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ENTROPY FORMULATION FOR PARABOLIC DEGENERATE EQUATIONS WITH GENERAL DIRICHLET BOUNDARY CONDITIONS AND APPLICATION TO THE CONVERGENCE OF FV METHODS*

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Abstract. This paper is devoted to the analysis and the approximation of parabolic hyperbolic degenerate problems defined on bounded domains with *nonhomogeneous* boundary conditions. It consists of two parts. The first part is devoted to the definition of an original notion of entropy solutions to the continuous problem, which can be adapted to define a notion of measure-valued solutions, or entropy process solutions. The uniqueness of such solutions is established. In the second part, the convergence of the finite volume method is proved. This result relies on (weak) estimates and on the theorem of uniqueness of the first part. It also entails the existence of a solution to the continuous problem.

Key words. parabolic degenerate equations, boundary conditions, finite volume methods

AMS subject classifications. 35K65, 35F30, 35K35, 65M12

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1. Introduction.

Let Ω be an open bounded polyhedral subset of $\mathbb{R}^d$ and $T \in \mathbb{R}_+^*$. Let us denote by Q the set $(0,T)\times\Omega$, and by Σ the set $(0,T)\times\partial\Omega$.

We consider the following parabolic-hyperbolic problem:

$$\begin{cases} u_t + \operatorname{div}(F(t,x,u)) - \Delta\varphi(u) = 0, \quad (t,x)\in Q,\\ u(0,x) = u_0(x), \quad x\in\Omega,\\ u(t,x) = \bar{u}(t,x), \quad (t,x)\in\Sigma\,. \end{cases} \tag{1}$$

Such an equation of quasilinear advection with degenerate diffusion governs the evolution of the saturation of the wetting fluid in the study of diphasic flow in porous media [GMT96], [Mic01], [EHM01]. In that case, the function φ can be expressed using the capillary pressure and the relative mobilities. The function φ is only supposed to be a nondecreasing Lipschitz continuous function. In particular, the study of problem (1) includes the study of nonlinear hyperbolic problems (cases where $\varphi' = 0$).

The analysis of the approximation of nonlinear hyperbolic problems via the finite volume (FV) method began in the mid 1980s, involving several authors including, for example, Cockburn, Coquel, and LeFloch [CCL95], Szepessy [Sze91], Vila [Vil94], Kröner, Rokyta, and Wierse [KRW96], and Eymard, Gallouët, and Herbin [EGH00]. Results on the convergence of FV schemes for degenerate problems in general came to light in more recent years [EGHM02], [Ohl01]. See also [BGN00], [EK00] for other methods of approximation.

When the function φ is strictly increasing, problem (1) is of parabolic type. In that case, the existence of a unique weak solution is well known. In the case where $\varphi' = 0$, problem (1) is a nonlinear hyperbolic problem, the uniqueness of a weak

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solution is not ensured, and one has to define a notion of entropy solutions to recover uniqueness [Kru70]. Therefore, it is quite difficult to define a notion of solution in the case where φ is merely a nonincreasing function. In fact, as far as the Cauchy problem in the whole space is concerned, such a definition has been done for a long time, since Volpert and Hudjaev [VH69], but uniqueness with nonlinear parabolic terms has only been proved recently by Carrillo [Car99] (see also [KO01], [KR00]).

Another difficulty in the study of degenerate parabolic problems is analysis of the boundary conditions (see [LBS93], [RG99]). It is not always easy to give a correct formulation of the boundary conditions, or of the way they have to be taken into account. In the case where the function φ is strictly increasing, the classical framework of variational solutions of parabolic equations is enough to satisfy this wish. In the case where $\varphi' = 0$, things are completely different. Even if the (entropy) solution u of problem (1) admits a trace (say, γu) on Σ, the equality $\gamma u = \overline{u}$ on Σ does not necessarily hold. Actually, a condition on Σ can be given, which is known as the BLN condition [BLN79]: this is the right way to formulate boundary conditions in the study of scalar hyperbolic problems. However, the notion of entropy solution to nonlinear Cauchy–Dirichlet hyperbolic problems given by Bardos, LeRoux, and Nédélec is not really suitable to the study of FV schemes since it requires that the solution u be in a space BV (because the trace of u is involved in the formulation of the BLN condition), and it is known that it is difficult to get BV estimates on the numerical approximations given by the FV method on non-Cartesian grids. Actually, Otto gave an integral formulation of entropy solutions to scalar hyperbolic problems with boundary conditions [Ott96], and this indeed allows us to prove the convergence of the FV method [Vov02].

To our knowledge, the problem that we deal with (convergence of the FV method for degenerate parabolic equations with nonhomogeneous boundary conditions) has never been considered before. Nevertheless, in [MPT02], the authors give a definition of entropy solution for which uniqueness and consistency with the parabolic approximation are proved. This definition is not completely in integral form and therefore not suitable for proving the convergence of the FV method, since only poor compactness results are available on the numerical approximation. That is why we give an original definition of the problem (see Definition 3.1). This complete integral formulation includes the definition of Otto but not exactly the one of Carrillo (see the comments that follow Definition 3.1). It is well suited to the study of the convergence of several approximations of problem (1) and is used, for example, in [GMT02] to prove the convergence of a discrete Bhatnagar–Gross–Krook (BGK) model (see also [MPT02] for the parabolic approximation).

Notice that some particular cases have been fully treated: in [EGHM02], the authors prove the convergence of the FV method in the case where $F(x,t,s) = \mathbf{q}(x,t)f(s)$, $\operatorname{div}(\mathbf{q}) = 0$, with $\mathbf{q} \cdot \mathbf{n} = 0$ on Σ. In that case, the boundary condition does not act on the hyperbolic part of the equation. From a technical point of view, this means that the influence of the boundary condition appears in the terms related to the parabolic degenerate part of the equation. These parabolic degenerate terms are estimated by following the methods of Carrillo in [Car99], who deals with homogeneous boundary conditions. On the other hand, in [Vov02], the author proves the convergence of an FV method in the case where $\varphi' = 0$, adapting the ideas of Otto [Ott96]. In that case, the effects of the boundary condition in the hyperbolic equation are the center of the work. In this paper we mix these two precedent approaches to deal with the parabolic degenerate problem with general boundary conditions.

We will make the following assumptions on the data:

(H1) $F : (t, x, s) \mapsto F(t, x, s) \in C^1(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R})\,,\quad \mathrm{div}_x F = 0,$
$\dfrac{\partial F}{\partial s}$ is locally Lipschitz continuous uniformly with respect to (t, x);

(H2) $\varphi : s \mapsto \varphi(s)$ is a nondecreasing Lipschitz continuous function;

(H3) $u_0 : x \mapsto u_0(x) \in L^\infty(\Omega)$; and

(H4) the function $\bar{u} : (x, t) \mapsto \bar{u}(x, t) \in L^\infty(\Sigma)$ and is the trace of a function $\bar{u} \in L^\infty(Q)$with $\varphi(\bar{u}) \in L^2(0, T; H^1(\Omega))\,.$

To prove the convergence of the FV method, we will also assume that the boundary datum satisfies

(H5) the function $\bar{u} : (x, t) \mapsto \bar{u}(x, t) \in L^\infty(\Sigma)$ and is the trace of a function $\bar{u} \in L^\infty(Q)$ with $\varphi(\bar{u}) \in L^2(0, T; H^1(\Omega))\,,\ \nabla\bar{u} \in L^2(Q)\,,\ \bar{u}_t \in L^1(Q)\,.$

In the course of the proof of uniqueness of the entropy process solution (Theorem 4.1), additional hypotheses on the boundary datum are required. Using the notation defined in subsection 4.1, they read

$$\text{(H6) } \overline{u}_\Sigma \in W^{1,1}((0, T) \times B \cap Q) \quad \text{and} \quad \Delta\varphi(\overline{u}_\Sigma) \in L^1((0, T) \times B \cap Q)\,.$$

Remark 1.1. As suggested by Porretta [MPT02], hypothesis (H6) may be relaxed as

$$\begin{aligned}&\text{(H6Bis) } \overline{u}_\Sigma \in W^{1,1}((0, T) \times B \cap Q) \quad \text{and}\\ &\Delta\varphi(\overline{u}_\Sigma) \text{ is a bounded Radon measure on } (0, T) \times \Pi\,.\end{aligned}$$

We do not give a justification of this assertion now. Indeed, hypothesis (H6) is involved in the proof of Lemma 4.2, and we have waited until Remark 4.1, just after this proof, to specify to what extent hypothesis (H6Bis) is admissible.

Under assumptions (H3)–(H4), there exists $(A, B) \in \mathbb{R}^2$ such that

$$A \le \min\left(\operatorname*{ess\,inf}_\Omega(u_0), \operatorname*{ess\,inf}_Q(\bar{u})\right) \le \max\left(\operatorname*{ess\,sup}_\Omega(u_0), \operatorname*{ess\,sup}_Q(\bar{u})\right) \le B, \tag{2}$$

and we set

$$M = \max\left\{\left|\frac{\partial F}{\partial s}(t, x, s)\right|\,,\ (t, x, s) \in Q \times [A, B]\right\}\,.$$

We introduce the function ζ defined by $\zeta' = \sqrt{\varphi'}$. (This makes sense in view of (H2).) We will derive $L^2(0, T; H^1)$ estimates on nonlinear quantities such as $\zeta(u)$. A simple explanation for this fact is the following. Consider the equation $u_t - \Delta\varphi(u) = 0$ on $(0, T) \times \Omega$. Multiply it by u, and sum the result with respect to $x \in \Omega$. The formal identity $\int_\Omega \nabla\varphi(u) \cdot \nabla u = \int_\Omega |\nabla\zeta(u)|^2$ then leads to $\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla\zeta(u)|^2 \le 0$, from which can be derived an energy estimate.

Notice that the hypothesis $\mathrm{div}_x F = 0$ can be relaxed, and source terms can be considered in the right-hand side of (1).

The assumption that $\bar{u}$ is the trace of an L^∞ function $\bar{u}$ such that $\varphi(\bar{u}) \in L^2(0, T; H^1(\Omega))$ is a necessary condition for the existence of solutions to problem (1);

the additional hypotheses introduced in (H5) are involved in the proofs of different estimates on the approximate solution, defined thanks to the FV method.

As implied at the beginning of this introduction, one of the main points in the study of problem (1) is the definition of a notion of solution suitable for the classical techniques of convergence of FV schemes. This point is specified in section 2. In section 3, we introduce and define a notion of entropy process solutions (a concept similar to the concept of measure-valued solutions), and in section 4 we prove the uniqueness of such solutions (see Theorem 4.1). Section 5 is devoted to the FV scheme used to approximate problem (1); a priori estimates are derived and the convergence is proved.

2. Entropy weak solution.

Here, as in the study of purely hyperbolic problems, the concept of weak solutions is not sufficient since the uniqueness of such solutions may fail. Thus, we turn to the notion of weak entropy solutions. The entropy-flux pairs considered in the definition of this solution are the so-called Kruzhkov semi entropy-flux pairs $(\eta_\kappa^\pm, \Phi_\kappa^\pm)$ (see [Car99], [Ser96], [Vov02]). They are defined by the formula

$$\begin{cases}\eta_\kappa^+(s) = (s-\kappa)^+ = s\top\kappa - \kappa,\\ \eta_\kappa^-(s) = (s-\kappa)^- = \kappa - s\perp\kappa,\end{cases} \qquad \begin{cases}\Phi_\kappa^+(t,x,s) = (s-\kappa)^+ = F(t,x,s\top\kappa) - F(t,x,\kappa),\\ \Phi_\kappa^-(t,x,s) = (s-\kappa)^- = F(t,x,\kappa) - F(t,x,s\perp\kappa),\end{cases}$$

with $a\top b = \max(a,b)$ and $a\perp b = \min(a,b)$. Notice that, in the case where κ is considered as a variable, for example when the doubling variable technique of Kruzhkov is used, the entropy-fluxes will be written

$$\Phi^+(x,t,s,\kappa) = \Phi_\kappa^+(t,x,s) \quad \text{and} \quad \Phi^-(x,t,s,\kappa) = \Phi_\kappa^-(t,x,s).$$

Definition 2.1 (entropy weak solution). *A function u of $L^\infty(Q)$ is said to be an entropy weak solution to problem* (1) *if it is a weak solution of problem* (1), *that is, if $\varphi(u) - \varphi(\overline{u}) \in L^2(0,T;H_0^1(\Omega))$ and*

$$\forall\theta \in \mathcal{C}_c^\infty([0,T)\times\Omega),$$
$$\int_Q u\,\theta_t + (F(t,x,u) - \nabla\varphi(u))\cdot\nabla\theta \;\, dx\,dt + \int_\Omega u_0\,\theta(0,x)\,dx = 0, \tag{3}$$

and if it satisfies the following entropy inequalities for all $\kappa \in [A,B]$, for all $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R}^d)$ such that $\psi \ge 0$ and $\operatorname{sgn}^\pm(\varphi(\overline{u}) - \varphi(\kappa))\psi = 0$ a.e. on Σ:

$$\int_Q \eta_\kappa^\pm(u)\,\psi_t + (\,\Phi_\kappa^\pm(t,x,u) - \nabla\,(\varphi(u)-\varphi(\kappa))^\pm\,)\cdot\nabla\psi \;\, dx\,dt + \int_\Omega \eta_\kappa^\pm(u_0)\,\varphi(0,x)\,dx$$
$$+ M\int_\Sigma \eta_\kappa^\pm(\bar{u})\psi\,d\gamma(x)\,dt \ge 0. \tag{4}$$

Notice that the weak equation (3) is superfluous, for it is a consequence of (4). However, if the function φ were (strictly) increasing, (3) would be enough to define a notion of the solution of problem (1) for which existence and uniqueness hold: in that case, problem (1) would merely be a nonlinear parabolic problem. For general φ, the uniqueness of the solution will be a consequence of the entropy inequalities (4); indeed, the class of Kruzhkov semi entropy-flux pairs is wide enough to ensure the uniqueness of the weak entropy solution, while—and we stress this fact—the class of classical Kruzhkov entropy-flux pairs $s \mapsto |s-\kappa|$ is not.

Also notice that, first, in the homogeneous case $\bar{u} = 0$, the previous definition is slightly different from the original definition given by Carrillo [Car99] and that, second, if $\varphi' = 0$ (problem (1) becomes hyperbolic), then the previous definition of the entropy solution coincides with the definition of a solution suitable for hyperbolic problems; see Otto [Ott96] and [Vov02]. A notion of an entropy solution for degenerate parabolic problems with nonhomogeneous boundary conditions has also been defined by Mascia, Porretta, and Terracina in [MPT02]. It is interesting to notice that, in their definition, they directly require that the entropy condition satisfy the entropy condition on the boundary (14) as stated in Proposition 4.1. We prove that this property (14) is, in fact, a consequence of the entropy inequalities (4) and then follow the main lines of the uniqueness theorem proved in [MPT02].

3. Entropy process solution. The proof of the existence of a weak entropy solution to problem (1) lies in the study of the numerical solution u_D defined by an FV method for problem (1) (see section 5.2). Theorem 5.1 states that the numerical solution satisfies approximate entropy inequalities (see (50)), but the bounds on u_D (a bound in $L^\infty(Q)$ and a bound on the discrete H^1-norm of $\varphi(u_D)$) do not give strong compactness, only weak compactness. Therefore, in order to be able to take the limit of the nonlinear terms of u_D (as $\Phi_\kappa^\pm(u_D)$, in particular), we have to turn to the notion of measure-valued solutions (see DiPerna [DiP85], Szepessy [Sze91]) or, equivalently, to the notion of entropy process solution defined by Eymard, Gallouët, and Herbin [EGH00]. In light of the following theorem, it appears that the notion of entropy process solution is indeed well suited to compensate for the weakness of the compactness estimates on the approximate solution u_D and to deal with nonlinear expressions of u_D.

THEOREM 3.1 (nonlinear convergence for the weak-$\star$ topology). *Let $\mathcal{O}$ be a Borel subset of $\mathbb{R}^m$, R be positive, and (u^n) be a sequence of $L^\infty(\mathcal{O})$ such that, for all $n \in \mathbb{N}$, $||u^n||_{L^\infty} \le R$. Then there exists a subsequence, still denoted by (u^n) and $\mu \in L^\infty(\mathcal{O} \times (0,1))$, such that*

$$\forall g \in \mathcal{C}(\mathbb{R}), \quad g(u^n) \longrightarrow \int_0^1 g(\mu(.,\alpha))\, d\alpha \quad \text{in } L^\infty(\mathcal{O}) \text{ weak-}\star\,.$$

Now the notion of an entropy process solution can be defined.

DEFINITION 3.1 (weak entropy process solution). *Let u be in $L^\infty(Q \times (0,1))$. The function u is said to be an entropy process solution to problem* (1) *if*

$$\varphi(u) - \varphi(\overline{u}) \in L^2(0,T; H_0^1(\Omega)) \tag{5}$$

and if u satisfies the following entropy inequalities for all $\kappa \in [A,B]$, for all $\psi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{R}^d)$ such that $\psi \ge 0$ and $\operatorname{sgn}^\pm(\varphi(\overline{u}) - \varphi(\kappa))\psi = 0$ a.e. on Σ:

$$\begin{aligned}\int_Q \int_0^1 & \eta_\kappa^\pm(u(t,x,\alpha))\, \psi_t(t,x) \\ & + \left(\Phi_\kappa^\pm(t,x,u(t,x,\alpha)) - \nabla\left(\varphi(u)(t,x) - \varphi(\kappa)\right)^\pm\right) \cdot \nabla\psi(t,x) d\alpha dx dt \\ & + \int_\Omega \eta_\kappa^\pm(u_0)\, \psi(0,x)\, dx + M \int_\Sigma \eta_\kappa^\pm(\bar{u})\psi\, d\gamma(x) dt \ge 0\,.\end{aligned} \tag{6}$$

Notice that if the function u is an entropy process solution of problem (1), then it satisfies condition (5), which means in particular that $\varphi(u)$ does not depend on the last variable α and is denoted by $\varphi(u)(t,x)$.

Notation. We set $\mathcal{Q} = Q \times (0,1)$.

We will now show that any entropy *process* solution actually reduces to an entropy *weak* solution.

4. Uniqueness of the entropy process solution.

THEOREM 4.1 (uniqueness of the entropy process solution). *Let $u,\ v \in L^\infty(Q \times (0,1))$ be two entropy process solutions of problem* (1) *in accordance with Definition* 3.1. *Suppose that Ω is either a polyhedral open subset of $\mathbb{R}^d$ or a strong $\mathcal{C}^{1,1}$ open subset of $\mathbb{R}^d$, and assume hypotheses* (H1), (H2), (H3), (H4), *and* (H6) *(or* (H6Bis)*). Then there exists a function $w \in L^\infty(Q)$ such that*

$$u(t,x,\alpha) \,=\, w(t,x) \,=\, v(t,x,\beta) \ \ for\ \ almost\ every\ (t,x,\alpha,\beta) \in Q \times (\,0,1)^2\,.$$

COROLLARY 4.1 (uniqueness of the weak entropy solution). *If Ω is either a polyhedral open subset of $\mathbb{R}^d$ or a strong $\mathcal{C}^{1,1}$ open subset of $\mathbb{R}^d$, and under hypotheses* (H1), (H2), (H3), (H4), *and* (H6) *(or* (H6Bis)*), problem* (1) *admits at most one weak entropy solution.*

In the case where Ω is a polyhedral open subset of $\mathbb{R}^d$, the proof of Theorem 4.1 is slightly more complicated than the proof in the case where Ω is a strong $\mathcal{C}^{1,1}$ open subset of $\mathbb{R}^d$. Besides, although the study of the FV method applied to (1) relies on Theorem 4.1 only in the case of Ω polyhedral, we wish to specify the validity of Theorem 4.1 when Ω is $\mathcal{C}^{1,1}$. Indeed, problem (1) may of course be posed on such an open set, and, in that case, Theorem 4.1 would be one of the major steps in the proof of the convergence of such an approximation, as for the vanishing viscosity approximation, for example.

We therefore explain the proof of Theorem 4.1 in the case where Ω is $\mathcal{C}^{1,1}$ and then indicate how to adapt it to the case where Ω is a polyhedral open subset of $\mathbb{R}^d$ (see subsection 4.6).

4.1. Proof of Theorem 4.1: Definitions and notation.

4.1.1. Localization near the boundary. We suppose that Ω is a strong $\mathcal{C}^{1,1}$ open subset of $\mathbb{R}^d$. In that case, there exists a finite open cover $(B_\nu)_{0,\dots,N}$ of $\overline{\Omega}$ and a partition of unity $(\lambda_\nu)_{0,\dots,N}$ on $\overline{\Omega}$ subordinate to $(B_\nu)_{0,\dots,N}$ such that, for $\nu \geq 1$, up to a change of coordinates represented by an orthogonal matrix A_ν, the set $\Omega \cap B_\nu$ is the epigraph of a $\mathcal{C}^{1,1}$-function $f_\nu : \mathbb{R}^{d-1} \to \mathbb{R}$; that is,

$$\Omega \cap B_\nu = \{x \in B_\nu \ ;\ (A_\nu\, x)_d > f_\nu(\overline{A_\nu\, x})\} \quad \text{ and}$$
$$\partial\Omega \cap B_\nu = \{x \in B_\nu \ ;\ (A_\nu\, x)_d = f_\nu(\overline{A_\nu\, x})\}\,,$$

where $\overline{y}$ stands for $(y_i)_{1,d-1}$ if $y \in \mathbb{R}^d$.

Until the end of the proof of Theorem 4.1, the problem will be localized with the help of a function λ_ν. We drop the index ν and, for the sake of clarity, suppose that the change of coordinates is trivial: $A = I_d$. We denote by $\Pi = \{\bar{x}, x \in \Omega \cap B\} \subset \mathbb{R}^{d-1}$ the projection of $B \cap \Omega$ onto the $(d-1)$ first components, and $\Pi_\lambda = \{\bar{x}, x \in \operatorname{supp}(\lambda) \cap \Omega\}$ (see Figure 1). If a function ψ is defined on Σ, we denote by ψ_Σ the function defined on $[0,T) \times B \cap Q$ by $\psi_\Sigma(t,x) = \psi(t,\bar{x},f(\overline{x}))$. Notice that the function ψ_Σ does not depend on x_d and that, by abusing the notation, we shall also denote by ψ_Σ the restriction of ψ_Σ to $[0,T) \times \Pi$. In the same way, if L_i is defined on $[0,T] \times \Pi$, we also denote by L_i the function defined on $[0,T) \times B \cap Q$ by $L_i(t,x) = L_i(t,\overline{x})$.

4.1.2. Weak notion of trace. An important step in the proof of the uniqueness of entropy process solutions is the derivation of the condition satisfied by any entropy

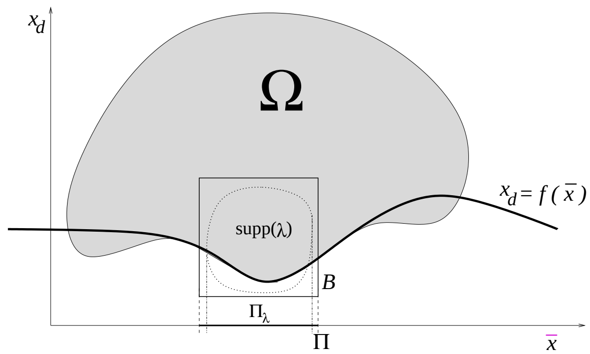

FIG. 1. *Localization by λ in the ball B.*

process solution on the boundary of the domain. This condition is the matter of Proposition 4.1. (It can be viewed as a kind of BLN condition [BLN79], balanced by second order terms issued from the degenerate parabolic part of the equation of (1).) In the course of the proof of Proposition 4.1, we need to define the normal trace of certain fluxes $(\Phi_\kappa^+(t,x,u) - \nabla(\varphi(u)-\varphi(\kappa))^+$, among others, for example) and, more precisely, to ensure the consistency of this definition of the normal trace with different approximations. For that purpose, we turn to the work of Chen and Frid [CF02]. Adapted to our context, the main theorem of [CF02] is the following.

THEOREM 4.2 (see Chen and Frid [CF02]). *Recall that $Q=(0,T)\times\Omega$, and denote by ν the outward unit normal to Q. Let $\mathcal{F}\in(L^2(Q))^{d+1}$ be such that* $\mathrm{div}\mathcal{F}$ *is a bounded Radon measure on Q. Then there exists a linear functional $\mathcal{T}_\nu$ on $W^{1/2,2}(\partial Q)\cap\mathcal{C}(\partial Q)$ which represents the normal traces $\mathcal{F}\cdot\nu$ on ∂Q in the sense that, first, the following Gauss–Green formula holds: for all $\psi\in\mathcal{C}_c^\infty(\overline{Q})$,*

$$\langle \mathcal{T}_\nu,\psi\rangle = \int_Q \psi\,\mathrm{div}\mathcal{F} + \int_Q \nabla\psi\cdot\mathcal{F}. \tag{7}$$

Second, $\langle\mathcal{T}_\nu,\psi\rangle$ depends only on $\psi_{|\partial Q}$, while, third, if (B,λ,f) is as above (subsection localization near the boundary), then for all $\psi\in\mathcal{C}_c^\infty([0,T)\times\overline{\Omega})$,

$$\langle\mathcal{T}_\nu,\psi\lambda\rangle = -\lim_{s\to 0}\frac{1}{s}\left(\int_s^T\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s}\mathcal{F}\cdot\begin{pmatrix}-\nabla f(\overline{x})\\ 1\\ 0\end{pmatrix}\psi\lambda\,dx_d\,d\overline{x}\,dt \right.$$
$$\left. + \int_0^s\int_\Omega \mathcal{F}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\psi\lambda\,dx\,dt\right). \tag{8}$$

Let u be an entropy weak solution of problem (1). The entropy inequality (4) shows that the divergence of the field

$$\mathcal{F}_\kappa^+(t,x) = \begin{pmatrix}(u-\kappa)^+\\ \Phi_\kappa^+(t,x,u)-\nabla(\varphi(u)-\varphi(\kappa))^+\end{pmatrix}$$

is a bounded Radon measure on Q. This field belongs to $(L^2(Q))^{d+1}$, and according to the previous theorem, there exists a linear functional $\mathcal{T}_{\nu,\kappa}^+$ on $W^{1/2,2}(\partial Q)\cap\mathcal{C}(\partial Q)$ which represents $\mathcal{F}_\kappa^+(t,x)\cdot\nu$. Then, to define a notion of the normal trace of the flux

$\Phi^+_\kappa(t,x,u) - \nabla(\varphi(u)-\varphi(\kappa))^+$, we set

$$\langle \mathcal{T}^+_{n,\kappa}, \psi\rangle = \langle \mathcal{T}^+_{\nu,\kappa}, \psi\rangle + \int_\Omega (u_0-\kappa)^+\psi(0,x)\,dx \qquad \forall \psi \in \mathcal{C}^\infty_c([0,T)\times\overline{\Omega})\,. \tag{9}$$

This definition makes sense because the entropy weak solution assumes the values of the initial data u_0:

$$\lim_{s\to 0}\frac{1}{s}\int_0^s\int_\Omega (u-\kappa)^+\psi\,dx\,dt = \int_\Omega (u_0-\kappa)^+\psi(0,x)\,dx\,,$$

as can be seen by choosing $\frac{s-t}{s}\chi_{(0,s)}(t)\psi$ as a test-function in (4). In particular, $\langle \mathcal{T}^+_{n,\kappa}, \psi\lambda\rangle$ depends only on $\psi_{|\Sigma}$, and from (8) we can derive the formula

$$\begin{aligned}&\langle \mathcal{T}^+_{n,\kappa}, \psi\lambda\rangle\\ &= -\lim_{s\to 0}\frac{1}{s}\int_s^T\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s}(\Phi^+_\kappa(t,x,u)-\nabla(\varphi(u)-\varphi(\kappa))^+)\cdot\begin{pmatrix}-\nabla f(\overline{x})\\ 1\end{pmatrix}\psi\lambda\,dx_d\,d\overline{x}\,dt\end{aligned}$$

for all $\psi\in\mathcal{C}^\infty_c([0,T)\times\overline{\Omega})$.

4.1.3. Mollifiers ρ_n and the cut-off function ω_ε. Technically, the heart of the proof of uniqueness is the doubling of variables. This technique involves mollifiers, which are defined as $\rho_n(t) = n\rho(nt)$, where ρ is a nonnegative function of $\mathcal{C}^\infty_c(-1,0)$ such that $\int_{-1}^0 \rho(t)\,dt = 1$. (Notice that the the support of the function ρ is located to the left of zero.) For ε a positive number, ρ_ε naturally denotes the map $t\mapsto \frac{1}{\varepsilon}\rho(\frac{t}{\varepsilon})$, and we define $R_n(t) = \int_{-\infty}^{-t}\rho_n(s)\,ds$. Since the technique of doubling of variables interferes with a certain evaluation of the boundary behavior of the entropy process solution (described by (14)), we need to define a cut-off function ω_ε built upon the sequence of mollifiers. We set

$$\omega_\varepsilon(x) = \int_{f(\overline{x})-x_d}^0 \rho_\varepsilon(z)\,dz = \int_{\frac{f(\overline{x})-x_d}{\varepsilon}}^0 \rho(z)\,dz. \tag{10}$$

On $\Omega\cap B$, the function ω_ε vanishes in a neighborhood of $\partial\Omega$ and equals 1 if $\mathrm{dist}(x,\partial\Omega) > \varepsilon$; in particular, $\omega_\varepsilon\to 1$ in $L^1(\Omega\cap B)$ and, if $\psi\in H^1(\Omega)$, then

$$\int_\Omega \lambda\,\psi\cdot\nabla\omega_\varepsilon = -\int_\Omega \mathrm{div}(\lambda\,\psi)\,\omega_\varepsilon \overset{\varepsilon\to 0}{\to} -\int_\Omega \mathrm{div}(\lambda\,\psi) = -\int_{\partial\Omega}\lambda\,\psi\cdot\mathbf{n}\,.$$

Roughly speaking, if $F:\Omega\to\mathbb{R}^d$, then $-F\cdot\nabla\omega_\varepsilon$ approaches the normal trace $F\cdot\mathbf{n}$. To make this idea more precise, for the field $F = \Phi^+_\kappa(t,x,u)-\nabla(\varphi(u)-\varphi(\kappa))^+$ we call upon the notion of normal trace defined above (subsection 4.1.2). Let $\psi\in\mathcal{C}^\infty_c([0,T)\times\overline{\Omega})$. Since $\psi=\psi(1-\omega_\varepsilon)$ on Σ, $\langle\mathcal{T}^+_{n,\kappa},\psi\lambda\rangle = \langle\mathcal{T}^+_{n,\kappa},\psi\lambda(1-\omega_\varepsilon)\rangle$. The definition of $\mathcal{T}^+_{n,\kappa}$ (see (9)) and the Gauss–Green formula (7) yield

$$\langle\mathcal{T}^+_{n,\kappa},\psi\lambda\rangle = \int_Q \psi(1-\omega_\varepsilon)\lambda\,\mathrm{div}\mathcal{F}^+_\kappa + \int_Q \nabla(\psi(1-\omega_\varepsilon)\lambda)\cdot\mathcal{F}^+_\kappa + \int_\Omega (u_0-\kappa)^+\psi(1-\omega_\varepsilon)\lambda\,dx.$$

Since $0\le 1-\omega_\varepsilon\le 1$ and $\omega_\varepsilon(x)\to 1$ for all $x\in\Omega\cap B$, the dominated convergence theorem ensures that $\lim_{\varepsilon\to 0}\int_Q\psi(1-\omega_\varepsilon)\lambda\,\mathrm{div}\mathcal{F}^+_\kappa = 0$ and

$$\langle\mathcal{T}^+_{n,\kappa},\psi\lambda\rangle = -\lim_{\varepsilon\to 0}\int_Q [\Phi^+_\kappa(t,x,u)-\nabla(\varphi(u)-\varphi(\kappa))^+]\cdot\nabla\omega_\varepsilon\,\psi\lambda\,dx\,dt\,.$$

4.1.4. Otto entropy-fluxes. Let $u \in L^\infty(Q \times (0,1))$ be an entropy process solution of problem (1) and $\kappa \in [A,B]$. Set $\Phi = \Phi^+ + \Phi^-$. We denote by $\mathcal{G}_x(t,x,u,\kappa)$ the quantity

$$\mathcal{G}_x(t,x,u,\kappa) = \Phi(t,x,u(t,x,\alpha),\kappa) - \nabla_x|\varphi(u)(t,x) - \varphi(\kappa)|\,. \tag{11}$$

For $w \in \mathbb{R}$, the function $\mathcal{F}_\varphi$ is defined by the formula

$$\mathcal{F}_\varphi(t,x,u,\kappa,w) = \mathcal{G}_x(t,x,u,\kappa) + \mathcal{G}_x(t,x,u,w) - \mathcal{G}_x(t,x,\kappa,w)\,. \tag{12}$$

4.2. A result of approximation.

Lemma 4.1. *Let U be a bounded open subset of $\mathbb{R}^q$, $q \geq 1$. If $f \in L^\infty \cap BV(U)$, then, given $\varepsilon > 0$, there exists $g \in \mathcal{C}(\overline{U})$ such that*

$$g \geq f \quad \textit{a.e. on } U \qquad \textit{and} \qquad \int_U (g(x) - f(x))\,dx < \varepsilon\,.$$

This result may be false if $f \notin BV(U)$ (consider $f = \mathbb{1}_{\mathbb{Q}\cap(0,1)}$ on $U = (0,1)$), but this is not a necessary condition, because, on $U = (0,1)$, the function $f = \mathbb{1}_K$, where K is the triadic Cantor, can be approximated in $L^1(0,1)$ by continuous functions g such that $g \geq f$ a.e. Indeed, we claim that, if E is a measurable subset of U, then $f = \mathbb{1}_E$ satisfies the conclusion of Lemma 4.1 if and only if

$$m(E) = \inf\{m(K)\,;\ E \subset K\,,\ K \text{ compact}\}\,. \tag{13}$$

(Here, m denotes the Lebesgue measure on $\mathbb{R}^q$.)

Before proving Lemma 4.1, let us justify this assertion. If (13) holds, then, given $\varepsilon > 0$, there exists a compact K of U such that $E \subset K$ and $m(K \setminus E) < \varepsilon$. Since the Lebesgue measure is regular, there exists an open subset V of U such that $K \subset V \subset \overline{V} \subset U$ and $m(V \setminus K) < \varepsilon$. Then the function $g : x \mapsto d(x, \mathbb{R}^q \setminus V)/(d(x,K) + d(x, \mathbb{R}^q \setminus V))$ is continuous on $\mathbb{R}^q$, $g \geq \mathbb{1}_E$, and $\int_U (g - \mathbb{1}_E) < 2\varepsilon$.

Conversely, suppose that, given $\varepsilon > 0$, there exists $g \in \mathcal{C}(\overline{U})$ such that $g \geq \mathbb{1}_E$ and $\int_U (g - \mathbb{1}_E) < \varepsilon$. Then $K = \{x \in \overline{U}\,;\ g(x) \geq 1\}$ is compact, $E \subset K$, and $m(K \setminus E) < \varepsilon$.

Proof of Lemma 4.1. Notice that, if E is a measurable subset of U such that $m(\partial E) = 0$, then (13) holds (consider the compact $\overline{E}$). If E is a level set of a BV function, then E has almost surely a finite perimeter and, consequently, $m(\partial E) = 0$, which ensures that $\mathbb{1}_E$ satisfies the conclusion of Lemma 4.1. This result may be seen as the heart of the proof. Indeed, first suppose that $0 \leq f(x) \leq 1$ for every $x \in U$. For $t \in [0,1]$, set $E_t = \{x \in U\,;\ f(x) < t\}$. Then, for almost every t, E_t is a set with finite perimeter since $f \in BV(U)$. Let (t_n) be a sequence of reals dense in $[0,1]$ and such that $t_1 = 1$; E_{t_n} is a set with finite perimeter for every n. We will define a sequence of simple functions $\theta_n = \sum_{i=1}^n \alpha_i^n \mathbb{1}_{A_i^n}$ which approximate f from above and such that each set A_i^n is built upon the level sets E_{t_i}. To that purpose, first define $\theta_1(x) = 1$ for all $x \in U$. If $n > 1$, let $\{k_1, \dots, k_n\}$ be an enumeration of $\{1, \dots, n\}$ such that $t_{k_1} > \cdots > t_{k_n}$. Set

$$\begin{aligned} &A_i^n = E_{t_{k_i}} \setminus E_{t_{k_{i+1}}} \quad \text{if} \quad 1 \leq i < n\,, \\ &A_n^n = E_{t_{k_n}} \end{aligned}$$

and $\theta_n = \sum_{i=1}^n t_{k_i} \mathbb{1}_{A_i^n}$. Notice that $(A_i^n)_{1\leq i\leq n}$ is a partition of U and that $A_i^n \subset E_{t_{k_i}}$; therefore, if $x \in U$, say $x \in A_i^n$, then $\theta_n(x) = t_{k_i} > f(x)$ and $\theta_n \geq f$. Besides, the sequence $(E_{t_{k_i}})_{1\leq i\leq n}$ is decreasing, and this, together with the definition of A_i^n, ensures that $\theta_n(x) \leq t_i$ if $x \in E_{t_i}$ for $1 \leq i \leq n$. Now, given $x \in U$ and $\varepsilon > 0$, there exists n_0 such that $f(x)+\varepsilon > t_{n_0} > f(x)$. Then, for every $n \geq n_0$, $x \in E_{t_{n_0}}$ and, consequently, $\theta_n(x) \leq t_{n_0} < f(x)+\varepsilon$. Thus, (θ_n) converges to f everywhere on U (in fact, the convergence is monotone, but we do not prove this fact), and, since $0 \leq \theta_n \leq 1$, the dominated convergence theorem shows that

$$\lim_{n\to+\infty}\int_U \theta_n - f = 0\,.$$

However, for each fixed n, the function θ_n satisfies the conclusion of the lemma. Indeed, let $\varepsilon > 0$ be fixed. Since $E_{t_{k_{i+1}}} \subset E_{t_{k_i}}$, we have $1\!\!1_{A_i^n} = 1\!\!1_{E_{t_{k_i}}} - 1\!\!1_{E_{t_{k_{i+1}}}}$. The functions $1\!\!1_{E_{t_{k_i}}}$ and $1\!\!1_{E_{t_{k_{i+1}}}}$ are in $BV(U)$, by the definition of a set with finite perimeter. Thus $1\!\!1_{A_i^n}$ is BV too, and A_i^n is a set with finite perimeter. As noticed in the beginning of the proof, A_i^n satisfies (13), and there exists $g_i \in \mathcal{C}(\overline{U})$ such that $g_i \geq t_{k_i}\,1\!\!1_{A_i^n}$ and $\int_U (g_i - t_{k_i}\,1\!\!1_{A_i^n}) < \varepsilon/n$. Moreover, we can suppose that $g_i \leq t_{k_i}$ for every i. Set $g = \max_{1\leq i\leq n} g_i$. The function g is continuous on $\overline{U}$, and $g \geq \theta_n$ on U by construction. It remains to compute $||g-\theta_n||_{L^1(U)}$. If $x \in A_i^n$, then $g_i(x) = t_{k_i}$, and the condition $g_j \leq t_{k_j}$ enforces the maximum of the $g_j(x)$ to be reached for $j \in \{i,\dots,n\}$. We then have

$$(g-\theta_n)(x) = g_j(x) - t_{k_i} \leq g_j(x) - t_{k_j}\,1\!\!1_{A_j^n}(x).$$

Indeed, if $j = i$, this is obvious, and if $j > i$, we have $1\!\!1_{A_j^n}(x) = 0$, while $t_{k_i} \geq 0$. Consequently, $(g-\theta_n)(x) \leq \sum_{i=1}^n (g_j - t_{k_j}\,1\!\!1_{A_j^n})(x)$ and $\int_U (g-\theta_n) < n\times\varepsilon/n = \varepsilon$. If n has been chosen such that $\int_U(\theta_n - f) < \varepsilon$, then g is relevant to the conclusion of the lemma.

We suppose that $0 \leq f(x) \leq 1$ for every $x \in U$. For a general function $f \in L^\infty \cap BV(U)$, we can suppose, after an adequate modification of the function on a set of negligible measure, that $-M \leq f(x) \leq M$ for every $x \in U$, where $M = ||f||_{L^\infty(U)}$. Then we consider the function $f_1 = (f+M)/(2M)$. Given $\varepsilon > 0$, there exists $g_1 \in \mathcal{C}(\overline{U})$ such that $g_1(x) \geq f_1(x)$ and $||g_1 - f_1||_{L^1(U)} < \varepsilon/(2M)$ and $g = 2Mg_1 - M$ is convenient. $\square$

4.3. Proof of Theorem 4.1 (preliminary): Boundary condition.

PROPOSITION 4.1 (boundary condition). *Let $u \in L^\infty(Q\times(0,1))$ be an entropy process solution of problem* (1), *and let $\mathcal{F}_\varphi$ be defined by* (12). *Assume hypotheses* (H1), (H2), (H3), (H4), *and* (H6) *(or* (H6Bis)*). Then, for all $\kappa \in [A,B]$, for all nonnegative $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R}^d)$,*

$$\lim_{\varepsilon\to0}\int_{\mathcal{Q}} \mathcal{F}_\varphi(t,x,u(t,x,\alpha),\kappa,\overline{u}_\Sigma(t,x))\cdot\nabla\omega_\varepsilon(x)\,\psi(t,x)\,\lambda(x)\,d\alpha\,dx\,dt \;\leq 0\,. \tag{14}$$

In the case of a purely hyperbolic problem ($\varphi' = 0$), inequality (14) is the boundary condition written by Otto [Ott96], equivalent to the BLN condition [BLN79] for BV solutions. If the problem is strictly parabolic (that is, $\varphi'(u) \geq \Phi_{min} > 0$), then inequality (14) is trivially satisfied by any weak solution of the problem (1). In [MPT02], the condition (14) is listed among the conditions that an entropy solution should satisfy *by definition*. We refer to [MPT02] for a complete discussion of (14).

Proof of Proposition 4.1. We first aim to prove the following result: for every $\tilde\kappa \in [A,B]$, for every nonnegative $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0}\int_Q [\Phi^+(t,x,u,\tilde\kappa\top\overline{u}_\Sigma) - \nabla(\varphi(u)-\varphi(\tilde\kappa\top\overline{u}_\Sigma))^+]\cdot\nabla\omega_\varepsilon(x)\,\psi(t,x)\,\lambda(x)\,d\alpha\,dx\,dt \ \leq 0. \tag{15}$$

Fix $\tilde\kappa \in [A,B]$. In subsections 4.1.2 and 4.1.3, we defined a notion of normal trace for the flux $\Phi^+_\kappa(t,x,u) - \nabla(\varphi(u)-\varphi(\kappa))^+$ when u is an entropy weak solution of problem (1). Of course, the same can be done when u is an entropy process solution of problem (1); this time just consider the field $\mathcal{F}^+_\kappa$ defined by

$$\mathcal{F}^+_\kappa = \begin{pmatrix} \int_0^1 (u-\kappa)^+\,d\alpha \\ \int_0^1(\Phi^+_\kappa(t,x,u) - \nabla(\varphi(u)-\varphi(\kappa))^+)d\alpha \end{pmatrix}.$$

Moreover, if $\mathcal{T}^+_{n,\kappa}$ still denotes the normal trace of the spatial part of $\mathcal{F}^+_\kappa$, for all $\psi \in \mathcal{C}_c^\infty([\,0,T)\times\mathbb{R}^d)$,

$$\begin{aligned}&\langle \mathcal{T}^+_{n,\kappa},\psi\lambda\rangle \\ &= -\lim_{s\to 0}\frac{1}{s}\int_s^T\!\!\int_0^1\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s}(\Phi^+_\kappa(t,x,u) - \nabla(\varphi(u)-\varphi(\kappa))^+)\cdot\begin{pmatrix}-\nabla f(\overline{x})\\ 1\end{pmatrix}\psi\lambda\,dx_d\,d\overline{x}\,dt\,d\alpha\end{aligned} \tag{16}$$

and

$$\langle \mathcal{T}^+_{n,\kappa},\psi\lambda\rangle = -\lim_{\varepsilon\to 0}\int_Q[\Phi^+_\kappa(t,x,u) - \nabla(\varphi(u)-\varphi(\kappa))^+]\cdot\nabla\omega_\varepsilon\,\psi\lambda\,dx\,dt\,d\alpha\,. \tag{17}$$

Therefore, if ψ is a nonnegative function of $\mathcal{C}_c^\infty([\,0,T)\times\mathbb{R}^d)$ such that $\operatorname{sgn}^+(\varphi(\bar u) - \varphi(\kappa))\overline{\psi} = 0$ a.e. on $(0,T)\times\partial\Omega$, then, choosing $\psi(1-\omega_\varepsilon)$ as a test-function in (6), we get

$$-\langle \mathcal{T}^+_{n,\kappa},\psi\lambda\rangle \leq M\int_\Sigma(\overline{u}-\kappa)^+\,\psi\,\lambda\,d\gamma(x)\,dt\,. \tag{18}$$

Now, we intend to define a notion of normal trace for the flux $\Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde\kappa) - \nabla(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde\kappa))^+$. To that purpose, we set

$$\overline{\mathcal{F}}^+ = \begin{pmatrix} \int_0^1(u-\overline{u}_\Sigma\top\tilde\kappa)^+\,d\alpha \\ \int_0^1(\Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde\kappa) - \nabla(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde\kappa))^+)d\alpha\end{pmatrix}, \tag{19}$$

and we prove the following lemma.

Lemma 4.2. *Let $u \in L^\infty(Q\times(\,0,1))$ be an entropy process solution of problem* (1)*, and let the field $\overline{\mathcal{F}}^+ \in (L^2((0,T)\times B\cap Q))^{d+1}$ be defined by* (19)*. Assume hypotheses* (H1), (H2), (H3), (H4), *and* (H6) *(or* (H6Bis)*). Then, for every open subset D of B such that $\overline{D}\subset B$, the divergence of $\overline{\mathcal{F}}^+$ is a bounded Radon measure on* $(0,T)\times D\cap Q$.

Proof of Lemma 4.2. Set $g = \partial_t\,\overline{u}_\Sigma + \operatorname{div}_x F(t,x,\overline{u}_\Sigma) - \Delta\,\varphi(\overline{u}_\Sigma)$. From hypothesis (H6) we have $g \in L^1((0,T)\times B\cap Q)$, and the function $\overline{u}_\Sigma$ (which, we recall, belongs to $W^{1,1}((0,T)\times B\cap Q)))$ can be seen as an entropy solution of the equation $\partial_t\,w +$

$\operatorname{div}_x F(t,x,w) - \Delta\,\varphi(w) = g$ with unknown w. The identity $(\overline{u}_\Sigma\top\tilde{\kappa} - \kappa)^- = (\overline{u}_\Sigma - \kappa)^- - (\overline{u}_\Sigma - \tilde{\kappa}\bot\kappa)^-$ ensures that the function $\overline{u}_\Sigma\top\tilde{\kappa}$ satisfies the entropy inequality

$$\int_{\mathcal{Q}} \left[(\overline{u}_\Sigma\top\tilde{\kappa} - \kappa)^-\,\theta_t + [\Phi^-(t,x,\overline{u}_\Sigma\top\tilde{\kappa},\kappa) - \nabla(\varphi(\overline{u}_\Sigma\top\tilde{\kappa}) - \varphi(\kappa))^-]\cdot\nabla\theta\right]\,d\alpha\,dx\,dt$$
$$+ \int_\Omega (\overline{u}_\Sigma\top\tilde{\kappa}(0,x) - \kappa)^-\,\theta(0)\,dx + \int_{\mathcal{Q}} \operatorname{sgn}^-(\overline{u}_\Sigma\top\tilde{\kappa} - \kappa)\,g\,\theta\,dx\,dt\,d\alpha \ge 0$$

for every $\kappa \in [A,B]$ and nonnegative function $\theta \in \mathcal{C}_c^\infty([0,T)\times B\cap Q)$. Now we use a result of comparison and assert that, for any nonnegative function $\theta \in \mathcal{C}_c^\infty([0,T)\times B\cap Q)$, we have

$$\int_{\mathcal{Q}} \left[(u - \overline{u}_\Sigma\top\tilde{\kappa})^+\,\theta_t + [\Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde{\kappa}) - \nabla(\varphi(u) - \varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+]\cdot\nabla\theta\right]\,d\alpha\,dx\,dt$$
$$+ \int_\Omega (u_0 - \overline{u}_\Sigma\top\tilde{\kappa}(0,x))^+\,\theta(0)\,dx + \int_{\mathcal{Q}} \operatorname{sgn}^+(u - \overline{u}_\Sigma\top\tilde{\kappa})\,g\,\theta\,dx\,dt\,d\alpha \ge 0. \tag{20}$$

This result of comparison, proved in [Car99] for entropy weak solution, remains true when applied to entropy process solutions. Notice that we state a result of comparison *inside* $[0,T)\times\Omega$ (the previous function θ vanishes on $[0,T)\times\partial\Omega$); this point is crucial. A result of comparison on the whole domain Q is the object of Theorem 4.1, which we are actually proving. As a matter of fact, we would like to rule out the hypothesis that θ vanishes on $[0,T)\times\partial\Omega$. Toward that end, first notice that (20) is still true if $\theta \in \mathcal{C}_c^1([0,T)\times(B\cap\overline{\Omega}))$ and $\theta = 0$ on $[0,T)\times(B\cap\partial\Omega)$. Let $\tilde{\theta} \in \mathcal{C}_c^\infty([0,T)\times(B\cap\overline{\Omega}))$, define, for $s>0$, $h_s(x) = \min([x_d - f(\overline{x})]/s, 1)$, and choose $\theta = \tilde{\theta}\,h_s$ in (20) to get

$$\int_{\mathcal{Q}} \left[(u - \overline{u}_\Sigma\top\tilde{\kappa})^+\,\tilde{\theta}_t + [\Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde{\kappa}) - \nabla(\varphi(u) - \varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+]\cdot\nabla\tilde{\theta}\right] h_s\,d\alpha\,dx\,dt$$
$$+ \int_\Omega (u_0 - \overline{u}_\Sigma\top\tilde{\kappa}(0,x))^+\,\tilde{\theta}(0)\,h_s\,dx + \int_{\mathcal{Q}} \operatorname{sgn}^+(u - \overline{u}_\Sigma\top\tilde{\kappa})\,g\,\tilde{\theta}\,h_s\,dx\,dt\,d\alpha \ge A_s + B_s, \tag{21}$$

where

$$A_s = -\frac{1}{s}\int_0^T\!\!\int_0^1\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} \Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde{\kappa})\cdot\nabla_x(x_d - f(\overline{x}))\,\tilde{\theta}\,dx_d\,d\overline{x}\,dt\,d\alpha,$$
$$B_s = \frac{1}{s}\int_0^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} \nabla(\varphi(u) - \varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+\cdot\nabla_x(x_d - f(\overline{x}))\,\tilde{\theta}\,dx_d\,d\overline{x}\,dt.$$

Let C be a bound of $\Phi^+(t,x,z,w)\cdot\nabla_x(x_d - f(\overline{x}))$ in $L^\infty(Q\times[A,B]^2)$. Such a bound exists and, for every s,

$$A_s \ge -C\,T\,|\Pi|\,||\tilde{\theta}||_{L^\infty([0,T)\times(B\cap\overline{\Omega}))}\,. \tag{22}$$

On the other hand, the term B_s can be decomposed as $B_s = \overline{B_s} + B_s^d$, where

$$\overline{B_s} = -\frac{1}{s}\int_0^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} \nabla_{\overline{x}}(\varphi(u) - \varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+\cdot\nabla_{\overline{x}}f(\overline{x})\,\tilde{\theta}\,dx_d\,d\overline{x}\,dt,$$
$$B_s^d = \frac{1}{s}\int_0^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} \partial_{x_d}(\varphi(u) - \varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+\,\tilde{\theta}\,dx_d\,d\overline{x}\,dt.$$

Integration by parts with respect to $\overline{x}$ in $\overline{B_s}$ and integration by parts with respect to x_d in B_s^d (we use the fact that $\varphi(u)(t,\overline{x},f(\overline{x}))=\varphi(\overline{u}_\Sigma)(t,\overline{x})$) yields the following: for almost every positive s (small enough),

$$\begin{aligned}
\overline{B_s} &= \frac{1}{s}\int_0^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s}(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+\mathrm{div}_{\overline{x}}\nabla_{\overline{x}}f(\overline{x})\,\tilde{\theta}\,dx_d\,d\overline{x}\,dt,\\
B_s^d &= -\frac{1}{s}\int_0^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s}(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+\,\partial_{x_d}\tilde{\theta}\,dx_d\,d\overline{x}\,dt\\
&\quad+\frac{1}{s}\int_0^T\!\!\int_\Pi(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+(t,\overline{x},f(\overline{x})+s)\,\tilde{\theta}(\overline{x},f(\overline{x})+s)d\overline{x}\,dt.
\end{aligned}$$

Notice that, first, the second term on the right-hand side of the previous equality in nonnegative; that, second, $\lim_{s\to0}\overline{B_s}=0$ and $\lim_{s\to0}\frac{1}{s}\int_0^T\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s}(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+\,\partial_{x_d}\tilde{\theta}\,dx_d\,d\overline{x}\,dt\;=0$ (because the trace of $\varphi(u)$ is $\varphi(\overline{u})$); and that, third, h_s converge to 1 in $L^1(B\cap\Omega)$. Consequently, letting s go to zero on both sides of inequality (21) yields

$$\begin{aligned}
&\int_{\mathcal{Q}}\left[(u-\overline{u}_\Sigma\top\tilde{\kappa})^+\,\tilde{\theta}_t+[\Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde{\kappa})-\nabla(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+]\cdot\nabla\tilde{\theta}\right]\,d\alpha\,dx\,dt\\
&+\int_\Omega(u_0-\overline{u}_\Sigma\top\tilde{\kappa}(0,x))^+\,\tilde{\theta}(0)\,dx\;+\int_{\mathcal{Q}}\mathrm{sgn}^+(u-\overline{u}_\Sigma\top\tilde{\kappa})\,g\,\tilde{\theta}\,dx\,dt\,d\alpha\;\geq\liminf_{s\to0}A_s.
\end{aligned}$$

Let D be an open subset of B whose closure is a subset of B too. From (22), it appears that $\liminf_{s\to0}A_s$ can be viewed as the action of a certain distribution A_∞ on $\tilde{\theta}$ and that A_∞ is a bounded Radon measure on $[0,T)\times D\cap Q$. Since $\int_0^1\mathrm{sgn}^+(u-\overline{u}_\Sigma)\,g\,d\alpha$ and $(u_0-\overline{u}_\Sigma\top\tilde{\kappa}(0,x))^+\delta_{t=0}$ are bounded Radon measures on $[0,T)\times D\cap Q$, the previous inequality shows that the divergence of the field $\overline{\mathcal{F}}^+$ is a bounded Radon measure on $[0,T)\times D\cap Q$. This ends the proof of Lemma 4.2. □

Remark 4.1. If $\overline{u}_\Sigma$ satisfies (H6Bis) instead of (H6), then $\overline{u}_\Sigma$ can be seen as the entropy solution of the equation $\partial_t\,w+\mathrm{div}_x\,F(t,x,w)-\Delta\,\varphi(w)=g$, with a source term g which is a bounded Radon measure on $(0,T)\times B\cap Q$. In the proof of the previous lemma we used a theorem of comparison of Carrillo (Theorem 8 in [Car99]) between two entropy solutions u_i ($i\in\{1,2\}$) of the equation

$$\partial_t u_i+\mathrm{div}F(t,x,u_i)-\Delta\varphi(u_i)=f_i$$

(where $f_i\in L^1$) to derive the inequality (20). A careful study of the proof of the result of comparison given by Carrillo shows that it still holds if $f_1=0$ and f_2 is a bounded Radon measure. Consequently, inequality (20) remains true under hypothesis (H6Bis) and Lemma 4.2 also.

As a consequence of this lemma, we can define a functional $\overline{\mathcal{T}}^+_{n,\tilde{\kappa}}$, which represents the normal trace of the flux $\Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde{\kappa})-\nabla(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde{\kappa}))^+$ on $(0,T)\times(\partial\Omega\cap D)$ and satisfies the analogue of the relations (16) and (17), where κ has been replaced by $\overline{u}_\Sigma\top\tilde{\kappa}$ in these latter. (We use the fact that there exists an open set D such that $\mathrm{supp}(\lambda)\subset D\subset\overline{D}\subset B$ to ensure that these limits make sense.)

Now, denote by $\overline{\mathcal{S}}$ the set of all the functions $v:(0,T)\times\Pi\to\mathbb{R}$ satisfying

$$v(t,x)=\sum_{i=1}^{N_v}w_i\,L_i(t,x),\tag{23}$$

where

$$\forall i, \quad w_i \in \mathbb{R},\ L_i \in \mathcal{C}^\infty([0,T]\times\Pi),\ L_i \geq 0,\ \text{and} \ \sum_{i=1}^{N_v} L_i = 1 \quad \text{on } [0,T]\times\Pi_\lambda. \tag{24}$$

We say that $v \in \overline{\mathcal{S}}^+$ if $v \in \overline{\mathcal{S}}$ and admits a decomposition as (23) such that $w_i \geq \overline{u}_\Sigma$ a.e. on $\operatorname{supp}(L_i)$ for all i. If $v \in \overline{\mathcal{S}}$ and satisfies (23), we set

$$\langle \mathcal{T}^+_{n,v\top\tilde\kappa}, \psi\lambda\rangle \;=\; \sum_{i=1}^{N_v} \langle \mathcal{T}^+_{n,w_i\top\tilde\kappa}, L_i\psi\lambda\rangle.$$

Notice that this is a *notation* and not a *definition*, because the decomposition (23) with w_i, L_i satisfying (24) is not unique. An immediate consequence of (18) is the following: if $v \in \overline{\mathcal{S}}^+$, then

$$-\langle \mathcal{T}^+_{n,v\top\tilde\kappa}, \psi\lambda\rangle \leq 0 \quad \forall \psi \in \mathcal{C}^\infty_c([0,T)\times\mathbb{R}^d),\ \psi \geq 0. \tag{25}$$

Furthermore, we claim that, if $v \in \overline{\mathcal{S}}^+$, then

$$\langle \mathcal{T}^+_{n,v\top\tilde\kappa} - \overline{\mathcal{T}}^+_{n,\tilde\kappa}, \psi\lambda\rangle \leq M\sqrt{1+||\nabla_{\overline{x}} f||^2_\infty}\ \sum_{i=1}^{N_v}\int_0^T\!\!\int_\Pi |w_i - \overline{u}_\Sigma|\,\psi\lambda\, L_i\, d\overline{x}\, dt. \tag{26}$$

Let us prove this result: from (16) we have $\langle \mathcal{T}^+_{n,v\top\tilde\kappa} - \overline{\mathcal{T}}^+_{n,\tilde\kappa}, \psi\lambda\rangle = -\lim_{s\to 0}\sum_{i=1}^{N_v}(H_i(s) + P_i(s))$, where

$$H_i(s) = \frac{1}{s}\int_s^T\!\!\int_0^1\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} \big(\, \Phi^{\,+}(t,x,u,w_i\top\tilde\kappa) \\ - \Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde\kappa)\big)\cdot \begin{pmatrix} -\nabla f(\overline{x}) \\ 1 \end{pmatrix} L_i\psi\lambda\, dx_d\, d\overline{x}\, dt\, d\alpha,$$

$$P_i(s) = \frac{1}{s}\int_s^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} \nabla\big((\,\varphi\,(u) - \varphi(\overline{u}_\Sigma\top\tilde\kappa))^+ \\ - (\varphi(u) - \varphi(w_i\top\tilde\kappa))^+\big)\cdot \begin{pmatrix} -\nabla f(\overline{x}) \\ 1 \end{pmatrix} L_i\psi\lambda\, dx_d\, d\overline{x}\, dt.$$

Since the function $\Phi^+(t,x,u,v)$ is M-Lipschitz continuous with respect to v, uniformly with respect to $(t,x,u) \in Q\times[A,B]$, we have

$$\begin{aligned} H_i(s) &\geq -\frac{1}{s}M\sqrt{1+||\nabla_{\overline{x}} f||^2_\infty}\int_0^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} |w_i\top\tilde\kappa - \overline{u}_\Sigma\top\tilde\kappa|\, L_i\psi\lambda\, dx_d\, d\overline{x}\, dt \\ &\geq -\frac{1}{s}M\sqrt{1+||\nabla_{\overline{x}} f||^2_\infty}\int_0^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} |w_i - \overline{u}_\Sigma|\, L_i\psi\lambda\, dx_d\, d\overline{x}\, dt\,. \end{aligned}$$

Consequently,

$$\sum_{i=1}^{N_v} H_i(s) \geq -\frac{1}{s}M\sqrt{1+||\nabla_{\overline{x}} f||^2_\infty}\sum_{i=1}^{N_v}\int_0^T\!\!\int_\Pi\int_{f(\overline{x})}^{f(\overline{x})+s} |w_i - \overline{u}_\Sigma|\,\psi\lambda\, L_i\, dx_d\, d\overline{x}dt,$$

and the limit of the right-hand side of this latter inequality can be explicitly computed since the function $\psi\lambda\, L_i$ is smooth:

$$\lim_{s\to 0}\sum_{i=1}^{N_v} H_i(s) \geq -M\sqrt{1+||\nabla_{\overline{x}} f||_\infty^2}\sum_{i=1}^{N_v}\int_0^T\!\!\int_\Pi |w_i-\overline{u}_\Sigma|\,\psi\lambda\, L_i\, d\overline{x}\, dt\,. \tag{27}$$

On the other hand, we have $\limsup_{s\to 0} P_i(s) \geq 0$. We will not detail the proof of this result, for it is identical to the justification of the fact that $\limsup_{s\to 0} B_s \geq 0$ in the proof of Lemma 4.2. Together with (27), the result $\limsup_{s\to 0} P_i(s) \geq 0$ yields (26). Furthermore, (26) combined with (25) shows that, if $v \in \overline{\mathcal{S}}^+$($v$ satisfies (23), with $w_i \geq \overline{u}_\Sigma$ a.e. on $\mathrm{supp}(L_i)$), then

$$-\langle \overline{\mathcal{T}}^+_{n,\tilde\kappa}, \psi\lambda\rangle \leq M\sqrt{1+||\nabla_{\overline{x}} f||_\infty^2}\sum_{i=1}^{N_v}\int_0^T\!\!\int_\Pi (w_i-\overline{u}_\Sigma)\,\psi\lambda\, L_i\, d\overline{x}\, dt. \tag{28}$$

Since

$$\langle \overline{\mathcal{T}}^+_{n,\tilde\kappa}, \psi\lambda\rangle = -\lim_{\varepsilon\to 0}\int_Q [\Phi^+(t,x,u,\overline{u}_\Sigma\top\tilde\kappa) - \nabla(\varphi(u)-\varphi(\overline{u}_\Sigma\top\tilde\kappa))^+]\cdot\nabla\omega_\varepsilon\,\psi\lambda\, dx\, dt\, d\alpha\,,$$

our first aim, which is the proof of (15), will be reached if the right-hand side of (28) can be made as small as desired. Let us prove this fact: $\varepsilon > 0$. Since $\overline{u}_\Sigma \in L^\infty \cap W^{1,1}((0,T)\times\Pi)$ (hypothesis (H6)), we have $\overline{u}_\Sigma \in L^\infty\cap BV((0,T)\times\Pi)$, and Lemma 4.1 shows that there exists $g \in \mathcal{C}([0,T]\times\overline{\Pi})$ such that $g \geq \overline{u}_\Sigma$ a.e. on $(0,T)\times\Pi$ and $\int_{(0,T)\times\Pi} g - \overline{u}_\Sigma < \varepsilon$. Let η be a modulus of uniform continuity of g on $[0,T]\times\overline{\Pi}$. The set $(0,T)\times\Pi$ (with compact closure) can be covered by a finite number of balls with radius η centered in $(0,T)\times\Pi$, say $V_1,\ldots,V_Q$. Let $(L_i)_{1,Q}$ be a regular partition of unity subordinate to the open coverage (V_i) of $[0,T]\times\overline{\Pi}$. For a certain $(t_i,\overline{x}_i)\in V_i$, set $w_i = g(t_i,\overline{x}_i)+\varepsilon$ and define $v = \sum_{i=1}^Q w_i\, L_i$. Then $v \in \overline{\mathcal{S}}^+$ and

$$\begin{aligned}\sum_{i=1}^{Q}\int_0^T\!\!\int_\Pi (w_i-\overline{u}_\Sigma)\,\psi\lambda\, L_i\, d\overline{x}\, dt &= \int_0^T\!\!\int_\Pi (v-\overline{u}_\Sigma)\,\psi\lambda\, d\overline{x}\, dt\\ &= \int_0^T\!\!\int_\Pi (v-g)\,\psi\,\lambda\, d\overline{x}\, dt + \int_0^T\!\!\int_\Pi (g-\overline{u}_\sigma)\,\psi\,\lambda\, d\overline{x}\, dt\\ &\leq 2||\psi\,\lambda||_\infty\, T\,|\Pi|\,\varepsilon.\end{aligned}$$

This completes the proof of (15). Similarly, we can prove

(29)

$$\lim_{\varepsilon\to 0}\int_Q [\Phi^-(t,x,u,\tilde\kappa\bot\overline{u}_\Sigma) - \nabla(\varphi(u)-\varphi(\tilde\kappa\bot\overline{u}_\Sigma))^-]\cdot\nabla\omega_\varepsilon(x)\,\psi(t,x)\,\lambda(x)\, d\alpha\, dx\, dt \leq 0$$

for every $\tilde\kappa\in[A,B]$ and for every nonnegative $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R}^d)$. Then Proposition 4.1 follows from the formula

$$\begin{aligned}\mathcal{F}_\varphi(t,x,u,\kappa,w) = &\left[\,\Phi^{\,+}(t,x,u,\kappa\top w) - \nabla(\varphi(u)-\varphi(\kappa\top w))^+\right]\\ &+\left[\Phi^-(t,x,u,\kappa\bot w) - \nabla(\varphi(u)-\varphi(\kappa\bot w))^-\right]. \qquad \square\end{aligned}$$

4.4. Proof of Theorem 4.1 (step 1): Inner comparison. Let u and $v \in L^\infty(Q \times (0,1))$ be two entropy process solutions of problem (1). The following result of comparison between u and v involving test-functions which *vanish* on the boundary of Ω can be proved (see [Car99] or [EGHM02]).

PROPOSITION 4.2 (inner comparison). *Let u and $v \in L^\infty(Q \times (0,1))$ be two entropy process solutions of problem* (1). *Assume hypotheses* (H1), (H2), (H3), *and* (H4). *Let ζ be a nonnegative function of $\mathcal{C}^\infty([0,T) \times \mathbb{R}^d \times [0,T) \times \mathbb{R}^d)$ such that*

$$\begin{cases} \forall (s,y) \in Q\,,\ (t,x) \longmapsto \zeta(t,x,s,y) \in \mathcal{C}_c^\infty([0,T) \times \Omega), \\ \forall (t,x) \in Q\,,\ (s,y) \longmapsto \zeta(t,x,s,y) \in \mathcal{C}_c^\infty([0,T) \times \Omega). \end{cases}$$

Then we have

$$\begin{aligned} &\int_Q\int_Q \begin{bmatrix} |u(t,x,\alpha) - v(s,y,\beta)|(\zeta_t + \zeta_s) \\ + \mathcal{G}_x(t,x,u(t,x,\alpha), v(s,y,\beta)) \cdot \nabla_x \zeta \\ + \mathcal{G}_y(s,y,v(s,y,\beta), u(t,x,\alpha)) \cdot \nabla_y \zeta \\ - \nabla_x |\varphi(u)(t,x) - \varphi(v)(s,y)| \cdot \nabla_y \zeta \\ - \nabla_y |\varphi(u)(t,x) - \varphi(v)(s,y)| \cdot \nabla_x \zeta \end{bmatrix} d\alpha dx dt d\beta dy ds \\ &+ \int_Q \int_\Omega |u_0(x) - v(s,y,\beta)|\, \zeta(0,x,s,y)\, dx\, d\beta\, dy\, ds \\ &+ \int_Q \int_\Omega |u_0(y) - u(t,x,\alpha)|\, \zeta(t,x,0,y)\, dy\, d\alpha\, dx\, dt \ \geq 0. \end{aligned} \tag{30}$$

4.5. Proof of Theorem 4.1 (step 2): General test-function. We now follow the lines of the proof of uniqueness given by Mascia, Porretta, and Terracina in [MPT02].

First, we would like to consider test-functions which do not necessarily vanish on $\partial\Omega$ and are localized into the ball B. For $\overline{x} \in \mathbb{R}^{d-1}$, set $\rho_m(\overline{x}) = \rho_m(x_1) \cdots \rho_m(x_{d-1})$ and define the function ξ by

$$\xi(t,s,x,y) = \psi(t,x)\, \rho_l(t-s)\, \rho_m(\overline{x} - \overline{y})\, \rho_n(x_d - y_d)\,. \tag{31}$$

We took care to choose ρ satisfying $\mathrm{supp}(\rho) \subset [-1,0)$ to ensure

$$\begin{aligned} &\forall (t,x) \in Q\,,\ (s,y) \longmapsto \xi(t,s,x,y) \in \mathcal{C}_c^\infty(Q)\,, \\ &\forall (t,s,x) \in [0,T) \times [0,T) \times \mathrm{supp}\,(\lambda),\ \mathrm{supp}_y\, \xi(t,s,x,\cdot) \subset B. \end{aligned} \tag{32}$$

For $\varepsilon > 0$ define ζ to be the function

$$\zeta \,:\, (t,s,x,y) \longmapsto \omega_\varepsilon(x)\, \xi(t,s,x,y)\, \lambda(x).$$

Then, for m large enough compared with n, the assumptions of Proposition 4.2 are satisfied, and, with this particular choice of function ζ, inequality (30) turns into the inequality

$$\int_{\mathcal{Q}}\int_{\mathcal{Q}}\left[\begin{array}{l} \qquad\qquad |u-\widehat{v}|\,\omega_\varepsilon(x)\,((\xi\lambda)_t+(\xi\lambda)_s) \\ +\Big(\mathcal{G}_x(t,x,u,\widehat{v})\cdot\nabla_x(\xi\,\lambda) \\ \qquad\qquad +\mathcal{G}_y(t,y,\widehat{v},u)\cdot\nabla_y(\xi\,\lambda)\Big)\,\omega_\varepsilon(x) \\ -\Big(\nabla_x|\varphi(u)-\varphi(\widehat{v})|\cdot\nabla_y(\xi\,\lambda) \\ \qquad\qquad +\nabla_y|\varphi(u)-\varphi(\widehat{v})|\cdot\nabla_x(\xi\,\lambda)\Big)\,\omega_\varepsilon(x) \end{array}\right] dxdtd\alpha dydsd\beta$$
$$+\int_{\mathcal{Q}}\int_{\mathcal{Q}} \mathcal{G}_x(t,x,u,\widehat{v})\cdot\nabla\omega_\varepsilon(x)\;\xi\,\lambda\,d\alpha\,dx\,dt\;\,d\beta\,dy\,ds$$
$$-\int_{\mathcal{Q}}\int_{\mathcal{Q}} \nabla_y|\varphi(u)-\varphi(\widehat{v})|\cdot\nabla\omega_\varepsilon(x)\,\xi\,\lambda\,dx\,dt\,d\alpha\,dy\,ds\,d\beta$$
$$+\int_{\Omega}\int_{\mathcal{Q}} |u_0(x)-\widehat{v}|\,(\xi\,\lambda)(0,x,y)\,\omega_\varepsilon(x)\,dx\delta\beta\,dy\,ds\ge 0,$$

where

$$u=u(t,x,\alpha)\quad\text{and}\quad \widehat{v}=v(s,y,\beta).$$

Using formula (12), this inequality can be rewritten as

$$\int_{\mathcal{Q}}\int_{\mathcal{Q}}\left[\begin{array}{l} \qquad\qquad |u-\widehat{v}|\,\omega_\varepsilon(x)\,((\xi\,\lambda)_t+(\xi\lambda)_s) \\ +\left(\mathcal{G}_x(t,x,u,\widehat{v})\cdot\nabla_x(\xi\,\lambda)+\mathcal{G}_y(t,y,\widehat{v},u)\cdot\nabla_y(\xi\,\lambda)\right)\omega_\varepsilon(x) \\ -\left(\nabla_x|\varphi(u)-\varphi(\widehat{v})|\cdot\nabla_y(\xi\,\lambda)\right. \\ \quad \left.+\nabla_y|\varphi(u)-\varphi(\widehat{v})|\cdot\nabla_x(\xi\,\lambda)\right)\omega_\varepsilon(x) \end{array}\right] d\alpha dxdtd\beta dyds$$
$$+\int_{\mathcal{Q}}\int_{\mathcal{Q}} \mathcal{F}_\varphi(t,x,u,\widehat{v},\overline{u}_\Sigma)\cdot\nabla\omega_\varepsilon(x)\,\xi\,\lambda\,d\alpha\,dx\,dt\,d\beta\,dy\,ds$$
$$+\int_{\Omega}\int_{\mathcal{Q}} |u_0(x)-\widehat{v}|\,(\xi\,\lambda)(0,x,y)\,\omega_\varepsilon(x)\,dx\,dy\,ds\,d\beta\;\ge\;A+B+C, \tag{33}$$

where

$$A=\int_Q\int_Q \nabla_y|\varphi(u)-\varphi(\widehat{v})|\cdot\nabla\omega_\varepsilon(x)\,\xi\,\lambda\,dx\,dt\,dy\,ds,$$
$$B=-\int_{\mathcal{Q}}\int_{\mathcal{Q}} \mathcal{G}_x(t,x,\widehat{v},\overline{u}_\Sigma)\cdot\nabla\omega_\varepsilon(x)\,\xi\,\lambda\,d\alpha\,dx\,dt\,d\beta\,dy\,ds,$$
$$C=\int_{\mathcal{Q}}\int_{\mathcal{Q}} \mathcal{G}_x(t,x,u,\overline{u}_\Sigma)\cdot\nabla\omega_\varepsilon(x)\,\xi\,\lambda\,d\alpha\,dx\,dt\,d\beta\,dy\,ds.$$

Using Proposition 4.1 and taking the limit of both sides of the previous inequality with respect to ε then yields

$$\int_{\mathcal{Q}}\int_{\mathcal{Q}}\left[\begin{array}{c} |u-\widehat{v}|\,((\xi\,\lambda)_t+(\xi\,\lambda)_s) \\ +\mathcal{G}_x(t,x,u,\widehat{v})\cdot\nabla_x(\xi\,\lambda)+\mathcal{G}_y(t,y,\widehat{v},u)\cdot\nabla_y(\xi\,\lambda) \\ -\nabla_x|\varphi(u)-\varphi(\widehat{v})|\cdot\nabla_y(\xi\,\lambda)+\nabla_y|\varphi(u)-\varphi(\widehat{v})|\cdot\nabla_x(\xi\,\lambda) \end{array}\right] d\alpha\,dx\,dt\,d\beta\,dy\,ds$$
$$+\int_{\Omega}\int_{\mathcal{Q}} |u_0(x)-\widehat{v}|\,(\xi\,\lambda)(0,x,y)\,d\beta\,dy\,ds\,dx\ge\lim_{\varepsilon\to 0}\,(A+B+C\,),$$

or (using formula (11))

$$\int_{\mathcal{Q}}\int_{\mathcal{Q}}\left[\begin{array}{c}|u-\widehat{v}|\,((\xi\,\lambda)_t+(\xi\,\lambda)_s)\\ +\,\Phi(t,x,u,\widehat{v})\cdot\nabla_x(\xi\,\lambda)+\Phi(t,y,\widehat{v},u)\cdot\nabla_y(\xi\,\lambda)\\ -\,(\nabla_x|\varphi(u)-\varphi(\widehat{v})|+\nabla_y|\varphi(u)-\varphi(\widehat{v})|)\cdot(\nabla_y+\nabla_x)(\xi\,\lambda)\end{array}\right]d\alpha\,dx\,dt\,d\beta\,dy\,ds$$
$$+\int_{\Omega}\int_{\mathcal{Q}}|u_0(x)-\widehat{v}|\,(\xi\,\lambda)(0,x,y)\,d\beta\,dy\,ds\,dx\geq\lim_{\varepsilon\to 0}\,(A+B+C\,). \tag{34}$$

Now, we intend to pass to the limit on l, m, and n in the previous inequality. We will do so (on l and m and, eventually, on n), but notice that the study of the behavior of A, B, and C as $[\varepsilon\to 0]$ and the doubling variable technique itself interfere with each other.

Using the definition of ξ from (31), it appears that C does not depend on l, m, and n:

$$C=\int_{\mathcal{Q}}\mathcal{G}_x(t,x,u,\overline{u}_\Sigma)\cdot\nabla\omega_\varepsilon(x)\,\psi\,\lambda\,d\alpha\,dx\,dt.$$

Moreover, inequality (34) can be rewritten as

$$\int_{\mathcal{Q}}\int_{\mathcal{Q}}\left[\begin{array}{c}|u-\widehat{v}|\,\rho_l\,\rho_m\,\rho_n(\psi\,\lambda)_t\\ +\,\Phi(t,x,u,\widehat{v})\cdot\nabla_x(\psi\,\lambda)\rho_l\,\rho_m\,\rho_n\\ -\,(\nabla_x|\varphi(u)-\varphi(\widehat{v})|+\nabla_y|\varphi(u)-\varphi(\widehat{v})|)\cdot\nabla_x(\psi\,\lambda)\,\rho_l\,\rho_m\,\rho_n\end{array}\right]d\alpha\,dx\,dt\,d\beta\,dy\,ds$$
$$+\int_{\Omega}\int_{\Omega}|u_0(x)-u_0(y)|\,(\psi\,\lambda)(0,x)\,\rho_m\,\rho_n\,dx\,dy\,\geq\,\lim_{\varepsilon\to 0}\,(A+B+C\,)+D+E, \tag{35}$$

where

$$D=-\int_{\mathcal{Q}}\int_{\mathcal{Q}}[\Phi(t,x,u,\widehat{v})-\Phi(t,y,u,\widehat{v})]\cdot\nabla_x(\rho_l\,\rho_m\,\rho_n)\,\psi\,\lambda\,d\alpha\,dx\,dt\,d\beta\,dy\,ds,$$

$$E=\int_{\Omega}\int_{\mathcal{Q}}|u_0(y)-\widehat{v}|\,(\psi\,\lambda)(0,x)\,\rho_l(-s)\,\rho_m\,\rho_n\;d\beta\,dy\,ds\,dx.$$

The term E can be estimated by using the fact that the solution v completely satisfies the initial condition, which means, for example, that $\operatorname{ess\,lim}_{s\to 0^+}\int_\Omega\int_0^1|v(s,y,\alpha)-u_0(y)|\,d\beta\,dy=0$. On the other hand, if the flux function F does not depend on the (t,x)-variables, then $D=0$, and more generally, one can prove (see [CH99]) $D+E\geq H$, where

$$H=-C(F,\psi)\sup\left\{\int_{\mathcal{Q}}|v(s,\overline{y},y_d,\beta)-v(s+\sigma,\overline{y}+\overline{h},y_d+k,\beta)|ds\,d\overline{y}\,dy_d\,d\beta\,;\right.$$
$$\left.|\sigma|\leq\frac{1}{l},|\overline{h}|\leq\frac{1}{m},|k|\leq\frac{1}{n}\right\}. \tag{36}$$

Notice that, by continuity of the translations in L^1, we have $\lim_{l,m,n\to+\infty}H=0$.

4.5.1. Study of $A+B$. Going back to the study of A, B, we write $A+B = I + J^y + J^x$, where

$$\begin{aligned}
I &= -\int_{\mathcal{Q}}\int_{\mathcal{Q}} (\Phi(t,x,\widehat{v},\overline{u}_\Sigma(t,\overline{x}))\cdot\nabla\omega_\varepsilon(x)\,\xi\,\lambda\,d\alpha\,dx\,dt\,d\beta\,dy\,ds,\\
J^y &= \int_Q\int_Q \nabla_y|\varphi(u)(t,x)-\varphi(v)(s,y)|\cdot\nabla\omega_\varepsilon(x)\,\xi\,\lambda\,dx\,dt\,dy\,ds,\\
J^x &= \int_Q\int_Q \nabla_x|\varphi(\widehat{v})-\varphi(\overline{u}_\Sigma(t,\overline{x}))|\cdot\nabla\omega_\varepsilon(x)\,\xi\,\lambda\,dx\,dt\,dy\,ds.
\end{aligned}$$

Recall that

$$\nabla\omega_\varepsilon(x) = \rho_\varepsilon(f(\overline{x})-x_d)\begin{pmatrix}-\nabla f(\overline{x})\\ 1\end{pmatrix},$$

so that

$$\begin{aligned}
\widetilde{I} &= \lim_{\varepsilon\to 0} I\\
&= -\int_{\mathcal{Q}}\int_{[0,T)\times\Pi\times(0,1)} (\Phi(t,\overline{x},f(\overline{x}),\widehat{v},\overline{u}_\Sigma(t,\overline{x}))\cdot\begin{pmatrix}-\nabla f(\overline{x})\\ 1\end{pmatrix}(\xi\,\lambda)_{\Sigma_x}d\alpha d\overline{x}dtd\beta dyds,
\end{aligned}$$

where the index Σ_x indicates that the transformation concerns only the x variable. Here, for example, $(\xi\,\lambda)_{\Sigma_x}(t,x,y) = \xi(t,\overline{x},f(\overline{x}),y)\lambda(\overline{x},f(\overline{x}))$. To study the term J^x, we notice that the function $\overline{u}_\Sigma$ does not depend on x_d, and thus

$$\widetilde{J^x} = \lim_{\varepsilon\to 0} J^x = -\int_{[0,T)\times\Pi}\int_Q \nabla_{\overline{x}}\,|\varphi(\widehat{v})-\varphi(\overline{u}_\Sigma(t,\overline{x}))|\cdot\nabla f(\overline{x})\,(\xi\,\lambda)_{\Sigma_x}\,d\overline{x}\,dt\,dy\,ds.$$

Integration by parts with respect to $\overline{x}$ in $\widetilde{J^x}$ yields $\widetilde{J^x} = \widetilde{J^x_f} + \widetilde{J^x_\psi} + \widetilde{J^x_{\rho_m}} + \widetilde{J^x_{\rho_n}}$, where

$$\begin{aligned}
\widetilde{J^x_f} &= \int_{[0,T)\times\Pi}\int_Q |\varphi(\widehat{v})-\varphi(\overline{u}_\Sigma)|\Delta f(\overline{x})\,(\psi\,\lambda)_{\Sigma_x}\,\rho_l(t-s)\\
&\qquad\times\rho_m(\overline{x}-\overline{y})\,\rho_n(f(\overline{x})-y_d)\,d\overline{x}\,dt\,dy\,ds,\\
\widetilde{J^x_\psi} &= \int_{[0,T)\times\Pi}\int_Q |\varphi(\widehat{v})-\varphi(\overline{u}_\Sigma)|\nabla f(\overline{x})\\
&\qquad\cdot\nabla_{\overline{x}}\,((\psi\,\lambda)_{\Sigma_x})\rho_l(t-s)\,\rho_m(\overline{x}-\overline{y})\rho_n(f(\overline{x})-y_d)d\overline{x}dtdyds,\\
\widetilde{J^x_{\rho_m}} &= \int_{[0,T)\times\Pi}\int_Q |\varphi(\widehat{v})-\varphi(\overline{u}_\Sigma)|\nabla f(\overline{x})\\
&\qquad\cdot\nabla_{\overline{x}}\,\rho_m(\overline{x}-\overline{y})\,\rho_n(f(\overline{x})-y_d)\,\rho_l(t-s)\,\psi\,\lambda\,d\overline{x}\,dt\,dy\,ds,\\
\widetilde{J^x_{\rho_n}} &= \int_{[0,T)\times\Pi}\int_Q |\varphi(\widehat{v})-\varphi(\overline{u}_\Sigma)||\nabla f(\overline{x})|^2\rho_l(t-s)\\
&\qquad\times\rho_m(\overline{x}-\overline{y})\,\rho_n'(f(\overline{x})-y_d)\,(\psi\,\lambda)_{\Sigma_x}d\overline{x}dtdyds.
\end{aligned}$$

On the other hand, via integration by parts in J^y with respect to y, and recalling that the boundary condition $\varphi(u)=\varphi(\overline{u})$ on Σ is strongly satisfied according to Definition 3.1, we get

$$\begin{aligned}
\widetilde{J^y} &= \lim_{\varepsilon\to 0} J^y\\
&= -\int_{[0,T)\times\Pi}\int_Q |\varphi(\overline{u}_\Sigma(t,\overline{x}))-\varphi(\widehat{v})|\begin{pmatrix}-\nabla f(\overline{x})\\ 1\end{pmatrix}\cdot\nabla_y(\xi\,\lambda)(t,s,\overline{x},f(\overline{x}),y)\,d\overline{x}\,dt\,dy\,ds,
\end{aligned}$$

and, developing the scalar product,

$$\begin{aligned}\widetilde{J^y} &= \int_{[0,T)\times\Pi}\int_Q |\varphi(\overline{u}_\Sigma(t,\overline{x})) - \varphi(\widehat{v})|\nabla f(\overline{x})\cdot\nabla_{\overline{y}}(\xi\,\lambda)(t,s,\overline{x},f(\overline{x}),y)dy\,d\overline{x}\,dt\,ds\\ &\quad - \int_{[0,T)\times\Pi}\int_Q |\varphi(\overline{u}_\Sigma(t,\overline{x})) - \varphi(\widehat{v})|\partial_{y_d}(\xi\,\lambda)(t,s,\overline{x},f(\overline{x}),y)\,dy\,d\overline{x}\,dt\,ds\\ &= -\widetilde{J^x_{\rho_m}} + \int_{[0,T)\times\Pi}\int_Q |\varphi(\overline{u}_\Sigma) - \varphi(\widehat{v})|\rho_l(t-s)\\ &\qquad\qquad \times\ \rho_m(\overline{x}-\overline{y})\,\rho_n'(f(\overline{x})-y_d)\,(\psi\,\lambda)_{\Sigma_x} dy\,d\overline{x}\,dt\,ds,\end{aligned}$$

so that

$$\begin{aligned}\widetilde{J^x} + \widetilde{J^y} = \widetilde{J^x_f} + \widetilde{J^x_\psi} + \int_{[0,T)\times\Pi}\int_\Omega |\varphi(\overline{u}_\Sigma) - \varphi(\widehat{v})|\,(1+|\nabla f(\overline{x})|^2)\\ \times\ \rho_l(t-s)\,\rho_m(\overline{x}-\overline{y})\rho_n'(f(\overline{x})-y_d)\,(\psi\,\lambda)_{\Sigma_x}\,d\overline{x}\,dt\,dy\,ds.\end{aligned}$$

In particular, no derivatives of the functions ρ_m or ρ_l appear in $J^x + J^y$. Hence, summing up by $\widetilde{v}$ the quantity $v(t,\overline{x},y_d,\beta)$ and passing to the limit $[l,m\to+\infty]$ in $\lim_{\varepsilon\to0}(A+B) = \widetilde{I} + \widetilde{J^x} + \widetilde{J^y}$, we get

$$\lim_{l,m\to+\infty}\lim_{\varepsilon\to0}(A+B) = \overline{I} + \overline{J_f} + \overline{J_\psi} + \overline{J_{\rho_n}}\,,$$

with

$$\begin{aligned}\overline{I} = -\int_{[0,T)\times\Pi\times(0,1)}\int_0^\infty\int_0^1 \Phi(t,\overline{x},f(\overline{x}),\widetilde{v},\overline{u}_\Sigma)\\ \cdot\begin{pmatrix}-\nabla f(\overline{x})\\ 1\end{pmatrix}\rho_n(f(\overline{x})-y_d)(\psi\,\lambda)_{\Sigma_x}d\overline{x}dtd\alpha dy_d d\beta,\end{aligned}$$

$$\overline{J_f} = \int_{[0,T)\times\Pi}\int_0^\infty |\varphi(\widetilde{v}) - \varphi(\overline{u}_\Sigma)|\Delta f(\overline{x})\,(\psi\,\lambda)_{\Sigma_x}\,\rho_n(f(\overline{x})-y_d)\,d\overline{x}\,dt\,dy_d,$$

$$\overline{J_\psi} = \int_{[0,T)\times\Pi}\int_0^\infty |\varphi(\widetilde{v}) - \varphi(\overline{u}_\Sigma)|\nabla f(\overline{x})\cdot\nabla_{\overline{x}}\left((\psi\,\lambda)_{\Sigma_x}\right)\rho_n(f(\overline{x})-y_d)\,d\overline{x}\,dt\,dy_d,$$

$$\overline{J_{\rho_n}} = \int_{[0,T)\times\Pi}\int_0^\infty |\varphi(\widetilde{v}) - \varphi(\overline{u}_\Sigma)|\,(1+|\nabla f(\overline{x})|^2)\,\rho_n'(f(\overline{x})-y_d)\,(\psi\,\lambda)_{\Sigma_x}\,d\overline{x}\,dt\,dy_d.$$

To compute the limit as n tends to $+\infty$ of the four preceding terms, first recall that $\mathrm{trace}((\varphi(v)) - \varphi(\overline{u}_\Sigma)) = 0$, and that, consequently,

$$\lim_{n\to+\infty}\overline{J_f} = 0 \quad\text{and}\quad \lim_{n\to+\infty}\overline{J_\psi} = 0.$$

Besides, we note that

$$\Delta\,\omega_{1/n}(x) = -\rho_n'(f(\overline{x})-x_d)\,(1+|\nabla f(\overline{x})|^2) + \rho_n(f(\overline{x})-x_d)\,\Delta f(\overline{x}),$$

so that, replacing y_d by x_d in $\overline{J_{\rho_n}}$, we have

$$\begin{aligned}\overline{J_{\rho_n}} &= -\int_Q |\varphi(v) - \varphi(\overline{u}_\Sigma(t,\overline{x}))|\,\Delta\,\omega_{1/n}(x)\,(\psi\,\lambda)(t,\overline{x},f(\overline{x}))\,dx\,dt + \overline{J_f}\\ &= \int_Q \nabla|\varphi(v) - \varphi(\overline{u}_\Sigma(t,\overline{x}))|\,\nabla\,\omega_{1/n}(x)\,(\psi\,\lambda)(t,\overline{x},f(\overline{x}))\,dx\,dt + \overline{\varepsilon^1_n}\,.\end{aligned}$$

Here, the quantity $\overline{\varepsilon_n^1} = \overline{J_f} + \int_Q |\varphi(v) - \varphi(\overline{u}_\Sigma(t,\overline{x}))|\,\nabla\omega_{1/n}(x)\cdot\nabla(\psi\,\lambda)_{\Sigma_x}\,dx\,dt$ tends to zero when $n \to +\infty$. Moreover,

$$\overline{I} = -\int_{\mathcal{Q}} \Phi(t,x,v,\overline{u}_\Sigma)\cdot\nabla\omega_{1/n}(x)\,(\psi\,\lambda)_{\Sigma_x}\,d\beta\,dx\,dt\; + \overline{\varepsilon_n^2}\,,$$

where $\overline{\varepsilon_n^2} = \int_{\mathcal{Q}} (\Phi(t,x,v,\overline{u}_\Sigma) - \Phi(t,\overline{x},f(\overline{x}),v,\overline{u}_\Sigma))\cdot\nabla\omega_{1/n}(x)\,(\psi\,\lambda)_{\Sigma_x} d\beta\,dx\,dt$ tends to zero when $n \to +\infty$.

Using formula (11), we get

$$\begin{aligned}&\liminf_{n\to+\infty}\lim_{l,m\to+\infty}\lim_{\varepsilon\to 0}(A+B)\\ &\qquad = -\limsup_{n\to+\infty}\int_{\mathcal{Q}} \mathcal{G}_x(t,x,v(t,x,\beta),\overline{u}_\Sigma)\cdot\nabla\omega_{1/n}(x)\,(\psi\,\lambda)_\Sigma\,dx\,dt\,d\beta\,.\end{aligned}$$

Starting from inequality (35) and taking the limit with respect to l, m, then the limit with respect to n of both sides yields

$$\begin{aligned}&\int_Q\int_0^1\int_0^1 \left[|u-v|\,(\psi\lambda)_t + \mathcal{G}_x(t,x,u,v)\cdot\nabla(\psi\lambda)\right]\,d\beta\,d\alpha\,dx\,dt\\ &\geq \left[\begin{array}{l} -\lim\limits_{n\to+\infty}\displaystyle\int_Q\int_0^1 \mathcal{G}_x(t,x,v(t,x,\beta),\overline{u}_\Sigma(t,\overline{x}))\cdot\nabla\omega_{1/n}\,(\psi\,\lambda)(t,\overline{x},f(\overline{x}))\,d\beta\,dx\,dt\\ +\lim\limits_{\varepsilon\to 0}\displaystyle\int_Q\int_0^1 \mathcal{G}_x(t,x,u,\overline{u}_\Sigma(t,\overline{x}))\cdot\nabla\omega_\varepsilon(x)\,(\psi\,\lambda)(t,\overline{x},f(\overline{x}))\,d\alpha\,dx\,dt\\ +\lim\limits_{n\to+\infty}\lim\limits_{l,m\to+\infty} H\end{array}\right].\end{aligned}\tag{37}$$

Since $\lim_{n\to+\infty}\lim_{l,m\to+\infty} H = 0$ (see (36)), the right-hand side of (37) is an antisymmetric function in (u,v), while the left-hand side of (37) is a symmetric function of (u,v). We therefore have

$$\int_{\mathcal{Q}}\int_0^1 \left[|u-v|\,(\psi\lambda)_t + \mathcal{G}_x(t,x,u,v)\cdot\nabla(\psi\lambda)\right]\,d\beta\,d\alpha\,dx\,dt \geq 0.\tag{38}$$

Now, recall that $\lambda = \lambda_\alpha$ is an element of the partition of unity $(\lambda_\alpha)_{0\leq\alpha\leq N}$; summing the previous inequality over $\alpha \in 0,\ldots,N$ yields

$$\int_{\mathcal{Q}}\int_0^1 \left[|u-v|\,\psi_t + \mathcal{G}_x(t,x,u,v)\cdot\nabla\psi\right]\,d\beta\,d\alpha\,dx\,dt \geq 0.\tag{39}$$

We define the nonnegative function ψ_0 by $\psi_0(t,x) = \psi_0(t) = (T-t)\chi_{(0,T)}(t)$, and apply (39) with ψ_0 as a test-function to get

$$\int_0^T\int_\Omega\int_0^1\int_0^1 |u(t,x,\alpha) - v(t,x,\beta)|\,d\beta\,d\alpha\;dx\,dt \leq 0.$$

Consequently, we have $u(t,x,\alpha) = v(t,x,\beta)$ for a.e. $(t,x,\alpha,\beta) \in Q\times(0,1)\times(0,1)$. Defining the function w by the formula

$$w(t,x) = \int_0^1 u(t,x,\alpha)\,d\alpha$$

and accounting for the product structure of the measurable space $Q\times(0,1)\times(0,1)$, we conclude

$$u(t,x,\alpha) \;=\; w(t,x) \;=\; v(t,x,\beta) \text{ for a.e. } (t,x,\alpha,\beta)\in Q\times(0,1)^2. \qquad \square$$

4.6. Proof of Theorem 4.1 for Ω a bounded polyhedral subset. Let d be the Euclidean distance on $\mathbb{R}^d$. Denote by $(\partial\Omega_i)_{i=1,\ldots,N}$ the faces of Ω, and by $\mathbf{n}_i$ the outward unit normal to Ω along $\partial\Omega_i$. For $\varepsilon > 0$ small, let B_i^ε be the subset of all $x \in \Omega$ such that $d(x, \partial\Omega_i) < \varepsilon$ and $d(x, \partial\Omega_i) < d(x, \partial\Omega_j)$ if $i \neq j$; define G_i^ε to be the largest cylinder generated by $\mathbf{n}_i$ included in B_i^ε, and set $\Delta_i^\varepsilon = B_i^\varepsilon \setminus G_i^\varepsilon$, $\Omega_\varepsilon = \Omega \setminus (\cup_{1,N}\Delta_i^\varepsilon)$, and $b^\varepsilon = \mathbb{1}_{\Omega_{\varepsilon/2}} \star \rho_{\varepsilon/4}$. We have $\mathrm{meas}(\Omega \setminus \Omega_\varepsilon) \leq C\varepsilon^2$. If $\lambda_i \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is such that $\mathrm{supp}(\lambda_i) \cap \partial\Omega \subset \partial\Omega_i$ and such that the orthogonal projection of $\mathrm{supp}(\lambda_i)$ on the affine hyperplane determined by $\partial\Omega_i$ is included in $\partial\Omega_i$, then of course the whole previous proof explained in the case where Ω is $\mathcal{C}^{1,1}$ applies here (we look at a half-space), to give a result of comparison on $\mathrm{supp}(\lambda)$. Otherwise, for such a choice of function λ_i, (38) is true. Equation (38) is also still true if $\lambda = \lambda_0$, where $\lambda_0 \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\mathrm{supp}(\lambda_0) \subset \Omega$ (use Proposition 4.2). Since the function b^ε can be written as $b^\varepsilon = \sum_{i=0,N} \lambda_i$ for functions λ_i as above, we have

$$\int_{\mathcal{Q}} \int_0^1 \left[|u - v| \, (\psi b^\varepsilon)_t + \mathcal{G}_x(t, x, u, v) \cdot \nabla(\psi b^\varepsilon) \right] d\beta \, d\alpha \, dx \, dt \geq 0. \tag{40}$$

Equation (40) can be rewritten as

$$\int_{\mathcal{Q}} \int_0^1 \left[|u - v| \, \psi_t + \mathcal{G}_x(t, x, u, v) \cdot \nabla\psi \right] d\beta \, d\alpha \, dx \, dt \geq \alpha_\varepsilon,$$

where $\alpha_\varepsilon = \int_{\mathcal{Q}} \int_0^1 \mathcal{G}_x(t, x, u, v) \cdot \nabla b^\varepsilon \psi \, d\beta \, d\alpha \, dx \, dt$ tends to zero when $\varepsilon \to 0$. Indeed, we have $\nabla b^\varepsilon = 0$ on Ω_ε, so that, setting $\mathcal{R}_\varepsilon = (0, T) \times (\Omega \setminus \Omega_\varepsilon) \times (0, 1)^2$, we have

$$\begin{aligned}
\alpha_\varepsilon &\leq ||\psi||_{L^\infty} ||\mathcal{G}_x(t, x, u, v)||_{L^1(\mathcal{R}_\varepsilon)} ||\nabla b^\varepsilon||_{L^\infty(\mathcal{R}_\varepsilon)} \\
&\leq ||\psi||_{L^\infty} \mathrm{meas}(\mathcal{R}_\varepsilon)^{1/2} ||\mathcal{G}_x(t, x, u, v)||_{L^2(\mathcal{R}_\varepsilon)} || \mathbb{1}_{\Omega_{\varepsilon/2}} ||_{L^\infty(\mathcal{R}_\varepsilon)} ||\nabla\rho_{\varepsilon/4}||_{L^1(\mathcal{R}_\varepsilon)} \\
&\leq C(T, \psi) \;\; \varepsilon \cdot ||\mathcal{G}_x(t, x, u, v)||_{L^2(\mathcal{R}_\varepsilon)} \cdot \tfrac{1}{\varepsilon},
\end{aligned}$$

and we conclude by using $||\mathcal{G}_x(t, x, u, v)||_{L^2(\mathcal{R}_\varepsilon)} \to 0$ when $\varepsilon \to 0$. We thus obtain (39), from which Theorem 4.1 follows. ∎

5. The FV scheme. The mesh used to discretize problem (1) has to be regular enough to ensure the consistency of the numerical fluxes, mainly because a second order problem is considered (at least when the function φ is not constant). This is specified in the following section.

5.1. Assumptions and notation. We set d to be the Euclidean distance on $\mathbb{R}^d$ and denote by γ the $(d-1)$-Hausdorff measure on $\partial\Omega$.

Definition 5.1 (admissible mesh of Ω). *An admissible mesh of Ω consists of a set $\mathcal{T}$ of open bounded polyhedral convex subsets of Ω called control volumes, a family $\mathcal{E}$ of subsets of $\bar{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$ with positive measure, and a family of points (the "centers" of control volumes) satisfying the following properties:*

(i) *The closure of the union of all control volumes is $\bar{\Omega}$.*

(ii) *For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bar{K} \backslash K = \cup_{\sigma \in \mathcal{E}_K} \bar{\sigma}$. Furthermore, $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$.*

(iii) *For any $(K, L) \in \mathcal{T}^2$ with $K \neq L$, either the "length" (i.e., the $(d-1)$-dimensional Lebesgue measure) of $\bar{K} \cap \bar{L}$ is 0 or $\bar{K} \cap \bar{L} = \bar{\sigma}$ for some $\sigma \in \mathcal{E}$. In the latter case, we shall write $\sigma = K|L$ and $\mathcal{E}_{int} = \{\sigma \in \mathcal{E}, \exists (K, L) \in \mathcal{T}^2, \sigma = K|L\}$. For any $K \in \mathcal{T}$, we shall denote by $\mathcal{N}_K$ the set of neighbor control volumes of K, i.e., $\mathcal{N}_K = \{L \in \mathcal{T}, K|L \in \mathcal{E}_K\}$.*

(iv) *The family of points* $(x_K)_{K\in\mathcal{T}}$ *is such that* $x_K \in K$ *(for all* $K \in \mathcal{T}$*), and, if* $\sigma = K|L$, *it is assumed that the straight line* (x_K, x_L) *is orthogonal to* σ.

Given a control volume $K \in \mathcal{T}$, *we will denote by* $m(K)$ *its measure and by* $\mathcal{E}_{ext,K}$ *the subset of the edges of* K *included in the boundary* $\partial\Omega$. *If* $L \in \mathcal{N}_K$, $m(K|L)$ *will denote the measure of the edge between* K *and* L, *and* $T_{K|L}$ *the "transmissibility" through* $K|L$, *defined by* $T_{K|L} = \frac{m(K|L)}{d(x_K,x_L)}$. *Similarly, if* $\sigma \in \mathcal{E}_{ext,K}$, *we will denote by* $m(\sigma)$ *its measure and by* τ_σ *the "transmissibility" through* σ, *defined by* $\tau_\sigma = \frac{m(\sigma)}{d(x_K,\sigma)}$. *One also denotes by* $\mathcal{E}_{ext}$ *the union of the edges included in the boundary of* Ω: $\cup_{K\in\mathcal{T}}\mathcal{E}_{ext,K}$. *The size of the mesh* $\mathcal{T}$ *is defined by*

$$\mathrm{size}(\mathcal{T}) = \max_{K\in\mathcal{T}} \mathrm{diam}(K),$$

and we introduce the following geometrical factor, linked with the regularity of the mesh, defined by

$$\mathrm{reg}(\mathcal{T}) = \min_{K\in\mathcal{T},\sigma\in\mathcal{E}_K} \frac{d(x_K,\sigma)}{\mathrm{diam}(K)}.$$

Remark 5.1. Some examples of meshes satisfying these assumptions are the triangular meshes, which verify the acute angle condition (in fact this condition may be weakened to the Delaunay condition), the rectangular meshes, or the Voronoï meshes; see [EGH99] or [EGH00] for more details.

Definition 5.2 (time discretization of $(0,T)$). *A time discretization of* $(0,T)$ *is given by an integer value* N *and by an increasing sequence of real values* $(t^n)_{n\in[\![0,N+1]\!]}$ *with* $t^0 = 0$ *and* $t^{N+1} = T$. *The time steps are then defined by* $\delta t^n = t^{n+1} - t^n$, *for* $n \in [\![0,N]\!]$.

Definition 5.3 (space-time discretization of Q). *A finite volume discretization* D *of* Q *is a family* $D = (\mathcal{T}, \mathcal{E}, (x_K)_{K\in\mathcal{T}}, N, (t^n)_{n\in[\![0,N]\!]})$, *where* $\mathcal{T}$, $\mathcal{E}$, $(x_K)_{K\in\mathcal{T}}$ *is an admissible mesh of* Ω *according to Definition* 5.1 *and* N, $(t^n)_{n\in[\![0,N+1]\!]}$ *is a time discretization of* $(0,T)$ *according to Definition* 5.2. *For a given FV discretization* D, *one defines*

$$\mathrm{size}(D) = \max(\mathrm{size}(\mathcal{T}), (\delta t^n)_{n\in[\![0,N]\!]}) \quad \textit{and} \quad \mathrm{reg}(D) = \mathrm{reg}(\mathcal{T}).$$

5.2. The FV scheme. We may now define the FV discretization of (1). Let D be a FV discretization of Q according to Definition 5.3. First, the initial and boundary data are discretized by setting

$$U_K^0 = \frac{1}{m(K)} \int_K u_0(x)dx \quad \forall K \in \mathcal{T} \tag{41}$$

and

$$\bar{U}_\sigma^{n+1} = \frac{1}{\delta t^n \ m(\sigma)} \int_{t^n}^{t^{n+1}} \int_\sigma \bar{u}(t,x) d\gamma(x) dt \quad \forall \sigma \in \mathcal{E}_{ext}, \forall n \in [\![0,N]\!]. \tag{42}$$

An *implicit FV scheme* for the discretization of problem (1) is given by the following set of nonlinear equations with unknowns $U_D = (U_K^{n+1})_{K\in\mathcal{T},n\in[\![0,N]\!]}$: $\forall K \in \mathcal{T}, \forall n \in [\![0,N]\!]$,

(43)
$$\frac{U_K^{n+1} - U_K^n}{\delta t^n} \ m(K) + \sum_{\sigma\in\mathcal{E}_K} m(\sigma) F_{K,\sigma}^{n+1}(U_K^{n+1}, U_{K_\sigma}^{n+1}) - \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma(\varphi(U_{K_\sigma}^{n+1}) - \varphi(U_K^{n+1})) = 0,$$

where

$$U_{K_\sigma}^{n+1} = \begin{cases} U_L^{n+1} & \textit{if } \sigma = K|L, \\ \bar{U}_\sigma^{n+1} & \textit{if } \sigma \in \mathcal{E}_{ext}, \end{cases} \tag{44}$$

and where the function $F_{K,\sigma}^{n+1}$ is a monotonous flux consistent with the function F, which means that

- for all $v \in \mathbb{R}$, $u \mapsto F_{K,\sigma}^{n+1}(u,v)$ is a nondecreasing function and for all $u \in \mathbb{R}$, $v \to F_{K,\sigma}^{n+1}(u,v)$ is a nonincreasing function,
- $F_{K,\sigma}^{n+1}(u,v) = -F_{K,\sigma}^{n+1}(v,u)$ for all $(u,v) \in \mathbb{R}^2$,
- $F_{K,\sigma}^{n+1}$ is M-Lipschitz continuous with respect to each variable,
- $F_{K,\sigma}^{n+1}(s,s) = \frac{1}{\delta t_n} \frac{1}{m(\sigma)} \int_{t^n}^{t^{n+1}} \int_\sigma F(x,t,s) \cdot \mathbf{n}_{K,\sigma} d\gamma(x)\, dt.$

The Godunov scheme and the splitting flux scheme of Osher may be the most common examples of schemes with monotone fluxes.

We call an *approximate solution* the piecewise constant function u_D defined a.e. on Q by

$$u_D(t,x) = U_K^{n+1}\,, \quad t \in (t_n, t_{n+1}),\ x \in K. \tag{45}$$

5.3. Monotony of the scheme and direct consequences. As already said in the introduction, it is a necessity to select a physically admissible solution by means of the entropy inequalities. The schemes with monotonous fluxes are well known to add numerical viscosity to the equations. They are L^∞ stable, and they are monotonous so that they respect discrete entropy inequalities. In other words, continuous entropy inequalities have their discrete analogue, and they are respected by any solution of (41)–(44). This is summarized in the following proposition.

PROPOSITION 5.1 (monotony). *Assume hypotheses* (H1), (H2), (H3), *and* (H4). *Then there exists a unique solution to the scheme. Moreover, this solution satisfies the following maximum principle and discrete entropy inequalities:* $\forall K \in \mathcal{T}, \forall n \in [\![0,N]\!]$,

$$A \le U_K^{n+1} \le B, \tag{46}$$

$$\frac{\eta_\kappa^\pm(U_K^{n+1}) - \eta_\kappa^\pm(U_K^n)}{\delta t^n} m(K) + \sum_{\sigma \in \mathcal{E}_K} m(\sigma) \Phi_{K,\sigma,\kappa}^{\pm,n+1}(U_K^{n+1}, U_{K_\sigma}^{n+1}) - \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma \big(\eta_\kappa^\pm(\varphi(U_{K_\sigma}^{n+1})) - \eta_\kappa^\pm(\varphi(U_K^{n+1}))\big) \le 0, \tag{47}$$

where $\Phi_{K,\sigma,\kappa}^{+,n+1}$ *and* $\Phi_{K,\sigma,\kappa}^{-,n+1}$ *are the numerical entropy-fluxes defined by*

$$\begin{aligned} \Phi_{K,\sigma,\kappa}^{+,n+1}(u,v) &= F_{K,\sigma}^{n+1}(u \top \kappa, v \top \kappa) - F_{K,\sigma}^{n+1}(\kappa,\kappa) \quad \textit{and} \\ \Phi_{K,\sigma,\kappa}^{-,n+1}(u,v) &= F_{K,\sigma}^{n+1}(\kappa,\kappa) - F_{K,\sigma}^{n+1}(u \perp \kappa, v \perp \kappa). \end{aligned} \tag{48}$$

Proof. We give only some elements of the proof of this proposition because it consists of rewriting the proofs of three lemmas that can be found in [EGHM02] (Lemmas 3.1, 3.3, and 3.4 there) in the case where the convective flux $\mathbf{q}(x,t)f(u)$ is replaced by a more general flux $F(x,t,u)$ and the Kruzhkov entropies are replaced by the semi-Kruzhkov entropies as in the work of Vovelle [Vov02].

We follow the classical framework of implicit FV schemes for conservation laws (see [EGH00]). The function U_D is defined in an implicit way, so we first show, using the monotony of the scheme, that if a function U_D is a solution to the scheme, then it satisfies the discrete inequalities (47). Then we derive the maximum principle (46) that provides a result of existence by use of the Leray–Schauder theorem. Uniqueness of U_D is proved by using a method analogous to the one used to prove the discrete entropy inequalities. ∎

5.4. A priori estimates. The inequalities derived from the properties of monotony and local conservation are L^∞ and L^1 estimates. We will prove now L^2 estimates. We introduce a discretization $\bar{U}_D = (\bar{U}_K^{n+1})_{\{K\in\mathcal{T}, n\in[\![0,N]\!]\}}$ of $\bar{u}$ defined by

$$\bar{U}_K^{n+1} = \frac{1}{\delta t_n}\frac{1}{m(K)}\int_{t^n}^{t^{n+1}}\int_K \bar{u}\,dx\,dt \quad \forall K\in\mathcal{T},\ \forall n\in[\![0,N]\!].$$

PROPOSITION 5.2 ($L^2(0,T,H^1(\Omega))$ and weak BV estimate). *Assume hypotheses* (H1), (H2), (H3), (H4), *and* (H5). *Let u_D be the approximate solution defined by* (41)–(44), *and assume that $reg(D)\geq\xi$, where $\xi>0$. Then there exists a constant C depending only on ξ, T, Ω, $Lip(\varphi)$, M, $\bar{u}$, A, B such that*

$$\big(\mathcal{N}_D(\zeta(u_D))\big)^2 = \sum_{n=0}^{N}\delta t_n\sum_{K\in\mathcal{T}}\left(\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{int,K}}\tau_\sigma(\zeta(U_K^{n+1})-\zeta(U_{K_\sigma}^{n+1}))^2 + \sum_{\sigma\in\mathcal{E}_{ext,K}}\tau_\sigma(\zeta(U_K^{n+1})-\zeta(U_{K_\sigma}^{n+1}))^2\right)\leq C$$

and

$$\sum_{n=0}^{N}\delta t_n\sum_{K\in\mathcal{T}}\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{int,K}} m(\sigma)\max_{U_K^{n+1}\leq c\leq d\leq U_{K_\sigma}^{n+1}}\big((F_{K,\sigma}^{n+1}(d,c)-F_{K,\sigma}^{n+1}(d,d))^2 + (F_{K,\sigma}^{n+1}(d,c)-F_{K,\sigma}^{n+1}(c,c))^2\big)\leq C. \tag{49}$$

Remark 5.2. The inequality (49) is called the "weak BV inequality." See [EGH00], [CGH93], or [CH99].

Proof. As for Proposition 5.1, the proof has already been done in a simpler case in [EGHM02] (Proposition 3.1). The details of the proof differ only by some arguments that can be found in [Vov02].

These estimates are discrete energy estimates. They are obtained by multiplying (41)–(44) by $\delta t_n(U_K^{n+1}-\bar{U}_K^{n+1})$ and summing over $K\in\mathcal{T}$ and $n\in[\![0,N]\!]$. In the proof, we separate terms that contain only U_D from terms containing U_D and $\bar{U}_D$. Then we use the Cauchy–Schwarz inequality and regularity hypotheses (H5) on $\bar{u}$ to control the second type of terms. To get a bound on $\mathcal{N}_D(\bar{U}_D)$, which is a discrete $L^2(0,T,H^1)$-norm for $\bar{U}_D$, we use the following inequality proved in [EGH99]:

$$\mathcal{N}_D(\bar{u})\leq C(reg(D))\|\nabla\bar{u}\|_{L^2(Q)}.$$

This is a consequence of the local conservativity of the scheme combined with the consistency of the numerical fluxes.

The last ingredient is the assumption $\mathrm{div}_x(F(x,t,u))=0$, which ensures that the boundary terms in the discrete integrations-by-parts concerning the hyperbolic terms can be controlled. The constant C depends on ξ, $m(\Omega)$, T, B, A, $Lip(F_{K,\sigma}^{n+1})$, $\|\bar{u}_t\|_{L^1(Q)}$, and on $\|\nabla\bar{u}\|_{L^2(Q)}$. ∎

5.5. Continuous entropy inequalities. From the discrete entropy inequalities we deduce continuous approximate entropy inequalities. The following theorem is central in the proof of the convergence of the scheme.

THEOREM 5.1 (continuous approximate entropy inequalities). *Assume hypotheses* (H1)*,* (H2)*,* (H3)*,* (H4)*, and* (H5)*. Let D be an admissible discretization of Q, and let u_D be the corresponding approximate solution defined above. Then u_D satisfies the following approximate entropy inequalities: for all $\kappa \in \mathbb{R}$, for all $\psi \in \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\psi \geq 0$ and $(\varphi(\bar{u}) - \varphi(\kappa))^{\pm}\,\psi = 0$ a.e. on Σ,*

$$\int_Q \eta_\kappa^\pm(u_D)\psi_t + \Phi_\kappa^\pm(t,x,u_D)\cdot\nabla\psi + \eta_{\varphi(\kappa)}^\pm(\varphi(u_D))\Delta\psi \; dxdt - \int_\Sigma \eta_{\varphi(\kappa)}^\pm(\varphi(\bar{u}))\nabla\psi\cdot\mathbf{n}d\gamma(x)dt$$
$$+\int_\Omega \eta_\kappa^\pm(u_0)\psi(0)dx + M\int_\Sigma \eta_\kappa^\pm(\bar{u})\psi\, d\gamma(x)dt \geq -\mathcal{E}_D^\pm(\psi). \tag{50}$$

Also assume that a uniform CFL condition $\delta t_n \leq C\mathrm{size}(\mathcal{T})$ for all n holds true (with a CFL number C that can be as large as desired). Then, for a given ψ, $\mathcal{E}_D^\pm(\psi)$ tends to zero when the size of the discretization tends to zero.

Proof. The proof of Theorem 5.1 is quite similar to the proof of Theorem 5.1 in [EGHM02], except for the boundary terms, which require extra care. We will therefore stress the analysis of these terms and make reference to [EGHM02] when needed. Of course, we can also limit ourselves to giving the proof of (50) when the nonnegative Kruzhkov entropy pairs are under consideration.

Let $\kappa \in \mathbb{R}$, and let $\psi \in \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ be a nonnegative function satisfying $(\varphi(\bar{u}) - \varphi(\kappa))^+\psi = 0$ a.e. on Σ. We define discrete values of ψ with respect to the mesh as

$$\Psi_K^0 = \psi(0, x_K) \quad \forall K \in \mathcal{T},$$
$$\Psi_K^{n+1} = \frac{1}{\delta t_n}\int_{t^n}^{t^{n+1}} \psi(t, x_K)dt \quad \forall K \in \mathcal{T}, \forall n \in [\![0, N]\!],$$
$$\psi_\sigma^{n+1} = \frac{1}{\delta t_n}\int_{t^n}^{t^{n+1}} \psi(t, x_\sigma)dt \quad \forall \sigma \in \mathcal{E}_{ext}, \forall n \in [\![0, N]\!]$$

and set $\Psi_{K,\sigma}^{n+1} = \Psi_L^{n+1}$ if $\sigma = K|L$ and $\Psi_{K,\sigma}^{n+1} = \psi_\sigma^{n+1}$ if $\sigma \in \mathcal{E}_{ext,K}$.

The definition of the numerical flux $\Phi_{K,\sigma,\kappa}^{+,n+1}$ (see (48)) ensures that it is a conservative flux, consistent with the function Φ_κ^+. Therefore, we have

$$\sum_{\sigma\in\mathcal{E}_K} m(\sigma)\Phi_{K,\sigma,\kappa}^{+,n+1}(U_K^{n+1}, U_K^{n+1}) = 0 \quad \forall K \in \mathcal{T}, n \in [\![0, N]\!],$$

and the discrete entropy inequality (47) can then be rewritten as

$$\frac{\eta_\kappa^+(U_K^{n+1}) - \eta_\kappa^+(U_K^n)}{\delta t^n} m(K) + \sum_{\sigma\in\mathcal{E}_K} m(\sigma)(\Phi_{K,\sigma,\kappa}^{+,n+1}(U_K^{n+1}, U_{K_\sigma}^{n+1}) - \Phi_{K,\sigma,\kappa}^{+,n+1}(U_K^{n+1}, U_K^{n+1}))$$
$$- \sum_{\sigma\in\mathcal{E}_K} \tau_\sigma\big(\eta_{\varphi(\kappa)}^+(\varphi(U_{K_\sigma}^{n+1})) - \eta_{\varphi(\kappa)}^+(\varphi(U_K^{n+1}))\big) \leq 0. \tag{51}$$

Multiplying (51) by $\delta t_n \Psi_K^{n+1}$ and summing over $K \in \mathcal{T}$ and $n \in [\![0, N]\!]$ yields

$$A1 + A2 + A3 \leq 0,$$

where

$$A1 = \sum_{n=0}^{N} \sum_{K\in\mathcal{T}} m(K)(\eta^+_\kappa(U_K^{n+1}) - \eta^+_\kappa(U_K^n))\Psi_K^{n+1}\,,$$

and, summing over the edges, $A2 = A2\text{int} + A2\text{ext}$, with

$$A2\text{int} = \sum_{n=0}^{N} \delta t_n \sum_{K\in\mathcal{T}} \frac{1}{2} \sum_{\sigma\in\mathcal{E}_{int,K}} m(\sigma)\big(\Psi_K^{n+1}(\Phi^{+,n+1}_{K,\sigma,\kappa}(U_K^{n+1}, U_{K_\sigma}^{n+1}) - \Phi^{+,n+1}_{K,\sigma,\kappa}(U_K^{n+1}, U_K^{n+1})) \\ - \Psi_{K,\sigma}^{n+1}(\Phi^{+,n+1}_{K,\sigma,\kappa}(U_K^{n+1}, U_{K_\sigma}^{n+1}) - \Phi^{+,n+1}_{K,\sigma,\kappa}(U_{K_\sigma}^{n+1}, U_{K_\sigma}^{n+1}))\big)$$

and

$$A2\text{ext} = \sum_{n=0}^{N} \delta t_n \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{ext,K}} m(\sigma)\Psi_K^{n+1}\big(\Phi^{+,n+1}_{K,\sigma,\kappa}(U_K^{n+1}, U_{K_\sigma}^{n+1}) - \Phi^{+,n+1}_{K,\sigma,\kappa}(U_K^{n+1}, U_K^{n+1})\big).$$

Similarly, $A3$ admits the decomposition $A3 = A3\text{int} + A3\text{ext}$, with

$$A3\text{int} = \sum_{n=0}^{N} \delta t_n \sum_{K\in\mathcal{T}} \frac{1}{2} \sum_{\sigma\in\mathcal{E}_{int,K}} \tau_\sigma\big(\eta^+_{\varphi(\kappa)}(\varphi(U_K^{n+1})) - \eta^+_{\varphi(\kappa)}(\varphi(U_{K_\sigma}^{n+1}))\big)(\Psi_K^{n+1} - \Psi_{K,\sigma}^{n+1})$$

and

$$A3\text{ext} = \sum_{n=0}^{N} \delta t_n \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{ext,K}} \tau_\sigma\big(\eta^+_{\varphi(\kappa)}(\varphi(U_K^{n+1})) - \eta^+_{\varphi(\kappa)}(\varphi(U_{K_\sigma}^{n+1}))\big)\Psi_K^{n+1}\,.$$

Now, set

$$\begin{aligned}
I1 &= -\int_Q \eta^+_\kappa(u_D)\psi_t\,dx\,dt - \int_\Omega \eta^+_\kappa(u_0)\psi(0,x)\,dx,\\
I2 &= -\int_Q \Phi^+_\kappa(t,x,u_D)\cdot\nabla\psi\,dx\,dt - M\int_\Sigma \eta^+_\kappa(\bar{u})\psi\,d\gamma(x)dt,\\
I3 &= -\int_Q \eta^+_{\varphi(\kappa)}(\varphi(u_D))\Delta\psi\,dx\,dt + \int_\Sigma \eta^+_{\varphi(\kappa)}(\varphi(\bar{u}))\nabla\psi\cdot\mathbf{n}\,d\gamma(x)\,dt\,.
\end{aligned}$$

We aim at proving the estimate $I1+I2+I3 \leq \mathcal{E}^+_D(\psi)$ and, to that purpose, compare $I1$ to $A1$, $I2$ to $A2$, and $I3$ to $A3$, respectively.

A discrete integration by parts leads to $|I1 - A1| \leq \mathcal{E}_{1,D}(\psi)$, with $\mathcal{E}_{1,D}(\psi) \to 0$ as $\text{size}(D) \to 0$ (see [EGHM02]).

Using integration by parts in $I2$ and the fact that u_D is piecewise constant, we obtain

$$I2 = I2\text{int} + I2\text{ext},$$

where $I2\text{ext}$ is the boundary term and $I2\text{int}$ gathers the sums on the internal edges. Precisely, we have

$$I2\text{int} = -\sum_{n=0}^{N} \sum_{K\in\mathcal{T}} \frac{1}{2} \sum_{\sigma\in\mathcal{E}_{int,K}} \left(\int_{t^n}^{t^{n+1}}\int_\sigma \Phi^+_\kappa(t,x,U_K^{n+1})\cdot\mathbf{n}_{K,\sigma}\psi\,d\gamma(x)\,dt \right. \\ \left. - \int_{t^n}^{t^{n+1}}\int_\sigma \Phi^+_\kappa(t,x,U_{K_\sigma}^{n+1})\cdot\mathbf{n}_{K,\sigma}\psi\,d\gamma(x)\,dt\right)$$

and

$$I2\text{ext} = -\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{ext,K}}\int_{t^n}^{t^{n+1}}\!\!\int_{\sigma}\Phi_\kappa^+(t,x,U_K^{n+1})\cdot\mathbf{n}_{K,\sigma}\psi d\gamma(x)dt - M\int_{\Sigma}\eta_\kappa^+(\bar{u})\psi d\gamma(x)dt.$$

As in [EGHM02], we prove $|I2\text{int} - A2\text{int}| \le \mathcal{E}_{2,D}^{\text{int}}(\psi)$, with $\mathcal{E}_{2,D}^{\text{int}}(\psi) \to 0$ as $\text{size}(D) \to 0$.

The comparison of $I2$ext with $A2$ext involves a term corresponding to the consistency error, and three terms related to the approximation of the boundary data:

$$I2\text{ext} - A2\text{ext} \le \mathcal{E}_{2,D}^{c1,\text{ext}}(\psi) + \mathcal{E}_{2,D}^{b1,\text{ext}}(\psi) + \mathcal{E}_{2,D}^{b2,\text{ext}}(\psi) - T_{2,D}^{b2,\text{ext}}(\psi),$$

where

$$\mathcal{E}_{2,D}^{c1,\text{ext}}(\psi) = \sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}\text{ext}\kappa}\left|\int_{t^n}^{t^{n+1}}\!\!\int_{\sigma}(\Psi_K^{n+1}-\psi)\Phi_\kappa^+(\cdot,\cdot,U_K^{n+1})\cdot\mathbf{n}_{K,\sigma}\,d\gamma(x)\,dt\right|,$$

$$\mathcal{E}_{2,D}^{b1,\text{ext}}(\psi) = \sum_{n=0}^{N}\delta t_n\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}\text{ext}\kappa} m(\sigma)|(\Psi_K^{n+1}-\psi_\sigma^{n+1})\Phi_{K,\sigma,\kappa}^{+,n+1}(U_K^{n+1},U_{K_\sigma}^{n+1})|,$$

and

$$\begin{aligned}\mathcal{E}_{2,D}^{b2,\text{ext}}(\psi) = M\sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}\text{ext}\kappa}\Bigg|\int_{t^n}^{t^{n+1}}\int_{\sigma}(\bar{u}-\kappa)^+\psi\,d\gamma(x)\,dt \\ - \delta t_n\, m(\sigma)(U_{K_\sigma}^{n+1}-\kappa)^+\psi_\sigma^{n+1}\Bigg|\end{aligned}$$

are three terms converging to zero when $\text{size}(D) \to 0$ and

$$T_{2,D}^{b2,\text{ext}}(\psi) = \sum_{n=0}^{N}\delta t_n\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}\text{ext}\kappa} m(\sigma)\psi_\sigma^{n+1}\left(\Phi_{K,\sigma,\kappa}^{+,n+1}(U_K^{n+1},U_{K_\sigma}^{n+1}) + M(U_{K_\sigma}^{n+1}-\kappa)^+\right).$$

From the definition of $\Phi_{K,\sigma,\kappa}^{+,n+1}$ (see (48)) and from the monotony of the scheme,

$$\Phi_{K,\sigma,\kappa}^{+,n+1}(a,b) = F_{K,\sigma}^{n+1}(a\top\kappa, b\top\kappa) - F_{K,\sigma}^{n+1}(\kappa,\kappa) \ge -Lip(F_{K,\sigma}^{n+1})(b-\kappa)^+$$

follows, and this entails $T_{2,D}^{b2,\text{ext}}(\psi) \ge 0$.

Now, to compare $I3$ to $A3$ we make the distinction between the different contributions of the terms (inside and on the boundary of Ω). Indeed, since the approximate solution u_D is piecewise constant, the term $I3$ reads as $I3 = I3\text{int} + I3\text{ext}$, where

$$I3\text{int} = \sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\frac{1}{2}\sum_{\sigma\in\mathcal{E}\text{int}\kappa}(\eta_{\varphi(\kappa)}^+(\varphi(U_{K_\sigma}^{n+1})) - \eta_{\varphi(\kappa)}^+(\varphi(U_K^{n+1})))\int_{t^n}^{t^{n+1}}\int_{\sigma}\nabla\psi\cdot\mathbf{n}_{K,\sigma}\,d\gamma(x)\,dt$$

and

$$I3\text{ext} = \sum_{n=0}^{N}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}\text{ext}\kappa}\int_{t^n}^{t^{n+1}}\int_{\sigma}(\eta_{\varphi(\kappa)}^+(\varphi(\bar{u})) - \eta_{\varphi(\kappa)}^+(\varphi(U_K^{n+1})))\nabla\psi\cdot\mathbf{n}_{K,\sigma}\,d\gamma(x)\,dt.$$

A consistency error term controls the proximity of $A3$int to $I3$int:

$$|I3\text{int} - A3\text{int}| \le \mathcal{E}_{3,D}^{c,\text{int}}(\psi),$$

with $\mathcal{E}_{3,D}^{c,\text{int}}(\psi) \to 0$ when size$(D) \to 0$ [EGHM02].

In order to compare $I3$ext and $A3$ext, rearrange the term $I3$ext, up to consistency or approximation errors, to get

$$I3\text{ext} \le \sum_{n=0}^{N} \delta t_n \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}\text{ext}\kappa} \tau_\sigma \left(\eta^+_{\varphi(\kappa)}(\varphi(U_{K_\sigma}^{n+1})) - \eta^+_{\varphi(\kappa)}(\varphi(U_K^{n+1}))\right)(\Psi_{K,\sigma}^{n+1} - \Psi_K^{n+1}) \\ + \mathcal{E}_{3,D}^{c,\text{ext}}(\psi) + \mathcal{E}_{3,D}^{b1,\text{ext}}(\psi),$$

where

$$\mathcal{E}_{3,D}^{c,\text{ext}}(\psi) \\ = \sum_{n=0}^{N} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}\text{ext}\kappa} 2 \max_{u\in[A,B]} \eta^+_{\varphi(\kappa)}(\varphi(u)) \left| \int_{t^n}^{t^{n+1}} \int_\sigma \left(\nabla\psi\cdot\mathbf{n} - \frac{\psi_\sigma^{n+1} - \Psi_K^{n+1}}{d_K,\sigma}\right) d\gamma(x)\,dt \right|,$$

$$\mathcal{E}_{3,D}^{b1,\text{ext}}(\psi) = \sum_{n=0}^{N} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}\text{ext}\kappa} \int_{t^n}^{t^{n+1}} \int_\sigma \left|\varphi(\bar u) - \varphi(\bar U_\sigma^{n+1})\right|\left|\nabla\psi\cdot\mathbf{n}\right| d\gamma(x)\,dt.$$

Then we have

$$I3\text{ext} - A3\text{ext} = \sum_{n=0}^{N} \delta t_n \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}\text{ext}\kappa} \tau_\sigma(\eta^+_{\varphi(\kappa)}(\varphi(U_{K_\sigma}^{n+1})) - \eta^+_{\varphi(\kappa)}(\varphi(U_K^{n+1})))\Psi_{K,\sigma}^{n+1} \\ + \mathcal{E}_{3,D}^{c,\text{ext}}(\psi) + \mathcal{E}_{3,D}^{b1,\text{ext}}(\psi)\,.$$

Now, either $\eta^+_{\varphi(\kappa)}(\varphi(U_{K_\sigma}^{n+1})) = 0$, and in that case

$$(\eta^+_{\varphi(\kappa)}(\varphi(U_{K_\sigma}^{n+1})) - \eta^+_{\varphi(\kappa)}(\varphi(U_K^{n+1}))\Psi_{K,\sigma}^{n+1}) \le 0,$$

or $\eta^+_{\varphi(\kappa)}(\varphi(U_{K_\sigma}^{n+1})) > 0$. In the latter case, the condition $(\varphi(\bar u) - \varphi(\kappa))^+\,\psi = 0$ a.e. on Σ ensures that there exists $(t,x) \in [t^n, t^{n+1}] \times \sigma$ such that $\psi(t,x) = 0$. Consequently, we have

$$\Psi_{K,\sigma}^{n+1} \le Lip(\psi)(\delta t_n + diam(\sigma))\,.$$

This estimate, combined with the inequality

$$\eta^+_{\varphi(\kappa)}(\varphi(U_{K_\sigma}^{n+1})) - \eta^+_{\varphi(\kappa)}(\varphi(U_K^{n+1})) \le (\eta^+_{\varphi(\kappa)})'(\varphi(\bar U_\sigma^{n+1}))(\varphi(\bar U_\sigma^{n+1}) - \varphi(U_K^{n+1})),$$

which is consequence of the convexity of the function $\eta^+_{\varphi(\kappa)}$, leads to

$$I3\text{ext} - A3\text{ext} \le \mathcal{E}_{3,D}^{b2,\text{ext}}(\psi) + \mathcal{E}_{3,D}^{c,\text{ext}}(\psi) + \mathcal{E}_{3,D}^{b1,\text{ext}}(\psi),$$

where

$$\mathcal{E}_{3,D}^{b2,\text{ext}}(\psi) = \sum_{n=0}^{N} \delta t_n \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}\text{ext}\kappa} \tau_\sigma Lip(\psi)(\delta t_n + diam(\sigma))|\varphi(\bar U_\sigma^{n+1}) - \varphi(U_K^{n+1})|\,.$$

Using the Cauchy–Schwarz inequality, together with the $L^2(0,T;H_0^1(\Omega))$ estimate of Proposition 5.2 and the inequality $\varphi(a)-\varphi(b) \le \sqrt{Lip(\varphi)}(\zeta(a)-\zeta(b))$, yields

$$\mathcal{E}_{3,D}^{b2,\text{ext}}(\psi) \le C\sum_{n=0}^{N}\delta t_n \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}\text{ext}\kappa} \tau_\sigma(\delta t_n + diam(\sigma))^2\,.$$

Therefore, a simple way to ascertain that $\mathcal{E}_{3,D}^{b2,\text{ext}}(\psi)$ converges to zero is to suppose a uniform CFL condition such as $\delta t_n \le C\text{size}(\mathcal{T})$ for all n (where the CFL number C can be as large as desired). Then we conclude the proof of Theorem 5.1 by defining $\mathcal{E}_D^+(\psi)$ as the sum of the errors $\mathcal{E}_{1,D}(\psi)$, $\mathcal{E}_{2,D}^{\text{int}}(\psi)$, $\mathcal{E}_{2,D}^{c1,\text{ext}}(\psi)$, $\mathcal{E}_{2,D}^{b1,\text{ext}}(\psi)$, $\mathcal{E}_{2,D}^{b2,\text{ext}}(\psi)$, $\mathcal{E}_{3,D}^{c,\text{int}}(\psi)$, $\mathcal{E}_{3,D}^{c,\text{ext}}(\psi)$, $\mathcal{E}_{3,D}^{b1,\text{ext}}(\psi)$, and $\mathcal{E}_{3,D}^{b2,\text{ext}}(\psi)$. $\square$

5.6. Convergence of the scheme. Let D_n be a sequence of discretizations, such that size(D_n) tends to zero. We wish to prove the convergence of u_{D_n} to an entropy solution of problem (1). For that purpose, in view of the uniqueness Theorem 4.1, it suffices to show that, up to a subsequence, u_{D_n} tends in the nonlinear weak-$\star$ sense to an entropy process solution of (1). We obtain compactness properties using estimates on u_{D_n} derived from discrete estimates on U_{D_n}, then pass to the limit in inequality (50).

5.6.1. Nonlinear weak-$\star$ compactness. The maximum principle ensures that (u_{D_n}) is bounded in $L^\infty(Q)$. Consequently, there exist $u \in L^\infty(Q\times(0,1))$ such that, up to a subsequence, u_{D_n} tends to u in the nonlinear weak-$\star$ sense.

5.6.2. Compactness in $L^2(Q)$. From discrete estimates obtained in Proposition 5.2 we easily deduce (see, e.g., [EGH00]) the following inequalities on $z_D = \zeta(u_D) - \zeta(\bar{u}_D)$.

PROPOSITION 5.3 (space translation estimates). *Assume hypotheses* (H1), (H2), (H3), (H4), *and* (H5). *There exists a constant C_1 such that*

$$\forall y\in\mathbb{R}^d,\quad \int_0^T\!\!\int_{\Omega_y} (z_D(t,x+y)-z_D(t,x))^2 dxdt \le C_1|y|(|y|+\text{size}(\mathcal{T})),$$

where $\Omega_y = \{x\in\Omega, [x,x+y]\subset\Omega\}$.

The hypothesis (H5) includes the assumption $\bar{u}_t \in L^1(Q)$, while the discrete evolution equation (43) relates the discrete time derivative of u_D to its discrete space derivative. Therefore the following time translation estimate on z_D is available.

PROPOSITION 5.4 (time translation estimates). *Assume hypotheses* (H1), (H2), (H3), (H4), *and* (H5). *There exists a constant C_2 such that*

$$\forall s>0,\quad \int_0^{T-s}\!\!\int_\Omega (z_D(t+s,x)-z_D(t,x))^2 dxdt \le C_2\, s.$$

Since the function z_D vanishes on Σ, it can be extended by zero out of Q. Then using the Fréchet–Kolmogorov theorem (see, e.g., [Bre83]), we get the existence of a function $z\in L^2(0,T,H^1(\Omega))$ such that, up to a subsequence, $z_{D_n}\to z$ in $L^2(Q)$. Besides, since $z_D = \zeta(u_D)-\zeta(\bar{u}_D)$ and $\zeta(\bar{u}_D)$ converges to $\zeta(\bar{u})$ in $L^2(Q)$, we get the convergence of $\zeta(u_{D_n})$ in $L^2(Q)$ (to $\zeta(\bar{u})+z$). On the other hand, the nonlinear weak-$\star$ convergence of (u_{D_n}) shows that $\zeta(u_{D_n})$ converges also to $\zeta(u)$ weakly in $L^\infty(Q)$, so that $\zeta(\bar{u})+z=\zeta(u)$. In particular, $\zeta(u)$ does not depend on the last argument α, and the trace of $\zeta(u)$ on Σ is $\zeta(\bar{u})$. See [EGHM02] for more details on this step of the proof.

5.6.3. Conclusion. It remains to pass to the limit in the continuous entropy inequalities to prove that u is an entropy process solution. The uniqueness Theorem 4.1 proves that u does not depend on α and is the unique entropy weak solution of problem (1). Besides, the whole sequence u_{D_n} is convergent (u is the unique possible limit), and by definition of the nonlinear weak-$\star$ convergence, $(u_{D_n})^2$ also converges weakly to $(u)^2$ so that u_{D_n} converges to u in $L^2(Q)$ (strong), and in all $L^p(Q)$, for $1 \le p < +\infty$. Therefore, we have proved the following theorem.

THEOREM 5.2. *Let D_n be a sequence of discretizations, such that size(D_n) tends to zero. Assume hypotheses* (H1), (H2), (H3), (H4), (H5), *and* (H6) *(or* (H6bis)*). Then, for every $1 \le p < +\infty$, (u_{D_n}) converges to the unique entropy solution of problem* (1) *in $L^p(Q)$.*

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