The determinant of the Malliavin matrix and the determinant of the covariance matrix for multiple integrals

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Abstract

A well-known problem in Malliavin calculus concerns the relation between the determinant of the Malliavin matrix of a random vector and the determinant of its covariance matrix. We give an explicit relation between these two determinants for couples of random vectors of multiple integrals. In particular, if the multiple integrals are of the same order and this order is at most 4, we prove that two random variables in the same Wiener chaos either admit a joint density, either are proportional and that the result is not true for random variables in Wiener chaoses of different orders.

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1 Introduction

The original motivation of the Malliavin calculus was to study the existence and the regularity of the densities of random variables. In this research direction, the determinant of the so-callled Malliavin matrix plays a crucial role.

We give here an explicit formula that connects the determinant of the Malliavin matrix and the determinant of the covariance matrix of a couple of multiple stochastic integrals. This is related to two open problems stated in [1]. In this reference, the authors showed that, if $F = (F_1, .., F_d)$ is a random vector whose components belong to a finite sum of Wiener chaoses, then the law of F is not absolutely continuous with respect to the Lebesque measure if and only if $E \det \Lambda = 0$. Here Λ denotes the Malliavin matrix of the vector F. In particular, they proved that a couple of multiple integrals of order 2 either admits a density or its components are proportional.

They stated two open questions (Questions 6.1 and 6.2 in [1], arXiv version): if C is the covariance matrix and Λ the Malliavin matrix of a vector of multiple stochastic integrals,

- is there true that $E \det \Lambda \geq c \det C$, with $c > 0$ an universal constant?
- is there true that the law of a vector of multiple integrals with components in the same Wiener chaos is either absolutely continuous with respect to the Lesque measure or its components are proportional?

We make a first step in order to answer to these two open problems. Actually, we find an explicit relation that connects the two determinants. In particular, if the multiple integrals are of the same order and this order is at most 4, we prove that two random variables in the same Wiener chaos either admit a joint density, either are proportional. The basic idea is to write the Malliavin matrix as a sum of squares and to compute the dominant term of its determinant.

We organized our paper as follows. Section 2 contains some preliminaries on analysis on Wiener chaos. Section 3 is devoted to express the Malliavin matrix as the sum of the squares of some random variables and in Section 4 we derive an explicit formula for the determinant of Λ which also involves the determinant of the covariance matrix. In Section 5 we discuss the existence of the joint density of a vector of multiple integrals.

2 Preliminaries

We briefly describe the tools from the analysis on Wiener space that we will need in our work. For complete presentations, we refer to [4] or [2]. Let H be a real and

separable Hilbert space and consider $(W(h), h \in H)$ an isonormal process. That is, $(W(h), h \in H)$ is a family of centered Gaussian random variables on the probability space $(\Omega, \mathcal{F}, P)$ such that $EW(h)W(g) = \langle f, g \rangle_H$ for every $h, g \in H$. Assume that the σ-algebra $\mathcal{F}$ is generated by W.

Denote, for $n \geq 0$, by $\mathcal{H}_n$ the nth Wiener chaos generated by W. That is, $\mathcal{H}_n$ is the vector subspace of $L^2(\Omega)$ generated by $(H_n(W(h)), h \in H, \|h\| = 1)$ where H_n the Hermite polynomial of degree n. For any $n \geq 1$, the mapping $I_n(h^{\otimes n}) = H_n(W(h))$ can be extended to an isometry between the Hilbert space $H^{\otimes n}$ endowed with the norm $\sqrt{n!}\|\cdot\|_{H^{\otimes n}}$ and the nth Wiener chaos $\mathcal{H}_n$. The random variable $I_n(f)$ is called the multiple Wiener Itô integral of f with respect to W.

Consider $(e_j)_{j\geq 1}$ a complete orthonormal system in H and let $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ be two symmetric functions with $n, m \geq 1$. Then

$$f = \sum_{j_1,..,j_n \geq 1} \lambda_{j_1,..,j_n} e_{j_1} \otimes ... \otimes e_{j_n} \tag{1}$$

and

$$g = \sum_{k_1,..,k_m \geq 1} \beta_{k_1,..,k_m} e_{k_1} \otimes .. \otimes e_{k_m} \tag{2}$$

where the coefficients λ_i and β_j satisfy $\lambda_{j_{\sigma(1)},...j_{\sigma(n)}} = \lambda_{j_1,..,j_n}$ and $\beta_{k_{\pi(1)},...,k_{\pi(m)}} = \beta_{k_1,..,k_m}$ for every permutation σ of the set $\{1, ..., n\}$ and for every permutation π of the set $\{1, .., m\}$. Actually $\lambda_{j_1,..,j_n} = \langle f, e_{j_1} \otimes ... \otimes e_{j_n} \rangle$ and $\beta_{k_1,..,k_m} = \langle g, e_{k_1} \otimes .. \otimes e_{k_m} \rangle$ in (1) and (2). Note that, throughout the paper we will use the notation $\langle \cdot, \cdot \rangle$ to indicate the scalar product in $H^{\otimes k}$, independently of k.

If $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ are symmetric given by (1), (2) respectively, then the contraction of order r of F and g is given by

$$\begin{aligned} f \otimes_r g \;\; = \;\; & \sum_{i_1,..,i_r \geq 1} \sum_{j_1,..,j_{n-r} \geq 1} \sum_{k_1,..,k_{m-r} \geq 1} \lambda_{i_1,..,i_r,j_1,..,j_{n-r}} \beta_{i_1,..,i_r,k_1,..,k_{m-r}} \\ & \times \left(e_{j_1} \otimes .. \otimes e_{j_{n-r}}\right) \otimes \left(e_{k_1} \otimes .. \otimes e_{k_{m-r}}\right) \end{aligned} \tag{3}$$

for every $r = 0, .., m \wedge n$. In particular $f \otimes_0 g = f \otimes g$. Note that $f \otimes_r g$ belongs to $H^{\otimes(m+n-2r)}$ for every $r = 0, .., m \wedge n$ and it is not in general symmetric. We will denote by $f \tilde{\otimes}_r g$ the symmetrization of $f \otimes_r g$. In the particular case when $H = L^2(T, \mathcal{B}, \mu)$ where μ is a sigma-finite measure without atoms, (3) becomes

$$\begin{aligned} & (f \otimes_r g)(t_1, .., t_{m+n-2r}) \\ = \;\; & \int_{T^r} d\mu(u_1)..d\mu(u_r) f(u_1, .., u_r, t_1, .., t_{n-r}) g(u_1, .., u_r, t_{n-r+1}, .., t_{m+n-2r}) \end{aligned} \tag{4}$$

An important role will be played by the following product formula for multiple Wiener-Itô integrals: if $f \in H^{\otimes n}$, $g \in H^{\otimes m}$ are symmetric, then

$$I_n(f)I_m(g) = \sum_{r=0}^{m\wedge n} r!C_m^r C_n^r I_{m+n-2r}\left(f\tilde{\otimes}_r g\right). \tag{5}$$

We will need the concept of Malliavin derivative D with respect to W, but we will use only its action on Wiener chaos. In order to avoid too many details, we will just say that, if f is given by (1) and $I_n(f)$ denotes its multiple integral of order n with respect to W, then

$$DI_n(f) = n\sum_{j_1,..,j_n\geq 1} \lambda_{j_1,..,j_n} I_{n-1}\left(e_{j_2}\otimes ..\otimes e_{j_n}\right) e_{j_1}.$$

If F, G are two random variables which are differentiable in the Malliavin sense, we will denote throughout the paper by C the covariance matrix and by Λ the Malliavin matrix of the random vector (F, G). That is,

$$\Lambda = \begin{pmatrix} \|DF\|_H^2 & \langle DF, DG\rangle_H \\ \langle DF, DG\rangle_H & \|DF\|_H^2 \end{pmatrix}.$$

3 The Malliavin matrix as a sum of squares

In this section we will express the determinant of the Malliavin matrix of a random couple as a sum of squares of certain random variables. This will be useful in order to derive the exact formula for the determinant of the Malliavin matrix and its connection with the determinant of the covariance matrix for a given random vector of dimension 2.

Let $f \in H^{\otimes n}$ and $g \in H^{\otimes m}$ be given by (1) and (2) respectively, with $n, m \geq 1$. Let $F = I_n(f), G = I_m(g)$ denote the multiple Wiener-Itô integrals of f and g with respect to W respectively. Then

$$I_n(f) = \sum_{j_1,..,j_n\geq 1} \lambda_{j_1,..,j_n} I_n\left(e_{j_1}\otimes ...\otimes e_{j_n}\right) \tag{6}$$

and

$$I_m(g) = \sum_{k_1,..,k_m\geq 1} \beta_{k_1,..,k_m} I_m\left(e_{k_1}\otimes ..\otimes e_{k_m}\right). \tag{7}$$

From (6) and (7) we have

$$DF = n\sum_{j_1,..,j_n\geq 1} \lambda_{j_1,..,j_n} I_{n-1}\left(e_{j_2}\otimes ..\otimes e_{j_n}\right) e_{j_1}$$

and

$$DG = m \sum_{k_1,..,k_m \geq 1} \beta_{k_1,..,k_m} I_{m-1}\left(e_{k_2} \otimes .. \otimes e_{k_m}\right) e_{k_1}.$$

This implies

$$\|DF\|_H^2 = n^2 \sum_{i \geq 1} \sum_{j_2,..,j_n \geq 1} \sum_{k_1,..,k_n \geq 1} \lambda_{i,j_2,..,j_n} \lambda_{i,k_2,..,k_n} I_{n-1}\left(e_{j_2} \otimes .. \otimes e_{j_n}\right) I_{n-1}\left(e_{k_2} \otimes .. \otimes e_{k_n}\right)$$

and

$$\|DG\|_H^2 = m^2 \sum_{l \geq 1} \sum_{l,j_2,..,j_n \geq 1} \sum_{l,k_2,..,k_n \geq 1} \beta_{l,j_2,..,j_m} \beta_{l,k_2,..,k_m} I_{m-1}\left(e_{j_2} \otimes .. \otimes e_{j_m}\right) I_{m-1}\left(e_{k_2} \otimes .. \otimes e_{k_m}\right)$$

and

$$\langle DF, DG \rangle_H = nm \sum_{i \geq 1} \sum_{j_2,..,j_n \geq 1} \sum_{k_1,..,k_m \geq 1} \lambda_{i,j_2,..,j_n} \beta_{i,j_1,..,j_m} I_{n-1}\left(e_{j_2} \otimes .. \otimes e_{j_n}\right) I_{m-1}\left(e_{k_2} \otimes .. \otimes e_{k_m}\right).$$

Let us make the following notation. For every $i \geq 1$, let

$$S_{i,f} = n \sum_{i \geq 1} \sum_{j_2,..,j_n \geq 1} \lambda_{i,j_2,..,j_n} I_{n-1}\left(e_{j_2} \otimes .. \otimes e_{j_n}\right) \tag{8}$$

and

$$S_{i,g} = m \sum_{i \geq 1} \sum_{i,k_2,..,k_m \geq 1} \beta_{i,k_2,..,k_m} I_{m-1}\left(e_{k_2} \otimes .. \otimes e_{k_m}\right). \tag{9}$$

We can write

$$\|DF\|_H^2 = \sum_{i \geq 1} S_{i,f}^2, \quad \|DG\|_H^2 = \sum_{l \geq 1} S_{l,g}^2, \langle DF, DG \rangle = \sum_{i \geq 1} S_{i,f} S_{i,g}$$

and

$$det(\Lambda) \quad = \quad \|DF\|_H^2 \|DG\|_H^2 - \langle DF, DG \rangle_H^2 = \sum_{i,l \geq 1} S_{i,f}^2 S_{l,g}^2 - \left(\sum_{i \geq 1} S_{i,f} S_{i,g} \right)^2.$$

A key observation is that

$$\sum_{i,l \geq 1} S_{i,f}^2 S_{l,g}^2 - \left(\sum_{i \geq 1} S_{i,f} S_{i,g} \right)^2 \quad = \quad \frac{1}{2} \sum_{i,l \geq 1} \left(S_{i,f} S_{l,g} - S_{l,f} S_{i,g} \right)^2. \tag{10}$$

We obtained

Proposition 1 *The determinant of the Malliavin matrix Λ of the vector $(F,G) = (I_n(f), I_m(g))$ can be expressed as*

$$det\Lambda = \frac{1}{2}\sum_{i,l\geq 1}\left(S_{i,f}S_{l,g} - S_{l,f}S_{i,g}\right)^2$$

where $S_{i,f}, S_{i,g}$ are given by (8) and (9) respectively.

Corollary 1 *The determinant of the Malliavin matrix Λ of the vector $(F,G) = (I_n(f), I_m(g))$ can be expressed as*

$$det\Lambda = \frac{1}{2}\sum_{i,l\geq 1}\left(\langle DF, e_i\rangle\langle DG, e_l\rangle - \langle DF, e_l\rangle\langle DG, e_i\rangle\right)^2$$

Proof: *This comes from Proposition 1 and the relations*

$$S_{i,f} = \langle DF, e_i\rangle, \quad S_{i,g} = \langle DG, e_i\rangle$$

for every $i \geq 1$. ■

4 The determinant of the Malliavin matrix on Wiener chaos

Fix $n, m \geq 1$ and f, g in $H^{\otimes n}, H^{\otimes m}$ respectively defined by (1) and (2). Consider the random vector $(F,G) = (I_n(f), I_m(g))$ and denote by Λ its Malliavin matrix and by C its covariance matrix.

Let us compute $E\det\Lambda$. Denote, for every $i, l \geq 1$

$$s_{i,f} = n\sum_{j_2,..,j_m\geq 1}\lambda_{i,j_2,..,j_n}e_{j_2}\otimes ..\otimes e_{j_n} \tag{11}$$

and

$$s_{l,g} = m\sum_{k_2,..,k_m\geq 1}\beta_{l.k_2,..,k_m}e_{k_2}\otimes ..\otimes e_{k_m}. \tag{12}$$

Clearly, for every $i, l \geq 1$

$$S_{i,f} = I_{n-1}(s_{i,f}), \quad S_{i,g} = I_{m-1}(s_{i,g}). \tag{13}$$

The following lemma plays a key role in our construction.

Lemma 1 *If $f \in H^{\otimes n}$ and $g \in H^{\otimes m}$ are given by (1) and (2) respectively and $s_{i,f}, s_{i,g}$ by (11), (12) respectively, then for every $r = 0, .., n-1$*

$$f \otimes_{r+1} g = \frac{1}{nm} \sum_{i \geq 1} \left(s_{i,f} \otimes_r s_{i,g} \right).$$

Proof: Consider first $r = 0$. Clearly, by (3)

$$\begin{aligned} f \otimes_1 g &= \sum_{i \geq 1} \sum_{j_2,..,j_n \geq 1} \sum_{k_2,..,k_m \geq 1} \lambda_{i,j_2,..,j_n} \beta_{i,k_2,..,k_m} e_{j_2} \otimes .. \otimes e_{j_n} \otimes e_{k_2} \otimes ..e_{k_m} \\ &= \frac{1}{nm} \sum_{i \geq 1} \left(s_{i,f} \otimes s_{i,g} \right). \end{aligned}$$

The same argument applies for every $r = 1, .., n-1$. Indeed,

$$\begin{aligned} & f \otimes_{r+1} g \\ = & \left(\sum_{j_1,..,j_n \geq 1} \lambda_{j_1,..,j_n} e_{j_1} \otimes .. \otimes e_{j_n} \right) \otimes_r \left(\sum_{k_1,..,k_m \geq 1} \beta_{k_1,..,k_m} e_{k_1} \otimes .. \otimes e_{k_m} \right) \\ = & \sum_{i_1,..,i_{r+1}} \sum_{j_{r+2},..,j_n} \sum_{k_{r+2},..,k_m} \lambda_{i_1,..,i_{r+1},j_{r+2},..,j_n} \beta_{i_1,..,i_{r+1},k_{r+2},..,k_m} \left(e_{j_{r+2}} \otimes .. \otimes e_{j_n} \right) \otimes \left(e_{k_{r+2}}, .., e_{k_m} \right) \end{aligned}$$

and by (3) again

$$\begin{aligned} & \sum_{i \geq 1} s_{i,j} \otimes_r s_{i,g} \\ = & nm \sum_{i \geq 1} \left(\sum_{j_2,..,j_n \geq 1} \lambda_{i,j_2,..,j_n} e_{j_2} \otimes .. \otimes e_{j_n} \right) \otimes_r \left(\sum_{k_2,..,k_m \geq 1} \beta_{i,k_2,..,k_m} e_{k_2} \otimes .. \otimes e_{k_m} \right) \\ = & nm \sum_{i \geq 1} \sum_{i_2,..,i_{r+1}} \sum_{j_{r+2},..,j_n} \sum_{k_{r+2},..,k_m} \lambda_{i,i_2,..,i_{r+1},j_{r+2},..,j_n} \beta_{i,i_2,..,i_{r+1},k_{r+2},..,k_m} \\ & \times \left(e_{j_{r+2}} \otimes .. \otimes e_{j_n} \right) \otimes \left(e_{k_{r+2}}, .., e_{k_m} \right) \\ = & nm \sum_{i_1,..,i_{r+1}} \sum_{j_{r+2},..,j_n} \sum_{k_{r+2},..,k_m} \lambda_{i_1,..,i_{r+1},j_{r+2},..,j_n} \beta_{i_1,..,i_{r+1},k_{r+2},..,k_m} \left(e_{j_{r+2}} \otimes .. \otimes e_{j_n} \right) \otimes \left(e_{k_{r+2}}, .., e_{k_m} \right) \\ = & f \otimes_{r+1} g. \end{aligned}$$

■

We make a first step to compute $E \det \Lambda$.

Lemma 2 *Let $f \in H^{\otimes n}, g \in H^{\otimes m}$ be symmetric and denote by Λ the Malliavin matrix of the vector $(F,G) = (I_n(f), I_m(g))$. Then we have*

$$E \det \Lambda = \sum_{k=0}^{(n-1)\wedge(m-1)} T_k$$

where we denote, for $k = 0, .., (m-1) \wedge (n-1)$,

$$T_k := \frac{1}{2} \sum_{i,l \geq 1} k!^2 \left(C_{m-1}^k\right)^2 \left(C_{m-1}^k\right)^2 (m+n-2-2k)! \| s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g} \|^2 \quad (14)$$

and $s_{i,f}, s_{i,g}$ are given by (11), (12) for $i \geq 1$.

Proof: By Proposition 1 and relation (13)

$$\begin{aligned} 2 \det \Lambda &= \sum_{i,l \geq 1} \left(I_{n-1}(s_{i,f}) I_{m-1}(s_{l,g}) - I_{n-1}(s_{l,f}) I_{m-1}(s_{i,g}) \right)^2 \\ &= \sum_{i,l \geq 1} \left(\sum_{k=0}^{(m-1)\wedge(n-1)} k! C_{m-1}^k C_{n-1}^k I_{m+n-2-2k} \left(s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g} \right) \right)^2 \end{aligned}$$

where we used the the product formula (5). Consequently, from the isometry of multiple stochastic integrals,

$$\begin{aligned} E \det \Lambda &= \frac{1}{2} \sum_{i,l \geq 1} \sum_{k=0}^{(n-1)\wedge(m-1)} k!^2 \left(C_{m-1}^k\right)^2 \left(C_{n-1}^k\right)^2 (m+n-2-2k)! \| s_{i,f} \tilde{\otimes}_k s_{l,g} - s_{l,f} \tilde{\otimes}_k s_{i,g} \|^2 \\ &= \sum_{k=0}^{(n-1)\wedge(m-1)} T_k. \end{aligned}$$

■

For every $n, m \geq 1$ let us denote by

$$R_{n,m} := \sum_{k=1}^{(n-1)\wedge(m-1)} T_k, \qquad R_n := R_{n,n}. \quad (15)$$

Remark 1 *Obviously all the terms T_k above are positive, for $k = 0, .., (n-1)\wedge(n-1)$.*

We will need two more auxiliary lemmas.

Lemma 3 *Assume $f_1, f_3 \in H^{\otimes n}$ and $f_2, f_4 \in H^{\otimes m}$ are symmetric functions. Then for every $r = 0, .., (m-1) \wedge (n-1)$ we have*

$$\langle f_1 \otimes_{n-r} f_3, f_2 \otimes_{m-r} f_4 \rangle = \langle f_1 \otimes_r f_2, f_3 \otimes_r f_4 \rangle.$$

Proof: The case $r = 0$ is trivial, so assume $r \geq 1$. Without any loss of the generality, assume that H is $L^2(T; \mu)$ where μ is a sigma-finite measure without atoms. Then, by (4)

$$\begin{aligned}
& \langle f_1 \otimes_{n-r} f_3, f_2 \otimes_{m-r} f_4 \rangle \\
& \int_{T^r} d\mu^r(t_1, .., t_r) \int_{T^r} d\mu^r(s_1, .., s_r) \\
& \left(\int_{T^{n-r}} d\mu^{n-r}(u_1, .., u_{n-r}) f_1(u_1, .., u_{n-r}, t_1, .., t_r) f_3(u_1, .., u_{n-r}, s_1, .., s_r) \right) \\
& \left(\int_{T^{m-r}} d\mu^{m-r}(v_1, .., v_{m-r}) f_2(v_1, .., v_{m-r}, t_1, .., t_r) f_4(v_1, .., v_{m-r}, s_1, .., s_r) \right) \\
= & \int_{T^{n-r}} d\mu^{n-r}(u_1, .., u_{n-r}) \int_{T^{m-r}} d\mu^{m-r}(v_1, .., v_{m-r}) \\
& (f_1 \otimes_r f_2)(u_1, .., u_{n-r}, v_1, .., v_{m-r}) (f_3 \otimes_r f_4)(u_1, .., u_{n-r}, v_1, .., v_{m-r}) \\
= & \langle f_1 \otimes_r f_2, f_3 \otimes_r f_4 \rangle.
\end{aligned}$$

■

Lemma 4 *Suppose $f_1, f_4 \in H^{\otimes n}$, $f_2, f_3 \in H^{\otimes m}$ are symmetric functions. Then*

$$\langle f_1 \tilde{\otimes} f_2, f_3 \tilde{\otimes} f_4 \rangle = \frac{m!n!}{(m+n)!} \sum_{r=0}^{m \wedge n} C_n^r C_m^r \langle f_1 \otimes_r f_3, f_4 \otimes_r f_2 \rangle.$$

Proof: This has been stated and proven in [3] in the case $m = n$. Exactly the same lines of the proofs apply for $m \neq n$. ■

We first compute the term T_0 obtained for $k = 0$ in (14).

Proposition 2 *Let T_0 be given by (14) with $k = 0$.*

$$T_0 = \sum_{r=0}^{(n-1) \wedge (m-1)} mnm!n! C_{n-1}^r C_{m-1}^r \left[\|f \otimes_r g\|^2 - \|f \otimes_{r+1} g\|^2 \right].$$

Proof: From (14),

$$\begin{aligned} T_0 &= \frac{1}{2}(m+n-2)! \sum_{i,l\geq 1} \|s_{i,f}\tilde{\otimes} s_{l,g} - s_{l,f}\tilde{\otimes} s_{i,g}\|^2 \\ &= \frac{1}{2}(m+n-2)! \sum_{i,l\geq 1} \left[\|s_{i,f}\tilde{\otimes} s_{l,g}\|^2 + \|s_{l,f}\tilde{\otimes} s_{i,g}\|^2 - 2\langle s_{i,f}\tilde{\otimes} s_{l,g}, s_{l,f}\tilde{\otimes} s_{i,g}\rangle\right]. \end{aligned}$$

Let us apply Lemma 4 to compute these norms and scalar products. We obtain, by letting $f_1 = s_{i,f} = f_4$ and $f_2 = s_{l,g} = f_3$ (note that $s_{i,f}, s_{i,g}$ are symmetric functions in $H^{\otimes n}, H^{\otimes m}$ respectively)

$$\begin{aligned} (m+n-2)!\langle s_{i,f}\tilde{\otimes} s_{l,g}, s_{i,f}\tilde{\otimes} s_{l,g}\rangle &= (m+n-2)!\langle s_{i,f}\tilde{\otimes} s_{l,g}, s_{l,g}\tilde{\otimes} s_{i,f}\rangle \\ &= (m-1)!(n-1)! \sum_{r=0}^{(n-1)\wedge(m-1)} C_{n-1}^r C_{m-1}^r \langle s_{i,f}\otimes_r s_{l,g}, s_{i,f}\otimes_r s_{l,g}\rangle \\ &= (m-1)!(n-1)! \sum_{r=0}^{(n-1)\wedge(m-1)} C_{n-1}^r C_{m-1}^r \|s_{i,f}\otimes_r s_{l,g}\|^2. \end{aligned}$$

Analogously, for $f_1 = s_{l,f} = f_4$ and $f_2 = s_{i,g} = f_3$ in Lemma 4 we get

$$\begin{aligned} &(m+n-2)!\langle s_{l,f}\tilde{\otimes} s_{i,g}, s_{l,f}\tilde{\otimes} s_{i,g}\rangle \\ &= \sum_{r=0}^{(n-1)\wedge(m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \|s_{l,f}\otimes_r s_{i,g}\|^2. \end{aligned}$$

Next, with $f_1 = s_{i,f}, f_2 = s_{l,g}, f_4 = s_{l,f}, f_3 = s_{i,g}$

$$\begin{aligned} &(m+n-2)!\langle s_{i,f}\tilde{\otimes} s_{l,g}, s_{l,f}\tilde{\otimes} s_{i,g}\rangle \\ = \;&(m+n-2)!\langle s_{i,f}\tilde{\otimes} s_{l,g}, s_{i,g}\tilde{\otimes} s_{l,f}\rangle \\ = \;&\sum_{r=0}^{(n-1)\wedge(m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \langle s_{i,f}\otimes_r s_{i,g}, s_{l,f}\otimes_r s_{l,g}\rangle. \end{aligned}$$

Then

$$
\begin{aligned}
& (m+n-2)! \sum_{i,l\geq 1} \|s_{i,f} \tilde{\otimes} s_{l,g} - s_{l,f} \tilde{\otimes} s_{i,g}\|^2 \\
= & \sum_{r=0}^{(n-1)\wedge(m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \\
& \sum_{i,l\geq 1} \left[\|s_{l,f} \otimes_r s_{i,g}\|^2 + \|s_{i,f} \otimes_r s_{l,g}\|^2 - 2\langle s_{i,f} \otimes_r s_{i,g}, s_{l,f} \otimes_r s_{l,g} \rangle \right] \\
= & 2 \sum_{r=0}^{(n-1)\wedge(m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \sum_{i,l\geq 1} \left[\|s_{i,f} \otimes_r s_{l,g}\|^2 - \langle s_{i,f} \otimes_r s_{i,g}, s_{l,f} \otimes_r s_{l,g} \rangle \right] \\
= & 2 \sum_{r=0}^{(n-1)\wedge(m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \\
& \times \left[\sum_{i,l\geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 - \langle \sum_{i\geq 1} s_{i,f} \otimes_r s_{i,g}, \sum_{l\geq 1} s_{l,f} \otimes_r s_{l,g} \rangle \right] \\
= & 2 \sum_{r=0}^{(n-1)\wedge(m-1)} (n-1)!(m-1)! C_{n-1}^r C_{m-1}^r \left[\sum_{i,l\geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 - \| \sum_{i\geq 1} s_{i,f} \otimes_r s_{i,g}\|^2 \right] . \quad (16)
\end{aligned}
$$

Notice that, by Lemma 1, for every $r = 0, .., n-1$

$$
\| \sum_{i\geq 1} s_{i,f} \otimes_r s_{i,g}\|^2 = n^2 m^2 \| f \otimes_{r+1} g\|^2. \quad (17)
$$

We apply now Lemma 3 and we get

$$
\begin{aligned}
\sum_{i,l\geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 &= \sum_{i,l\geq 1} \langle s_{i,f} \otimes_r s_{l,g}, s_{i,f} \otimes_r s_{l,g} \rangle \\
&= \sum_{i,l\geq 1} \langle s_{i,f} \otimes_{n-1-r} s_{i,f}, s_{l,g} \otimes_{m-1-r} s_{l,g} \rangle \\
&= \langle \sum_{i\geq 1} \langle s_{i,f} \otimes_{n-1-r} s_{i,f}, \sum_{l\geq 1} s_{l,g} \otimes_{m-r-1} s_{l,g} \rangle
\end{aligned}
$$

and by Lemma 1 and Lemma 3, this equals

$$
\begin{aligned}
\sum_{i,l\geq 1} \|s_{i,f} \otimes_r s_{l,g}\|^2 &= n^2 m^2 \langle f \otimes_{n-r} f, g \otimes_{m-r} g \rangle \\
&= n^2 m^2 \| f \otimes_r g\|^2. \quad (18)
\end{aligned}
$$

By replacing (17) and (18) in (16) we obtain

$$\begin{aligned} T_0 &= \frac{1}{2}(m+n-2)! \sum_{i,l\geq 1} \|s_{i,f}\tilde{\otimes} s_{l,g} - s_{l,f}\tilde{\otimes} s_{i,g}\|^2 \\ &= \sum_{r=0}^{(n-1)\wedge(m-1)} mnm!n! C_{n-1}^r C_{m-1}^r \left[\|f\otimes_r g\|^2 - \|f\otimes_{r+1} g\|^2\right]. \end{aligned}$$

■

As a consequence of the above proof, we obtain

Corollary 2 *For every $r = 0, .., (m-1)\wedge(n-1)$ and if $s_{i,f}, s_{i,g}$ are given by (11), (12), it holds that*

$$n^2m^2 \sum_{r=0}^{n-1} C_{m-1}^r C_{m-1}^r \left[\|f\otimes_r g\|^2 - \|f\otimes_{r+1} g\|^2\right] = \sum_{i,l\geq 1} \|s_{i,f}\tilde{\otimes} s_{l,g} - s_{l,f}\tilde{\otimes} s_{i,g}\|^2.$$

As a consequence, for every $r = 0, .., (m-1)\wedge(n-1)$ we have

$$\sum_{r=0}^{n-1} C_{m-1}^r C_{m-1}^r \left[\|f\otimes_r g\|^2 - \|f\otimes_{r+1} g\|^2\right] \geq 0. \tag{19}$$

Proof: It is a consequence of the proof of Proposition 2. ■

Let us state the main results of this section.

Theorem 1 *Let $f \in H^{\otimes n}, g \in H^{\otimes m} (n, m \geq 1)$ be symmetric and denote by Λ the Malliavin matrix of the vector $(F, G) = (I_n(f), I_m(g))$. Then*

$$\det\Lambda = \sum_{r=0}^{(n-1)\wedge(m-1)} mnm!n! C_{n-1}^r C_{m-1}^r \left[\|f\otimes_r g\|^2 - \|f\otimes_{r+1} g\|^2\right] + R_{n,m}$$

where for every $n, m \geq 1$, $R_{n,m}$ is given by (15). Note that $R_{n,m} \geq 0$ for every $n, m \geq 1$.

Proof: It follows from Proposition 2 and Lemma 2. ■

In the case when the two multiple integrals live in the same Wiener chaos, we have a nicer expression.

Theorem 2 *Under the same assumptions as in Theorem 1 but with $m = n$, we have*

$$\det \Lambda = m^2 \det C + (mm!)^2 \sum_{r=1}^{\left[\frac{m-1}{2}\right]} \left((C_{m-1}^r)^2 - (C_{m-1}^{r-1})^2\right) \left(\|f \otimes_r g\|^2 - \|f \otimes_{n-r} g\|^2\right) + R_m$$

with R_m given by (15). Here $[x]$ denotes the integer part of x.

Proof: Suppose $n \leq m$ and that m is odd. The case m even is similar. From Theorem 1 we have

$$\begin{aligned}
\det \Lambda &= (mm!)^2 \left[\sum_{r=0}^{(m-1)} \left(C_{m-1}^r\right)^2 \|f \otimes_r g\|^2 - \sum_{r=0}^{(m-1)} \left(C_{m-1}^r\right)^2 \|f \otimes_{r+1} g\|^2 \right] \\
&= (mm!)^2 \left[\sum_{r=0}^{\frac{m-1}{2}} \left(C_{m-1}^r\right)^2 \|f \otimes_r g\|^2 - \sum_{r=\frac{m-1}{2}}^{(m-1)} \left(C_{m-1}^r\right)^2 \|f \otimes_{r+1} g\|^2 \right] \\
&\quad + (mm!)^2 \left[\sum_{\frac{m-1}{2}}^{m-1} \left(C_{m-1}^r\right)^2 \|f \otimes_r g\|^2 - \sum_{r=0}^{\frac{m-1}{2}} \left(C_{m-1}^r\right)^2 \|f \otimes_{r+1} g\|^2 \right] \\
&= (mm!)^2 \sum_{r=0}^{\frac{m-1}{2}} \left(C_{m-1}^r\right)^2 \left[\|f \otimes_r g\|^2 - \|f \otimes_{n-r} g\right] \\
&\quad + (mm!)^2 \sum_{r=1}^{\frac{m-1}{2}} \left(C_{m-1}^r\right)^2 \left[\|f \otimes_{n-r} g\|^2 - \|f \otimes_r g\|^2\right]
\end{aligned}$$

where we made the change of index $r' = n-1-r$ in the second and third sum above. Finally, noticing that for $r = 0$ we have

$$m^2 m!^2 \left(C_{m-1}^0\right)^2 \left[\|f \otimes_0 g\|^2 - \|f \otimes_n g\|\right] = m^2 \det C$$

we obtain the conclusion. ■

Example 1 *Suppose $m = n = 2$. Then*

$$\begin{aligned}
\det \Lambda &= 16 \left[\|f \otimes g\|^2 - \|f \otimes_2 g\|^2\right] + R_2 \\
&= 4 \det C + R_2.
\end{aligned}$$

We retrieve the formula in [1] with $R_2 = 32 \left(\|f \otimes_1 g\|^2 - \|f \tilde{\otimes}_1 g\|^2\right)$.

Assume $m = n = 3$. Then

$$\begin{aligned}\det\Lambda &= 9\times 36\left[\left(\|f\otimes g\|^2 - \|f\otimes_3 g\|^2\right) + 9\times 36\times\left((C_2^1)^2-1\right)\left(\|f\otimes_1 g\|^2 - \|f\otimes_2 g\|^2\right)\right] + R_3\\ &= 9\det C + 9\times 36\times 3\left(\|f\otimes_1 g\|^2 - \|f\otimes_2 g\|^2\right) + R_3.\end{aligned}$$

Suppose $m = n = 4$. Then

$$\det\Lambda = 16\det C + 16\times 4!\times 4!\left((C_3^1)^2-1\right)\left(\|f\otimes_1 g\|^2 - \|f\otimes_3 g\|^2\right) + R_4.$$

5 Densities of vectors of multiple integrals

Let us discuss when a couple of multiple stochastic integrals has a law which is absolutely continuous with respect to the Lebesque measure. The situations when the components of the vector are in the same chaos of in chaoses of different orders need to be separated.

Let us first discuss the case of variables in the same chaos. In order better understand the relation between $\det\Lambda$ and $\det C$ we need more information on the terms R_m in Theorem 2. It is actually possible to compute the last term T_{m-1} in (14).

Proposition 3 *Suppose $m = n$ and let T_{m-1} be the term obtained in (14) for $r = m-1$. Then*

$$T_{m-1} = m^2 m!^2\left[\|f\otimes_{m-1} g\|^2 - \langle f\otimes_1 g, g\otimes_1 f\rangle\right].$$

Proof: From (14),

$$\begin{aligned}T_{m-1} &= \frac{1}{2}\sum_{i,l\geq 1}(m-1)!^2\||s_{i,f}\tilde{\otimes}_{m-1}s_{l,g} - s_{l,f}\tilde{\otimes}_{m-1}s_{i,g}\|^2\\ &= \frac{1}{2}\sum_{i,l\geq 1}(m-1)!^2\||s_{i,f}\otimes_{m-1} s_{l,g} - s_{l,f}\otimes_{m-1} s_{i,g}\|^2\\ &= \frac{1}{2}\sum_{i,l\geq 1}(m-1)!^2\left[\langle s_{i,f}, s_{l,g}\rangle - \langle s_{l,f}, s_{i,g}\rangle\right]^2\\ &= (m-1)!^2\left[\sum_{i,l\geq 1}\langle s_{i,f}, s_{l,g}\rangle^2 - \sum_{i,l\geq 1}\langle s_{i,f}, s_{l,g}\rangle\langle s_{l,f}, s_{i,g}\rangle\right]\\ &= (m-1)!^2\left[\sum_{i,l\geq 1}\langle s_{i,f}\otimes s_{i,f}, s_{l,g}\otimes s_{l,g}\rangle - \sum_{i,l\geq 1}\langle s_{i,f}\otimes s_{i,g}, s_{l,g}\otimes s_{l,f}\rangle\right]\\ &= (m-1)!^2 m^4\left[\langle f\otimes_1 f, g\otimes_1 g\rangle - \langle f\otimes_1 g, g\otimes_1 f\rangle\right]\\ &= m^2 m!^2\left[\|f\otimes_{m-1} g\|^2 - \langle f\otimes_1 g, g\otimes_1 f\rangle\right]\end{aligned}$$

where we applied Lemmas 3 and 1. ■.

We first answer the open problem 6.2 in [1] for chaoses of order lesser than five.

Theorem 3 *Let $m \leq 4$ and let $f, g \in H^{\otimes m}$ be symmetric. Then the random vector $(F, G) = (I_m(f), I_m(g))$ does not admit a density if and only if*

$$\det C = 0.$$

In other words, the vector (F, G) does not admit a density if and only if its components are proportional.

Proof: The case $m = n = 1$ is obvious and the case $m = n = 2$ follows from [1] (it also follows from Example 1). Suppose $m = n = 3$. Then

$$\begin{aligned} \det \Lambda &= 9 \det C + 9 \times 36 \times \left((C_2^1)^2 - 1\right) \left[\|f \otimes_1 g\|^2 - \|f \otimes_2 g\|^2\right] \\ &\quad +9 \times 36 \left[\|f \otimes_2 g\|^2 - \langle f \otimes_2 g, g \otimes_2 f\rangle\right] + R_3' \end{aligned}$$

where R_3' is the term with $k = 1$ in (14). Using $\langle f \otimes_1 g, g \otimes_1 f\rangle = \langle f \otimes_2 g, g \otimes_2 f\rangle$ (Lemma 3) we get

$$\begin{aligned} \det \Lambda &= 9 \det C + 9 \times 36 \times 3 \left[\|f \otimes_1 g\|^2 - \langle f \otimes_1 g, g \otimes_1 f\rangle\right] \\ &\quad -9 \times 36 \times 2 \left[\|f \otimes_2 g\|^2 - \langle f \otimes_2 g, g \otimes_2 f\rangle\right] + R_3'. \end{aligned}$$

Suppose $\det \Lambda = 0$. Then T_0, T_1, T_2 from (14) vanish. In particular $T_2 = 0$ in (14) and so

$$\|f \otimes_2 g\|^2 - \langle f \otimes_2 g, g \otimes_2 f\rangle = 0.$$

This implies

$$9 \det C + 9 \times 36 \times 3 \left[\|f \otimes_1 g\|^2 - \langle f \otimes_1 g, g \otimes_1 f\rangle\right] = 0$$

and therefore $\det C = 0$ because $\|f \otimes_1 g\|^2 - \langle f \otimes_1 g, g \otimes_1 f\rangle$ is positive by Cauchy-Schwarz.

Suppose $m = n = 4$.

$$\begin{aligned} \det \Lambda &= 16 \det C + 16 \times 4!^2 \left((C_3^1)^2 - 1\right) \left[\|f \otimes_1 g\|^2 - \|f \otimes_3 g\|^2\right] \\ &\quad +16 \times 4!^2 \left[\|f \otimes_3 g\|^2 - \langle f \otimes_3 g, g \otimes_3 f\rangle\right] + R_4' \end{aligned}$$

where R_4' is the sum of terms obtained for $k=1$ and $k=2$ in (14). Since $\langle f\otimes_3 g, g\otimes_3 f\rangle = \langle f\otimes_1 g, g\otimes_1 f\rangle$(Lemma 3) we get

$$\begin{aligned}\det\Lambda &= 16\det C + 16\times 4!^2\left((C_3^1)^2-1\right)\left[\|f\otimes_1 g\|^2 - \langle f\otimes_1 g, g\otimes_1 f\rangle\right]\\ &\quad -16\times 4!^2\left((C_3^1)^2-2\right)\left[\|f\otimes_3 g\|^2 - \langle f\otimes_3 g, g\otimes_3 f\rangle\right] + R_4'.\end{aligned}$$

Assume $\det\Lambda = 0$. Then in particular T_3 from 14) vanishes. So

$$\|f\otimes_3 g\|^2 - \langle f\otimes_3 g, g\otimes_3 f\rangle = 0$$

and this implies $\det C = 0$.

Remark 2 *For $m=n\geq 5$, we have*

$$\begin{aligned}\det\Lambda &= 25\det C + 25\times 5!^2\left((C_4^1)^2-1\right)\left[\|f\otimes_1 g\|^2 - \|f\otimes_4 g\|^2\right]\\ &\quad +25\times 5!^2\left((C_4^2)^2-1\right)\left[\|f\otimes_2 g\|^2 - \|f\otimes_3 g\|^2\right]\\ &\quad +25\times 5!^2\left[\|f\otimes_4 g\|^2 - \langle f\otimes_4 g, g\otimes_4 f\rangle\right] + R_5'\end{aligned}$$

If $\det\Lambda = 0$ then, since T_4 vanishes, we get that $\|f\otimes_4 g\|^2 - \langle f\otimes_4 g, g\otimes_4 f\rangle$ vanishes. But this is not enough. We need some additional information in order to handle the difference $\|f\otimes_2 g\|^2 - \|f\otimes_3 g\|^2$. One possibility is to look to the terms T_1, T_2, T_3 in (14) but these terms cannot be written in a closed form, since they involve more complicated contractions (some "contractions of contractions").

Let us finish by some comments concerning the case of variables in chaoses of different orders. Consider $(F,G) = (I_n(f), I_m(g))$ with $n\neq m$. First, let us note that $E\det\Lambda = 0$ does not imply $\det C = 0$. This can be viewed by considering the following example.

Example 2 *Take $F = I_2(f)$ and $G = I_2(h^{\otimes 2})$ where $\|h\| = 1$. In this case*

$$\det C = 2 \text{ and } \det\Lambda = 0.$$

One can also choose $F = I_n(h^{\otimes n})$ and $G = I_m(h^{\otimes m})$ with $m\neq n$ and $\|h\| = 1$.

In the case $(I_n(f), I_1(g))$ there is only one term in (14) obtained for $k=0$. It reads

$$T_0 = nn!\left[\|f\otimes_2 g\|^2 - \|f\otimes_1 g\|^2\right].$$

and therefore the condition for the existence of the joint density is $\|f\otimes_2 g\|^2 - \|f\otimes_1 g\|^2 > 0$.

The case $(I_n(f), I_2(g))$ is more complicated and needs new ideas in order to obtain the if and only if condition for the existence of the density of the vector. Even the "last term"in (14) (that is, the term obtained for $k = (m-1)\wedge(n-1)$ cannot be written is a nice form.

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