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Abstract. The use of Lyapunov conditions for proving functional inequalities was initiated in [5]. It was shown in [4, 30] that there is an equivalence between a Poincaré inequality, the existence of some Lyapunov function and the exponential integrability of hitting times. In the present paper, we close the scheme of the interplay between Lyapunov conditions and functional inequalities by

- showing that strong functional inequalities are equivalent to Lyapunov type conditions;
- showing that these Lyapunov conditions are characterized by the finiteness of generalized exponential moments of hitting times.

We also give some complement concerning the link between Lyapunov conditions and integrability property of the invariant probability measure and as such transportation inequalities, and we show that some "unbounded Lyapunov conditions" can lead to uniform ergodicity, and coming down from infinity property.

Key words : Lyapunov functions, hitting times, uniform ergodicity, F-Sobolev inequalities, Poincaré inequality, ultracontractivity.

MSC 2010 : 26D10, 39B62, 47D07, 60G10, 60J60

1. Introduction

Let D be some smooth open domain in $\mathbb{R}^d$. In this paper, we will mainly consider the differential operator defined for smooth functions $f \in C^\infty(D)$ by

$$Lf = \frac{1}{2}\sum_{ij}(\sigma\,\sigma^*)_{ij}(x)\,\frac{\partial^2_{ij}f}{\partial x_{ij}} + \sum_i b_i(x)\frac{\partial_i f}{\partial x_i}\,,$$

where σ is an $\mathbb{R}^{d\times m}$ smooth and bounded (for simplicity $C_b^\infty(\bar{D})$) matrix field and b a $C^\infty(\bar{D})$ vector field.
We may see L as the infinitesimal generator of a diffusion process associated to the stochastic differential equation (SDE)

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt \quad , \quad X_0 = x\,,$$

Date: April 18, 2016.

where B_t is an usual $\mathbb{R}^m$-Brownian motions when $D = \mathbb{R}^d$, or to the reflected SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt + dN_t, \quad \int_0^t \mathbf{1}_{\partial D}(X_s)dN_s = N_t \quad X_0 = x\,,$$

if D is some smooth subdomain.
The domain $\mathcal{D}(L)$ of L (viewed as a generator) is thus some extension of the set of smooth and compactly supported functions $C_c^\infty(\bar{D})$ such that the normal derivative $\frac{\partial f}{\partial n}$ vanishes on ∂D (if ∂D is non void). This corresponds to normal reflection or to a Neumann condition on the boundary. We also define P_t the associated semi-group

$$P_t f(x) = \mathbb{E}_x(f(X_t))$$

which is defined for bounded functions f.
In order to use classical results in PDE theory we will also assume that L is uniformly elliptic, i.e.

$$\sigma\,\sigma^* \;\geq\; a\,Id$$

in the sense of quadratic forms for some $a > 0$, or more generally that L is uniformly strongly hypo-elliptic in the sense of Bony (see [15]) and that the boundary is non characteristic. For details we refer to [17].

We will also assume (though it should be a consequence of some of our assumptions) that there exists a probability measure

$$\mu(dx) = e^{-V(x)}\,dx$$

which is an invariant measure for the process (or the semi-group) i.e. for all bounded and smooth function $f \in \mathcal{D}(L)$,

$$\int Lf\,d\mu = 0 \quad \text{ or equivalently for all } t, \quad \mathbb{E}_\mu(f(X_t)) = \int f\,d\mu\,.$$

P_t then extends to a contraction semi-group on $\mathbb{L}^p(\mu)$ for $1 \leq p \leq +\infty$. We shall say that μ is symmetric, or that P_t is μ symmetric if for smooth f and g in the domain of L,

$$\int f\,Lg\,d\mu = \int Lf\,g\,d\mu\,.$$

The standard example of μ-symmetric semi-group is obtained for $\sigma = \sqrt{2}\,Id$ and $b = -\nabla V$ (provided V is smooth enough). In all cases our ellipticity assumptions imply that this measure is unique and ergodic.

Among the most fascinating recent developments at the border of analysis and probability theory, a lot of work has been devoted to the study of the relationship between

- geometric properties of the measure μ, for instance concentration properties,
- functional inequalities (the study of weighted Sobolev or Orlicz-Sobolev spaces associated to L and μ) like the Poincaré (Wirtinger) inequality or the Gross logarithmic Sobolev inequality,
- transportation inequalities like the T_2 Talagrand's inequality,
- the rate of convergence to equilibrium for the semi-group P_t in various functional spaces, $\mathbb{L}^2(\mu)$, Orlicz spaces related to μ,
- the rate of convergence of its dual P_t^* (i.e. the distribution of the process at time t) in total variation or in Wasserstein distance,

- the existence of Lyapunov functions,
- and finally some properties of the stochastic process $X_.$ in particular existence of general moments for hitting times of some subsets (for instance the control of how the process comes down from infinity in the ultracontractive situation).

We refer to the monographs of Davies [33], Ledoux [45], Wang [58] and Bakry-Gentil-Ledoux [7], the surveys by Gross [39] and Ledoux [44], the collective book [3] and the papers [2, 6, 9, 8, 10, 11, 14, 13, 12, 18, 22, 20, 43, 55] among many others, for the first four items.
For the last three items we refer to the monographs of Hasminskii [40] and Meyn-Tweedie [52] and the papers [16, 35, 36, 37, 53, 54, 57] among many others.

The link between both approaches was done in [5] for the first time, up to our knowledge. It was extended in [4, 24, 27, 28, 29]. One can see the (now outdated) survey [25].

To be a little bit more precise, let us recall the following result from [30] Theorem 2.3 (also see [42] for a similar statement)

Theorem 1.1. *Consider the following properties*

(HP1) *There exists a Lyapunov function W, i.e. a smooth function $W : D \to \mathbb{R}$, s.t. $W \geq w > 0$, and there exist a constant $\lambda > 0$ and an open connected bounded subset U such that $\frac{\partial W}{\partial n} = 0$ on ∂D and*

$$LW \leq -\lambda W + \mathbf{1}_{\bar{U}}\,.$$

(HP2) *There exist an open connected bounded subset U and a constant $\theta > 0$ such that for all x,*

$$\mathbb{E}_x\left(e^{\theta\, T_U}\right) < +\infty\,,$$

where T_U denotes the hitting time of U.

(HP3) *The process is geometrically ergodic, i.e. there exist constants $\beta > 0$ and $C > 0$ and a function $W \geq 1$ belonging to $\mathbb{L}^1(\mu)$ such that for all x*

$$\|P_t(x,.) - \mu\|_{TV} \leq C\, W(x)\, e^{-\beta\, t}\,,$$

where $P_t(x,.)$ denotes the distribution of X_t (when $X_0 = x$) and $\|.\|_{TV}$ denotes the total variation distance.

(HP4) *μ satisfies a Poincaré inequality, i.e. there exists a constant $C_P(\mu)$ such that for all smooth $f \in \mathcal{D}(L)$,*

$$Var_\mu(f) \leq C_P(\mu)\, \mathcal{E}(f)\,,$$

where

$$\mathcal{E}(f) = \int -Lf\, f\, d\mu = \frac{1}{2}\int |\sigma\, \nabla f|^2\, d\mu\,.$$

(HP5) *There exists a constant $\lambda_P(\mu)$ such that for all $f \in \mathbb{L}^2(\mu)$,*

$$Var_\mu(P_t f) \leq e^{-\lambda_P(\mu)\, t}\, Var_\mu(f)\,.$$

Then $(HP5) \Leftrightarrow (HP4)$, $(HP4) \Rightarrow (HP2)$, $(HP2) \Leftrightarrow (HP1)$ and $(HP1) \Rightarrow (HP3)$. Actually $(HP4)$ implies $(HP2)$ for all (non-empty) open connected and bounded subset U.
If in addition μ is symmetric, then

$$(HP1) \Leftrightarrow (HP2) \Leftrightarrow (HP3) \Leftrightarrow (HP4) \Leftrightarrow (HP5)\,.$$

When μ is not symmetric, examples are known (kinetic diffusions) where $(HP1)$ (hence $(HP3)$) is satisfied but $(HP4)$ is not. In the case of kinetic diffusions, it is evident that $(HP4)$ cannot hold as the Dirichlet form is degenerate. It is however also possible to build "monster" diffusion where the invariant probability measure has some polynomial tail but the diffusion (with identity diffusion matrix) may converge exponentially fast and thus a Lyapunov condition holds.

The first equivalence is well known (see [3]), the second one is a simple application of Ito's calculus and PDE results (see [30] and [17]), the final implication is a consequence of the Meyn-Tweedie theory. The implication $(HP4) \Rightarrow (HP2)$ is shown in [30] by using the deviation results for the occupation measure obtained in [23] (using a beautiful deviation result obtained in [59]). We shall see in the next section another much more direct approach. Finally, in the symmetric case the converse implications are obtained by using the method in [4].

Some extensions of this theorem to polynomial ergodicity are discussed in [30] in connexion with weak Poincaré inequalities. A deeper study of this situation is done in [50, 49, 48].

The questions we shall address in the present paper are not concerned with weakening but with reinforcing of the assumptions, that is, does it exist similar results as in Theorem 1.1 if we replace the Poincaré inequality by stronger inequalities, for instance F-Sobolev inequalities in the spirit of [1, 9, 10] ?

Partial answers are known: an F-Sobolev inequality is equivalent to exponential stabilization in some Orlicz space (see [56]) and in the symmetric situation, reinforced Lyapunov conditions imply super-Poincaré inequalities or F-Sobolev inequalities (see [27, 28, 29]). Recently, Liu ([47, 46]) proposed some new ideas in order to directly link Lyapunov conditions on one hand, concentration properties or functional inequalities on the other hand. Though some aspects of his proofs are a little bit obscure for us, we shall follow his main idea in order to deduce a Lyapunov condition from a functional inequality, and then get equivalent results in terms of hitting times. This yields the following result written here for the logarithmic Sobolev inequality which is the best known F-Sobolev

Theorem 1.2. *Assume that D is not bounded. Consider the following properties*

(HLS1) There exists a Lyapunov function W, i.e. there exists a smooth function $W : D \to \mathbb{R}$, with $W \geq w > 0$, and there exist constants $\lambda > 0$ and $b > 0$ such that $\frac{\partial W}{\partial n} = 0$ on ∂D and

$$LW(x) \leq -\lambda |x|^2 W(x) + b .$$

(HLS1') There exists a Lyapunov function W, i.e. there exists a smooth function $W : D \to \mathbb{R}$, with $W \geq w > 0$, and there exist constants $\lambda > 0$ and $b > 0$ such that $\frac{\partial W}{\partial n} = 0$ on ∂D and

$$LW(x) \leq -\lambda V(x) W(x) + b .$$

(HLS2) There exist an open connected bounded subset U and a constant $\theta > 0$ such that for all x,

$$\mathbb{E}_x \left(\exp \left(\int_0^{T_U} \theta |X_s|^2 \, ds \right) \right) < +\infty ,$$

where T_U denotes the hitting time of U.

(HLS2') *There exist an open connected bounded subset U and a constant $\theta > 0$ such that for all x,*

$$\mathbb{E}_x \left(\exp \left(\int_0^{T_U} \theta\, V(X_s)\, ds \right) \right) < +\infty\,,$$

where T_U denotes the hitting time of U.

(HLS4) *μ satisfies a logarithmic-Sobolev inequality, i.e. there exists a constant $C_{LS}(\mu)$ such that for all smooth $f \in \mathcal{D}(L)$,*

$$Ent_\mu(f^2) := \int f^2 \ln \left(\frac{f^2}{\int f^2\, d\mu} \right) d\mu \leq C_{LS}(\mu)\, \mathcal{E}(f)\,,$$

where

$$\mathcal{E}(f) = \int -Lf\, f\, d\mu = \frac{1}{2} \int |\sigma\, \nabla f|^2\, d\mu\,.$$

(HLS5) *There exists a constant $C_E(\mu)$ such that for all $f^2 \in \mathbb{L} \ln \mathbb{L}(\mu)$ s.t. $\int f^2 d\mu = 1$,*

$$Ent_\mu(P_t(f^2)) \leq e^{-C_E(\mu)\, t}\, Ent_\mu(f^2)\,.$$

Then

1) *$(HLS5) \Leftrightarrow (HLS4)$, $(HLS4) \Rightarrow (HLS1)$ and $(HLS2) \Leftrightarrow (HLS1)$.*

1') *Assume that V goes to infinity at infinity and that there exists some $a > 0$ such that $\mu(e^{aV}) < +\infty$.*
Then $(HLS5) \Leftrightarrow (HLS4)$, $(HLS4) \Rightarrow (HLS1')$ and $(HLS2') \Leftrightarrow (HLS1')$.

Actually $(HLS4)$ implies in both cases $(HLS2)$ or $(HLS2')$ for all open nice subset U.

Assume in addition that μ is symmetric and that $\sigma.\sigma^*$ is uniformly elliptic.

2') *Assume that V goes to infinity at infinity, that $|\nabla V(x)| \geq v > 0$ for $|x|$ large enough and that there exists some $a > 0$ such that $\mu(e^{aV}) < +\infty$.*
Then $(HLS1') \Leftrightarrow (HLS2') \Leftrightarrow (HLS4) \Leftrightarrow (HLS5)$.

2) *Assume the curvature condition $Ric + HessV \geq -C > -\infty$ where Ricci and Hess are related to the riemanian metric defined by σ.*
Then $(HLS1) \Leftrightarrow (HLS2) \Leftrightarrow (HLS4) \Leftrightarrow (HLS5)$.

Except $(HLS5) \Leftrightarrow (HLS4)$ which is well known, we will prove this Theorem (and actually more general results) in section 3. Part of this result can be extended to general F-Sobolev inequalities, this is done in section 4.

The next two sections are devoted to the rate of convergence of $P_t(\nu, .)$, the distribution at time t of the process with initial distribution ν, to the invariant distribution μ for the total variation distance. In section 5 Theorem 5.4 we get the following: under some natural assumptions on μ, almost any F-Sobolev inequality combined with the Poincaré inequality provides an exponential convergence

$$\|P_t(\nu, .) - \mu\|_{TV} \leq C(\nu)\, e^{-\beta\, t}\,,$$

for all ν absolutely continuous w.r.t. μ such that $\frac{d\nu}{d\mu}$ belongs to $\mathbb{L}^p(\mu)$ for some $p > 1$. The remarkable fact here is that β does not depend on the integrability property (the p) of $d\nu/d\mu$.

If this and more general results were proved in [24], the proof given here is particularly simple and understandable.
In the next section 6 we study $\mathbb{L}^\infty$ properties of the Lyapunov functions, in relation with the property of "coming down from infinity" for the process. In particular we show that if the ultra boundedness property of the semi-group implies the "coming down from infinity" property for the process, the converse is not true. All these notions are particularly relevant in the study of quasi-stationnary distributions.

Finally in the last section we directly rely general Lyapunov condition to the existence of some exponential moments for the measure μ, extending the results in [47].

2. Back to the Poincaré inequality.

As we said in the introduction, we shall give here a new direct proof of $(HP4) \Rightarrow (HP1)$ in Theorem 1.1.

Theorem 2.1. *Assume that μ satisfies a Poincaré inequality with constant $C_P(\mu)$. Then for all open subset A, there exists a smooth Lyapunov function $W \in \mathcal{D}(L)$ i.e. a smooth function satisfying $W \geq w > 0$ on A^c and $LW \leq -cW$ on A^c with*

$$c = \mu(A) \min\left(\frac{1}{4C_P(\mu)}\,,\,\frac{1}{8}\right)\,.$$

It is easily seen that if A is smooth an bounded, we can modify W to get $(HP1)$ (with a λ smaller than the c in the theorem).

Proof. Let start with a simple lemma.

Lemma 2.2. *Assume that μ satisfies a Poincaré inequality with constant $C_P(\mu)$. Then for all subset A such that $\mu(A) > 0$ and all f in $H^1(\mu) = \mathcal{D}(\mathcal{E})$ it holds*

$$\int f^2\,d\mu \leq \frac{2C_P(\mu)}{\mu(A)}\,\mathcal{E}(f) \,+\, \frac{4}{\mu(A)}\int_A f^2\,d\mu\,.$$

Proof. Using Poincaré inequality and the elementary

$$(a+b)^2 \leq (1+\lambda)a^2 + \left(1+\frac{1}{\lambda}\right)b^2 \quad \text{for all } \lambda > 0$$

we can write

$$\begin{aligned}
\int f^2\,d\mu &\leq C_P(\mu)\,\mathcal{E}(f) \,+\, \mu^2(f) \\
&\leq C_P(\mu)\,\mathcal{E}(f) \,+ \mu^2(f\mathbf{1}_A + f\mathbf{1}_{A^c}) \\
&\leq C_P(\mu)\,\mathcal{E}(f) \,+ (1+\lambda)\,\mu^2(f\mathbf{1}_A) + \frac{1+\lambda}{\lambda}\,\mu^2(f\mathbf{1}_{A^c}) \\
&\leq C_P(\mu)\,\mathcal{E}(f) \,+ (1+\lambda)\,\mu(A)\int_A f^2\,d\mu + \frac{1+\lambda}{\lambda}\,\mu(f^2)\mu(A^c)
\end{aligned}$$

where we have used Cauchy-Schwartz in the final inequality. Hence provided

$$\mu(A)(1+\lambda) - 1 \,>\, 0\,,$$

we have obtained

$$\int f^2\,d\mu \le \frac{\lambda\, C_P(\mu)}{\mu(A)(1+\lambda)-1}\,\mathcal{E}(f) + \frac{\lambda(1+\lambda)\,\mu(A)}{\mu(A)(1+\lambda)-1}\int_A f^2\,d\mu\,.$$

The result follows by choosing $\lambda = 2(1-\mu(A))/\mu(A)$. □

Now define

$$\phi(x) = -c + \mathbf{1}_A(x)\,, \tag{2.3}$$

and introduce for all smooth $u \in \mathcal{D}(L)$,

$$Hu = -Lu + \phi\, u\,.$$

On one hand, it holds

$$\mu(u.Hu) \le \mathcal{E}(u) + \int_A u^2\,d\mu\,.$$

On the other hand, applying the previous lemma we have

$$\begin{aligned}
\mu(u.Hu) &= \mathcal{E}(u) + \int_A u^2\,d\mu - c\mu(u^2) \\
&\ge \mathcal{E}(u) + \int_A u^2\,d\mu - c\left(\frac{2C_P(\mu)}{\mu(A)}\,\mathcal{E}(u) + \frac{4}{\mu(A)}\int_A u^2\,d\mu\right) \\
&\ge \frac{1}{2}\left(\mathcal{E}(u) + \int_A u^2\,d\mu\right)
\end{aligned}$$

if we choose

$$c = \mu(A)\min\left(\frac{1}{4C_P(\mu)}\,,\,\frac{1}{8}\right)\,.$$

Now we will linearize $\mu(u.Hu)$. If $v \in \mathcal{D}(L)$ and $u \in H^1(\mu)$, $\mathcal{H}(u,v) = \mu(u.Hv)$ is well defined, and using an integration by parts (or the Green-Rieman formula since the normal derivative of v at the boundary vanishes) can be written as a (non necessarily symmetric) bilinear form on $H^1(\mu)$. It is easily seen that this bilinear form $\mathcal{H}$ is continuous on $H^1(\mu)$ equipped with the (usual) hilbertian norm $\left(\mathcal{E}(u) + \int u^2\,d\mu\right)^{\frac{1}{2}}$, hence equipped with the hilbertian norm $\left(\mathcal{E}(u) + \int_A u^2\,d\mu\right)^{\frac{1}{2}}$ which is equivalent according to Lemma 2.2. But according to what precedes, $\mathcal{H}$ is also coercive for this norm.
Hence, we may apply the Lax-Milgram theorem which tells us that for any $g \in H^1(\mu)$ there exists some $v \in H^1(\mu)$ such that for all u, $\mathcal{H}(u,v) = \langle u, g\rangle$. We will use this result with $g \equiv 1$.
First of all, the previous relation with u compactly supported in D shows that $Hv = g$ in $\mathcal{D}'(D)$, so that thanks to ellipticity (or hypo-ellipticity), $v \in C^\infty(D)$ and satisfies $Hv = g$ in the usual sense in D. When $D = \mathbb{R}^d$ this is enough. Otherwise, since the boundary is non characteristic, v admits sectional traces on ∂D of any order (see [17] Theorem 4.6) and using the results in [17] section 4, one can see that $v \in C^\infty(\bar{D})$ and satisfies $\partial_n v = 0$ on ∂D. Since $Hv \in \mathbb{L}^2(\mu)$, it follows that $v \in \mathcal{D}(H)$.

As a routine, defining $v_- = \min(v, 0)$, one can check using integration by parts or the Green-Rieman formula, that $\mu(v_-.Lv) = \mathcal{E}(v_-)$ so that, using the previous lower bound we obtain,

$$\begin{aligned}\mu(v_-.g) &= \mathcal{H}(v_-, v) = \mu(v_-.Hv) = \mu(v_-.Lv) + \mu(\phi.v.v_-) = -\mathcal{E}(v_-) - \mu(\phi.v_-^2) \\ &\leq -\frac{1}{2}\left(\mathcal{E}(v_-) + \int_A v_-^2\, d\mu\right)\end{aligned}$$

and $v_- = 0$ μ since $g > 0$. So $v \geq 0$ almost surely.
One should now use the maximum principle but we prefer use Ito's formula. Assume that A is open and bounded. Since $\phi = -c$ on A^c, we get for any $x \in A^c$,

$$\begin{aligned}v(x) &= \mathbb{E}_x(v(X_{t\wedge T_A}) + \mathbb{E}_x\left(\int_0^t \mathbf{1}_{s\leq T_A}\,(Hv - \phi\, v)(X_s)\, ds\right) \\ &\geq \mathbb{E}_x(t \wedge T_A) > 0\end{aligned}$$

since $T_A > 0$ if $x \in A^c$. Replacing A by $A_\varepsilon = \{y, d(y, A) < \varepsilon\}$ we may finally let t go to infinity, use the fact that v is bounded from below by a positive constant $v(A, \varepsilon)$ on ∂A_ε which is compact and obtain that $v(x) \geq v(A, \varepsilon)$ for all $x \in A_\varepsilon^c$ using the previous inequality (which actually furnishes exactly the minimum principle). For a general open set A just take the intersection with a large ball to get a bounded subset. □

Remark 2.4. As soon as we know the existence of a Lyapunov function satisfying $(HP1)$ it immediately follows using Ito's formula with the function $(t, y) \mapsto e^{\lambda t}\, W(y)$ that for all x,

$$W_U(x) = \mathbb{E}_x(e^{\lambda\, T_U}) < +\infty\,,$$

and conversely, if the exponential moment is finite, W_U is a Lyapunov function (see [30]).
Also notice that the proof of Theorem 2.1 furnishes a Lyapunov function in $H^1(\mu)$, hence in $\mathcal{D}(L)$ since $Lv = (\mathbf{1}_A - c)v - 1$ implies $Lv \in \mathbb{L}^2(\mu)$. Hence the previous W_U belongs to $\mathcal{D}(L)$. Conversely if W_U is finite, $LW_U = -\lambda W_U$ in U^c. Replacing λ by $\lambda/2$ if necessary, we may assume that W_U is in $\mathbb{L}^2(\mu)$ so that LW_U is square integrable too (at least in U^c). If U is relatively compact it is easy to see that one can modify U and W_U to get a smooth function everywhere that belongs to $\mathcal{D}(L)$. ♢

3. The logarithmic Sobolev inequality.

We start with an analogue of Theorem 2.1.

Proposition 3.1. *Assume that μ satisfies the logarithmic Sobolev inequality $(HLS4)$. Let h be a non-negative continuous function such that $b = 2\mu(e^h) < +\infty$. For $\varepsilon > 0$, define $U_\varepsilon(h) = \{(1-\varepsilon)h > b\}$.*
Then there exists a Lyapunov function $W \in \mathcal{D}(L)$ such that $W(x) \geq w_\varepsilon > 0$ on $U_\varepsilon(h)$ and

$$(HLh) \quad LW \leq -\frac{\varepsilon}{2\, C_{LS}}\, h\, W \qquad \text{on } U_\varepsilon(h).$$

Proof. We follow and modify the proof in [46]. Assume that h is a non-negative function such that $\mu(e^h) < +\infty$. We follow the proof of Theorem 2.1 and define, for $2\,\rho \leq \frac{1}{C_{LS}}$ so that $2\rho\, Ent_\mu(u^2) \leq \mathcal{E}(u)$,

$$\phi(x) = \rho\,(-\,h(x) + \,b)\,, \tag{3.2}$$

with $b = 2\mu(e^h)$ and introduce for all smooth $u \in \mathcal{D}(L)$,

$$Hu = -Lu + \phi\, u\,.$$

On one hand, it holds

$$\mu(u.Hu) \leq \mathcal{E}(u) + \rho\, b\, \mu(u^2)\,.$$

On the other hand, applying this time Young's inequality and LSI, we get for a smooth $u \in \mathcal{D}(L)$,

$$\begin{aligned}
\mu(u.Hu) &= \mathcal{E}(u) + \rho\, b\, \mu(u^2) - \rho\, \mu(hu^2) \\
&\geq \mathcal{E}(u) + \rho\, b\, \mu(u^2) - \rho\, \mu(u^2)\, \mu\left(e^h - \frac{u^2}{\mu(u^2)} + \frac{u^2}{\mu(u^2)} \ln\left(\frac{u^2}{\mu(u^2)}\right)\right) \\
&\geq \mathcal{E}(u) + \rho\, b\, \mu(u^2) - \rho\, \frac{b}{2}\, \mu(u^2) + \rho\, \mu(u^2) - \rho\, Ent(u^2) \\
&\geq \frac{1}{2}\, \left(\mathcal{E}(u) + \rho\, b\, \mu(u^2)\right)\,.
\end{aligned}$$

We can then follow the proof of Theorem 2.1, and thus apply again the Lax-Milgram theorem to get the existence of a non-negative smooth function $v \in H^1(\mu)$ satisfying

$$Lv = \phi\, v - 1 = -1 - \rho\, (b - h)\, v\,.$$

If in addition

$$h(x) > b \quad , \forall x \in U^c$$

we obtain that v is bounded from below by a positive constant in U_ε^c. The proof is completed. □

This result is in particular interesting when D is not bounded and h goes to infinity at infinity. Two cases are mainly relevant, due to the converse statements we will prove below

Corollary 3.3. *Assume that μ satisfies the logarithmic Sobolev inequality $(HLS4)$ and that D is not bounded. Then*

1) *for all $x_0 \in D$, there exists a Lyapunov function $W \in \mathcal{D}(L)$ with $W(x) \geq w > 0$ for all $x \in D$ satisfying*

$$LW(x) \leq -\lambda\, d^2(x, x_0)\, W(x) + b\,,$$

for some λ and b strictly positive;

2) *if in addition V goes to infinity at infinity and $e^{aV} \in \mathbb{L}^1(\mu)$ for some $a > 0$, there exists a Lyapunov function $W \in \mathcal{D}(L)$ with $W(x) \geq w > 0$ for all $x \in D$ satisfying*

$$LW(x) \leq -\lambda\, V(x)\, W(x) + b\,,$$

for some λ and b strictly positive.

We may replace b by $b\mathbf{1}_A$ for some well chosen bounded subset A of D.

Proof. In case 2), just apply the previous proposition with $h = a|V|$ and modify W in the corresponding level set A of V. For case 1), recall that the logarithmic-Sobolev inequality implies that there exists some $c > 0$ such that $\mu(e^{cd^2(.,x_0)}) < +\infty$ and conclude as before with $h = cd^2(., x_0)$. □

Introducing the process

$$dH_t = h(X_t)\,dt$$

and applying Ito's formula to $(H,x) \mapsto e^{\frac{\varepsilon}{2\,C_{LS}}\,H}$ it is easy to show that (HLh) implies for all x,

$$W_{h,\varepsilon}(x) \,=\, \mathbb{E}_x\left(\exp\left(\int_0^{T_{U_{\varepsilon(h)}}} \frac{\varepsilon}{2\,C_{LS}}\,h(X_s)\,ds\right)\right) < +\infty\,. \tag{3.4}$$

Conversely, if $W_{h,\varepsilon}(x)$ is finite for all x, using the arguments in [17] one can prove that it satisfies (HLh) with an equality instead of an inequality. Notice that once again we may apply the arguments in Remark 2.4.

To complete the proof of Theorem 1.2, it remains to look at the converse statements in the symmetric situation.
The first case is the case $h(x) = cd^2(x,x_0)$. Statement 2) in Theorem 1.2 under the curvature assumption is proved in [28] using transportation inequalities. An alternative method of proof was recently proposed in [46] (with some points to be corrected). For the second case $h = aV$ we will use the results in [27] based on super-Poincaré inequalities.

Proposition 3.5. *Assume that μ is symmetric and that $\sigma.\sigma^*$ is uniformly elliptic. Assume in addition that V goes to infinity at infinity, that $|\nabla V(x)| \geq v > 0$ for $|x|$ large enough and that $e^{aV} \in \mathbb{L}^1(\mu)$ for some $a > 0$.*
If there exists a Lyapunov function W with $W(x) \geq w > 0$ for all $x \in D$, $\frac{\partial W}{\partial n} = 0$ on ∂D and satisfying

$$LW(x) \leq -\lambda\,V(x)\,W(x) \,+\, b\,,$$

for some λ and b strictly positive, then μ satisfies a logarithmic-Sobolev inequality.

Proof. We follow the method in [27] Theorem 2.1 (itself inspired by [4]). Let $A_r = \{V \leq r\}$. For r_0 large enough and some $\lambda' < \lambda$ we have

$$LW(x) \leq -\lambda'\,V(x)\,W(x) \,+\, b\,\mathbf{1}_{A_{r_0}}\,,$$

so that we may assume that

$$\frac{LW}{W}(x) \leq -\,\lambda\,V(x) \quad \text{ for } x \in A_r^c \text{ and all } r \text{ large enough.}$$

Denote by $M = \sup(-V)$. We have for $s \leq s_0$ and $r > r_0$,

$$\begin{aligned}
\int f^2\,d\mu \;&=\; \int_{A_r} f^2\,d\mu \,+\, \int_{A_r^c} f^2\,d\mu \\
&\leq\; e^M\left(1+\frac{b}{\lambda r_0}\right)\int_{A_r} f^2\,dx \,+\, \frac{1}{\lambda r}\int \lambda V(x)\,f^2 d\mu, \\
&\leq\; e^M\left(1+\frac{b}{\lambda r_0}\right)\int_{A_r} f^2\,dx \,+\, \frac{1}{\lambda\,r}\int f^2\left(\frac{-LW}{W}\right)d\mu \\
&\leq\; e^M\left(1+\frac{b}{\lambda r_0}\right)\left(s\int_{A_r}|\nabla f|^2\,dx \,+\, \frac{C}{s^{d/2}}\left(\int_{A_r}|f|\,dx\right)^2\right) \,+\, \frac{1}{\lambda\,r}\int |\sigma.\nabla f|^2\,d\mu\,.
\end{aligned}$$

The first part of the last bound is obtained by using (3.1.4) in [27] (it is here that we are using the assumption on $|\nabla V|$), while the second bound is obtained using integration by

parts or the Green-Rieman formula (see [27] (2.2)). Using uniform ellipticity we thus obtain, denoting $c = e^M \left(1 + \frac{b}{\lambda r_0}\right)$

$$\mu(f^2) \leq \left(\frac{s\,c}{a} + \frac{1}{\lambda\, r}\right) \int |\sigma.\nabla f|^2 \, d\mu + C\, s^{-d/2}\, c\, e^{2r} \left(\int |f|\, d\mu\right)^2 . \tag{3.6}$$

Hence choosing $r = c'/s$ we get the following super-Poincaré inequality for small s,

$$\mu(f^2) \leq s \int |\sigma.\nabla f|^2 \, d\mu + C'\, e^{\tilde{c}/s} \left(\int |f|\, d\mu\right)^2 ,$$

which is known to be equivalent to a defective logarithmic Sobolev inequality (see the introduction of [27]). But the Lyapunov condition being stronger than $(HP1)$, we know that μ satisfies a Poincaré inequality, hence using Rothaus lemma, that it satisfies a (tight) log-Sobolev inequality.

□

Remark 3.7. With some additional effort one should replace (3.1.4) in [27] directly by a similar statement with $\sigma.\nabla f$ instead of ∇f, even in the strongly hypo-elliptic case, replacing the arguments in [27] by the Jerison and Sanchez-Calle estimates for such operators, up to a modification of the power $s^{-d/2}$ replaced by s^{-m} where m depends on the dimension of the graded Lie algebra. We do not want to go further into details here, that is why we choosed to only consider the uniformly elliptic situation. ♢

4. F-Sobolev inequalities.

We will extend the results of the previous section to general F-Sobolev inequalities introduced by Aida ([1]) and studied in [58, 9, 10]. Actually we will not be as complete as for the log-Sobolev inequality, because for general functions F instead of the logarithm, results are much more intricate. In particular the reader will find in [56] convergence results in Orlicz spaces (replacing $(HLS5)$ we shall not give here.

We are mainly interested here with the following version of (defective) F-Sobolev inequalities for a nice F defined on $\mathbb{R}^+$:

$$(HFS4defect) \quad \int f^2\, F\left(\frac{f^2}{\int f^2\, d\mu}\right) \leq C_F(\mu)\, \mathcal{E}(f) + D_F\, \mu(f^2)\,.$$

When $D_F = 0$ one say that the inequality is tight and simply denote it by $(HFS4)$. The relationship between F-Sobolev and super-Poincaré inequalities is due to Wang ([58] Theorem 3.3.1 and Theorem 3.3.3). Recall some basic facts on these inequalities

Proposition 4.1. *We have:*

- *(see* [58]*.) A super-Poincaré inequality*

  $$\mu(f^2) \leq s\, \mathcal{E}(f) + \beta(s)\, (\mu(|f|))^2$$

  implies $(HFS4defect)$ *for*

  $$F(u) = \frac{1}{u} \int_0^u \xi(t/2)\, dt \quad , \quad \xi(t) = \sup_{a>0} \left(\frac{1}{a} - \frac{\beta(a)}{ta}\right) .$$

- *(see* [58]*.) If* $(HFS4defect)$ *holds true for* F *such that* $\int_\cdot^{+\infty} \frac{1}{uF(u)}\,du < +\infty$*, then the semi-group* P_t *is ultra-bounded, i.e. for all* $t>0$ *there exists* $C(t)$ *such that,*

$$\|P_t f\|_\infty \le C(t)\,\|f\|_{\mathbb{L}^1(\mu)}$$

so that if in addition a Poincaré inequality holds,

$$\|P_t f - \mu(f)\|_\infty \le M\,e^{-Ct}\,\|f-\mu(f)\|_{\mathbb{L}^2(\mu)}$$

for some $C>0$ *(one can replace* $\mathbb{L}^2$ *by any* $\mathbb{L}^p$ *for* $p>1$ *just changing* C *using interpolation results, see e.g.* [24, 26]*).*
- *(see* [9] *lemma 8.) If* $F(1)=0$*,* F *is* C^2 *in a neighborhood of* 1 *and* $2F'(1)+F''(1) = c>0$*,* $(HFS4)$ *implies the Poincaré inequality* $(HP4)$ *with* $C_P(\mu)=1/(2c)$*.*
- *(see* [9] *Remark 22.) If* $F\ge 0$ *and* $F(u)\ge c>0$ *for* $u\ge 2$*, then* $(HFS4)$ *implies the Poincaré inequality* $(HP4)$*.*
- *(see* [9] *lemma 9.) If* F *is concave, non-decreasing, growths to infinity and satisfies* $F(1)=0$ *and* $uF'(u)\le M$*, then* $(HFSdefect)$ *and the Poincaré inequality imply* $(HFS4)$*.*

We thus have two situations: either $F \le \log$ (interpolating between Poincaré and log-Sobolev), in which case (with additional structural conditions on F) F-Sobolev inequalities are equivalent to an exponential convergence in some Orlicz space ([56]), we still have some Rothaus(-Orlicz) lemma allowing us to tight a defective F-Sobolev inequality and a lot of additional properties connected with Orlicz hyperboundedness, concentration and isoperimetry (see [9, 10]), or $\int_\cdot^{+\infty} \frac{1}{uF(u)}\,du < +\infty$ in which case exponential convergence holds in $\mathbb{L}^\infty$, with a very small gap between both classes of F.

We will now prove the analogue of Proposition 3.1. To this end we need to introduce some convexity notions.

Definition 4.2. *Assume that* $u\mapsto uF(u)=G(u)$ *is convex. We define* G^* *as the Fenchel-Legendre dual function of* G *i.e.* $G^*(u)=\sup_{t>0}(ut-G(t))$*.*
For instance if $F(u)$ *behaves like* $\ln_+^\beta(u)$ *at infinity for some* $\beta>0$*, then* $G^*(t)$ *behaves like* $\beta\, t^{(\beta-1)/\beta}\, e^{t^{\frac{1}{\beta}}}$ *at infinity (see* [9] *subsection 7.1 to see the correct* F *to be chosen).*

Proposition 4.3. *Assume that* μ *satisfies the F-Sobolev inequality* $(HFS4defect)$ *and that* $G(s)=sF(s)$ *is convex. Let* h *be a non-negative continuous function such that* $b=2(D_F+\mu(G^*(h)))<+\infty$*. For* $\varepsilon>0$*, define* $U_\varepsilon(h)=\{(1-\varepsilon)h>b\}$*.*
Then there exists a Lyapunov function $W\in\mathcal{D}(L)$ *such that* $W(x)\ge w_\varepsilon>0$ *on* $U_\varepsilon(h)$ *and*

$$LW \le -\frac{\varepsilon}{2\,C_F}\,h\,W \qquad \text{on } U_\varepsilon(h).$$

Proof. The proof mimic the one of proposition 3.1 with the following modifications: take $\phi=\rho(-h+b)$ with $b=2(D_F+\mu(G^*(h)))$ and $\rho\,C_F=\frac12$, use Young's inequality $st\le G(s)+G^*(t)$. □

We thus obtain, for a particular choice of F:

Theorem 4.4. *Assume that* D *is not bounded, that* V *goes to infinity at infinity and that* $e^{aV}\in\mathbb{L}^1(\mu)$ *for some* $a>0$*. Consider the following properties*

(HFS1) *There exists a Lyapunov function W, i.e. there exists a smooth function $W : D \to \mathbb{R}$ with $W \geq w > 0$, and there exist constants $\lambda > 0$ and $b > 0$ such that $\frac{\partial W}{\partial n} = 0$ on ∂D and*

$$LW(x) \leq -\lambda |V|^{\beta}(x) W(x) + b .$$

(HFS2) *There exist an open connected bounded subset U and a constant $\theta > 0$ such that for all x,*

$$\mathbb{E}_x \left(\exp \left(\int_0^{T_U} \theta |V|^{\beta}(X_s)\, ds \right) \right) < +\infty ,$$

where T_U denotes the hitting time of U.

$(HFS\beta)$ *μ satisfies $(HFS4defect)$ with $F(s) = \ln_+^{\beta}(s)$.*

Then $(HFS\beta) \Rightarrow (HFS1)$ and $(HFS1) \Leftrightarrow (HFS2)$.

If in addition μ is symmetric, $\sigma.\sigma^*$ is uniformly elliptic and $|\nabla V(x)| \geq v > 0$ for $|x|$ large enough, then

$$(HFS\beta) \Leftrightarrow (HFS1) \Leftrightarrow (HFS2) .$$

For $\beta \leq 1$ we may replace $(HFS\beta)$ by its tight version.

Proof. For the first part we use the previous proposition with $h = a|V|^{\beta}$ for some a small enough. In the symmetric situation, we mimic the proof of proposition 3.5 yielding a super-Poincaré inequality with $\beta(s) = e^{c/s^{\frac{1}{\beta}}}$ hence the corresponding defective F-Sobolev inequality using Proposition 4.1. But $(HFS1)$ implies $(HP1)$ hence a Poincaré inequality and we can use the final statement of Proposition 4.1 to get a tight version when $\beta \leq 1$. □

We know in particular (see [9]) that for $V(x) = |x|^{\alpha}$, $\alpha \geq 1$, μ satisfies $(HFS\beta)$ with

$$\beta = 2 \left(1 - \frac{1}{\alpha} \right) .$$

The main reason for writing Theorem 4.4 only in the case $F \sim \ln_+^{\beta}$ is the converse part $(HFS1) \Rightarrow (HFS\beta)$ for which the argument is easy since we have an explicit expression of G^*. Of course Proposition 4.3 contains much more general situations. Here is one which will be useful in the sequel

Theorem 4.5. *Assume that D is not bounded, that μ satisfies both a Poincaré inequality and the F-Sobolev inequality $(HFS4defect)$, that $G(s) = sF(s)$ is convex, non decreasing at infinity and that $(G^*)^{-1}$ (the inverse function of G^*) growths to infinity at infinity.*
Then for all $x_0 \in D$ there exist a and θ two positive constants, such that, defining

$$h(x) = (G^*)^{-1}(e^{ad(x,x_0)}) ,$$

we have for all x and all non empty, open and bounded subset U,

$$W_{\theta,U,h}(x) = \mathbb{E}_x \left(\exp \left(\int_0^{T_U} \theta\, h(X_s)\, ds \right) \right) < +\infty ,$$

where T_U denotes the hitting time of U. Actually $W_{\theta,U,h} \in \mathbb{L}^1(\mu)$.

Proof. Since μ satisfies a Poincaré inequality, it is known that there exists $a > 0$ such that

$$\mu(G^*(h)) = \mu(e^{ad(x,x_0)}) < +\infty\,.$$

In addition h goes to infinity at infinity so that its level sets are compact. It remains to apply Proposition 4.3 to get the Lyapunov function $W_{\theta,U,h}$ and consequently the result. As usual since $LW_{\theta,U,h} + \theta h W_{\theta,U,h} \leq 0$ outside of a compact set, we get that $h\,W_{\theta,U,h}$ is integrable and since h goes to infinity that $W_{\theta,U,h}$ is integrable too. □

5. $\mathbb{L}^p$ geometric ergodicity and functional inequalities.

Come back to the geometric ergodicity property $(HP3)$. If we replace the initial distribution δ_x by some initial probability distribution ν, we have

$$\|P_t(\nu,.) - \mu\|_{TV} \leq C\,e^{-\beta\,t}\,,$$

provided

$$(LW\nu) \qquad\qquad W \in \mathbb{L}^1(\nu)\,. \tag{5.1}$$

If ν is absolutely continuous w.r.t. μ and $\frac{d\nu}{d\mu} \in \mathbb{L}^p(\mu)$ a sufficient condition is thus

$$(LWq) \qquad\qquad W \in \mathbb{L}^q(\mu)$$

for $\frac{1}{p} + \frac{1}{q} = 1$. It is thus interesting to study the property (LWq).
As shown in [30], once $(HP1)$ is satisfied, $(LW1)$ is satisfied too. It follows that for some $\theta > 0$, $W_\theta(x) = \mathbb{E}_x(e^{\theta T_U})$ is finite, hence satisfies $LW_\theta = -\theta W_\theta$ on U^c, so that enlarging a little bit U (say U_ε) we can modify the previous W_θ in U_ε in order to get a new Lyapunov function, still denoted W_θ for simplicity satisfying

$$LW_\theta \leq -\theta W_\theta + b\mathbf{1}_{\bar{U}}\,.$$

Hence $W_\theta \in \mathbb{L}^1(\mu)$. It follows that for every $p \geq 1$, defining $W_{\theta,p}(x) = \mathbb{E}_x(e^{\frac{\theta}{p}T_U})$, we have first that $W_{\theta,p} \in \mathbb{L}^p(\mu)$, second that (after similar modifications) $W_{\theta,p}$ is a Lyapunov function with θ replaced by θ/p. Since $\bar{U}$ is compact, these modifications do not modify the integrability properties of $W_{\theta,p}$. Hence we have obtained

Proposition 5.2. *If $(HP1)$ is satisfied, for all $p > 1$, one can find another Lyapunov function (associated to a different λ and a different U) $W_p \in \mathbb{L}^p(\mu)$. Hence there exists some β_p such that as soon as $\frac{d\nu}{d\mu} \in \mathbb{L}^q(\mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\|P_t(\nu,.) - \mu\|_{TV} \leq C(\nu)\,e^{-\beta_p\,t}\,.$$

In the symmetric case the situation is better understood. Indeed, $(HP1)$ implies the geometric ergodicity in $\mathbb{L}^2(\mu)$ $(HP5)$, so that using Riesz-Thorin interpolation theorem in an appropriate way (see [26]) we have that provided $\frac{d\nu}{d\mu} \in \mathbb{L}^q(\mu)$ for some $1 < q \leq 2$,

$$\left\| P_t\left(\frac{d\nu}{d\mu}\right) - 1\right\|_{\mathbb{L}^q(\mu)} \leq K_q\,e^{-\frac{(q-1)}{q}\,\lambda_P(\mu)\,t}\,\left\|\frac{d\nu}{d\mu} - 1\right\|_{\mathbb{L}^q(\mu)}\,.$$

In this situation we thus have geometric convergence for a stronger topology.

But the discussion preceding Proposition 5.2 furnishes a stronger result. Indeed $(HP1)$ yields $(HP3)$

$$H(x) = \|P_t(x,.) - \mu\|_{TV} \leq C\, W(x)\, e^{-\beta\, t}\,, \tag{5.3}$$

so that $H(x)$ converges to 0 for all x at a geometric rate. But we may replace W by W_p and β by β_p and get that actually H converges to 0 in all $\mathbb{L}^p(\mu)$ for $1 \leq p < +\infty$, with a geometric rate depending on p.

Assume from now on that μ satisfies some F-Sobolev inequality, for some smooth F. Does it improve the previous results ? In this situation, the rate of convergence to equilibrium in total variation distance was studied in [24]. The results we proved in the previous sections allow us to give a new and substantially simpler proof of some results contained in [24]. Here is a result in this direction:

Theorem 5.4. *Under the assumptions of Theorem 4.5, there exists $\lambda > 0$ such that for all non-empty, open and bounded set U,*

$$W(.) = \mathbb{E}_{.}\left(e^{\lambda\, T_U}\right) \in \bigcap_{1 \leq q < +\infty} \mathbb{L}^q(\mu)\,.$$

It thus follows that there exists some $\beta > 0$ such that for all ν absolutely continuous w.r.t. μ such that $\frac{d\nu}{d\mu}$ belongs to $\mathbb{L}^p(\mu)$ for some $p > 1$,

$$\|P_t(\nu,.) - \mu\|_{TV} \leq C(\nu)\, e^{-\beta\, t}\,.$$

Actually H defined in (5.3) *converges to* 0 *in all the $\mathbb{L}^p$'s with a rate $e^{-\beta\, t}$.*

Proof. Let h as in Theorem 4.5. The level sets $H_R = \{h \leq R\}$ of h are smooth (since F is smooth) compact sets. Denote by T_R the hitting time of H_R. For R large enough, H_R contains U. Let $\lambda > 0$ be such that $W(x) = \mathbb{E}_x(e^{\lambda T_U}) < +\infty$ for all x. Such a λ exists since μ satisfies a Poincaré inequality. If $y \in H_R$, $W(y) \leq K(R) < +\infty$ using the regularity of W. If $x \notin H_R$, we have

$$\mathbb{E}_x\left(e^{\lambda T_U}\right) = \mathbb{E}_x\left(e^{\lambda\, T_R}\, \mathbb{E}_{X_{T_R}}\left(e^{\lambda\, T_U}\right)\right) \leq K(R)\, \mathbb{E}_x\left(e^{\lambda\, T_R}\right)\,.$$

But now for $\theta R > q\, \lambda$,

$$\begin{aligned} W^q(x) &\leq K^q(R)\, \mathbb{E}_x^q\left(e^{\lambda\, T_R}\right) \leq K^q(R)\, \mathbb{E}_x\left(e^{q\, \lambda\, T_R}\right) \\ &\leq K^q(R)\, \mathbb{E}_x\left(e^{\int_0^{T_R}\, \theta\, h(X_s)\, ds}\right) = K^q(R)\, W_{\theta,U,h}(x)\,. \end{aligned}$$

The first part of the Theorem follows from Theorem 4.5. The second part is immediate. □

Remark 5.5. If $\frac{d\nu}{d\mu} \in \mathbb{L}^1(\mu)$, La Vallée-Poussin theorem implies that there is some Young function $\phi(u) = u\psi(u)$ with ψ growing to infinity, such that $\frac{d\nu}{d\mu}$ belongs to the Orlicz space $\mathbb{L}_\phi$. Hence the second part of the Theorem will follow from the first one and the Hölder-Orlicz inequality, once $W \in \mathbb{L}_{\phi^*}(\mu)$. But for the previous proof to work we need something like $\phi^*(e^{\lambda u}) \leq C\, e^{\phi^{**}(\lambda)\, u}$ which is not true for ϕ^* growing faster than a power function. That is why the result is only stated for ν with a density belonging to some $\mathbb{L}^p$ space. ♢

This result is nor new nor surprising. For instance, when a logarithmic Sobolev inequality holds true, Theorem 2.13 in [24] shows that there is exponential convergence to the equilibrium in total variation distance as soon as ν belongs to the space $\mathbb{L}\ln\mathbb{L}$. It is then a simple consequence of Pinsker's inequality and the entropic convergence to equimibrium. More generally, the courageous reader will find in the jungle of section 3 of [24] similar results for general F-Sobolev inequalities. It should be interesting to recover these results by using Lyapunov functions.

Actually we should describe the problem as follows: we know that reinforcing functional inequalities from Poincaré to F-Sobolev, reinforces the Lyapunov condition and conversely in the symmetric case. Does a reinforced functional inequality reinforce the integrability of the Lyapunov function and conversely in the symmetric case ? In particular can we characterize (at least in the symmetric case) a functional inequality through integrability properties of (some) Lyapunov function ?

Example 5.6. Look at the simple symmetric case $L = \Delta - \nabla V.\nabla$ in the whole $\mathbb{R}^d$. Then it is easily seen that the following holds: there is an equivalence between

- $x.\nabla V(x) \geq \alpha\,|x|^2$ for large $|x|$,
- $W_e(x) = e^{\alpha\,|x|^2/2}$ is a Lyapunov function, i.e. satisfies $(HP1)$,
- $W_2(x) = |x|^2$ also satisfies $(HP1)$.

The first item is of course equivalent to the fact that V is uniformly convex. We see that the behavior of various Lyapunov functions can be very different. They also imply various integrability properties and functional inequalities. Notice however that thanks to what we said previously we can directly make the following reasoning: if V is uniformly convex, W_2 is a Lyapunov function and admits some exponential moment so that convergence to the equilibrium in total variation distance holds as soon as ν belongs to the space $\mathbb{L}\ln\mathbb{L}$ as expected. Indeed, we have by $(HP1)$ that for some δ and some r (denoting B_r the euclidean ball of radius r, and $x \in B_r^c$

$$\mathbb{E}_x(e^{\delta T_{B_r}}) \leq W_2(x).$$

Note however that in fact W_e satisfies a stronger Lyapunov condition, i.e.

$$LW_e(x) \leq -\lambda|x|^2\,W_e(x) + b1_{B_r}$$

so that by the previous result on logarithmic , we have a stroonger integrability property linked to hitting times:

$$\mathbb{E}_x\left(e^{\delta\int_0^{T_{B_r}} X_s^2}\right) \leq W_e(x).$$

However, it does not seem possible to pass from this last control to the previous one.
Hence, finally, these results do not give any precise idea of the dependence in x of $\mathbb{E}_x(e^{\lambda T_U})$ for a given bounded U. Actually this is a very difficult problem for which results are only known in the gaussian case (i.e. for the Ornstein-Uhlenbeck process). ♢

In the next section we shall look more into details at the case where one can find a bounded Lyapunov function.

6. Coming down from infinity, uniform geometric ergodicity and Lyapunov functions.

If $d\nu/d\mu$ only belongs to $\mathbb{L}^1$, it is interesting to look at a bounded Lyapunov function. What precedes allows us to state

Proposition 6.1. *The following statements are equivalent:*

- *there exists $\lambda > 0$ such that*
$$\sup_{x\in\bar{D}} \mathbb{E}_x\left(e^{\lambda T_U}\right) < +\infty$$
for one (or all) non empty, open and bounded set U,
- *there exists a bounded Lyapunov function satisfying $(HP1)$,*
- *the process is uniformly geometrically ergodic, i.e. there exist $\beta > 0$ and $C > 0$ such that*
$$\sup_{x\in\bar{D}} \|P_t(x,.) - \mu\|_{TV} \leq C\, e^{-\beta\, t}\,.$$

In this case of course, for any initial probability measure ν,
$$\|P_t(\nu,.) - \mu\|_{TV} \leq C\, e^{-\beta\, t}\,.$$

There exists a stronger form of uniform exponential integrability, the notion of "coming down from infinity" which is used by people who are studying quasi-stationary distributions or more precisely Yaglom limits (see e.g. the recent book [32]). We shall use the following definition

Definition 6.2. *We say that the process comes down from infinity if for all $a > 0$ there exists some open, bounded subset U_a such that $\sup_x \mathbb{E}_x\left(e^{a\, T_{U_a}}\right) < +\infty$.*

In one dimension, this property was related to the uniqueness of quasi-stationary distributions (QSD) and to the fact that ∞ is an entrance boundary, in [19]. Uniqueness of a (QSD) also follows from the ultraboundedness property of the semi-group, even in higher dimension (see e.g. [31]). In [30] Proposition 5.3, we claimed that ultraboundedness is actually equivalent to coming down from infinity for one dimensional diffusion processes with generator $\Delta - \nabla V.\nabla$ satisfying some extra condition. D. Loukianova pointed out to us that the proof of this proposition in fact needs slightly more stringent assumptions (the function $z \mapsto F(z)/z$ therein is not necessarily non-increasing) for this equivalence to hold true.

Nevertheless, part of this result is true and we shall give a direct and simple proof.

Indeed, assume that W satisfies $(HP1)$. We have seen that we can always assume that $W \in \mathbb{L}^2(\mu)$ and then $W \in \mathcal{D}(L)$). It follows that
$$L(P_t W) = P_t(LW) \leq -\lambda\, P_t W + P_t(\mathbf{1}_{\bar{U}})\,. \tag{6.3}$$
Now assume that $R \leq d(x,\bar{U}) \leq 2R$. We have
$$\begin{aligned} P_t(\mathbf{1}_{\bar{U}})(x) &= \mathbb{E}_x\left(\mathbf{1}_{X_t\in\bar{U}}\right) \\ &\leq \mathbb{E}_x\left(\mathbf{1}_{T_{\bar{U}}<t}\right) \leq \sup_{d(y,\bar{U})\geq R} \mathbb{Q}_y(T_{\bar{U}} < t) \end{aligned}$$
where Q_y denotes the law of the process $Y_.$ with the same generator L but reflected on $d(z,\bar{U}) = 2R$ (which can be assumed to be smooth) and starting from y. Indeed if the

process $X_.$ hits $\bar U$ before to leave $\{d(z,\bar U)\le 2R\}$, it coincides with $Y_.$ (with the same starting point) up to $T_{\bar U}$. If not, $X_.$ leaves $\{d(z,\bar U)\le 2R\}$ before $T_{\bar U}$, but in order to hit $\bar U$ it has to come back to $\{d(z,\bar U)\le 2R\}$ first, so that using the Markov property we may apply the same argument as before this time starting from X_{T_R} where T_R is the hitting time of $\{d(z,\bar U)\le 2R\}$. That is why the final upper bound contains the supremum over y.
Now, since all coefficients are smooth, they are bounded with bounded derivatives of any order in $\{d(z,\bar U)\le 2R\}$ which is compact. It is then well known that

$$\sup_{d(y,\bar U)\ge R} \mathbb{Q}_y(T_{\bar U}<t)\le C\,e^{-cR/t}$$

for some constants C and c only depending on these bounds. Hence

$$P_t(\mathbf{1}_{\bar U})(x)\le C\,e^{-cR/t}\,,$$

as soon as $R\le d(x,\bar U)\le 2R$. If $d(x,\bar U)>2R$, $T_U>T_R$, and we may apply again the Markov property to get the same upper bound.
Pick some $R>0$ once for all and choose $t>0$ in such a way that

$$Ce^{-cR/t}<\frac{1}{2}\,\lambda\, w\,.$$

We thus have

$$\begin{aligned} L(P_tW)(x) &\le -\lambda\,P_tW(x)+\mathbf{1}_{d(x,\bar U)\le R}+Ce^{-cR/t}\,\mathbf{1}_{d(x,\bar U)\ge R}\\ &\le -\frac{\lambda}{2}\,P_tW(x)+\mathbf{1}_{d(x,\bar U)\le R}\,,\end{aligned}$$

so that P_tW is a new Lyapunov function with $\lambda/2$ and $\{d(x,\bar U)\le R\}$ in place of λ and $\bar U$ (of course P_tW belongs to $\mathcal{D}(L)$ and satisfies $P_tW\ge w>0$).
We deduce immediately

Theorem 6.4. *Assume that $(HP1)$ is satisfied (for example μ satisfies a Poincaré inequality) and that the semi-group $P_.$ is ultra-bounded, i.e. that P_t maps continuously $\mathbb{L}^1$ into $\mathbb{L}^\infty$ for any $t>0$. Then the process comes down from infinity.*

Proof. $(HP1)$ together with ultra-boundedness imply that the semi-group is hyper-contractive so that μ satisfies a logarithmic Sobolev inequality according to Gross theorem. Hence, we may apply Corollary 3.3 and find a Lyapunov function W such that

$$LW\le -\lambda\,d^2(.,x_0)\,W+b\,\mathbf{1}_{\bar U}$$

for some bounded open subset U. Hence for all large enough $a>0$,

$$LW\le -2a\,W\,+b_a\,\mathbf{1}_{U_a}$$

with $U_a=\{x\,;\,\lambda\,d^2(x,x_0)\le 2a\}$ and $b_a=\sup_{x\in U_a}LW$, a being large enough for $\bar U\subset U_a$. According to the previous discussion there exists a new Lyapunov function $W_a=P_tW$ for some adequat t which is bounded and satisfies $LW_a(x)\le -aW_a(x)$ for $x\in V_a=\{d(x,U_a)>R\}$, so that

$$\sup_x \mathbb{E}_x\left(e^{aT_a}\right)<+\infty \quad \text{for } T_a \text{ the hitting time of } V_a^c.$$

□

Remark 6.5. The fact that $P_t W$ is still a Lyapunov function is interesting by itself, but except in the ultra-bounded situation, it does not furnish new results. For instance if we want to get a Lyapunov function that belongs to all the $\mathbb{L}^p$ spaces, we have to assume that $P_.$ is immediately hyper bounded, which is stronger than log-Sobolev, while Theorem 5.4 gives the result under a simple F-Sobolev condition. Working a little bit more, one can extend the previous result to discrete valued Markov process, which are ultracontractive (birth-death processes,...). ♢

The previous proof indicates a way to prove "coming down from infinity" using Lyapunov functions, more precisely nested Lyapunov conditions

Definition 6.6.

(SLC) *We shall say that a Super-Lyapunov condition is satisfied if there exist a sequence* $W_k \geq 1$*, a sequence of increasing bounded sets* B_k *growing to* $\mathbb{R}^n$*, an increasing sequence* $\lambda_k > 0$ *growing to infinity and a sequence* $b_k > 0$ *such that*

$$LW_k \leq -\lambda_k W_k + b_k \mathbf{1}_{B_k}.$$

For instance in the situation of Theorem 4.5, (SLC) is satisfied with the same Lyapunov function W, i.e. $W = W_k$, similarly as what we have done (in the case of log-Sobolev) in the previous proof. Of course if all the W_k are bounded, (SLC) is equivalent to "coming down from infinity".

Assume that (SLC) is satisfied. Denote by T_k the hitting time of B_k and choose some $\delta > 0$. As we did in the proof of Theorem 5.4, for $k \geq k_0$ and $\lambda_{k_0} > \delta$ we have for $x \notin B_k$,

$$\begin{aligned}
\mathbb{E}_x\left(e^{\delta T_{k_0}}\right) &= \mathbb{E}_x\left(e^{\delta T_k}\, \mathbb{E}_{X_{T_k}}\left(e^{\delta T_{k_0}}\right)\right) \leq \mathbb{E}_x\left(e^{\delta T_k}\right) \sup_{y\in\partial B_k} \mathbb{E}_y\left(e^{\delta T_{k_0}}\right) \\
&\leq \left(\mathbb{E}_x\left(e^{\lambda_k T_k}\right)\right)^{\delta/\lambda_k} \sup_{y\in\partial B_k} \mathbb{E}_y\left(e^{\delta T_{k_0}}\right) \\
&\leq (W_k(x))^{\delta/\lambda_k} \sup_{y\in\partial B_k} \mathbb{E}_y\left(e^{\delta T_{k_0}}\right).
\end{aligned}$$

Define

$$w_k = \sup_{y\in B_{k+1}-B_k} W_k(y)\,.$$

Proceeding by induction we have for all $x \notin B_{k_0+1}$

$$\mathbb{E}_x\left(e^{\lambda T_{k_0}}\right) \leq C_{k_0} \prod_{k=k_0}^{+\infty} (w_k)^{\frac{\delta}{\lambda_k}}. \tag{6.7}$$

As we see, if the sequence λ_k goes to infinity, δ has no role in the convergence of the previous infinite product, as well as the value of k_0. We thus have

Theorem 6.8. *If* (SLC) *is satisfied, and for some* k_0*,*

$$\sum_{k=k_0}^{+\infty} \frac{\ln w_k}{\lambda_k} < +\infty\,,$$

the process comes down from infinity. Hence (SLC) *is satisfied with another sequence* (λ_k, W_k, B_k) *where all the* W_k *are bounded.*

Example 6.9. In one dimension consider (for $\beta > 0$) the potential

$$V_\beta(x) = (1+x^2)\,\ln^\beta(1+x^2)\,.$$

For $W(x) = e^{x^2/2}$ it holds for $|x|$ large enough,

$$L_\beta W(x) \leq -\,(1-\varepsilon)\,x^2\,\ln^\beta(1+x^2)\,W(x)\,.$$

Hence if $B_k = B(0,R_k)$ we have $\ln(w_k) \leq \frac{1}{2}\,R_k^2$, while for $|x| > R_k$, we may choose $\lambda_k = c\,R_k^2\,\ln^\beta(R_k)$. It follows that the process comes down from infinity for any $\beta > 0$ by choosing for instance $R_k = \exp k^{2/\beta}$ in Theorem 6.8.
It is known however that the semi-group is ultra-bounded if and only if $\beta > 1$ (see e.g [41]). Hence we have some examples of processes coming down from infinity for which the semi-group is not ultra-bounded (but is immediately hyper-contractive as shown in [41]).

7. Integrability and Lyapunov conditions.

In this section we will study the interplay between Lyapunov conditions and integrability conditions. So it is not a restriction to look at $L = \Delta - \nabla V.\nabla$ such that μ is symmetric (or reversible).
Let us recall that integrability conditions are related to transportation inequalities. Indeed, if μ satisfies a Gaussian integrability condition, i.e.

$$\exists \delta > 0, x_0 \qquad \int e^{\delta d(x,x_0)^2} d\mu < \infty$$

then

$$(T1) \quad \forall \nu \in \mathcal{P}_1, \qquad W_1(\nu,\mu) \leq \sqrt{2CH(\nu|\mu)}$$

for some explicit C (see [34]), where W_1 and H denote respectively the 1-Wasserstein distance and the relative entropy (or Kullback-Leibler information). The converse, $(T1)$ implies gaussian integrability is also true and due to K. Marton [51]. Gozlan [38] has generalized the approach to various type of $(T1)$ type transportation inequalities, getting that integrability property

$$\exists \delta > 0, x_0 \qquad \int e^{\alpha(\delta d(x,x_0))} d\mu < \infty$$

for α convex (quadratic near 0) is equivalent to $\alpha(W_1(\nu,\mu) \leq C\,H(\nu|\mu)$ for very ν. The links between functional inequalities and integrability properties are usually consequences of concentration properties of lipschitzian function (as Herbst argument for logarithmic Sobolev inequality). A more direct approach is taken in [22], where in particular an equivalence between a Poincaré inequality and a weak form of a $(T2)$ inequality (involving W_2) is shown. Note however that we won't require here some local inequality, so that we cannot hope to get better functional inequalities than $(T1)$ like.

We shall now investigate the relationship between the existence of Lyapunov functions and integrability properties. In the sequel we will thus assume the following

Definition 7.1. *Let ϕ be a (strictly) positive function. We say that $(\phi - Lyap)$ is satisfied if there exist $W > 0$, $b \geq 0$ and a bounded open subset C such that*

$$LW \leq -\,\phi^2\,W + b\,\mathbf{1}_C\,.$$

If we define $U = \ln(W)$ it is immediately seen that

$$LU \,+\, |\nabla U|^2 \,\leq\, -\phi^2 + \,\frac{b}{\min_{\bar{C}} W}\,\mathbf{1}_C\,. \tag{7.2}$$

Using this inequality it is shown in [28] lemma 3.4 that for any smooth h,

$$\int h^2\,\phi^2 d\mu \,\leq\, \mathcal{E}(h) \,+\, \frac{b}{\min_{\bar{C}} W}\int_C h^2\,d\mu\,. \tag{7.3}$$

Of course, if $\int_C h\,d\mu = 0$, and provided C is smooth enough (which is not a restriction), we may apply the Holley-Stroock perturbation argument and get

$$\int h^2\,\phi^2 d\mu \,\leq\, \left(1 + \frac{b}{\min_{\bar{C}} W}\,e^{Osc_{\bar{C}}V}\,C_P(C,dx)\right)\mathcal{E}(h)$$

where $Osc_U(V)$ denotes the oscillation of V on the subset U and $C_P(U,dx)$ the Poincaré constant of the uniform measure on U. If ϕ goes to infinity at infinity, the previous inequality is thus stronger than the Poincaré inequality (as expected). It is well known that the Poincaré inequality implies the exponential integrability of $c\,d(.,x_0)$ for some small enough positive c, as the logarithmic Sobolev inequality implies the exponential integrability of $c\,d^2(.,x_0)$ for some small enough positive c (we previously recalled this result).

In [47], Yuan Liu proved that $(\phi - Lyap)$ with $\phi(x) = a\,d(x,x_0)$ implies the exponential integrability of $c\,d^2(.,x_0)$ for $c < a$. This result is not surprising since in this case $(\phi - Lyap)$ is exactly $(HLS1)$ which is equivalent to the logarithmic-Sobolev inequality, at least if the curvature is bounded from below according to Theorem 1.2, hence implies as a consequence the quoted exponential integrability.
We shall here follow and generalize Liu's argument in order to answer the following question: what are sufficient conditions on ψ for $c\,\psi^2$ to be exponentially integrable when $(\phi - Lyap)$ is satisfied ?
A particularly interesting example is the case when $\phi(x) = a\,d^p(x,x_0)$. Indeed, for $p = 0$ we know that a Poincaré inequality is satisfied and for $p = 1$ a log-Sobolev inequality is satisfied under a curvature assumption. The use of curvature is very specific and strongly connected to $p = 1$. Hence for $0 < p < 1$ we do not know whether $(\phi - Lyap)$ implies some natural functional inequality or not, even for bounded from below curvature. We shall see however that $c\,d^{p+1}(.,x_0)$ is exponentially integrable for some small enough positive c and all $0 \leq p$, so that a generalized transportation inequality holds.

For $\psi \geq 0$, introduce

$$\beta_n := \int \psi^{2n} d\mu\,.$$

We will use (7.3) to initiate a recurrence on β_n using the notation $\bar{b} = \frac{b}{\min_{\bar{C}} W}$,

$$\begin{aligned}\beta_n &= \int \frac{\psi^{2n}}{\phi^2}\phi^2 d\mu \\ &\leq \int \left|\nabla\left(\frac{\psi^n}{\phi}\right)\right|^2 d\mu + \bar{b}\int_C \frac{\psi^{2n}}{\phi^2} d\mu.\end{aligned}$$

Let us focus on the first term

$$\begin{aligned}\int \left|\nabla\left(\frac{\psi^n}{\phi}\right)\right|^2 d\mu &= \int \left|n\psi^{n-1}\frac{\nabla\psi}{\phi} - \psi^n\frac{\nabla\phi}{\phi^2}\right|^2 d\mu \\ &= n^2\int \frac{\psi^2|\nabla\psi|^2}{\phi^2}\psi^{2(n-2)} + 2n\int \frac{\psi\nabla\phi.\nabla\psi}{\phi^3}\psi^{2(n-1)}d\mu + \int\frac{|\nabla\phi|^2}{\phi^4}\psi^{2n}d\mu.\end{aligned}$$

Let us assume then that there exists $\alpha,\beta,\gamma>0$ and $0<\delta<1$, (at least outside a compact K, and if so choose ψ to be 0 on K) such that

$$\frac{\psi^2|\nabla\psi|^2}{\phi^2}\le\alpha,\quad \left|\frac{\psi\nabla\phi.\nabla\psi}{\phi^3}\right|\le\beta,\quad \frac{|\nabla\phi|^2}{\phi^4}\le\delta,\quad \sup_C\frac{\psi^2}{\phi^2}\le\gamma\,. \tag{7.4}$$

Under these assumptions, we get that

$$\beta_n\le\frac{\alpha}{1-\delta}n^2\beta_{n-2}+\frac{2n\beta+\gamma\bar b}{1-\delta}\beta_{n-1}\,.$$

Combined with a direct consequence of Cauchy-Schwarz inequality, we obtain

$$\beta_n\le\sqrt{\beta_{n+1}\beta_{n-1}}\le\left(\frac{\alpha}{1-\delta}(n+1)^2\beta_{n-1}+\frac{2(n+1)\beta+\gamma\bar b}{1-\delta}\beta_n\right)^{\frac12}\beta_{n-1}^{\frac12}.$$

We then easily deduce that

$$\beta_n\le\frac12\left(\frac{2(n+1)\beta+\gamma\bar b}{1-\delta}+\sqrt{\left(\frac{2(n+1)\beta+\gamma\bar b}{1-\delta}\right)^2+4\frac{\alpha}{1-\delta}(n+1)^2}\right)\beta_{n-1}\le an\,\beta_{n-1}$$

for

$$a>\frac12\left(\frac{2\beta}{1-\delta}+\sqrt{\frac{4\beta^2}{(1-\delta)^2}+4\frac{\alpha}{1-\delta}}\right)$$

and n large enough. It then follows that for some c

$$\beta_n\le ca^n n!$$

and thus we have that for $a'<a^{-1}$

$$\int e^{a'\psi^2}d\mu\le\frac{c}{1-a'a}.$$

Now come back to (7.4). I f we assume that ϕ is bounded from below by some positive constant in C, the condition on γ is satisfied as soon as ϕ and ψ are, say, continuous. The condition on δ says that $1/\phi$ is a contraction outside some compact set. Assuming this condition we see that both conditions on α and β are the same, i.e. $|\nabla(\psi^2)|/\phi$ is bounded. Thus

Theorem 7.5. *Assume that ($\phi-Lyap$) is satisfied for some function ϕ such that ϕ is bounded from below by a positive constant on C and $1/\phi$ is η-Lipschitz for some $\eta<1$. Then for all function ψ^2 such that $|\nabla(\psi^2)|/\phi$ is bounded, there exists $a'>0$ such that $\int e^{a'\psi^2}d\mu<+\infty$.*

Example 7.6. (1) For a constant ϕ we recover the fact that any Lipschitz function has an exponential moment, hence the usual concentration result once a Poincaré inequality is satisfied. For $\phi(x)=ad^2(x,x_0)$, we recover the gaussian nature of the tails once a log-Sobolev inequality is satisfied.

(2) If we take $\phi(x) = ad^p(x, x_0)$ for $p \geq 0$ we obtain exponential integrability of functions g such that $|\nabla g| \leq C\,|x|^p$, hence for instance for $g(x) = d^{p+1}(x, x_0)$. As for the Poincaré or the log-Sobolev case, this exponential integrability is sharp, since for $\mu(dx) = Z_p\, e^{-|x|^{p+1}}$, $(\phi - Lyap)$ is satisfied with $\phi(x) = c|x|^p$ and $W(x) = e^{a|x|^p}$ for a small enough a.

(3) One may of course also consider $\phi^2(x) = ad^{-1}(x, x_0)$, if $a > 1$, which can be obtained for Cauchy type measures (see [21] for details). In this case we may take ψ^2 behaving as $\log(|x|)$ at infinity recovering polynomial integrability (also quite sharply). Note also that in [21, Th. 5.1], it is shown how a converse Poincaré inequality (obtained by a $(\phi - Lyap)$ condition and a local Poincaré inequality) plus an involved integrability condition implies a weak Poincaré inequality. This integrability condition can be checked using Th. 7.5 for example for Cauchy type measure (via tedious computations), so that only a Lyapunov condition and local inequality are also sufficient for weak Poincaré inequality.

Of course there is no converse statement for Theorem 7.5, since for instance exponential integrability for the distance cannot imply a Poincaré inequality (disconnected domains for example).

Finally recall that that $(\phi - Lyap)$ is equivalent to the following

$$\mathbb{E}_x\left(e^{\int_0^{T_C}\,\phi^2(X_s)\,ds}\right) < +\infty\,,$$

where in our situation

$$X_t = X_0 + \sqrt{2}\,W_t - \int_0^t \nabla V(X_s)\,ds\,.$$

It would be particularly interesting to show that weak Poincaré inequality implies back Lyapunov condition as done in section 2 for Poincaré inequality.

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