# Adoption dynamics: sequential or synchronous modelling-version2

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# Adoption dynamics: sequential or synchronous modelling

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**Abstract:** This paper deals with the choice of dynamics in spatial simulation and modelling. In economical context, $N$ agents choose between two technological standards according to a local assignment rule. The adoption dynamics is sequential if the choices are made one after the other; it is synchronous or partially synchronous if all or some part of the agents choose simultaneously. This paper points out differences among the three dynamics, especially in their evolution and limit configurations.

*Key words:* standard adoption, sequential dynamics, synchronous dynamics, partial parallelism, Markov chain, ergodicity.

## 1 Introduction

In many applications, we are interested in the study of the evolution of a system, in space and in time. For instance it may concern competition between different species in ecology, diffusion of technological innovation involving social behaviour, particles system in physics, fluid spread model, disease propagation, image sequences. Some models are deterministic such as cellular automata ([5]), differential equations for instance, while others are based on a stochastic framework, like Gibbs dynamics, probabilistic automata , or particle systems.

Our framework is standard's adoption but the results may apply to other applications where the entourage plays a major role in one's decision making process (see for instance [2], [3]). We consider a finite set of sites $S = \{1, 2, \cdots, N\}$. Each site $i$ is associated to an agent who makes a choice $X_i$ in a state space $E$. The state space $E$ can be finite or not. In all the following, we will assume for simplicity $E = \{-1, +1\}$, which is associated to a choice between two competitive technologies (see for instance [1]). When this choice $X_i$ depends of the local context, we say that there is spatial coordination, the spatial dependency being positive if there is cooperation between the agents, and negative in case of competition. Guyon and Hardouin proposed in [10] tests for spatial coordination allowing to distinguish between "independent" and influenced choices. We consider in this work probabilistic assignment rules depending on the neighbourhood, and compare different adoption dynamics: sequential, synchronous or partially synchronous. We do not provide new theoretical results but we gather and clarify some results, hoping to shed light on different modellings, offering information tools about the ins and outs. Indeed, the Gibbs sampler theory is well known, synchronous parallelism a little bit less, and partial parallelism is not much investigated in the literature, though the three dynamics are considered in standard's adoption. The main result gives the limit behaviour of these dynamics; first, whatever the dynamics, sequential, synchronous or partially synchronous, it is ergodic; the proof involves standard

Markov chains properties; Second, we show that the limit distribution differs with the dynamics; consequences are important, directly linked with the choice of the model for standards adoption or other economic application; one has to figure out that this primary choice will lead to different economic situations. The invariant distribution is the generating distribution for the sequential dynamics, but it remains generally unknown for the other dynamics. However, its characterization is possible for the synchronous dynamics in the particular case of the Ising model; and we can apply some simulated annealing properties to the partially synchronous dynamics.

A *scan* or a *sweep* of $S$ is a tour of all the sites. When the scans are sequential and indefinitely repeated, the agents make their decision one by one; then we get the well known Gibbs sampler and it is possible to characterize the probability distribution of the limit configuration. When the dynamics is synchronous, all the agents make their decision simultaneously, there is still ergodicity but it is difficult to explicit the limit distribution (See [8] for a full description). Finally, partial synchronous dynamics run step by step, a significant part of the changes happening simultaneously at each step.

Our purpose is not to discuss about the choice of the dynamics; this depends only on the economic situation to be modeled. We just want to point out that, for a same local assignment rule, the configurations of the systems can differ widely according to a synchronous or sequential course.

In section 2, we briefly describe the difference between deterministic and probabilistic assignment rules, through standard examples. Then we present the sequential, synchronous and partially synchronous dynamics in section 3, followed by their ergodic properties in section 4. Some illustrating examples are given in section 5.

## 2 Assignment rules

Let us specify the model and give some notations. $S$ is equipped with a symmetric graph $\mathcal{G}$ and $\langle i,j\rangle$ denotes a pair of neighbouring sites $i$ and $j$. If $A$ is a subset of $S$, we denote $\partial A = \{i \in S, i \notin A$ and $\exists j \in A$ s.t. $i$ and $j$ are neighbouring sites$\}$ the neighbourhood of $A$, and $\partial i = \partial\{i\}$. Let us note $x = (x_1, x_2, \cdots, x_N)$ a realization of $X = (X_1, X_2, \cdots, X_N)$ in $\Omega = E^S$; for a subset $A \subset S$, $x_A$ (resp. $x^A$) is the configuration $x$ on $A$ (resp. outside of $A$), and $x^i = x^{\{i\}}$. Finally, $|A|$ denotes the cardinal number of $A$.

The agent $i$ makes his choice according to a local assignment rule $\pi_i(. \mid x_{\partial i})$ depending on $x_{\partial i}$. We give below two commonly used examples of deterministic and probabilistic rules.

**Example 1** *Deterministic Majority choice*

Let $S = \{1, 2, ..., n\}^2$ be a square lattice of size $n \times n$, with the 4 nearest neighbours system. The agent $i$ chooses the state $+1$ (resp. $-1$) if $+1$ (resp. $-1$) is majority among his neighbours, and makes a choice at random in case of equality. If we add the assumption that the agent also takes into account his own advice, or private information, then there is always a majority state among the 5 sites of $\partial i \cup \{i\}$ and the rule is deterministic; the system is a kind of cellular automata ([6]). For those two rules, general consensus (same state everywhere) is an absorbing state. That means that if the number of scans is large, one of the technological standards will emerge and dominate, ending unique. The only point is to know which one of the standards will disappear, and the necessary number of scans to determine the winner. The answer depends on the rule, that is the kind of majority, and on the initial rates of the two standards.

**Example 2** *Probabilistic Ising rule*
Let us consider an Ising type model; the Ising model was introduced by physicists in ferromagnetism to represent spins configurations. It is frequently used for standards' adoption or voter models (see for instance [7]). We note $N_i(x) = N(x_{\partial i}) = \sum_{j \in \partial i} x_j$; then agent $i$ chooses state $+1$ with probability:

$$\pi_i(x_i \mid x_{\partial i}) = \frac{\exp x_i(\alpha + \beta N_i(x))}{\exp(\alpha + \beta N_i(x)) + \exp -(\alpha + \beta N_i(x))} \; . \tag{1}$$

This probability is nothing but the conditional distribution probability of a Gibbs field on $\Omega$ with a joint distribution given by:

$$\pi(x) = Z^{-1} \exp\{\alpha \sum_{i \in S} x_i + \beta \sum_{\langle i,j \rangle} x_i x_j\} \tag{2}$$

The normalization constant $Z$ of the joint distribution, which is often computationally intractable, does not occur in the expression of conditional local distributions (1); that's the reason why we work with these ones.

In this example, the choice of the agent depends on two parameters $\alpha$ and $\beta$ setting the marginal distributions and spatial correlation.The parameter $\alpha$ is a measure of the global frequency of $+1$ and $-1$; $\alpha > 0$ strengthens states $+1$ while $\alpha < 0$ increases the number of states $-1$, and $\alpha = 0$ balances the two standards. The parameter $\beta$ (the inverse temperature) determines the resemblance or dissimilarity between neighbouring sites. There is cooperation between neighbouring sites if $\beta > 0$, while $\beta < 0$ ensures competition. If $\beta = 0$, the assignment is independent of the neighbourhood.

If we set $\alpha = 0$ and $\beta > 0$, $\beta$ rather large, the rule meets the previous majority choice. Then, a deterministic rule can be approximated by a probabilistic rule.

In all the following, we consider the case of a probabilistic rule in terms of conditional distributions; this implies to define properly the underlying model. Let us consider the general positive distribution $\pi$ on the configuration set $\Omega$:

$$\pi = \{\pi(x), x \in \Omega\}, \text{ with } \pi(x) > 0 \text{ for all } x \text{ and } \sum_{\Omega} \pi(x) = 1 \tag{3}$$

The positivity condition allows us to define, for all $A$ and $x^A$, the conditional probabilities $\pi_A(. \mid x^A)$, particularly the conditional distributions $\{\pi_i(. \mid x^i), i \in S\}$. When $\pi_A(. \mid x^A)$ depends only on $x_{\partial A}$, $\pi$ refers to a Markov random field, as in the example above (2) (see for instance [9]).

## 3 Sequential, synchronous dynamics

Let us describe the three dynamics involved in standards adoption. We start with an initial state denoted $x = (x_1, x_2, \cdots, x_N)$ which represents the initial standards adopted by agents $1, ..., N$. Then, one agent (sequential dynamics), or all agents (synchronous dynamics), or some of the agents (partially synchronous dynamics) choose a new standard, each agent $i$ making his choice according to a local probabilistic rule $\pi_i(x_i \mid x_{\partial i})$.

**The sequential dynamics**

A sequential dynamics is defined by a sequence of scans of the set of sites $S$. For instance, we browse the sites 1 to $N$, sequentially, in this order.

◦ step 1: the initial state is $x = (x_1, x_2, \cdots, x_N)$.

◦ step 2: we browse the sites 1 to $N$; first, agent 1 turns from $x_1$ to $y_1$ according to $\pi_1(y_1 \mid x_2, \cdots, x_N)$; then agent 2 turns from $x_2$ to $y_2$ according to $\pi_2(y_2 \mid y_1, x_3, \cdots, x_N)$; and so on; at the $k$-th site, agents 1 to $k-1$ have made a choice and their state is $(y_1, y_2, ..., y_{k-1})$ while agents $k+1$ to $N$ are still in the state $(x_{k+1}, ..., x_N)$; we (uniquely) relax (modify) the $k$-th value according to the local conditional assignment rule, and conditionally to the previous configuration;

$$x_k \mapsto y_k \text{ according to } \pi_k(y_k \mid y_1, y_2, \cdots, y_{k-1}, x_{k+1}, \cdots, x_N).$$

◦ step 3: we come back to step 1 with the new initial state $y = (y_1, y_2, \cdots, y_N)$.

Then a scan of $S$ changes the configuration $x$ to the new one $y = (y_1, y_2, \cdots, y_N)$ in $N$ steps.

Some variants are possible:

1. the route to visit all sites can be different from one scan to the other;
2. an individual site can be visited several times during the scan, the important point being to visit all sites;
3. the order of the visits can be chosen at random;
4. the release can also be done by groups of sites, one group followed by another one, with $S$ being the union of the groups, and each group evolving internally sequentially.

As we will see, all those sequential procedures are asymptotically equivalent, leading to the same stationary distribution ([8], [9]).

**The synchronous dynamics**

Synchronous dynamics is also called total parallelism; in fact, all the sites are relaxed simultaneously and a scan of $S$ is realized in 1 step.

◦ step 1 : the initial state is $x = (x_1, x_2, \cdots, x_N)$.

◦ step 2: we release simultaneously all the states, getting $x$ to $y = (y_1, y_2, \cdots, y_N)$ with the simultaneous rule on each site $k$,

$$x_k \mapsto y_k \text{ according to } \pi_k(y_k \mid x^k) = \pi_k(y_k \mid x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots, x_N),\ k \in S.$$

◦ step 3 : we come back to step 1 with the new initial state $y$.

**Partial parallelism.** Between sequential and synchronous dynamics, we can define partially synchronous dynamics where $M$ of the $N$ agents choose simultaneously.

Let $M$ be an integer, $1 \leq M \leq N$.

$\circ$ step 1 : the initial state is $x = (x_1, x_2, \cdots, x_N)$.

$\circ$ step 2: we choose a subset $A$ of $S$ with $M$ elements ($|A| = M$) and we simultaneously modify the values of the sites in $A$, while the other sites remain unchanged, getting $x = (x_A, x^A)$ to $y = (y_A, x^A)$ with the simultaneous rule on each site of $A$,

$$x_k \mapsto y_k \text{ according to } \pi_k(y_k \mid x^k) = \pi_k(y_k \mid x_1, x_2, \cdots, x_{k-1}, x_{k+1}, \cdots, x_N),\ k \in A,$$
$$x_k \mapsto x_k,\ \ k \notin A.$$

$\circ$ step 3 : we come back to step 1 with the new initial state $y = (y_A, x^A)$.

The ratio $\tau = \frac{M}{N}$ is called the parallelism rate; $M = 1$ corresponds to the sequential dynamics, while $M = N$ defines the synchronous one. Let us precise that iterating the dynamics, we choose a new subset $A$ at each step 2. This hybrid dynamics depends on the way of choosing $A$. It can be chosen at random, for instance with a uniform distribution giving the same weight to the $\binom{N}{M}$ subsets of $S$ with $M$ elements, or not, for instance we fix a covering of $S$ with subsets of cardinal $M$.

More generally, we can consider several rates of active sites; let $A_1, ..., A_n$ be some subsets of $S$ such that $\cup_i A_i = S$; at each step, we choose a subset $A_i$ of $S$ with probability $\gamma(A_i) > 0$ and we update the sites of $A_i$.

We distinguish a particular case of partial parallelism; let us assume that there is a neighbourhood graph on $S$; a coding subset $C$ is a subset of $S$ such that any two sites of $C$ are not neighbours with respect to this graph. Then, let us consider a partition of coding subsets $\{C_1, C_2, ...C_K\}$ of $S$; if we run the previous partial algorithm with these coding subsets, then it meets the sequential dynamics.

# 4 Ergodicity

## 4.1 General results

Let us consider the same generating distribution $\pi$ for each dynamics; we assume that, for each dynamics, we repeat the scans a large number of times; the following result shows that the final configurations differ from each other; specifically, in the case of sequential dynamics, the generating distribution $\pi$ is stationary, whereas it is not the case for other dynamics. It underlines the importance of the kind of dynamics when choosing such an agent-based model. The result is obtained writing the dynamics in terms of Markov chains.

Let us note $\sigma_k$ the $k-$th scan, $x = X(k) = (X_1(k), X_2(k), \cdots, X_N(k))$ and $y = X(k+1)$ the configurations before and after the $k^{th}$ scan; let us write $P = (P(x,y))_{x,y\in\Omega}$, the dynamics' transition matrix for the scan $\sigma_k$ defined by:

$$P_{\sigma_k}(x,y) = P(X(k+1) = y \mid X(k) = x),\ x, y \in \Omega$$

The following properties hold for $X = (X(k), k \geq 0)$, the evolution of these configurations.

**Proposition 1** *Let $X = (X(k), k \geq 0)$ be the dynamics generated by probability $\pi$.*

*(1) $X$ is an ergodic Markov chain on $\Omega$.*

*(2) For a sequential dynamics, the invariant distribution is $\pi$.*

*(3) For a synchronous dynamics, the invariant distribution is $\nu$, and $\nu$ differs from $\pi$.*

*(4) For a partially synchronous dynamics with $M \geq 2$, $\tau = \frac{M}{N}$, the invariant distribution is $\lambda_\tau$, and $\lambda_\tau$ differs from $\pi$.*

We give hereafter the main lines of the proof and refer the reader for instance to [4] or [11], [16] for general results on Markov chains. We complete the results with properties on the limit distributions characterization.

## 4.2 Sequential dynamics

The transition is for one scan

$$P_\sigma(x, y) = \prod_{i=1}^{N} \pi_i(y_i \mid y_1, y_2, \cdots, y_{i-1}, x_{i+1}, \cdots, x_N),$$

and $P_\sigma(x, y) > 0$ for every $x, y$. If we have different ways of scanning, we note $P_k = P_{\sigma_k}$ and $\mu$ the distribution of $X(0)$; $X$ is an inhomogeneous chain ([4]) and the distribution of $X(k)$ is

$$X(k) \sim \mu P_1 P_2 P_3 \cdots P_k.$$

On the other hand, if the scanning order is always the same, $\sigma_k \equiv \sigma$ for all $k \geq 1$, the chain is homogeneous with transition probabilities $P = P_\sigma$, and $X(k) \sim \mu P^k$.

It is easy to see that $\pi$ is invariant for each $P_\sigma$ which is strictly positive; therefore $\pi$ is the stationary distribution. For instance in the homogeneous case, we write

$$\forall x \in \Omega, \quad P^k(x, y) \underset{k \to \infty}{\longrightarrow} \pi(y).$$

Hence, if we repeat the scans a large number of times, the "final" layout of the standards depends on $\pi$ and its parameters.

Application: this result enables one to simulate any law $\pi$; it suffices to use it as generating distribution in the sequential dynamics.This procedure is the well-known Gibbs sampler ([16]).

## 4.3 Synchronous dynamics

### 4.3.1 The general case

Let us write the transition

$$Q(x, y) = \prod_{i=1}^{N} \pi_i(y_i \mid x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N) = \prod_{i=1}^{N} \pi_i(y_i \mid x^i) \ > 0.$$

Again, this expression is strictly positive for every $x, y$, which ensures the ergodicity of the dynamics:

$$\forall x \in \Omega, \quad Q^k(x,y) \underset{k\to\infty}{\longrightarrow} \nu(y)$$

But the stationary distribution $\nu$ is different from $\pi$ (the generating distribution which is also the invariant distribution for the sequential case), and moreover in most cases, not explicit. Indeed, $\pi$ is no more invariant for $Q$.

$\nu$ verifies $\nu Q = \nu$; then $\nu$ is a left eigenvector associated to $Q$ and the eigenvalue 1. The search of this eigenvector is difficult because of the high dimension of the matrix $Q$ (originated by the large cardinal number of $\Omega$) while in the sequential case, this search of the eigenvector is trivially solved since $\pi P = \pi$.

However, if the $\pi_i$ are the conditional distributions of the nearest neighbours Ising model, it is possible to write $\nu$; we detail this in the following example.

### 4.3.2 The Ising model

We consider the torus $S = \{1, 2, \cdots, n\}^2$, $n$ being even, equipped with the 4 nearest neighbours system. The generating distribution $\pi$ is the trimmed Gibbs law (1) :

$$\pi_i(x_i \mid x_{\partial i}) = \frac{\exp x_i(\alpha + \beta N_i(x))}{2\cosh(\alpha + \beta N_i(x))},$$

with $N_i(x) = \sum_{j:|i-j|=1} x_j$.

Then the transition for one synchronous scan is

$$Q(x,y) = \prod_{i\in S} \frac{\exp y_i(\alpha + \beta N_i(x)}{2\cosh(\alpha + \beta N_i(x))}$$

One can show that the invariant distribution $\nu$ for the transition $Q$ is

$$\nu(x) = \Gamma^{-1} \exp\{\alpha \sum_{i\in S} x_i\} \prod_{i\in S} \cosh(\alpha + \beta N_i(x))$$

where $\Gamma$ is a normalization constant.

Indeed, let's write

$$\nu(x)Q(x,y) = \Gamma^{-1} \exp\{\alpha \sum_{i\in S} x_i\} \exp\{\alpha \sum_{i\in S} y_i\} \prod_{i\in S} \exp \beta y_i N_i(x) \ ;$$

since for instance for $i = (s,t)$, $\sum y_{s,t}(x_{s-1,t} + x_{s+1,t}) = \sum x_{s,t}(y_{s-1,t} + y_{s+1,t})$, the previous equality is also

$$\nu(x)Q(x,y) = \Gamma^{-1} \exp\{\alpha \sum_{i\in S} x_i\} \exp\{\alpha \sum_{i\in S} y_i\} \prod_{i\in S} \exp \beta x_i N_i(y),$$

that is $\nu(x)Q(x,y) = \nu(y)Q(y,x)$. The transition matrix $Q$ is then $\nu-$reversible, which implies that $\nu$ is $Q-$invariant;

From this explicit expression for $\nu$, let us underline two important differences between the sequential and synchronous dynamics:

(i) We have seen that the $\pi_i$ are the conditional distribution probability of a Gibbs field on $\Omega$ with a joint distribution given by

$$\pi(x) = Z^{-1} \exp\{\sum_{i \in S} \alpha x_i + \beta \sum_{i \in S} \sum_{j:|i-j|=1} x_i x_j\}.$$

From this form, we see that $\pi$ is a Markov distribution and the cliques of the associated neighbourhood graph (the four nearest neighbours) are of order 1 and 2, made up with singletons and pairs of sites at distance 1, which are of two types, horizontal or vertical.

Similarly, we write

$$\nu(x) = \Gamma^{-1} \exp\{\sum_{i \in S} \alpha x_i + \sum_{i \in S} \log\{\cosh(\alpha + \beta \sum_{j:|i-j|=1} x_j)\}\}.$$

Then $\nu$ characterizes a Markov field, like $\pi$. But the neighbourhood system is quite different: the cliques are the singletons and the four nearest neighbours for $\pi$, while they are the singletons and the squares of sites at distance $\sqrt{2}$ for $\nu$.

(ii) Let us denote $S^+$ the subset of sites $i = (u, v)$ with $u + v$ even (the black fields on a chequer board), $S^-$ the complementary subset (the white fields), and $x^+$ (resp. $x^-$) the configuration on $S^+$ (resp. $S^-$).

We define

$$\nu^+(x^+) = \Gamma^{-\frac{1}{2}} \exp\{a \sum_{i \in S^+} x_i^+\} \prod_{i \in S^-} \cosh(a + bN_i(x^+)), \text{ and}$$
$$\nu^-(x^-) = \Gamma^{-\frac{1}{2}} \exp\{a \sum_{i \in S^-} x_i^-\} \prod_{i \in S^+} \cosh(a + bN_i(x^-)).$$

We have $\nu(x) = \nu^+(x^+)\nu^-(x^-)$: contrary to the sequential dynamics, the synchronous evolutions on $S^+$ and $S^-$ are independent from each other.

Figures 1 and 2 in section 5 illustrate the difference between these dynamics.

## 4.4 Partially synchronous dynamics

Let us denote $R$ the transition matrix of this dynamics. We also denote $\mathbf{1}$ the indicator function.

### 4.4.1 The general case

We choose the subset $A$ at random, for instance uniformly in the set of the subsets of cardinal number $M$ (with $\tau = \frac{M}{N}$). Therefore,

$$R(x, y) = \binom{N}{M}^{-1} \sum_{A \subset S:|A|=M} \left\{ \mathbf{1}(x^A = y^A) \prod_{k \in A} \pi_k(y_k \mid x^A) \right\}.$$

This transition is positive for every $x, y$. Hence the partially synchronous dynamics is ergodic, with the stationary distribution $\lambda_\tau$,

$$\forall x \in \Omega, R^k(x, y) \stackrel{k \to \infty}{\longrightarrow} \lambda_\tau(y).$$

If $M \geq 2$, we verify that $\pi$ is not invariant for $R$, and then $\lambda_\tau \neq \pi$.

Again, $\lambda_\tau$ is not explicit but we some properties established for simulated annealing apply.

**Proposition 2** *([15] Theorem 2.7 and [14])*
*(i) If the parallelism rate tends to zero, and for a fix value of the interaction parameter $\beta > 0$, we get a continuity property with*

$$\lim_{\tau \to 0} \lambda_\tau = \pi \ . \tag{4}$$

*(ii) On the other hand, if the interaction parameter $\beta$ tends to infinity, and for a fix value of the parallelism rate $\tau$, $0 < \tau < 1$, we get a limit distribution that does not depend anymore on the parallelism rate:*

$$\lim_{\beta \to +\infty} \lambda_\tau(\beta) = \lambda_\tau(\infty) = \lambda(\infty) \ \textit{if}\ 0 < \tau < 1 \tag{5}$$

*This is no more true for $\tau = 1$. Therefore we can have discontinuity for large $\beta$.*

These properties are illustrated by Figure 4 in the next section.

More generally, let us consider $A_1, ..., A_n$ some subsets of $S$; at each step, we choose a subset $A_i$ of $S$ with probability $\gamma(A_i) > 0$ and we update the sites of $A_i$; the parallelism rate is no longer fixed but determined by $\gamma$; the dynamics is ergodic with limit distribution $\lambda_\gamma$ if and only if $\cup_i A_i = S$ ([13]).

#### 4.4.2 The coding case

Let us assume that $\pi$ has a Markov property with respect to a neighbourhood graph. A subset $C$ of $S$ is called a coding subset if for all $i, j$ of $C$, $i \neq j$, $i$ and $j$ are not neighbours for the markovian structure. Let $\{C_1, C_2, ...C_K\}$ be a partition of coding subsets of $S$; at each step we choose a coding subset $C_k$ of $S$ and we update its $|C_k|$ sites. We define for $k = 1, ..., K$,

$$R_{C_k}(x, y) = \mathbf{1}(x^{C_k} = y^{C_k}) \prod_{s \in C_k} \pi_s(y_s^{C_k} \mid x).$$

Then the transition matrix is $R(x, y) = R_{C_1} ... R_{C_K}(x, y)$; but for each $k$ and each $C_k = \{s_1, ...s_{|C_k|}\}$ there is no interaction between the sites of $C_k$ that all simultaneously change, and all conditional probabilities $\pi_s(y_s \mid x^{C_k}) = \pi_s(y_s \mid x_{\partial s})$, $s \in C_k$, depend only of the values of $\overline{C_k}$; this leads to write $R_{C_k}(x, y) = \pi_{s_1}(y_{s_1} \mid x_{\partial s_1}) \pi_{s_2}(y_{s_2} \mid x_{\partial s_2}) ... \pi_{s_{|C_k|}}(y_{s_{|C_k|}} \mid x_{\partial s_{|C_k|}})$; finally $R$ coincides with the transition probability $P$ for a sequential sweep of $S$.

As for a simple example, consider the square lattice for $S$ with the four nearest neighbours system, $C$ is the subset of the "black" nodes, and $\overline{C}$ is the white ones. Obviously, $C$ and $\bar{C}$ are coding subsets and $S = C \cup \overline{C}$; changing simultaneously the black sites then the white ones leads to the same result as changing all the sites one by one.

# 5 Some illustrating examples

We propose a simulation experiment to illustrate the differences of the previous dynamics. As for the generating distribution we consider the Ising model defined in the previous section with the four or eight nearest neighbours system, with conditional distributions

$$\pi_i(x_i \mid x_{\partial i}) = \frac{\exp x_i(\alpha + \beta V_i(x) + \gamma W_i(x))}{2\cosh(a + \beta V_i(x) + \gamma W_i(x))}$$

where $V_i(x)$ is the sum of the four nearest neighbours (at distance 1) of site $i$ and $W_i(x)$ is the sum of the four diagonal neighbours (at distance $\sqrt{2}$) of site $i$.If $\gamma = 0$, then we come back to the four nearest neighbours. We will consider $\alpha = 0$, that is +1 and -1 occur with the same probability.

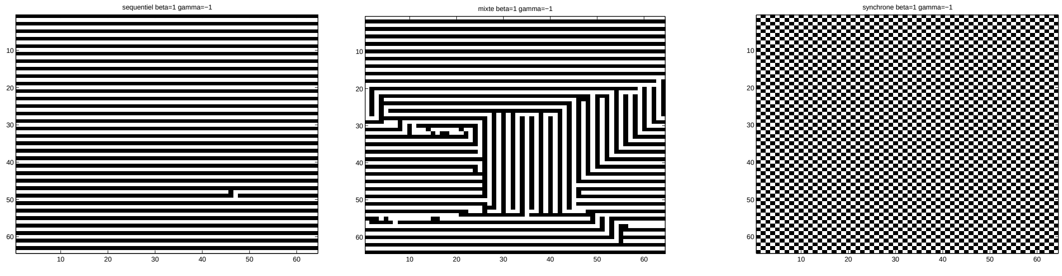
We initialize at random and for one initialization we simulate the three dynamics described above: the sequential, synchronous and partially synchronous dynamics. Each simulation of the sequential and synchronous dynamics is obtained running 600 scans on a square toric lattice of size $64 \times 64$. Of course, similar results occur for non toric lattices but we avoid here some disturbances which may be caused by edge effects.

In the case of partial parallelism, we iterate the scans until each site has been visited at least 600 times. We also compute the empirical spatial correlation $\hat{\rho}_1$ at distance 1 (based on the four nearest neighbours) and $\hat{\rho}_8$ based on the eight nearest neighbours.

The parameters are $\beta$ and $\gamma$, and the parallelism rate $\tau$. First we present below some examples of realizations obtained from the same generating distribution $\pi$ with the three algorithms, for different values of $\beta$ and $\gamma$ (the parallelism rate is fixed to $\tau = 50\%$). Figure 1 present resulting configurations which strongly visually differ, illustrating Proposition 1 ($\pi \neq \nu \neq \lambda_{0.5}$). We get horizontal stripes for the sequential algorithm; we still meet these stripes together with vertical ones in the partially synchronous case; but the totally synchronous layout is completely different and features a chessboard.

Fig. 1: Simulations of fields with parameters $\beta = 1,\ \gamma = -1$ for different dynamics

Sequential, $\tau = 0$ | Partially synchronous $\tau = 50\%$ | Synchronous, $\tau = 1$

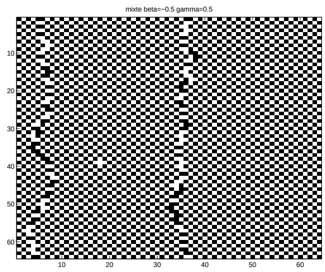


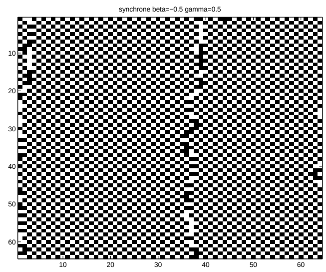

$\hat{\rho}_1 = 0,\ \hat{\rho}_8 = -0.4993$ | $\hat{\rho}_1 = 0.0083,\ \hat{\rho}_8 = -0.4624$ | $\hat{\rho}_1 = -1,\ \hat{\rho}_8 = 0$

On the other hand, for other sets of parameters, we may have similar final configurations; for instance we present in Figure 2 perfect or nearly chessboard images for $\pi$, $\nu$, and $\lambda_{0.5}$ for parameters $\beta = -0.5$ and $\gamma = +0.5$.

Fig. 2: Simulations of fields with parameters $\beta = -0.5,\ \gamma = 0.5$ for different dynamics

Sequential, $\tau = 0$ Partially synchronous $\tau = 50\%$ Synchronous, $\tau = 1$

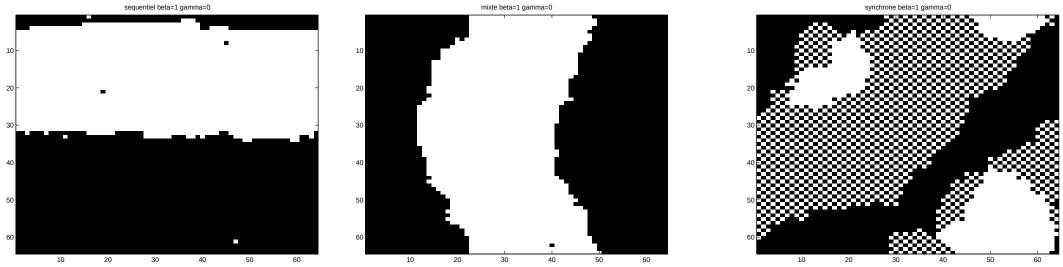

$\hat{\rho}_1 = -1,\ \hat{\rho}_8 = 0$ $\hat{\rho}_1 = -0.9497,\ \hat{\rho}_8 = -0.0083$ $\hat{\rho}_1 = -0.9497,\ \hat{\rho}_8 = -0.0076$

Most often in the context of standards' adoption, interaction parameters are positive, agents inclining to imitate their "neighbours"; in this case, we obtain similar configurations ensuing from the sequential and partially synchronous dynamics, unlike the synchronous case; Figure 3 illustrates this case with the set of parameters $\beta = 1,\ \gamma = 0$; similar phenomena occur for $\beta = \gamma = 0.5$ or $\beta = 0.5,\ \gamma = 0$, or $\beta = 0.3,\ \gamma = 0$; in each case, we get similar images for sequential and partially synchronous dynamics, the clustering being more or less emphasised with respect to the interaction parameter value; but the synchronous realization is different. It is not so clear for small values of $\beta$; for instance, Figure 3 (last row) presents realizations issued for $\beta = 0.2,\ \gamma = 0$; it seems now that the three configurations visually look like each other; and the final frequencies of sites +1 are very close, between 49.44% and 50.49%; however, the empirical correlations highlights the differences and isolates the synchronous case.

Fig. 3: Simulations of fields with positive interaction parameters for different dynamics

$\beta = 1,\ \gamma = 0$

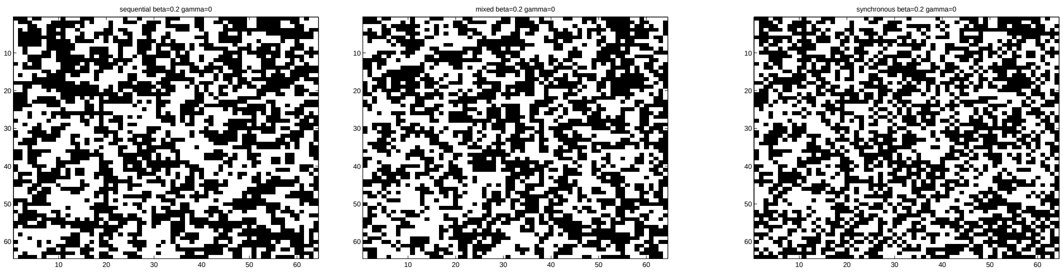
Sequential, $\tau = 0$ | Partially synchronous $\tau = 50\%$ | Synchronous, $\tau = 1$




$\hat{\rho}_1 = 0.9594,\ \hat{\rho}_8 = 0.9468$ | $\hat{\rho}_1 = 0.9520,\ \hat{\rho}_8 = 0.9419$ | $\hat{\rho}_1 = -0.1173,\ \hat{\rho}_8 = 0.3976$

$\beta = 0.2,\ \gamma = 0$

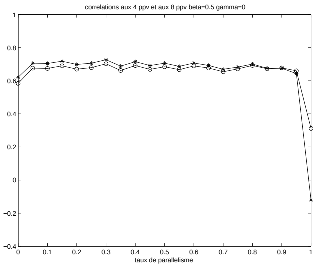
Sequential, $\tau = 0$ | Partially synchronous $\tau = 50\%$ | Synchronous, $\tau = 1$



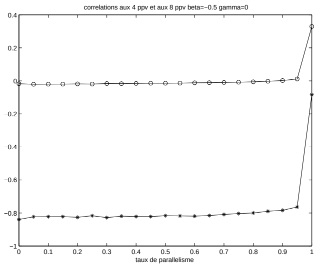

$\hat{\rho}_1 = 0.2411,\ \hat{\rho}_8 = 0.1679$ | $\hat{\rho}_1 = 0.1718,\ \hat{\rho}_8 = 0.1325$ | $\hat{\rho}_1 = 0.0243,\ \hat{\rho}_8 = 0.0597$

Then we consider only the partially synchronous dynamics. Coming back to the previous section, both limit distributions $\pi$ and $\nu$ are known for the Ising model (sequential and synchronous dynamics), but we can't characterize $\lambda_\tau$ for the partially synchronous dynamics. We repeat hundred times each simulation and compute the mean of the empirical correlations $\hat{\rho}_1$ and $\hat{\rho}_8$ We draw the evolution of those two correlations when $\tau$ increases from 0 to 1 by steps of 5%. We know from property (5) that if $\beta$ tends to infinity, the stationary distribution doesn't depend of $\tau,\ 0 < \tau < 1$; in practice, this means that we should observe some constancy for large values of $\beta$; in fact, we observe that the correlations are constant from 5% to 95% in the cases $\beta > 0$ ($\beta \geq 0.5$ is large enough) and $\gamma = 0$, but also if $\beta < 0$ ($\beta \leq -0.5$) and $\gamma = 0$. In both cases we observe constancy or a light gap between the sequential dynamics with $\tau = 0$ and the partial dynamics with $\tau = 5\%$, which illustrates property (4); most of all, we observe discontinuity between $\tau = 0.95$ and $\tau = 1$, as foreseen again by (5). The same thing occurs if we permute parameters $\beta$ and $\gamma$ (for instance $\beta = 0,\ \gamma = 1$ or $\beta = 0,\ \gamma = -1$), see Figure 4.

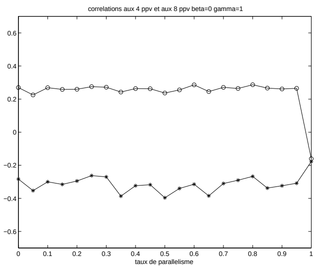
Fig. 4: Correlations $\hat{\rho}_1 - * -$ and $\hat{\rho}_8 - o -$



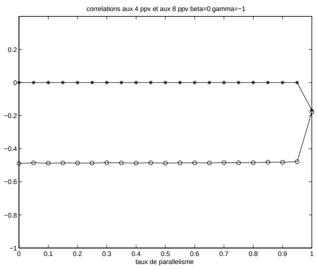

$\beta = 0.5,\ \gamma = 0$




$\beta = -0.5,\ \gamma = 0$



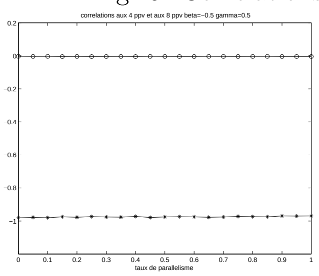

$\beta = 0,\ \gamma = 1$



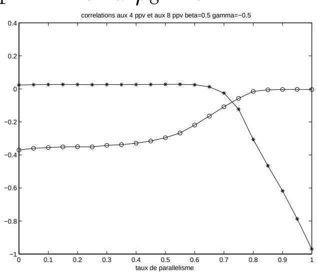

$\beta = 0,\ \gamma = -1$

Finally, if the parameters have different signs, we observe different behaviours; property (5) doesn't hold anymore. We observe constant correlations in the case $\beta = -0.5,\ \gamma = 0.5$; in fact the observed images at different parallelism rates are all similar to the sequential and synchronous dynamics cases, leading to chessboard like configurations (see Figure 5). On the other hand, if $\beta > 0$, and $\gamma < 0$, the behaviour of the correlations is quite different; the correlations are quite equal for small parallelism rate and then slowly meet the total synchronous correlations values. It seems there is a threshold rate from which the configurations change, from the correlations point of view (see Figure 5).

Fig. 5: Correlations $\hat{\rho}_1 - * -$ and $\hat{\rho}_8 - o -$




$\beta = -0.5,\ \gamma = 0.5$




$\beta = 0.5,\ \gamma = -0.5$

In conclusion, our simulations allow us to illustrate the theoretical results; we have shown that the choice of the dynamics is very important in standards adoption context, as well as in other application fields. Moreover, we point out that except in specific examples, the limit ergodic distributions remain unknown for the synchronous and partially synchronous choices; they coincide or strongly differ for specific values of the dynamics parameters, as presented in the previous examples. We may suppose that the partial synchronous case coincides with the sequential dynamics for positive interaction parameters, whenever the parallelism rate remains much less than 1 (less than more or less 90% to 95% in our simulations).

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