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A Bhatnagar–Gross–Krook approximation to scalar conservation laws with discontinuous flux

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We study the Bhatnagar–Gross–Krook (BGK) approximation to first-order scalar conservation laws with a flux which is discontinuous in the space variable. We show that the Cauchy problem for the BGK approximation is well posed and that, as the relaxation parameter tends to 0, it converges to the (entropy) solution of the limit problem.

1. Introduction

We consider the equation

$$\partial_t f^\varepsilon + \partial_x(k(x)a(\xi)f^\varepsilon) = \frac{\chi_{u^\varepsilon} - f^\varepsilon}{\varepsilon}, \quad t > 0,\ x \in \mathbb{R},\ \xi \in \mathbb{R}, \tag{1.1}$$

with the initial condition

$$f^\varepsilon|_{t=0} = f_0 \quad \text{in } \mathbb{R}_x \times \mathbb{R}_\xi. \tag{1.2}$$

Here k is given by

$$k = k_{\mathrm{L}}\mathbf{1}_{(-\infty,0)} + k_{\mathrm{R}}\mathbf{1}_{(0,+\infty)},$$

where $\mathbf{1}_B$ is the characteristic function of a set B, $\xi \mapsto a(\xi)$ is a continuous function on $\mathbb{R}$ such that, for all $u \in [0,1]$,

$$\int_0^u a(\xi)\,\mathrm{d}\xi \geqslant 0, \qquad \int_0^1 a(\xi)\,\mathrm{d}\xi = 0 \tag{1.3}$$

and, in (1.1), χ_{u^ε}, the so-called *equilibrium function* associated to f^ε is defined by

$$u^\varepsilon(t,x) = \int_{\mathbb{R}} f^\varepsilon(t,x,\xi)\,\mathrm{d}\xi, \qquad \chi_\alpha(\xi) = \mathbf{1}_{]0,\alpha[}(\xi) - \mathbf{1}_{]\alpha,0[}(\xi)$$

for $t > 0$, $x \in \mathbb{R}$, $\xi \in \mathbb{R}$ and $\alpha \in \mathbb{R}$.

Equation (1.1) is the so-called Bhatnagar–Gross–Krook (BGK) approximation to the scalar conservation law

$$\partial_t u + \partial_x(k(x)A(u)) = 0, \quad A(u) = \int_0^u a(\xi)\,\mathrm{d}\xi. \tag{1.4}$$

The flux $(x, u) \mapsto k(x)A(u)$ is discontinuous with respect to $x \in \mathbb{R}$; actually, (1.4) is a prototype of the scalar (first-order) conservation law with discontinuous flux function. In the last 10 years, scalar conservation laws with discontinuous flux functions have been studied extensively. We refer the reader to [6] for a comprehensive introduction to the subject and a complete list of references (see also [10]). Let us simply mention that the discontinuous character of the flux function gives rise to a multiplicity of weak solutions, even if traditional entropy conditions are imposed in the spatial domain away from the discontinuity. An additional criterion therefore has to be given in order to select solutions in a unique way. For the scalar conservation law in the general form $\partial_t u + \partial_x(B(x, u)) = 0$, where the function B is discontinuous with respect to x, several criteria are possible [1]. For $B(x, u) = k(x)A(u)$ as above, the choice of entropy solution is unambiguous [1, remark 4.4] and we consider here the selection criterion first given in [13]. A kinetic formulation (in the spirit of [9]) equivalent to the entropy formulation in [13] has been given in [3]. In particular, solutions given by this criterion are limits (almost everywhere (a.e.) and in L^1) of the solutions obtained by *monotone* regularization of the coefficient k in (1.4). For example,

$$k_\varepsilon(x) = k_{\mathrm{L}}\mathbf{1}_{x<-\varepsilon}(x) + \left(\frac{k_{\mathrm{R}} - k_{\mathrm{L}}}{2\varepsilon}x + \frac{k_{\mathrm{R}} + k_{\mathrm{L}}}{2}\right)\mathbf{1}_{-\varepsilon\leqslant x\leqslant\varepsilon} + k_{\mathrm{R}}\mathbf{1}_{\varepsilon<x}, \quad \varepsilon > 0.$$

The kinetic formulation of scalar conservation laws is well adapted to the analysis of the (Perthame–Tadmor) BGK approximation of scalar conservation laws. Developed in [12], this equation is a continuous version of the transport-collapse method of Brenier [4, 5]. BGK models have also been used for gas dynamics and the construction of numerical schemes. See, for example, [11] for a survey of this field.

Our purpose here is to apply the kinetic formulation of [3] to show the convergence of the BGK approximation. To this end we first study the BGK equation itself in § 2. In § 3 we introduce the kinetic formulation for the limit problem. We also introduce a notion of the generalized (kinetic) solution in definition 3.3. We show that any generalized solution reduces to a mere solution, i.e. a solution in the sense of definition 3.1. This theorem of 'reduction' is theorem 3.4. Then, in § 4 we show that the BGK model converges to a generalized solution of (1.4) and, using theorem 3.4, deduce the strong convergence of the BGK model to a solution of (1.4), theorem 4.1.

A key step in the whole proof of convergence is the result of the reduction of theorem 3.4. Its proof, given in § 3.2, is close to the proof of uniqueness of solutions given in [3]. A minor difference is that we deal here with generalized solutions instead of 'kinetic process solutions'. There is also a minor error in the proof given in [3] (specifically, the remainder terms $R_{\alpha,\varepsilon,\delta}$ and $Q_{\beta,\nu,\sigma}$ in equations (3.17) and (3.18) of the present paper are missing in [3]). We have therefore given a complete proof of theorem 3.4.

We end this introduction with two remarks.

REMARK 1.1. The BGK model provides an approximation of the entropy solutions to (1.4) by relaxation of the kinetic equation corresponding to (1.4). A relaxation scheme of the Jin–Xin-type applied directly to the original equation (1.4) has been developed in [8].

REMARK 1.2. The kinetic formulation of scalar conservation laws with discontinuous spatial dependence of the form $\partial_t u + \partial_x(B(x,u)) = 0$ (which are more general than (1.4)) is derived in the last chapter of [2]. We indicate (this would have to be proved rigorously) that, in the case where our approach via the BGK approximation was applied to this problem, the solutions obtained would be the type of entropy solutions considered in [7].

Notation. For $p,q \in [1,+\infty]$ we denote the space $L^p(\mathbb{R}_x; L^q(\mathbb{R}_\xi))$ by $L^p_x L^q_\xi$ and we denote the space $L^q(\mathbb{R}_\xi; L^p(\mathbb{R}_x))$ by $L^q_\xi L^p_x$.

We also set $\operatorname{sgn}_+(s) = \mathbf{1}_{\{s>0\}}$, $\operatorname{sgn}_-(s) = -\mathbf{1}_{\{s\leqslant 0\}}$ and $\operatorname{sgn} = \operatorname{sgn}_+ + \operatorname{sgn}_-$, $s \in \mathbb{R}$.

2. The BGK equation

2.1. The balance equation

By the change of variables $\tilde{f}^\varepsilon(t,x,\xi) = \mathrm{e}^{t/\varepsilon} f^\varepsilon(t,x,\xi)$, equation (1.1) can be rewritten as the balance equation

$$\partial_t \tilde{f}^\varepsilon + \partial_x(k(x)a(\xi)\tilde{f}^\varepsilon) = \frac{\mathrm{e}^{t/\varepsilon}}{\varepsilon}\chi_{u^\varepsilon}$$

with (unknown dependent) source term $(\mathrm{e}^{t/\varepsilon}/\varepsilon)\chi_{u^\varepsilon}$. Hence, we first consider the following Cauchy problem for the balance equation:

$$\partial_t f + \partial_x(k(x)a(\xi)f) = g, \quad t>0,\ x\in\mathbb{R},\ \xi\in\mathbb{R}, \tag{2.1}$$

$$f|_{t=0} = f_0 \quad \text{in } \mathbb{R}_x \times \mathbb{R}_\xi. \tag{2.2}$$

PROPOSITION 2.1. *Suppose that $k_{\mathrm{R}}k_{\mathrm{L}} > 0$. Then problem (2.1), (2.2) is well posed in $L^1_\xi L^p_x$, $1 \leqslant p < +\infty$: for all $f_0 \in L^1_\xi L^p_x$, $T>0$ and $g \in L^1(]0,T[; L^1_\xi L^p_x)$, there exists a unique $f \in C([0,T]; L^1_\xi L^p_x)$ solving (2.1) in $\mathcal{D}'(]0,T[\times\mathbb{R}_x\times\mathbb{R}_\xi)$ such that $f(0)=f_0$. Additionally, we have*

$$\|f(t)\|_{L^1_\xi L^p_x} \leqslant M_k\biggl(\|f_0\|_{L^1_\xi L^p_x} + \int_0^t \|g(s)\|_{L^1_\xi L^p_x}\,\mathrm{d}s\biggr), \tag{2.3}$$

where $M_k = \max(k_{\mathrm{L}}/k_{\mathrm{R}}, k_{\mathrm{R}}/k_{\mathrm{L}})$.

Proof. Since (2.1) is linear, it is sufficient to solve the case when $g=0$. The general case will follow from Duhamel's formula. Assume without loss of generality that $k_{\mathrm{R}}, k_{\mathrm{L}} > 0$. Let $A_+ := \{\xi\in\mathbb{R};\ a(\xi)>0\}$. Then, for fixed $\xi \in A_+$, and although k is a discontinuous function, the ordinary differential equation

$$\dot{X}(t,s,x,\xi) = k(X(t,s,x,\xi))a(\xi), \quad t\in\mathbb{R}, \tag{2.4}$$

with datum $X(s,s,x,\xi)=x$ has an obvious solution for $x\neq 0$, given by

$$X(t,s,x,\xi)=x+(t-s)k_{\mathrm{R}}a(\xi),\quad t>s,\ \text{when } x>0,$$

and by

$$X(t,s,x,\xi)=\begin{cases} x+(t-s)k_{\mathrm{L}}a(\xi) & \text{if } t<s+\dfrac{|x|}{k_{\mathrm{L}}a(\xi)},\\[2mm] \dfrac{k_{\mathrm{R}}}{k_{\mathrm{L}}}x+(t-s)k_{\mathrm{R}}a(\xi) & \text{if } t>s+\dfrac{|x|}{k_{\mathrm{L}}a(\xi)}, \end{cases}$$

when $x<0$. Denoting the positive and negative parts of $s\in\mathbb{R}$ by $s^+=\max(s,0)$ and $s^-=s^+-s$, respectively, and introducing

$$\alpha_k(x)=\mathbf{1}_{\{x>0\}}+\frac{k_{\mathrm{R}}}{k_{\mathrm{L}}}\mathbf{1}_{\{x<0\}},$$

this can be summed up as

$$X(t,s,x,\xi)=\{\alpha_k(x)x+(t-s)k_{\mathrm{R}}a(\xi)\}^+-\{x+(t-s)k_{\mathrm{L}}a(\xi)\}^-,\quad t>s. \tag{2.5}$$

Similarly, we have, for the resolution of (2.4) backward in time,

$$X(t,s,x,\xi)=\{x+(t-s)k_{\mathrm{R}}a(\xi)\}^+-\{\beta_k(x)x+(t-s)k_{\mathrm{L}}a(\xi)\}^-,\quad t<s, \tag{2.6}$$

where

$$\beta_k(x)=\frac{k_{\mathrm{L}}}{k_{\mathrm{R}}}\mathbf{1}_{\{x>0\}}+\mathbf{1}_{\{x<0\}}.$$

A similar computation in the case where $a(\xi)\leqslant 0$ gives the solution to (2.4) by (2.5) for $(t-s)a(\xi)\geqslant 0$, and by (2.6) for $(t-s)a(\xi)\leqslant 0$. For the transport equation $(\partial_t+k(x)a(\xi)\partial_x)\varphi^*=0$, interpreted as

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^*(t,X(t,s,x,\xi),\xi)=0,$$

this yields the solution

$$\varphi^*(t,x,\xi)=\psi(X(T,t,x,\xi),\xi),$$

which satisfies the terminal condition $\varphi^*(T)=\psi$. We suppose in what follows that ψ is independent of ξ, compactly supported and Lipschitz continuous. Then a simple change of variable shows that, for every $t\in[0,T]$, for almost every $\xi\in\mathbb{R}$,

$$\|\varphi^*(t,\cdot,\xi)\|_{L^q_x}\leqslant M_k\|\psi\|_{L^q_x},\quad M_k=\max\left(\frac{k_{\mathrm{L}}}{k_{\mathrm{R}}},\frac{k_{\mathrm{R}}}{k_{\mathrm{L}}}\right),\ 1\leqslant q\leqslant+\infty. \tag{2.7}$$

If $f\in C([0,T];L^1_\xi L^p_x)$ solves (2.1), (2.2), then, by duality (note that φ^* is Lipschitz continuous and compactly supported in x if ψ is) we have, for $t\in[0,T]$, for almost every $\xi\in\mathbb{R}$,

$$\int_{\mathbb{R}}f(T,x,\xi)\psi(x,\xi)\,\mathrm{d}x=\int_{\mathbb{R}}f_0(x,\xi)\varphi^*(0,x,\xi)\,\mathrm{d}x. \tag{2.8}$$

In particular, for almost every $\xi\in\mathbb{R}$, the estimate (2.7), where q is the conjugate exponent of p, gives

$$\|f(T,\cdot,\xi)\|_{L^p_x}\leqslant M_k\|f_0(\cdot,\xi)\|_{L^p_x},$$

and then by Duhamel's principle, for $g \neq 0$,

$$\|f(T,\cdot,\xi)\|_{L^p_x} \leqslant M_k\biggl(\|f_0(\cdot,\xi)\|_{L^p_x} + \int_0^T \|g(t,\cdot,\xi)\|_{L^p_x}\,\mathrm{d}t\biggr). \tag{2.9}$$

The estimate (2.3) and uniqueness of the solution to (2.1), (2.2) readily follows. Existence follows from (2.5), (2.6) and (2.8), from which one derives the explicit formula

$$f(t,x,\xi) = J(t,x,\xi)f_0(X(0,t,x,\xi),\xi),$$

the coefficient $J(t,x,\xi)$ being given by

$$J(t,x,\xi) = \mathbf{1}_{\{x<0\}\cup\{x>tk_{\mathrm{R}}a(\xi)\}} + \frac{k_{\mathrm{L}}}{k_{\mathrm{R}}}\mathbf{1}_{\{0<x<tk_{\mathrm{R}}a(\xi)\}}$$

if $a(\xi) > 0$ and

$$J(t,x,\xi) = \mathbf{1}_{\{x<tk_{\mathrm{L}}a(\xi)\}\cup\{x>0\}} + \frac{k_{\mathrm{R}}}{k_{\mathrm{L}}}\mathbf{1}_{\{tk_{\mathrm{L}}a(\xi)<x<0\}}$$

if $a(\xi) \leqslant 0$. □

2.2. The BGK equation

Denote by $\mathcal{T}(t)f_0$ the solution to (2.1), (2.2) with $g = 0$, i.e.

$$\mathcal{T}(t)f_0(x,\xi) = J(t,x,\xi)f_0(X(0,t,x,\xi),\xi),$$

with X given by (2.5), (2.6).

Definition 2.2. Let $f_0 \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$, $T > 0$. A function $f^\varepsilon \in C([0,T];L^1(\mathbb{R}_x \times \mathbb{R}_\xi))$ is said to be a solution to (1.1), (1.2) if

$$f^\varepsilon(t) = \mathrm{e}^{-t/\varepsilon}\mathcal{T}(t)f_0 + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-s/\varepsilon}\mathcal{T}(s)\chi_{u^\varepsilon(t-s)}\,\mathrm{d}s, \quad u^\varepsilon = \int_{\mathbb{R}} f^\varepsilon(\xi)\,\mathrm{d}\xi, \tag{2.10}$$

for all $t \in [0,T]$.

Theorem 2.3. *Assume that $k_{\mathrm{R}}\cdot k_{\mathrm{L}} > 0$. Let $f_0 \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$, $T > 0$. There exists a unique solution $f^\varepsilon \in C([0,T];L^1(\mathbb{R}_x \times \mathbb{R}_\xi))$ to (1.1), (1.2). Denoting this solution by $S_\varepsilon(t)f_0$, we have:*

(i) $\|(S_\varepsilon(t)f_0^\natural - S_\varepsilon(t)f_0^\flat)^+\|_{L^1(\mathbb{R}_x\times\mathbb{R}_\xi)} \leqslant M_k\|(f_0^\natural - f_0^\flat)^+\|_{L^1(\mathbb{R}_x\times\mathbb{R}_\xi)}$*;*

(ii) $0 \leqslant \operatorname{sgn}(\xi)f_0(x,\xi) \leqslant 1$ *a.e.* $\Rightarrow 0 \leqslant \operatorname{sgn}(\xi)S_\varepsilon(t)f_0(x,\xi) \leqslant 1$ *a.e.;*

(iii) *if $f_0 = \chi_{u_0}$, $u_0 \in L^\infty(\mathbb{R})$, $0 \leqslant u_0 \leqslant 1$ a.e., then $0 \leqslant S_\varepsilon(t)f_0 \leqslant \chi_1$.*

Proof. The change of variable $(t',x') = \varepsilon(t,x)$ reduces (1.1) to the same equation with $\varepsilon = 1$. We then have to solve $f = F(f)$ for

$$F(f)(t) := \mathrm{e}^{-t}\mathcal{T}(t)f_0 + \int_0^t \mathrm{e}^{-s}\mathcal{T}(s)\chi_{u(t-s)}\,\mathrm{d}s, \quad u = \int_{\mathbb{R}} f(\xi)\,\mathrm{d}\xi.$$

By (2.3) and the identity

$$\int_{\mathbb{R}} |\chi_u - \chi_v|(\xi)\,\mathrm{d}\xi = |u - v|, \quad u, v \in \mathbb{R},$$

we have $F\colon C([0,T]; L^1_{x,\xi}) \to C([0,T]; L^1_{x,\xi})$ and F is a $(1 - \mathrm{e}^{-T})$ contraction for the norm

$$\|f\| = \sup_{t\in[0,T]} \|f(t)\|_{L^1(\mathbb{R}_x\times\mathbb{R}_\xi)}.$$

Indeed, we compute

$$\begin{aligned}
\|F(f^\natural)(t) - F(f^\flat)(t)\|_{L^1_{x,\xi}} &\leqslant \int_0^t \mathrm{e}^{-s}\|\mathcal{T}(s)(\chi_{u^\natural(t-s)} - \chi_{u^\flat(t-s)})\|_{L^1_{x,\xi}}\,\mathrm{d}s \\
&= \int_0^t \mathrm{e}^{-s}\|\chi_{u^\natural(t-s)} - \chi_{u^\flat(t-s)}\|_{L^1_{x,\xi}}\,\mathrm{d}s \\
&= \int_0^t \mathrm{e}^{-s}\|u^\natural(t-s) - u^\flat(t-s)\|_{L^1_x}\,\mathrm{d}s \\
&\leqslant \int_0^t \mathrm{e}^{-s}\|f^\natural(t-s) - f^\flat(t-s)\|_{L^1_{x,\xi}}\,\mathrm{d}s \\
&\leqslant \int_0^t \mathrm{e}^{-s}\,\mathrm{d}s\|f^\natural - f^\flat\|.
\end{aligned}$$

By the Banach fixed-point theorem we obtain the existence and uniqueness of the solution to (1.1), (1.2). Since $0 \leqslant \operatorname{sgn}(\xi)\chi_u(\xi) \leqslant 1$ a.e., we have

$$0 \leqslant \operatorname{sgn}(\xi)F(f)(t,x,\xi) \leqslant 1 \text{ a.e.}$$

if $0 \leqslant \operatorname{sgn}(\xi)f_0(x,\xi) \leqslant 1$ a.e. This proves part (ii) of theorem 2.3. Part (i) follows from the following inequality:

$$\int_{\mathbb{R}} \operatorname{sgn}_+(f-g)(Q(f) - Q(g))\,\mathrm{d}\xi \leqslant 0, \quad f, g \in L^1(\mathbb{R}_\xi),\ Q(f) := \chi_{\int f\,\mathrm{d}\xi} - f,$$

which is easy to check, and from the identity

$$f(t) = \mathcal{T}(t)f_0 + \int_0^t \mathcal{T}(s)Q(f)(t-s)\,\mathrm{d}s$$

for the solution to (1.1), (1.2). If $f_0 = \chi_{u_0}$, $0 \leqslant u_0 \leqslant 1$ a.e., then $0 = \chi_0 \leqslant f_0 \leqslant \chi_1$. Hence, part (iii) of theorem 2.3 follows from part (i) and the fact that any constant equilibrium function χ_α, $\alpha \in \mathbb{R}$, is a solution to (1.1). □

3. The limit problem

Assume $f_0 = \chi_{u_0}$ with $u_0 \in L^\infty(\mathbb{R})$, $0 \leqslant u_0 \leqslant 1$ a.e. Set

$$A(u) = \int_0^u a(\xi)\mathbf{1}_{[0,1]}(\xi)\,\mathrm{d}\xi. \tag{3.1}$$

Note that by (1.3) we have $A \geqslant 0$ and A vanishes outside the interval $[0,1]$. We expect the solution f^{ε} to (1.1), (1.2) to converge to the solution u of the first-order scalar conservation law

$$\partial_t u + \partial_x(k(x)A(u)) = 0, \quad t > 0, \ x \in \mathbb{R}, \tag{3.2}$$

with initial datum

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}. \tag{3.3}$$

For a fixed $T > 0$, set $Q =]0,T[\,\times \mathbb{R}_x$.

Definition 3.1 (solution). Let $u_0 \in L^{\infty}(\mathbb{R})$, $0 \leqslant u_0 \leqslant 1$ a.e. A function $u \in L^{\infty}(Q)$ is said to be a (kinetic) solution to (3.2), (3.3) if there exist non-negative measures $m_{\pm}$ on $[0,T] \times \mathbb{R} \times \mathbb{R}$ such that

(i) m_+ is supported in $[0,T] \times \mathbb{R} \times]-\infty, 1]$, m_- is supported in $[0,T] \times \mathbb{R} \times [0,+\infty[$,

(ii) for all $\psi \in C_c^{\infty}([0,T[\,\times \mathbb{R} \times \mathbb{R})$,

$$\begin{aligned}\int_Q \int_{\mathbb{R}} h_{\pm}(\partial_t \psi + k(x)a(\xi)\partial_x \psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \int_{\mathbb{R}} h_{0,\pm}\psi(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ - (k_{\mathrm{L}} - k_{\mathrm{R}})^{\pm} \int_0^T \int_{\mathbb{R}} a(\xi)\psi(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t \\ = \int_Q \int_{\mathbb{R}} \partial_{\xi}\psi\,\mathrm{d}m_{\pm}(t,x,\xi),\end{aligned} \tag{3.4}$$

where $h_{\pm}(t,x,\xi) = \operatorname{sgn}_{\pm}(u(t,x) - \xi)$ and $h_{0,\pm}(x,\xi) = \operatorname{sgn}_{\pm}(u_0(x) - \xi)$.

Proposition 3.2 (bound in L^{∞}). *Let $u_0 \in L^{\infty}(\mathbb{R})$, $0 \leqslant u_0 \leqslant 1$ a.e. If $u \in L^{\infty}(Q)$ is a kinetic solution to (3.2), (3.3), then $0 \leqslant u \leqslant 1$ a.e.*

Proof. Consider the kinetic formulation (3.4) for h_+ with a test function

$$\psi(t,x,\xi) = \varphi(t,x)\mu(\xi).$$

If μ is supported in $]1,+\infty[$, two terms cancel as follows:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h_{0,+}\psi(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{1 \geqslant u_0(x) > \xi}\varphi(0,x)\mathbf{1}_{\xi > 1}\mu(\xi)\,\mathrm{d}\xi\,\mathrm{d}x = 0$$

and

$$\int_Q \int_{\mathbb{R}} \partial_{\xi}\psi\,\mathrm{d}m_+(t,x,\xi) = 0$$

by the hypothesis on the support of m_+. Hence, we have

$$\begin{aligned}\int_Q \int_{\mathbb{R}} h_+(\partial_t \varphi + k(x)a(\xi)\partial_x \varphi)\mu(\xi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\ - (k_{\mathrm{L}} - k_{\mathrm{R}})^{+} \int_0^T \int_{\mathbb{R}} a(\xi)\varphi(t,0)\mu(\xi)\,\mathrm{d}\xi\,\mathrm{d}t = 0.\end{aligned}$$

A step of approximation and regularization shows that we can take $\mu(\xi) = \mathbf{1}_{\xi>1}$ in this equation. Since

$$\int_1^{+\infty} a(\xi)\,\mathrm{d}\xi = A(+\infty) - A(1) = 0 - 0 = 0,$$

$$\int_1^{+\infty} h_+(t,x,\xi)\,\mathrm{d}\xi = \int_1^{+\infty} \mathbf{1}_{\xi<u(t,x)}\,\mathrm{d}\xi = (u(t,x)-1)^+,$$

$$\begin{aligned}\int_1^{+\infty} h_+(t,x,\xi)a(\xi)\,\mathrm{d}\xi &= \int_1^{+\infty} \mathbf{1}_{\xi<u(t,x)}a(\xi)\,\mathrm{d}\xi \\ &= \mathrm{sgn}_+(u(t,x)-1)\int_1^{u(t,x)} a(\xi)\,\mathrm{d}\xi \\ &= \mathrm{sgn}_+(u(t,x)-1)(A(u(t,x)) - A(1)),\end{aligned}$$

we obtain

$$\int_Q (u-1)^+\partial_t\varphi + k(x)\,\mathrm{sgn}_+(u-1)(A(u)-A(1))\partial_x\varphi\,\mathrm{d}t\,\mathrm{d}x = 0.$$

It is then classical to deduce that $(u-1)^+ = 0$ a.e. (see the end of the proof of proposition 3.8 after (3.25)); that is, $u \leqslant 1$ a.e. Similarly, we show $u \geqslant 0$ a.e. □

Our aim is to prove the uniqueness of the solution to (3.2), (3.3). Actually, more than mere uniqueness of the solution to (3.2), (3.3), we will show a result of reduction/uniqueness (see theorem 3.4) of a generalized kinetic solution. To this end, let us recall that a Young measure $Q \to \mathbb{R}$ is a measurable mapping $(t,x) \mapsto \nu_{t,x}$ from Q into the space of probability (Borel) measures on $\mathbb{R}$. The mapping is measurable in the sense that, for each Borel subset A of $\mathbb{R}$, $(t,x) \mapsto \nu_{t,x}(A)$ is measurable $Q \to \mathbb{R}$. Let us also introduce the following notation: if $f \in L^1(Q\times\mathbb{R})$, we set

$$f_\pm(y,\xi) = f(y,\xi) - \mathrm{sgn}_\mp(\xi), \quad y \in Q,\ \xi \in \mathbb{R}.$$

This is consistent with the notation used in definition 3.1 in the case where $f = \chi_u$.

Definition 3.3 (generalized solution). Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leqslant u_0 \leqslant 1$ a.e. A function $f \in L^1(Q \times \mathbb{R}_\xi)$ is said to be a generalized (kinetic) solution to (3.2), (3.3) if

$$0 \leqslant f \leqslant \chi_1 \text{ a.e.}, \ -\partial_\xi f_+ \text{ is a Young measure } Q \to \mathbb{R},$$

and if there exists non-negative measures $m_\pm$ on $[0,T]\times\mathbb{R}\times\mathbb{R}$ such that

(i) m_+ is supported in $[0,T]\times\mathbb{R}\times\,]-\infty,1]$, m_- is supported in $[0,T]\times\mathbb{R}\times[0,+\infty[$,

(ii) for all $\psi \in C_c^\infty([0,T[\,\times\mathbb{R}\times\mathbb{R})$,

$$\begin{aligned}\int_Q\int_{\mathbb{R}} f_\pm(\partial_t\psi + k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,\pm}\psi(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ - (k_{\mathrm{L}} - k_{\mathrm{R}})^\pm \int_0^T\int_{\mathbb{R}} a(\xi)\psi(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t \\ = \int_Q\int_{\mathbb{R}} \partial_\xi\psi\,\mathrm{d}m_\pm(t,x,\xi), \quad (3.5)\end{aligned}$$

where $f_{0,\pm}(x,\xi) = \mathrm{sgn}_\pm(u_0(x)-\xi)$.

THEOREM 3.4 (reduction, uniqueness). *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leqslant u_0 \leqslant 1$ a.e. Problem (3.2), (3.3) admits at most one solution. Additionally, any generalized solution is actually a solution: if $f \in L^1(Q \times \mathbb{R}_\xi)$ is a generalized solution to (3.2), (3.3), then there exists $u \in L^\infty(Q)$ such that $f = \chi_u$.*

To prepare the proof of theorem 3.4 we first have to analyse formulation (3.5) and the behaviour of f at $t = 0$ and $x = 0$.

3.1. Weak traces

Introduce the cut-off function

$$\omega_\varepsilon(s) = \int_0^{|s|} \rho_\varepsilon(r)\,\mathrm{d}r, \quad \rho_\varepsilon(s) = \varepsilon^{-1}\rho(\varepsilon^{-1}s), \quad s \in \mathbb{R}, \tag{3.6}$$

where $\rho \in C_{\mathrm{c}}^\infty(\mathbb{R})$ is a non-negative function with total mass 1 compactly supported in $]0,1[$. We have the following proposition.

PROPOSITION 3.5 (weak traces). *Let $f \in L^\infty(Q \times \mathbb{R}_\xi)$ be a generalized solution to (3.2) and (3.3). There exists $f_\pm^{\tau_0} \in L^2(\mathbb{R}\times\mathbb{R})$, $F_\pm \in L^2(\,]0,T[\times\mathbb{R})$ and a sequence $(\eta_n) \downarrow 0$ such that, for all $\varphi \in L_{\mathrm{c}}^2(\mathbb{R}\times\mathbb{R})$ and for all $\theta \in L_{\mathrm{c}}^2(\,]0,T[\times\mathbb{R})$ (the subscript* 'c' *denotes compact support),*

$$\int_Q \int_{\mathbb{R}} f_\pm(t,x,\xi)\omega'_{\eta_n}(t)\varphi(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \to \int_{\mathbb{R}}\int_{\mathbb{R}} f_\pm^{\tau_0}(x,\xi)\varphi(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x, \tag{3.7}$$

$$\int_Q \int_{\mathbb{R}} f_\pm(t,x,\xi)k(x)a(\xi)\omega'_{\eta_n}(x)\theta(t,\xi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \to \int_0^T \int_{\mathbb{R}} F_\pm(t,\xi)\theta(t,\xi)\,\mathrm{d}\xi\,\mathrm{d}t \tag{3.8}$$

as $n \to +\infty$. In addition, there exist non-negative measures $m_\pm^{\tau_0}$ and $\bar{m}_\pm$ on $\mathbb{R}^2$ and $[0,T]\times\mathbb{R}$, respectively, such that

(i) $m_+^{\tau_0}$ (and, respectively, $\bar{m}_+$) is supported in $\mathbb{R}\times\,]-\infty,1]$ (and, respectively, $[0,T]\times\,]-\infty,1\,]$), $m_-^{\tau_0}$ (respectively, $\bar{m}_-$) is supported in $\mathbb{R}\times[0,+\infty[$ (respectively, $[0,T]\times[0,+\infty[\,$),

(ii) for all $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^2)$, $\theta \in C_{\mathrm{c}}^\infty([0,T[\times\mathbb{R})$,

$$\int_{\mathbb{R}^2} f_\pm^{\tau_0}\varphi\,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^2} f_{0,\pm}\varphi\,\mathrm{d}x\,\mathrm{d}\xi - \int_{\mathbb{R}^2} \partial_\xi\varphi\,\mathrm{d}m_\pm^{\tau_0}(x,\xi), \tag{3.9}$$

$$\int_0^T \int_{\mathbb{R}} F_\pm\theta\,\mathrm{d}\xi\,\mathrm{d}t = -(k_{\mathrm{L}} - k_{\mathrm{R}})^\pm \int_0^T \int_{\mathbb{R}} a(\xi)\theta\,\mathrm{d}\xi\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}} \partial_\xi\theta\,\mathrm{d}\bar{m}_\pm(t,\xi). \tag{3.10}$$

Proof. The first part of the proposition does not use the fact that f is a solution. Indeed, since $|f_\pm| \leqslant 2$, we have

$$\left|\int_0^T f_\pm(t,x,\xi)\omega'_\eta(t)\,\mathrm{d}t\right| \leqslant 2\int_0^T |\omega'_\eta(t)|\,\mathrm{d}t = 2\int_0^T \rho_\eta(t)\,\mathrm{d}t \leqslant 2$$

for all $(x,\xi)\in\mathbb{R}^2$. This gives, in particular, a bound in $L^2(K)$, with K a compact solution of $\mathbb{R}^2$ on

$$\int_0^T f_\pm(t,\cdot)\omega'_\eta(t)\,\mathrm{d}t,$$

hence the existence of a subsequence that converges weakly in $L^2(K)$. Writing $\mathbb{R}^2$ as an increasing countable union of compact sets and using a diagonal process, we obtain (3.7). The proof of (3.8) is similar. To obtain (3.9), apply formulation (3.5) to $\psi(t,x,\xi)=\varphi(x,\xi)(1-\omega_{\eta_n}(t))$. We obtain (3.9) by using (3.7) and setting

$$\int_{\mathbb{R}^2}\varphi\,\mathrm{d}m_\pm^{\tau_0}(x,\xi)=\lim_{n\to+\infty}\int_Q\int_{\mathbb{R}}\varphi(x,\xi)(1-\omega_{\eta_n}(t))\,\mathrm{d}m_\pm(t,x,\xi)$$

for all non-negative $\varphi\in C_{\mathrm{c}}(\mathbb{R}^2)$: the limit is well defined since the argument is monotone in n and it defines a non-negative functional on $C_{\mathrm{c}}(\mathbb{R}^2)$ which is represented by a non-negative Radon measure. Similarly, applying formulation (3.5) to $\psi(t,x,\xi)=\theta(t,\xi)(1-\omega_{\eta_n}(x))$, we obtain (3.10) with

$$\int_0^T\int_{\mathbb{R}}\theta\,\mathrm{d}\bar{m}_\pm(t,\xi)=\lim_{n\to+\infty}\int_Q\int_{\mathbb{R}}\theta(t,\xi)(1-\omega_{\eta_n}(x))\,\mathrm{d}m_\pm(t,x,\xi)$$

for all non-negative $\theta\in C_{\mathrm{c}}([0,T]\times\mathbb{R})$. □

REMARK 3.6. Since $0\leqslant f\leqslant\chi_1$, (3.7) shows that $f_+^{\tau_0}$ (respectively, $f_-^{\tau_0}$) is supported in $\mathbb{R}\times\,]-\infty,1]$ (respectively, $\mathbb{R}\times[0,+\infty[\,$). Similarly, F_+ (respectively, F_-) is supported in $[0,T]\times\,]-\infty,1]$ (respectively, $[0,T]\times[0,+\infty[\,$). We use this remark to show the following corollary.

COROLLARY 3.7. *For all $\varphi_-\in L^\infty(\mathbb{R}^2)$ supported in $[-R,R]\times[-R,+\infty[\,$, $R>0$, such that $\partial_\xi\varphi_-\leqslant 0$ (in the sense of distributions), we have*

$$\lim_{n\to+\infty}\int_Q\int_{\mathbb{R}}f_+\omega'_{\eta_n}(t)\varphi_-(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x\geqslant\int_{\mathbb{R}^2}f_{0,+}\varphi_-\,\mathrm{d}x\,\mathrm{d}\xi. \tag{3.11}$$

For all $\theta_-\in L^\infty(\,]0,T[\,\times\mathbb{R})$ supported in $[0,T]\times[-R,+\infty[\,$, $R>0$, such that $\partial_\xi\theta_-\leqslant 0$ (in the sense of distributions), we have

$$\lim_{n\to+\infty}\int_Q\int_{\mathbb{R}}f_+k(x)a(\xi)\omega'_{\eta_n}(x)\theta_-(t,\xi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x\geqslant-(k_{\mathrm{L}}-k_{\mathrm{R}})^+\int_0^T\int_{\mathbb{R}}a(\xi)\theta_-\,\mathrm{d}\xi\,\mathrm{d}t. \tag{3.12}$$

Proof. Note first that each term in (3.11) is well defined by the remark above and that, by (3.7),

$$\lim_{n\to+\infty}\int_Q\int_{\mathbb{R}}f_+(t,x,\xi)\omega'_{\eta_n}(t)\varphi_-(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x=\int_{\mathbb{R}}\int_{\mathbb{R}}f_+^{\tau_0}\varphi_-\,\mathrm{d}\xi\,\mathrm{d}x.$$

By regularization (parameter ε) and truncation (parameter M), we have

$$\int_{\mathbb{R}^2}(f_+^{\tau_0}-f_{0,+})\varphi_-\,\mathrm{d}x\,\mathrm{d}\xi=\int_{\mathbb{R}^2}(f_+^{\tau_0}-f_{0,+})\varphi_-^{\varepsilon,M}\,\mathrm{d}x\,\mathrm{d}\xi+\eta(\varepsilon,M),$$

where $\lim_{\varepsilon\to 0,\, M\to+\infty}\eta(\varepsilon,M)=0$. More precisely, we set

$$\varphi_-^{\varepsilon,M}=(\varphi_- * \psi_\varepsilon)\times\chi_M,$$

where ψ_ε is a (smooth, compactly supported) approximation of the unit on $\mathbb{R}^2$ and χ_M is a smooth, non-increasing function such that $\chi_M\equiv 1$ on $]-\infty,M]$, $\chi_M\equiv 0$ on $[M+1,+\infty[$. Apply (3.9) to $\varphi_-^{\varepsilon,M}$ to obtain

$$\int_{\mathbb{R}^2}(f_+^{\tau_0}-f_{0,+})\varphi_-\,\mathrm{d}x\,\mathrm{d}\xi=-\int_{\mathbb{R}^2}\partial_\xi\varphi_-^{\varepsilon,M}\,\mathrm{d}m_+^{\tau_0}(x,\xi)+\eta(\varepsilon,M).$$

For $M>R+1$ and $\varepsilon<1$, we have $\varphi_-^{\varepsilon,M}=\varphi_- * \psi_\varepsilon$; hence $\partial_\xi\varphi_-^{\varepsilon,M}\leqslant 0$. It follows that

$$\int_{\mathbb{R}^2}(f_+^{\tau_0}-f_{0,+})\varphi_-\,\mathrm{d}x\,\mathrm{d}\xi\geqslant\eta(\varepsilon,M)$$

for $M>R+1$, $\varepsilon<1$. At the limit $M\to+\infty$, $\varepsilon\to 0$, we obtain (3.11). The proof of (3.12) is similar. □

3.2. Proof of theorem 3.4

Our aim is to show the following.

Proposition 3.8. *Let $u_0,v_0\in L^\infty(\mathbb{R})$, $0\leqslant u_0,v_0\leqslant 1$ a.e., and let f (respectively, g) be a generalized solution to (3.2), (3.3) with datum u_0 (respectively, v_0). Let $M=\sup_{x\in\mathbb{R},\,\xi\in[0,1]}|k(x)a(\xi)|$. Then we have, for $R>0$,*

$$\frac{1}{T}\int_0^T\int_{\{|x|<R\}}\int_{\mathbb{R}}-f_+g_-\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\leqslant\int_{\{|x|<R+MT\}}(u_0-v_0)^+\,\mathrm{d}x. \tag{3.13}$$

Remark 3.9. In the case where $f=\chi_u$, $g=\chi_v$, we have

$$\int_{\mathbb{R}}-f_+g_-\,\mathrm{d}\xi=(u-v)^+,$$

and hence (3.13) gives the uniqueness of the solution to (3.2), (3.3). More precisely, it gives the L^1-contraction with averaging in time and the comparison result

$$u_0\leqslant v_0\text{ a.e. }\iff u\leqslant v\text{ a.e.}$$

Remark 3.10. To obtain the second part of theorem 3.4, we apply (3.13) with $g=f$ to obtain

$$\int_0^T\int_{\{|x|<R\}}\int_{\mathbb{R}}-f_+f_-\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t\leqslant 0. \tag{3.14}$$

Since $0\leqslant f\leqslant\chi_1$, we have $f_+\geqslant 0$ a.e. and $f_-\leqslant 0$ a.e. We deduce from (3.14) that $f_+f_-=0$ a.e. Let $\nu_{t,x}$ denote the Young measure $-\partial_\xi f_+$. We have $\partial_\xi f_-=\partial_\xi f-\delta_0=\partial_\xi f_+$ and, by examination of the values at $\xi=\pm\infty$ of $f_\pm$, for almost every $(t,x)\in Q$,

$$f_+(t,x,\xi)=\nu_{t,x}(\xi,+\infty),\qquad f_-(t,x,\xi)=-\nu_{t,x}(-\infty,\xi).$$

But then the relation $f_+f_-=0$ implies that $\nu_{t,x}$ is a Dirac mass at, say, $u(t,x)$. By measurability of ν, u is measurable and $f=\chi_u$.

Proof of proposition 3.8. Since f_+ and g_- satisfy

$$\int_Q \int_{\mathbb{R}} f_+(\partial_t\psi + k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,+}\psi(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ - (k_{\mathrm{L}} - k_{\mathrm{R}})^+ \int_0^T \int_{\mathbb{R}} a(\xi)\psi(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t \\ = \int_Q \int_{\mathbb{R}} \partial_\xi\psi\,\mathrm{d}m_+(t,x,\xi) \tag{3.15}$$

and

$$\int_Q \int_{\mathbb{R}} g_-(\partial_t\psi + k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} g_{0,-}\psi(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ - (k_{\mathrm{L}} - k_{\mathrm{R}})^- \int_0^T \int_{\mathbb{R}} a(\xi)\psi(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t \\ = \int_Q \int_{\mathbb{R}} \partial_\xi\psi\,\mathrm{d}p_-(t,x,\xi) \tag{3.16}$$

for all $\psi \in C_{\mathrm{c}}^\infty([0,T[\times\mathbb{R}\times\mathbb{R})$ (here, $g_{0,-} = \mathrm{sgn}_-(v_0 - \xi)$ and p_- is a non-negative measure on $[0,T]\times\mathbb{R}\times\mathbb{R}$ supported in $[0,T]\times\mathbb{R}\times[0,+\infty[$). It is possible to obtain an estimate for $-f_+g_-$ by setting $\psi = -g_-\varphi$ in (3.15) and $\psi = f_+\varphi$ in (3.16) (φ being a given test function) and adding the result. First, however, this requires a regularization step.

Step 1 (regularization). Let $\rho_{\alpha,\varepsilon,\delta}$ denote the approximation of the unit on $\mathbb{R}^3$ given by

$$\rho_{\alpha,\varepsilon,\delta}(t,x,\xi) = \rho_\alpha(t)\rho_\varepsilon(x)\rho_\delta(\xi), \quad (t,x,\xi)\in\mathbb{R}^3,$$

where ρ_ε is defined as in (3.6). Let $\psi \in C_{\mathrm{c}}^\infty([0,T[\times\mathbb{R}\times\mathbb{R})$ be compactly supported in $]0,T[\times\mathbb{R}\backslash\{0\}\times\mathbb{R}$. Use $\psi*\rho_{\alpha,\varepsilon,\delta}$ as a test function in (3.15) and Fubini's theorem to obtain

$$\int_Q \int_{\mathbb{R}} f_+^{\alpha,\varepsilon,\delta}(\partial_t\psi + k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\ + \int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,+}\psi*\rho_{\alpha,\varepsilon,\delta}(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ - (k_{\mathrm{L}} - k_{\mathrm{R}})^+ \int_0^T \int_{\mathbb{R}} a(\xi)\psi*\rho_{\alpha,\varepsilon,\delta}(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t \\ = \int_Q \int_{\mathbb{R}} \partial_\xi\psi\,\mathrm{d}m_+^{\alpha,\varepsilon,\delta}(t,x,\xi) + R_{\alpha,\varepsilon,\delta}(\psi),$$

where $f_+^{\alpha,\varepsilon,\delta} := f_+ * \check{\rho}_{\alpha,\varepsilon,\delta}$, $m_+^{\alpha,\varepsilon,\delta} := m_+ * \check{\rho}_{\alpha,\varepsilon,\delta}$ and

$$R_{\alpha,\varepsilon,\delta}(\psi) = \int_Q \int_{\mathbb{R}} f_+[k(x)a(\xi)(\partial_x\psi)*\rho_{\alpha,\varepsilon,\delta} - (k(x)a(\xi)\partial_x\psi)*\rho_{\alpha,\varepsilon,\delta}]\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x.$$

Here we have defined $\check{\rho}(t,x,\xi)=\rho(-t,-x,-\xi)$. Also observe that, implicitly, we have extended f_+ by 0 outside $[0,T]$ since, for example,

$$\begin{aligned}\int_0^T f_+(t)\psi*\rho_\alpha(t)\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}} f_+(t)\psi(s)\rho_\alpha(t-s)\,\mathrm{d}s\,\mathrm{d}t\\ &= \int_{\mathbb{R}}\psi(s)\int_0^T f_+(t)\check{\rho}_\alpha(s-t)\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

Since ψ is supported in $]0,T[\times\mathbb{R}\setminus\{0\}\times\mathbb{R}$ we have, for sufficiently small α,ε,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,+}\psi*\rho_{\alpha,\varepsilon,\delta}(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x=0,$$

$$\int_0^T\int_{\mathbb{R}} a(\xi)\psi*\rho_{\alpha,\varepsilon,\delta}(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t=0$$

and

$$R_{\alpha,\varepsilon,\delta}(\psi)=\int_Q\int_{\mathbb{R}} f_+k(x)[a(\xi)(\partial_x\psi)*\rho_{\alpha,\varepsilon,\delta}-(a(\xi)\partial_x\psi)*\rho_{\alpha,\varepsilon,\delta}]\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x.$$

We deduce

$$\int_Q\int_{\mathbb{R}} f_+^{\alpha,\varepsilon,\delta}(\partial_t\psi+k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x = \int_Q\int_{\mathbb{R}}\partial_\xi\psi\,\mathrm{d}m_+^{\alpha,\varepsilon,\delta}(t,x,\xi)+R_{\alpha,\varepsilon,\delta}(\psi). \tag{3.17}$$

A similar procedure on g_- gives

$$\int_Q\int_{\mathbb{R}} g_-^{\beta,\nu,\sigma}(\partial_t\psi+k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x = \int_Q\int_{\mathbb{R}}\partial_\xi\psi\,\mathrm{d}p_-^{\beta,\nu,\sigma}(t,x,\xi)+Q_{\beta,\nu,\sigma}(\psi), \tag{3.18}$$

where

$$Q_{\beta,\nu,\sigma}(\psi)=\int_Q\int_{\mathbb{R}} g_-k(x)[a(\xi)(\partial_x\psi)*\rho_{\beta,\nu,\sigma}-(a(\xi)\partial_x\psi)*\rho_{\beta,\nu,\sigma}]\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x.$$

Step 2 (equation for $-f_+^{\alpha,\varepsilon,\delta}g_-^{\beta,\nu,\sigma}$). Let $\varphi\in C_c^\infty([0,T[\times\mathbb{R})$ be non-negative and compactly supported in $]0,T[\times\mathbb{R}\setminus\{0\}$. Note that φ does not depend on ξ. Set $\psi=-\varphi g_-^{\beta,\nu,\sigma}$ in (3.17) and $\psi=-\varphi f_+^{\alpha,\varepsilon,\delta}$ in (3.18). Since

$$f\partial_t(\varphi g)+g\partial_t(\varphi f)=fg\partial_t\varphi+\partial_t(\varphi fg),$$

we obtain, by addition of the resulting equations,

$$\begin{aligned}&\int_Q\int_{\mathbb{R}} -f_+^{\alpha,\varepsilon,\delta}g_-^{\beta,\nu,\sigma}(\partial_t\varphi+k(x)a(\xi)\partial_x\varphi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x\\ &\quad= -\int_Q\varphi\int_{\mathbb{R}}\partial_\xi f_+^{\alpha,\varepsilon,\delta}\,\mathrm{d}p_-^{\beta,\nu,\sigma}(t,x,\xi)+\partial_\xi g_-^{\beta,\nu,\sigma}\,\mathrm{d}m_+^{\alpha,\varepsilon,\delta}(t,x,\xi)\\ &\qquad+R_{\alpha,\varepsilon,\delta}(-\varphi g_-^{\beta,\nu,\sigma})+Q_{\beta,\nu,\sigma}(-\varphi f_+^{\alpha,\varepsilon,\delta}).\end{aligned}$$

Note that the term

$$-\int_Q \varphi \int_{\mathbb{R}} \partial_\xi f_+^{\alpha,\varepsilon,\delta}\,\mathrm{d}p_-^{\beta,\nu,\sigma}(t,x,\xi) + \partial_\xi g_-^{\beta,\nu,\sigma}\,\mathrm{d}m_+^{\alpha,\varepsilon,\delta}(t,x,\xi)$$

is well defined since the intersection of the supports of the functions $f_+^{\alpha,\varepsilon,\delta}$ and $p_-^{\beta,\nu,\sigma}$ (respectively, $f_-^{\beta,\nu,\sigma}$ and $m_+^{\alpha,\varepsilon,\delta}$) is compact. Actually, this term is non-negative since $p_-^{\beta,\nu,\sigma}, m_+^{\alpha,\varepsilon,\delta} \geqslant 0$ and $\partial_\xi f_+^{\alpha,\varepsilon,\delta}, \partial_\xi g_-^{\beta,\nu,\sigma} \leqslant 0$. We thus have

$$\int_Q \int_{\mathbb{R}} -f_+^{\alpha,\varepsilon,\delta} g_-^{\beta,\nu,\sigma}(\partial_t\varphi + k(x)a(\xi)\partial_x\varphi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \geqslant R_{\alpha,\varepsilon,\delta}(-\varphi g_-^{\beta,\nu,\sigma}) + Q_{\beta,\nu,\sigma}(-\varphi f_+^{\alpha,\varepsilon,\delta}). \tag{3.19}$$

It is easily checked that

$$R_{\alpha,\varepsilon,\delta}(-\varphi j_-^{\beta,\nu,\sigma}) = \mathcal{O}(\nu^{-1}\delta), \qquad Q_{\beta,\nu,\sigma}(-\varphi h_+^{\alpha,\varepsilon,\delta}) = \mathcal{O}(\varepsilon^{-1}\sigma);$$

hence,

$$\lim_{\delta,\sigma\to 0} R_{\alpha,\varepsilon,\delta}(-\varphi g_-^{\beta,\nu,\sigma}) + Q_{\beta,\nu,\sigma}(-\varphi f_+^{\alpha,\varepsilon,\delta}) = 0.$$

At the limit $\delta,\sigma \to 0$ in (3.19), we conclude that

$$\int_Q \int_{\mathbb{R}} -f_+^{\alpha,\varepsilon} g_-^{\beta,\nu}(\partial_t\varphi + k(x)a(\xi)\partial_x\varphi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \geqslant 0. \tag{3.20}$$

Step 3 (traces). Suppose that $k_{\mathrm{L}} < k_{\mathrm{R}}$. We then pass to the limit $\varepsilon,\alpha \to 0$ in (3.20) to obtain

$$\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}(\partial_t\varphi + k(x)a(\xi)\partial_x\varphi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \geqslant 0. \tag{3.21}$$

Note that in the opposite case $k_{\mathrm{L}} > k_{\mathrm{R}}$, and with our method of proof we would *first* pass to the limit on β,ν. Let us now remove the hypothesis that φ vanishes at $t=0$: suppose that $\psi \in C_{\mathrm{c}}^\infty([0,T[\times\mathbb{R})$ is non-negative and supported in $[0,T[\times\mathbb{R}\setminus\{0\}$ and apply (3.21) to $\varphi(t,x) = \psi(t,x)\omega_{\eta_n}(t)$. We have

$$\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}\omega_{\eta_n}(t)(\partial_t\psi + k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}\psi(t,x)\omega'_{\eta_n}(t)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \geqslant 0. \tag{3.22}$$

By (3.11) applied with $\varphi_-(x,\xi) = g_-^{\beta,\nu}(0,x,\xi)\psi(0,x)$, we obtain

$$\lim_{n\to+\infty} \int_Q \int_{\mathbb{R}} f_+ g_-^{\beta,\nu}(0,x,\xi)\psi(0,x)\omega'_{\eta_n}(t)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \geqslant \int_{\mathbb{R}} \int_{\mathbb{R}} f_{0,+} g_-^{\beta,\nu}(0,x,\xi)\psi(0,x)\,\mathrm{d}\xi\,\mathrm{d}x.$$

Now $f_+(t,x,\xi)g_-^{\beta,\nu}(t,x,\xi)\psi(t,x)$ has a compact support, say in

$$[0,T]\times[-R,R]\times[-R,R],$$

and thus $\varphi_-(t,x,\xi) = g_-^{\beta,\nu}(t,x,\xi)\psi(t,x)$ is uniformly continuous on this compact support. Therefore, for $\mu > 0$, there exists $\gamma > 0$ such that

$$|\varphi_-(t,x,\xi) - \varphi_-(0,x,\xi)| \leqslant \frac{\mu}{8R^2}$$

for any $0 \leqslant t < \gamma$ and any $x, \xi \in [-R, R]$, and then for large n we have $\eta_n < \gamma$ and

$$\begin{aligned}\bigg|\int_Q \int_{\mathbb{R}} f_+(t,x,\xi)(g_-^{\beta,\nu}(t,x,\xi)\psi(t,x) - g_-^{\beta,\nu}(0,x,\xi)\psi(0,x))\omega'_{\eta_n}(t)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x\bigg| \\ \leqslant \int_Q \int_{\mathbb{R}} |f_+(t,x,\xi)|\rho_{\eta_n}(t)\frac{\mu}{8R^2}\mathbf{1}_{(x,\xi)\in[-R,R]^2}\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\ \leqslant \mu \int \rho_{\eta_n}(t)\,\mathrm{d}t = \mu.\end{aligned} \tag{3.23}$$

Thus, we obtain, at the limit $n \to +\infty$ in (3.22),

$$\begin{aligned}\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}(\partial_t\psi + k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\ + \int_{\mathbb{R}} \int_{\mathbb{R}} -f_{0,+} g_-^{\beta,\nu}(0,x,\xi)\psi(0,x)\,\mathrm{d}\xi\,\mathrm{d}x \geqslant 0.\end{aligned}$$

The next step is to remove the hypothesis that ψ vanishes at $x = 0$ by setting $\psi(t,x) = \theta(t,x)\omega_{\eta_n}(x)$, where $\theta \in C_{\mathrm{c}}^\infty([0,T[\times\mathbb{R})$ is a non-negative test function. We have

$$\begin{aligned}\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}\omega_{\eta_n}(x)(\partial_t\theta + k(x)a(\xi)\partial_x\theta)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\ + \int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}\theta(t,x)k(x)a(\xi)\omega'_{\eta_n}(x)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\ + \int_{\mathbb{R}} \int_{\mathbb{R}} -f_{0,+} g_-^{\beta,\nu}(0,x,\xi)\theta(0,x)\omega_{\eta_n}(x)\,\mathrm{d}\xi\,\mathrm{d}x \geqslant 0.\end{aligned}$$

By (3.12) with $\theta_-(t,\xi) = g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)$,

$$\begin{aligned}\lim_{n\to+\infty} \int_Q \int_{\mathbb{R}} f_+ k(x)a(\xi)\omega'_{\eta_n}(x)g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\ \geqslant -(k_{\mathrm{L}} - k_{\mathrm{R}})^+ \int_0^T \int_{\mathbb{R}} a(\xi)g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)\,\mathrm{d}\xi\,\mathrm{d}t,\end{aligned}$$

and, by an argument similar to (3.23), the limit as $[n \to +\infty]$ of the term

$$\int_Q \int_{\mathbb{R}} f_+ k(x)a(\xi)\omega'_{\eta_n}(x)(g_-^{\beta,\nu}(t,x,\xi)\theta(t,x) - g_-^{\beta,\nu}(t,0,\xi)\theta(t,0))\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x$$

is zero. We therefore have

$$\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}(\partial_t\theta + k(x)a(\xi)\partial_x\theta)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + (k_\mathrm{L}-k_\mathrm{R})^+ \int_0^T \int_{\mathbb{R}} a(\xi) g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)\,\mathrm{d}\xi\,\mathrm{d}t + \int_{\mathbb{R}} \int_{\mathbb{R}} -f_{0,+} g_-^{\beta,\nu}(0,x,\xi)\theta(0,x)\,\mathrm{d}\xi\,\mathrm{d}x \geqslant 0.$$

Since $(k_\mathrm{L}-k_\mathrm{R})^+ = 0$, we actually have

$$\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}(\partial_t\theta + k(x)a(\xi)\partial_x\theta)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \int_{\mathbb{R}} -f_{0,+} g_-^{\beta,\nu}(0,x,\xi)\theta(0,x)\,\mathrm{d}\xi\,\mathrm{d}x \geqslant 0.$$

Take $\beta = \eta_n$ where (η_n) is given in proposition 3.5. First at the limit $\nu \to 0$, then at $n \to +\infty$, we obtain

$$\int_Q \int_{\mathbb{R}} -f_+ g_-(\partial_t\theta + k(x)a(\xi)\partial_x\theta)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \limsup_{n\to+\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} -f_{0,+} g_-^{\eta_n}(0,x,\xi)\theta(0,x)\,\mathrm{d}\xi\,\mathrm{d}x \geqslant 0. \tag{3.24}$$

Observe that

$$\begin{aligned} g_-^{\eta_n}(0,x,\xi) &= \int_0^T g_-(t,x,\xi)\rho_{\eta_n}(t)\,\mathrm{d}t \\ &= \int_0^T g_-(t,x,\xi)\omega'_{\eta_n}(t)\,\mathrm{d}t. \end{aligned}$$

By (3.11) (transposed to g_- tested against a function φ_+), we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} -f_{0,+} g_-^{\eta_n}(0,x,\xi)\theta(0,x)\,\mathrm{d}\xi\,\mathrm{d}x \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} -f_{0,+} g_{0,-}\theta(0,x)\,\mathrm{d}\xi\,\mathrm{d}x.$$

Since

$$\int_{\mathbb{R}} -f_{0,+} g_{0,-}\,\mathrm{d}\xi = \int_{\mathbb{R}} -\operatorname{sgn}_+(u_0-\xi)\operatorname{sgn}_-(v_0-\xi)\,\mathrm{d}\xi = (u_0-v_0)^+,$$

we obtain, by (3.24),

$$\int_Q \int_{\mathbb{R}} -f_+ g_-(\partial_t\theta + k(x)a(\xi)\partial_x\theta)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} (u_0-v_0)^+\theta(0,x)\,\mathrm{d}x \geqslant 0. \tag{3.25}$$

It is then classical to conclude to (3.13): let $M > 0$, $R > MT$, let $\eta > 0$ and let r be a non-negative, non-increasing function such that $r \equiv 1$ on $[0,R]$, $r \equiv 0$ on $[R+\eta, +\infty[$. Set $\theta(t,x) = (T-t)r(|x|+Mt)/T$ in (3.25) to obtain

$$\frac{1}{T}\int_Q \int_{\mathbb{R}} -f_+ g_- r(|x|+Mt)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \leqslant \int_{\{|x|\leqslant R+\eta\}} (u_0-v_0)^+\,\mathrm{d}x + \mathrm{I},$$

where the remainder term is

$$\mathrm{I} = \int_Q \int_{\mathbb{R}} -f_+ g_- \frac{T-t}{T} r'(|x| + Mt)(M + k(x)a(\xi)\,\mathrm{sgn}(x))\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t.$$

By definition of M, $\mathrm{I} \leqslant 0$, and since $r(|x| + Mt) = 1$ for $|x| \leqslant R - MT$, $0 \leqslant t \leqslant T$, we obtain

$$\frac{1}{T} \int_0^T \int_{|x|<R-MT} \int_{\mathbb{R}} -f_+ g_-\,\mathrm{d}\xi\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_{\{|x| \leqslant R+\eta\}} (u_0 - v_0)^+\,\mathrm{d}x.$$

Replacing R by $R + MT$ and letting $\eta \to 0$ gives (3.13). □

4. Convergence of the BGK approximation

THEOREM 4.1. *Let $u_0 \in L^1 \cap L^\infty(\mathbb{R})$, $0 \leqslant u_0 \leqslant 1$ a.e. When $\varepsilon \to 0$, the solution f^ε to (1.1) with initial datum $f_0 = \chi_{u_0}$ converges in $L^p(Q \times \mathbb{R}_\xi)$, $1 \leqslant p < +\infty$, to χ_u, where u is the unique solution to (3.2), (3.3).*

Proof. For $f \in L^1(\mathbb{R}_\xi)$, set

$$m_f(\xi) = \int_{-\infty}^{\xi} (\chi_u - f)(\zeta)\,\mathrm{d}\zeta, \quad u = \int_{\mathbb{R}} f(\xi)\,\mathrm{d}\xi.$$

It is easy to check that $m_f \geqslant 0$ if $0 \leqslant \mathrm{sgn}(\xi) f(\xi) \leqslant 1$ for almost every ξ (see equation (29) of [5]). In our context, we have $0 \leqslant f^\varepsilon \leqslant \chi_1$ hence, $m^\varepsilon := 1/\varepsilon m_{f^\varepsilon} \geqslant 0$. Viewed as a measure, m^ε is supported in $[0,T] \times \mathbb{R}_x \times [0,1]$. Integration with respect to ξ in (1.1) gives

$$m^\varepsilon(\xi) = \partial_t \biggl(\int_0^\xi f^\varepsilon(\zeta)\,\mathrm{d}\zeta \biggr) + \partial_x \biggl(k(x) \int_0^\xi a(\zeta) f^\varepsilon(\zeta)\,\mathrm{d}\zeta \biggr)$$

in $\mathcal{D}'(\,]0,T[\, \times \mathbb{R}_x)$. Summing over $(t,x) \in [0,T] \times [x_1,x_2]$, $\xi \in]0,1[\,$, we obtain the estimate

$$\begin{aligned} m^\varepsilon([0,T] \times [x_1,x_2] \times [0,1]) = \int_{x_1}^{x_2} \int_0^1 (1-\xi)(f^\varepsilon(T,x,\xi) - f^\varepsilon(0,x,\xi))\,\mathrm{d}\xi\,\mathrm{d}x \\ + \biggl[\int_0^T \int_0^1 (1-\xi) k(x) a(\xi) f^\varepsilon(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}t \biggr]_{x_1}^{x_2}. \end{aligned} \tag{4.1}$$

Since $f^\varepsilon(t) \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$, there exist sequences $(x_1^n) \downarrow -\infty$ and $(x_2^n) \uparrow +\infty$ such that the last term of the right-hand side of (4.1) tends to 0 when $n \to +\infty$. Since $f^\varepsilon \geqslant 0$ and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f^\varepsilon(T,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{u_0}\,\mathrm{d}\xi\,\mathrm{d}x = \|u_0\|_{L^1(\mathbb{R})},$$

we obtain the uniform estimate

$$m^\varepsilon([0,T] \times \mathbb{R} \times [0,1]) \leqslant \|u_0\|_{L^1(\mathbb{R})}. \tag{4.2}$$

We also have

$$0 \leqslant f^\varepsilon \leqslant \chi_1, \quad -\partial_\xi f_+^\varepsilon(t,x,\xi) = \nu_{t,x}^\varepsilon(\xi) + \mathcal{O}(\varepsilon), \tag{4.3}$$

where $\nu_{t,x}^\varepsilon(\xi) := \delta_{u^\varepsilon(t,x)}(\xi)$ and the identity is satisfied in $\mathcal{D}'(]0,T[\times\mathbb{R}_x\times\mathbb{R}_\xi)$. Indeed, by (1.1),

$$f^\varepsilon = \chi_{u^\varepsilon} + \varepsilon(\partial_t f^\varepsilon + \partial_x(k(x)a(\xi)f^\varepsilon)) = \chi_{u^\varepsilon} + \mathcal{O}(\varepsilon);$$

hence,

$$-\partial_\xi f_+^\varepsilon = -\partial_\xi f^\varepsilon + \delta_0(\xi) = -\partial_\xi \chi_{u^\varepsilon} + \delta_0(\xi) + \mathcal{O}(\varepsilon) = \delta_{u^\varepsilon}(\xi) + \mathcal{O}(\varepsilon).$$

Note that, for almost every (t,x), $\nu_{t,x}^\varepsilon$ is supported in the fixed compact subset $[0,1]$ of $\mathbb{R}_\xi$. We deduce from (4.2), (4.3) that, up to a subsequence, there exists a non-negative measure m on $\mathbb{R}^3$ supported in $[0,T]\times\mathbb{R}_x\times[0,1]$, a function $f \in L^\infty(]0,T[;L^1(\mathbb{R}_x\times\mathbb{R}_\xi))$ such that $0 \leqslant f \leqslant \chi_1$, $-\partial_\xi f_+(t,x,\xi) = \nu_{t,x}(\xi)$, where ν is a Young measure $Q \to \mathbb{R}_\xi$ and such that $m^\varepsilon \rightharpoonup m$ weakly in the sense of measures (i.e. $\langle m^\varepsilon - m, \varphi\rangle \to 0$ for every continuous compactly supported φ on $\mathbb{R}^3$) and $f^\varepsilon \rightharpoonup f$ in $L^\infty(Q\times\mathbb{R}_\xi)$ weak*. Besides, since $f^\varepsilon \in C([0,T];L^1_{x,\xi})$ satisfies $f^\varepsilon(0) = f_0$ and the BGK equation

$$\partial_t f^\varepsilon + \partial_x(k(x)a(\xi)f^\varepsilon) = \partial_\xi m^\varepsilon,$$

it satisfies the weak formulation: for all $\psi \in C_c^\infty([0,T[\times\mathbb{R}\times\mathbb{R})$,

$$\begin{aligned}\int_Q\int_{\mathbb{R}} f^\varepsilon(\partial_t\psi + k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} f_0\psi(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ = \int_Q\int_{\mathbb{R}} \partial_\xi\psi\,\mathrm{d}m^\varepsilon(t,x,\xi).\end{aligned}$$

In particular, we have

$$\begin{aligned}\int_Q\int_{\mathbb{R}} f_\pm^\varepsilon(\partial_t\psi + k(x)a(\xi)\partial_x\psi)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,\pm}\psi(0,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ = -\int_Q\int_{\mathbb{R}} \operatorname{sgn}_\mp(\xi)k(x)a(\xi)\partial_x\psi\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x + \int_Q\int_{\mathbb{R}}\partial_\xi\psi\,\mathrm{d}m^\varepsilon(t,x,\xi) \\ = (k_{\mathrm{R}} - k_{\mathrm{L}})\int_0^T\int_{\mathbb{R}} \operatorname{sgn}_\mp(\xi)a(\xi)\psi(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t + \int_Q\int_{\mathbb{R}}\partial_\xi\psi\,\mathrm{d}m^\varepsilon(t,x,\xi) \\ = (k_{\mathrm{L}} - k_{\mathrm{R}})^\pm\int_0^T\int_{\mathbb{R}} a(\xi)\psi(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t + \int_Q\int_{\mathbb{R}}\partial_\xi\psi\,\mathrm{d}m_\pm^\varepsilon(t,x,\xi),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\langle m_\pm^\varepsilon, \partial_\xi\psi\rangle := \langle m^\varepsilon, \partial_\xi\psi\rangle - \int_0^T\int_{\mathbb{R}} a(\xi)[(k_{\mathrm{L}} - k_{\mathrm{R}})\operatorname{sgn}_\mp(\xi) \\ + (k_{\mathrm{L}} - k_{\mathrm{R}})^\pm]\psi(t,0,\xi)\,\mathrm{d}\xi\,\mathrm{d}t.\end{aligned} \tag{4.5}$$

More precisely, we set

$$m_+^\varepsilon = m^\varepsilon + \int_\xi^{+\infty} a(\zeta)[(k_{\mathrm{L}} - k_{\mathrm{R}})^+ \operatorname{sgn}_+(\zeta) - (k_{\mathrm{L}} - k_{\mathrm{R}})^- \operatorname{sgn}_-(\zeta)] \, \mathrm{d}\zeta \delta(x = 0)$$

and

$$m_-^\varepsilon = m^\varepsilon + \int_{-\infty}^{\xi} a(\zeta)[(k_{\mathrm{L}} - k_{\mathrm{R}})^+ \operatorname{sgn}_+(\zeta) - (k_{\mathrm{L}} - k_{\mathrm{R}})^- \operatorname{sgn}_-(\zeta)] \, \mathrm{d}\zeta \delta(x = 0).$$

Note that in both cases, and since $A(\xi) \geqslant 0$ for any ξ, we have added a non-negative quantity to m^ε. At the limit $\varepsilon \to 0$ we thus obtain $m_\pm^\varepsilon \rightharpoonup m_\pm$, where $m_\pm$ is a non-negative measure. Examination of the support of $m_\pm^\varepsilon$ shows that m_+ (respectively, m_-) is supported in $[0, T] \times \mathbb{R}_x \times]{-\infty}, 1]$ (respectively, $[0, T] \times \mathbb{R}_x \times [0, +\infty[$). At the limit $\varepsilon \to 0$ we thus obtain the kinetic formulation (3.5). We conclude that f is a generalized solution to (3.2), (3.3). By theorem 3.4, $f = \chi_u$, where $u \in L^\infty(Q)$ is a solution to (3.2), (3.3). By uniqueness, the whole sequence (f^ε) converges (in L^∞ weak*) to χ_u. Actually, the convergence is strong since

$$\begin{aligned} \int_Q \int_{\mathbb{R}} |f^\varepsilon - \chi_u|^2 \, \mathrm{d}\xi \, \mathrm{d}t \, \mathrm{d}x &= \int_Q \int_{\mathbb{R}} |f^\varepsilon|^2 - 2 f^\varepsilon \chi_u + \chi_u \, \mathrm{d}\xi \, \mathrm{d}t \, \mathrm{d}x \\ &\leqslant \int_Q \int_{\mathbb{R}} f^\varepsilon - 2 f^\varepsilon \chi_u + \chi_u \, \mathrm{d}\xi \, \mathrm{d}t \, \mathrm{d}x. \end{aligned} \tag{4.6}$$

We have used the fact that $0 \leqslant f^\varepsilon \leqslant 1$. The right-hand side of (4.6) tends to 0 when $\varepsilon \to 0$ since $1, \chi_u \in L^\infty$ can be taken as test functions. Hence $f^\varepsilon \to \chi_u$ in $L^2(Q \times \mathbb{R})$. The convergence in $L^p(Q \times \mathbb{R})$, $1 \leqslant p < +\infty$, follows from the uniform bound on f^ε in $L^1 \cap L^\infty(Q \times \mathbb{R})$. □

Remark 4.2. It is possible to relax the assumption that the initial datum for (1.1) is at equilibrium and independent of ε in theorem 4.1. Indeed, the conclusion of theorem 4.1 remains valid under the hypothesis that the initial datum f_0^ε for (1.1) satisfies

$$0 \leqslant f_0^\varepsilon \leqslant \chi_1, \quad f_0^\varepsilon \rightharpoonup f_0, \quad u_0(x) := \int_{\mathbb{R}} f_0(x, \xi) \, \mathrm{d}\xi, \tag{4.7}$$

where $f_0^\varepsilon \rightharpoonup f_0$ in (4.7) denotes weak convergence in $L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$. Indeed, the proof of theorem 4.1 remains unchanged under the following modification: passing to the limit in (4.4), we obtain that f is a generalized solution to (3.2) with an initial datum f_0 that is not necessary at equilibrium. However, we have (see equation (29) of [5])

$$f_0 - \operatorname{sgn}_\mp(\xi) = \operatorname{sgn}_\pm(u_0 - \xi) - \partial_\xi m_\pm^0,$$

where m_+^0 (respectively, m_-^0) is a non-negative measure supported in $[0, T] \times \mathbb{R} \times]{-\infty}, 1]$ (respectively, $[0, T] \times \mathbb{R} \times [0, +\infty[$). Consequently, up to a modification of the kinetic measure $m_\pm$, we obtain that f is indeed a generalized solution to (3.2), (3.3). The rest of the proof is similar.

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