# Discussion of “Exact Analytical Solutions of the Colebrook-White Equation” by Yozo Mikata and Walter S. Walczak

Dejan Brkić

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## Discussion of "Exact Analytical Solutions of the Colebrook-White Equation" by Yozo Mikata and Walter S. Walczak, J. Hydraul. Eng. 04015050; doi. 10.1061/(ASCE)HY.1943-7900.0001074

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This discussion communicates comparisons, practical applications and accuracy check related to the closed-form solutions of the Colebrook-White equation (Colebrook 1939, Colebrook and White 1937) showed in the discussed paper and provides comments and references to already available similar works.

The first exact closed-form solution of the Colebrook-White equation in terms of Lambert W-function appears in Keady (1998) [Eq. (9) of the discussed paper]. Sonnad and Goudar (2004), confirmed also by Brkić (2012a), show constraints for the use of Keady's equation because it contains exponential form as the argument of the Lambert W-function (as defined in Hayes (2005), "W" represents the Lambert function). Very high numerical values caused by exponential form in Keady's case very often make impossible calculation using computers (these values are too high to be processed by registers of memory and processor unit). These cases are associated in engineering practice with high speed of flow through very rough pipes (higher numerical values of the Reynolds (R) and higher values the relative roughness of inner pipe surface (ε/D) occurred simultaneously in pairs). Rollman and Spindler (2015), referred to general properties of the Lambert W-function (Corless et al. 1996), give their solution in order to avoid such exponential form in the transformed Colebrook-White equation (similar as Eq. (14) of the discussed paper). To overwhelm such inconvenience Brkić (2011a) also transforms the Colebrook-White equation using Lambert W-function avoiding exponential term (1):

$$\frac{1}{\sqrt{f}} = -2 \cdot \log_{10}\left( \frac{2 \cdot 2.51 \cdot W\left[\frac{\ln 10}{2} \cdot \frac{R}{2.51}\right]}{R \cdot \ln 10} + \frac{\varepsilon}{3.71 \cdot D} \right) = -2 \cdot \log_{10}\left( 10^{\frac{-W\left[\frac{\ln 10}{2} \cdot \frac{R}{2.51}\right]}{\ln 10}} + \frac{\varepsilon}{3.71 \cdot D} \right) \qquad (1)$$

To develop (1), Brkić (2011a) use limiting case of the Colebrook-White equation under smooth pipe law of flow as showed in Appendix I of the discussed paper, where $\varepsilon/D \rightarrow 0$ (Goudar and Sonnad 2003, Sonnad and Goudar 2006, Brkić 2011b; 2012b). To make Eq. (1) more applicable for engineering practice, Brkić (2011c) replaces Lambert W-function with approximate calculus. Further for the purpose of this discussion, numerical values of the parameters from this explicit approximation are optimized using genetic algorithms developed by Ćojbašić and Brkić (2013) where the relative error, $\delta(\%) = (|f - f_0| / f_0) \cdot 100\%$, decreases to 1.28% (before optimization it was 2.2% and 3.16%, respectively referred to Eq. (2)):

$$\left.\begin{aligned} &\frac{1}{\sqrt{f}} \approx -2.013 \cdot \log_{10}\left( \frac{2.261 \cdot A}{R} + \frac{1}{3.71} \cdot \frac{\varepsilon}{D} \right) \approx -2.013 \cdot \log_{10}\left( 10^{-0.43 \cdot A} + \frac{1}{3.71} \cdot \frac{\varepsilon}{D} \right) \\ &A \approx \ln \frac{R}{2.479 \cdot \ln\left( \frac{1.1 \cdot R}{\ln(1 + 1.1 \cdot R)} \right)} \end{aligned}\right\} \qquad (2)$$

Regarding soft computation techniques such as optimization through genetic algorithms, it should be noted that the Colebrook-White equation can be simulated very accurately using Artificial Neural Networks as showed in Brkić and Ćojbašić (2016).

The *n*th formula is developed (Eq. (26) of the discussed paper) using similar approach as in Brkić (2011a, 2012a). In this case, the Boyd's "shifted" Lambert W-function (Boyd 1998) is used which is noted as the Y function the discussed paper. In addition to the solution based on Boyd's function, Brkić (2011a, 2012a,c) presents some further solutions based on works of Barry et al. (2000) and

Winitzki (2003). Also, Brkić (2011a, 2012a,c) uses the series expansion of the Lambert W-function in a similar way as it is done for of Y function (Eq. (44) of the discussed paper).

Approximations of the Colebrook-White equation based on the *n*th formula, Eq (21) of the discussed paper, are given by (3).

$$\left.\begin{aligned}
\frac{1}{\sqrt{f_{n=1}}} &\approx \frac{2}{\ln 10}\cdot\left[\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)-\ln\left(x_1-\ln x_1\right)\right]\\
\frac{1}{\sqrt{f_{n=2}}} &\approx \frac{2}{\ln 10}\cdot\left[\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)-\ln\left(x_1-\ln\left(x_1-\ln x_1\right)\right)\right]\\
\frac{1}{\sqrt{f_{n=3}}} &\approx \frac{2}{\ln 10}\cdot\left[\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)-\ln\left(x_1-\ln\left(x_1-\ln\left(x_1-\ln x_1\right)\right)\right)\right]\\
&\vdots\\
\frac{1}{\sqrt{f_{n}}} &\approx \frac{2}{\ln 10}\cdot\left[\ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)-\ln\left(x_1-\ln\left(x_1-\ln\left(x_1-\underbrace{\cdots-\ln x_1\cdots}_{n-1}\right)\right)\right)\right]\\
x_1 &= \ln\left(\frac{\ln 10}{2}\cdot\frac{R}{2.51}\right)+\frac{\ln 10}{2}\cdot\frac{R}{2.51}\cdot\frac{\varepsilon}{3.71\cdot D}
\end{aligned}\right\} \quad (3)$$

Using methodology from Brkić (2011d), the maximal percentage relative error is evaluated. It is 1.1877% for n=1, 0.1826% for n=2, 0.0278% for n=3, 0.004249% for n=4, $6.48\cdot 10^{-4}$% for n=4, $9.89\cdot 10^{-5}$% for n=5, etc. The error aggregates in the zone of small values of the Reynolds number (R) and relative roughness (ε/D) and it decreases with geometric progression. On the other hand, these approximations are computationally demanded since they contain many logarithmic terms (Clamond 2009, Giustolisi et al. 2011).

**Notation**

The following symbols are used in this discussion:

R – Reynolds number (dimensionless)

ε/D – Relative roughness of inner pipe surface (dimensionless)

f – Darcy (Moody) flow friction factor (dimensionless)

"=" – exactly equivalent to the Colebrook-White equation

"≈"– approximately equivalent to the Colebrook-White equation

$f_0$ – Darcy (Moody) flow friction factor (dimensionless); obtained from the Colebrook-White equation using iterative procedure and hence treated as accurate

$\delta(\%)=(|f-f_0|/f_0)\cdot 100\%$ - relative error (%)

W – Lambert function

$X_1$-parameter defined by Eq. (14) of the discussed paper (here in Eq. 3)

A-auxiliary term used in Eq. 2

'ln' denotes the natural log function



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