Convergence of the Finite Volume Method for scalar conservation laws with multiplicative noise: an approach by kinetic formulation

Sylvain Dotti, Julien Vovelle

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July 9, 2019

Abstract

Under a standard CFL condition, we prove the convergence of the explicit-in-time Finite Volume method with monotone fluxes for the approximation of scalar first-order conservation laws with multiplicative, compactly supported noise. In [9], a framework for the analysis of the convergence of approximations to stochastic scalar first-order conservation laws has been developed, on the basis of a kinetic formulation. Here, we give a kinetic formulation of the numerical method, analyse its properties, and explain how to cast the problem of convergence of the numerical scheme into the framework of [9]. This uses standard estimates (like the so-called "weak BV estimate", for which we give a proof using specifically the kinetic formulation) and an adequate interpolation procedure.

Keywords: Finite Volume Method, stochastic scalar conservation law, kinetic formulation
MSC Number: 65M08 (35L60 35L65 35R60 60H15 65M12)

Contents

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1 Introduction

Stochastic first-order scalar conservation law. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis and let $T > 0$. Consider the first-order scalar conservation law with stochastic forcing

$$du(x,t) + \operatorname{div}(A(u(x,t)))dt = \Phi(x, u(x,t))dW(t), \quad x \in \mathbb{T}^N, t \in (0,T). \tag{1.1}$$

Equation (1.1) is periodic in the space variable: $x \in \mathbb{T}^N$ where $\mathbb{T}^N$ is the N-dimensional torus. The flux function A in (1.1) is supposed to be of class C^2: $A \in C^2(\mathbb{R}; \mathbb{R}^N)$. We assume that A and its derivatives have at most polynomial growth. Without loss of generality, we will assume also that $A(0) = 0$. The right-hand side of (1.1) is a stochastic increment in infinite dimension. It is defined as follows (see [7] for the general theory): W is a cylindrical Wiener process, $W = \sum_{k\geq 1} \beta_k e_k$, where the coefficients β_k are independent Brownian processes and $(e_k)_{k\geq 1}$ is a complete orthonormal system in a Hilbert space H. For each $x \in \mathbb{T}^N$, $u \in \mathbb{R}$, $\Phi(x,u) \in L_2(H, \mathbb{R})$ is defined by $\Phi(x,u)e_k = g_k(x,u)$ where $g_k(\cdot, u)$ is a regular function on $\mathbb{T}^N$. Here, $L_2(H,K)$ denotes the set of Hilbert-Schmidt operators from the Hilbert space H to an other Hilbert space K . Since $K = \mathbb{R}$ in our case, this set is isomorphic to H, thus we may also define

$$\Phi(x, u) = \sum_{k \geq 1} g_k(x, u) e_k,$$

the action of $\Phi(x, u)$ on $e \in H$ being given by $\langle \Phi(x, u), e \rangle_H$. We assume $g_k \in C(\mathbb{T}^N \times \mathbb{R})$, with the bounds

$$\mathbf{G}^2(x, u) = \|\Phi(x, u)\|_H^2 = \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0(1 + |u|^2), \tag{1.2}$$

$$\|\Phi(x, u) - \Phi(y, v)\|_H^2 = \sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D_1(|x - y|^2 + |u - v| h(|u - v|)), \tag{1.3}$$

where $x, y \in \mathbb{T}^N$, $u, v \in \mathbb{R}$, and h is a continuous non-decreasing function on $\mathbb{R}_+$ such that $h(0) = 0$. We assume also $0 \leq h(z) \leq 1$ for all $z \in \mathbb{R}_+$.

Notation: in what follows, we will use the convention of summation over repeated indices k. For example, we write $W = \beta_k e_k$.

Compactly supported multiplicative noise. In this paper, we study the numerical approximation of (1.1): our aim is to prove the convergence of the Finite Volume method with monotone fluxes, see our main result, Theorem 7.4. Our analysis will be restricted to the case of *multiplicative noise with compact support*. Indeed, from Section 3 to Section 7, we will work under the following hypothesis: there exists $a, b \in \mathbb{R}$, $a < b$, such that $g_k(x, u) = 0$ for all u outside the compact $[a, b]$, for all $x \in \mathbb{T}^N$, $k \geq 1$. For simplicity, we will take $a = -1$, $b = 1$. We will assume therefore that

$$\text{for all } u \in \mathbb{R}, \ |u| \geq 1 \Rightarrow g_k(x, u) = 0, \tag{1.4}$$

for all $x \in \mathbb{T}^N$, $k \geq 1$, and consider initial data with values in $[-1, 1]$. The solution of the continuous equation (1.1) then takes values in $[-1, 1]$ almost surely (see [9, Theorem 22]). There is no loss in generality in considering that A is globally Lipschitz continuous then:

$$\mathrm{Lip}(A) := \sup_{\xi \in \mathbb{R}} |A'(\xi)| < +\infty. \tag{1.5}$$

In that framework, we will build a stable and convergent approximation to (1.1) by an explicit-in-time Finite Volume method. Under (1.4), it is also natural to assume

$$\mathbf{G}^2(x, u) = \|\Phi(x, u)\|_H^2 = \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0, \tag{1.6}$$

which is of course stronger than (1.2). In what follows, we will assume that (1.6) is satisfied.

Let us do some comments about our hypotheses. Consider the general framework where the flux A is only locally Lipschitz continuous and the noise is not compactly supported. In that situation, the Courant-Friedrichs-Lewy (CFL) condition depends effectively on the L^∞-norm of the numerical approximation (see Remark 4.2). We do not know how to control this L^∞-norm (for the continuous problem, only L^p-norms, with an arbitrary finite p are controlled). This is why we require the flux A to be globally Lipschitz continuous. We may actually assume solely (1.5), without considering that the noise is compactly supported. However, if (1.4) is a reasonable assumption in our opinion, (1.5) without much specification seems to us too stringent. It rules out the case of the Burgers equation for example.

Numerical approximation. Let us give a very brief summary of the theory for (1.1) and of its approximation by numerical methods. Different approximation schemes to stochastically forced first-order conservation laws have already been analysed: time-discrete schemes, [17, 1, 18], space-discrete scheme [20], and space-time Finite Volume discrete schemes. For this latter kind of numerical approximation (space-time discrete schemes), there already exists some results of convergence of the method in the literature in various frameworks:

- in space dimension 1, with strongly monotone fluxes, [21],
- in space dimension $N \geq 1$, by a flux-splitting scheme, [2],
- in space dimension $N \geq 1$, for general schemes with monotone fluxes, [3].

The Cauchy or the Cauchy-Dirichlet problem associated to the continuous problem (1.1) have been studied in [10, 19, 12, 32, 8, 6, 4, 5, 18].

The approximation of scalar conservation laws with stochastic flux has also been considered in [15] (time-discrete scheme) and [28] (space discrete scheme). For the corresponding Cauchy Problem, see [23, 22, 24, 14, 13, 16].

The reference [3] gives a result very close to the convergence statement in Theorem 7.4. In [3] the flux A, which may also depend on (t,x), is supposed to be globally Lipschitz continuous. The noise is restricted to one mode ($g_k = 0$ if $k \geq 2$). The convergence of the Finite Volume method with monotone fluxes is obtained under a CFL condition of the type: time step $\Delta t = o(h)$, where h is the space step. In the present paper, we are able to establish the convergence of the Finite Volume Method under the standard CFL condition $\Delta t = \mathcal{O}(h)$, thus relaxing the $o(h)$ to a $\mathcal{O}(h)$, see Theorem 7.4. In our opinion, this is the essential difference between our result and the one obtained in [3]. The fact that the flux may depend on (t,x), or the number of modes affected by the noise are not decisive here. Let us give some additional comments on those two different CFL conditions. First note that they can hardly be distinguished numerically: h and $h/|\ln(|\ln(h)|)|$ are quasi similar for very small values of h. However, the fact is that our approach, which uses the kinetic formulation of the Finite Volume scheme, contrary to [3], where an approach based on entropy characterization is used, is able to yield the expected convergence result under the standard CFL condition $\Delta t = \mathcal{O}(h)$. Eventually, a last methodological difference between this present paper and the paper [3] has to be emphasized: in [3], Young measures and weak convergence with respect to the whole set of variables (ω, t, x) are used. In the paper [9], which we use to establish our convergence result, the weak mode of convergence with respect to $\omega \in \Omega$ that is used is the usual weak probabilistic mode, that is convergence in law.

Kinetic formulation. To prove the convergence of the Finite Volume method with monotone fluxes, we will use the companion paper [9] and a kinetic formulation of the Finite Volume scheme. The subject of [9] is the convergence of approximations to (1.1) in the context of the kinetic formulation of scalar conservation laws. Such kinetic formulations have been developed in [25, 26, 27, 29, 30]. In [27], a kinetic formulation of Finite Volume E-schemes is given (and applied in particular to obtain sharp CFL criteria). For Finite Volume schemes with monotone fluxes, the kinetic formulation is simpler, we give it explicitly in Proposition 4.1. Based on the kinetic formulation and an energy estimate, we derive some a priori bounds on the numerical approximation (these are "weak BV estimates" in the terminology of [11, Lemma 25.2]), see Section 5. These estimates are used in the proof of consistency of the scheme when we show that it gives rise to an approximate solution to (1.1) in the sense of Definition 2.6. There is a difference between the approach of [3] and our approach that we would like to emphasize here. The use of the kinetic formulation and of the theory in [9] gives some flexibility insofar as the sequence of approximate generalized solutions (see Definition 2.6) generated by the numerical scheme may not be at equilibrium (here, we refer to Definition 2.4). We take profit of this observation in particular in (4.27), when we build the interpolation f_δ. Compare to the interpolation [3, Section 5.1]. Let us also highlight the fact that we give a proof of the "weak BV estimate" slightly different from the one in [11, Lemma 25.2], since it is based solely on the kinetic formulation of the scheme. See Section 5.2.

Plan of the paper. The plan of the paper is the following one. In the preliminary section 2, we give a brief summary of the notion of solution and approximate solution to (1.1) developed in [9]. In Section 3, we describe the kind of approximation to (1.1) by the Finite Volume method that we consider here. In Section 4 we establish the kinetic formulation of the scheme. This numerical kinetic formulation is analysed as follows: energy estimates are derived in Section 5, then we show in Section 6 that this gives rise to an approximate generalized solution in the sense of Definition 2.6. We show some additional estimates and then conclude to the convergence of the scheme in Section 7. This result is stated in Theorem 7.4.

2 Generalized solutions, approximate solutions

The object of this section is to recall several results concerning the solutions to the Cauchy Problem associated to (1.1) and their approximations. We give the main statements, without much explanations or comments; those latter can be found in [9]: we give the precise references when needed.

2.1 Solutions

Definition 2.1 (Random measure)**.** Let $\mathcal{M}_b^+\left(\mathbb{T}^N \times [0,T] \times \mathbb{R}\right)$ be the set of bounded Borel non-negative measures. If m is a map from Ω to $\mathcal{M}_b^+\left(\mathbb{T}^N \times [0,T] \times \mathbb{R}\right)$ such that, for each continuous and bounded function ϕ on $\mathbb{T}^N \times [0,T] \times \mathbb{R}$, $\langle m, \phi\rangle$ is a random variable, then we say that m is a random measure on $\mathbb{T}^N \times [0,T] \times \mathbb{R}$.

A random measure m is said to have a finite first moment if

$$\mathbb{E}m(\mathbb{T}^N \times [0,T] \times \mathbb{R}) < +\infty. \tag{2.1}$$

Definition 2.2 (Solution)**.** Let $u_0 \in L^\infty(\mathbb{T}^N)$. An $L^1(\mathbb{T}^N)$-valued stochastic process $(u(t))_{t\in[0,T]}$ is said to be a solution to (1.1) with initial datum u_0 if u and $\mathtt{f} := \mathbf{1}_{u>\xi}$ have the following properties:

1. $u \in L^1_{\mathcal{P}}(\mathbb{T}^N \times [0,T] \times \Omega)$,
2. for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, almost surely, $t \mapsto \langle \mathtt{f}(t), \varphi\rangle$ is càdlàg,
3. for all $p \in [1, +\infty)$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|u(t)\|^p_{L^p(\mathbb{T}^N)}\right) \leq C_p, \tag{2.2}$$

4. there exists a random measure m with finite first moment (2.1), such that for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, for all $t \in [0,T]$,

$$\begin{aligned}\langle \mathtt{f}(t), \varphi\rangle = \langle \mathtt{f}_0, \varphi\rangle &+ \int_0^t \langle \mathtt{f}(s), a(\xi)\cdot\nabla\varphi\rangle ds \\ &+ \sum_{k\geq1}\int_0^t \int_{\mathbb{T}^N} g_k(x,u(x,s))\varphi(x,u(x,s))dx d\beta_k(s) \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{T}^N} \partial_\xi\varphi(x,u(x,s))\mathbf{G}^2(x,u(x,s))dxds - m(\partial_\xi\varphi)([0,t]),\end{aligned} \tag{2.3}$$

a.s., where $\mathtt{f}_0(x,\xi) = \mathbf{1}_{u_0(x)>\xi}$, $\mathbf{G}^2 := \sum_{k=1}^\infty |g_k|^2$ and $a(\xi) := A'(\xi)$.

In item 1, the index $\mathcal{P}$ in $u \in L^1_{\mathcal{P}}(\mathbb{T}^N \times [0,T] \times \Omega)$ means that u is predictable. See [9, Section 2.1.1]. The function denoted $\mathtt{f} := \mathbf{1}_{u>\xi}$ is given more precisely by

$$(x,t,\xi) \mapsto \mathbf{1}_{u(x,t)>\xi}.$$

This is the characteristic function of the subgraph of u. To study the stability of solutions, or the convergence of approximate solutions (these are two similar problems), we have to consider the stability of this property, the fact of being the " characteristic function of the subgraph of a function". If (u_n) is a sequence of functions, say on a finite measure space X, $p \in (1,\infty)$ and (u_n) is bounded in $L^p(X)$, then there is a subsequence still denoted (u_n) which converges to a function u in $L^p(X)$-weak. Up to a subsequence, the sequence of equilibrium functions $\mathtt{f}_n := \mathbf{1}_{u_n>\xi}$ is converging to a function f in $L^\infty(X\times\mathbb{R})$-weak star. The limit f is equal to $\mathtt{f} := \mathbf{1}_{u>\xi}$ if, and only if, (u_n) is converging strongly, see [9, Lemma 2.6]. When strong convergence remains a priori unknown, the limit f still keeps some structural properties. This is a kinetic function in the sense of Definition 2.4 below, [9, Corollary 2.5]. Our notion of generalized solution is based on this notion.

2.2 Generalized solutions

Definition 2.3 (Young measure)**.** Let $(X,\mathcal{A},\lambda)$ be a finite measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on $\mathbb{R}$. We say that a map $\nu\colon X\to\mathcal{P}_1(\mathbb{R})$ is a Young measure on X if, for all $\phi\in C_b(\mathbb{R})$, the map $z\mapsto\nu_z(\phi)$ from X to $\mathbb{R}$ is measurable. We say that a Young measure ν vanishes at infinity if, for every $p\geq 1$,

$$\int_X\int_{\mathbb{R}}|\xi|^p d\nu_z(\xi)d\lambda(z)<+\infty. \tag{2.4}$$

Definition 2.4 (Kinetic function)**.** Let $(X,\mathcal{A},\lambda)$ be a finite measure space. A measurable function $f\colon X\times\mathbb{R}\to[0,1]$ is said to be a kinetic function if there exists a Young measure ν on X that vanishes at infinity such that, for λ-a.e. $z\in X$, for all $\xi\in\mathbb{R}$,

$$f(z,\xi)=\nu_z(\xi,+\infty).$$

We say that f is an equilibrium if there exists a measurable function $u\colon X\to\mathbb{R}$ such that $f(z,\xi)=\mathtt{f}(z,\xi)=\mathbf{1}_{u(z)>\xi}$ a.e., or, equivalently, $\nu_z=\delta_{\xi=u(z)}$ for a.e. $z\in X$.

Definition 2.5 (Generalized solution)**.** Let $f_0\colon\mathbb{T}^N\times\mathbb{R}\to[0,1]$ be a kinetic function. An $L^\infty(\mathbb{T}^N\times\mathbb{R};[0,1])$-valued process $(f(t))_{t\in[0,T]}$ is said to be a generalized solution to (1.1) with initial datum f_0 if $f(t)$ and $\nu_t := -\partial_\xi f(t)$ have the following properties:

1. for all $t\in[0,T]$, almost surely, $f(t)$ is a kinetic function, and, for all $R>0$, $f\in L^1_{\mathcal{P}}(\mathbb{T}^N\times(0,T)\times(-R,R)\times\Omega)$,
2. for all $\varphi\in C_c^\infty(\mathbb{T}^N\times\mathbb{R})$, almost surely, the map $t\mapsto\langle f(t),\varphi\rangle$ is càdlàg,
3. for all $p\in[1,+\infty)$, there exists $C_p\geq 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|\xi|^p d\nu_{x,t}(\xi)dx\right)\leq C_p, \tag{2.5}$$

4. there exists a random measure m with first moment (2.1), such that for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, for all $t \in [0,T]$, almost surely,

$$\begin{aligned}\langle f(t), \varphi\rangle =& \langle f_0, \varphi\rangle + \int_0^t \langle f(s), a(\xi)\cdot\nabla_x\varphi\rangle ds \\ &+ \int_0^t \int_{\mathbb{T}^N}\int_{\mathbb{R}} g_k(x,\xi)\varphi(x,\xi)d\nu_{x,s}(\xi)dxd\beta_k(s) \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{T}^N}\int_{\mathbb{R}} \mathbf{G}^2(x,\xi)\partial_\xi\varphi(x,\xi)d\nu_{x,s}(\xi)dxds - m(\partial_\xi\varphi)([0,t]).\end{aligned} \tag{2.6}$$

The following statement is Theorem 3.2. in [9]. Note that we use the following terminology, what we call *solution* is a stochastic process u as in Definition 2.2. What we call *generalized* solution is a stochastic process f as in Definition 2.5.

Theorem 2.1 (Uniqueness, Reduction). *Let $u_0 \in L^\infty(\mathbb{T}^N)$. Assume (1.2)-(1.3). Then we have the following results:*

1. *there is at most one solution u with initial datum u_0 to (1.1).*
2. *If f is a generalized solution to (1.1) with initial datum f_0* at equilibrium: *$f_0 = \mathbf{1}_{u_0>\xi}$, then there exists a solution u to (1.1) with initial datum u_0 such that $f(x,t,\xi) = \mathbf{1}_{u(x,t)>\xi}$ a.s., for a.e. (x,t,ξ).*
3. *if u_1, u_2 are two solutions to (1.1) associated to the initial data $u_{1,0}$, $u_{2,0} \in L^\infty(\mathbb{T}^N)$ respectively, then*

$$\mathbb{E}\|(u_1(t)-u_2(t))^+\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E}\|(u_{1,0}-u_{2,0})^+\|_{L^1(\mathbb{T}^N)}. \tag{2.7}$$

This implies the L^1-contraction property, and comparison principle for solutions.

2.3 Approximate solutions

In [9], we give a general method and sufficient conditions for the convergence of sequences of approximations of (1.1). The solutions of these approximate problems give rise to approximate solutions and, more precisely, to approximate generalized solutions, according to the following definition (see Definition 4.1 and Section 5 in [9]).

Definition 2.6 (Approximate generalized solutions)**.** For each $n \in \mathbb{N}$, let $f_0^n : \mathbb{T}^N \times \mathbb{R} \to [0,1]$ be a kinetic function. Let $\left((f^n(t))_{t\in[0,T]}\right)_{n\in\mathbb{N}}$ be a sequence of $L^\infty(\mathbb{T}^N \times \mathbb{R}; [0,1])$-valued processes. Assume that the functions $f^n(t)$, and the associated Young measures $\nu_t^n = -\partial_\xi f^n(t)$ are satisfying item 1, 2, 3, in Definition 2.5 and Equation (2.6) up to an error term, *i.e.*: for each $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R}), n \in \mathbb{N}$, there exists an adapted process $\varepsilon^n(t,\varphi)$, with $t \mapsto \varepsilon^n(t,\varphi)$ almost surely continuous. Furthermore, assume that the sequence $(\varepsilon^n(t,\varphi))_{n\in\mathbb{N}}$ satisfies

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} |\varepsilon^n(t,\varphi)| = 0 \text{ in probability,} \tag{2.8}$$

and that there exists some random measures m^n with first moment (2.1), such that, for all n, for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, for all $t \in [0,T]$, almost surely,

$$\begin{aligned}\langle f^n(t),\varphi\rangle =& \varepsilon^n(t,\varphi) + \langle f_0^n,\varphi\rangle + \int_0^t \langle f^n(s), a(\xi)\cdot\nabla_x\varphi\rangle ds \\ &+ \int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} g_k(x,\xi)\varphi(x,\xi)d\nu^n_{x,s}(\xi)dxd\beta_k(s) \\ &+ \frac{1}{2}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} \mathbf{G}^2(x,\xi)\partial_\xi\varphi(x,\xi)d\nu^n_{x,s}(\xi)dxds - m^n(\partial_\xi\varphi)([0,t]).\end{aligned} \tag{2.9}$$

Then we say that (f^n) is a sequence of approximate generalized solutions to (1.1) with initial datum f_0^n.

Consider a sequence (f_n) of approximate solutions to (1.1) satisfying the following (minimal) bounds.

1. There exists $C_p \geq 0$ independent of n such that $\nu^n := -\partial_\xi f^n$ satisfies

$$\mathbb{E}\left[\sup_{t\in[0,T]}\int_{\mathbb{T}^N}\int_{\mathbb{R}}|\xi|^p d\nu^n_{x,t}(\xi)dx\right] \leq C_p, \tag{2.10}$$

2. the measures $\mathbb{E}m^n$ satisfy the bound

$$\sup_n \mathbb{E}m^n(\mathbb{T}^N\times[0,T]\times\mathbb{R}) < +\infty, \tag{2.11}$$

   and the following tightness condition: if $B_R^c = \{\xi\in\mathbb{R}, |\xi|\geq R\}$, then

$$\lim_{R\to+\infty}\sup_n \mathbb{E}m^n(\mathbb{T}^N\times[0,T]\times B_R^c) = 0. \tag{2.12}$$

We give in [9] the proof of the following convergence result, see Theorem 40 in [9].

Theorem 2.2 (Convergence)**.** *Suppose that there exists a sequence of approximate generalized solutions (f^n) to (1.1) with initial datum f_0^n satisfying (2.10), (2.11) and the tightness condition (2.12) and such that (f_0^n) converges to the equilibrium function $\mathtt{f}_0(\xi) = \mathbf{1}_{u_0>\xi}$ in $L^\infty(\mathbb{T}^N\times\mathbb{R})$-weak-*, where $u_0 \in L^\infty(\mathbb{T}^N)$. Then*

1. *there exists a unique solution $u \in L^1(\mathbb{T}^N\times[0,T]\times\Omega)$ to (1.1) with initial datum u_0;*
2. *let*

$$u^n(x,t) = \int_{\mathbb{R}}\xi d\nu^n_{x,t}(\xi) = \int_{\mathbb{R}}(f^n(x,t,\xi) - \mathbf{1}_{0>\xi})\,d\xi.$$

   Then, for all $p\in[1,\infty[$, (u^n) is converging to u with the following two different modes of convergence: $u_n \to u$ in $L^p(\mathbb{T}^N\times(0,T)\times\Omega)$ and almost surely, for all $t\in[0,T]$, $u_n(t)\to u(t)$ in $L^p(\mathbb{T}^N)$.

In the next section, we define the numerical approximation to (1.1) by the Finite Volume method. To prove the convergence of the method, we will show that the hypotheses of Theorem 2.2 are satisfied. The most difficult part in this programme is to prove that the numerical approximations generate a sequence of approximate generalized solutions, see Section 6.

3 The finite volume scheme

Mesh. A mesh of $\mathbb{T}^N$ is a family $\mathcal{T}_\#$ of disjoint connected open subsets $K \in (0,1)^N$ which form a partition of $(0,1)^N$ up to a negligible set. We denote by $\mathcal{T}$ the mesh

$$\{K + l; l \in \mathbb{Z}^N, K \in \mathcal{T}_\#\}$$

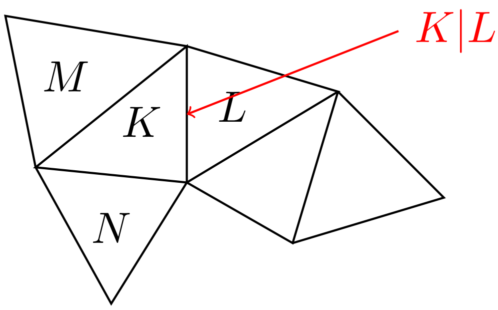
deduced on $\mathbb{R}^N$. For all distinct $K, L \in \mathcal{T}$, we assume that $\overline{K} \cap \overline{L}$ is contained in an hyperplane; the interface between K and L is denoted $K|L := \overline{K} \cap \overline{L}$. The set of neighbours of K is

$$\mathcal{N}(K) = \{L \in \mathcal{T}; L \neq K,\ K|L \neq \emptyset\}.$$

We use also the notation

$$\partial K = \bigcup_{L \in \mathcal{N}(K)} K|L.$$

In general, there should be no confusion between ∂K and the topological boundary $\overline{K} \setminus K$.



We also denote by $|K|$ the N-dimensional Lebesgue Measure of K and by $|\partial K|$ (respectively $|K|L|$) the $(N-1)$-dimensional Haussdorff measure of ∂K (respectively of $K|L$) (the $(N-1)$-dimensional Haussdorff measure is normalized to coincide with the $(N-1)$-dimensional Lebesgue measure on hyperplanes).

Scheme Let $(A_{K \to L})_{K \in \mathcal{T}, L \in \mathcal{N}(K)}$ be a family of monotone, Lipschitz continuous numerical fluxes, consistent with A. We assume that each function $A_{K \to L} : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following properties.

- Monotonicity: $A_{K \to L}(v, w) \leq A_{K \to L}(v', w)$ for all v, v', $w \in \mathbb{R}$ with $v \leq v'$ and $A_{K \to L}(v, w) \geq A_{K \to L}(v, w')$ for all v, w, $w' \in \mathbb{R}$ with $w \leq w'$.

- Lipschitz regularity: there exists $L_A \geq 0$ such that

$$|A_{K\to L}(v,w) - A_{K\to L}(v',w')| \leq |K|L|L_A(|v-v'| + |w-w'|), \tag{3.1}$$

  for all v, v', w, $w' \in \mathbb{R}$.

- Consistency:

$$A_{K\to L}(v,v) = \int_{K|L} A(v)\cdot n_{K,L} d\mathcal{H}^{N-1} = |K|L|A(v)\cdot n_{K,L}, \tag{3.2}$$

  for all $v \in \mathbb{R}$, where $n_{K,L}$ is the outward unit normal to K on $K|L$.

- Conservative symmetry :

$$A_{K\to L}(v,w) = -A_{L\to K}(w,v), \tag{3.3}$$

  for all $K, L \in \mathcal{T}$, $v, w \in \mathbb{R}$.

The conservative symmetry property ensures that the numerical flux $Q^n_{K\to L}$ defined below in (3.6) satisfies $Q^n_{K\to L} = -Q^n_{L\to K}$ for all K, L.

Let $0 = t_0 < ... < t_n < t_{n+1} < ... < t_{N_T} = T$ be a partition of the time interval $[0,T]$, with $N_T \in \mathbb{N}^*$. Given two discrete times t_n and t_{n+1}, we define $\Delta t_n = t_{n+1} - t_n$ for each $n \in \{0, ..., N_T - 1\}$. Knowing v^n_K, an approximation of the value of the solution u to (1.1) in the cell K at time t_n, we compute v^{n+1}_K, the approximation to the value of u in K at the next time step t_{n+1}, by the formula

$$|K|(v^{n+1}_K - v^n_K) + \Delta t_n \sum_{L\in\mathcal{N}(K)} Q^n_{K\to L} = |K|(\Delta t_n)^{1/2} g_{k,K}(v^n_K) X^{n+1}_k, \tag{3.4}$$

where $Q^n_{K\to L}$, $g_{k,K}$ and X^{n+1}_k are defined below in (3.6), (3.8) and (3.7). The initialization is given by the formula

$$v^0_K = \frac{1}{|K|}\int_K u_0(x)dx, \quad K \in \mathcal{T}. \tag{3.5}$$

In (3.4), $\Delta t_n Q^n_{K\to L}$ is the numerical flux at the interface $K|L$ on the range of time $[t_n, t_{n+1}]$, where $Q^n_{K\to L}$ is given by

$$Q^n_{K\to L} = A_{K\to L}(v^n_K, v^n_L). \tag{3.6}$$

We have also defined

$$X^{n+1}_k = \frac{\beta_k(t_{n+1}) - \beta_k(t_n)}{(\Delta t_n)^{1/2}}. \tag{3.7}$$

Then, the $(X^{n+1}_k)_{k\geq 1, n\in\mathbb{N}}$ are independent random variables, normally distributed with mean 0 and variance 1. Besides, for each $n \geq 1$, the sequence $(X^{n+1}_k)_{k\geq 1}$ is independent of $\mathcal{F}_n$, the sigma-algebra generated by $\{X_k^{m+1}; k \geq 1, m < n\}$. The numerical functions $g_{k,K}$ are defined by the average

$$g_{k,K}(v) = \frac{1}{|K|} \int_K g_k(x, v) dx. \tag{3.8}$$

Then, in virtue of (1.6) we have

$$\mathbf{G}_K^2(v) := \sum_{k \geq 1} |g_{k,K}(v)|^2 \leq D_0, \tag{3.9}$$

where $v \in \mathbb{R}$, $K \in \mathcal{T}$. We deduce (3.9) from (1.6) and Jensen's Inequality. Similarly, we deduce from (1.3) and Jensen's Inequality that

$$\sum_{k \geq 1} |g_{k,K}(\xi) - g_k(y, \xi)|^2 \leq D_1 \frac{1}{|K|} \int_K |x - y|^2 dx,$$

for all $y \in \mathbb{T}^N$. In particular, assuming $\operatorname{diam}(K) \leq h$ (this figures in the hypotheses (4.22), which we will assume next), we have the following consistency estimate

$$\sum_{k \geq 1} |g_{k,K}(\xi) - g_k(x, \xi)|^2 \leq D_1 h^2, \tag{3.10}$$

for all $x \in K$, which will be used later (see (6.40) for example).

Remark 3.1 (Approximation in law). In effective computations, the random variables X_k^{n+1} are drawn at each time step. They are i.i.d. random variables with normalized centred normal law $\mathcal{N}(0, 1)$. In this situation, we will prove the convergence in law of the Finite Volume scheme to the solution of (1.1), see Remark 7.2 after Theorem 7.4.

Remark 3.2 (Global Lipschitz Numerical Flux). We assume in (3.1) that the numerical fluxes $A_{K \to L}$ are globally Lipschitz continuous. This is consistent with (1.5). Both (3.1) and (1.5) are strong assumptions, except if a priori L^∞-bounds are known on the solutions to (1.1), which is the case here, due to the hypothesis of compact support (1.4). Without loss of generality, we will assume that $\operatorname{Lip}(A) \leq L_A$.

4 The kinetic formulation of the finite volume scheme

4.1 Discussion on the kinetic formulation of the finite volume scheme

The kinetic formulation of the Finite Volume method has been introduced by Makridakis and Perthame in [27]. The principle is the following one. For linear transport equations, which corresponds to a linear flux function $A(u) = au$, $a \in \mathbb{R}^N$, the upwind numerical flux $A_{K \to L}$ in (3.6) is given by

$$A_{K \to L}(v, w) = [a_{K \to L}^*]^+ v - [a_{K \to L}^*]^- w, \tag{4.1}$$

where

$$a^*_{K\to L} := \int_{K|L} a\cdot n_{K|L} d\mathcal{H}^{N-1} = |K|L| a\cdot n_{K,L},$$

with the usual notation $v^+ = \max(v,0)$, $w^- = (-w)^+$. The discrete approximation of the kinetic version of the transport equation

$$\partial_t f(x,t,\xi) + a(\xi)\cdot\nabla_x f(x,t,\xi) = 0$$

by the Finite Volume method is therefore

$$|K|(f_K^{n+1}(\xi) - f_K^n(\xi)) + \Delta t_n \sum_{L\in\mathcal{N}(K)} a^n_{K\to L}(\xi) = 0, \tag{4.2}$$

where

$$a^n_{K\to L}(\xi) = [a^*_{K\to L}(\xi)]^+ f^n_K(\xi) - [a^*_{K\to L}(\xi)]^- f^n_L(\xi)$$

and

$$a^*_{K\to L}(\xi) := \int_{K|L} a(\xi)\cdot n_{K|L} d\mathcal{H}^{N-1} = |K|L| a(\xi)\cdot n_{K,L}. \tag{4.3}$$

Recall that $\mathbf{G}_K$ is defined by (3.9). One may think that a kinetic formulation of (3.4) consistent with (4.2) should be, for instance, the following identity

$$\begin{aligned} &|K|(\mathtt{f}_K^{n+1}(\xi) - \mathtt{f}_K^n(\xi)) + \Delta t_n \sum_{L\in\mathcal{N}(K)} a^n_{K\to L}(\xi) \\ &\quad = |K|(\Delta t_n)^{1/2}\delta_{v_K^n=\xi}\, g_{k,K}(\xi) X_k^{n+1} + |K|\Delta t_n \partial_\xi\Big(m^n_K(\xi) - \frac{1}{2}\mathbf{G}^2_K(\xi)\delta_{v^n_K=\xi}\Big), \end{aligned} \tag{4.4}$$

where

$$\mathtt{f}^n_K(\xi) := \mathbf{1}_{v^n_K>\xi}. \tag{4.5}$$

However, we do not know how to find a formulation (4.4)-(4.5) satisfying both the constraint $m^n_K(\xi)\geq 0$ for all $K\in\mathcal{T}$, $n\in\mathbb{N}$, $\xi\in\mathbb{R}$, and some satisfactory consistency conditions for $a^n_{K\to L}(\xi)$. The fact is that such a kinetic formulation as (4.4)-(4.5) is not necessary to our purpose. Indeed, the kinetic formulation authorizes general kinetic functions (Definition 2.4). It is not mandatory to work with sole equilibrium functions as in (4.5). We will exploit this flexibility of the approach by kinetic formulation and follow a procedure into three steps:

- Split the Finite Volume scheme into a step of deterministic evolution and a step of stochastic evolution (Section 4.2),
- Associate a discrete kinetic formulation to the deterministic evolution (Section 4.3),
- Gather the deterministic evolution (in its kinetic form) and the stochastic evolution to build an adequate discrete kinetic function (Section 4.4).

Once this is done, we explain in Section 4.5 to what extent the discrete kinetic function build at the end of Section 4.4 does satisfy a satisfactory discrete kinetic formulation.

4.2 Splitting

For $K \in \mathcal{T}$ and $n \in \mathbb{N}$, let us define $v_K^{n+1/2}$ as the solution to

$$|K|(v_K^{n+1/2} - v_K^n) + \Delta t_n \sum_{L\in\mathcal{N}(K)} A_{K\to L}(v_K^n, v_L^n) = 0. \tag{4.6}$$

Then $v_K^{n+1/2}$ is the state reached after a step of deterministic evolution, by the discrete approximation of the equation $u_t + \operatorname{div}(A(u)) = 0$.

4.3 Kinetic formulation of the deterministic evolution

We claim that, to (4.6) corresponds the kinetic formulation

$$|K|\left(\mathtt{f}_K^{n+1/2}(\xi) - \mathtt{f}_K^n(\xi)\right) + \Delta t_n \sum_{L\in\mathcal{N}(K)} a_{K\to L}^n(\xi) = |K|\Delta t_n\, \partial_\xi m_K^n(\xi), \tag{4.7}$$

where $\mathtt{f}_K^m(\xi) = \mathbf{1}_{v_K^m>\xi}$, $m \in \{n, n+1/2\}$ and

$$m_K^n \geq 0. \tag{4.8}$$

In (4.7), $a_{K\to L}^n(\xi)$ is a function

$$a_{K\to L}^n(\xi) = a_{K\to L}(\xi, v_K^n, v_L^n), \tag{4.9}$$

where $(\xi, v, w) \mapsto a_{K\to L}(\xi, v, w)$ satisfies the following consistency conditions:

$$\int_{\mathbb{R}} [a_{K\to L}(\xi, v, w) - a_{K\to L}^*(\xi)\mathbf{1}_{0>\xi}]\, d\xi = A_{K\to L}(v, w), \tag{4.10}$$

$$a_{K\to L}(\xi, v, v) = a_{K\to L}^*(\xi)\mathbf{1}_{v>\xi}, \tag{4.11}$$

for all ξ, v, $w \in \mathbb{R}$, where $a_{K\to L}^*$ is defined by (4.3). Let us state and prove the existence of the kinetic formulation (4.7)-(4.8)-(4.10)-(4.11).

Proposition 4.1 (Kinetic formulation of the Finite Volume method)**.** *Set*

$$a_{K\to L}(\xi, v, w) = a_{K\to L}^*(\xi)\mathbf{1}_{\xi<v\wedge w} + [\partial_2 A_{K\to L}(v, \xi)\mathbf{1}_{v\leq\xi\leq w} + \partial_1 A_{K\to L}(\xi, w)\mathbf{1}_{w\leq\xi\leq v}] \tag{4.12}$$

and

$$m_K^n(\xi) = -\frac{1}{\Delta t_n}\left[(v_K^{n+1/2} - \xi)^+ - (v_K^n - \xi)^+\right] - \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)} \int_\xi^{+\infty} a_{K\to L}^n(\zeta)d\zeta. \tag{4.13}$$

Let us also assume that

$$\Delta t_n \frac{|\partial K|}{|K|} L_A \leq 1, \quad \forall K \in \mathcal{T}, \tag{4.14}$$

for all $n \in \mathbb{N}$, $K \in \mathcal{T}$. Then the equations (4.7)-(4.8)-(4.10)-(4.11) are satisfied. Additionally, we have

$$a_{K\to L}(\xi, v, w) = 0, \quad \text{when } \xi \geq v \vee w, \tag{4.15}$$

for all $K, L \in \mathcal{T}$.

Remark 4.1 (Support of m_K^n). By (4.13), the definition (4.12) of $a_{K\to L}$ and the equation (4.6), $\xi \mapsto m_K^n(\xi)$ is compactly supported in the convex envelope of the points $v_K^{n+1/2}$, v_K^n, $\{v_L^n; L \in \mathcal{N}(K)\}$.

Proof of Proposition 4.1. We check at once (4.7) and (4.10), (4.11), (4.15). To show that $m_K^n(\xi) \geq 0$, let us introduce

$$\Phi_{K\to L}(\xi, v, w) = \int_\xi^{+\infty} a_{K\to L}(\zeta, v, w) d\zeta \tag{4.16}$$

$$\Phi_{K\to L}^n(\xi) = \Phi_{K\to L}(\xi, v_K^n, v_L^n) = \int_\xi^{+\infty} a_{K\to L}^n(\zeta) d\zeta. \tag{4.17}$$

A simple computation gives the formula

$$\Phi_{K\to L}^n(\xi) = A_{K\to L}(v_K^n, v_L^n) - A_{K\to L}(v_K^n \wedge \xi, v_L^n \wedge \xi). \tag{4.18}$$

By comparison with the identity $(v-\xi)^+ = v - v \wedge \xi$, the quantities $\Phi_{K\to L}^n(\xi)$ appears, in virtue of (4.18), as the numerical entropy fluxes associated to the entropy $\eta(v) := (v-\xi)^+$. Then $m_K^n(\xi) \geq 0$ is equivalent to the discrete entropy inequality

$$\frac{1}{\Delta t_n}\left[\eta(v_K^{n+1/2}) - \eta(v_K^n)\right] + \frac{1}{|K|}\sum_{L\in\mathcal{N}(K)} \Phi_{K\to L}^n(\xi) \leq 0. \tag{4.19}$$

It is a classical fact that, under the CFL condition (4.14), the deterministic Finite Volume scheme (4.6) has the following monotonicity property: $v_K^{n+1/2}$ in (4.6) is a non-decreasing function of each of the entries v_K^n, v_L^n, $L \in \mathcal{N}(K)$. This implies (4.19) then. See Lemma 25.1 and Lemma 27.1 in [11]. □

Remark 4.2. Consider the general framework where the flux A is only locally Lipschitz and the noise is not compactly supported. Then the constant L_A in the CFL condition (4.14) has to be replaced (up to a factor $|K||L|$) by the Lipschitz constant of the numerical fluxes $A_{K\to L}$ over the compact $[m_K^n, M_K^n]$, where

$$m_K^n = \min_{L\in\mathcal{N}(K)\cup\{K\}} v_L^n, \quad M_K^n = \max_{L\in\mathcal{N}(K)\cup\{K\}} v_L^n.$$

This puts a restriction on the time-step Δt_n, which is governed by the L^∞-norm of the numerical solution at time t_n. The issue then, is that, in the stochastic context of (3.4), no control on this L^∞-norm is known. This is why we assume that the flux A is globally Lipschitz, which, to repeat ourselves, is relevant if an a priori bound on the L^∞-norm of the solution to the continuous equation (1.1) is known. The condition of compact support (1.4) provides this a priori bound.

4.4 Construction of the discrete kinetic unknown

For a fixed final time $T > 0$, we denote by $\mathfrak{d}_T$ the set of admissible space-step and time-steps, defined as follows: if $h > 0$ and $(\Delta t) = (\Delta t_0, \ldots, \Delta t_{N_T - 1})$, $N_T \in \mathbb{N}^*$, then we say that $\delta := (h, (\Delta t)) \in \mathfrak{d}_T$ if

$$\frac{1}{h} \in \mathbb{N}^*, \quad t_{N_T} := \sum_{n=0}^{N_T-1} \Delta t_n = T, \quad \sup_{0 \le n < N_T} \Delta t_n \le 1. \tag{4.20}$$

We say that $\delta \to 0$ if

$$|\delta| := h + \sup_{0 \le n < N_T} \Delta t_n \to 0. \tag{4.21}$$

For a given mesh parameter $\delta = (h, (\Delta t)) \in \mathfrak{d}_T$, we assume that a mesh $\mathcal{T}$ is given, with the following properties:

$$\operatorname{diam}(K) \le h, \quad \alpha_N h^N \le |K|, \quad |\partial K| \le \frac{1}{\alpha_N} h^{N-1}, \tag{4.22}$$

for all $K \in \mathcal{T}$, where

$$\operatorname{diam}(K) = \max_{x,y \in K} |x - y|$$

is the diameter of K and α_N is a given positive absolute constant depending on the dimension N only. Note the following consequence of (4.22):

$$h|\partial K| \le \alpha_N^{-2} |K|, \tag{4.23}$$

for all $K \in \mathcal{T}$. We introduce then the discrete unknown $v_\delta(t)$ defined a.e. by

$$v_\delta(x, t) = v_K^n, \quad x \in K, t_n \le t < t_{n+1}. \tag{4.24}$$

We will also need the intermediary discrete function

$$v_\delta^\flat(x, t_{n+1}) = v_K^{n+1/2}, \quad x \in K, \tag{4.25}$$

defined for $n \in \mathbb{N}$, and then the linear interpolation $v_\delta^\sharp(x, t)$, corresponding to the stochastic evolution, given by

$$v_\delta^\sharp(x, t) = v_K^{n+1/2} + g_{k,K}(v_K^n)(\beta_k(t) - \beta_k(t_n)), \quad t_n \le t < t_{n+1}, x \in K. \tag{4.26}$$

Indeed, $v_K^\sharp(t) := v_\delta^\sharp(x, t)$, $x \in K$ is, for $t \in [t_n, t_{n+1})$, an interpolation between $v_K^{n+1/2}$ and v_K^{n+1}. Eventually, for $t \in [t_n, t_{n+1})$, we define the discrete kinetic unknown $f_\delta(t)$ by the interpolation formula

$$f_\delta(x, t, \xi) = \frac{t - t_n}{\Delta t_n} \mathbf{1}_{v_\delta^\sharp(x,t) > \xi} + \frac{t_{n+1} - t}{\Delta t_n} \mathbf{1}_{v_\delta(x,t) > \xi}, \quad \xi \in \mathbb{R}, \ x \in \mathbb{T}^N. \tag{4.27}$$

4.5 Discrete kinetic formulation

We state two results in this section. In Lemma 4.2 below, we compare f_δ to the piecewise constant equilibrium function $\mathtt{f}_\delta$ defined by

$$\mathtt{f}_\delta(x,t,\xi)=\mathtt{f}_K^n=\mathbf{1}_{v_\delta(x,t)>\xi},\quad x\in K,t\in[t_n,t_{n+1}).\tag{4.28}$$

Then, in Proposition 4.3, we give the discrete kinetic formulation satisfied by f_δ.

Lemma 4.2. *Let $u_0\in L^\infty(\mathbb{T}^N)$, $T>0$. Assume that (1.3), (1.4), (1.6), (3.1), and (4.29) are satisfied. For $\delta\in\mathfrak{d}_T$, assume (4.22). Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(4.24) and let f_δ, $\mathtt{f}_\delta$ be defined by (4.27)-(4.28). Assume that the CFL condition*

$$\Delta t_n\leq(1-\theta)\frac{\alpha_N^2}{2L_A}h,\quad 0\leq n<N_T,\tag{4.29}$$

where $\theta\in(0,1)$, is satisfied. We have then

$$\begin{aligned}\mathbb{E}\int_0^T\int_{\mathbb{T}^N}&\left[\int_{\mathbb{R}}|f_\delta(x,t,\xi)-\mathtt{f}_\delta(x,t,\xi)|d\xi\right]^2dxdt\\&\leq\left[\theta^{-1}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)}+D_0T(1+\theta^{-1})\right]\left[\sup_{0\leq n<N_T}\Delta t_n\right].\end{aligned}\tag{4.30}$$

To f_δ we will associate the Young measure

$$\nu^\delta_{x,t}(\xi):=-\partial_\xi f_\delta(x,t,\xi)=\frac{t-t_n}{\Delta t_n}\delta(\xi=v^\sharp_\delta(x,t))+\frac{t_{n+1}-t}{\Delta t_n}\delta(\xi=v_\delta(x,t)),\tag{4.31}$$

We also denote by m_δ the discrete random measure given by

$$dm_\delta(x,t,\xi)=\sum_{n=0}^{N_T-1}\sum_{K\in\mathcal{T}}\mathbf{1}_{K\times[t_n,t_{n+1})}(x,t)m_K^n(\xi)\,dxdtd\xi.\tag{4.32}$$

Proposition 4.3 (Discrete kinetic equation)**.** *Let $u_0\in L^\infty(\mathbb{T}^N)$, $T>0$. Assume that (1.3), (1.4), (1.6), (3.1) and (4.29) are satisfied. For $\delta\in\mathfrak{d}_T$, assume (4.22). Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(4.24) and let f_δ, ν^δ, m_δ be defined by (4.27), (4.31), (4.32) respectively. Then f_δ satisfies the following discrete kinetic formulation: for all $t\in[t_n,t_{n+1}]$, $x\in K$, for all $\psi\in C_c^\infty(\mathbb{R})$,*

$$\begin{aligned}&\langle f_\delta(x,t),\psi\rangle-\langle f_\delta(x,t_n),\psi\rangle\\&=-\frac{1}{|K|}\int_{t_n}^t\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)}a^n_{K\to L}(\xi)\psi(\xi)d\xi ds-\int_{t_n}^t\int_{\mathbb{R}}\partial_\xi\psi(\xi)m_K^n(\xi)d\xi ds\\&\quad+\frac{t-t_n}{\Delta t_n}\int_{t_n}^t g_{k,K}(v_K^n)\psi(v^\sharp_\delta(x,s))d\beta_k(s)+\frac{1}{2}\frac{t-t_n}{\Delta t_n}\int_{t_n}^t\mathbf{G}^2_K(v^n_K)\partial_\xi\psi(v^\sharp_\delta(x,s))ds.\end{aligned}\tag{4.33}$$

In (4.33), $\langle f_\delta(x,t),\psi\rangle$ stands for the product

$$\int_{\mathbb{R}} f_\delta(x,t,\xi)\psi(\xi)d\xi.$$

The proof of Lemma 4.2 and Proposition 4.3 is reported to Section 6. Indeed, it uses some estimates that are established in the following Section 5.

5 Energy estimates

The Finite Volume scheme (3.4) may be compared to the stochastic parabolic equation

$$du^\varepsilon(x,t)+\operatorname{div}(A(u^\varepsilon(x,t)))dt-\varepsilon\Delta u^\varepsilon(x,t)dt=\Phi(x,u^\varepsilon(x,t))dW(t). \tag{5.1}$$

For (5.1), we have the energy estimate

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\|u^\varepsilon(t)\|^2_{L^2(\mathbb{T}^N)}+\varepsilon\mathbb{E}\|\nabla u^\varepsilon(t)\|^2_{L^2(\mathbb{T}^N)}=\frac{1}{2}\mathbb{E}\|\mathbf{G}(\cdot,u^\varepsilon(\cdot,t))\|^2_{L^2(\mathbb{T}^N)}. \tag{5.2}$$

(Recall that $\mathbf{G}$ is defined by (1.2)). In the following Proposition 5.1, we obtain an analogous result for the Finite Volume scheme (3.4). To state Proposition 5.1, we need first some notations.

5.1 Notations

Let us define the conjugate function $\bar{f}=1-f$. We introduce the following conjugate quantities:

$$\bar{a}_{K\to L}(\xi,v,w)=a^*_{K\to L}(\xi)-a_{K\to L}(\xi,v,w),\quad \overline{\Phi}_{K\to L}(\xi,v,w)=\int_{-\infty}^{\xi}\bar{a}_{K\to L}(\zeta,v,w)d\zeta. \tag{5.3}$$

We compute

$$\overline{\Phi}_{K\to L}(\xi,v,w)=A_{K\to L}(\xi,\xi)-A_{K\to L}(v\wedge\xi,w\wedge\xi). \tag{5.4}$$

We recognize in (5.4) a numerical flux associated to the entropy

$$v\mapsto(v-\xi)^-=\xi-v\wedge\xi.$$

From the explicit formula (4.12), we obtain the identity

$$\begin{aligned}\bar{a}_{K\to L}(\xi,v,w)=a^*_{K\to L}(\xi)\mathbf{1}_{\xi>v\vee w}+(a^*_{K\to L}(\xi)-\partial_2A_{K\to L}(v,\xi))\mathbf{1}_{v\le\xi\le w}\\+(a^*_{K\to L}(\xi)-\partial_1A_{K\to L}(\xi,w))\mathbf{1}_{w\le\xi\le v}.\end{aligned} \tag{5.5}$$

Note that, for $a_{K\to L}$ defined by (4.12), we have (using the fact that $\operatorname{Lip}(A)\le L_A$),

$$\sup\{|a_{K\to L}(\xi,v,w)|;\xi,v,w\in\mathbb{R}\}\le L_A|K|L|. \tag{5.6}$$

Formula (5.5) gives the estimate

$$\sup\{|\bar{a}_{K\to L}(\xi,v,w)|;\xi,v,w\in\mathbb{R}\}\leq 2L_A|K|L|, \tag{5.7}$$

which is not optimal as (5.6) may be, since it has an additional factor 2. Consequently, we will use a slightly different formulation for $\overline{\Phi}_{K\to L}$:

$$\overline{\Phi}_{K\to L}(\xi,v,w)=\int_{-\infty}^{\xi}\bar{b}_{K\to L}(\zeta,\xi,v,w)d\zeta, \tag{5.8}$$

where

$$\bar{b}_{K\to L}(\zeta,\xi,v,w):=a^*_{K\to L}(\xi)\mathbf{1}_{\xi>v\vee w}+\partial_1 A_{K\to L}(\zeta,\xi)\mathbf{1}_{v\leq\xi\leq w}+\partial_2 A_{K\to L}(\xi,\zeta)\mathbf{1}_{w\leq\xi\leq v}. \tag{5.9}$$

We also introduce

$$\bar{b}^n_{K\to L}(\zeta,\xi)=\bar{b}_{K\to L}(\zeta,\xi,v^n_K,v^n_L). \tag{5.10}$$

Now for $\bar{b}_{K\to L}$, we have an estimate similar to (5.6):

$$\sup\{|\bar{b}_{K\to L}(\xi,v,w)|;\xi,\zeta,v,w\in\mathbb{R}\}\leq L_A|K|L|. \tag{5.11}$$

5.2 Energy estimate and controls by the dissipation

Proposition 5.1 (Energy estimate for the Finite Volume Scheme). *Let $u_0\in L^\infty(\mathbb{T})$, $T>0$ and $\delta\in\mathfrak{d}_T$. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(4.24). Set*

$$\mathcal{E}(T)=\sum_{n=0}^{N_T-1}\Delta t_n\sum_{K\in\mathcal{T}_\#}|K|\int_{\mathbb{R}}m^n_K(\xi)d\xi. \tag{5.12}$$

Then, under the CFL condition (4.14), we have the energy estimate

$$\frac{1}{2}\mathbb{E}\|v_\delta(T)\|^2_{L^2(\mathbb{T}^N)}+\mathbb{E}\mathcal{E}(T)=\frac{1}{2}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)}+\frac{1}{2}\mathbb{E}\sum_{n=0}^{N_T-1}\Delta t_n\sum_{K\in\mathcal{T}_\#}|K|\sum_{k\geq 1}|g_{k,K}(v^n_K)|^2. \tag{5.13}$$

In the following proposition we derive various estimates, where the right-hand side is controlled by the dissipation term $\mathcal{E}(T)$ introduced in (5.12).

Proposition 5.2 (Control by the dissipation). *Let $u_0\in L^\infty(\mathbb{T}^N)$, $T>0$ and $\delta\in\mathfrak{d}_T$. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(4.24). Let $v^\flat_\delta$ be defined by (4.25). Then, under the CFL condition*

$$2\Delta t_n\frac{|\partial K|}{|K|}\sup_{\xi\in\mathbb{R}}\frac{|a^n_{K\to L}(\xi)|}{|K|L|}\leq(1-\theta),\quad 0\leq n<N,\ K,L\in\mathcal{T}, \tag{5.14}$$

where $\theta \in (0,1)$, we have the following control:

$$\sum_{n=0}^{N_T-1} \Delta t_n \sum_{K\in\mathcal{T}_\#} \sum_{L\in\mathcal{N}(K)} \int_{\mathbb{R}} (\bar{\mathtt{f}}_L^n - \bar{\mathtt{f}}_K^n)\Phi_{K\to L}^n(\xi)d\xi$$
$$\leq \frac{2}{\theta} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K\in\mathcal{T}_\#} |K| \int_{\mathbb{R}} \bar{\mathtt{f}}_K^n(\xi) m_K^n(\xi) d\xi, \tag{5.15}$$

and

$$\sum_{n=0}^{N_T-1} \left\| \left[v_\delta^\flat(t_{n+1}) - v_\delta(t_n) \right]_+ \right\|_{L^2(\mathbb{T}^N)}^2 \leq \frac{2}{\theta} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K\in\mathcal{T}_\#} |K| \int_{\mathbb{R}} \bar{\mathtt{f}}_K^n(\xi) m_K^n(\xi) d\xi. \tag{5.16}$$

Under the CFL condition

$$2\Delta t_n \frac{|\partial K|}{|K|} \sup_{\xi\in\mathbb{R}} \frac{|\bar{b}_{K\to L}^n(\xi,\xi)|}{|K|L|} \leq (1-\theta), \quad 0 \leq n < N,\ K, L \in \mathcal{T}, \tag{5.17}$$

where $\theta \in (0,1)$, (and where $\bar{b}_{K\to L}^n$ is defined by (5.10)) we have the following control:

$$\sum_{n=0}^{N_T-1} \Delta t_n \sum_{K\in\mathcal{T}_\#} \sum_{L\in\mathcal{N}(K)} \int_{\mathbb{R}} (\mathtt{f}_L^n - \mathtt{f}_K^n)\overline{\Phi}_{K\to L}^n(\xi)d\xi$$
$$\leq \frac{2}{\theta} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K\in\mathcal{T}_\#} |K| \int_{\mathbb{R}} \mathtt{f}_K^n(\xi) m_K^n(\xi) d\xi, \tag{5.18}$$

and

$$\sum_{n=0}^{N_T-1} \left\| \left[v_\delta^\flat(t_{n+1}) - v_\delta(t_n) \right]_- \right\|_{L^2(\mathbb{T}^N)}^2 \leq \frac{2}{\theta} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K\in\mathcal{T}_\#} |K| \int_{\mathbb{R}} \mathtt{f}_K^n(\xi) m_K^n(\xi) d\xi. \tag{5.19}$$

Eventually, as a corollary to Proposition 5.2, we obtain the following estimates.

Corollary 5.3 (Weak derivative estimates)**.** *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{d}_T$. Assume that (1.4), (1.6), (3.1), (4.22) and (4.29) are satisfied. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(4.24). Let $v_\delta^\flat$ be defined by (4.25). Then we have the spatial estimate*

$$\mathbb{E} \sum_{n=0}^{N_T-1} \Delta t_n \sum_{K\in\mathcal{T}_\#} \sum_{L\in\mathcal{N}(K)} \int_{\mathbb{R}} \left[(\bar{\mathtt{f}}_L^n - \bar{\mathtt{f}}_K^n)\Phi_{K\to L}^n(\xi) + (\mathtt{f}_L^n - \mathtt{f}_K^n)\overline{\Phi}_{K\to L}^n(\xi) \right] d\xi$$
$$\leq \frac{1}{\theta} \|v_\delta(0)\|_{L^2(\mathbb{T}^N)}^2 + \frac{D_0 T}{\theta}, \tag{5.20}$$

and the two following temporal estimates:

$$\mathbb{E}\sum_{n=0}^{N_T-1}\left\|v_\delta^\flat(t_{n+1})-v_\delta(t_n)\right\|^2_{L^2(\mathbb{T}^N)}\leq\frac{1}{\theta}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)}+\frac{D_0T}{\theta},\tag{5.21}$$

and

$$\mathbb{E}\sum_{n=0}^{N_T-1}\|v_\delta(t_{n+1})-v_\delta(t_n)\|^2_{L^2(\mathbb{T}^N)}\leq\frac{1}{\theta}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)}+\frac{2D_0T}{\theta}.\tag{5.22}$$

5.3 Proof of Proposition 5.1, Proposition 5.2, Corollary 5.3

Proof of Proposition 5.1. We multiply first (4.7) by ξ and sum the result over $K\in\mathcal{T}_\#$ and $\xi\in\mathbb{R}$ to get the following balance equation

$$\frac{1}{2}\|v_\delta^\flat(t_{n+1})\|^2_{L^2(\mathbb{T}^N)}+\Delta t_n\sum_{K\in\mathcal{T}_\#}|K|\int_{\mathbb{R}}m_K^n(\xi)d\xi=\frac{1}{2}\|v_\delta(t_n)\|^2_{L^2(\mathbb{T}^N)}.\tag{5.23}$$

We have used Remark 4.1 to justify the integration by parts in the term with the measure m_K^n. The term

$$\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}a^n_{K\to L}(\xi)\tag{5.24}$$

related to the flux term in (4.7) has vanished. Indeed, (5.24) is equal to

$$\frac{1}{2}\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}a^n_{K\to L}(\xi)+a^n_{L\to K}(\xi)\tag{5.25}$$

by relabelling of the indexes of summation. All the arguments in (5.25) cancel individually in virtue of the conservative symmetry property (3.3) of $A_{K\to L}(v,w)$. Indeed, one can check that $a_{K\to L}$ inherits this property, *i.e.*

$$a_{K\to L}(\xi,v,w)=-a_{L\to K}(\xi,w,v),\quad K,L\in\mathcal{T},\ v,w\in\mathbb{R},\tag{5.26}$$

owing to the explicit formula (4.12). To obtain the equation for the balance of energy corresponding to the stochastic forcing, we use the equation

$$v_K^{n+1}=v_K^{n+1/2}+(\Delta t_n)^{1/2}g_{k,K}(v_K^n)X_k^{n+1},\tag{5.27}$$

which follows from the equation of the scheme (3.4) and the definition of $v_K^{n+1/2}$ by (4.6). Taking the square of both sides of (5.27) and using the independence of X_k^{n+1} and $v_K^{n+1/2}$, we obtain the identity

$$\frac{1}{2}\mathbb{E}\|v_\delta(t_{n+1})\|^2_{L^2(\mathbb{T}^N)}=\frac{1}{2}\mathbb{E}\|v_\delta^\flat(t_{n+1})\|^2_{L^2(\mathbb{T}^N)}+\frac{\Delta t_n}{2}\mathbb{E}\sum_{K\in\mathcal{T}_\#}|K|\sum_{k\geq1}|g_{k,K}(v_K^n)|^2.\tag{5.28}$$

Adding (5.23) to (5.28) gives (5.13). □

Remark 5.1. Note that (5.27) also gives

$$\frac{1}{2}\mathbb{E}\|v_\delta(t_{n+1})-v_\delta^\flat(t_{n+1})\|^2_{L^2(\mathbb{T}^N)}=\frac{\Delta t_n}{2}\mathbb{E}\sum_{K\in\mathcal{T}_\#}|K|\sum_{k\geq 1}|g_{k,K}(v_K^n)|^2, \tag{5.29}$$

for all $0\leq n\leq N_T$.

Proof of Proposition 5.2. We begin with the proof of the estimates (5.15) and (5.16). Multiplying Equation (4.7) by $\bar{\mathtt{f}}_K^n:=1-\mathtt{f}_K^n$, we obtain

$$|K|\bar{\mathtt{f}}_K^n(\xi)\mathtt{f}_K^{n+1/2}(\xi)+\Delta t_n\bar{\mathtt{f}}_K^n(\xi)\sum_{L\in\mathcal{N}(K)}a^n_{K\to L}(\xi)=|K|\Delta t_n\,\bar{\mathtt{f}}_K^n(\xi)\partial_\xi m_K^n(\xi). \tag{5.30}$$

Next, we multiply (5.30) by $(\xi-v_K^n)$ and sum the result over ξ, K. We use the first identity

$$\int_{\mathbb{R}}(\xi-v_K^n)\bar{\mathtt{f}}_K^n(\xi)\partial_\xi m_K^n(\xi)d\xi=\int_{\mathbb{R}}(\xi-v_K^n)_+\partial_\xi m_K^n(\xi)d\xi=-\int_{\mathbb{R}}\bar{\mathtt{f}}_K^n(\xi)m_K^n(\xi)d\xi, \tag{5.31}$$

(once again, we use the fact that m_K^n is compactly supported to do the integration by parts in (5.31), *cf.* Remark 4.1) and the second identity

$$\int_{\mathbb{R}}(\xi-v_K^n)\bar{\mathtt{f}}_K^n(\xi)\mathtt{f}_K^{n+1/2}(\xi)d\xi=\frac{1}{2}(v_K^{n+1/2}-v_K^n)_+^2,$$

to obtain

$$\begin{aligned}\frac{1}{2}\left\|\left[v_\delta^\flat(t_{n+1})-v_\delta(t_n)\right]_+\right\|^2_{L^2(\mathbb{T}^N)}&+\Delta t_n\sum_{K\in\mathcal{T}_\#}|K|\int_{\mathbb{R}}\bar{\mathtt{f}}_K^n(\xi)m_K^n(\xi)d\xi\\&=-\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\int_{\mathbb{R}}(\xi-v_K^n)_+a^n_{K\to L}(\xi)d\xi.\end{aligned} \tag{5.32}$$

We transform the right-hand side of (5.32) by integration by parts in ξ: this gives, by (4.16)-(4.17), the term

$$-\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\int_{\mathbb{R}}\bar{\mathtt{f}}_K^n(\xi)\Phi^n_{K\to L}(\xi)d\xi. \tag{5.33}$$

Then we can relabel the indices in (5.33) and use the conservative symmetry relation (consequence of (5.26))

$$\Phi_{K\to L}(\xi,v,w)=-\Phi_{L\to K}(\xi,w,v), \tag{5.34}$$

to see that

$$\begin{aligned}\frac{1}{2}\left\|\left[v_\delta^\flat(t_{n+1})-v_\delta(t_n)\right]_+\right\|^2_{L^2(\mathbb{T}^N)}&+\Delta t_n\sum_{K\in\mathcal{T}_\#}|K|\int_{\mathbb{R}}\bar{\mathtt{f}}_K^n(\xi)m_K^n(\xi)d\xi\\&=\frac{1}{2}\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\int_{\mathbb{R}}(\bar{\mathtt{f}}_L^n(\xi)-\bar{\mathtt{f}}_K^n(\xi))\Phi^n_{K\to L}(\xi)d\xi.\end{aligned} \tag{5.35}$$

Note that the integrand $(\bar{\mathtt{f}}^n_L(\xi)-\bar{\mathtt{f}}^n_K(\xi))\Phi^n_{K\to L}(\xi)$ is non-negative, due to the monotonicity properties of $A_{K\to L}$ and (4.18). At this stage, in order to deduce (5.15) from (5.35), we have to prove that, under the CFL condition (5.14), a fraction of the right-hand side of (5.35) controls the term

$$\frac{1}{2}\left\|\left[v^\flat_\delta(t_{n+1})-v_\delta(t_n)\right]_+\right\|^2_{L^2(\mathbb{T})},$$

(see the estimate (5.40) below). To this end, we integrate Equation (5.30) over $\xi\in\mathbb{R}$. This gives

$$|K|\left[v^{n+1/2}_K-v^n_K\right]_+ +\Delta t_n\sum_{L\in\mathcal{N}(K)}\int_{\mathbb{R}}\bar{\mathtt{f}}^n_K(\xi)a^n_{K\to L}(\xi)d\xi\leq 0, \tag{5.36}$$

which reads also

$$|K|\left[v^{n+1/2}_K-v^n_K\right]_+\leq -\Delta t_n\sum_{L\in\mathcal{N}(K)}\Phi^n_{K\to L}(v^n_K)$$

by (4.17) (note that it is also the discrete entropy inequality (4.19) with $\xi=v^n_K$). Taking the square, using the Cauchy-Schwarz Inequality and summing over $K\in\mathcal{T}_\#$, we deduce that

$$\left\|\left[v^\flat_\delta(t_{n+1})-v_\delta(t_n)\right]_+\right\|^2_{L^2(\mathbb{T}^N)}\leq\Delta t_n\sum_{K\in\mathcal{T}_\#}\Delta t_n\frac{|\partial K|}{|K|}\sum_{L\in\mathcal{N}(K)}\frac{|\Phi^n_{K\to L}(v^n_K)|^2}{|K|L|}. \tag{5.37}$$

Next, we note that $|\Phi^n_{K\to L}(v^n_K)|^2$ is non-trivial only if $v^n_K<v^n_L$. In that case, it can be decomposed as

$$|\Phi^n_{K\to L}(v^n_K)|^2=-2\int_{v^n_K}^{v^n_L}\Phi^n_{K\to L}(\xi)\partial_\xi\Phi^n_{K\to L}(\xi)d\xi=2\int_{v^n_K}^{v^n_L}\Phi^n_{K\to L}(\xi)a^n_{K\to L}(\xi)d\xi, \tag{5.38}$$

which is bounded by

$$2\sup_{\xi\in\mathbb{R}}|a^n_{K\to L}(\xi)|\int_{v^n_K}^{v^n_L}|\Phi^n_{K\to L}(\xi)|d\xi=2\sup_{\xi\in\mathbb{R}}|a^n_{K\to L}(\xi)|\int_{\mathbb{R}}(\bar{\mathtt{f}}^n_L-\bar{\mathtt{f}}^n_K)\Phi^n_{K\to L}(\xi)d\xi. \tag{5.39}$$

Under the CFL condition (5.14), the estimate (5.37) can be completed into

$$\begin{aligned}\frac{1}{2}&\left\|\left[v^\flat_\delta(t_{n+1})-v_\delta(t_n)\right]_+\right\|^2_{L^2(\mathbb{T}^N)}\\&\leq(1-\theta)\frac{1}{2}\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\int_{\mathbb{R}}(\bar{\mathtt{f}}^n_L(\xi)-\bar{\mathtt{f}}^n_K(\xi))\Phi^n_{K\to L}(\xi)d\xi.\end{aligned} \tag{5.40}$$

Using (5.35) then, we deduce the two estimates (5.15)-(5.16).

To prove the estimates (5.18) and (5.19), we proceed similarly: we start from the following equation on $\bar{\mathtt{f}}^n_K$, which is equivalent to (4.7):

$$|K|(\bar{\mathtt{f}}^{n+1/2}_K(\xi)-\bar{\mathtt{f}}^n_K(\xi))+\Delta t_n\sum_{L\in\mathcal{N}(K)}\bar{a}^n_{K\to L}(\xi)=-|K|\Delta t_n\,\partial_\xi m^n_K(\xi). \tag{5.41}$$

Then we multiply Eq. (5.41) by $\mathtt{f}^n_K$, to obtain

$$|K|\mathtt{f}^n_K\bar{\mathtt{f}}^{n+1/2}_K(\xi)+\Delta t_n\sum_{L\in\mathcal{N}(K)}\mathtt{f}^n_K\bar{a}^n_{K\to L}(\xi)=-|K|\Delta t_n\,\mathtt{f}^n_K\partial_\xi m^n_K(\xi), \tag{5.42}$$

which is the analogue to (5.30). In a first step, we multiply (5.42) by $(v^n_K-\xi)$ and sum the result over $\xi\in\mathbb{R}$, $K\in\mathcal{T}_\#$. This gives (compare to (5.32)-(5.35))

$$\begin{aligned}\frac{1}{2}\left\|\left[v^\flat_\delta(t_{n+1})-v_\delta(t_n)\right]_-\right\|^2_{L^2(\mathbb{T}^N)}+\Delta t_n\sum_{K\in\mathcal{T}_\#}|K|\int_{\mathbb{R}}\mathtt{f}^n_K(\xi)m^n_K(\xi)d\xi\\
=-\,\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\mathtt{f}^n_K\overline{\Phi}^n_{K\to L}(\xi)\\
=\frac{1}{2}\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}(\mathtt{f}^n_L-\mathtt{f}^n_K)\overline{\Phi}^n_{K\to L}(\xi).\end{aligned} \tag{5.43}$$

To conclude to (5.18)-(5.19) under the CFL condition (5.17), we proceed as in (5.36)-(5.40) above, with the minor difference that, instead of the identity $\partial_\xi\overline{\Phi}^n_{K\to L}(\xi)=\bar{a}^n_{K\to L}(\xi)$, we use the formula $\partial_\xi\overline{\Phi}^n_{K\to L}(\xi)=\bar{b}^n_{K\to L}(\xi,\xi)$ (see (5.10)) when we develop $|\overline{\Phi}^n_{K\to L}(v^n_K)|^2$. □

Remark 5.2. A slight modification of the lines (5.38)-(5.39) in the proof above shows that

$$|\Phi^n_{K\to L}(\xi\vee v^n_K)|^2\leq 2\sup_{\xi\in\mathbb{R}}|a^n_{K\to L}(\xi)|\int_{\mathbb{R}}(\bar{\mathtt{f}}^n_L-\bar{\mathtt{f}}^n_K)\Phi^n_{K\to L}(\xi)d\xi, \tag{5.44}$$

for all $\xi\in\mathbb{R}$. This estimate will be used in the proof of Lemma 6.2 below.

Proof of Corollary 5.3. Assume that (4.29) is satisfied. It is clear, in virtue of the estimate (4.23) and the bound (5.6) and (5.11) on $a^n_{K\to L}$ and $\bar{b}^n_{K\to L}$, that (4.29) implies the CFL conditions (5.14) and (5.17). Besides, due to (3.9), we have the bound

$$\sum_{K\in\mathcal{T}_\#}|K|\sum_{k\geq 1}|g_{k,K}(v^n_K)|^2\leq D_0. \tag{5.45}$$

This gives

$$\sum_{n=0}^{N_T-1}\Delta t_n\sum_{K\in\mathcal{T}_\#}|K|\sum_{k\geq 1}|g_{k,K}(v^n_K)|^2\leq D_0T,$$

which, inserted in the energy estimate (5.13), shows that

$$\mathbb{E}\mathcal{E}(T) \leq \frac{1}{2}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)} + \frac{1}{2}D_0 T.$$

By addition of the estimates (5.15)-(5.18) and (5.16)-(5.19) respectively, we obtain therefore (5.20) and (5.21). There remains to prove (5.22). For that purpose, we use (5.29) and (5.45) to obtain

$$\mathbb{E}\|v_\delta(t_{n+1}) - v_\delta^\flat(t_{n+1})\|^2_{L^2(\mathbb{T}^N)} = \mathbb{E}\|v_\delta(t_{n+1})\|^2_{L^2(\mathbb{T}^N)} - \mathbb{E}\|v_\delta^\flat(t_{n+1})\|^2_{L^2(\mathbb{T}^N)} \leq D_0\Delta t_n. \tag{5.46}$$

Summing (5.46) over $0 \leq n < N_T$ and using (5.21) yields (5.22). □

6 Approximate kinetic equation

Recall that f_δ is defined by (4.27). We will show in this section that f_δ generates a sequence of approximate solutions: see Proposition 6.1. First, we will use the results of Section 5 to give the proofs of Lemma 4.2 and Proposition 4.3.

6.1 Proof of Lemma 4.2

Since

$$f_\delta(t) - \mathtt{f}_\delta(t) = \frac{t-t_n}{\Delta t_n}(\mathbf{1}_{v_\delta^\sharp(t)>\xi} - \mathbf{1}_{v_\delta(t)>\xi}),$$

for $t \in [t_n, t_{n+1})$ and since the factor $\frac{t-t_n}{\Delta t_n}$ is less than 1, the quantity we want to estimate is bounded by the following L^2-norm:

$$\mathbb{E}\int_0^T \int_{\mathbb{T}^N} \left|\int_{\mathbb{R}} |f_\delta(x,t,\xi) - \mathtt{f}_\delta(x,t,\xi)| d\xi\right|^2 dxdt \leq \mathbb{E}\int_0^T \|v_\delta^\sharp(t) - v_\delta(t)\|^2_{L^2(\mathbb{T}^N)} dt. \tag{6.1}$$

By definition of $v_\delta^\sharp(t)$ and independence and (3.9), we obtain

$$\begin{aligned}\mathbb{E}\int_0^T \int_{\mathbb{T}^N} \left|\int_{\mathbb{R}} |f_\delta(x,t,\xi) - \mathtt{f}_\delta(x,t,\xi)| d\xi\right|^2 dxdt &\leq D_0 \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} |t-t_n| dt \\ &\quad + \mathbb{E}\sum_{n=0}^{N_T-1} \Delta t_n \left\|v_\delta^\flat(t_{n+1}) - v_\delta(t_n)\right\|^2_{L^2(\mathbb{T}^N)}.\end{aligned}$$

Using the temporal estimate (5.21), we deduce (4.30). □

Remark 6.1. Note for a future use (*cf.* (6.38)) that we have just proved the estimate

$$\mathbb{E}\int_0^T \|v_\delta^\sharp(t) - v_\delta(t)\|^2_{L^2(\mathbb{T}^N)} dt \leq \left[\theta^{-1}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)} + D_0 T(1+\theta^{-1})\right]\left[\sup_{0\leq n<N_T} \Delta t_n\right]. \tag{6.2}$$

6.2 Proof of Proposition 4.3

Let Ψ be a primitive function of the function ψ and let $x \in K$, $t \in [t_n, t_{n+1})$. By definition of f_δ, see Equation (4.27), we have

$$\langle f_\delta(x,t), \psi\rangle - \langle f_\delta(x,t_n), \psi\rangle = \frac{t-t_n}{\Delta t_n}\left[\Psi(v^\sharp(x,t)) - \Psi(v_\delta(x,t))\right],$$

which we decompose as the sum of two terms:

$$\frac{t-t_n}{\Delta t_n}\left[\Psi(v^\sharp(x,t)) - \Psi(v^\flat(x,t_{n+1}))\right], \tag{6.3}$$

and

$$\frac{t-t_n}{\Delta t_n}\left[\Psi(v^\flat(x,t_{n+1})) - \Psi(v_\delta(x,t))\right]. \tag{6.4}$$

We use first the deterministic kinetic formulation (4.7), which we multiply by $\psi(\xi)$. By integration over $\xi \in \mathbb{R}$, it gives

$$(6.4) = -\frac{1}{|K|}\int_{t_n}^t \int_{\mathbb{R}} \sum_{L\in\mathcal{N}(K)} a^n_{K\to L}(\xi)\psi(\xi)d\xi ds - \int_{t_n}^t \int_{\mathbb{R}} \partial_\xi\psi(\xi) m^n_K(\xi)d\xi ds. \tag{6.5}$$

By Itō's Formula on the other hand (*cf.* (4.26)), the term (6.3) is equal to

$$\frac{t-t_n}{\Delta t_n}\int_{t_n}^t g_{k,K}(v^n_K)\psi(v^\sharp_\delta(x,s))d\beta_k(s) + \frac{1}{2}\frac{t-t_n}{\Delta t_n}\int_{t_n}^t \mathbf{G}^2_K(v^n_K)\partial_\xi\psi(v^\sharp_\delta(x,s))ds. \tag{6.6}$$

Summing (6.5) and (6.6), we obtain (4.33). □

6.3 Approximate kinetic equation

We will prove now that the Finite Volume scheme (3.4) is consistent with (1.1). Indeed, we will show, using the estimates obtained in Section 5, that an approximate kinetic equation for f_δ in the sense of (2.9) can be deduced from the discrete kinetic formulation (4.33).

Proposition 6.1 (Approximate kinetic equation)**.** *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$. Assume that (1.3), (1.4), (1.6), (3.1) and (4.29) are satisfied. For $\delta \in \mathfrak{d}_T$, assume (4.22). Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(4.24) and let f_δ, ν^δ, m_δ be defined by (4.27), (4.31), (4.32) respectively. If (δ_m) is a sequence in $\mathfrak{d}_T$ which tends to zero according to (4.21), then (f_{δ_m}) is a sequence of approximate generalized solutions to (1.1). Besides, $(f_{\delta_m}(0))$ converges to the equilibrium function $\mathtt{f}_0 = \mathbf{1}_{u_0>\xi}$ in $L^\infty(\mathbb{T}^N\times\mathbb{R})$-weak-*.*

Proof of Proposition 6.1. The last assertion is clear: $(f_{\delta_m}(0))$ converges to the equilibrium function $\mathtt{f}_0 = \mathbf{1}_{u_0>\xi}$ in $L^\infty(\mathbb{T}^N\times\mathbb{R})$-weak-* since, by (3.5), $v^{\delta_m}(0) \to u_0$ a.e.

on $\mathbb{T}^N$. We will show that, for all $t \in [0,T]$, for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$,

$$\begin{aligned}\langle f_\delta(t), \varphi\rangle =& \langle f_\delta(0), \varphi\rangle - \iiint_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} \partial_\xi\varphi(x,\xi)dm_\delta(x,s,\xi) + \varepsilon^\delta(t,\varphi) \\ &+ \int_0^t \langle f_\delta(s), a(\xi)\nabla_x\varphi\rangle ds \qquad (6.7)\\ &+ \int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} g_k(x,\xi)\varphi(x,\xi)d\nu_{x,s}^\delta(\xi)dxd\beta_k(s) \qquad (6.8)\\ &+ \frac{1}{2}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} \mathbf{G}^2(x,\xi)\partial_\xi\varphi(x,\xi)d\nu_{x,s}^\delta(\xi)dxds, \qquad (6.9)\end{aligned}$$

where the error term $\varepsilon^\delta(t,\varphi)$ satisfies

$$\lim_{\delta\to0}\mathbb{E}\left[\sup_{t\in[0,T]}|\varepsilon^\delta(t,\varphi)|^2\right] = 0, \qquad (6.10)$$

for all $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$. Note that the convergence in probability (2.8) follows from (6.10). Given $\varphi \in C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, we introduce the averages over the cells $K \in \mathcal{T}$

$$\varphi_K(\xi) = \frac{1}{|K|}\int_{|K|}\varphi(x,\xi)dx, \quad \xi \in \mathbb{R}. \qquad (6.11)$$

To prove (6.9), we apply the discrete kinetic equation (4.33) to $\xi \mapsto \varphi(x,\xi)$ for a fixed $x \in K$. Then we sum the result over $x \in \mathbb{T}^N$. By the telescopic formula

$$\langle f_\delta(x,t),\varphi\rangle - \langle f_\delta(x,0),\varphi\rangle = \sum_{n=0}^{N_T-1}\langle f_\delta(x,t\wedge t_{n+1}),\varphi\rangle - \langle f_\delta(x,t\wedge t_n),\varphi\rangle,$$

we obtain

$$\begin{aligned}\langle f_\delta(t), \varphi\rangle =& \langle f_\delta(x,0), \varphi\rangle - \iiint_{\mathbb{T}^N\times[0,t]\times\mathbb{R}} \partial_\xi\varphi(\xi)dm_\delta(x,s,\xi) \\ &- \sum_{n=0}^{N_T-1}\sum_{K\in\mathcal{T}_\#}\int_{t\wedge t_n}^{t\wedge t_{n+1}}\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)} a_{K\to L}^n(\xi)\varphi_K(\xi)dsd\xi \qquad (6.12)\\ &+ \int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}} g_{k,\delta}(x,\xi)\varphi(x,\zeta)d\mu_{x,s,t}^\delta(\xi,\zeta)d\beta_k(s) \qquad (6.13)\\ &+ \frac{1}{2}\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}} \mathbf{G}_\delta^2(x,\xi)\partial_\xi\varphi(x,\zeta)d\mu_{x,s,t}^\delta(\xi,\zeta)ds, \qquad (6.14)\end{aligned}$$

where the measure $\mu_{x,s,t}^\delta$ on $\mathbb{R}\times\mathbb{R}$ is defined by

$$\langle\mu_{x,s,t}^\delta,\psi\rangle = \sum_{n=0}^{N_T-1}\frac{t\wedge t_{n+1} - t\wedge t_n}{\Delta t_n}\mathbf{1}_{[t_n,t_{n+1})}(s)\psi(v_\delta(x,s), v_\delta^\sharp(x,s)), \quad \psi \in C_b(\mathbb{R}^2), \qquad (6.15)$$

and the discrete coefficient $g_{k,\delta}(x,\xi)$ is equal to $g_{k,K}(\xi)$ (*cf.* (3.8)) when $x \in K$ (similarly, $\mathbf{G}_\delta(x,\xi) := \mathbf{G}_K(\xi)$, $x \in K$). Note that $\mu^\delta_{x,s,t}$ is simply the Dirac mass at $(v_\delta(x,s), v^\sharp_\delta(x,s))$, except when $t_l \leq s \leq t$ (where l is the index such that $t_l \leq t < t_{l+1}$), in which case it is the same Dirac mass, with an additional multiplicative factor $\frac{t-t_l}{\Delta t_l}$.

The term (6.12) is a discrete space derivative: we will show that it is an approximation of the term (6.7). The two terms (6.13) and (6.14) are close to (6.8) and (6.9) respectively. We analyse those terms separately (see Section 6.3.1, Section 6.3.2). The conclusion of the proof of Proposition 6.1 is given in Section 6.3.3.

6.3.1 Space consistency

Lemma 6.2. *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{d}_T$. Assume that (1.4), (1.6), (3.1), (4.22) and (4.29) are satisfied. Then, for all $\varphi \in C^\infty_c(\mathbb{T}^N \times \mathbb{R})$, we have*

$$\begin{aligned}\int_{\mathbb{T}^N}\int_0^t\int_{\mathbb{R}} a(\xi)\cdot\nabla_x\varphi f_\delta(s)dxdsd\xi \\ = -\sum_{n=0}^{N_T-1}\sum_{K\in\mathcal{T}_\#}\int_{t\wedge t_n}^{t\wedge t_{n+1}}\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)} a^n_{K\to L}(\xi)\varphi_K(\xi)d\xi + \varepsilon^\delta_{\mathrm{space},0}(t,\varphi) + \varepsilon^\delta_{\mathrm{space},1}(t,\varphi),\end{aligned} \tag{6.16}$$

for all $t \in [0,T]$, with the estimates

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}|\varepsilon^\delta_{\mathrm{space},0}(t,\varphi)|^2 \\ \leq T|L_A|^2\|\nabla_x\varphi\|^2_{L^\infty_{x,\xi}}\left[\frac{1}{\theta}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)} + D_0T\left(1+\frac{1}{\theta}\right)\right]\sup_{0\leq n<N_T}\Delta t_n,\end{aligned} \tag{6.17}$$

and, for all compact $\Lambda \subset \mathbb{R}$, for all $\varphi \in C^\infty_c(\mathbb{T}^N \times \mathbb{R})$ supported in $\mathbb{T}^N \times \Lambda$,

$$\mathbb{E}\sup_{t\in[0,T]}|\varepsilon^\delta_{\mathrm{space},1}(t,\varphi)|^2 \leq \frac{16L_AT}{\alpha_N^2}|\Lambda|^2\|\partial_\xi\nabla_x\varphi\|^2_{L^\infty_{x,\xi}}\left[\frac{1}{\theta}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)} + \frac{2D_0T}{\theta}\right] h. \tag{6.18}$$

Proof of Lemma 6.2. To begin with, we replace f_δ by $\mathtt{f}_\delta$ in the left-hand side of (6.16). This accounts for the first error term

$$\varepsilon^\delta_{\mathrm{space},0}(t,\varphi) = \int_{\mathbb{T}^N}\int_0^t\int_{\mathbb{R}} a(\xi)\cdot\nabla_x\varphi(f_\delta(s) - \mathtt{f}_\delta(s))dxdsd\xi.$$

By Lemma 4.2, we have the estimate (6.17) for $\varepsilon^\delta_{\mathrm{space},0}(t,\varphi)$. Then, we use the following development:

$$\begin{aligned}\int_{\mathbb{T}^N}\int_0^t\int_{\mathbb{R}} a(\xi)\cdot\nabla_x\varphi\mathtt{f}_\delta(s))dxdsd\xi \\ = \sum_{n=0}^{N_T-1}\int_{t\wedge t_n}^{t\wedge t_{n+1}}\int_{\mathbb{R}}\sum_{K\in\mathcal{N}(K)}\int_K a(\xi)\cdot\nabla_x\varphi\mathtt{f}_\delta(s)dxdsd\xi,\end{aligned} \tag{6.19}$$

Since $\mathtt{f}_\delta(s)$ has a constant value $\mathtt{f}_K^n$ in $K\times[t_n,t_{n+1})$, we obtain, using the Stokes formula,

$$\int_{\mathbb{T}^N}\int_0^t\int_{\mathbb{R}} a(\xi)\cdot\nabla_x\varphi\mathtt{f}_\delta(s)dxdsd\xi \\ =\sum_{n=0}^{N_T-1}\int_{t\wedge t_n}^{t\wedge t_{n+1}}\int_{\mathbb{R}}\sum_{K\in\mathcal{N}(K)}\sum_{L\in\mathcal{N}(K)} a^*_{K\to L}(\xi)\varphi_{K|L}\mathtt{f}_K^n dsd\xi, \tag{6.20}$$

where $a^*_{K\to L}(\xi)$ is defined by (4.3) and $\varphi_{K|L}$ is the mean-value of φ over $K|L$:

$$\varphi_{K|L}(\xi)=\frac{1}{|K|L|}\int_{K|L}\varphi(x,\xi)d\mathcal{H}^{N-1}(x).$$

We add a corrective term to (6.20) to obtain

$$\int_{\mathbb{T}^N}\int_0^t\int_{\mathbb{R}} a(\xi)\cdot\nabla_x\varphi\mathtt{f}_\delta(s)dxdsd\xi \\ =\sum_{n=0}^{N_T-1}\int_{t\wedge t_n}^{t\wedge t_{n+1}}\int_{\mathbb{R}}\sum_{K\in\mathcal{N}(K)}\sum_{L\in\mathcal{N}(K)} a^*_{K\to L}(\xi)(\varphi_{K|L}-\varphi_K)\mathtt{f}_K^n dsd\xi. \tag{6.21}$$

Equation (6.21) follows indeed from (6.20) by the anti-symmetry property (5.26) of $a_{K\to L}$. Note that Equation (6.21) is more natural than Equation (6.20) (when one has in mind the decomposition of a volume integral over each cell K), by use of the correspondence

$$a(\xi)\cdot\nabla_x\varphi\simeq\sum_{L\in\mathcal{N}(K)} a^*_{K\to L}(\xi)(\varphi_{K|L}-\varphi_K)\text{ in }K.$$

By (5.26), the discrete convective term in (6.12) is

$$\sum_{n=0}^{N_T-1}\sum_{K\in\mathcal{T}_\#}\int_{t\wedge t_n}^{t\wedge t_{n+1}}\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)} a^n_{K\to L}(\xi)(\varphi_{K|L}-\varphi_K)dsd\xi. \tag{6.22}$$

To estimate how close is the right-hand side of (6.21) to (6.22), we have to compare $a^n_{K\to L}(\xi)$ and $a^*_{K\to L}(\xi)\mathtt{f}_K^n(\xi)$. Let $\gamma\in W^{1,1}(\mathbb{R}_\xi)$. If $v_K^n\leq v_L^n$, then

$$\int_{\mathbb{R}}\gamma(\xi)\left[a^*_{K\to L}(\xi)\mathtt{f}_K^n(\xi)-a^n_{K\to L}(\xi)\right]d\xi=-\int_{\mathbb{R}}\gamma(\xi)\bar{\mathtt{f}}_K^n(\xi)a^n_{K\to L}(\xi)d\xi \tag{6.23}$$

by the consistency hypothesis (4.11) and the support condition (4.15). Using an integration by parts and (4.16), we obtain

$$\int_{\mathbb{R}}\gamma(\xi)\left[a^*_{K\to L}(\xi)\mathtt{f}_K^n(\xi)-a^n_{K\to L}(\xi)\right]d\xi=\int_{\mathbb{R}}\gamma'(\xi)\Phi^n_{K\to L}(\xi\vee v_K^n)d\xi. \tag{6.24}$$

Similarly, if $v_L^n \leq v_K^n$, then

$$\int_{\mathbb{R}} \gamma(\xi)\left[a^*_{K\to L}(\xi)\mathtt{f}^n_K(\xi) - a^n_{K\to L}(\xi)\right] d\xi = -\int_{\mathbb{R}} \gamma'(\xi)\overline{\Phi}^n_{K\to L}(\xi \wedge v_K^n) d\xi. \tag{6.25}$$

We deduce that (6.16) is satisfied with an error term

$$\begin{aligned}\varepsilon^\delta_{\mathrm{space},1}(t,\varphi) := &\int_{\mathbb{T}^N}\int_0^t\int_{\mathbb{R}} a(\xi)\cdot\nabla_x\varphi\,\mathtt{f}_\delta(s)dxdsd\xi \\ &+ \sum_{n=0}^{N_T-1}\sum_{K\in\mathcal{T}_\#}\int_{t\wedge t_n}^{t\wedge t_{n+1}}\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)} a^n_{K\to L}(\xi)\varphi_K(\xi)d\xi,\end{aligned} \tag{6.26}$$

which is bounded as follows:

$$\begin{aligned}|\varepsilon^\delta_{\mathrm{space},1}(t,\varphi)| \leq &\sum_{n=0}^{N_T-1}\int_{t\wedge t_n}^{t\wedge t_{n+1}}\sum_{K\in\mathcal{T}_\#}\int_{\mathbb{R}}\sum_{L\in\mathcal{N}(K)}|\partial_\xi\varphi_{K|L} - \partial_\xi\varphi_K| \\ &\times\left[\mathbf{1}_{v_K^n\leq v_L^n}|\Phi^n_{K\to L}(\xi\vee v_K^n)| + \mathbf{1}_{v_L^n<v_K^n}|\overline{\Phi}^n_{K\to L}(\xi\wedge v_K^n)|\right]d\xi,\end{aligned} \tag{6.27}$$

By (4.22), we have

$$|\partial_\xi\varphi_{K|L}(\xi) - \partial_\xi\varphi_K(\xi)| \leq \|\partial_\xi\nabla_x\varphi(\cdot,\xi)\|_{L^\infty_x} h,$$

for all $\xi\in\mathbb{R}$. If φ is compactly supported in $\mathbb{T}^N\times\Lambda$, we obtain thus the bound

$$|\varepsilon^\delta_{\mathrm{space},1}(t,\varphi)| \leq \|\partial_\xi\nabla_x\varphi\|_{L^\infty_{x,\xi}}|\Lambda| B_{\mathrm{space}}\, h, \tag{6.28}$$

where B_{space} is equal to

$$\sum_{n=0}^{N_T-1}\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\left[\mathbf{1}_{v_K^n\leq v_L^n}\sup_{\xi\in\mathbb{R}}|\Phi^n_{K\to L}(\xi\vee v_K^n)| + \mathbf{1}_{v_L^n<v_K^n}\sup_{\xi\in\mathbb{R}}|\overline{\Phi}^n_{K\to L}(\xi\wedge v_K^n)|\right].$$

We seek for a bound of order $h^{-1/2}$ on B_{space}. For notational convenience we will estimate only the first part

$$B^1_{\mathrm{space}} := \sum_{n=0}^{N_T-1}\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\mathbf{1}_{v_K^n\leq v_L^n}\sup_{\xi\in\mathbb{R}}|\Phi^n_{K\to L}(\xi\vee v_K^n)|,$$

since the bound on the second part in B_{space} will be similar. By the Cauchy Schwarz inequality, we have

$$\begin{aligned}\mathbb{E}|B^1_{\mathrm{space}}|^2 \leq &\sum_{n=0}^{N_T-1}\Delta t_n\sum_{K\in\mathcal{T}_\#}|\partial K| \\ &\times\mathbb{E}\sum_{n=0}^{N_T-1}\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\frac{\mathbf{1}_{v_K^n\leq v_L^n}}{|K|L|}\sup_{\xi\in\mathbb{R}}|\Phi^n_{K\to L}(\xi\vee v_K^n)|^2.\end{aligned}$$

We use the estimate (5.44), which gives

$$|\Phi^n_{K\to L}(\xi\vee v^n_K)|^2 \le 2L_A|K|L|\int_{\mathbb{R}}(\bar{\mathfrak{f}}^n_L-\bar{\mathfrak{f}}^n_K)\Phi^n_{K\to L}(\xi)d\xi,$$

due to (5.6). We also use (4.23), and get

$$\mathbb{E}|B^1_{\text{space}}|^2 \le \frac{2L_AT}{\alpha_N^2 h}\mathbb{E}\sum_{n=0}^{N_T-1}\Delta t_n\sum_{K\in\mathcal{T}_\#}\sum_{L\in\mathcal{N}(K)}\int_{\mathbb{R}}(\bar{\mathfrak{f}}^n_L-\bar{\mathfrak{f}}^n_K)\Phi^n_{K\to L}(\xi)d\xi.$$

With (5.20) and (6.28), we conclude with (6.18). □

6.3.2 Stochastic terms

Lemma 6.3. *Let $u_0\in L^\infty(\mathbb{T}^N)$, $T>0$ and $\delta\in\mathfrak{d}_T$. Assume that (1.4), (1.6), (3.1) and (4.29) are satisfied. Then, for all $\varphi\in C^\infty_c(\mathbb{T}^N\times\mathbb{R})$, we have*

$$\begin{aligned}&\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}} g_{k,\delta}(x,\xi)\varphi(x,\zeta)d\mu^\delta_{x,s,t}(\xi,\zeta)d\beta_k(s)\\ &=\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} g_k(x,\xi)\varphi(x,\xi)d\nu^\delta_{x,s}(\xi)dxd\beta_k(s)+\varepsilon^\delta_{\mathrm{W},1}(t,\varphi)+\varepsilon^\delta_{\mathrm{W},2}(t,\varphi),\end{aligned}\tag{6.29}$$

and

$$\begin{aligned}&\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}} \mathbf{G}^2_\delta(x,\xi)\partial_\xi\varphi(x,\zeta)d\mu^\delta_{x,s,t}(\xi,\zeta)ds\\ &=\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}} \mathbf{G}^2(x,\xi)\partial_\xi\varphi(x,\xi)d\nu^\delta_{x,s}(\xi)dxds+\varepsilon^\delta_{\mathrm{W},3}(t,\varphi)+\varepsilon^\delta_{\mathrm{W},4}(t,\varphi),\end{aligned}\tag{6.30}$$

where

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\varepsilon^\delta_{\mathrm{W},1}(t,\varphi)|^2\right]\le 4D_1T\|\varphi\|^2_{L^\infty_{x,\xi}}h^2+2D_0\|\varphi\|^2_{L^\infty_{x,\xi}}\left[\sup_{0\le n<N_T}\Delta t_n\right],\tag{6.31}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\varepsilon^\delta_{\mathrm{W},3}(t,\varphi)|^2\right]\le 4D_1T\|\partial_\xi\varphi\|^2_{L^\infty_{x,\xi}}h^2+2D_0\|\partial_\xi\varphi\|^2_{L^\infty_{x,\xi}}\left[\sup_{0\le n<N_T}\Delta t_n\right],\tag{6.32}$$

and

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}|\varepsilon^\delta_{\mathrm{W},2}(t,\varphi)|^2\right]\le& 2D_0\|\varphi\|^2_{L^\infty_{x,\xi}}\left[\sup_{0\le n<N_T}\Delta t_n\right]\\ &+8\left\{D_1\|\varphi\|^2_{L^\infty_{x,v}}+D_0\|\partial_\xi\varphi\|^2_{L^\infty_{x,\xi}}\right\}\\ &\times\left[\frac{1}{\theta}\|v_\delta(0)\|^2_{L^2(\mathbb{T})}+\frac{3D_0T}{\theta}\right]\left[\sup_{0\le n<N_T}\Delta t_n\right]^{1/2}.\end{aligned}\tag{6.33}$$

Eventually, $\varepsilon^\delta_{\mathrm{W},4}(t,\varphi)$ satisfies the same estimate as $\varepsilon^\delta_{\mathrm{W},2}(t,\varphi)$ with $\partial_\xi\varphi$ instead of φ in the right-hand side of (6.33).

Proof of Lemma 6.3. Define

$$\varepsilon^{\delta}_{\mathrm{W},1}(t,\varphi)=\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}}\left[g_{k,\delta}(x,\xi)-g_k(x,\xi)\right]\varphi(x,\zeta)d\mu^{\delta}_{x,s,t}(\xi,\zeta)dxd\beta_k(s),$$

and let $\varepsilon^{\delta}_{\mathrm{W},2}(t,\varphi)$ be equal to

$$\int_0^t\int_{\mathbb{T}^N}\left[\int_{\mathbb{R}\times\mathbb{R}}g_k(x,\xi)\varphi(x,\zeta)d\mu^{\delta}_{x,s,t}(\xi,\zeta)-\int_{\mathbb{R}}g_k(x,\xi)\varphi(x,\xi)d\nu^{\delta}_{x,s}(\xi)\right]dxd\beta_k(s).$$

Then (6.29) is satisfied. Note that $n\mapsto\varepsilon^{\delta}_{\mathrm{W},1}(t_n,\varphi)$ is a $(\mathcal{F}_{t_n})$-martingale. By Doob's Inequality, Jensen's Inequality (note that $\mu^{\delta}_{x,s,t}(\mathbb{R}\times\mathbb{R})\leq 1$) and (3.10), we deduce

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq n<N_T}|\varepsilon^{\delta}_{\mathrm{W},1}(t_n,\varphi)|^2\right]\\
\leq&4\mathbb{E}\int_0^{t_{N_T}}\left|\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}}\left[g_{k,\delta}(x,\xi)-g_k(x,\xi)\right]\varphi(x,\zeta)d\mu^{\delta}_{x,s,t_{N_T}}(\xi,\zeta)\right|^2dxds\\
\leq&4\mathbb{E}\int_0^{t_{N_T}}\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}}|g_{k,\delta}(x,\xi)-g_k(x,\xi)|^2\,d\mu^{\delta}_{x,s,t}(\xi,\zeta)dxds\|\varphi\|^2_{L^\infty_{x,\xi}}\\
\leq&4D_1T\|\varphi\|^2_{L^\infty_{x,\xi}}h^2.\end{aligned}$$

Besides, we see, using Itō's Isometry, and (1.2), (3.9), that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[t_n,t_{n+1})}|\varepsilon^{\delta}_{\mathrm{W},1}(t,\varphi)-\varepsilon^{\delta}_{\mathrm{W},1}(t_n,\varphi)|^2\right]\\
\leq&\mathbb{E}\int_{t_n}^{t_{n+1}}\left|\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}}\left[g_{k,\delta}(x,v_\delta(s,x))-g_k(x,v_\delta(s,x))\right]\varphi(x,v^\sharp_\delta(s,x))\right|^2dxds\\
\leq&2D_0\|\varphi\|^2_{L^\infty_{x,\xi}}\left[\sup_{0\leq n<N_T}\Delta t_n\right].\end{aligned}$$

Similarly, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\varepsilon^{\delta}_{\mathrm{W},2}(t,\varphi)|^2\right]\leq 2D_0\|\varphi\|^2_{L^\infty_{x,\xi}}\left[\sup_{0\leq n<N_T}\Delta t_n\right]+\mathbb{E}\left[\sup_{0\leq n<N_T}|\varepsilon^{\delta}_{\mathrm{W},2}(t_n,\varphi)|^2\right].$$

Using Doob's Inequality, we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\varepsilon^{\delta}_{\mathrm{W},2}(t,\varphi)|^2\right]\leq 2D_0\|\varphi\|^2_{L^\infty_{x,\xi}}\left[\sup_{0\leq n<N_T}\Delta t_n\right]+4\mathbb{E}|\varepsilon^{\delta}_{\mathrm{W},2}(t_{N_T},\varphi)|^2.$$

By Itō's Formula, $\mathbb{E}|\varepsilon^{\delta}_{\mathrm{W},2}(t_{N_T},\varphi)|^2$ is bounded from above by

$$\mathbb{E}\int_0^{t_{N_T}}\int_{\mathbb{T}^N}\sum_{k\geq1}\left|\int_{\mathbb{R}\times\mathbb{R}}g_k(x,\xi)\varphi(x,\zeta)d\mu^{\delta}_{x,s,t_{N_T}}(\xi,\zeta)-\int_{\mathbb{R}}g_k(x,\xi)\varphi(x,\xi)d\nu^{\delta}_{x,s}(\xi)\right|^2dxds. \tag{6.34}$$

We have, for $t \in [0,T)$, $t \in [t_n, t_{n+1})$, $n < N_T$, and $\psi \in C_b(\mathbb{R}\times\mathbb{R})$,

$$\begin{aligned}\langle \mu^\delta_{x,t,t_{N_T}}, \psi\rangle =& \langle \nu_{x,t}\otimes\nu_{x,t}, \psi\rangle - \frac{t-t_n}{\Delta t_n}\left[\psi(v^\sharp_\delta(x,t), v^\sharp_\delta(x,t)) - \psi(v_\delta(x,t), v^\sharp_\delta(x,t))\right] \\ &- \frac{t_{n+1}-t}{\Delta t_n}\left[\psi(v_\delta(x,t), v_\delta(x,t)) - \psi(v_\delta(x,t), v^\sharp_\delta(x,t))\right].\end{aligned}$$

We estimate therefore (6.34) by the two terms

$$2\mathbb{E}\sum_{0\le n<N_T}\int_{t_n}^{t_{n+1}}\int_{\mathbb{T}^N}\sum_{k\ge1}\left|g_k(x, v^\sharp_\delta(x,t)) - g_k(x, v_\delta(x,t))\right|^2 |\varphi(x,\xi)|^2 dxdt, \tag{6.35}$$

and

$$2\mathbb{E}\sum_{0\le n<N_T}\int_{t_n}^{t_{n+1}}\int_{\mathbb{T}^N}\sum_{k\ge1}\left|\varphi(x, v^\sharp_\delta(x,t)) - \varphi(x, v_\delta(x,t))\right|^2 |g_k(x, v_\delta(x,t))|^2 dxdt. \tag{6.36}$$

Note that (1.3) gives, for all $\eta > 0$, and $\overline{v}$, $v \in \mathbb{R}$,

$$\sum_{k\ge1}|g_k(x,\overline{v}) - g_k(x,v)|^2 \le D_1|\overline{v}-v| \le D_1\left(\eta + \frac{1}{\eta}|\overline{v}-v|^2\right). \tag{6.37}$$

In virtue of (6.37), we can bound (6.35) by

$$2D_1\|\varphi\|^2_{L^\infty_{x,v}}\left[\eta + \frac{1}{\eta}\mathbb{E}\int_0^T \|v^\sharp_\delta(t) - v_\delta(t)\|^2_{L^2(\mathbb{T}^N)^2}dt\right].$$

Using (6.2) and taking $\eta = \left[\sup_{0\le n<N_T}\Delta t_n\right]^{1/2}$, we deduce that (6.35) is bounded by

$$2D_1\|\varphi\|^2_{L^\infty_{x,\xi}}\left[\frac{1}{\theta}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)} + \frac{3D_0T}{\theta}\right]\left[\sup_{0\le n<N_T}\Delta t_n\right]^{1/2}. \tag{6.38}$$

An estimate on (6.36) is obtained as follows: (6.36) is bounded by

$$2D_0\|\partial_\xi\varphi\|^2_{L^\infty_{x,\xi}}\mathbb{E}\int_0^T \|v^\sharp_\delta(t) - v_\delta(t)\|^2_{L^2(\mathbb{T}^N)^2}dt.$$

Using (6.2) gives an estimate on (6.36) from above by

$$2D_0\|\partial_\xi\varphi\|^2_{L^\infty_{x,\xi}}\left[\frac{1}{\theta}\|v_\delta(0)\|^2_{L^2(\mathbb{T}^N)} + \frac{3D_0T}{\theta}\right]\left[\sup_{0\le n<N_T}\Delta t_n\right]. \tag{6.39}$$

Next, we denote by $\varepsilon^\delta_{\mathrm{W},3}(t,\varphi)$ and $\varepsilon^\delta_{\mathrm{W},4}(t,\varphi)$ the error terms

$$\int_0^t\int_{\mathbb{T}^N}\int_{\mathbb{R}\times\mathbb{R}}\left[\mathbf{G}^2_{k,\delta}(x,\xi) - \mathbf{G}^2_k(x,\xi)\right]\partial_\xi\varphi(x,\zeta)d\mu^\delta_{x,s,t}(\xi,\zeta)dxds,$$

$$\int_0^t\int_{\mathbb{T}^N}\left[\int_{\mathbb{R}\times\mathbb{R}}\mathbf{G}^2_k(x,\xi)\partial_\xi\varphi(x,\zeta)d\mu^\delta_{x,s,t}(\xi,\zeta) - \int_{\mathbb{R}}\mathbf{G}^2_k(x,\xi)\partial_\xi\varphi(x,\xi)d\nu^\delta_{x,s}(\xi)\right]dxds.$$

We have, for $x\in K$, $\eta>0$,

$$\begin{aligned}|\mathbf{G}^2_{k,\delta}(x,\xi)-\mathbf{G}^2_k(x,\xi)|&=\left|\sum_{k\geq 1}(g_{k,K}(\xi)-g_k(x,\xi))(g_{k,K}(\xi)+g_k(x,\xi))\right|\\&\leq\frac{1}{2\eta}\sum_{k\geq 1}|g_{k,K}(\xi)-g_k(x,\xi)|^2+\eta\sum_{k\geq 1}|g_{k,K}(\xi)|^2+|g_k(x,\xi)|^2.\end{aligned}$$

Using (3.10), (1.2) and (3.9) and taking $\eta=h$, we see that

$$|\mathbf{G}^2_{k,\delta}(x,\xi)-\mathbf{G}^2_k(x,\xi)|\leq(D_0+D_1)h. \tag{6.40}$$

This is sufficient to obtain (6.32) and the last statement of the lemma (estimate on $\varepsilon^\delta_{\mathrm{W},4}(t,\varphi)$). □

6.3.3 Conclusion

To conclude, let us set

$$\varepsilon_\delta(\varphi)=\sum_{j=1}^4\varepsilon^\delta_{\mathrm{W},j}(t,\varphi)-\varepsilon^\delta_{\mathrm{space},0}(t,\varphi)-\varepsilon^\delta_{\mathrm{space},1}(t,\varphi).$$

Then the approximate kinetic equation (6.9) follows from the discrete kinetic equation (6.14) and from the consistency estimates (6.16)-(6.29)-(6.30). Since $\|v_\delta(0)\|_{L^2(\mathbb{T}^N)}\leq\|u_0\|_{L^2(\mathbb{T}^N)}$ (the projection (3.5) onto piecewise-constant functions is an orthogonal projection in $L^2(\mathbb{T}^N)$), it follows from the error estimates (6.17), (6.18), (6.31), (6.32), (6.33) and from the CFL condition (4.29) that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\varepsilon^\delta(t,\varphi)|^2\right]\leq C(\varphi)|\delta|^{1/2},$$

where $C(\varphi)$ is a constant that depends on $\|u_0\|_{L^2(\mathbb{T}^N)}$, on D_0, D_1, L_A, on the parameter θ in (4.29), on T, on $|\Lambda|$, where Λ is the support of φ, and on the norms $\|\partial^{j_i}_{x_i}\partial^k_\xi\varphi\|_{L^\infty(\mathbb{T}^N\times\mathbb{R})}$ with $j_i+k\leq 2$.

7 Convergence

To apply Theorem 2.2 on the basis of Proposition 6.1, we need to establish some additional estimates on the numerical Young measure ν^δ and on the numerical random measure m^δ. This is done in Section 7.1. We conclude to the convergence of the Finite Volume method in Section 7, Theorem 7.4.

7.1 Additional estimates

7.1.1 Tightness of (ν^δ)

Lemma 7.1 (Tightness of (ν^δ))**.** *Let $u_0 \in L^\infty(\mathbb{T}^N)$, $T > 0$ and $\delta \in \mathfrak{d}_T$. Assume that (1.4), (1.6), (3.1) and (4.29) are satisfied. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(4.24) and let ν^δ be defined by (4.31). Let $p \in [1, +\infty)$. We have*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} (1 + |\xi|^p) d\nu^\delta_{x,t}(\xi) dx\right) \leq C_p, \tag{7.1}$$

where C_p is a constant depending on D_0, p, T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$ only.

Proof of Lemma 7.1. It is sufficient to do the proof for $p \in 2\mathbb{N}^*$ since $1 + |\xi|^p \leq 2(1 + |\xi|^q)$ for all $\xi \in \mathbb{R}$ if $q \geq p$. Note that

$$\int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu^\delta_{x,t}(\xi) dx = \frac{t - t_n}{\Delta t_n} \|v^\sharp_\delta(t)\|^p_{L^p(\mathbb{T}^N)} + \frac{t_{n+1} - t}{\Delta t_n} \|v_\delta(t)\|^p_{L^p(\mathbb{T}^N)},$$

for $t \in [t_n, t_{n+1})$. Recall also that $v^\sharp_\delta$ is defined by (4.26). Let

$$\varphi_p(\xi) = p\xi^{p-1} = \partial_\xi \xi^p \quad \xi \in \mathbb{R}.$$

We multiply Equation (4.7) by $\varphi_p(\xi)$ and sum the result over K, ξ. We obtain then, using (5.26),

$$\|v^\flat_\delta(t_{n+1})\|^p_{L^p(\mathbb{T}^N)} + p(p-1)\Delta t_n \sum_{K\in\mathcal{T}_\#} |K| \int_{\mathbb{R}} \xi^{p-2} m^n_K(\xi) d\xi = \|v_\delta(t_n)\|^p_{L^p(\mathbb{T}^N)}. \tag{7.2}$$

In particular, we have the L^p estimate

$$\|v^\flat_\delta(t_{n+1})\|^p_{L^p(\mathbb{T}^N)} \leq \|v_\delta(t_n)\|^p_{L^p(\mathbb{T}^N)}. \tag{7.3}$$

Let us now estimate the increase of L^p-norm due to the stochastic evolution. By Itō's Formula and (4.26), we have

$$\begin{aligned}
&|v^{n+1}_K|^p \\
&= |v^{n+1/2}_K|^p + p\int_{t_n}^{t_{n+1}} v^\sharp_K(t)^{p-1} g_{k,K}(v^n_K) d\beta_k(t) + \frac{1}{2}p(p-1)\int_{t_n}^{t_{n+1}} v^\sharp_K(t)^{p-2} \mathbf{G}^2_K(v^n_K) dt,
\end{aligned}$$

and thus

$$\begin{aligned}
\|v_\delta(t_{n+1})\|^p_{L^p(\mathbb{T}^N)} = \|v^\flat_\delta(t_{n+1})\|^p_{L^p(\mathbb{T}^N)} &+ p\int_{t_n}^{t_{n+1}} \langle v^\sharp_\delta(t)^{p-1}, \gamma^n_k\rangle_{L^2(\mathbb{T}^N)} d\beta_k(t) \\
&+ \frac{1}{2}p(p-1)\int_{t_n}^{t_{n+1}} \langle v^\sharp_\delta(t)^{p-2}, \Gamma^n\rangle_{L^2(\mathbb{T}^N)} dt,
\end{aligned} \tag{7.4}$$

where

$$\gamma_k^n(x) = g_{k,K}(v_K^n), \quad \Gamma^n(x) = \mathbf{G}_K^2(v_K^n), \quad x \in K.$$

Using (7.3) and induction, we obtain

$$\|v_\delta(T)\|_{L^p(\mathbb{T}^N)}^p \leq \|v_\delta(0)\|_{L^p(\mathbb{T}^N)}^p + M_{N_T} + B_{N_T}, \tag{7.5}$$

where (M_N) is the martingale

$$M_N = p \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \langle v_\delta^\sharp(t)^{p-1}, \gamma_k^n \rangle_{L^2(\mathbb{T}^N)} d\beta_k(t)$$

and

$$B_N = \frac{1}{2} p(p-1) \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \langle v_\delta^\sharp(t)^{p-2}, \Gamma^n \rangle_{L^2(\mathbb{T}^N)} dt. \tag{7.6}$$

Note that the argument $\langle v_\delta^\sharp(t)^{p-2}, \Gamma^n \rangle_{L^2(\mathbb{T}^N)}$ in B_N is non-negative since $\Gamma^n \geq 0$ and $p-2 \in 2\mathbb{N}$. Consequently, $\mathbb{E} \sup_{0 \leq n \leq N_T} B_n = \mathbb{E} B_{N_T}$ and we deduce the following bound

$$\mathbb{E} \sup_{0 \leq n \leq N_T} B_n \leq \frac{1}{2} p(p-1) D_0 \sum_{n=0}^{N_T-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} dt. \tag{7.7}$$

We have used (3.9) to obtain (7.7). If $p = 2$, then $\mathbb{E}\|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} = 1$ and is therefore bounded. To estimate $\mathbb{E}\|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2}$ when $p \geq 4$, note that $v_\delta^\sharp(t) = v_\delta^\flat(t_{n+1}) + z_\delta^n(t)$ for $t \in (t_n, t_{n+1})$, where $z_\delta^n(x,t) := \gamma_k^n(x)(\beta_k(t) - \beta_k(t_n))$ is, conditionally to $\mathcal{F}_n$, a Gaussian random variable with variance, for $x \in K$,

$$\mathbb{E}\left[|z_K^n(t)|^2 | \mathcal{F}_n\right] = (t - t_n) \mathbf{G}_K^2(v_K^n) \leq D_0 \Delta t_n$$

by (3.9). In particular, we have the bound

$$\begin{aligned} \mathbb{E}\|z_\delta^n(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} &= \sum_{K \in \mathcal{T}_\#} |K| \mathbb{E}\left(\mathbb{E}\left[|z_K^n(t)|^{p-2} | \mathcal{F}_n\right]\right) \\ &= C(p) \sum_{K \in \mathcal{T}_\#} |K| \mathbb{E}\left(\mathbb{E}\left[|z_K^n(t)|^2 | \mathcal{F}_n\right]\right)^{(p-2)/2} \leq C(p)(D_0 \Delta t_n)^{(p-2)/2}, \end{aligned}$$

where $C(p)$ is a constant depending on p. It follows, using (7.3), that we have the estimate

$$\begin{aligned} \mathbb{E}\|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} &\leq C(p, D_0)\left(1 + \mathbb{E}\|v_\delta^\flat(t_{n+1})\|_{L^{p-2}(\mathbb{T}^N)}^{p-2}\right) \\ &\leq C(p, D_0)\left(1 + \mathbb{E}\|v_\delta(t_n)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2}\right), \end{aligned}$$

where $C(p, D_0)$ is a constant depending on p and D_0. In particular, we have

$$\sup_{0 \leq n < N_T} \sup_{t \in (t_n, t_{n+1})} \mathbb{E}\|v_\delta^\sharp(t)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2} \leq C(p, D_0)\left(1 + \sup_{0 \leq n < N_T} \mathbb{E}\|v_\delta(t_n)\|_{L^{p-2}(\mathbb{T}^N)}^{p-2}\right). \tag{7.8}$$

By (7.7), we conclude that

$$\mathbb{E} \sup_{0\le n\le N_T} B_n \le C(p,D_0)T\left(1+\sup_{0\le n<N_T}\mathbb{E}\|v_\delta(t_n)\|^{p-2}_{L^{p-2}(\mathbb{T}^N)}\right), \tag{7.9}$$

for possibly a different constant $C(p,D_0)$. Let us now turn to the estimate of the quantity $\mathbb{E}\sup_{0\le n\le N_T}|M_n|$. The martingale (M_N) can be rewritten as a stochastic integral (with an integrand which is a simple function). Consequently, the quadratic variation of M_{N_T} is

$$\begin{aligned}
\langle M_{N_T}\rangle &= p^2\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}}\sum_k |\langle |v_\delta^\sharp(t)|^{p-1},\gamma_k^n\rangle_{L^2(\mathbb{T}^N)}|^2 dt \\
&\le p^2\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}}\sum_k \||v_\delta^\sharp(t)|^{p-1}\|^2_{L^2(\mathbb{T}^N)}\|\gamma_k^n\|^2_{L^2(\mathbb{T}^N)} dt \\
&= p^2\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}} \|v_\delta^\sharp(t)\|^{2(p-1)}_{L^{2(p-1)}(\mathbb{T}^N)}\|\Gamma^n\|_{L^1(\mathbb{T}^N)} dt \\
&\le p^2 D_0\sum_{n=0}^{N_T-1}\int_{t_n}^{t_{n+1}} \|v_\delta^\sharp(t)\|^{2(p-1)}_{L^{2(p-1)}(\mathbb{T}^N)} dt,
\end{aligned}$$

by (3.9). Using (7.8) (with $2p$ instead of p) gives thus

$$\mathbb{E}\langle M_{N_T}\rangle \le p^2 D_0 T C(2p,D_0)\left(1+\sup_{0\le n<N_T}\mathbb{E}\|v_\delta(t_n)\|^{2p-2}_{L^{2p-2}(\mathbb{T}^N)}\right). \tag{7.10}$$

By Burkholder - Davis - Gundy's Inequality, there exists a constant C_{BDG} such that

$$\mathbb{E}\sup_{0\le n\le N_T}|M_n| \le C_{\mathrm{BDG}}\mathbb{E}\langle M_{N_T}\rangle^{1/2}.$$

By Jensen's Inequality and the estimate (7.10), we obtain

$$\begin{aligned}
\mathbb{E}\sup_{0\le n\le N_T}|M_n| &\le C_{\mathrm{BDG}}(\mathbb{E}\langle M_{N_T}\rangle)^{1/2} \\
&\le C_{\mathrm{BDG}}p(D_0TC(2p,D_0))^{1/2}\left(1+\sup_{0\le n<N_T}\mathbb{E}\|v_\delta(t_n)\|^{2p-2}_{L^{2p-2}(\mathbb{T}^N)}\right)^{1/2}.
\end{aligned} \tag{7.11}$$

We can conclude now. Since $\mathbb{E}M_{N_T}=0$, taking expectation in (7.5) (where we replace N_T by n) gives

$$\mathbb{E}\|v_\delta(t_n)\|^p_{L^p(\mathbb{T}^N)} \le \|v_\delta(0)\|^p_{L^p(\mathbb{T}^N)} + \mathbb{E}B_n.$$

Note (see Section 6.3.3) that

$$\|v_\delta(0)\|_{L^p(\mathbb{T}^N)} \le \|u_0\|_{L^p(\mathbb{T}^N)} \le \|u_0\|_{L^\infty(\mathbb{T}^N)}.$$

By (7.9), this gives

$$\sup_{0\le n<N_T} \mathbb{E}\|v_\delta(t_n)\|^p_{L^p(\mathbb{T}^N)} \le \|u_0\|^p_{L^\infty(\mathbb{T}^N)} + C(p,D_0)T\left(1+\sup_{0\le n<N_T}\mathbb{E}\|v_\delta(t_n)\|^{p-2}_{L^{p-2}(\mathbb{T}^N)}\right).$$

By iteration on $p\in 2\mathbb{N}^*$, we deduce, for every such p, that

$$\sup_{0\le n<N_T} \mathbb{E}\|v_\delta(t_n)\|^p_{L^p(\mathbb{T}^N)} \le C_p, \tag{7.12}$$

where the constant C_p depends on p, D_0, T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$. Denote generally by C_p any such constant, possibly different from line to line, depending only on p, D_0, T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$. By (7.12) with $2p-2$ instead of p, we have $\mathbb{E}\sup_{0\le n\le N_T}|M_n|\le C_p$. Then we use (7.9) with $p-2$ instead of p to obtain $\mathbb{E}\sup_{0\le n\le N_T} B_n\le C_p$. By (7.5), we deduce

$$\mathbb{E}\sup_{0\le n<N_T} \|v_\delta(t_n)\|^p_{L^p(\mathbb{T}^N)} \le C_p, \tag{7.13}$$

which concludes the proof of the lemma. □

Remark 7.1. Summing over n and taking expectation in (7.4) gives the estimate

$$\mathbb{E}\sum_{0\le n<N_T} \|v_\delta(t_{n+1})\|^p_{L^p(\mathbb{T}^N)} - \|v_\delta^\flat(t_{n+1})\|^p_{L^p(\mathbb{T}^N)} = \mathbb{E}B_{N_T}.$$

A corollary of (7.9) and (7.12) is the bound

$$\mathbb{E}\sum_{0\le n<N_T} \|v_\delta(t_{n+1})\|^p_{L^p(\mathbb{T}^N)} - \|v_\delta^\flat(t_{n+1})\|^p_{L^p(\mathbb{T}^N)} \le C_p, \tag{7.14}$$

where C_p is a constant depending on D_0, p, T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$ only

7.1.2 Tightness of (m_δ)

Lemma 7.2 (Tightness of (m_δ))**.** *Let $u_0\in L^\infty(\mathbb{T}^N)$, $T>0$ and $\delta\in\mathfrak{d}_T$. Assume that (1.4), (1.6), (3.1) and (4.29) are satisfied. Let $(v_\delta(t))$ be the numerical unknown defined by (3.4)-(3.5)-(4.24) and let m_δ be defined by (4.32). Then, for all $p\ge 1$, we have*

$$\mathbb{E}\iiint_{\mathbb{T}^N\times[0,T)\times\mathbb{R}} (1+|\xi|^p)dm_\delta(x,t,\xi)\le C_p, \tag{7.15}$$

where C_p is a constant depending on D_0, p, T and $\|u_0\|_{L^\infty(\mathbb{T}^N)}$ only.

Proof of Lemma 7.2. Let $p\in 2\mathbb{N}^*$. By (7.2), we have

$$\frac{1}{2}p(p-1)\mathbb{E}\sum_{0\le n<N_T}\Delta t_n\sum_{K\in\mathcal{T}_\#}|K|\int_{\mathbb{R}}\xi^{p-2}m_K^n(\xi)d\xi \\ = \mathbb{E}\sum_{0\le n<N_T}\|v_\delta(t_n)\|^p_{L^p(\mathbb{T}^N)} - \|v_\delta^\flat(t_{n+1})\|^p_{L^p(\mathbb{T}^N)}.$$

This gives

$$\frac{1}{2}p(p-1)\mathbb{E}\iiint_{\mathbb{T}^N\times[0,T)\times\mathbb{R}}\xi^{p-2}dm_\delta(x,t,\xi)\leq\mathbb{E}\|v_\delta(0)\|^p_{L^p(\mathbb{T}^N)}$$
$$+\mathbb{E}\sum_{0\leq n<N_T-1}\|v_\delta(t_{n+1})\|^p_{L^p(\mathbb{T}^N)}-\|v_\delta^\flat(t_{n+1})\|^p_{L^p(\mathbb{T}^N)}.$$

The bound (7.14) gives the desired conclusion. □

7.2 Convergence

We may now apply the theorem 2.2, to obtain the following results.

Theorem 7.3. *Let $u_0\in L^\infty(\mathbb{T}^N)$, $T>0$. Assume that the hypotheses (1.3), (1.4), (1.6), are satisfied. Then there exists a unique solution u to (1.1) with initial datum u_0, in the sense of Definition 2.2. Besides, for all $1\leq p<+\infty$, almost surely, $u\in C([0,T];L^p(\mathbb{T}^N))$.*

Theorem 7.4. *Let $u_0\in L^\infty(\mathbb{T}^N)$, $T>0$. Assume that the hypotheses (1.3), (1.4), (1.6), (3.1), (4.22) and (4.29) are satisfied. Let u be the solution to (1.1) with initial datum u_0 and let v_δ be the solution to the Finite Volume scheme (3.4)-(3.5)-(3.6)-(3.7). Then we have the convergence*

$$\lim_{\delta\to0}\mathbb{E}\|v_\delta-u\|^p_{L^p(\mathbb{T}^N\times(0,T))}=0, \tag{7.16}$$

for all $p\in[1,\infty)$.

Remark 7.2. If the X_k^{n+1} are merely i.i.d. random variables with normalized centred normal law $\mathcal{N}(0,1)$, then (v_δ) is converging to u in $L^p(\mathbb{T}^N\times(0,T))$ *in law* when $\delta\to0$. Indeed, the identity (3.7) is only satisfied in law now, hence v_δ has the same law as the function $\tilde v_\delta$ defined by (3.4)-(3.5)-(3.6), with X_k^{n+1} replaced by the right-hand side of (3.7). We apply the conclusion of Theorem 7.4 to $\tilde v_\delta$. As a corollary, we obtain the convergence in law of $(\tilde v_\delta)$ to u in $L^p(\mathbb{T}^N\times(0,T))$. A slightly different manner of expressing the same thing is to notice that, when the discrete increments (X_k^{n+1}) are some given normal law $\mathcal{N}(0,1)$, then we can construct a set of Brownian motions $\tilde\beta_k(t)$ such that

$$X_k^{n+1}=\frac{\tilde\beta_k(t_{n+1})-\tilde\beta_k(t_n)}{(\Delta t_n)^{1/2}}. \tag{7.17}$$

Indeed, without loss of generality, we can restrict ourselves to the case $\Delta t_n=1$ in (7.17) and use the Lévy-Ciesielski construction of the Brownian motion, [31, Section 3.2] on $[0,1]$ as follows: we define (*cf.* [31, Formula (3.1)])

$$G_0=X_k^{n+1},\ G_1=X_1^{n+1},\dots,G_{k-1}=X_{k-1}^{n+1},\ G_k=X_{k+1}^{n+1},\ G_{k+1}=X_{k+2}^{n+1},\dots$$

and we set

$$\tilde\beta(t)=\sum_{p=0}^\infty G_p\langle\mathbf{1}_{[0,t]},H_p\rangle,$$

where the H_p's are the Haar basis of $L^2(0,1)$. Then (7.17) follows from the fact that

$$\int_0^1 H_p(t)dt = \langle H_p, H_0\rangle = \delta_{p0}.$$

Remark 7.3. Theorem 7.3 has already been proved in [8] (see also Section 5 in [9]) under less restrictive hypotheses (having a compactly supported noise is unnecessary). We give the statement together with Theorem 7.4, however, to emphasize the fact that the convergence of the Finite Volume method, in the framework which we use, give both the existence-uniqueness of the solution to the limit continuous problem, and the convergence of the numerical method to this solution. It is not necessary to provide the existence of the solution u to (1.1) by an external means.

Proof of Theorem 7.3 and Theorem 7.4. Let us first prove the theorem 7.4. We take the existence of u, solution to (1.1) with initial datum u_0 for granted. By Proposition 6.1, Lemma 7.1 and Lemma 7.2, we may apply Theorem 2.2 to f_δ: we obtain (7.16) with z_δ instead of v_δ, where

$$z_\delta(x,t) := \int_{\mathbb{R}} \xi d\nu^\delta_{x,t}(\xi) = \frac{t-t_n}{\Delta t_n} v^\sharp_\delta(x,t) + \frac{t_{n+1}-t}{\Delta t_n} v_\delta(x,t),$$

for $t\in[t_n,t_{n+1}]$. By (6.2), we have the estimate

$$\mathbb{E}\int_0^T \|z_\delta - v_\delta\|^2_{L^2(\mathbb{T}^N)} dt = \mathcal{O}(|\delta|)$$

on the difference between z_δ and v_δ. This gives (7.16) for $p\leq 2$. If $p>2$, we use the inequality

$$\begin{aligned}\mathbb{E}\|v_\delta - u\|^p_{L^p(\mathbb{T}^N\times(0,T))} &\leq \|v_\delta - u\|_{L^2(\Omega\times\mathbb{T}^N\times(0,T))}\|v_\delta - u\|^{p-1}_{L^{2(p-1)}(\Omega\times\mathbb{T}^N\times(0,T))}\\ &\leq \frac{1}{\eta}\mathbb{E}\|v_\delta - u\|^2_{L^2(\mathbb{T}^N\times(0,T))} + \eta\mathbb{E}\|v_\delta - u\|^{2(p-1)}_{L^{2(p-1)}(\mathbb{T}^N\times(0,T))}, \end{aligned} \tag{7.18}$$

where η is a positive parameter. Due to the uniform bounds (2.2)-(7.13), we can choose η independent on δ to have the second term in (7.18) smaller than an arbitrary threshold. By the convergence result for $p=2$ the first term in (7.18) is then also small for δ close to 0. This concludes the proof of theorem 7.4. To prove 7.3, we just need to construct an approximation scheme satisfying (3.1), (4.22) and (4.29) and to compute v_δ by (3.4)-(3.5)-(3.6)-(3.7). We can use a cartesian grid for this purpose: let $h_m = \frac{1}{m}$, where $m\in\mathbb{N}^*$. Let $\mathcal{T}_\#$ be the set of open hypercubes of length h_m obtained by translates of the original hypercube $(0,h_m)^N$ by vectors $h_m x$, x having components in $\{0, l\ldots, m-1\}$. Then (4.22) is satisfied with $\alpha_N = 2^{-N}$ since a hypercube has 2^N sides. We can choose the Godunov numerical fluxes, defined as follows:

$$A_{K\to L}(v,w) = \begin{cases} |K|L| \min\limits_{v\leq\xi\leq w} A(\xi)\cdot n_{K,L} & \text{if } v\leq w,\\ |K|L| \max\limits_{w\leq\xi\leq v} A(\xi)\cdot n_{K,L} & \text{if } w\leq v.\end{cases}$$

These fluxes $A_{K\to L}$ are monotone (non-increasing in the first variable, non-increasing in the second variable). They satisfy the hypotheses of regularity, consistency (3.1) and (3.2) respectively with $L_A = \mathrm{Lip}(A)$. The conservative symmetry property (3.3) is also satisfied. At last, to ensure (4.29), we just need to take a uniform time step Δt like

$$\Delta t = \frac{1}{2}\frac{\alpha_N^2}{2L_A} h_m.$$

We have then (4.29) with $\theta = \frac{1}{2}$. At this stage, Proposition 6.1 provides a sequence of approximate generalized solutions (f_m). by the uniform bounds established in Section 7.1, we can apply Theorem 2.2: this gives the existence of a unique solution u to (1.1) with initial datum u_0. By Corollary 3.3 in [9], we have $u \in C([0,T];L^p(\mathbb{T}^N))$ almost surely. □

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