k}$.

Case #1: If $\overline{y}/\overline{k} > (g+\delta)/s$, then all the production functions f_t cross the $\mathscr{D}$ line over their final flat half-line $[A_{t,t}\infty)$. $i(t)$ must be equal to t all over the BGPFT. This case is represented on figure 6. For all dates $t \geq T$, y_t must then be equal to $\overline{y}_t = (1+\lambda)^t \overline{y}$. Thus, y_t grows at the rate λ. So $g = \lambda$.

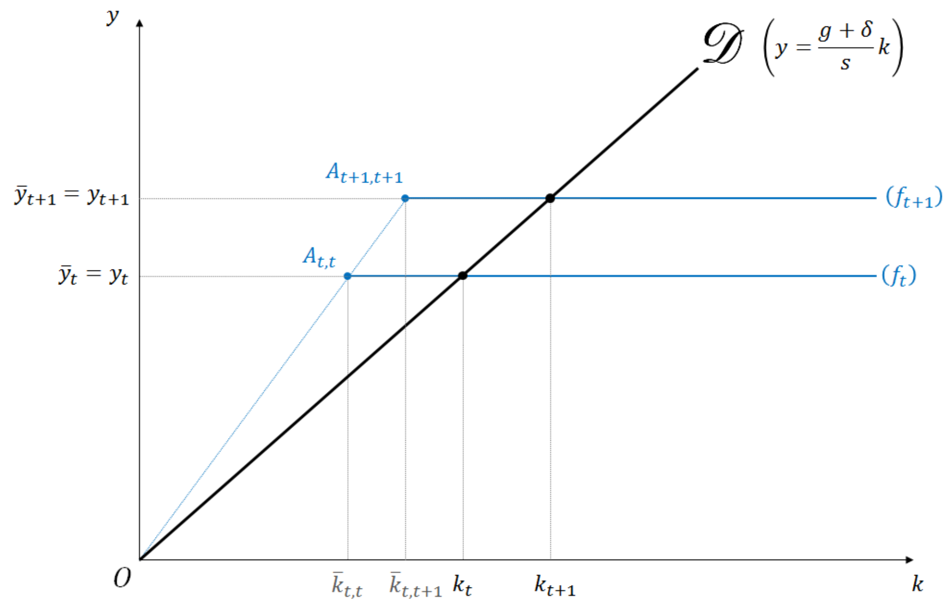


Figure 6: Proof of proposition 2 if $i(t) = t$ along the BGPFT.

Case #2: If $\overline{y}/\overline{k} > (g+\delta)/s$, then all points $(OA_{t,t})$ lie at the right of line $\mathscr{D}$. Since we assumed the model economy to be on a BGPFT, the production functions f_t must cross the $\mathscr{D}$ line for all $t \geq T$. Let X_t be that point at date t. X_t must then belong to some line segment $[A_{t,i^*(t)}A_{t,i^*(t)+1}]$ for some $i^*(t) \in \{0, \ldots, t-1\}$, $i^*(t)$. For all $t \geq T$, $i^*(t)$ is the integer such that $A_{t,i^*(t)}$ lies at the left of $\mathscr{D}$, while $A_{t,i^*(t)+1}$ lies at the right of $\mathscr{D}$. Equivalently, $i^*(t)$ is such that line $(OA_{t,i^*(t)})$ is steeper than $\mathscr{D}$ and such that line $\mathscr{D}$ is steeper than line $(OA_{t,i^*(t)+1})$. Since the slope of line $(OA_{t,i^*(t)})$ is the same as the slope of line $(OA_{t+1,i^*(t)+1})$, and that the slope of line $(OA_{t,i^*(t)+1})$ is the same as the slope of line $(OA_{t+1,i^*(t)+2})$, we can conclude that for all $t \geq T$, $i^*(t+1) = i^*(t)$. This situation is depicted in figure 7.

The two lines $(A_{t,i^*(t)}A_{t,i^*(t)+1})$ and $(A_{t+1,i^*(t+1)}A_{t+1,i^*(t+1)+1})$ then both have slope $\lambda(1+\mu)^{t-i^*(t)}/((1+\lambda)(1+\mu)-1) * \overline{y}/\overline{k}$, so lines $(A_{t,i^*(t)}A_{t,i^*(t)+1})$ and $(A_{t+1,i^*(t+1)}A_{t+1,i^*(t+1)+1})$ are parallel.

Consequently, triangle $OA_{t+1,i^*(t+1)}A_{t+1,i^*(t+1)+1}$ is the image of triangle $OA_{t,i^*(t)}A_{t,i^*(t)+1}$ by the homothetic transformation of center O and of factor $1+\lambda$. Since points O, X_t, X_{t+1} are aligned, and since $X_t \in (A_{t,i^*(t)}A_{t,i^*(t)+1})$ and $X_{t+1} \in (A_{t+1,i^*(t+1)}A_{t+1,i^*(t+1)+1})$, X_{t+1} is the image of X_t by the homothetic transformation of center O and of factor $1+\lambda$. So $k_{t+1} = (1+\lambda)k_t$ and $y_{t+1} = (1+\lambda)y_t$ for all $t \geq T$. So $g = \lambda$.

□

Like in the canonical Solow model, the growth rate is only related to technical change and not to s nor δ. Furthermore, proposition 2 establishes that the growth rate is equal to the rate of LATC. Although KATC expands the production possibility frontier,[9] KATC cannot increase the long-run growth rate of the economy.

[9] It is straightforward to see that a higher μ leads to global production functions f_t that is more efficient for all capital-labor ratios than the ones

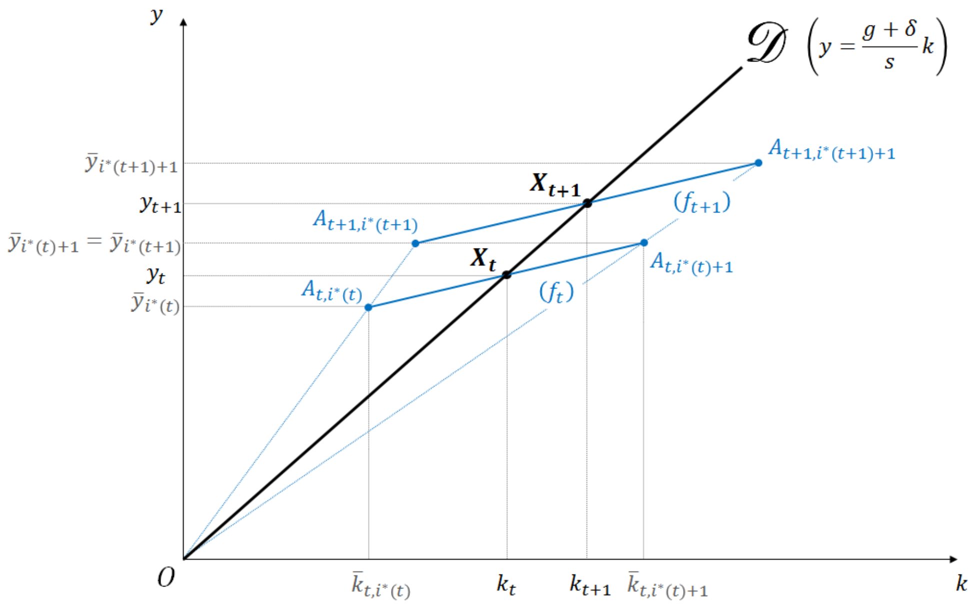


Figure 7: Proof of proposition 2 if $i(t) \leq t-1$ along the BGPFT.

2.2 Existence and features of a BGPFT

We can now establish the first central property of our growth model.

Theorem 1. *For any parameters* $(\overline{y}, \overline{k}, \lambda, \mu, s, \delta)$, *there exists some initial capital-labor ratio* $k_0^* > 0$ *such that, if* $k_0 = k_0^*$, *the sequence of* $(k_t, y_t)_{t\geq 0}$ *generated by the dynamic system* (2) *and* (3) *is a BGPFT.*

Proof. As per proposition 2, let $\mathscr{D}$ be the line of equation $y = ((\lambda+\delta)/s) * k$ in the (k, y) plane.

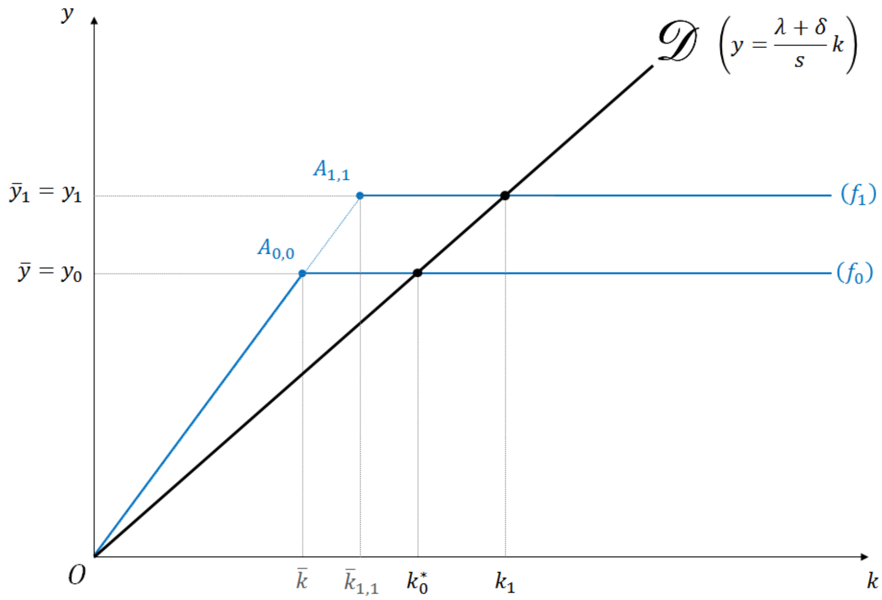


Figure 8: Proof of theorem 1 if $\overline{y}/\overline{k} > (\lambda+\delta)/s$.

Case #1: If $\overline{y}/\overline{k} > (\lambda+\delta)/s$, then the initial production function f_0 crosses line $\mathscr{D}$ at some point of the final flat half-line of f_0, i.e $[A_{0,0}\infty)$. Let k_0^* be the horizontal coordinate of this point. The situation is illustrated in figure 8.

corresponding to a lower μ.

Then, if $k_0 = k_0^*$, $k_1 = (1+\lambda)k_0^*$ since point (k_0, y_0) lies on line $\mathscr{D}$. Since $k_0^* > \overline{k}$, $k_1 > (1+\lambda)\overline{k} = \overline{k}_{1,1}$. So $y_1 = f_1(k_1) = \overline{y}_1 = (1+\lambda)\overline{y}$. So point (k_1, y_1) lies on line $\mathscr{D}$. Along the same lines, it is immediate to see that for all $t \geq 0$, point (k_t, y_t) lies on line $\mathscr{D}$, so that the growth rates of k_t and y_t are constant and equal to λ from $t = 0$ onwards.

Thus, if If $\overline{y}/\overline{k} > (\lambda+\delta)/s$, there exists some $k_0^* > 0$ such that, if $k_0 = k_0^*$, the model economy is on a BGP.

Case #2: If $\overline{y}/\overline{k} < (\lambda+\delta)/s$, then the initial production function f_0 only crosses line $\mathscr{D}$ at point O. After some time $T \geq 1$, the production function f_T crosses the $\mathscr{D}$ line at some point of the $\left[A_{T,0}A_{T,1}\right]$ line segment. Let $T \geq 1$ be the only integer such that point $A_{T,0} = \left(\overline{k}/(1+\mu)^T, \overline{y}\right)$ lies at the left of line $\mathscr{D}$, while point $A_{T,1} = \left(\overline{k} * (1+\lambda)/(1+\mu)^{T-1}, (1+\lambda)\overline{y}\right)$ lies at the right of line $\mathscr{D}$.[10] Since T must satisfy the inequalities $(1+\mu)^{T-1}\frac{\bar{y}}{\bar{k}} < \frac{\lambda+\delta}{s} < (1+\mu)^T\frac{\bar{y}}{\bar{k}}$, an exact formula for T is:

$$T = 1 + \left\lfloor \frac{\ln\left(\frac{\lambda+\delta}{s}\frac{\bar{k}}{\bar{y}}\right)}{\ln(1+\mu)} \right\rfloor \geq 1$$

where $\lfloor\cdot\rfloor$ is the floor function.

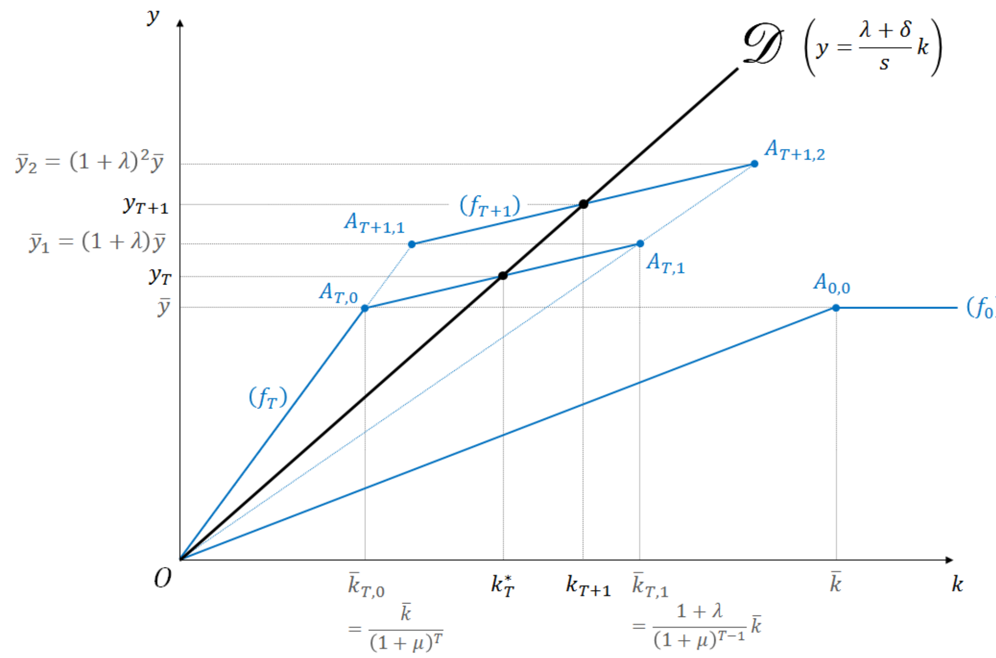


Figure 9: Proof of theorem 1 if $\overline{y}/\overline{k} < (\lambda+\delta)/s$.

The situation is depicted in figure 9. Let $k_T^* \in \left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)$ be the horizontal coordinate of the point of intersection of f_T and line $\mathscr{D}$. If $k_T = k_T^*$, then the point (k_T, y_T) lies on line $\mathscr{D}$, and so $k_{T+1} = (1+\lambda)k_T$.

Besides, $k_T \in \left(\overline{k}_{T,0}, \overline{k}_{T,1}\right) \Rightarrow i(T) = 0$, and so:

$$y_T = f_T(k_T) = \overline{y} + \underbrace{\left(\frac{\overline{y}_1 - \overline{y}}{\overline{k}_{T,1} - \overline{k}_{T,0}}\right)}_{\text{slope of line } (A_{T,0}A_{T,1})} \times \left(k_T - \overline{k}_{T,0}\right) = \overline{y} + \frac{\lambda(1+\mu)^T}{(1+\lambda)(1+\mu)-1}\frac{\overline{y}}{\overline{k}}\left(k_T - \frac{\overline{k}}{(1+\mu)^T}\right)$$

Remark that:

$$\overline{k}_{T,0} < k_T^* < \overline{k}_{T,1} \implies (1+\lambda)\overline{k}_{T,0} < (1+\lambda)k_T^* < (1+\lambda)\overline{k}_{T,1} \implies \overline{k}_{T+1,1} < k_{T+1} < \overline{k}_{T+1,2}$$

Consequently, $i(T+1) = 1$ and so:

$$\begin{aligned} y_{T+1} &= f_{T+1}(k_{T+1}) \\ &= \overline{y}_1 + \left(\frac{\overline{y}_2 - \overline{y}_1}{\overline{k}_{T+1,2} - \overline{k}_{T+1,1}}\right) \times \left(k_{T+1} - \overline{k}_{T+1,1}\right) \\ &= (1+\lambda)\overline{y} + \frac{\lambda(1+\mu)^T}{(1+\lambda)(1+\mu)-1}\frac{\overline{y}}{\overline{k}} \times \left((1+\lambda)k_T - \frac{\overline{k}(1+\lambda)}{(1+\mu)^T}\right) \\ &= (1+\lambda)y_T \end{aligned}$$

[10] Equivalently, $T \geq 1$ is defined as the only integer such that $A_{T,0}$ lies at the left of line $\mathscr{D}$ while $A_{T-1,0} = \left(\overline{k}/(1+\mu)^{T-1}, \overline{y}\right)$ lies at the right of line $\mathscr{D}$.

It is straightforward to show recursively that if $k_T = k_T^*$, then for all $t \geq T$, $i(t) = t - T$, $k_{t+1} = (1+\lambda)k_t$ and $y_{t+1} = (1+\lambda)y_t$.

There remains to prove that there exists some $k_0^* > 0$ such that, if $k_0 = k_0^*$, then $k_T = k_T^*$.

For all $t \in \{0,\ldots,T-1\}$ let κ_t be the function of $\mathbb{R}_+$ into itself such that for all $x \geq 0$, $\kappa_t(x) = s f_t(x) + (1-\delta)x$. Let $k_0 > 0$ and let $(k_t, y_t)_{t=0,\ldots,T}$ be the time series generated by our model, i.e. by equations (2) and (3). Then $k_T = (\kappa_{T-1} \circ \cdots \circ \kappa_0)(k_0)$. Functions $\kappa_.$ all being strictly increasing and continuous, function $(\kappa_{T-1} \circ \cdots \circ \kappa_0)$ is also strictly increasing and continuous. So there exists some $k_0^* > 0$ such that $k_T^* = (\kappa_{T-1} \circ \cdots \circ \kappa_0)\left(k_0^*\right)$.

Thus, if $\overline{y}/\overline{k} < (\lambda+\delta)/s$, there exists some $k_0^* > 0$ such that, if $k_0 = k_0^*$, the model economy is on a a BGPFT starting at date $T \geq 1$ where $T = 1 + \left\lfloor \frac{\ln\left(\frac{\lambda+\delta}{s}\frac{\overline{k}}{\overline{y}}\right)}{\ln(1+\mu)} \right\rfloor$. □

Theorem 1 establishes the existence of a BGPFT for any values of the parameters. The growth rate of is λ and the output-capital ratio is $(\lambda+\delta)/s$ along any BGPFT.

This result might seem to contradict the Robinson-Uzawa theorem, which states that balanced growth requires purely labor-augmenting technical change. Indeed, our form of technical change is *not* purely labor-augmenting. If we denote by F_t the extensive form of production function f_t, there exists no sequence of numbers $(A_t)_{t\geq 0}$ such that for all $(K,L) \in (\mathbb{R}_+)^2$ and for all $t \geq 0$, $F_t(K,L) = F_0(K, A_t L)$.

The reason for this apparent contradiction is actually simple. As Jones and Scrimgeour (2008) showed, balanced growth requires technical change to be representable as labor-augmenting only at the relative inputs *effectively used* along this BGP; and this is actually the case in our model:

- Case #1: If $\overline{y}/\overline{k} > (\lambda+\delta)/s$, then the BGP highlighted in theorem 1 corresponds to a sequence of capital-labor ratios $\left(k_t^*\right)_{t\geq 0}$ which is such that for all $t \geq 0$, $f_t\left(k_t^*\right) = (1+\lambda)^t \overline{y} = (1+\lambda)^t f_0\left(k_0^*\right)$. But $k_t^* = (1+\lambda)^t k_0^*$; hence $f_t\left(k_t^*\right) = (1+\lambda)^t f_0\left(k_t^*/(1+\lambda)^t\right)$. In extensive form, the last equation implies that $F_t\left(K_t^*, L\right) = F_0\left(K_t^*, (1+\lambda)^t L\right)$.
- Case #2: If $\overline{y}/\overline{k} < (\lambda+\delta)/s$, then the BGPFT starting at date $T \geq 1$ highlighted in theorem 1 is such that for all $t \geq T$, $f_t\left(k_t^*\right) = (1+\lambda)^{t-T}\overline{y} + r^*\left(k_t^* - \overline{k}_{t,t-T}\right)$ where $r^* = \frac{\lambda(1+\mu)^T}{(1+\lambda)(1+\mu)-1}\frac{\overline{y}}{\overline{k}}$. Besides, since $k_t^* = (1+\lambda)^{t-T} k_T^*$, then $f_t\left(k_t^*\right) = (1+\lambda)^{t-T} f_T\left(k_T^*\right)$. In extensive form, this last equation implies that $F_t\left(K_t^*, L\right) = F_T\left(K_t^*, (1+\lambda)^{t-T} L\right)$.

So without *being* labor-augmenting, technical change *appears as* labor-augmenting along the BGP or BGPFT, and there is no contradiction between our result and the Robinson-Uzawa theorem.

We now investigate the stability of our model. The assumption $\overline{y}/\overline{k} < (\lambda+\delta)/s$ generates BGPFT where the marginal product of capital and the competitive capital share are non-degenerated. Consequently, we focus on the growth paths where this assumption is valid in the rest of the paper.[11]

2.3 Stability of BGPFT

In this section, we prove the asymptotic balance of all growth paths. We proceed in two stages. We first show that the model exhibits local stability, and then extend the proof to global stability.

Lemma 1. *Let $\left(\overline{y}, \overline{k}, \lambda, \mu, s, \delta\right)$ be some set of parameters. We assume that $\overline{y}/\overline{k} < (\lambda+\delta)/s$. Let k_0^* the value of the initial capital-labor ratio that generates a BGPFT.*

Then, there exists an interval $\left(\underline{k}_0, \overline{k}_0\right)$ containing k_0^* and such that if $k_0 \in \left(\underline{k}_0, \overline{k}_0\right)$, then the economy is on an ABGP where $k_{t+1}/k_t \xrightarrow{\infty} 1+\lambda$, $y_{t+1}/y_t \xrightarrow{\infty} 1+\lambda$ and $y_t/k_t \xrightarrow{\infty} (\lambda+\delta)/s$.

Proof. Let T be the only strictly positive integer such that $(1+\mu)^{T-1}\frac{\overline{y}}{\overline{k}} < \frac{\lambda+\delta}{s} < (1+\mu)^T\frac{\overline{y}}{\overline{k}}$. Let $k_T^* > 0$ be the date-T capital-labor ratio that results in a BGPFT starting at date T, defined by the equation $f_T\left(k_T^*\right)/k_T^* = (\lambda+\delta)/s$. Then, as seen in the proof of theorem 1, $k_T^* \in \left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)$ where $\overline{k}_{T,0} = \overline{k}/(1+\mu)^T$ and $\overline{k}_{T,1} = \overline{k} * (1+\lambda)/(1+\mu)^{T-1}$.

[11] All the results of subsection 2.3 remain valid if $\overline{y}/\overline{k} > (\lambda+\delta)/s$.

Let $k_0 > 0$ and let $(k_t, y_t)_{t\geq 0}$ be sequences of capital-labor ratios and levels of output per worker generated by the model for initial condition k_0.

We first prove that the image of the interval $\left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)$ by date-T capital accumulation function $\kappa_T(x) = s f_T(x) + (1-\delta)x$ is included in interval $\left(\overline{k}_{T+1,1}, \overline{k}_{T+1,2}\right)$. By the inequalities defining T, we have:

$$\begin{array}{lclclclcl}
\kappa_T\left(\overline{k}_{T,0}\right) & = & s\overline{y}+(1-\delta)\frac{\overline{k}}{(1+\mu)^T} & > & (\lambda+\delta)\frac{\overline{k}}{(1+\mu)^T}+(1-\delta)\frac{\overline{k}}{(1+\mu)^T} & = & \frac{1+\lambda}{(1+\mu)^T}\overline{k} & = & \overline{k}_{T+1,1} \\
\kappa_T\left(\overline{k}_{T,1}\right) & = & s(1+\lambda)\overline{y}+(1-\delta)\frac{(1+\lambda)\overline{k}}{(1+\mu)^{T-1}} & < & (\lambda+\delta)\frac{(1+\lambda)\overline{k}}{(1+\mu)^{T-1}}+(1-\delta)\frac{(1+\lambda)\overline{k}}{(1+\mu)^{T-1}} & = & \frac{(1+\lambda)^2}{(1+\mu)^{T-1}}\overline{k} & = & \overline{k}_{T+1,2}
\end{array}$$

Since function κ_T is continuous and increasing, these two inequalities prove that $\kappa_T\left(\left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)\right) \subset \left(\overline{k}_{T+1,1}, \overline{k}_{T+1,2}\right)$. It is also immediate to extend this reasoning recursively to prove that if $k_T \in \left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)$, then for all $t \geq T$ $k_t \in \left(\overline{k}_{t,t-T}, \overline{k}_{t,t-T+1}\right)$.

Let's call $r^* = \frac{\lambda(1+\mu)^T}{(1+\lambda)(1+\mu)-1}\frac{\overline{y}}{\overline{k}}$ the slope of function f_t over the interval $\left(\overline{k}_{t,t-T}, \overline{k}_{t,t-T+1}\right)$ for $t \geq T$. Then, for all $t \geq T$ and for all $k \in \left(\overline{k}_{t,t-T}, \overline{k}_{t,t-T+1}\right)$, $f_t(k) = r\left(k - \overline{k}_{t,t-T}\right) + \overline{y}_{t-T}$. Thus, for all $t \geq T$:

$$y_{t+1} = r^*\left(k_{t+1} - \overline{k}_{t+1,t-T+1}\right) + \overline{y}_{t-T+1} = r^*\left(k_{t+1} - \frac{(1+\lambda)^{t-T+1}}{(1+\mu)^T}\overline{k}\right) + \overline{y}(1+\lambda)^{t-T+1} \tag{4}$$

$$y_t = r^*\left(k_t - \overline{k}_{t,t-T}\right) + \overline{y}_{t-T} = r^*\left(k_t - \frac{(1+\lambda)^{t-T}}{(1+\mu)^T}\overline{k}\right) + \overline{y}(1+\lambda)^{t-T} \tag{5}$$

When equation (5) is multiplied by $(1+\lambda)$ and subtracted from equation (4) we get:

$$y_{t+1} - (1+\lambda)y_t = r^*\left(k_{t+1} - (1+\lambda)k_t\right) \tag{6}$$

Let $x_t \equiv y_t/k_t$. From equation (6) and from the equation of capital accumulation (3), we have:

$$x_{t+1} = \frac{y_{t+1}}{k_{t+1}} = \frac{r^*\left(k_{t+1} - (1+\lambda)k_t\right) + (1+\lambda)y_t}{s y_t + (1-\delta)k_t} = \frac{r^*\left(s y_t + (1-\delta)k_t - (1+\lambda)k_t\right) + (1+\lambda)y_t}{s y_t + (1-\delta)k_t} \tag{7}$$

Dividing the numerator and the denominator of the right-hand side of (7) by k_t, it comes:

$$x_{t+1} = \frac{r^*(s x_t - \lambda - \delta) + (1+\lambda)x_t}{s x_t + 1 - \delta} \tag{8}$$

For $k_t \in \left(\overline{k}_{t,t-T}, \overline{k}_{t,t-T+1}\right)$, $\frac{y_t}{k_t} \in \left(\frac{\overline{y}_{t-T+1}}{\overline{k}_{t,t-T+1}}, \frac{\overline{y}_{t-T}}{\overline{k}_{t,t-T}}\right)$. Thus, if $k_T \in \left(\overline{k}_{t,t-T}, \overline{k}_{t,t-T+1}\right)$, then for all $t \geq T$, $x_t \in \left((1+\mu)^{T-1}\overline{y}/\overline{k}, (1+\mu)^T\overline{y}/\overline{k}\right)$.

Let ψ be the function of $\left((1+\mu)^{T-1}\overline{y}/\overline{k}, (1+\mu)^T\overline{y}/\overline{k}\right)$ into itself defined by:

$$\psi(x) = \frac{r^*(s x - \lambda - \delta) + (1+\lambda)x}{s x + 1 - \delta}$$

Then, for all $t \geq T$, $x_{t+1} = \psi(x_t)$. As expected, ψ has a unique fixed point which is $x^* = (\lambda+\delta)/s$. ψ is also increasing and concave. By replacing x^* by its value which is $(\lambda+\delta)/s$, it comes:

$$\psi'(x^*) = \frac{r^* s + 1 - \delta}{1+\lambda}$$

Since the segment of f_t that crosses line $\mathscr{D}$ is less steep than line $\mathscr{D}$, then $r^* < (\lambda+\delta)/s$. Thus, $\psi'(x^*) < 1$. Consequently, x_t converges to x^* for any initial $k_T \in \left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)$. The situation is depicted in figure 10.

We have proved that for all k_0 such that $k_T \in \left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)$, the output-capital ratio y_t/k_t converges to $x^* = (\lambda+\delta)/s$ as t tends to infinity. From the capital accumulation equation (3), the fact that y_t/k_t converges to $(\lambda+\delta)/s$ implies that k_{t+1}/k_t converges to $1+\lambda$. Since y_t/k_t converges to some strictly positive limit and k_{t+1}/k_t converges to $1+\lambda$, then y_{t+1}/y_t converges to $1+\lambda$.

So for all $k_0 \in \left((\kappa_{T-1} \circ \cdots \circ \kappa_0)^{-1}\left(\overline{k}_{T,0}\right), (\kappa_{T-1} \circ \cdots \circ \kappa_0)^{-1}\left(\overline{k}_{T,1}\right)\right)$, the model economy is on an ABGP.

□

Using lemma 1, we are now in a position to prove the global stability of our growth model.

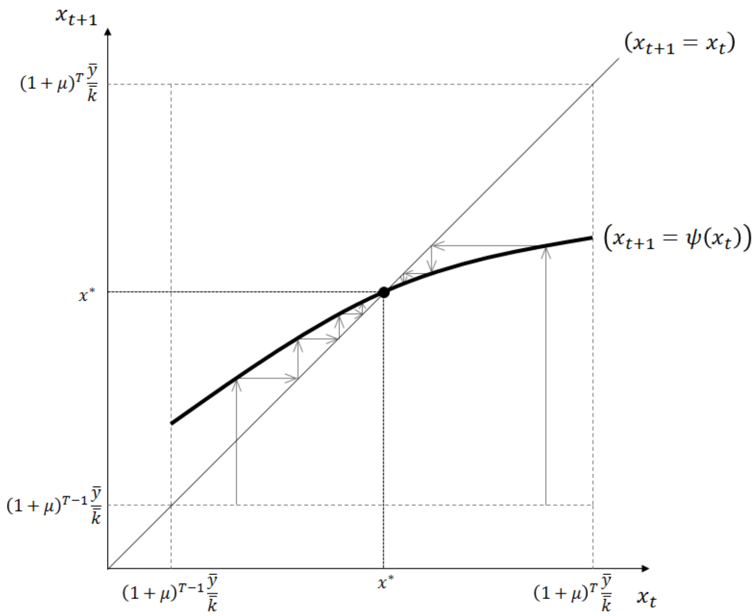


Figure 10: Convergence of x_t if $k_T \in \left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)$.

Theorem 2. *Let* $\left(\overline{y}, \overline{k}, \lambda, \mu, s, \delta\right)$ *be some set of parameters. We assume that* $\overline{y}/\overline{k} < (\lambda+\delta)/s$.

Then, for all $k_0 > 0$, *the model economy is on an ABGP where* $k_{t+1}/k_t \xrightarrow{\infty} 1+\lambda$, $y_{t+1}/y_t \xrightarrow{\infty} 1+\lambda$ *and* $y_t/k_t \xrightarrow{\infty} (\lambda+\delta)/s$.

Proof. Let $T = 1 + \left\lfloor \frac{\ln\left(\frac{\lambda+\delta}{s}\frac{\overline{k}}{\overline{y}}\right)}{\ln(1+\mu)} \right\rfloor \geq 1$. Let $k_0 > 0$ and let $(k_t, y_t)_{t\geq 0}$ be sequences of capital-labor ratios and levels of output per worker generated by the model for initial condition k_0. We call $\left(k_t^*\right)_{t\geq T}$ the sequence of BGPFT capital-labor ratios.

Lemma 1 proves that it suffices to prove that for some $t \geq T$, k_t lies within the interval $\left(\overline{k}_{t,t-T}, \overline{k}_{t,t-T+1}\right)$ to prove that the model economy is on an ABGP.

– If $k_T \in \left(\overline{k}_{T,0}, \overline{k}_{T,1}\right)$, lemma 1 applies and so the model economy is on an ABGP.

– Suppose that $k_T < \overline{k}_{T,0}$. Then, for $t \geq T$, as long as $k_t < \overline{k}_{t,t-T}$ it holds that $y_t/k_t > (1+\mu)^T \overline{y}/\overline{k}$. Consequently, as long as $k_t < \overline{k}_{t,t-T}$:

$$\frac{k_{t+1}}{k_t} = s\frac{y_t}{k_t} + 1 - \delta > s(1+\mu)^T \overline{y}/\overline{k} + 1 - \delta \tag{9}$$

Remark that the right-hand side of inequality (9) does not depend on t. Let call $\overline{g} = s(1+\mu)^T \overline{y}/\overline{k} - \delta$. The definition of T implies that $(1+\mu)^T \overline{y}/\overline{k} > (\lambda+\delta)/s$, and so (9) implies that for all $t \geq T$ such that $k_t < \overline{k}_{t,t-T}$, it holds that:

$$\frac{k_{t+1}}{k_t} > 1 + \overline{g} > 1 + \lambda$$

$(1+\lambda)$ is the constant growth rate of $\left(\overline{k}_{t,t-T}\right)_{t\geq T}$. By a simple the comparison of geometric sequences with different common ratios, we conclude that there exists some date $\tau \geq T$ such that $k_\tau < \overline{k}_{\tau,\tau-T}$ and $k_{\tau+1} > \overline{k}_{\tau+1,\tau-T+1}$. Besides, since $k_\tau < \overline{k}_{\tau,\tau-T} < k_\tau^*$, it also holds that $k_{\tau+1} = \kappa_\tau(k_\tau) < \kappa_\tau\left(k_\tau^*\right) = k_{\tau+1}^* < \overline{k}_{\tau+1,\tau-T+2}$.

So $k_{\tau+1} \in \left(\overline{k}_{\tau+1,\tau-T+1}, \overline{k}_{\tau+1,\tau-T+2}\right)$. In virtue of lemma 1, the model economy is then on an ABGP.

– Suppose that $k_T > \overline{k}_{T,1}$. Then, for $t \geq T$, as long as $k_t > \overline{k}_{t,t-T+1}$ it holds that $y_t/k_t < (1+\mu)^{T-1}\overline{y}/\overline{k}$. Consequently, as long as $k_t > \overline{k}_{t,t-T+1}$:

$$\frac{k_{t+1}}{k_t} = s\frac{y_t}{k_t} + 1 - \delta < s(1+\mu)^{T-1} \overline{y}/\overline{k} + 1 - \delta \tag{10}$$

Remark that the right-hand side of inequality (10) does not depend on t. Let call $\underline{g} = s(1+\mu)^{T-1} \overline{y}/\overline{k} - \delta$. The definition of T implies that $(1+\mu)^{T-1}\overline{y}/\overline{k} < (\lambda+\delta)/s$, and so (10) implies that for all $t \geq T$ such that $k_t > \overline{k}_{t,t-T+1}$, it holds that:

$$\frac{k_{t+1}}{k_t} < 1 + \underline{g} < 1 + \lambda$$

$(1+\lambda)$ is the constant growth rate of $\left(\overline{k}_{t,t-T+1}\right)_{t\geq T}$. By a simple the comparison of geometric sequences with different common ratios, we conclude that there exists some date $\tau' \geq T$ such that $k_{\tau'} > \overline{k}_{\tau',\tau'-T+1}$ and $k_{\tau'+1} < \overline{k}_{\tau'+1,\tau'-T+2}$. Besides, since $k_{\tau'} > \overline{k}_{\tau',\tau'-T+1} > k^*_{\tau}$, it also holds that $k_{\tau'+1} = \kappa_{\tau'}(k_{\tau'}) > \kappa_{\tau'}\left(k^*_{\tau'}\right) = k^*_{\tau'+1} > \overline{k}_{\tau'+1,\tau'-T+1}$. So $k_{\tau'+1} \in \left(\overline{k}_{\tau'+1,\tau'-T+1}, \overline{k}_{\tau'+1,\tau'-T+2}\right)$. In virtue of lemma 1, the model economy is then on an ABGP.

□

So our growth model admits a stable BGPFT for any values of the parameters and is thus immune to any razor-edge issue. The proof of theorem 1 makes it clear that the marginal productivity of capital ('MPK') is constant along any BGPFT. The steady-state MPK – which is actually reached in finite time – is equal to:

$$r^* = \begin{cases} 0 & \text{if } \frac{\overline{y}}{\overline{k}} > \frac{\lambda+\delta}{s} \\ \frac{\lambda(1+\mu)^T}{(1+\lambda)(1+\mu)-1}\frac{\overline{y}}{\overline{k}} & \text{if } \frac{\overline{y}}{\overline{k}} < \frac{\lambda+\delta}{s} \text{ and where } T = 1 + \left\lfloor \frac{\ln\left(\frac{\lambda+\delta}{s}\frac{\overline{k}}{\overline{y}}\right)}{\ln(1+\mu)} \right\rfloor \end{cases}$$

The competitive capital share $\alpha^* \equiv r^* k/y$ is also constant along any BGPFT and equal to:

$$\alpha^* = \begin{cases} 0 & \text{if } \frac{\overline{y}}{\overline{k}} > \frac{\lambda+\delta}{s} \\ \frac{\lambda(1+\mu)^T}{(1+\lambda)(1+\mu)-1}\frac{\overline{y}}{\overline{k}}\frac{s}{\lambda+\delta} & \text{if } \overline{y}/\overline{k} < (\lambda+\delta)/s \text{ and where } T = 1 + \left\lfloor \frac{\ln\left(\frac{\lambda+\delta}{s}\frac{\overline{k}}{\overline{y}}\right)}{\ln(1+\mu)} \right\rfloor \end{cases}$$

3 Conclusion

In this paper, we have set what appears to us as a minimal economic growth model where technical change determines not only the growth rate but also steady-state variables linked to the first derivatives of the production function. Leontief local production functions are not an obstacle to differentiability of the global production function, and the equilibrium factor prices are well determined except in a finite number of capital-labor ratios at each date. It came out that although technical change is not specified as factor-augmenting, it is purely labor-augmenting along the balanced growth path. We believe that this property of our model to be unexpected in light of the Robinson-Uzawa theorem. In a nutshell, the theory presented here proves that building a global production function from a finite set of local, Leontief production functions permits controlling for the curvature of the global production function.

The assumption of factor-augmenting technical change inside some core production function of some specified form leads to some dead ends. In contrast, our view of technical change can accommodate a wide range of medium- to long-run scenarios. There is no need for the elasticity of substitution to be above unity to account for a rise in the capital share; our model proves that all that is needed is to reconsider the assumption of purely factor-augmenting technical change.

One of our model's significant defects is that the steady-states marginal product of capital and capital share are not monotonic functions of our technical change metrics. We believe that this property is due to the discrete-time structure that we assumed. We leave the treatment of the continuous-time equivalent of our model for future work.

A Python plot of the production function

```
import matplotlib.pyplot as plt

def plot_f(lambd, mu, y0_bar, k0_bar, n):
    list_k_limit = [(1+lambd)**(i)*(1/(1+mu))**(n-i)*k0_bar for i in range(n)]
    list_y_limit = [(1+lambd)**(i)*y0_bar for i in range(n)]
    print(list_k_limit)
    print(list_y_limit)
    plt.plot([0]+list_k_limit, [0]+list_y_limit, 'r')
    plt.scatter(list_k_limit, list_y_limit, c='r')
    plt.hlines(y=list_y_limit[-1], xmin=list_k_limit[-1], xmax = list_k_limit[-1] + 1, colors='r')
    plt.show()

plot_f(lambd=0.02, mu=0.04, y0_bar=1, k0_bar=5, n=101)
```

B Python simulation of the Solow model

```
import numpy as np
import matplotlib.pyplot as plt

def remove_duplicate(l):
    final_list = []
    for e in l :
        if e not in final_list:
            final_list.append(e)
    return final_list

def f(lambd, mu, s, delta, k0,y0_bar, k0_bar, n, liste=True):
    y0 = np.interp(k0, [k0_bar], [y0_bar])
    k1_bar_1 = (1+lambd)*k0_bar
    k1_bar_2 = (1/(1+mu))*k0_bar
    list_k_limit = [k1_bar_1, k1_bar_2]
    list_y_limit = [(1+lambd)**(i)*y0_bar for i in range(n)]
    k1 = s*y0+(1-delta)*k0
    y1 = np.interp(k1, list_k_limit, list_y_limit[:2])
    list_k_limit.clear()
    list_y=[y0, y1]
    list_k=[k0, k1]
    for i in range(n-1):
        k0=k1
        y0=y1
        k1 = s*y0+(1-delta)*k0
        list_k_limit = [(1+lambd)**(j)*(1/(1+mu))**(i+1-j)*k0_bar for j in range(i+1)]
        y1 = np.interp(k1, list_k_limit, list_y_limit[:i+1], right = list_y_limit[-1])
        list_k.append(k1)
        list_y.append(y1)
        list_k_limit.clear()
    if liste==True :
        return list_k, list_y
    else :
        return list_k[n-1], list_y[n-1]


list_k, list_y = f(lambd = 0.02, mu = 0.04, s = 0.2, delta = 0.05, k0=4, y0_bar = 1, k0_bar = 5, n=1001, liste=True)
list_quo = [ list_k[i]/list_y[i] for i in range(len(list_k))]

plt.plot(list_k, label="Dynamics of k")
plt.plot(list_y, label="Dynamics of y")
plt.plot(list_quo, label="Dynamics of k/y")
```

```
plt.legend()
plt.show()
```

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