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Abstract. The aim of this paper is to establish the range of p's for which the expansion of a function $f \in L^p$ in a generalized prolate spheroidal wave function (PSWFs) basis converges to f in L^p. Two generalizations of PSWFs are considered here, the circular PSWFs introduced by D. Slepian and the weighted PSWFs introduced by Wang and Zhang. Both cases cover the classical PSWFs for which the corresponding results has been previously established by Barceló and Cordoba.

To establish those results, we prove a general result that allows to extend mean convergence in a given basis (e.g. Jacobi polynomials or Bessel basis) to mean convergence in a second basis (here the generalized PSWFs).

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1. Introduction

In their seminal work from the 70s, Landau, Pollak and Slepian [LP1, LP2, SP] have shown that the orthonormal basis that is best concentrated in the time-frequency plane is given by the Prolate Spheroidal Wave Functions (PSFWs). This basis therefore provides an efficient tool for signal processing. Since then, the PSFWs have proven useful in many applications ranging from random matrix theory (*e.g.* [dCM, Me, Dy]) to numerical analysis (*e.g.* [XRY, Wa2]). While taking naturally place in an L^2 setting, one may also consider the behavior of expansions of functions in the PSFW basis in the L^p-setting. This has been done by Barcelo and Cordoba for the usual PSFWs. Our aim here is to extend this work to two natural generalizations of the PSFWs, namely, the Hankel-PSFWs introduced by Slepian [Sl1] and the weighted PSFWs recently introduced by Wang and Zhang [WZ].

Let us now be more precise with the results in this paper. First let us recall that the prolate spheroidal wave functions $(\psi_{n,c})_{n\geq 0}$ are eigenvectors of an integral operator. Using the min-max theorem, they can thus be obtained as solutions of an extremal problem: for $c>0$, recall that the Paley-Wiener space $PW_c=\{f\in L^2(\mathbb{R})\,:\,\operatorname{supp}\widehat{f}\subset[-c,c]\}$ where $\widehat{f}$ stands for the Fourier transform of f. Then one sets

$$\psi_{n,c}=\operatorname{argmax}\left\{\frac{\|f\|_{L^2(I)}}{\|f\|_{L^2(\mathbb{R})}}\,:\,f\in PW_c,\ f\in\operatorname{span}\{\psi_{k,c},k<n\}^{\perp}\right\}.$$

A fundamental fact discovered by Landau, Pollak and Slepian is that they are also eigenfunctions of a Sturm-Liouville operator, a fact tagged as a "happy miracle" by Slepian [Sl3].

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Another key fact for our purpuses is that $(\psi_{n,c})_{n\geq 0}$ is an orthonormal basis of PW_c and this basis is the best concentrated in the time domain.

In this paper, we are interested in two generalizations of the PSFWs. For both cases, the basis is constructed as a set of eigenvectors of an integral operator, the happy miracle occurs so that they are also eigenvectors of a Sturm-Liouville operator and, more important for us, they form an orthonormal basis of a Paley-Wiener type of space.

The first basis we consider was introduced by Slepian [Sl1]. It is an analogue of the classical PSFWs adapted to higher dimensional *radial* Fourier analysis. To introduce them, we need some further notation. First, we replace the Fourier transform by the *Hankel transform* defined for $f\in L^1(0,+\infty)$ by

$$\mathcal{H}^\alpha f(x)=\int_0^{+\infty}\sqrt{xy}J_\alpha(xy)f(y)\,\mathrm{d}y$$

where J_α is the Bessel function and $\alpha>-1/2$. Like the usual Fourier transform, the Hankel transform extends into a unitary operator on $L^2(0,+\infty)$. The corresponding Paley-Wiener space is then denoted by

$$HB_c^{(\alpha)}=\left\{f\in L^2(0,\infty);\operatorname{supp}\mathcal{H}^\alpha(f)\subseteq[0,c]\right\}.$$

Finally, the Circular (Hankel) Prolate Spheroidal Wave Functions (CPSWFs) are defined by

$$\psi_{n,c}^\alpha=\operatorname{argmax}\left\{\frac{\|f\|_{L^2(0,1)}}{\|f\|_{L^2(0,+\infty)}}\ :f\in HB_c^{(\alpha)},\ f\in\operatorname{span}\{\psi_{k,c}^\alpha,k<n\}^\perp\right\}.$$

Then $(\psi_{n,c}^\alpha)_{n\geq 0}$ is an orthonormal basis of $HB_c^{(\alpha)}$. Note also that when $\alpha=0$, these are usual PSFWs, more precisely, $\psi_{n,c}^0=\psi_{2n,c}$.

The second basis we consider, the Weighted Prolate Spheroidal Wawe Functions (WPSFWs), is defined in a similar fashion. We first introduce the weighted Paley-Wiener spaces

$$wPW_c^{(\alpha)}=\left\{f\in L^2(\mathbb{R});\operatorname{supp}\widehat{f}\subseteq[-c,c],\ \widehat{f}\in L^2\big((-c,c),(1-x^2/c^2)^{-\alpha}\,\mathrm{d}x\big)\right\}.$$

The WPSFWs are defined by

$$\Psi_{n,c}^\alpha=\operatorname{argmax}\left\{\frac{\|f\|_{L^2\left((-1,1),(1-x^2)^\alpha\,\mathrm{d}x\right)}}{\left\|\widehat{f}\right\|_{L^2\left((-c,c),(1-x^2/c^2)^{-\alpha}\,\mathrm{d}x\right)}}\ :f\in wPW_c^{(\alpha)},\ f\in\operatorname{span}\{\Psi_{k,c}^\alpha,k<n\}^\perp\right\}.$$

Again, $\Psi_{n,c}^\alpha$ is an orthonormal basis of $wPW_c^{(\alpha)}$ and $\Psi_{n,c}^0=\psi_{n,c}$.

The aim of this paper is to characterize the range of p's for which prolate spheroidal wave functions converge in L^p. The subject of the L^p-convergence (also called mean convergence of order p) of orthogonal series, is a central subject in harmonic analysis. This kind of convergence is briefly described as follows. Let $1<p<\infty$, $a,b\in\overline{\mathbb{R}}$, $I=(a,b)$, and $\{\phi_n\}$ an orthonormal set of the weighted Hilbert space $L^2(I,\omega)$-space, where ω is a positive weight function. We define the kernel

$$K_N(x,y)=\sum_{n=0}^N\phi_n(x)\overline{\phi_n(y)}$$

so that the orthogonal projection of $f \in L^2(I,\omega)$ on the span of $\{\phi_0,\dots,\phi_N\}$ is given by

$$\mathcal{K}_N(f)(x) = \int_I K_N(x,y)f(y)\omega(y)\,\mathrm{d}y = \sum_{n=0}^{N} a_n(f)\phi_n(x)$$

with

$$a_n(f) = \int_I f(y)\overline{\phi_n(y)}\omega(x)\,\mathrm{d}y.$$

Now, this last expression may be well defined even for $f \in L^p(I,\omega)$, $p \neq 2$ and then $\mathcal{K}_N(f)$ is also well defined. This happens for instance if $\phi_n \in L^p(I,\omega)$ for every p which is often the case in practice. The orthonormal set $\{\phi_n\}$ is said to have mean convergence of order p, or L^p-convergence over the Banach space $L^p(I,d\omega)$ if for every $f \in L^p(I,d\omega)$, $\mathcal{K}_N(f)$ is well defined and

$$\lim_{N\to\infty} \|f - \mathcal{K}_N f\|_{L^p(I,\omega)} = \lim_{N\to\infty} \left[\int_a^b |f(x) - \mathcal{K}_N(f)(x)|^p \omega(x)\,\mathrm{d}x\right]^{1/p} = 0.$$

This concept of mean convergence is also valid on a subspace $\mathcal{B}$, rather than the whole Banach space $L^p(I,d\omega)$.

To the best of our knowledge, M. Riesz was the first in the late 1920's, to investigate this problem in the special case of the trigonometric Fourier series over $L^p(\mathbb{T})$, $1 \le p < +\infty$. More precisely, in [Ri], it has been shown that the Hilbert transform over the torus $\mathbb{T}$ is bounded on $L^p(\mathbb{T})$ if and only if $p > 1$. Further, the L^p-boundedness of the Hilbert transform is equivalent to the mean convergence of the Fourier series on $L^p(\mathbb{T})$. In the late 1940's, H. Pollard, in a series of papers [Po1, Po2, Po3], has studied the mean convergence of some classical orthogonal polynomials, such as Legendre and Jacobi polynomials. In particular, in the later case, he has shown that if $\alpha \ge -\frac{1}{2}$ and $\omega_\alpha(x) = (1-x^2)^\alpha$, $x \in I = [-1,1]$ is the Jacobi weight, then the mean convergence over $L^p(I,\omega_{\alpha,})$ of Jacobi series expansion holds true, whenever

$$m(\alpha) := 4\frac{\alpha+1}{2\alpha+3} < p < M(\alpha) := 4\frac{\alpha+1}{2\alpha+1}.$$

He has also shown that the previous conclusion fails if $p < m(\alpha)$ or $p > M(\alpha)$. In [MW], the authors have shown that the mean convergence of the Bessel series expansion over the space $L^p([0,1], x\,\mathrm{d}x)$ holds true whenever $4/3 < p < 4$. Later on, Newman and Rudin [NR] have shown that the mean convergence fails for the critical values of $p = m(\alpha), p = M(\alpha)$ in the Jacobi case and for $p = 4/3$, $p = 4$ for the Bessel case. More recently, in [Va] Varona has extended the mean convergence of Bessel series for $\alpha > -1/2$ over the Hankel Paley-Wiener space of functions from $L^p([0,\infty), x^\alpha\,\mathrm{d}x)$ with compactly supported Hankel transforms.

An other important extension has been given by Barcelo and Cordoba [BC] where they have shown that the series expansion in terms of the classical prolate spheroidal wave functions (PSWFs) has the mean convergence property over the previous Fourier Paley- Wiener space, holds true if and only if $4/3 < p < 4$. This is the main source of inspiration for our work, so let us detail the ideas behind [BC]. Barcelo and Cordoba first determine the expansion of PSFWs in a basis consisting of Bessel functions. It turns out that the kernel of the projection onto this second basis is given by a Christoffel-Darboux like formula so that it's mean convergence properties can be deduced from estimates for weighted Hilbert transforms. The last step of the proof is a sort of transference principle which allows to show that the PSWFs have the mean convergence property of order p exactly when the Bessel basis has this property.

Our first aim here is to formalize this transference principle. We consider two orthonormal bases $(\varphi_n)_{n\geq 0}$ and $(\psi_n)_{n\geq 0}$ of $L^2(\Omega,\mu)$. Then, we establish a fairly general principle giving several conditions on $(\varphi_n)_{n\geq 0}$ and $(\psi_n)_{n\geq 0}$ that will ensure the mutual mean convergence property of order p associated for the two bases.

The second part of the paper then consists in applying this principle to the two extensions of PSFWs mentionned above. For the Circular PSFWs the second basis consists again of a basis built from Bessel functions for which we have to adapt the proof of Barcelo-Cordoba to establish the range of p's for which mean convergence holds. The case of Weighted PSFWs is a bit simpler as the second basis consists of Jacobi polynomials for which the mean convergence property is already known. As this case is simpler, it will be treated first. We may now state our main result:

Theorem. *Let $\alpha > -1/2$, $c > 0$, $N \geq 0$. Let $I = (-1,1)$ and $\omega_\alpha(x) = (1-x^2)^\alpha$.*

- *Let $p_0 = 2 - \frac{1}{\alpha+3/2}$ and $p_0' = 2 + \dfrac{1}{\alpha+1/2}$. Let $(\Psi_{n,c}^{(\alpha)})_{n\geq 0}$ be the family of weighted prolate spheroidal wave functions. For a smooth function f on $I=(-1,1)$, define*

$$\Psi_N^{(\alpha)} f = \sum_{n=0}^{N} \left\langle f, \Psi_{n,c}^{(\alpha)} \right\rangle_{L^2(I,\omega_\alpha)} \Psi_{n,c}^{(\alpha)}.$$

  Then, for every $p \in (1,\infty)$, $\Psi_N^{(\alpha)}$ extends to a bounded operator $L^p(I,\omega_\alpha(x)\,\mathrm{d}x) \to L^p(I,\omega_\alpha(x)\,\mathrm{d}x)$. Further

$$\Psi_N^{(\alpha)} f \to f \qquad \text{in } L^p(I,\omega_\alpha(x)\,\mathrm{d}x)$$

  for every $f \in L^p(I,\omega_\alpha(x)\,\mathrm{d}x)$ if and only if $p \in (p_0, p_0')$.
- *Let $(\psi_{n,c}^{(\alpha)})_{n\geq 0}$ be the family of Hankel prolate spheroidal wave functions. For a smooth function f on $I=(0,\infty)$, define*

$$\Psi_N^{(\alpha)} f = \sum_{n=0}^{N} \left\langle f, \psi_{n,c}^{(\alpha)} \right\rangle_{L^2(0,\infty)} \psi_{n,c}^{(\alpha)}.$$

  Then, for every $p \in (1,\infty)$, $\Psi_N^{(\alpha)}$ extends to a bounded operator $L^p(0,\infty) \to L^p(0,\infty)$. Further

$$\Psi_N^{(\alpha)} f \to f \qquad \text{in } L^p(0,\infty)$$

  for every $f \in B_{c,p}^\alpha$ if and only if $p \in (4/3, 4)$.

This work is organized a follows. In section 2, we study a general principle that ensure the mutual L^p-convergence of two series expansion with respect to two different orthonormal bases of a Hilbert space $L^2(\mu)$. In section 3, we give a list of technical lemmas that ensure or simplify the conditions given in the general principle of the previous section. In section 4, we apply the results of sections 2 and 3 and check in detail that the conditions that we have established in the case of general principle hold true for the series expansion in the weighted PSWFs. Finally in section 5, we prove that this mean convergence property holds also true for circular PSWFs series.

2. The General principal

2.1. **The setting and the main result.** As already explained, to prove the L^p-convergence of the expansion in a prolate basis, we will expend the prolates in a second basis for which this L^p-convergence is easier to study. This idea is formalized in the following setting:

We consider a measure space (Ω,μ) and assume that, for every $1<p<\infty$, $L^p(\Omega,\mu)$ is infinite dimensional and separable. The dual index of p will be denoted by $p'=\dfrac{p}{p-1}$. We consider two orthonormal bases $(\varphi_n)_{n\geq 0}$ and $(\psi_n)_{n\geq 0}$ of $L^2(\Omega,\mu)$. For $N\geq 0$ we denote by Φ_N (resp. Ψ_N) both the orthogonal projection on $\operatorname{span}\{\varphi_0,\ldots,\varphi_N\}$ (resp. $\operatorname{span}\{\psi_0,\ldots,\psi_N\}$) and its kernel

$$\Phi_N(x,y)=\sum_{n=0}^N \varphi_n(x)\overline{\varphi_n(y)}\quad\text{resp.}\quad \Psi_N(x,y)=\sum_{n=0}^N \psi_n(x)\overline{\psi_n(y)}.$$

Our aim in this section is to define several conditions on φ_n,ψ_n that will ensure that, for any $1<p<\infty$, $\Phi_N f\to f$ in some L^p if and only if $\Psi_N f\to f$ in L^p. The first condition is of course that this makes sense. The second one is that some relation exists between the two bases. The other conditions are technical and are those that will be the most difficult to check in practice.

(L) For every $1<p<\infty$, and every n, $\varphi_n\in L^p(\mu)$. Further, we assume that there is a $0\leq\gamma_p<1$ such that

$$\|\varphi_n\|_{L^p(\mu)}\lesssim n^{\gamma_p}. \tag{2.1}$$

Finally, we assume that $0\alpha_p:=\gamma_p+\gamma_{p'}<1$ and that there is a p_0 such that if $p\in(p_0,p_0')$, $\alpha_p=0$. In other words, for $p\in(p_0,p_0')$,

$$\|\varphi_n\|_{L^p(\mu)}\|\varphi_n\|_{L^{p'}(\mu)}\lesssim 1 \tag{2.2}$$

while for $p\notin(p_0,p_0')$,

$$\|\varphi_n\|_{L^p(\mu)}\|\varphi_n\|_{L^{p'}(\mu)}\lesssim n^{\alpha_p},\qquad \alpha_p<1. \tag{2.3}$$

(R) Let $\alpha_k^n=\langle\psi_n,\varphi_k\rangle_{L^2(\mu)}$ so that $\psi_n=\sum_{n=0}^\infty \alpha_k^n\varphi_k$. We assume that there exists an integer n_0 and $\kappa,\kappa'>0$ two real numbers such that (α_k^n) satisfies a three term recursion formula

$$f(k,n)\alpha_k^n=a_k\alpha_{k-1}^n+a_{k+1}\alpha_{k+1}^n$$

where
1. $|a_k|\leq\frac12$,
2. for fixed n, there is a k_n such that $|f(k,n)|\gtrsim k^2$ when $k\geq k_n$,
3. there is an $n_0\geq 0$ such that, for $n\geq n_0$, and every $k\geq 0$, $|f(k,n)|\gtrsim k|k-n|$,
4.

$$\left|\frac{a_{n+1}}{f(n+1,n)}-\frac{a_{n+2}}{f(n+2,n+1)}\right|\lesssim n^{-2}. \tag{2.4}$$

(B) Let $\tilde\Phi_N(x,y)=\displaystyle\sum_{n=0}^N\varphi_n(x)\overline{\varphi_{n+1}(y)}$ and write also $\tilde\Phi_N$ for the corresponding integral operator. For every $1<p<\infty$, we assume that $\tilde\Phi_N$ defines a bounded linear operator on $L^p(\mu)$ and that there exists $\beta_p < 1$ such that, for every $f \in L^p(\mu)$

$$\left\|\tilde{\Phi}_N f\right\|_{L^p(\mu)} \lesssim N^{\beta_p}\|f\|_{L^p(\mu)}.$$

(C) There exists $1 < p_0 < 2$ such that $\Phi_N f \to f$ for every $f \in L^p(\Omega,\mu)$, with convergence in $L^p(\Omega,\mu)$, if and only if $p_0 < p < p_0'$.
(D) There exists a set $\mathcal{D}$ that is dense in every $L^p(\mu)$, $1 < p < \infty$, such that, for every $1 < p < \infty$ and every $f \in L^p(\mu)$, $\Phi_N f, \Psi_N f \to f$ in $L^p(\mu)$ when $N \to \infty$.

In this all of Section 2.1 we will use the above notation and assume that these conditions are fulfilled. Our main result is then:

Theorem 2.1. *With the above notation, and under conditions* (L), (R), (B), (C) *and* (D), *we have* $\Psi_N f \to f$ *for every* $f \in L^p(\Omega,\mu)$, *with convergence in* $L^p(\Omega,\mu)$, *if and only if* $p_0 < p < p_0'$.

Remark 2.2. Note that the adjoint $\tilde{\Phi}_N^*$ of $\tilde{\Phi}_N$ has kernel $\tilde{\Phi}_N^*(x,y) = \displaystyle\sum_{n=1}^{N+1} \varphi_n(x)\overline{\varphi_{n-1}(y)}$. Thus, if condition (B) holds, then for every $f \in L^p(\mu)$

$$\left\|\tilde{\Phi}_N^* f\right\|_{L^p(\mu)} \lesssim N^{\beta_{p'}}\|f\|_{L^p(\mu)}.$$

Condition (B) may be replaced by a slightly weaker condition, see Remark 2.7 below.
Also we state the various conditions with $1 < p < \infty$. It is enough that they hold for $p_1 < p < p_1'$ with $1 < p_1 < p_0$.

The remaining of this section is devoted to the proof of this result.

2.2. **Step 1: A simple lemma and an extension of the Banach-Steinhaus Theorem.** We will here formalize a result that has already been used in [BC]. To start, let us state the following simple and well known lemma that we prove for sake of completeness:

Lemma 2.3. *Let* $1 < p < \infty$ *and let* $K : \Omega \times \Omega \to \mathbb{C}$ *be such that*

$$\|K\|_{L^p(\mu)\otimes L^{p'}(\mu)} := \left(\int_\Omega \left(\int_\Omega |K(x,y)|^{p'}\,\mathrm{d}\mu(y)\right)^{p/p'} \mathrm{d}\mu(x)\right)^{1/p} < +\infty.$$

Then the integral operator K *defined by*

$$Kf(x) = \int_\Omega K(x,y)f(y)\,\mathrm{d}y$$

extends to a continuous operator $K : L^p \to L^p$ *with norm*

$$\|K\|_{L^p(\mu)\to L^p(\mu)} \leq \|K\|_{L^p(\mu)\otimes L^{p'}(\mu)}.$$

Proof. Indeed, using Hölder's inequality,

$$\begin{aligned} \|Kf\|_p^p &= \int_\Omega \left|\int_\Omega K(x,y)f(y)\,\mathrm{d}\mu(y)\right|^p \mathrm{d}\mu(x) \\ &\leq \int_\Omega \left(\int_\Omega |K(x,y)|^{p'}\,\mathrm{d}\mu(y)\right)^{p/p'} \int_\Omega |f(y)|^p\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\ &= \|K\|^p_{L^p\left(\mu)\otimes L^{p'}(\mu\right)} \|f\|^p_{L^p(\mu)} \end{aligned}$$

as claimed. □

With condition (L) we can now make sense of $\Phi_N f, \Psi_N f$ for every $f \in L^p(\mu)$. Moreover, according to the Banach-Steinhaus Theorem, the following are equivalent:

(i) for every $f \in L^p(\mu)$, $\Phi_N f \to f$ in $L^p(\mu)$;
(ii) there exists a dense set $\mathcal{D} \subset L^p(\mu)$ such that, for every $f \in \mathcal{D}$, $\Phi_N f \to f$ in $L^p(\mu)$ and for every $f \in L^p(\mu)$, $\|\Phi_N f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}$.

The statement hold of course with Φ_N replaced by Ψ_N.

Now, as we assume conditions (D), L^p-convergence of $\Phi_N f \to f$, $\Psi_N f \to f$, is equivalent to the uniform boundedness of $\|\Phi_N\|_{L^p(\mu)\to L^p(\mu)}$, and $\|\Psi_N\|_{L^p(\mu)\to L^p(\mu)}$. But then, under condition (C), the uniform boundedness of $\|\Phi_N\|_{L^p(\mu)\to L^p(\mu)}$ holds if and only if $p_0 < p < p_0'$. But if $\|\Phi_N - \Psi_N\|_{L^p(\mu)\to L^p(\mu)}$ is uniformly bounded for every p, then we also get that $\|\Psi_N\|_{L^p(\mu)\to L^p(\mu)}$ is uniformly bounded if and only if $p_0 < p < p_0'$. We may summarize this discussion in the following proposition:

Proposition 2.4. *With the notation of Section 2.1 and under conditions (L), (R), (B), (C) and (D), the following are equivalent:*

(i) *$\Psi_N f \to f$ for every $f \in L^p(\Omega,\mu)$, with convergence in $L^p(\Omega,\mu)$, if and only if $p_0 < p < p_0'$.*
(ii) *for every $1 < p < \infty$, there exists a constant C such that, for every $N \geq 0$ and every $f \in L^p(\mu)$,*

$$\|\Phi_N f - \Psi_N f\|_{L^p(\mu)} \leq C\|f\|_{L^p(\mu)}.$$

2.3. **Step 2: The behavior of the sequence α_k^n.**

Lemma 2.5. *Let n be an integer, (a_k), (f_k) be two sequences such that $a_1 \neq 0$, $|f_1| \lesssim n^2$ for every k, $|a_k| \leq \dfrac{1}{2}$, $|f_k| \gtrsim k|k-n|$. Let $(\alpha_k)_{k\geq 0}$ be a sequence such that*

– $\displaystyle\sum_{k=0}^{\infty} |\alpha_k|^2 = 1$;
– *$(\alpha_k)_{k\geq 0}$ satisfies a three term recursion formula*

$$f_k\alpha_k = a_k\alpha_{k-1} + a_{k+1}\alpha_{k+1}.$$

Then there exists κ, n_1 depending only on the constants appearing in the above $\lesssim$ and $\gtrsim$ inequalities such that, if $n \geq n_1$,

(i) $|\alpha_0| \lesssim (\kappa n)^{3-n}$,
(ii) *for $k \geq 1$, $|\alpha_k| \lesssim (Cn)^{-|k-n|}$*

(iii) $|\alpha_n|^2 = 1 - \eta$ *with* $0 < \eta \lesssim n^{-2}$.

Proof. First, as $\sum |\alpha_k|^2 = 1$, $|\alpha_k| \leq 1$ for every $k \geq 0$.

We will first prove the estimate for $k \geq 1$ and write $|f_k| \geq \kappa' k|n-k| \geq \kappa n$, $\kappa = \kappa'/4$. As $|a_k| \leq \frac{1}{2}$ for every k, we get $|f_k||\alpha_k| \leq 1$ thus $|\alpha_k| \leq (\kappa n)^{-1}$ if $|k-n| \geq 1$. Assume now that we have proven that, for $J \geq 1$, for every $k \geq 1$,

$$|\alpha_k| \leq (\kappa n)^{-\min(|k-n|,J)}.$$

Then, if $|k-n| \geq J+1$,

$$|f_k||\alpha_k| \leq (\kappa n)^{-J}.$$

As $|f_k| \geq \kappa n$, we obtain

$$|\alpha_k| \leq (\kappa n)^{-(J+1)}$$

as claimed. This induction does not allow to estimate α_0 for which we instead use the induction formula in a rougher way: we assumed that there is a constant $\tilde{C}$ such that $|f_1| \leq \tilde{C} n^2$

$$|a_1||\alpha_0| \leq |f_1||\alpha_1| + \frac{1}{2}|\alpha_2| \leq \tilde{C} n^2 (\kappa n)^{1-n} + \frac{1}{2}(\kappa n)^{2-n} \leq \left(\frac{\tilde{C}}{\kappa^2} + \frac{1}{2\kappa n}\right)(\kappa n)^{3-n}.$$

from a bound of the form $|\alpha_0| \geq \kappa(\kappa n)^{3-n}$ follows.

Finally, if $n > \max(4, 1/2\kappa)$

$$\begin{aligned}
|\alpha_n|^2 &= 1 - \sum_{|k-n|\geq 1} |\alpha_k|^2 \geq 1 - \kappa^2(\kappa n)^{6-2n} - 2\sum_{j\geq 1}(\kappa n)^{-2j} \\
&\geq 1 - \kappa^2(\kappa n)^{-2} - 2(\kappa n)^{-2}\big(1-(\kappa n)^{-2}\big) \\
&\geq 1 - (\kappa^2+4)(\kappa n)^{-2}
\end{aligned}$$

as claimed. □

Let us now state what this lemma implies on (α_k^n) satisfying condition (R).

According to Lemma 2.5, and up to replacing eventually n_0 by $\max(n_0, n_1)$, we may assume that, if $n \geq n_0$,

(i) for every n, $|\alpha_k^n| \lesssim k^{-2}$ (with a constant that depends on n);
(ii) if $n \geq n_0$,
  (a) $|\alpha_0^n| \lesssim n^{-2}$,
  (b) for $k \geq 1$, $|\alpha_k^n| \lesssim (\kappa n)^{-|k-n|}$
  (c) $|\alpha_n^n|^2 = 1 - \eta_n$ with $0 < \eta_n \lesssim n^{-2}$.

Let us show that this implies that (ψ_n) also satisfies condition (L):

Lemma 2.6. *With the notation of Section 2.1 and under conditions* (L), (R), *the sequence* $(\psi_n)_{n\geq 0}$ *also satisfies condition* (L).

Proof of Lemma 2.6. We write $\psi_n = \sum_{k=0}^{\infty} \alpha_k^n \varphi_k$ so that

$$\|\psi_n\|_{L^p(\mu)} \leq \sum_{k=0}^{\infty} |\alpha_k^n| \|\varphi_k\|_{L^p(\mu)} \lesssim \sum_{k=0}^{\infty} (1+k)^{-2+\gamma_p} < +\infty.$$

Further, if $n \geq n_0$,

$$\begin{aligned}
\|\psi_n\|_{L^p(\mu)} &\leq |\alpha_0^n|\|\varphi_0\|_{L^p(\mu)} + \sum_{k=1}^{n-1} |\alpha_k^n|\|\varphi_k\|_{L^p(\mu)} + \|\varphi_n\|_{L^p(\mu)} + \sum_{k=n+1}^{\infty} |\alpha_k^n|\|\varphi_k\|_{L^p(\mu)} \\
&\lesssim \left(n^{-2} + 2\sum_{k=1}^{\infty} (\kappa n)^{-|k-n|} + 1 \right) n^{\gamma_p} \lesssim n^{\gamma_p}
\end{aligned}$$

as claimed. □

2.4. **Step 3: The decomposition of Ψ_N.** In order to prove the theorem, we need to decompose Ψ_N in the basis $(\varphi_n)_{n\geq 0}$.

Recall that $\psi_n = \sum_{k=0}^{\infty} \alpha_k^n \varphi_n$ and that (α_k^n) satisfy (R).

The decomposition of Ψ_N is the following:

$$\begin{aligned}
\Psi_N(x,y) &= \sum_{n=0}^{N} \psi_n(x)\overline{\psi_n(y)} = \sum_{n=0}^{n_0} \psi_n(x)\overline{\psi_n(y)} + \sum_{n=n_0+1}^{N} \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \alpha_k^n \alpha_\ell^n \varphi_k(x)\overline{\varphi_\ell(y)} \\
&= \Phi_N(x,y) + K_1(x,y) - K_2(x,y) - K_3(x,y) + K_4(x,y) + K_5(x,y) + K_6(x,y) \qquad (2.5)
\end{aligned}$$

with

$$K_1(x,y) = \sum_{n=0}^{n_0} \psi_n(x)\overline{\psi_n(y)} \quad , \quad K_2(x,y) = \sum_{n=0}^{n_0} \varphi_n(x)\overline{\varphi_n(y)},$$

$$K_3(x,y) = \sum_{n=n_0+1}^{N} \eta_n \varphi_n(x)\overline{\varphi_n(y)},$$

$$K_4(x,y) = \sum_{n=n_0+1}^{N} \alpha_n^n \overline{\alpha_{n+1}^n} \varphi_n(x)\overline{\varphi_{n+1}(y)} \quad , \quad K_5(x,y) = \sum_{n=n_0+1}^{N} \alpha_{n+1}^n \overline{\alpha_n^n} \varphi_{n+1}(x)\overline{\varphi_n(y)}$$

and $K_6(x,y) = \tilde{K}_6(x,y) + \tilde{K}_6(y,x)$ where

$$\tilde{K}_6(x,y) = \sum_{n=n_0+1}^{N} \sum_{k} \sum_{|\ell-n|\geq 2} \alpha_k^n \alpha_\ell^n \varphi_k(x)\overline{\varphi_\ell(y)}.$$

Let us write $K_j f(x) = \displaystyle\int_\Omega K_j(x,y) f(y)\,\mathrm{d}\mu(y)$ for the corresponding integral operators. We want to bound $\|K_j\|_{L^p(\mu)\to L^p(\mu)}$ independently from N.

According to Lemma 2.3

$$\|K_1\|_{L^p(\mu)\to L^p(\mu)} \leq C_1 := \sum_{n=0}^{n_0} \|\psi_n\|_{L^p(\mu)} \|\psi_n\|_{L^{p'}(\mu)}$$

while

$$\|K_2\|_{L^p(\mu)\to L^p(\mu)} \leq C_2 := \sum_{n=0}^{n_0} \|\varphi_n\|_{L^p(\mu)} \|\varphi_n\|_{L^{p'}(\mu)}$$

and these two quantities are finite according to condition (L) and do not depend on N.

Further, thanks again to condition (L),

$$\|K_3\|_{L^p\to L^p}\leq\sum_{n=n_0+1}^N|\eta_n|\|\varphi_n\|_{L^p(\mu)}\|\varphi_n\|_{L^{p'}(\mu)}\lesssim\sum_{n=1}^\infty\frac{1}{n^{2-\alpha_p}}<+\infty.$$

For $\tilde K_6$ thus K_6 we will use Lemma 2.5 to see that, if $|\ell-n|\geq 2$, $|\alpha_\ell^n|\lesssim(\kappa n)^{-|\ell-n|}$ if $\ell\neq 0$. On the other hand $\|\varphi_\ell\|_{L^{p'}(\mu)}\lesssim k^{\gamma_{p'}}$ with $\gamma_{p'}<1$. Note also that if we denote by $S_j=\displaystyle\sum_{\ell=j}^\infty(\kappa n)^{-\ell}$ then $S_j\lesssim(\kappa n)^{-j}$. We then have to estimate

$$\begin{aligned}\sum_{\ell\geq n+2}|\alpha_\ell^n|\|\varphi_\ell\|_{L^{p'}(\mu)}&\lesssim\sum_{j\geq 2}(n+j)^{\gamma_{p'}}(\kappa n)^{-j}=\sum_{j\geq 2}(n+j)^{\gamma_{p'}}(S_j-S_{j+1})\\&=(n+2)^{\gamma_{p'}}S_2+\sum_{j=3}^\infty\big((n+j)^{\gamma_{p'}}-(n+j-1)^{\gamma_{p'}}\big)S_j\lesssim n^{\gamma_{p'}-2}\end{aligned}$$

and as $|\alpha_0^n|\lesssim n^{-2}$,

$$\sum_{0\leq\ell\leq n-2}|\alpha_\ell^n|\|\varphi_\ell\|_{L^{p'}(\mu)}\lesssim n^{-2}+n^{\gamma_p}\sum_{j\geq 2}(\kappa n)^{-j}\lesssim n^{\gamma_{p'}-2}.$$

Further, $|\alpha_{n\pm1}^n|,|\alpha_n^n|\leq 1$ and $\|\varphi_{n\pm1}\|_{L^p(\mu)}\|\varphi_n\|_{L^p(\mu)}\lesssim n^{\gamma_p}$ we thus also obtain that

$$\sum_{0k}|\alpha_k^n|\|\varphi_\ell\|_{L^p(\mu)}\lesssim\lesssim n^{\gamma_p}.$$

All together, this leads to

$$\begin{aligned}\left\|\tilde K_6\right\|_{L^p(\mu)\to L^p(\mu)}&\leq\sum_{n=n_0}^\infty\sum_k\sum_{|\ell-n|\geq 2}|\alpha_k^n||\alpha_\ell^n|\|\varphi_k\|_{L^p(\mu)}\|\varphi_\ell\|_{L^{p'}(\mu)}\\&\lesssim\sum_{n=n_0}^\infty n^{\gamma_p+\gamma_{p'}-2}=\sum_{n=n_0+1}^\infty n^{-2+\alpha_p}<+\infty\end{aligned}$$

since $\alpha_p<1$. By symmetry, $\|K_6\|_{L^p(\mu)\to L^p(\mu)}\lesssim 1$.

The terms K_4 and K_5 are the most difficult to treat. As they are similar, we will only show L^p-boundedness of the first one. To start, we use (R) to rewrite

$$\begin{aligned}K_4(x,y)&=\sum_{n=n_0+1}^N\alpha_n^n\overline{\alpha_{n+1}^n}\varphi_n(x)\overline{\varphi_{n+1}(y)}\\&=\sum_{n=n_0+1}^N\frac{\overline{a_{n+1}}}{f(n+1,n)}|\alpha_n^n|^2\varphi_n(x)\overline{\varphi_{n+1}(y)}+\sum_{n=n_0+1}^N\frac{\overline{a_{n+2}}}{f(n+1,n)}\alpha_n^n\overline{\alpha_{n+2}^n}\varphi_n(x)\overline{\varphi_{n+1}(y)}\\&=K_4^1(x,y)+K_4^2(x,y).\end{aligned}$$

Now

$$\begin{aligned}\left\|K_4^2\right\|^p_{L^p\left(\mu\right)\otimes L^{p'}(\mu)} &\lesssim \sum_{n=n_0+1}^{N}\frac{|a_{n+2}|}{|f(n+1,n)|}|\alpha_n^n||\alpha_{n+2}^n|\|\varphi_n\|_{L^p(\mu)}\|\varphi_n\|_{L^{p'}(\mu)}\\ &\lesssim \sum_{n=n_0+1}^{\infty} n^{-3+\alpha_p}<+\infty.\end{aligned}$$

since $|a_{n+2}|\lesssim 1$, $|f(n+1,n)|\gtrsim n$, $|\alpha_n^n|\leq 1$, $|\alpha_{n+2}^n|\lesssim n^{-2}$ and Property (L^p).

Next, writing $\tilde\alpha_n=\dfrac{\overline{a_{n+1}}}{f(n+1,n)}|\alpha_n^n|^2$ and using Abel summation, we get

$$K_4^1(x,y)=\sum_{n=n_0}^{N-1}\bigl(\tilde\alpha_n-\tilde\alpha_{n+1}\bigr)\tilde\Phi_n(x,y)+\tilde\alpha_N\tilde\Phi_{N-1}(x,y)$$

Note that $|\tilde\alpha_n|\lesssim n^{-1}$ so that, with (B), $\left\|\tilde\alpha_N\tilde\Phi_{N-1}\right\|_{L^p(\mu)\to L^p(\mu)}\lesssim 1$. Further $|\alpha_{n+1}^{n+1}|^2=|\alpha_n^n|^2+\eta_n-\eta_{n+1}$ thus

$$\tilde\alpha_{n+1}=\frac{\overline{a_{n+2}}}{f(n+2,n+1)}|\alpha_n^n|^2+O(n^{-3})$$

since $|\eta_n-\eta_{n+1}|\lesssim n^{-2}$, $|a_{n+2}|\lesssim 1$, $|f(n+2,n+1)|\gtrsim n$. Thus, using (2.4) we get $|\tilde\alpha_n-\tilde\alpha_{n+1}|\lesssim n^{-2}$. It follows that

$$\left\|\sum_{n=n_0}^{N-1}\bigl(\tilde\alpha_n-\tilde\alpha_{n+1}\bigr)\tilde\Phi_n\right\|_{L^p(\mu)\to L^p(\mu)}\lesssim\sum_{n=n_0}^{N-1}n^{-2}\left\|\tilde\Phi_n\right\|_{L^p(\mu)\to L^p(\mu)}\lesssim\sum_{n=n_0}^{\infty}n^{-2+\beta_p}<+\infty.$$

This shows that K_4^1 is also a bounded operator $L^p(\mu)\to L^p(\mu)$ with bound independent on N. The proof for K_5 being similar, we conclude that each term in (2.5) defines a bounded operator $L^p(\mu)\to L^p(\mu)$ with bound independent on N and the proof of the theorem is complete. □

Remark 2.7. By treating simultaneously the terms K_4 and K_5, it is enough to assume the following slightly weaker condition:

(B') Let $\hat\Phi_N(x,y)=\displaystyle\sum_{n=0}^{N}\varphi_n(x)\overline{\varphi_{n+1}(y)}+\varphi_{n+1}(x)\overline{\varphi_n(y)}$ and write also $\hat\Phi_N$ for the corresponding integral operator. For every $1<p<\infty$, we assume that $\hat\Phi_N$ defines a bounded linear operator on $L^p(\mu)$ and that there exists $\beta_p<1$ such that, for every $f\in L^p(\mu)$

$$\left\|\hat\Phi_N f\right\|_{L^p(\mu)}\lesssim N^{\beta_p}\|f\|_{L^p(\mu)}.$$

3. Preliminaries and technical Lemmas

In this section, we will gather some facts from the literature and some simple technical lemmas that will allow to easily establish the conditions of Theorem 2.1.

3.1. **Condition** (R).

Lemma 3.1. *Let $a,b,c,d,e,\ell\in\mathbb{R}$. Let $(e_{k,n})_{k,n\in\mathbb{N}}$ be a bounded sequence with $|e_{k,n}|\leq e$. Let*

$$f(k,n)=(an+b)(cn+d)-(ak+b)(ck+d)+e_{k,n}.$$

Let (a_n) be a sequence such that $a_n=\ell+\tilde a_n$ with $|\tilde a_n|\lesssim n^{-1}$. Then there exists n_0 such that

(i) *for fixed n, there exists k_n such that, if $k\geq k_n$, $|f(k,n)|\geq \dfrac{ac}{2}k^2$;*

(ii) *if $n\geq n_0$, $|f(k,n)|\geq \dfrac{ac}{2}n|n-k|$;*

(iii) *if $n\geq n_0$, $\left|\dfrac{a_{n+1}}{f(n+1,n)}-\dfrac{a_{n+2}}{f(n+2,n+1)}\right|\lesssim n^{-2}$.*

Proof. Up to replacing f by $-f$ we may assume that $ac>0$. The first part is trivial as, for fixed n, $k^{-2}f(k,n)\to ac$ when $k\to\infty$.

For the second part, the result is trivial for $k=n$ so let us first consider the case $k>n$ and write $k=n+p$, $p\geq 1$. Now

$$\begin{aligned} f(n+p,n) &= -acp^2-p\big(2acn+cb+ad\big)+e_{n+p,n}\\ &= -\frac{ac}{2}pn-p\left(ac(n+p)+\frac{ac}{4}n+cb+ad\right)-\left(\frac{ac}{4}pn-e_{n+p,n}\right). \end{aligned}$$

Now $ac(n+p)\geq 0$, $\dfrac{ac}{4}n+cb+ad\geq 0$ if $n\geq -\frac{4(cb+ad)}{ac}$ and $\frac{ac}{4}pn-e_{n+p,n}\geq \frac{ac}{4}n-e\geq 0$ if $n\geq \frac{4e}{ac}$. It follows that, if n is large enough $f(n+p,n)\leq -\dfrac{ac}{2}pn$.

Let us now turn to the case $0\leq k<n$ and write $k=n-p$ with $1\leq p\leq n$. Then

$$\begin{aligned} f(n-p,n) &= -acp^2+p\big(2acn+cb+ad\big)+e_{n-p,n}\\ &\geq \frac{ac}{2}np+p\left(ac(n-p)+\frac{ac}{4}n+cb+ad\right)+\frac{ac}{4}np+e_{n-p,n}. \end{aligned}$$

Now, $n\geq p$ thus $ac(n-p)\geq 0$, and the two other terms are treated as previously.

For the last assertion, first write

$$\frac{a_{n+1}}{f(n+1,n)}-\frac{a_{n+2}}{f(n+2,n+1)}=\frac{a_{n+1}f(n+2,n+1)-a_{n+2}f(n+1,n)}{f(n+1,n)f(n+2,n+1)}.$$

Next, note that $f(n+1,n)=-2acn+f_n$ with $f_n=-(ac+cb+ad)+e_{n+1,n}$ a bounded sequence, $|f_n|\leq f:=|ac+cb+ad|+e$ while $a_n=\ell+\tilde a_n$ with $|\tilde a_n|\leq Cn^{-1}$. But then

$$\begin{aligned} &a_{n+1}f(n+2,n+1)-a_{n+2}f(n+1,n)\\ &\qquad=(\ell+\tilde a_{n+1})(-2acn-2ac+f_{n+1})-(\ell+\tilde a_{n+2})(-2acn+f_n)\\ &\qquad\qquad=2acn(\tilde a_{n+2}-\tilde a_{n+1})+(\ell+\tilde a_{n+1})(-2ac+f_{n+1})-(\ell+\tilde a_{n+2})f_n \end{aligned}$$

which is bounded by $F:=ac(6C+2|\ell|)+2(|\ell|+C)f$. As for $n\geq f/ac$, $f(n+1,n)\leq -acn$ we obtain

$$\left|\frac{a_{n+1}}{f(n+1,n)}-\frac{a_{n+2}}{f(n+2,n+1)}\right|\leq \frac{F}{(ac)^2}n^{-2}$$

as claimed. □

3.2. **A simple criteria for condition** (D)**.** In the examples we have in mind, condition (D) will be very easy to check. Indeed, it will fall in the scope of the following simple lemma:

Lemma 3.2. *Assume that the following conditions hold:*

(i) $\Omega \subset \mathbb{R}^d$ *is an open set and the set of smooth compactly supported functions* $\mathcal{C}_c^\infty(\Omega)$ *is dense in every* $L^p(\Omega)$, $1 < p < \infty$;

(ii) *there exists a differential operators* L *(resp.* $\tilde{L}$*) such that each* φ_n *(resp.* ψ_n*'s) is an eigenfunctions of* L *(resp.* $\tilde{L}$*);*

(iii) *writing* $L\varphi_n = \lambda_n\varphi_n$ *(resp.* $\tilde{L}\psi_n = \tilde{\lambda}_n\psi_n$*) we further assume that there is an* $\alpha > 0$ *(resp.* $\tilde{\alpha} > 0$*) such that* $\lambda_n \gtrsim n^\alpha$ *(resp.* $\tilde{\lambda}_n \gtrsim n^\alpha$*) when* n *is big enough;*

(iv) (φ_n) *(resp.* ψ_n*) satisfy condition* (L).

Under the above conditions, $\Phi_N f \to f$ *(resp.* $\Psi_N f \to f$*) in* $L^p(\mu)$ *for every* $f \in \mathcal{C}_c^\infty(\Omega)$.

Proof. Indeed, if $f \in \mathcal{C}_c^\infty(\Omega)$ and n is big enough,

$$\langle \varphi_n, f\rangle_{L^2(\mu)} = \frac{1}{\lambda_n}\langle L\varphi_n, f\rangle_{L^2(\mu)} = \frac{1}{\lambda_n}\langle \varphi_n, L^* f\rangle_{L^2(\mu)} = \frac{1}{\lambda_n^k}\Big\langle \varphi_n, (L^*)^k f\Big\rangle_{L^2(\mu)}$$

by induction on k. But then $|\langle \varphi_n, f\rangle_{L^2(\mu)}| \lesssim n^{-k\alpha}\big\|(L^*)^k f\big\|_{L^2(\mu)}$. As $\|\varphi_n\|_{L^p(\mu)} \lesssim n^{\alpha_p}$ it is enough to take k big enough to have $-k\alpha + \alpha_p < -1$ to see that

$$\sum_{n\geq 0} \langle \varphi_n, f\rangle_{L^2(\mu)}\varphi_n$$

converges in $L^p(\mu)$. As the limit of this series in $L^2(\mu)$ is f, so is the limit in $L^p(\mu)$. The proof for ψ_n is the same. □

3.3. **The Hilbert transform on weighted** L^p **spaces.** In this section $1 < p < \infty$.

First, let us recall that $\omega : J \to \mathbb{R}_+$ (J an interval) is a Muckenhaupt A^p weight if

$$[\omega]_{A^p} := \sup_K \left(\frac{1}{|K|}\int_K \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|K|}\int_K \omega(x)^{-\frac{p'}{p}}\,\mathrm{d}x\right) < +\infty$$

where the supremum is taken over all intervals $K \subset J$. The quantity $[\omega]_{A^p}$ is called the A^p-characteristic of ω (or A^p norm, though it is not a norm).

Let us recall that the Hilbert transform is defined as

$$\mathcal{H}f(x) = \frac{1}{\pi}\int_J \frac{f(y)}{x-y}\,\mathrm{d}y$$

where the integral has to be taken in the principal value sense.

Hunt, Muckenhaupt and Wheeden [HMW] proved that the Hilbert transform extends into a bounded linear operator $L^p(J,\omega) \to L^p(J,\omega)$ if and only if ω is an A^p weight and the sharp dependence on the A^p characteristic has been obtained by Petermichl:

Theorem 3.3 (Petermichl)**.** *Let* $1 < p < +\infty$, J *an interval and let* ω *be an* A_p *weight, then*

$$\|\mathcal{H}\|_{L^p(J,\omega)\to L^p(J,\omega)} \lesssim [\omega]_{A^p}^{\max(1,(p-1)^{-1})}. \tag{3.6}$$

Let us now estimate some A^p characteristics that we will need in the sequel, when considering the Hankel prolates:

Lemma 3.4. *Let $1<p<\infty$, $,\alpha\in\mathbb{R}$ and $\mu\geq 1$. Let $\omega_{\alpha,\pm}$ be defined by*

$$\omega_{\alpha,\pm}(x)=x^\alpha\big(|c\sqrt{x}-\mu|+\mu^{\frac{1}{3}}\big)^{\pm\frac{p}{4}}. \tag{3.7}$$

Then $x^\alpha\in A^p[0,1]$ if and only if $x^\alpha\in A_p[1,+\infty]$ if and only if $-1<\alpha<p-1$. Moreover, $\omega_{\alpha,\pm}\in A^p[0,+\infty]$ if $-1+\frac{p}{8}<\alpha<\dfrac{7}{8}p-1$ and in this case

$$[\omega_{\alpha,\pm}]_{A^p}\lesssim\begin{cases}1 & \text{if } \frac{4}{3}<p<4\\ \mu^{3/4} & \text{otherwise}\end{cases}$$

with the implied constant depending on α.

Proof of Lemma 3.4. The first part is well known and left to the reader. Recall that if ω is an A^p weight, then so is $\lambda\omega(\mu x)$ with $[\lambda\omega_j(\mu x)]_{A^p}=[\omega]_{A^p}$.

Next, we have

$$\omega_{\alpha,+}(x)=(c^2\mu^{-2}x)^\alpha c^{-2\alpha}\mu^{2\alpha-p/4}\big(|\sqrt{c^2\mu^{-2}x}-1|+\mu^{-\frac{2}{3}}\big)^{\frac{p}{4}}=\lambda_3\tilde\omega_{\alpha,+}(c^2\mu^{-2}x)$$

with $\lambda_3=c^{-2\alpha}\mu^{2\alpha-p/4}$ and

$$\tilde\omega_{\alpha,+}(x)=x^\alpha\big(|\sqrt{x}-1|+\mu^{-\frac{2}{3}}\big)^{\frac{p}{4}}.$$

So it is enough to estimate $[\tilde\omega_{\alpha,+}]_{A^p}$. Similarly, we may replace $\omega_{\alpha,-}$ by

$$\tilde\omega_{\alpha,-}(x)=x^\alpha\big(|\sqrt{x}-1|+\mu^{-\frac{2}{3}}\big)^{-\frac{p}{4}}.$$

Next, on $[0,1/2]$, $\tilde\omega_{\alpha,\pm}(x)\simeq x^\alpha\in A^p$ since $-1<\alpha<p-1$. Note that constants here are independent on $\mu\geq 1$. On the other hand, on $[3/2,+\infty]$, $\tilde\omega_{\alpha,\pm}(x)\simeq x^{\alpha\pm p/8}\in A^p$ since $-1<\alpha\pm p/8<p-1$. Again, constants here are independent on $\mu\geq 1$.

Finally, on $[1/2,3/2]$,

$$\tilde\omega_{\alpha,\pm}(x)\simeq\big(|\sqrt{x}-1|+\mu^{-\frac{2}{3}}\big)^{\pm\frac{p}{4}}\simeq\omega_\pm(x):=\big(|x-1|+\mu^{-\frac{2}{3}}\big)^{\pm\frac{p}{4}}.$$

We thus want to estimate

$$[\omega_\pm]_{A^p}=\sup_I\left(\frac{1}{|I|}\int_I\big(|x-1|+\mu^{-\frac{2}{3}}\big)^{\pm\frac{p}{4}}\,\mathrm{d}x\right)\left(\frac{1}{|I|}\int_I\big(|x-1|+\mu^{-\frac{2}{3}}\big)^{\mp\frac{p'}{4}}\,\mathrm{d}x\right)^{p/p'}$$

where the sup runs over intervals $I\subset[1/2,3/2]$. Equivalently, we want to estimate

$$[\omega_\pm]_{A^p}\simeq\sup_I\left(\frac{1}{|I|}\int_I\big(|x|+\mu^{-\frac{2}{3}}\big)^{\pm\frac{p}{4}}\,\mathrm{d}x\right)\left(\frac{1}{|I|}\int_I\big(|x|+\mu^{-\frac{2}{3}}\big)^{\mp\frac{p'}{4}}\,\mathrm{d}x\right)^{p/p'}$$

where the sup runs over intervals $I \subset [-1/2, 1/2]$. It is enough to consider $I = [0, a]$ then, when $p \neq 4/3, 4$, we are looking at

$$[\omega_{\pm,p}]_{A^p} \simeq \sup_{a\in[0,1/2]} \left(\frac{\left(a+\mu^{-\frac{2}{3}}\right)^{1\pm\frac{p}{4}} - \left(\mu^{-\frac{2}{3}}\right)^{1\pm\frac{p}{4}}}{a}\right)\left(\frac{\left(a+\mu^{-\frac{2}{3}}\right)^{1\mp\frac{p'}{4}} - \left(\mu^{-\frac{2}{3}}\right)^{1\mp\frac{p'}{4}}}{a}\right)^{p/p'}$$

$$= \sup_{a\in[0,1/2]} \left(\frac{\left(1+a\mu^{\frac{2}{3}}\right)^{1\pm\frac{p}{4}} - 1}{a\mu^{\frac{2}{3}}}\right)\left(\frac{\left(1+a\mu^{\frac{2}{3}}\right)^{1\mp\frac{p'}{4}} - 1}{a\mu^{\frac{2}{3}}}\right)^{p/p'}$$

$$= \sup_{t\in[0,\mu^{\frac{2}{3}}/2]} \left(\frac{\left(1+t\right)^{1\pm\frac{p}{4}} - 1}{t}\right)\left(\frac{\left(1+t\right)^{1\mp\frac{p'}{4}} - 1}{t}\right)^{p/p'} := \sup_{t\in[0,\mu^{\frac{2}{3}}/2]} \varphi_\pm(t).$$

Note that $\varphi_\pm$ extends continuously at 0 and that, when $t \to +\infty$. Moreover,
— $\varphi_\pm(t) = O(1)$ for $p, q < 4$ that is $4/3 < p < 4$
— for $p > 4$, $\varphi_- = O\big(t^{\frac{p}{4}-1}\big)$, $\varphi_+ = O(1)$,
— for $p < 4$, $\varphi_+ = O\big(t^{\frac{p'}{4}-1}\big)$, $\varphi_- = O(1)$.

The computation has to be slightly modified for $p = 4/3$ to obtain $\varphi_+ = O(\log t)$ and $\varphi_- = O(1)$ while for $p = 4$ one gets $\varphi_- = O(\log t)$, $\varphi_+ = O(1)$. The result follows. $\square$

4. Application to weighted prolates

4.1. **Weighted prolates.** In this section, we will fix real numbers $c > 0$ and $\alpha > 0$. We denote by $I = [-1, 1]$ that will be endowed with the measure $\omega_\alpha(x)\,\mathrm{d}x$ with $\omega_\alpha(x) = (1-x^2)^\alpha$. We will simply write ω_α for the measure $\omega_\alpha(x)\,\mathrm{d}x$. The aim of this section is to consider the set of Weighted Prolate Spheroidal Wave Functions (WPSWFs) introduced in [KS1, KS2, WZ] and to study the $L^p(I, \omega_\alpha)$ convergence of the associated series.

More precisely, the WPSWFs are the eigenfunctions of the weighted finite Fourier transform operator $\mathcal{F}_c^{(\alpha)}$ defined by

$$\mathcal{F}_c^{(\alpha)} f(x) = \int_{-1}^{1} e^{icxy} f(y)\,\omega_\alpha(y)\,\mathrm{d}y. \tag{4.8}$$

It is well known, see [KS1, WZ] that the operator

$$\mathcal{Q}_c^{(\alpha)} = \frac{c}{2\pi}\mathcal{F}_c^{(\alpha)^*} \circ \mathcal{F}_c^{(\alpha)}$$

is defined on $L^2(I, \omega_\alpha)$ by

$$\mathcal{Q}_c^{(\alpha)} g(x) = \int_{-1}^{1} \frac{c}{2\pi}\mathcal{K}_\alpha(c(x-y))g(y)\omega_\alpha(y)\,\mathrm{d}y \tag{4.9}$$

with

$$\mathcal{K}_\alpha(x) = \sqrt{\pi}2^{\alpha+1/2}\Gamma(\alpha+1)\frac{J_{\alpha+1/2}(x)}{x^{\alpha+1/2}}$$

and $J_\alpha(\cdot)$ is the Bessel function of the first kind and order α.

It has been shown in [KS1, WZ] that the last two integral operators commute with the following Sturm-Liouville operator $\mathcal{L}_c^{(\alpha)}$ defined by

$$\mathcal{L}_c^{(\alpha)}(f)(x) = -\frac{d}{dx}\left[\omega_\alpha(x)(1-x^2)f'(x)\right] + c^2x^2\omega_\alpha(x)f(x). \tag{4.10}$$

Also, note that the $(n+1)-$th eigenvalue $\chi_n(c)$ of $\mathcal{L}_c^{(\alpha)}$ satisfies the following classical inequalities,

$$n(n+2\alpha+1) \leq \chi_n(c) \leq n(n+2\alpha+1) + c^2, \quad \forall n \geq 0. \tag{4.11}$$

We will denote by $(\Psi_{n,c}^{(\alpha)})_{n\geq 0}$ the set of common eigenfunctions of $\mathcal{F}_c^{(\alpha)}$, $\mathcal{Q}_c^{(\alpha)}$ and $\mathcal{L}_c^{(\alpha)}$ and call them *Weighted Prolate Spheroidal Wave Functions (WPSWFs).* Then $\{\psi_{n,c}^{(\alpha)}, n \geq 0\}$ is an orthogonal basis of $L^2(I,\omega_\alpha)$.

Our aim will be to apply Theorem 2.1 with the following setting: $\Omega = I$, $\mu = \omega_\alpha$, $\psi_n = \Psi_{n,c}^{(\alpha)}$. The first task will be to define the basis φ_n and then to show that it satisfies each of the desired properties.

4.2. **Some facts about Jacobi polynomials.**

4.2.1. *Jacobi polynomials.* In this section, we gather results on Jacobi polynomials[1] that will be used later. The Jacobi polynomials are defined as being the orthonormal family of polynomials with respect to the scalar product associated to $\|\cdot\|_{L^2(I,\omega_\alpha)}$ with leading coefficient being non-negative.

Alternatively, we define the (non-normalized) Jacobi polynomials $P_k^{(\alpha)}$ through the induction formula (see for example [AAR])

$$P_{k+1}^{(\alpha)}(x) = A_k x P_k^{(\alpha)}(x) - C_k P_{k-1}^{(\alpha)}(x), \quad x \in [-1,1], \tag{4.12}$$

where $P_0^{(\alpha)}(x) = 1, \quad P_1^{(\alpha)}(x) = (\alpha+1)x + \alpha$ and

$$A_k = \frac{(2k+2\alpha+1)(k+\alpha+1)}{(k+1)(k+2\alpha+1)} = 2 - \frac{1}{k} + O(k^{-2})\ ,\ C_k = \frac{(k+\alpha)(k+\alpha+1)}{(k+1)(k+2\alpha+1)} = 1 - \frac{1}{k} + O(k^{-2}).$$

We consider the normalized Jacobi polynomials $\widetilde{P}_k^{(\alpha)} = \left\|P_k^{(\alpha)}\right\|_{L^2(I,\omega_\alpha)}^{-1} P_k^{(\alpha)}$ which form an orthonormal basis of $L^2(I,\omega_\alpha)$. A cumbersome computation shows that

$$\widetilde{P}_k^{(\alpha)}(x) = \frac{1}{\sqrt{h_k^{(\alpha)}}} P_k^{(\alpha,\beta)}(x), \quad h_k^{(\alpha)} = \frac{2^{2\alpha+1}\Gamma(k+\alpha+1)^2}{k!(2k+2\alpha+1)\Gamma(k+2\alpha+1)}.$$

The normalized Jacobi polynomials satisfy the recursion formula

$$\widetilde{P}_{k+1}^{(\alpha)}(x) = \tilde{A}_k x \widetilde{P}_k^{(\alpha)}(x) - \tilde{C}_k \widetilde{P}_{k-1}^{(\alpha)}(x), \tag{4.13}$$

where

$$\tilde{A}_k = \sqrt{\frac{h_k^{(\alpha)}}{h_{k+1}^{(\alpha)}}} A_k = 2 + O(k^{-2}) \quad , \quad \tilde{C}_k = \sqrt{\frac{h_{k-1}^{(\alpha)}}{h_{k+1}^{(\alpha)}}} C_k = 1 - \frac{1}{2k} + O(k^{-2}) \tag{4.14}$$

[1]We only use a particular subfamily of Jacobi polynomials and may as well call them ultra-spherical or Gegenbauer polynomials.

since

$$\frac{h_k^{(\alpha)}}{h_{k+1}^{(\alpha)}} = \frac{(k+1)(2k+2\alpha+3)(k+2\alpha+1)}{(2k+2\alpha+1)(k+\alpha+1)^2} = 1 + \frac{1}{k} + O(k^{-2}).$$

Further, it has been shown that

$$|\widetilde{P}_n^{(\alpha)}(x)| \lesssim w_{n,\alpha}(x) := (\sqrt{1-x}+n^{-1})^{-\alpha-1/2}(\sqrt{1+x}+n^{-1})^{-\alpha-1/2} \tag{4.15}$$

uniformly over $(-1,1)$ where the constant involved is independent of n (*see e.g.* [Sz, Chapter 4]).

Moreover, let $p_0 = 2 - \frac{1}{\alpha+3/2}$ so that $p_0' = 2 + \dfrac{1}{\alpha+1/2}$ then, for $1 < p < \infty$, the L^p-norm of Jacobi polynomials is given by Aptekarev, Buyarov and Degeza [ABD] (*see also* [ADMF]):

$$\|\widetilde{P}_n^{(\alpha)}\|_{L^p(I,\omega_\alpha)} = \begin{cases} C(\alpha,p) + \circ(1) & \text{if } 1 < p < p_0' \\ C(\alpha,p)\log(n)(1+\circ(1)) & \text{when } p = p_0' \\ n^{(\alpha+1/2)(p-p_0')} & \text{when } p > p_0' \end{cases} \tag{4.16}$$

with $C(\alpha,p)$ is a generic constant depending only on α and p. Note that

$$L_n(\alpha) := \|\widetilde{P}_n^{(\alpha)}\|_{L^p(I,\omega_\alpha)}\|\widetilde{P}_n^{(\alpha)}\|_{L^{p'}(I,\omega_\alpha)} \approx \begin{cases} n^{(\alpha+1/2)(p'-p_0')} & \text{when } 1 < p < p_0 \\ \log n & \text{when } p = p_0 \text{ or } p = p_0' \\ 1 & \text{when } p_0 < p < p_0' \\ n^{(\alpha+1/2)(p-p_0')} & \text{when } p > p_0'. \end{cases} \tag{4.17}$$

In particular, $L_n(\alpha) = O(n^{\alpha_p})$ with $\alpha_p = 0$ if $p \in (p_0, p_0')$ and $\alpha_p < 1$ when $p \in (p_1, p_1')$ with $p_1' = p_0' + \dfrac{1}{\alpha+1/2} = 2 + \dfrac{2}{\alpha+1/2}$. It follows that Condition (L) of Theorem 2.1 is satisfied.

Further, the Jacobi polynomials are eigenfunctions of the differential operator

$$Lf := (1-x^2)f'' - (2\alpha+1)xf'$$

with eigenvalue $\lambda_n = -n(n+2\alpha+1)$. It follows from Lemma 3.2 that Condition (D) of Theorem 2.1 is also satisfied.

4.2.2. *The Projection on the span of Jacobi polynomials.* Let us now introduce

$$C_N^{(\alpha)}(x,y) = \sum_{k=0}^{N} \widetilde{P}_k^{(\alpha)}(x)\widetilde{P}_k^{(\alpha)}(y)$$

and, according to the Christofel Darboux Formula,

$$C_N^{(\alpha)}(x,y) = \frac{\beta_N}{\beta_{N+1}} \frac{\widetilde{P}_{N+1}^{(\alpha)}(x)\widetilde{P}_N^{(\alpha)}(y) - \widetilde{P}_{N+1}^{(\alpha)}(y)\widetilde{P}_N^{(\alpha)}(x)}{x-y}.$$

Pollard [Po2] proved that $C_N^{(\alpha)}$ defines a bounded operator $C_N^{(\alpha)} : L^p(I,\omega_\alpha) \to L^p(I,\omega_\alpha)$ and that the operators $C_N^{(\alpha)}$ are uniformly bounded in the range $p_0 < p < p_0'$. Further, he proved that the series $C_N^{(\alpha)}f$ may diverge if $p \notin [p_0, p_0']$ but did not provide a bound for $C_N^{(\alpha)}$. The divergence at the end points was proved later by Newman and Rudin [NR]. The key point in Pollard's proof is the following identity

$$\begin{aligned} C_N^{(\alpha)}f(x) &= U_n\widetilde{P}_{n+1}^{(\alpha)}(x)\int_{-1}^1 \frac{\widetilde{Q}_n^{(\alpha)}(y)f(y)\omega_\alpha(y)}{x-y}\,\mathrm{d}y \\ &\quad +V_n\widetilde{Q}_n^{(\alpha)}(x)\int_{-1}^1 \frac{\widetilde{P}_{n+1}^{(\alpha)}(y)f(y)\omega_\alpha(y)}{x-y}\,\mathrm{d}y \\ &\quad +W_n\left\langle f,\widetilde{P}_{n+1}^{(\alpha)}\right\rangle_{L^2(I,\omega_\alpha)}\widetilde{P}_{n+1}^{(\alpha)}(x) \\ &= C_N^{(\alpha,1)}f(x)+C_N^{(\alpha,2)}f(x)+C_N^{(\alpha,3)}f(x) \end{aligned}$$

where $U_n,V_n,W_n\to\frac{1}{2}$ and $\widetilde{Q}_n^{(\alpha)}$ is an other family of orthogonal polynomials.

Hölder's inequality and Lemma 2.3 show that $\left\|C_N^{(\alpha,3)}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)}\lesssim N^{\alpha_p}$ while Pollard showed that $\left\|C_N^{(\alpha,j)}f\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)}\lesssim 1$ for $j=1,2$.

Let us summarize the results from this section

Lemma 4.1. *Let $1<p<\infty$ and $\alpha>-1/2$, $\varepsilon>0$. Let $I=(-1,1)$, $\omega_\alpha(x)=(1-x^2)^\alpha$ and $\tilde{P}_n^{(\alpha)}$ be the Jacobi polynomials , i.e. the orthonormal family of polynomials in $L^2(I,\omega_\alpha)$ defined above. Let $C_N^{(\alpha)}$ be the orthogonal projection on the span of $\tilde{P}_0^{(\alpha)},\dots,\tilde{P}_N^{(\alpha)}$.*

Let $p_0=2-\frac{1}{\alpha+3/2}$ so that $p_0'=2+\dfrac{1}{\alpha+1/2}$. Define

$$\alpha_p=\begin{cases}(\alpha+1/2)(p'-p_0') & \text{when } 1<p<p_0\\ \varepsilon & \text{when } p=p_0 \text{ or } p_0'\\ 0 & \text{when } p\in(p_0,p_0')\\ (\alpha+1/2)(p-p_0') & \text{when } p>p_0'\end{cases}$$

so that $\alpha_p<1$ when $p\in(p_1,p_1')$ with $p_1'=p_0'+\dfrac{1}{\alpha+1/2}=2+\dfrac{2}{\alpha+1/2}$. Then

— Aptekarev, Buyarov and Degeza [ABD] *we have*

$$\|\widetilde{P}_n^{(\alpha)}\|_{L^p(I,\omega_\alpha)}\|\widetilde{P}_n^{(\alpha)}\|_{L^{p'}(I,\omega_\alpha)}\lesssim n^{\alpha_p}; \tag{4.18}$$

— Pollard [Po2] *the operators $C_N^{(\alpha)}$ extend to bounded operators $L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)$ with*

$$\left\|C_N^{(\alpha)}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)}\lesssim N^{\alpha_p}.$$

4.3. **Condition** (R)**.** The aim of this section is to establish condition (R) of Theorem 2.1.

The series expansion of the WPSWFs in the basis of Jacobi polynomials $(\widetilde{P}_n^{(\alpha)})$ which can be written in the form

$$\Psi_{n,c}^{(\alpha)}=\sum_{k\geq 0}\beta_k^n\widetilde{P}_k^{(\alpha)} \tag{4.19}$$

where $\beta_k^n = \left\langle \Psi_{n,c}^{(\alpha)}, \widetilde{P}_k^{(\alpha)} \right\rangle_{L^2(I,\omega_\alpha)}$. By replacing the expression (4.19) in the differential equation (4.10), one gets the following recursion formula satisfied by the β_k^n for $k \geq 2$

$$f(k,n)\beta_k^{(n)} = a_k^{(\alpha)}\beta_{k-2}^{(n)} + a_{k+2}^{(\alpha)}\beta_{k+2}^{(n)}, \tag{4.20}$$

where

$$\begin{aligned} f(k,n) &= \frac{\chi_n(c) - \left(k(k+2\alpha+1) + c^2 b_k^{(\alpha)}\right)}{c^2} \\ a_k^{(\alpha)} &= \frac{\sqrt{k(k-1)(k+2\alpha)(k+2\alpha-1)}}{(2k+2\alpha-1)\sqrt{(2k+2\alpha+1)(2k+2\alpha-3)}} \\ b_k^{(\alpha)} &= \frac{2k(k+2\alpha+1)+2\alpha-1}{(2k+2\alpha+3)(2k+2\alpha-1)}. \end{aligned} \tag{4.21}$$

This is not exactly of the desired form. To overcome this problem, first note that $\Psi_{n,c}^{(\alpha)}$ and $\widetilde{P}_n^{(\alpha)}$ have same parity as n, so that $\beta_k^{(n)} = 0$ if k and n have opposite parity. Next, we decompose

$$L^p(I,\omega_\alpha) = L_e^p(I,\omega_\alpha) \oplus L_o^p(I,\omega_\alpha)$$

where $L_e^p(I,\omega_\alpha)$, resp. $L_o^p(I,\omega_\alpha)$, is the set of even, resp. odd, functions in $L^p(I,\omega_\alpha)$.

Our aim is then to characterize for which p, for every $f \in L_e^p(I,\omega_\alpha)$ — resp. $f \in L_o^p(I,\omega_\alpha)$ — $\sum_{n\geq 0} \left\langle f, \Psi_{2n,c}^{(\alpha)} \right\rangle \Psi_{n,c}^{(\alpha)}$ — resp. $\sum_{n\geq 0} \left\langle f, \Psi_{2n+1,c}^{(\alpha)} \right\rangle \Psi_{n,c}^{(\alpha)}$ — converges to f in $L^p(I,\omega_\alpha)$. This can be done by applying Theorem 2.1. To do so, we will now establish condition (R).

First note that $a_k^{(\alpha)} \to 1/4$ and that we may write

$$a_k^{(\alpha)} = \frac{\sqrt{(1-k^{-1})(1+2\alpha k^{-1})\big(1+(2\alpha-1)k^{-1}\big)}}{\big(2+(2\alpha-1)k^{-1}\big)\sqrt{\big(2+(2\alpha+1)k^{-1}\big)\big(2+(2\alpha-3)k^{-1}\big)}}$$

from which it is obvious that $a_k^{(\alpha)} = 1/4 + O(k^{-1})$. As $b_k^{(\alpha)}$ is clearly bounded, all conditions of Lemma 3.1 are satisfied and $f(k,n)$ satisfies all requirements of condition (R).

It remains to establish the following:

Lemma 4.2. *For every $\alpha > -1/2$ and every $k \geq 2$, $|a_k^{(\alpha)}| \leq 1/2$.*

Proof. First

$$a_2^{(\alpha)} = \frac{2\sqrt{1+\alpha}}{(3+2\alpha)\sqrt{5+2\alpha}}$$

which is maximal for $\alpha = \sqrt{2} - 2 < -1/2$ and the maximal value is $\sqrt{\dfrac{64\sqrt{2}-52}{343}} \sim 0.335 < 1/2$.

Next, for $k \geq 3$ and $-1/2 < \alpha \leq 1/2$, we bound

$$a_k^{(\alpha)} \leq \frac{1}{4}\sqrt{\frac{k(k+1)}{(k-1)(k-2)}} \leq \frac{\sqrt{3}}{4} < \frac{1}{2}.$$

Finally, $2k+2\alpha+1 \geq k+2\alpha$, for $\alpha > 0$, $(2k+2\alpha-1) \geq 2\sqrt{k(k-1)}$, and $2k+2\alpha-3 \geq k+2\alpha-1$ when $k \geq 2$ thus

$$a_k^{(\alpha)} = \frac{\sqrt{k(k-1)(k+2\alpha)(k+2\alpha-1)}}{(2k+2\alpha-1)\sqrt{(2k+2\alpha+1)(2k+2\alpha-3)}} < \frac{1}{2}$$

as announced. $\square$

4.4. **Condition** (B')**.** We will now establish Condition (B') in Theorem 2.1. As we consider separately $L^p_e(I,\omega_\alpha)$ and $L^p_o(I,\omega_\alpha)$, we actually have to estimate the $L^p(I,\omega_\alpha) \to L^p(I,\omega_\alpha)$ norm of the operator with kernel

$$\Phi_N(x,y) = \sum_{n=0}^{N} \big(\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n+2}^{(\alpha)}(y) + \widetilde{P}_{n+2}^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)\big)$$

with $\widetilde{P}_{n+2}^{(\alpha)}$ instead of $\widetilde{P}_{n+1}^{(\alpha)}$. We will also write Φ_N for the associated operator on $L^p(I,\omega_\alpha)$. Note that the bound (4.18) together with Lemma 2.3 leads to

$$\|\Phi_N\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim N^{1+\alpha_p}$$

which is not good enough for our needs.

Using the recursion formula (4.13) twice, we get for $n \geq 2$,

$$\begin{aligned}
\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n+2}^{(\alpha)}(y) &= \widetilde{P}_n^{(\alpha)}(x)\big(\tilde{A}_{n+1}y\widetilde{P}_{n+1}^{(\alpha)}(y) - \tilde{C}_{n+1}\widetilde{P}_n^{(\alpha)}(y)\big)\\
&= y\tilde{A}_{n+1}\widetilde{P}_n^{(\alpha)}(x)\big(y\tilde{A}_n\widetilde{P}_n^{(\alpha)}(y) - \tilde{C}_n\widetilde{P}_{n-1}^{(\alpha)}(y)\big) - \tilde{C}_{n+1}\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)\\
&= y^2\tilde{A}_{n+1}\tilde{A}_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y) - y\tilde{A}_{n+1}\tilde{C}_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n-1}^{(\alpha)}(y) - \tilde{C}_{n+1}\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y).
\end{aligned}$$

Next note that

$$y\widetilde{P}_{n-1}^{(\alpha)}(y) = \frac{1}{\tilde{A}_{n-1}}\widetilde{P}_n^{(\alpha)}(y) + \frac{\tilde{C}_{n-1}}{\tilde{A}_{n-1}}\widetilde{P}_{n-2}^{(\alpha)}(y)$$

so that

$$\begin{aligned}
\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n+2}^{(\alpha)}(y) = y^2\tilde{A}_{n+1}\tilde{A}_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y) - \left(\tilde{C}_{n+1} + \frac{\tilde{A}_{n+1}\tilde{C}_n}{\tilde{A}_{n-1}}\right)\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)\\
- \frac{\tilde{A}_{n+1}\tilde{C}_n\tilde{C}_{n-1}}{\tilde{A}_{n-1}}\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n-2}^{(\alpha)}(y).
\end{aligned}$$

Let us define

$$\begin{aligned}
\kappa_n &= -\Big(\tilde{C}_{n+1} + \frac{\tilde{A}_{n+1}\tilde{C}_n}{\tilde{A}_{n-1}}\Big) = -1 + \frac{1}{n} + O(n^{-2})\\
\tilde{\kappa}_n &= 1 - \frac{\tilde{A}_{n+1}\tilde{C}_n\tilde{C}_{n-1}}{\tilde{A}_{n-1}} = \frac{1}{n} + O(n^{-2}).
\end{aligned}$$

Then

$$\begin{aligned}
\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n+2}^{(\alpha)}(y) + \widetilde{P}_n^{(\alpha)}(y)\widetilde{P}_{n+2}^{(\alpha)}(x) &= y^2\tilde{A}_{n+1}\tilde{A}_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y) + \kappa_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)\\
&\quad + \tilde{\kappa}_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n-2}^{(\alpha)}(y).
\end{aligned}$$

Summing over n, we conclude that

$$\begin{aligned}\Phi_N(x,y) &= \widetilde{P}_0^{(\alpha)}(x)\widetilde{P}_2^{(\alpha)}(y)+\sum_{n=2}^{N}\big(\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n+2}^{(\alpha)}(y)+\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n-2}^{(\alpha)}(y)\big)-\widetilde{P}_N^{(\alpha)}(x)\widetilde{P}_{N+2}^{(\alpha)}(y)\\ &= \widetilde{P}_0^{(\alpha)}(x)\widetilde{P}_2^{(\alpha)}(y)-\widetilde{P}_N^{(\alpha)}(x)\widetilde{P}_{N+2}^{(\alpha)}(y)+y^2\sum_{n=2}^{N}\tilde{A}_{n+1}\tilde{A}_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)\\ &\quad+\sum_{n=2}^{N}\kappa_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)+\sum_{n=2}^{N}\tilde{\kappa}_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n-2}^{(\alpha)}(y).\end{aligned}$$

Further, exchanging the roles of x and y and summing, we obtain $2\Phi_N(x,y)=\Phi_N^1(x,y)+\cdots+\Phi_N^6(x,y)$ where

$$\begin{aligned}\Phi_N^1(x,y) &= \widetilde{P}_0^{(\alpha)}(x)\widetilde{P}_2^{(\alpha)}(y)+\widetilde{P}_2^{(\alpha)}(x)\widetilde{P}_0^{(\alpha)}(y)\\ &\quad-(4x^2+4y^2-2)\big(\widetilde{P}_0^{(\alpha)}(x)\widetilde{P}_0^{(\alpha)}(y)+\widetilde{P}_1^{(\alpha)}(x)\widetilde{P}_1^{(\alpha)}(y)\big)\\ \Phi_N^2(x,y) &= -\widetilde{P}_N^{(\alpha)}(x)\widetilde{P}_{N+2}^{(\alpha)}(y)-\widetilde{P}_{N+2}^{(\alpha)}(x)\widetilde{P}_N^{(\alpha)}(y)\\ \Phi_N^3(x,y) &= (x^2+y^2)\sum_{n=2}^{N}\tilde{A}_{n+1}\tilde{A}_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)\\ \Phi_N^4(x,y) &= 2\sum_{n=2}^{N}\kappa_n\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)\\ \Phi_N^5(x,y) &= \sum_{n=2}^{N}\tilde{\kappa}_n\big(\widetilde{P}_n^{(\alpha)}(x)\widetilde{P}_{n-2}^{(\alpha)}(y)+\widetilde{P}_{n-2}^{(\alpha)}(x)\widetilde{P}_n^{(\alpha)}(y)\big).\end{aligned}$$

We also write Φ_N^j for the corresponding integral operators and will now estimate their norm as operators $L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)$.

Using the bound (4.18) together with Lemma 2.3 we get

$$\big\|\Phi_N^1\big\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)}\lesssim 1$$

and

$$\big\|\Phi_N^2\big\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)}\lesssim N^{\alpha_p}.$$

Using Abel summation, we can write

$$\begin{aligned}\Phi_N^3(x,y) &= -\tilde{A}_3\tilde{A}_2(x^2+y^2)C_1^{(\alpha)}(x,y)+(x^2+y^2)\sum_{n=2}^{N}\tilde{A}_{n+1}(\tilde{A}_n-\tilde{A}_{n+2})C_n^{(\alpha)}(x,y)\\ &\quad+\tilde{A}_{N+1}\tilde{A}_N(x^2+y^2)C_N^{(\alpha)}(x,y)\\ &= \Phi_N^{3,1}(x,y)+\Phi_N^{3,2}(x,y)+\Phi_N^{3,3}(x,y).\end{aligned}$$

Of course

$$\Big\|\Phi_N^{3,1}\Big\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)}\lesssim 1$$

while Lemma 4.1 shows that

$$\left\|\Phi_N^{3,2}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim \sum_{n=2}^{N}\frac{1}{n^2}n^{\alpha_p} \lesssim N^{\alpha_p-1}$$

since $|\tilde{A}_{n+1}(\tilde{A}_n-\tilde{A}_{n+2})| \lesssim n^{-2}$ and

$$\left\|\Phi_N^{3,3}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim N^{\alpha_p}.$$

Using Abel summation again, we write

$$\begin{aligned}\Phi_N^4(x,y) &= -2\kappa_2 C_1^{(\alpha)}(x,y)+2\sum_{n=2}^{N}(\kappa_n-\kappa_{n+1})C_n^{(\alpha)}(x,y)+2\kappa_N y^2 C_N^{(\alpha)}(x,y)\\ &= \Phi_N^{4,1}(x,y)+\Phi_N^{4,2}(x,y)+\Phi_N^{4,3}(x,y).\end{aligned}$$

Again

$$\left\|\Phi_N^{4,1}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim 1$$

while Lemma 4.1 shows that

$$\left\|\Phi_N^{4,2}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim \sum_{n=2}^{N}\frac{1}{n^2}n^{\alpha_p} \lesssim N^{\alpha_p-1}$$

since $|\kappa_n-\kappa_{n+1}| \lesssim n^{-2}$ and

$$\left\|\Phi_N^{4,3}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim N^{\alpha_p}.$$

A last use of Abel summation leads to

$$\begin{aligned}\Phi_N^5(x,y) &= -\tilde{\kappa}_2\Phi_1^{(\alpha)}(x,y)+\sum_{n=2}^{N}(\tilde{\kappa}_n-\tilde{\kappa}_{n+1})\Phi_n^{(\alpha)}(x,y)+\tilde{\kappa}_N\Phi_N^{(\alpha)}(x,y)\\ &= \Phi_N^{5,1}(x,y)+\Phi_N^{5,2}(x,y)+\Phi_N^{5,3}(x,y).\end{aligned}$$

Of course

$$\left\|\Phi_N^{5,1}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim 1.$$

For the two other terms, we will use the fact that $\|\Phi_N\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim N^{1+\alpha_p}$ and that $\tilde{\kappa}_n = n^{-1}+O(n^{-2})$, in particular $|\tilde{\kappa}_n-\tilde{\kappa}_{n+1}| \lesssim n^{-2}$. It follows that

$$\left\|\Phi_N^{5,2}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim \sum_{n=2}^{N} n^{-2}n^{1+\alpha_p} \lesssim N^{\alpha_p}$$

and

$$\left\|\Phi_N^{5,3}\right\|_{L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)} \lesssim N^{-1}N^{1+\alpha_p} \lesssim N^{\alpha_p}.$$

Summing all terms, Condition (B) of Theorem 2.1 is satisfied.

4.5. **Conclusion.** It remains to conclude, all conditions of Theorem 2.1 are satisfied. Therefore, the Weighted prolate spheroidal series converges in $L^p(I,\omega_\alpha)$ if and only if the Jacobi series converge. The later ones converge in $L^p(I,\omega_\alpha)$ if and only if $p\in(p_0,p_0')$. We have thus proved the following:

Theorem 4.3. *Let $\alpha>-1/2$ and $c>0$, $N\geq 0$. Let $p_0=2-\frac{1}{\alpha+3/2}$ so that $p_0'=2+\dfrac{1}{\alpha+1/2}$.*

Let $(\psi_{n,c}^{(\alpha)})_{n\geq 0}$ be the family of weighted prolate spheroidal wave functions. For a smooth function f on $I=(-1,1)$, define

$$\Psi_N^{(\alpha)}f=\sum_{n=0}^{N}\left\langle f,\psi_{n,c}^{(\alpha)}\right\rangle_{L^2(I,\omega_\alpha)}\psi_{n,c}^{(\alpha)}.$$

Then, for every $p\in(1,\infty)$, $\Psi_N^{(\alpha)}$ extends to a bounded operator $L^p(I,\omega_\alpha)\to L^p(I,\omega_\alpha)$. Further

$$\Psi_N^{(\alpha)}f\to f\qquad \text{in } L^p(I,\omega_\alpha)$$

for every $f\in L^p(I,\omega_\alpha)$ if and only if $p\in(p_0,p_0')$.

5. Application to circular prolate spheroidal wave functions

For two real numbers $c>0$ and $\alpha>-\frac{1}{2}$, the family of the circular prolate spheroidal wave functions (CPSWFs), introduced by D. Slepian [Sl2] and denoted by $\psi_{n,c}^{(\alpha)}$, are the eigenfunctions of the finite Hankel transform $\mathcal{H}_c^\alpha$, the operator on $L^2[0,1]$ with kernel given by $\mathcal{H}_c^\alpha(x,y)=\sqrt{cxy}J_\alpha(cxy)$. On other words

$$\mathcal{H}_c^\alpha f(x)=\int_0^1\sqrt{cxy}J_\alpha(cxy)f(y)\,\mathrm{d}y.$$

We denote by $\mu_{n,\alpha}(c)$ the family of the eigenvalues of the operator $\mathcal{H}_c^\alpha$, that is $\mathcal{H}_c^\alpha\psi_{n,c}^{(\alpha)}=\mu_{n,\alpha}(c)\psi_{n,c}^{(\alpha)}$. The functions $\psi_{n,c}^{(\alpha)}$ satisfy the following orthogonality relations:

$$\int_0^1\psi_{n,c}^\alpha(x)\psi_{m,c}^\alpha(x)\,\mathrm{d}x=\delta_{n,m}\quad\text{and}\quad\int_0^{+\infty}\psi_{n,c}^\alpha(x)\psi_{m,c}^\alpha(x)\,\mathrm{d}x=\frac{\delta_{n,m}}{c\mu_{n,\alpha}^2(c)}$$

and the $\psi_{n,c}^{(\alpha)}$'s constitute a complete orthonormal system in $L^2[0,1]$.

The $\psi_{n,c}^{(\alpha)}$'s are also related to the Hankel operator $\mathcal{H}^\alpha$, the integral operator on $L^2[0,+\infty[$ with kernel given by $\mathcal{H}^\alpha(x,y)=\sqrt{xy}J_\alpha(xy)$. More precisely,

$$\mathcal{H}^\alpha(\psi_{n,c}^\alpha)(x)=\frac{1}{c\mu_{n,\alpha}(c)}\psi_{n,c}^\alpha\left(\frac{x}{c}\right)\chi_{[0,c]}(x).$$

According to Plancherel's theorem, the family $\psi_{n,c}^{(\alpha)}=\sqrt{c}|\mu_{n,\alpha}(c)|\psi_{n,c}^{(\alpha)}$ constitute a complete orthonormal system in $\mathcal{B}_c^\alpha$ defined by:

$$\mathcal{B}_c^\alpha=\{f\in L^2(0,\infty);\ \operatorname{supp}(\mathcal{H}^\alpha(f))\subset[0,c]\}.\tag{5.22}$$

Fore more details, see for example [Sl2, BK]. Our first aim in this section is to prove that in the case of CPSWFs, we have mean convergence in the Hankel Paley-Wiener space $B_{c,p}^\alpha$

defined by

$$B_{c,p}^{\alpha} = \{f \in L^p(0,\infty); \operatorname{supp}(\mathcal{H}^{\alpha}(f)) \subseteq [0,c]\},$$

if and only if $4/3 < p < 4$.

5.1. **Some facts about Spherical Bessel function.**

5.1.1. *Spherical Bessel function.* The spherical Bessel function is defined as

$$j_{n,c}^{(\alpha)}(x) = \sqrt{2(2n+\alpha+1)}\frac{J_{2n+\alpha+1}(cx)}{\sqrt{cx}}. \tag{5.23}$$

Here, J_α is the Bessel function of the first kind and order α. The spherical Bessel functions satisfy the orthogonality relation,

$$\int_0^{+\infty} j_{n,c}^{(\alpha)}(x) j_{m,c}^{(\alpha)}(x)\,\mathrm{d}x = \delta_{n,m}.$$

Their Hankel transforms are given by, see for example [Sl2]

$$\mathcal{H}^{\alpha}(j_{n,c}^{(\alpha)})(x) = \frac{\sqrt{2(2n+\alpha+1)}}{c}\left(\frac{x}{c}\right)^{\alpha+\frac{1}{2}} P_n^{(\alpha,0)}\left(1-2\left(\frac{x}{c}\right)^2\right)\chi_{[0,c]}(x). \tag{5.24}$$

where $P_n^{(\alpha,0)}$ is the Jacobi polynomials of degree n and parameter α, normalized so that $P_n^{(\alpha,0)}(1) = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$. Introducing

$$T_{n,\alpha}(x) = (-1)^n\sqrt{2(2n+\alpha+1)}x^{\alpha+\frac{1}{2}}P_n^{(\alpha,0)}(1-2x^2). \tag{5.25}$$

We thus get $\mathcal{H}^{\alpha}(j_{n,c}^{(\alpha)})(x) = c^{-1}\chi_{[0,1]}(x/c)T_{n,\alpha}(x/c)$. Note that the orthogonality relations of the $j_{n,c}^{(\alpha)}$'s and the unitary character of $\mathcal{H}^{\alpha}$ imply that $(T_{n,\alpha})_{n\geq 0}$ is an orthonormal basis of $L^2[0,1]$ while the spherical Bessel functions $j_{n,c}^{(\alpha)}$ form a complete orthonormal system in $\mathcal{B}_c^{\alpha}$.

Further, using the induction property $\dfrac{2\beta}{x}J_\beta(x) = J_{\beta-1}(x) + J_{\beta+1}(x)$, we get the following induction formula

$$j_{n+1,c}^{(\alpha)} = \frac{2\sqrt{(2n+\alpha+2)(2n+\alpha+3)}}{cx} j_{n+1/2,c}^{(\alpha)} - \frac{\sqrt{2n+\alpha+3}}{\sqrt{2n+\alpha+1}} j_{n,c}^{(\alpha)}.$$

Moreover, for $1 < p < \infty$, we have

$$\left\|j_{n,c}^{(\alpha)}\right\|_{L^p(0,\infty)} \sim \begin{cases} n^{-\frac{1}{2}+\frac{1}{p}} & \text{when } 1 < p < 4 \\ n^{-\frac{1}{4}}\log n & \text{when } p = 4 \\ n^{-\frac{1}{3}+\frac{1}{3p}} & \text{when } p > 4 \end{cases}. \tag{5.26}$$

Note that, if $\frac{1}{p} + \frac{1}{q} = 1$, then for $\ell \in \mathbb{Z}$, we have

$$\left\|j_{n+\ell,c}^{(\alpha)}\right\|_{L^p(0,\infty)}\left\|j_{n,c}^{(\alpha)}\right\|_{L^q(0,\infty)} \sim \begin{cases} n^{\frac{2}{3p}-\frac{1}{2}} & \text{when } 1 < p < \frac{4}{3} \\ \log n & \text{when } p = \frac{4}{3} \text{ or } p = 4 \\ 1 & \text{when } \frac{4}{3} < p < 4 \\ n^{\frac{1}{6}-\frac{2}{3p}} & \text{when } p > 4 \end{cases} = o(n^{\frac{1}{6}}) \tag{5.27}$$

where the constants depend on ℓ, for more details see [BC]. We can now see that condition (L) is satisfied.

The expansion of $\psi_{n,c}^{(\alpha)}$ in the basis of the spherical Bessel functions is done as follows. First, by using (5.23), we calculate the scalar product

$$\begin{aligned}\left\langle \psi_{n,c}^{(\alpha)}, j_{n,c}^{(\alpha)} \right\rangle_{L^2[0,+\infty[} &= \int_0^{+\infty} \sqrt{c}|\mu_{n,\alpha}(c)|\psi_{n,c}^{\alpha}(x) j_{n,c}^{(\alpha)}(x)\,\mathrm{d}x \\ &= \sqrt{c}|\mu_{n,\alpha}(c)| \int_0^{+\infty} \psi_{n,c}^{\alpha}(x)\sqrt{2(2n+\alpha+1)}\frac{J_{2n+\alpha+1}(cx)}{\sqrt{cx}}\,\mathrm{d}x.\end{aligned}$$

Writing $\nu = \sqrt{c}\dfrac{|\mu_{n,\alpha}(c)|}{\mu_{n,\alpha}(c)}\sqrt{2(2n+\alpha+1)}$ and since

$$\psi_{n,c}^{\alpha}(x) = \frac{1}{\mu_{n,\alpha}(c)}\mathcal{H}_c^{\alpha}\psi_{n,c}^{\alpha}(x) = \frac{1}{\mu_{n,\alpha}(c)}\int_0^1 \sqrt{cxy}J_\alpha(cxy)\psi_{n,c}^{\alpha}(y)\,\mathrm{d}y,$$

then Fubini's theorem together with (5.24), we get

$$\begin{aligned}\left\langle \psi_{n,c}^{(\alpha)}, j_{n,c}^{(\alpha)} \right\rangle_{L^2[0,+\infty[} &= \nu \int_0^1 \sqrt{y}\psi_{n,c}^{\alpha}(y) \int_0^{+\infty} J_{2n+\alpha+1}(cx)J_\alpha(cxy)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{\nu}{c}\int_0^1 y^{\alpha+\frac{1}{2}}P_n^{(\alpha,0)}(1-2y^2)\psi_{n,c}^{\alpha}(y)\,\mathrm{d}y.\end{aligned}$$

We thus have

$$\left\langle \psi_{n,c}^{(\alpha)}, j_{n,c}^{(\alpha)} \right\rangle_{L^2[0,+\infty[} = (-1)^k \frac{|\mu_{n,\alpha}(c)|}{\sqrt{c}\mu_{n,\alpha}(c)}\left\langle \psi_{n,c}^{(\alpha)}, T_{n,\alpha} \right\rangle_{L^2[0,1]}$$

where $T_{n,\alpha}$ has been defined in (5.25). Writing $d_k^n = \left\langle \psi_{n,c}^{(\alpha)}, T_{k,\alpha} \right\rangle_{L^2[0,1]}$, we thus get the following expansion on $[0,+\infty)$:

$$\psi_{n,c}^{(\alpha)}(x) = \frac{|\mu_{n,\alpha}(c)|}{\sqrt{c}\mu_{n,\alpha}(c)}\sum_{k\geq 0}(-1)^k d_k^n j_{k,c}^{(\alpha)}(x).$$

Consider the differential operator given by

$$\mathcal{D}_c^{\alpha}(\phi)(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right]\phi(x) - \left(\frac{\frac{1}{4}-\alpha^2}{x^2} - c^2x^2\right)\phi(x).$$

We know from [Sl2] that the operators $\mathcal{D}_c^{\alpha}$ and $\mathcal{H}_c^{\alpha}$ commute so that $\psi_{n,c}^{(\alpha)}$ are eigenvectors of both operators and we denote by $\chi_{n,\alpha}(c)$ the corresponding eigenvalue of $\mathcal{D}_c^{\alpha}$, that is

$$\mathcal{D}_c^{\alpha}(\psi_{n,c}^{(\alpha)}) = \chi_{n,\alpha}(c)\psi_{n,c}^{(\alpha)}. \tag{5.28}$$

Further more, we have the following inequality see [Sl2]:

$$\left(\alpha+2n+\frac{1}{2}\right)\left(\alpha+2n+\frac{3}{2}\right) \leq \chi_{n,\alpha}(c) \leq \left(\alpha+2n+\frac{1}{2}\right)\left(\alpha+2n+\frac{3}{2}\right)+c^2. \tag{5.29}$$

According to [W], the Spherical Bessel functions are the eigenfunctions of the differential operator given by

$$\mathcal{D}_c(\omega) = \left(\frac{x}{c}\right)^2 \frac{d^2}{d^2x}(\omega) + \frac{(1+c)}{\sqrt{c}}\left(\frac{x}{c}\right)\frac{d}{dx}(\omega) + c^2x^2\omega$$

and the corresponding eigenvalues are given by $\left((2n+\alpha+1)^2 - \frac{c^2-2c}{2c^3}\right)\sqrt{2(2n+\alpha+1)}$. It follows from Lemma 3.2 that Condition (D) of Theorem 2.1 is also satisfied.

If we substitute the expression of $\psi_{n,c}^{(\alpha)}$ as a series of Jacobi polynomials into (5.28), we obtain the relations satisfied by the coefficients d_k^n. More precisely, from [Sl2], we obtain the three term recurrence relation

$$f(k,n,c,\alpha)d_k^n = a_{k,\alpha}d_{k-1}^n + a_{k+1,\alpha}d_{k+1}^n, \ \forall k \geq 0 \tag{5.30}$$

where $d_{-1}^n = 0$ and

$$\begin{aligned} f(k,n,c,\alpha) &= \frac{\chi_{n,\alpha}(c) - (\alpha+2k+\frac{1}{2})(\alpha+2k+\frac{3}{2}) - c^2b_{k,\alpha}}{c^2} \\ a_{k,\alpha} &= \frac{k(k+\alpha)}{(\alpha+2k)\sqrt{\alpha+2k+1}\sqrt{\alpha+2k-1}} \\ b_{k,\alpha} &= \frac{1}{2}\left[\frac{\alpha^2}{(\alpha+2k+1)(\alpha+2k)}+1\right]. \end{aligned} \tag{5.31}$$

5.1.2. *The projection on the span of spherical Bessel functions.* Let $1<p<\infty$ and $\alpha \geq -\frac{1}{2}$. For $n \geq 0$, let

$$P_n^{(\alpha)}(x,y) := \sum_{k=0}^n j_{k,c}^{(\alpha)}(x) j_{k,c}^{(\alpha)}(y)$$

and $\mathcal{P}_n^{(\alpha)}$ be the operator with kernel $P_n^{(\alpha)}(x,y)$. That is, $\mathcal{P}_n^{(\alpha)}$ is the projection on the span of $\{j_{0,c}^{(\alpha)}, \dots, j_{n,c}^{(\alpha)}\}$.

Proposition 5.1. *Let $1<p<\infty$, $\alpha > -1/2$. Then the following estimate holds for every n and every $f \in L^p(0,\infty)$*

$$\left\|\mathcal{P}_n^{(\alpha)}(f)\right\|_{L^p(0,\infty)} \lesssim \begin{cases} \|f\|_{L^p(0,\infty)} & \textit{if } \frac{4}{3} < p < 4 \\ n^{3/4}\|f\|_{L^p(0,\infty)} & \textit{otherwise} \end{cases}$$

with the implied constant independent of f and n.

Proof. The projection on the span of spherical Bessel functions has been studied by Varona [Va] with a different normalization. He considered

$$j_n^\alpha(x) = \sqrt{2n+\alpha+1}J_{2n+\alpha+1}(\sqrt{x})x^{-\alpha/2-1/2}$$

so that

$$j_{n,c}^{(\alpha)} = \sqrt{2}j_n^\alpha(c^2x^2)(cx)^{\alpha+1/2}.$$

Next, if we define

$$K_n(x,y) = \sum_{k=0}^n j_n^\alpha(x) j_n^\alpha(y)$$

(with Varona's notation) then $P_n^{(\alpha)}(x,y)=2c^{2\alpha+1}(xy)^{\alpha+1/2}K_n(c^2x^2,c^2y^2)$.

Using Varona's computation [Va, page 69] we get

$$\begin{aligned} P_n^{(\alpha)}(x,y) &= \frac{(xy)^{1/2}}{x^2-y^2}\{xJ_{\alpha+1}(cx)J_\alpha(cy)-yJ_{\alpha+1}(cy)J_\alpha(cx)\} \\ &+ \frac{(xy)^{1/2}}{x^2-y^2}\{xJ'_{\alpha+2n+2}(cx)J_{\alpha+2n+2}(cy)-yJ_{\alpha+2n+2}(cx)J'_{\alpha+2n+2}(cy)\}. \end{aligned} \tag{5.32}$$

Now, recalling that $\mathcal{H}$ denotes the Hilbert transform, it follows from (5.32) that

$$\mathcal{P}_n^{(\alpha)}(f)(x)=\Omega_1(f)(x)-\Omega_2(f)(x)+\Omega_3(f)(x)-\Omega_4(f)(x)$$

where

$$\begin{aligned} \Omega_1(f)(x) &= \int_0^\infty \frac{(xy)^{1/2}}{x^2-y^2}xJ_{\alpha+1}(cx)J_\alpha(cy)f(y)\,\mathrm{d}y \\ &= \frac{x^{\frac32}}{2}J_{\alpha+1}(cx)\mathcal{H}\big[y^{-1/4}J_\alpha(cy^{1/2})f(y^{1/2})\big](x^2). \\ \Omega_2(f)(x) &= \int_0^\infty \frac{(xy)^{1/2}}{x^2-y^2}yJ_\alpha(cx)J_{\alpha+1}(cy)f(y)\,\mathrm{d}y \\ &= \frac{x^{\frac12}}{2}J_\alpha(cx)\mathcal{H}\big[y^{1/4}J_{\alpha+1}(cy^{1/2})f(y^{1/2})\big](x^2). \\ \Omega_3(f)(x) &= \int_0^\infty \frac{(xy)^{1/2}}{x^2-y^2}xJ'_{\alpha+2n+2}(cx)J_{\alpha+2n+2}(cy)f(y)\,\mathrm{d}y \\ &= \frac{x^{\frac32}}{2}J'_{\alpha+2n+2}(cx)\mathcal{H}\big[y^{-1/4}J_{\alpha+2n+2}(cy^{1/2})f(y^{1/2})\big](x^2). \\ \Omega_4(f)(x) &= \int_0^\infty \frac{(xy)^{1/2}}{x^2-y^2}yJ_{\alpha+2n+2}(cx)J'_{\alpha+2n+2}(cy)f(y)\,\mathrm{d}y \\ &= \frac{x^{\frac12}}{2}J_{\alpha+2n+2}(cx)\mathcal{H}\big[y^{1/4}J'_{\alpha+2n+2}(cy^{1/2})f(y^{1/2})\big](x^2). \end{aligned}$$

Note that each of these operators is of the form

$$\Omega_j(f)(x)=G_j(x)\mathcal{H}\big[\varphi_j\big](x^2)$$

so that

$$\begin{aligned} \|\Omega_j(f)\|^p_{L_p(0,\infty)} &= \int_0^\infty |G_j(x)|^p|\mathcal{H}\big[\varphi_j\big](x^2)|^p\,\mathrm{d}x \\ &= \int_0^\infty \frac{|G_j(\sqrt{x})|^p}{2\sqrt{x}}|\mathcal{H}\big[\varphi_j\big](x)|^p\,\mathrm{d}x. \end{aligned}$$

But then, if we are able to find an upper bound $\omega_j\in A^p$ (see [Va]) of $\dfrac{|G_j(\sqrt{x})|^p}{2\sqrt{x}}\lesssim\omega_j(x)$, we obtain

$$\|\Omega_j(f)\|^p_{L_p(0,\infty)}\lesssim\big\|\mathcal{H}\big[\varphi_j\big]\big\|^p_{L_p\left((0,\infty),\omega_j(x)\,\mathrm{d}x\right)}\lesssim[\omega_j]_{A^p}^{\max(p,p')}\|\varphi_j\|^p_{L_p\left((0,\infty),\omega_j(x)\,\mathrm{d}x\right)}.$$

It then remains to prove that $\|\varphi_j\|^p_{L_p\big((0,\infty),w_j(x)\,\mathrm{d}x\big)} \lesssim \|f\|^p_{L_p(0,\infty)}$.

For Ω_1, $\varphi_1(y) = y^{-1/4}J_\alpha(cy^{1/2})f(x^{1/2})$. Further, we use the bound $|J_{\alpha+1}(t)| \leq C_\alpha t^{-1/2}$ which allows us to chose $\omega_1(y) = y^{\frac{p-1}{2}} \in A^p$ since $-1 < \dfrac{p-1}{2} < p-1$. Further

$$\begin{aligned}\|\varphi_1\|^p_{L^p(\mathbb{R}_+,\omega_1(y)\,\mathrm{d}y)} &= \left\|y^{-1/4}J_\alpha(cy^{1/2})f(y^{1/2})\right\|^p_{L^p(\mathbb{R}_+,\omega_1(y)\,\mathrm{d}y)} \lesssim \int_0^\infty \left|\frac{f(y^{1/2})}{(cy)^{1/2}}\right|^p x^{\frac{p-1}{2}}\,\mathrm{d}y \\ &\lesssim \|f\|^p_{L^p(0,+\infty)}.\end{aligned}$$

We will now take care of Ω_2. In this case with $\varphi_2(y) = y^{1/4}J_{\alpha+1}(cy^{1/2})f(y^{1/2})$ and the same bound on the Bessel function shows that we can chose $\omega_2(x) = x^{-\frac{1}{2}} \in A^p$.

$$\begin{aligned}\|\varphi_2\|^p_{L^p(\mathbb{R}_+,\omega_2(x)\,\mathrm{d}x)} &= \left\|x^{1/4}J_{\alpha+1}(cx^{1/2})f(x^{1/2})\right\|^p_{L^p(\mathbb{R}_+,\omega_2(x)\,\mathrm{d}x)} \lesssim \int_0^\infty \left|f(x^{1/2})\right|^p x^{-\frac{1}{2}}\,\mathrm{d}x \\ &\lesssim \|f\|^p_{L^p(0,+\infty)}.\end{aligned}$$

The same reasoning would apply to Ω_3, Ω_4 but with a bound that depends on n. We thus need a more refined estimate which follows from [BC]:

$$\begin{aligned}|J_\mu(x)| &\lesssim x^{-\frac{1}{4}}\big(|x-\mu| + \mu^{\frac{1}{3}}\big)^{-\frac{1}{4}} \\ |J'_\mu(x)| &\lesssim x^{-\frac{3}{4}}\big(|x-\mu| + \mu^{\frac{1}{3}}\big)^{\frac{1}{4}}.\end{aligned}$$

Set $\mu = \alpha + 2n + 2$. We may then take

$$\omega_3(x) = x^{\frac{3p}{8}-\frac{1}{2}}\big(|c\sqrt{x}-\mu| + \mu^{\frac{1}{3}}\big)^{\frac{p}{4}} \quad \text{and} \quad \omega_4(x) = x^{\frac{p}{8}-\frac{1}{2}}\big(|c\sqrt{x}-\mu| + \mu^{\frac{1}{3}}\big)^{-\frac{p}{4}}$$

By the Lemma (3.4), ω_3 and $\omega_4 \in A_p$ with $[\omega_j]_{A^p} \lesssim 1$ if $\dfrac{4}{3} < p < 4$ and $[\omega_j]_{A^p} \lesssim \mu^{3/4}$ otherwise.

Finally

$$\varphi_3(x) = x^{-1/4}J_{\alpha+2n+2}(cx^{1/2})f(x^{1/2}) \quad \text{and} \quad \varphi_4(x) = x^{1/4}J'_{\alpha+2n+2}(cx^{1/2})f(x^{1/2}).$$

Note that

$$|\varphi_3(x)| \lesssim x^{-3/8}\big(|c\sqrt{x}-\mu| + \mu^{\frac{1}{3}}\big)^{-\frac{1}{4}}|f(x^{1/2})| \quad \text{and} \quad |\varphi_4(x)| \lesssim x^{-1/8}\big(|c\sqrt{x}-\mu| + \mu^{\frac{1}{3}}\big)^{\frac{1}{4}}|f(x^{1/2})|.$$

so that

$$\|\varphi_j\|^p_{L^p(\mathbb{R}_+,\omega_j(x)\,\mathrm{d}x)} \lesssim \int_0^\infty x^{-\frac{1}{2}}|f(x^{1/2})|\,\mathrm{d}x \lesssim \|f\|^p_{L^p(0,+\infty)}.$$

It follows that $\|\Omega_j(f)\|_{L^p(0,\infty)} \lesssim 1$ if $\dfrac{4}{3} < p < 4$ and $\|\Omega_j(f)\|_{L^p(0,\infty)} \lesssim n^{3/4}\|f\|_{L^p(0,\infty)}$ for $1 < p \leq \dfrac{4}{3}$. Grouping all estimates, the same holds for $\mathcal{P}_n^{(\alpha)}$. Finally, as $\mathcal{P}_n^{(\alpha)}$ is self-adjoint, we also get the estimate $\left\|\mathcal{P}_n^{(\alpha)}(f)\right\|_{L^p(0,\infty)} \lesssim n^{3/4}\|f\|_{L^p(0,\infty)}$ for $p \geq 4$.

$\square$

5.2. **Condition** (R)**.** We will now show that conditions (R) of Theorem 2.2 are satisfied, this is done in three lemmas.

Lemma 5.2. *For every $k \geq 1$ and $\alpha > -\dfrac{1}{2}$, $0 \leq a_k^{(\alpha)} \leq \dfrac{1}{2}$.*

Proof. For $k=1$, $a_{1,\alpha} = \frac{\sqrt{1+\alpha}}{(2+\alpha)\sqrt{\alpha+3}}$ which is clearly $\leq 1/2$ when $\alpha \geq 0$. It is easy to see that $a_{1,\alpha}$ is increasing with $\alpha \in (-1/2, \alpha_0)$ and decreasing with $\alpha \in (\alpha_0, 0)$ where $\alpha_0 = \frac{-3+\sqrt{5}}{2}$. Finally, $a_{1,\alpha_0} \sim 0.3$ so that $a_{1,\alpha} \leq 1/2$ for every α. Write

$$|a_{k,\alpha}| = \frac{1}{4} \frac{1+\frac{2\alpha}{2k}}{\left(1+\frac{\alpha}{2k}\right)\sqrt{1+\frac{\alpha+1}{2k}}\sqrt{1+\frac{\alpha-1}{2k}}} = \frac{1}{4}\psi(1/2k),$$

where $\psi(x) = \dfrac{1+2\alpha x}{(1+\alpha x)\big(1+(\alpha-1)x\big)^{1/2}\big(1+(\alpha+1)x\big)^{1/2}}$. It is thus enough to show that $|\psi(x)| \leq 2$ for $x \in [0, 1/4]$. Note that ψ is non-negative for $\alpha > -1/2$ and $x \leq 1$. When $-1/2 < \alpha \leq 0$, as $1 + 2\alpha x \leq 1 + \alpha x$ and $1 + (\alpha+1)x \geq 1$,

$$\psi(x) \leq \frac{1}{\sqrt{1+(\alpha-1)x}} \leq \frac{1}{\sqrt{1-3x/2}} \leq \frac{1}{\sqrt{5/8}} < 2$$

when $x \leq 1/4$. When $\alpha > 0$, we first bound

$$\psi(x) \leq \frac{1+2\alpha x}{(1+\alpha x)\big(1+(\alpha-1)x\big)}$$

and it is enough to prove that, for $x > 0$, we have

$$1 + 2\alpha x \leq 2(1+\alpha x)\big(1+(\alpha-1)x\big) = 2 + (4\alpha - 2)x + 2\alpha(\alpha-1)x^2.$$

or, equivalently, $1 + 2(\alpha-1)x + 2\alpha(\alpha-1)x^2 \geq 0$. When $\alpha \geq 1$ this is obvious, while for $0 < \alpha < 1$, the roots of this equation are

$$-\frac{1-\alpha+\sqrt{1-\alpha^2}}{2\alpha(1-\alpha)} < 0 < \frac{\sqrt{1-\alpha^2}-(1-\alpha)}{2\alpha(1-\alpha)} = \frac{\sqrt{1+\alpha}-\sqrt{1-\alpha}}{2\alpha\sqrt{1-\alpha}}.$$

The inequality is therefore satisfied as soon as $x \leq 1/2k$ with

$$k \geq k_\alpha := \frac{\alpha\sqrt{1-\alpha}}{\sqrt{1+\alpha}-\sqrt{1-\alpha}} = \frac{\alpha(1-\alpha+\sqrt{1-\alpha^2})}{2}.$$

As $0 < \alpha < 1$, it is easy to see that $k_\alpha \leq 1$. □

Let us now estimate the $b_{k,\alpha}$'s:

Lemma 5.3. *For every k and every $\alpha > -1/2$, $b_{k,\alpha} = \frac{1}{2} + \tilde{\eta}_{k,\alpha}$ with $|\tilde{\eta}_{k,\alpha}| \leq \frac{1}{2}$.*

Proof. From the definition of $b_{k,\alpha}$, $\tilde{\eta}_{k,\alpha} = \frac{\alpha^2}{2(\alpha+2k+1)(\alpha+2k)}$. When $\alpha = 0$, $\eta = 0$ and when $\alpha > 0$ we directly get $\tilde{\eta}_{k,\alpha} \leq \frac{1}{2}$. When $-1/2 < \alpha < 0$, $\alpha + j > 1/2 > |\alpha|$ for every $j \geq 1$ thus $|\tilde{\eta}_{0,\alpha}| = \frac{|\alpha|}{2(\alpha+1)} \leq \frac{1}{2}$ while for $k \geq 1$ we directly get $0 \leq \tilde{\eta}_{k,\alpha} = \frac{1}{2}\frac{|\alpha|}{\alpha+2k}\frac{|\alpha|}{\alpha+2k+1} \leq \frac{1}{2}$. □

The last step consists in establishing the bounds for $|f(k,n,c,\alpha)|$. But from (5.29) and (5.31), it is straightforward to see that $f(k,n,c,\alpha)$ satisfies the conditions of Lemma 3.1. In summary

Lemma 5.4. *For every $\alpha>-1/2$, every $c>0$,*
— *for fixed n, $f(k,n,c,\alpha)\gtrsim k^2$, when k is large enough;*
— *for every $n\geq c^2/2$, $k\geq 0$, $k\neq n$, we have*

$$|f(k,n,c,\alpha)|\geq 4\frac{|k-n|k+c^2}{c^2};$$

— *for every $n\geq c^2/2$,*

$$\left|\frac{a_{n+1}^{(\alpha)}}{f(n+1,n,c,\alpha)}-\frac{a_{n+2}^{(\alpha)}}{f(n+2,n+1,c,\alpha)}\right|\lesssim n^{-2}.$$

5.3. **Condition** (B')**.** It remains to check condition (B') that is, to estimate the L^p norm of the operator with kernel

$$Q_N^{(\alpha)}(x,y)=\sum_{n=n_0}^{N}\big(j_{n,c}^{(\alpha)}(x)j_{n+1,c}^{(\alpha)}(y)+j_{n+1,c}^{(\alpha)}(x)j_{n,c}^{(\alpha)}(y)\big).$$

Lemma 5.5. *Let $1<p<\infty$ then, for every $f\in L^p(0,\infty)$*

$$\left(\int_0^\infty\left|\int_0^\infty Q_N^{(\alpha)}(x,y)f(y)\,\mathrm{d}y\right|^p\mathrm{d}x\right)^{1/p}\lesssim N^{2/3}\|f\|_{L^p(0,\infty)}.$$

and the implied constant is independent of N and f.

Proof. First, using the identity (see [W])

$$\frac{2\nu}{x}J_\nu(x)=J_{\nu+1}(x)+J_{\nu-1}(x)\tag{5.33}$$

twice, one gets

$$\begin{aligned}J_{2n+\alpha+3}(x)&=\frac{4(2n+\alpha+1)(2n+\alpha+2)}{x^2}J_{2n+\alpha+1}(x)\\&\quad-\frac{2(2n+\alpha+2)}{x}J_{2n+\alpha}(x)-J_{2n+\alpha+1}(x).\end{aligned}$$

so that

$$\begin{aligned}&J_{2n+\alpha+1}(x)J_{2n+\alpha+3}(y)+J_{2n+\alpha+1}(y)J_{2n+\alpha+3}(x)\\&\qquad=4(2n+\alpha+1)(2n+\alpha+2)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)J_{2n+\alpha+1}(x)J_{2n+\alpha+1}(y)\\&\qquad\quad-2(2n+\alpha+2)\left(\frac{1}{y}J_{2n+\alpha}(y)J_{2n+\alpha+1}(x)+\frac{1}{x}J_{2n+\alpha}(x)J_{2n+\alpha+1}(y)\right)\\&\qquad\quad-2J_{2n+\alpha+1}(x)J_{2n+\alpha+1}(y).\end{aligned}$$

Using again (5.33) for the middle term, we get

$$\begin{aligned}
&J_{2n+\alpha+1}(x)J_{2n+\alpha+3}(y)+J_{2n+\alpha+1}(y)J_{2n+\alpha+3}(x)\\
&\qquad= 4(2n+\alpha+1)(2n+\alpha+2)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)J_{2n+\alpha+1}(x)J_{2n+\alpha+1}(y)\\
&\qquad\quad -2\frac{(2n+\alpha+2)}{2n+\alpha}\Big(\big(J_{2k+\alpha+1}(y)+J_{2k+\alpha-1}(y)\big)J_{2n+\alpha+1}(x)\\
&\qquad\quad +\big(J_{2n+\alpha+1}(x)+J_{2n+\alpha-1}(x)\big)J_{2n+\alpha+1}(y)\Big)\\
&\qquad\quad -2J_{2n+\alpha+1}(x)J_{2n+\alpha+1}(y).
\end{aligned}$$

Next, since $j^{(\alpha)}_{n,c}(x)=\sqrt{2(2n+\alpha+1)}\frac{J_{2n+\alpha+1}(cx)}{\sqrt{cx}}$, one obtains

$$\begin{aligned}
&j^{(\alpha)}_{n,c}(x)j^{(\alpha)}_{n+1,c}(y)+j^{(\alpha)}_{n,c}(y)j^{(\alpha)}_{n+1,c}(x)\\
&\qquad=\frac{4}{c^2}\sqrt{(2n+\alpha+1)(2n+\alpha+3)}(2n+\alpha+2)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)j^{(\alpha)}_{n,c}(x)j^{(\alpha)}_{n,c}(y)\\
&\qquad\quad -2\frac{(2n+\alpha+2)}{2n+\alpha}\sqrt{\frac{2n+\alpha+3}{2n+\alpha-1}}\Big(j^{(\alpha)}_{n-1,c}(y)j^{(\alpha)}_{n,c}(x)+j^{(\alpha)}_{n-1,c}(x)j^{(\alpha)}_{n,c}(y)\Big)\\
&\qquad\quad -2\sqrt{\frac{2n+\alpha+3}{2n+\alpha+1}}\left(\frac{2(2n+\alpha+2)}{2n+\alpha}+1\right)j^{(\alpha)}_{n,c}(x)j^{(\alpha)}_{n,c}(y).
\end{aligned}\tag{5.34}$$

Now we write

$$\gamma_n=\frac{4}{c^2}\sqrt{(2n+\alpha+1)(2n+\alpha+3)}(2n+\alpha+2),$$

so that $|\gamma_n|\lesssim n^2$,

$$2\frac{(2n+\alpha+2)}{2n+\alpha}\sqrt{\frac{2n+\alpha+3}{2n+\alpha-1}}=2+\kappa_n,\qquad 0\le\kappa_n\lesssim n^{-1}$$

and

$$2\sqrt{\frac{2n+\alpha+3}{2n+\alpha+1}}\left(\frac{2(2n+\alpha+2)}{2n+\alpha}+1\right)=6+\tilde\kappa_n,\qquad 0\le\tilde\kappa_n\lesssim n^{-1}.$$

Then, summing (5.34) from $n=n_0$ to $n=N$ gives

$$\begin{aligned}
Q^{(\alpha)}_N(x,y) \;=\;& \left(\frac{1}{x^2}+\frac{1}{y^2}\right)\sum_{n=n_0}^{N}\gamma_n j^{(\alpha)}_{n,c}(x)j^{(\alpha)}_{n,c}(y)\\
&-(2+\kappa_{n_0})\Big(j^{(\alpha)}_{n_0-1,c}(y)j^{(\alpha)}_{n_0,c}(x)+j^{(\alpha)}_{n_0-1,c}(x)j^{(\alpha)}_{n_0,c}(y)\Big)\\
&-2Q^{(\alpha)}_N(x,y)-\sum_{n=n_0}^{N}\kappa_n\Big(j^{(\alpha)}_{n,c}(y)j^{(\alpha)}_{n+1,c}(x)+j^{(\alpha)}_{n,c}(x)j^{(\alpha)}_{n+1,c}(y)\Big)\\
&+(2+\kappa_N)\Big(j^{(\alpha)}_{N,c}(y)j^{(\alpha)}_{N+1,c}(x)+j^{(\alpha)}_{N,c}(x)j^{(\alpha)}_{N+1,c}(y)\Big)\\
&-6P^{(\alpha)}_N(x,y)+6P_{n_0-1}(x,y)-\sum_{n=n_0}^{N}\tilde\kappa_n j^{(\alpha)}_{n,c}(x)j^{(\alpha)}_{n,c}(y).
\end{aligned}$$

It follows that
$$Q_N^{(\alpha)}(x,y)=Q_{N,1}(x,y)+\cdots+Q_{N,7}(x,y).$$
Using (5.27) and Lemma 2.3 we get that
$$\|Q_{N,2}\|_{L^p(0,\infty)\to L^p(0,\infty)},\|Q_{N,6}\|_{L^p(0,\infty)\to L^p(0,\infty)}\lesssim 1$$
and
$$\|Q_{N,4}\|_{L^p(0,\infty)\to L^p(0,\infty)}\lesssim N^{1/6}$$
while
$$\|Q_{N,3}\|_{L^p(0,\infty)\to L^p(0,\infty)}\lesssim\sum_{n=n_0}^N\frac{1}{n}n^{1/6}\lesssim N^{1/6}$$
and
$$\|Q_{N,7}\|_{L^p(0,\infty)\to L^p(0,\infty)}\lesssim\sum_{n=n_0}^N\frac{1}{n}n^{1/6}\lesssim N^{1/6}.$$
We have seen in Proposition 5.1 that $\|Q_{N,5}\|_{L^p(0,\infty)\to L^p(0,\infty)}\lesssim N^{3/4}$.

Concerning $Q_{N,1}$, we will use the following equality, see [BC],
$$\left\|x^{-2}j_n^\alpha\right\|_{L^p(0,\infty)}\|j_n^\alpha\|_{L^q(0,\infty)}+\|j_n^\alpha\|_{L^p(0,\infty)}\left\|x^{-2}j_n^\alpha\right\|_{L^q(0,\infty)}=O(n^{-\frac{7}{3}})$$
from which we deduce that
$$\|Q_{N,1}\|_{L^p(0,\infty)\to L^p(0,\infty)}\lesssim\sum_{n=n_0}^N n^2n^{-7/3}\lesssim N^{2/3}.$$
By grouping all estimates, we obtain $\left\|Q_N^{(\alpha)}\right\|_{L^p(0,\infty)\to L^p(0,\infty)}\lesssim N^{2/3}$ as claimed. □

5.4. **Conclusion.** It remains to conclude. ll conditions of Theorem 2.1 are satisfied. Therefore, the Hankel prolate spheroidal series converges in $L^p(0,\infty)$ if and only if the Bessel series converge. The later ones converge in $L^p(0,\infty)$ if and only if $p\in(4/3,4)$. We have thus proved the following:

Theorem 5.6. *Let $\alpha>-1/2$ and $c>0$, $N\geq 0$.*

Let $(\psi_{n,c}^{(\alpha)})_{n\geq 0}$ be the family of circular prolate spheroidal wave functions. For a smooth function f on $I=(0,\infty)$, define
$$\Psi_N^{(\alpha)}f=\sum_{n=0}^N\left\langle f,\psi_{n,c}^{(\alpha)}\right\rangle_{L^2(0,\infty)}\psi_{n,c}^{(\alpha)}.$$
Then, for every $p\in(1,\infty)$, $\Psi_N^{(\alpha)}$ extends to a bounded operator $L^p(0,\infty)\to L^p(0,\infty)$. Further
$$\Psi_N^{(\alpha)}f\to f\qquad\text{in } L^p(0,\infty)$$
for every $f\in B_{c,p}^\alpha$ if and only if $p\in(4/3,4)$.

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