Supplement to “Weighted Multilevel Langevin Simulation of Invariant Measures”

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Submitted to the Annals of Applied Probability

SUPPLEMENT TO "WEIGHTED MULTILEVEL LANGEVIN SIMULATION OF INVARIANT MEASURES"

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This document contains several postponed proofs of the article "Weighted Multilevel Langevin Simulation of Invariant Measures".

1. Proof of [PP18, Lemma 2.1]. Prior to the proof of Lemma 2.1, we need to prove this first technical lemma which will be used to estimate in a precise way the coefficients $\widetilde{\mathbf{W}}_{R+1}$ and $\widetilde{\mathbf{W}}_{R+2}$ involved in the asymptotic mean square error of the **ML2Rgodic** estimator in Theorems 2.1 and 2.2 of [PP18].

Lemma 1.1. *Let $R \geq 2$ be an integer.and let $(x_r)_{r=1,\ldots,R}$ be pairwise distinct real numbers. Then the unique solution $(y_r)_{r=1,\ldots,R}$ to the solution to the $R \times R$-Vandermonde system*

$$\sum_{r=1}^{R} x_r^{\ell-1} y_r = c^{\ell-1},\ \ell = 1,\ldots,R,$$

is given by

$$y_r = \frac{\prod_{s=1,s\neq r}^{R}(x_r - c)}{\prod_{s=1,s\neq r}^{R}(x_r - x_s)}. \tag{1.1}$$

Moreover,

$$\sum_{r=1}^{R} y_r x_r^R = c^R - \prod_{r=1}^{R}(c - x_r) \quad \text{and} \tag{1.2}$$

$$\sum_{r=1}^{R} y_r x_r^R = c^R + \left(\sum_{r=1}^{R} x_r + c\right)\prod_{r=1}^{R}(c - x_r). \tag{1.3}$$

Proof. The above Vandermonde system $\mathrm{Vand}(x_r, r = 1:R)\mathbf{w} = [0^{\ell-1}]_{\ell=1,R}$ can be explicitly solved by the Cramer formulas since its right hand side is of the form $[c^{\ell-1}]_{1\leq \ell\leq R}$ for some $c\in\mathbb{R}$. Namely

$$y_r = \frac{\det(\mathrm{Vand}(x_1,\ldots,x_{r-1},c,x_{r+1},\ldots,x_R))}{\det(\mathrm{Vand}(x_s, s=1:R))},\ r = 1,\ldots,R$$

(the column vector $[c^{\ell-1}]_{1\le \ell\le R}$ replaces the r^{th} column of the original Vandermonde matrix). Then, elementary computations show that it yields the announced solutions.

To compute the next two sums, we start from the following canonical decomposition of the rational fraction

$$\frac{1}{\prod_{r=1}^{R}(X-\frac{1}{x_r-c})}=\sum_{r=1}^{R}\frac{1}{(X-\frac{1}{x_r-c})\prod_{s\neq r}(\frac{1}{x_r-c}-\frac{1}{x_s-c})}.$$

Setting $X=0$ yields after elementary computations

$$\sum_{r=1}^{R} y_r(x_r-c)^R=(-1)^R\prod_{r=1}^{R}(x_r-c).$$

Now, using that $(y_r)_{r=1,\dots,R}$ solves the above Vandermonde system, we get

$$\begin{aligned}\sum_{r=1}^{R} y_r(x_r-c)^R &= \sum_{r=1}^{R} y_r\sum_{k=0}^{R}\binom{R}{k}(-1)^{R-k}x_r^k c^{R-k}\\ &=\sum_{k=0}^{R}\binom{R}{k}(-1)^{R-k}c^{R-k}\underbrace{\sum_{r=1}^{R} y_r x_r^k}_{=c^k \text{if } k<R}=\sum_{r=1}^{R} y_r x_r^R+c^R\big((1-1)^R-1\big)\end{aligned}$$

so that

$$\sum_{r=1}^{R} y_r x_r^R=c^R-(-1)^R\prod_{r=1}^{R}(x_r-c)=c^R-\prod_{r=1}^{R}(c-x_r).$$

The second identity follows likewise by differentiating the above rational fraction with respect to X and then setting $X=0$ again. □

Proof of [PP18, Lemma 2.1]. (a) We introduce the auxiliary variables and parameters

$$(1.4)\quad \overline{\mathbf{W}}_r=\left(\frac{q_1}{q_{r+1}}\right)^a\frac{W_{r+1}}{M^{r-1}},\qquad x_r=M^{-(r-1)}\left(\frac{q_1}{q_{r+1}}\right)^a,\ r=1,\dots,R-1.$$

Then $(\mathbf{W}_r)_{1\le r\le R-1}$ is solution to the system (2.17) of [PP18] if and only if $(\overline{\mathbf{W}}_r)_{1\le r\le R-1}$ is solution to

$$\sum_{r=1}^{R-1}\overline{\mathbf{W}}_r x_r^{\ell-1}=\frac{1}{1-M^{-\ell}},\ \ell=1,\dots,R-1.$$

Expanding $\dfrac{1}{1-M^{-\ell}} = \displaystyle\sum_{k\geq 0} \frac{1}{M^k}\frac{1}{M^{k(\ell-1)}}$ yields by linearity of the above system that it suffices to solve the sequence of $(R-1)\times(R-1)$-Vandermonde systems.

$$(\mathcal{V}_k) \equiv \sum_{r=1}^{R-1} \overline{\mathbf{W}}_{k,r} x_r^{\ell-1} = M^{-k(\ell-1)},\ \ell = 1,\dots,R-1,\ \ k \geq 0.$$

As the x_r are pairwise distinct, $(\mathcal{V}_k)$ has a unique solutions given by

$$\overline{\mathbf{W}}_{k,r} = \prod_{s=1,s\neq r}^{R-1} \frac{x_s - M^{-k}}{x_s - x_r},\ r = 1,\dots,R-1.$$

with the usual convention $\prod_{\emptyset} = 1$ Consequently, for every $r = 2,\dots,R$,

$$\overline{\mathbf{W}}_r = \sum_{k\geq 0} \frac{1}{M^k}\overline{\mathbf{W}}_{k,r} = \sum_{k\geq 0} \frac{1}{M^k} \prod_{s=1,s\neq r}^{R-1} \frac{x_s - M^{-k}}{x_s - x_r},\ r = 1,\dots,R-1.$$

Coming back to the weights of interest finally yields the expected formula.

One derives from the definition of $\widetilde{\mathbf{W}}_{R+1}$ (see (2.18) of [PP18]), using the auxiliary variables, that

$$\widetilde{\mathbf{W}}_{R+1} = q_1^{-aR}\big(1 + (M^{-R}-1)\widetilde{\widetilde{\mathbf{W}}}_R\big) \quad \text{with} \quad \widetilde{\widetilde{\mathbf{W}}}_R = \sum_{r=1}^{R-1} \overline{\mathbf{W}}_r x_r^{R-1}$$

and the x_r are given by (1.4). Following the lines of (a), we derive that

$$\widetilde{\widetilde{\mathbf{W}}}_R = \sum_{k\geq 0} \frac{1}{M^k}\,\widetilde{\widetilde{\mathbf{W}}}_{R,k}$$

where the identity (1.2) established in the above lemma 1.1 yields

$$\widetilde{\widetilde{\mathbf{W}}}_{R,k} = M^{-k(R-1)} - \prod_{r=1}^{R-1}(M^{-k} - x_r).$$

Finally

$$\widetilde{\mathbf{W}}_{R+1} = q_1^{-aR}\left(1 + (M^{-R}-1)\sum_{k\geq 0}\frac{1}{M^{kR}}\Big(1 - \prod_{r=0}^{R-2}\Big(1 - M^{k-r}\Big(\frac{q_1}{q_{r+2}}\Big)^a\Big)\right).$$

Noting that $\sum_{k\geq 0}\frac{1}{M^{kR}}=\frac{1}{1-M^{-R}}$ completes the proof this claim. The computation of $\widetilde{\mathbf{W}}_{R+2}$ follows likewise, starting from the identity (1.3).

(b) In the starting system (2.17) of [PP18] for the weights $q_r^{a(\ell-1)}$ no longer depends on r and can be cancelled in each equation. This leads to the system

$$\mathbf{W}_1=1,\quad 1+(M^{-(\ell-1)}-1)\sum_{r=2}^{R}M^{-(r-2)(\ell-1)}\mathbf{W}_r=0,\ \ell=2,\dots,R.$$

After a standard Abel transform, we get that $\mathbf{W}_r=\mathbf{w}_r+\cdots+\mathbf{w}_R$ where the $\mathbf{w}_r$ are solution to the Vandermonde system

$$\sum_{r=1}^{R}M^{-(r-1)(\ell-1)}\mathbf{w}_r=0^{\ell-1},\ \ell=1,\dots,R.$$

Note that these weights corresponds to those coming out when dealing with $ML2R$ for regular Monte Carlo (see [LP17]) under a weak error expansion condition at rate $\alpha=1$.

As for the boundedness, first note that the "small" weights $\mathbf{w}_r$ read $\mathbf{w}_r=b_{R-r}/a_r$, $r=1,\dots,R$, with

$$a_r=\prod_{k=1}^{r}(1-M^{-k})\quad\text{ and }\quad b_r=(-1)^rM^{-\frac{r(r-1)}{2}}a_r^{-1}.$$

One straightforwardly checks that $a_r\downarrow a_\infty=\prod_{k\geq 1}(1-M^{-k})>0$ and $B_\infty=\sum_{r\geq 1}|b_r|<+\infty$. As a consequence

$$\forall\, R\in\mathbb{N}^*,\ \forall\, r\in\{1,\dots,R\},\quad |\mathbf{W}_r^{(R)}|\leq\frac{B_\infty}{a_\infty}<+\infty.$$

Finally, the same Abel transform shows that

$$\widetilde{\mathbf{W}}_{R+i}=R^{a(R+i)}\sum_{r=1}^{R}M^{-(r-1)(R+i-1)}\mathbf{w}_r,\ i=1,2,$$

and one concludes by formula (1.2) and (1.3) from Lemma 1.1. $\square$

2. An additional bias term. In this part of the appendix, we focus the bias induced by the approximation

$$\frac{\Gamma_{n_r}^{(\ell)}}{\Gamma_{n_r}}\approx q_r^{-a\ell}\frac{\Gamma_n^{(\ell)}}{\Gamma_n}\quad(\text{with }\gamma_n=\gamma_1n^{-a},\ a\in(0,1)),$$

that we use to build some universal weights $(\mathbf{W}_r^{(R)})_{r=1,\dots,R}$ (by universal, we mean that they do not depend on n). We have the following lemma:

Lemma 2.2. *Assume that* $\gamma_n = \gamma_1 n^{-a}$ *with* $a \in (0,1)$.

(a) Let $\chi \in (0,1)$ *and* $L \in \mathbb{N}$ *such that* $La < 1$. *Then, for every* $n \geq n_0 = \lceil \frac{6^{\frac{1}{1-a}}}{\chi} \rceil$,

$$
\begin{aligned}
\left| \frac{\Gamma^{(\ell)}_{\lfloor \chi n \rfloor}}{\Gamma_{\lfloor \chi n \rfloor}} - \chi^{-a(\ell-1)} \frac{\Gamma^{(\ell)}_n}{\Gamma_n} \right| \quad &\leq \quad 3\Big(1 + \frac{1-a}{1-a(R+1)}\Big) \frac{\gamma_1^{\ell-1}}{n^{1-a}} \frac{\chi^{-a\ell}}{\chi^{1-a} - 3n^{a-1}} \\
&\leq \quad \left(6\, \frac{2-aL}{1-aL} \gamma_1^{\ell-1} \chi^{-1-a(\ell-1)} \right) \frac{1}{n^{1-a}}. \qquad (2.5)
\end{aligned}
$$

(b) Set

$$
\begin{aligned}
&\mathrm{Bias}^{(1)}(a,R,q,n) \\
&= \sum_{\ell=2}^{R} \left[\Big[\frac{\Gamma^{(\ell)}_{n_1}}{\Gamma_{n_1}} - q_1^{-a(\ell-1)} \frac{\Gamma^{(\ell)}_n}{\Gamma_n} \Big] \mathbf{W}_1 + \sum_{r=2}^{R} m_{r,\ell} \mathbf{W}_r \Big[\frac{\Gamma^{(\ell)}_{n_r}}{\Gamma_{n_r}} - q_r^{-a(\ell-1)} \frac{\Gamma^{(\ell)}_n}{\Gamma_n} \Big] \right] c_\ell
\end{aligned}
$$

where $m_{r,\ell} = (M^{-(\ell-1)} - 1) M^{-(r-2)(\ell-1)}$. *We have:*

$$
|\mathrm{Bias}^{(1)}(a,R,q,n)| \leq \frac{C_{a,\mathbf{q},r}}{n^{1-a}},
$$

where

$$
C_{a,q,r} = 6\, \frac{2-a(R+1)}{1-a(R+1)} \|\mathbf{W}\|_\infty \, q_*^{-1} \sum_{\ell=2}^{R} (\gamma_1 q_*^{-a})^{\ell-1} \left[1 + \sum_{r=2}^{R} m_{r,\ell} \right] |c_\ell|
$$

with $q_* = \min_{1 \leq r \leq R} q_r$ *and* $\|\mathbf{W}\|_\infty = \sup_{r \in \{1\ldots,R\}, R \geq 2} \mathbf{W}^{(R)}_r$.

Furthermore, if $q_1 = \ldots = q_R = \frac{1}{R}$, *then* $\mathrm{Bias}^{(1)}(a,R,q,n) = 0$.

Remark 2.1. Note that since $a < 1/2$, $n^{1-a} = o(n^{-\frac{1}{2}})$ so that this term is negligible at the first and second orders of the expansions obtained in this paper. Finally, it is worth noting that this term is equal to 0 when the q_i are equal to $\frac{1}{R}$, case where, in addition, the $\mathbf{W}_r$, $r = 1\ldots,R$ have a simple closed form given by formulas (2.22) and (2.23) of [PP18, Lemma 2.1].

Proof. First, we derive by a comparison argument with integrals $\int_0^n x^{-a}dx$ and $\int_1^{n+1} x^{-a}dx$ that

$$
\frac{n^{1-a}-2}{1-a} \leq \sum_{k=1}^{n} k^{-a} \leq \frac{n^{1-a}}{1-a}, \; n \geq 1, a \in (0,1). \qquad (2.6)
$$

Elementary computations then show that, for every $a \in \left(0, \frac{1}{R}\right)$, $\chi \in (0,1)$, and every $n \geq 1$, every integer $\ell \in \{1,\ldots,R+1\}$

$$\left|\frac{\Gamma^{(\ell)}_{\lfloor \chi n\rfloor}}{\Gamma_{\lfloor \chi n\rfloor}} - \chi^{-a(\ell-1)}\frac{\Gamma^{(\ell)}_n}{\Gamma_n}\right| \leq \frac{3\gamma_1^\ell}{\Gamma_{\lfloor \chi n\rfloor}}\left(\frac{1}{1-a\ell}+\frac{\chi^{-a\ell}}{1-a}\right)$$

Using that $u \mapsto u^{1-a}$ is $(1-a)$-Hölder, we derive from the left inequality in (2.6) that $\Gamma_{\lfloor \chi n\rfloor} \geq \gamma_1 \frac{(\chi n)^{1-a}-3}{1-a}$ so that, for every $n \geq \frac{6^{\frac{1}{1-a}}}{\chi}$,

$$\begin{aligned}\left|\frac{\Gamma^{(\ell)}_{\lfloor \chi n\rfloor}}{\Gamma_{\lfloor \chi n\rfloor}} - \chi^{-a(\ell-1)}\frac{\Gamma^{(\ell)}_n}{\Gamma_n}\right| &\leq 3\Big(1+\frac{1-a}{1-a(R+1)}\Big)\frac{\gamma_1^{\ell-1}}{n^{1-a}}\frac{\chi^{-a\ell}}{\chi^{1-a}-3n^{a-1}}\\ &\leq 6\,\frac{2-a(R+1)}{1-a(R+1)}\frac{\gamma_1^{\ell-1}}{n^{1-a}}\chi^{-1-a(\ell-1)}. \qquad (2.7)\end{aligned}$$

Now, since $\|\mathbf{W}\|_\infty < +\infty$ (see [PP18, Lemma 2.1(b)]), we deduce by plugging the above inequality in $\mathrm{Bias}^{(1)}(a,R,q,n)$ that, for every $n \geq \frac{6^{\frac{1}{1-a}}}{q_*}$,

$$\begin{aligned}&|\mathrm{Bias}^{(1)}(a,R,q,n)|\\ &\qquad \leq 6\,\frac{2-a(R+1)}{1-a(R+1)}\frac{1}{n^{1-a}}\|\mathbf{W}\|_\infty\, q_*^{-1}\sum_{\ell=2}^{R}(\gamma_1 q_*^{-a})^{\ell-1}\left[1+\sum_{r=2}^{R} m_{r,\ell}\right]|c_\ell|.\end{aligned}$$

When $q_r = \frac{1}{R}$, $r = 1,\ldots,R$,

$$\mathrm{Bias}^{(1)}(a,R,q,n) = \sum_{\ell=2}^{R}\left[\frac{\Gamma^{(\ell)}_{n_1}}{\Gamma_{n_1}} - \bar{q}_1^{-a(\ell-1)}\frac{\Gamma^{(\ell)}_n}{\Gamma_n}\right]\left(\mathbf{W}_1+\sum_{r=2}^{R}\mathbf{W}_r m_{r,\ell}\right)c_\ell = 0$$

since $\mathbf{W}$ is solution to (2.17) of [PP18]. $\square$

3. Proof of of [PP18, Proposition 7.6]. *(i)* Setting $T = \frac{1}{\rho}\log(1/\varepsilon)$ leads to

$$|\mathbb{E}[f(X_T)] - \nu(f)| \leq c_1\varepsilon.$$

Thus, for such a T, it remains, for a given ε to show that this is possible to build $\Upsilon(T,R,\mathbf{N})$ in such a way that $\|\Upsilon(T,R,\mathbf{N}) - \mathbb{E}[f(X_T)]\|_2 \leq C\varepsilon$ (where C is independent of ε) with a cost of computation proportional to $\varepsilon^{-2}\log(\varepsilon^2)$ when $\beta = 1$ and to ε^{-2} when $\beta > 1$. This property is now classical but in our context in the Finite Horizon Multilevel literature (see *e.g.* [Gil08]), but here, we certainly have to take into account the dependence in T: we recall that

$$\|\Upsilon(T,R,\mathbf{N})-\mathbb{E}[f(X_T)]\|_2^2=\big(m_{h2^{1-R},T}\big)^2+\sum_{r=1}^{R}\frac{\mathrm{Var}(Y_1^{(r)})}{N_r}$$

where $m_{h,T}=\mathbb{E}[f(\bar{X}_T^h)]-\mathbb{E}[f(X_T)]$. Under the assumptions of the proposition, we have

$$|m_{h2^{1-R},T}|\le c_2h2^{1-R}$$

so that $R=(\log 2)^{-1}\log(1/\varepsilon)$ leads to $|m_{h2^{1-R},T}|\le\tilde{c}_2\varepsilon$ with $\tilde{c}_2=2c_2h$. Let us now consider the variance component in terms of the computational cost. Optimizing the choice can be made by minimizing the *effort* $\mathcal{E}(T,R,h,\mathbf{N})$, *i.e.* the product of the cost of simulation by the variance. Here,

$$\mathcal{E}(T,R,h,\mathbf{N})=\frac{T}{h}\left(N_1+3\sum_{r=2}^{R}2^{r-2}N_r\right)\left(\sum_{r=1}^{R}\frac{\mathrm{Var}(Y_1^{(r)})}{N_r}\right).$$

The equality case in the Schwarz Inequality shows that this above product is minimal when the terms in both sums are proportional, *i.e.* when there exists $\lambda>0$ such that

$$\forall r\in\{1,\dots,R\},\quad 2^rN_r=\lambda\frac{\mathrm{Var}(Y_1^{(r)})}{N_r}\quad\Longleftrightarrow\quad N_r=\sqrt{\lambda}2^{-\frac{r}{2}}\sqrt{\mathrm{Var}(Y_1^{(r)})}.$$

Hence,

$$\sum_{r=1}^{R}\frac{\mathrm{Var}(Y_1^{(r)})}{N_r}=\lambda^{-\frac12}\sum_{r=1}^{R}2^{\frac{r}{2}}\sqrt{\mathrm{Var}(Y_1^{(r)})}.$$

Under the third assumption(on the variance), $\sqrt{\mathrm{Var}(Y_1^{(r)})}\le(1+2^{\frac{\beta}{2}})c_3h^{\frac{\beta}{2}}2^{-\frac{\beta(r-1)}{2}}$ so that

$$\begin{aligned}\sum_{r=1}^{R}\frac{\mathrm{Var}(Y_1^{(r)})}{N_r}&\le(1+2^{\frac{\beta}{2}})c_3(2h)^{\frac{\beta}{2}}2^{\frac12}\lambda^{-\frac12}\sum_{r=1}^{R}2^{(r-1)\frac{1-\beta}{2}}\\&=(1+2^{\frac{\beta}{2}})c_3(2h)^{\frac{\beta}{2}}2^{\frac12}\lambda^{-\frac12}\frac{2^{R\frac{1-\beta}{2}}-1}{2^{\frac{1-\beta}{2}}-1}\\&\le C_{\beta,h}\lambda^{-\frac12}\big(2^{R\frac{1-\beta}{2}}\mathbf{1}_{\beta\neq1}+R\mathbf{1}_{\beta=1}\big)\end{aligned}$$

(with the usual convention $\frac{1^R-1}{1-1}=R$ when $\beta=1$). We may now fix λ in such a way that the variance contribution to the quadratic error be proportional to ε. Up to a constant only depending on h and β, we have

$$\sum_{r=1}^{R}\frac{\mathrm{Var}(Y_1^{(r)})}{N_r}\lesssim\varepsilon^2 \quad \text{if } \sqrt{\lambda}\lesssim\varepsilon^2 C_{\beta,h}\lambda^{-\frac{1}{2}}\big(2^{R\frac{1-\beta}{2}}\mathbf{1}_{\beta\neq 1}+R\mathbf{1}_{\beta=1}\big)^{-1}.$$

The above property combined with what precedes implies that the global MSE is bounded by $C\varepsilon^2$ if we set

$$N_r=\begin{cases}2^{-r\frac{\beta+1}{2}}\varepsilon^{-2}\log(1/\varepsilon) & \text{if } \beta=1\\ 2^{-r\frac{\beta+1}{2}}\varepsilon^{-2} & \text{if } \beta>1.\end{cases}$$

To complete the proof of this first part, it remains to note that with these choices of T, R and N_r, the computational cost for the simulation is equal to

$$\frac{T}{h}\left(N_1+3\sum_{r=2}^{R}2^{r-2}N_r\right)\lesssim\begin{cases}\alpha^{-1}\varepsilon^{-2}\log^3(1/\varepsilon)) & \text{if } \beta=1\\ \alpha^{-1}\varepsilon^{-2}\log(1/\varepsilon)) & \text{if } \beta>1.\end{cases}$$

(ii) We have to prove that the three conditions of (i) hold true under $(\mathbf{C_s})$.

First, applying Itô's formula to $e^{2\alpha t}\|X_t^x-X_t^y\|_S^2$, shows, owing to Assumption $(\mathbf{C_s})$, that

$$\mathbb{E}[\|X_T^x-X_T^y\|_S^2]\leq\|x-y\|_S^2e^{-2\alpha T}.$$

Using the Lipschitz continuity of f, the stationarity property and the previous inequality yield

$$\begin{aligned}\big|\mathbb{E}[f(X_T^x)]-\nu(f)\big| &= \Big|\int_{\mathbb{R}^d}\mathbb{E}[f(X_T^x)-f(X_T^y)]\nu(dy)\Big|\\ &\leq [f]_1\int_{\mathbb{R}^d}\sqrt{\mathbb{E}[\|X_T^x-X_T^y\|_S^2]}\nu(dy)\\ &\leq [f]_1e^{-\alpha T}\int_{\mathbb{R}^d}\|x-y\|_S\nu(dy)\end{aligned}$$

where $[f]_1$ denotes the Lipschitz constant of f.

Assumption (c) is also obtained by exploiting the contractive properties derived from $(\mathbf{C_s})$. The proof follows the lines of [Lem05], Theorem IV.1 applied with $p=2$ and the function $V(x):=|x|_S^2$. This yields in particular that $\beta=2$ and the time-independence of c_3. Note that finally, since $\beta=2$ and f is Lipschitz, we can control the weak error as follows:

$$|\mathbb{E}[f(X_T)]-\mathbb{E}[f(\xi_T^h)]|\leq[f]_1\sqrt{\mathbb{E}[\|X_T-\xi_T^h\|_S{}^2]}\leq c_3[f]_1h.$$

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