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# Adoption dynamics: sequential or synchronous modelling

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## Abstract

This paper deals with the choice of dynamics in spatial simulation and modelling. In economical context,  $N$  agents choose between two technological standards according to a local assignment rule. The adoption dynamics is sequential if the choices are made one after the other; it is synchronous or partially synchronous if all or some part of the agents choose simultaneously. This paper points out differences between the three dynamics, especially in their evolution.

*Key words:* standard adoption, sequential dynamics, synchronous dynamics, partial parallelism, Markov chain, ergodicity.

## 1 Introduction

In many applications, we are interested in the study of the evolution of a system, in space and in time. For instance it may concern competition between different species in ecology, diffusion of technological innovation involving social behaviour, particles system in physics, fluid spread model, disease propagation, image sequences. Some models are deterministic (cellular automata (Chopard et al. 2002), differential equations for instance), while others are based on a stochastic framework, like Gibbs dynamics, probabilistic automata, or particle systems (David and Foray 1993; Cox 1989; Galam 1997).

Our framework is standard's adoption (Arthur 1989; Banerjee 1992; Bikhchandani et al. 1998) but the results may apply to other applications. We do not provide new theoretical results but we hope to shed light on different modellings, offering information tools about the ins and outs.

We consider a finite set of sites  $S = \{1, 2, \dots, N\}$ . Each site  $i$  is associated to an agent who makes a choice  $X_i$  in a state space  $E$ . The state space  $E$  can be finite or not. In all the following, we will assume for simplicity  $E = \{-1, +1\}$ , which is associated to a choice between two competitive technologies. When this choice  $X_i$  depends of the local context, we say that there is spatial coordination, the spatial dependency being positive if there is cooperation between the agents, and negative in case of competition. We proposed in Guyon and Hardouin (2001) tests for spatial coordination allowing to distinguish between "independent" and influenced choices. We consider in this work probabilistic assignment rules depending on the neighbourhood, and compare different adoption dynamics: sequential, synchronous or partially synchronous.

A *scan* or a *sweep* of  $S$  is a tour of all the sites. We studied in Hardouin (2007) and Hardouin (2008) the case of a non iterative dynamics with a unique sequential scan of  $S$ , under the assumption of an initial occurrence of standards  $A$ . When the scans are sequential and indefinitely repeated, the agents make their decision one by one; then we get the well known Gibbs sampler and it is possible to characterize the probability distribution of the limit configuration. When the dynamics is synchronous, all the agents make their decision simultaneously, there is still ergodicity but it is difficult to explicit the limit distribution (See Geman and Geman 1984, Geman1990 for a full description). However this is possible in some cases and we present such an example. Finally, partial synchronous dynamics run step by step, a significant part of the changes happening simultaneously at each step; the latter has not been much studied though it is often used in economic applications; Winkler (2006) presents a result for conditional independent subsets; however, most of the results are provided for simulated annealing (Trouvé 1992; Trouvé 1993).

Our purpose is not to discuss about the choice of the dynamics. We just want to point out that, for a same local assignment rule, the configurations of the systems can differ widely according to a synchronous or sequential course.

In section 2, we briefly describe the difference between deterministic and probabilistic assignment rules, through standard examples. Then we present the sequential, synchronous and partially synchronous dynamics in section 3, followed by their ergodic properties in section 4. Some illustrating examples are given in section 6.

## 2 Assignment rules

Let us specify the model and give some notations.  $S$  is equipped with a symmetric graph  $\mathcal{G}$  and  $\langle i, j \rangle$  denotes a pair of neighbouring sites  $i$  and  $j$ . If  $A$  is a subset of  $S$ , we denote  $\partial A = \{i \in S, i \notin A \text{ and } \exists j \in A \text{ s.t. } i \text{ and } j \text{ are neighbouring sites}\}$  the neighbourhood of  $A$ , and  $\partial i = \partial\{i\}$ . Let us note  $x = (x_1, x_2, \dots, x_N)$  a realization of  $X = (X_1, X_2, \dots, X_N)$  in  $\Omega = E^S$ ; for a subset  $A \subset S$ ,  $x_A$  (resp.  $x^A$ ) is the configuration  $x$  on  $A$  (resp. outside of  $A$ ), and  $x^i = x^{\{i\}}$ . The agent  $i$  makes his choice according to a local assignment rule  $\pi_i(\cdot | x_{\partial i})$  depending on  $x_{\partial i}$ . We give below two commonly used examples of deterministic and probabilistic rules.

### Example 1 *Deterministic Majority choice*

Let  $S = \{1, 2, \dots, n\}^2$  be a square lattice of size  $n \times n$ , with the 4 nearest neighbours system. The agent  $i$  chooses the state  $+1$  (resp.  $-1$ ) if  $+1$  (resp.  $-1$ ) is majority among his neighbours, and makes a choice at random in case of equality. If we add the assumption that the agent also takes into account his own advice, or private information, then there is always a majority state among the 5 sites of  $\partial i \cup \{i\}$  and the rule is deterministic; the system is a kind of cellular automata. For those two rules, general consensus (same state everywhere) is an absorbing state. That means that if the number of scans is large, one of the technological standards will emerge and dominate, ending the only one. The only point is to know which one of the standards will disappear, and the necessary number of scans to determine the winner. The answer depends on the rule, that is the kind of majority, and on the initial rates of the two standards.

### Example 2 Probabilistic Ising rule

Let us consider an Ising type model; we note  $N_i(x) = N(x_{\partial i}) = \sum_{j \in \partial i} x_j$ ; then agent  $i$  chooses state +1 with probability:

$$\pi_i(x_i | x_{\partial i}) = \frac{\exp x_i(\alpha + \beta N_i(x))}{\exp(\alpha + \beta N_i(x)) + \exp(-(\alpha + \beta N_i(x)))}. \quad (1)$$

This probability is nothing but the conditional distribution probability of a Gibbs field on  $\Omega$  with a joint distribution given by:

$$\pi(x) = Z^{-1} \exp\left\{\alpha \sum_{i \in S} x_i + \beta \sum_{\langle i, j \rangle} x_i x_j\right\} \quad (2)$$

The normalization constant  $Z$  of the joint distribution, which is often computationally intractable, does not occur in the expression of local distributions.

In this example, the choice of the agent depends on two parameters  $\alpha$  and  $\beta$ . The parameter  $\alpha$  is a measure of the global frequency of +1 and -1;  $\alpha > 0$  strengthens states +1 while  $\alpha < 0$  increases the number of states -1, and  $\alpha = 0$  balances the two standards. The parameter  $\beta$  determines the resemblance or dissimilarity between neighbouring sites. There is cooperation between neighbouring sites if  $\beta > 0$ , while  $\beta < 0$  ensures competition. If  $\beta = 0$ , the assignment is independent of the neighbourhood.

If we set  $\alpha = 0$  and  $\beta > 0$ ,  $\beta$  rather large, the rule meets the previous majority choice. Then, a deterministic rule can be approximated by a probabilistic rule.

In all the following, we consider the case of a probabilistic rule in terms of conditional distributions; this implies to define properly the underlying model. Let us consider the general positive distribution  $\pi$  on the configuration set  $\Omega$ :

$$\pi = \{\pi(x), x \in \Omega\}, \text{ with } \pi(x) > 0 \text{ for all } x \text{ and } \sum_{\Omega} \pi(x) = 1 \quad (3)$$

The positivity condition allows us to define, for all  $A$  and  $x^A$ , the conditional probabilities  $\pi_A(\cdot | x^A)$ , particularly the conditional distributions  $\{\pi_i(\cdot | x^i), i \in S\}$ . When  $\pi_A(\cdot | x^A)$  depends only on  $x_{\partial A}$ ,  $\pi$  refers to a Markov random field, as in the example above (2).

## 3 Sequential, synchronous dynamics

Let us describe the three dynamics involved in standards adoption.

### The sequential dynamics

A sequential dynamics is defined by a sequence of scans of the set of sites  $S$ . For instance, we browse the sites 1 to  $N$ , sequentially, in this order.

◦ step 1: the initial state is  $x = (x_1, x_2, \dots, x_N)$ .

◦ step 2: we browse the sites 1 to  $N$ ; at the  $k$ -th site, we (uniquely) relax (modify) its value according to the local conditional assignment rule, and conditionally to the previous configuration,

$$x_k \mapsto y_k \text{ according to } \pi_k(y_k | y_1, y_2, \dots, y_{k-1}, x_{k+1}, \dots, x_N).$$

◦ step 3: we come back to step 1 with the new initial state  $y = (y_1, y_2, \dots, y_N)$ .  
 Then a scan of  $S$  changes the configuration  $x$  to the new one  $y = (y_1, y_2, \dots, y_N)$  in  $N$  steps.

Some variants are possible: (i) the route to visit all sites can be different from one scan to the other; (ii) an individual site can be visited several times during the scan, the important point being to visit all sites; (iii) the order of the visits can be chosen at random; (iv) the release can also be done by groups of sites, one group followed by another one, with  $S$  being the union of the groups, and each group evolving internally sequentially. As we will see, all those sequential procedures are asymptotically equivalent, leading to the same stationary distribution (Geman 1990; Guyon 1995).

### The synchronous dynamics

Synchronous dynamics is also called total parallelism; in fact, all the sites are relaxed simultaneously in one scan.

◦ step 1 : the initial state is  $x = (x_1, x_2, \dots, x_N)$ .

◦ step 2: we release simultaneously all the states, getting  $x$  to  $y = (y_1, y_2, \dots, y_N)$  with the simultaneous rules on each site  $k$ ,

$x_k \mapsto y_k$  according to  $\pi_k(y_k | x^k) = \pi_k(y_k | x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_N)$ ,  $k \in S$ .

◦ step 3 : we come back to step 1 with the new initial state  $y$ .

**Partial parallelism.** Between sequential and synchronous dynamics, we can define partially synchronous dynamics.

Let  $M$  be an integer,  $1 \leq M \leq N$ .

◦ step 1 : the initial state is  $x = (x_1, x_2, \dots, x_N)$ .

◦ step 2: we choose a subset  $A$  of  $S$  with  $M$  elements ( $|A| = M$ ) and we simultaneously modify the values of the sites in  $A$ , while the other sites remain unchanged, getting  $x = (x_A, x^A)$  to  $y = (y_A, x^A)$  with the simultaneous rules on each site of  $A$ ,

$x_k \mapsto y_k$  according to  $\pi_k(y_k | x^k) = \pi_k(y_k | x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_N)$ ,  $k \in A$ .

◦ step 3 : we come back to step 1 with the new initial state  $y = (y_A, x^A)$ .

The ratio  $\tau = \frac{M}{N}$  is called the parallelism rate;  $M = 1$  corresponds to the sequential dynamics, while  $M = N$  defines the synchronous one. Let us precise that iterating the dynamics, we choose a new subset  $A$  at each step 2. This hybrid dynamics depends on the way of choosing  $A$ . It can be chosen at random, for instance with a uniform distribution giving the same weight to the  $\binom{N}{M}$  subsets of  $S$  with  $M$  elements, or not, for instance we fix a covering of  $S$  with subsets of cardinal  $M$ .

More generally, we can consider several rates of active sites; let  $A_1, \dots, A_n$  be some subsets of  $S$  such that  $\cup_i A_i = S$ ; at each step, we choose a subset  $A_i$  of  $S$  with probability  $\gamma(A_i) > 0$  and we update the sites of  $A_i$ .

We distinguish a particular case of partial parallelism; let us assume that there is a neighbourhood graph on  $S$ ; a coding subset  $C$  is a subset of  $S$  such that any two sites of  $C$  are not neighbours with respect to this graph. Then, let us consider a partition of coding subsets  $\{C_1, C_2, \dots, C_k\}$  of  $S$ ; if we run

the previous partial algorithm with these coding subsets, then it meets the sequential dynamics. As for a simple example, consider the square lattice for  $S$  with the four nearest neighbours system,  $C$  is the subset of the “black” nodes, and  $\bar{C}$  is the white ones. Obviously,  $C$  and  $\bar{C}$  are coding subsets and  $S = C \cup \bar{C}$ ; changing simultaneously the black sites then the white ones leads to the same result as changing all the sites one by one in  $n^2$  steps. In fact, there is no interaction between the sites that simultaneously change.

## 4 Ergodicity

### 4.1 General results

Let us consider the same generating distribution  $\pi$  for each dynamics; we assume that, for each dynamics, we repeat the scans a large number of times; the following result shows that the final configurations differ from each other; specifically, in the case of sequential dynamics, the generating distribution  $\pi$  is stationary, whereas it is not the case for toher dynamics. The result is obtained writing the dynamics in terms of Markov chains.

Let us note  $\sigma_k$  the  $k$ -th scan,  $x = X(k) = (X_1(k), X_2(k), \dots, X_N(k))$  and  $y = X(k+1)$  the configurations before and after the  $k^{\text{th}}$  scan; let us write  $P = (P(x, y))_{x, y \in \Omega}$ , the dynamics’ transition matrix for the scan  $\sigma_k$  defined by:

$$P_{\sigma_k}(x, y) = P(X(k+1) = y \mid X(k) = x), \quad x, y \in \Omega$$

The following properties hold for  $X = (X(k), k \geq 0)$ , the evolution of these configurations.

**Proposition 3** *Let  $X = (X(k), k \geq 0)$  be the dynamics generated by probability  $\pi$ .*

- (1)  *$X$  is an ergodic Markov chain on  $\Omega$ .*
- (2) *For a sequential dynamics, the invariant distribution is  $\pi$ .*
- (3) *For a synchronous dynamics, the invariant distribution is  $\nu$ , and  $\nu$  differs from  $\pi$ .*
- (4) *For a partially synchronous dynamics with  $M \geq 2$ ,  $\tau = \frac{M}{N}$ , the invariant distribution is  $\lambda_\tau$ , and  $\lambda_\tau$  differs from  $\pi$ .*

We give hereafter the main lines of the proof and refer the reader for instance to Feller (1968) or Isaacson (1985), Kemeny and Snell (1960) for general results on Markov chains.

### 4.2 Sequential dynamics

The transition is for one scan

$$P_\sigma(x, y) = \prod_{i=1, N} \pi_i(y_i \mid y_1, y_2, \dots, y_{i-1}, x_{i+1}, \dots, x_N) > 0.$$

If we have different ways of scanning, we note  $P_k = P_{\sigma_k}$  and  $\mu$  the initial distribution of  $X(0)$ ;  $X$  is an inhomogeneous chain, and the distribution of  $X(k)$  is

$$X(k) \sim \mu P_1 P_2 P_3 \cdots P_k.$$

On the other hand, if the scanning order is always the same,  $\sigma_k \equiv \sigma$  for all  $k \geq 1$ , the chain is homogeneous with transition probabilities  $P = P_\sigma$ , and  $X(k) \sim \mu P^k$ .

It is easy to see that  $\pi$  is invariant for each  $P_\sigma$  which is strictly positive; therefore  $\pi$  is the stationary distribution. For instance in the homogeneous case, we write

$$\forall x \in \Omega, \quad P^k(x, y) \xrightarrow[k \rightarrow \infty]{} \pi(y).$$

Hence, if we repeat the scans a large number of times, the “final” layout of the standards depends on  $\pi$  and its parameters.

Application: this result enables one to simulate any law  $\pi$ ; it suffices to use it as generating distribution in the sequential dynamics. This procedure is the well-known Gibbs sampler.

### 4.3 Synchronous dynamics

#### 4.3.1 The general case

Let us write the transition

$$Q(x, y) = \prod_{i=1}^N \pi_i(y_i \mid x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_N) = \prod_{i=1}^N \pi_i(y_i \mid x^i).$$

Again, this expression is strictly positive, which ensures the ergodicity of the dynamics:

$$\forall x \in \Omega, \quad Q^k(x, y) \xrightarrow[k \rightarrow \infty]{} \nu(y)$$

But the stationary distribution  $\nu$  is different from  $\pi$  and in most cases, not explicit. Indeed,  $\pi$  is no more invariant for  $Q$ .

$\nu$  verifies  $\nu Q = \nu$ ; then  $\nu$  is an eigenvector associated to  $Q$  and the eigenvalue 1. The search of this eigenvector is difficult because of the high dimension of the matrix  $Q$  (originated by the large cardinal number of  $\Omega$ ) while in the sequential case, this search of the eigenvector is trivially solved since  $\pi P = \pi$ .

However, if the  $\pi_i$  are the conditional distributions of the nearest neighbours Ising model, it is possible to write  $\nu$ ; we detail this in the following example (Trouvé 1988).

#### 4.3.2 The Ising model

We consider the torus  $S = \{1, 2, \dots, n\}^2$ ,  $n$  being even, equipped with the 4 nearest neighbours system. The generating distribution  $\pi$  is the trimered Gibbs law (1) :

$$\pi_i(x_i \mid x_{\partial i}) = \frac{\exp x_i(\alpha + \beta N_i(x))}{2ch(\alpha + \beta N_i(x))},$$

with  $N_i(x) = \sum_{j:|i-j|=1} x_j$ .

Then the transition for one synchronous scan is

$$Q(x, y) = \prod_{i \in S} \frac{\exp y_i(\alpha + \beta N_i(x))}{2ch(\alpha + \beta N_i(x))}$$

One can show that the invariant distribution  $\nu$  for the transition  $Q$  is

$$\nu(x) = \Gamma^{-1} \exp\{\alpha \sum_{i \in S} x_i\} \prod_{i \in S} ch(\alpha + \beta N_i(x))$$

where  $\Gamma$  is a normalization constant.

Indeed, it is easily seen that the transition matrix  $Q$  is  $\nu$ -reversible, which implies that  $\nu$  is  $Q$ -invariant:

$$\begin{aligned} \nu(x)Q(x, y) &= \Gamma^{-1} \exp\{\alpha \sum_{i \in S} x_i\} \exp\{\alpha \sum_{i \in S} y_i\} \prod_{i \in S} \exp\{\beta y_i N_i(x)\} \\ &= \Gamma^{-1} \exp\{\alpha \sum_{i \in S} x_i\} \exp\{\alpha \sum_{i \in S} y_i\} \prod_{i \in S} \exp\{\beta x_i N_i(y)\} \\ &\quad (\text{since for instance for } i = (s, t), \sum y_{s,t}(x_{s-1,t} + x_{s+1,t}) = \sum x_{s,t}(y_{s-1,t} + y_{s+1,t})) \\ &\text{that is } \nu(x)Q(x, y) = \nu(y)Q(y, x). \end{aligned}$$

From this explicit expression for  $\nu$ , let us underline two important differences between the sequential and synchronous dynamics:

(i) We have seen that the  $\pi_i$  are the conditional distribution probability of a Gibbs field on  $\Omega$  with a joint distribution given by  $\pi(x) = Z^{-1} \exp\{\sum_{i \in S} \alpha x_i + \beta \sum_{i \in S} \sum_{j:|i-j|=1} x_i x_j\}$ . From this form, we see that  $\pi$  is a Markov distribution and the cliques of the associated neighbourhood graph (the four nearest neighbours) are of order 1 and 2, made up with singletons and pairs of sites at distance 1, which are of two types, horizontal or vertical.

Similarly, we write  $\nu(x) = \Gamma^{-1} \exp\{\sum_{i \in S} \alpha x_i + \sum_{i \in S} \log\{ch(\alpha + \beta \sum_{j:|i-j|=1} x_j)\}\}$ .

Then  $\nu$  is a Markov distribution, like  $\pi$ . But the neighbourhood system is quite different: the cliques are the singletons and the sets of four sites that are squares of side  $\sqrt{2}$ .

(ii) Let us denote  $S^+$  the subset of sites  $i = (u, v)$  with  $u + v$  even (the black fields on a chequer board),  $S^-$  the complementary subset (the white fields), and  $x^+$  (resp.  $x^-$ ) the configuration on  $S^+$  (resp.  $S^-$ ).

We define  $\nu^+(x^+) = \Gamma^{-\frac{1}{2}} \exp\{a \sum_{i \in S^+} x_i^+\} \prod_{i \in S^-} ch(a + b N_i(x^+))$  and  $\nu^-(x^-) = \Gamma^{-\frac{1}{2}} \exp\{a \sum_{i \in S^-} x_i^-\} \prod_{i \in S^+} ch(a + b N_i(x^-))$ .

We have  $\nu(x) = \nu^+(x^+) \nu^-(x^-)$ : contrary to the sequential dynamics, the synchronous evolutions on  $S^+$  and  $S^-$  are independent from each other. Figure 1 and Figure 2 in section 5 illustrate this difference.

## 4.4 Partially synchronous dynamics

Let us denote  $R$  the transition matrix of this dynamics.

### 4.4.1 The general case

We choose the subset  $A$  at random, for instance uniformly in the set of the subsets of cardinal number  $M$  (with  $\tau = \frac{M}{N}$ ). Therefore,

$$R(x, y) = \binom{N}{M}^{-1} \sum_{A \subset S: |A|=M} \left\{ \mathbf{1}(x^A = y^A) \prod_{k \in A} \pi_k(y_k | x^A) \right\}$$

This transition is positive. Hence the partially synchronous dynamics is ergodic, with the stationary distribution  $\rho_\tau$ ,

$$\forall x \in \Omega, R^m(x, y) \xrightarrow{m \rightarrow \infty} \lambda_\tau(y).$$

If  $M \geq 2$ , we verify that  $\pi$  is not invariant for  $R$ , and then  $\lambda_\nu \neq \pi$ .

More generally, let us consider  $A_1, \dots, A_n$  some subsets of  $S$  such that  $\gamma(A_i) > 0$  (3); the dynamics is ergodic with limit distribution  $\lambda_\gamma$  if and only if  $\cup_i A_i = S$  (Trouvé 1988).

Again,  $\lambda_\tau$  is not explicit but we have some properties:

If the parallelism rate tends to zero, and for a fix value of the interaction parameter  $\beta > 0$ , we get a continuity property with  $\lim_{\tau \rightarrow 0} \lambda_\tau = \pi$  (Trouvé 1993 Theorem 2.7).

On the other hand,  $\lim_{\beta \rightarrow +\infty} \lambda_\tau(\beta) = \lambda_\tau(\infty) = \lambda(\infty)$  if  $0 < \tau < 1$ , and the limit distribution  $\lambda_\tau(\infty)$  does not depend on  $\tau$  (Trouvé 1992); this is no more true for  $\tau = 1$ . Therefore we can have discontinuity for large  $\beta$ . This is illustrated by Figure 3 in the next section.

#### 4.4.2 The coding case

On the other hand, if  $\pi$  has a Markov property with respect to a neighbourhood graph, and if we choose a partition  $\{C_1, C_2, \dots, C_k\}$  of coding subsets of  $S$ , then

$$R_{C_k}(x, y) = \mathbf{1}(x^{C_k} = y^{C_k}) \prod_{s \in C_k} \pi_s(y_s | x^{C_k}), \quad k = 1 \text{ à } K$$

and  $R(x, y) = R_{C_1} \dots R_{C_K}(x, y)$ ; but for each  $k$  and each  $C_k = \{s_1, \dots, s_{|C_k|}\}$ ,  $R_{C_k}(x, y) = \pi_{s_1} \dots \pi_{s_{|C_k|}}(x, y)$ ; finally  $R$  coincides with the transition probability  $P$  for a sequential sweep of  $S$ .

## 5 Some illustrating examples

We propose a simulation experiment to illustrate the differences of the previous dynamics. As for the generating distribution we consider the Ising model defined in the previous section with the four or eight nearest neighbours system, with conditional distributions

$$\pi_i(x_i | x_{\partial i}) = \frac{\exp x_i(\alpha + \beta V_i(x) + \gamma W_i(x))}{2ch(\alpha + \beta V_i(x) + \gamma W_i(x))}$$

where  $V_i(x)$  is the sum of the four nearest neighbours (at distance 1) of site  $i$  and  $W_i(x)$  is the sum of the four diagonal neighbours (at distance  $\sqrt{2}$ ) of site  $i$ . If  $\gamma = 0$ , then we come back to the four nearest neighbours. We will consider  $\alpha = 0$ , that is +1 and -1 occur with the same probability.

We initialize at random and for one initialization we simulate the three dynamics described above: the sequential, synchronous and partially synchronous dynamics. Each simulation of the sequential and synchronous dynamics is obtained running 600 scans on a square toric lattice of size  $64 \times 64$ . In the case of

partial parallelism, we iterate the scans until each site has been visited at least 600 times. We also compute the empirical spatial correlation  $\rho_1$  at distance 1 (based on the four nearest neighbours) and  $\rho_8$  based on the eight nearest neighbours.

The parameters are  $\beta$  and  $\gamma$ , and the parallelism rate  $\tau$ . First we present below some examples of realizations of distributions  $\pi$ ,  $\nu$ , and  $\lambda_{0.5}$  obtained from the same generating distribution  $\pi$  for different values of  $\beta$  and  $\gamma$ . We can see that the resulting configurations may strongly visually differ. On the other hand, for other sets of parameters, we may have similar final configurations; for instance we get perfect or nearly chessboard images for  $\pi$ ,  $\nu$ , and  $\lambda_{0.5}$  for parameters  $\beta = -0.5$  and  $\gamma = +0.5$ , and configurations looking like the one of the sequential (and partially synchronous) case with  $\beta = 1$ ,  $\gamma = 0$  (see below) for the three  $\pi$ ,  $\nu$ , and  $\lambda_{0.5}$  and for parameters  $\beta = \gamma = 0.5$ .

Fig. 1: Simulations of fields with parameters  $\beta = 1$ ,  $\gamma = -1$  for different dynamics  
Sequential,  $\tau = 0$       Partially synchronous  $\tau = 50\%$       Synchronous,  $\tau = 1$

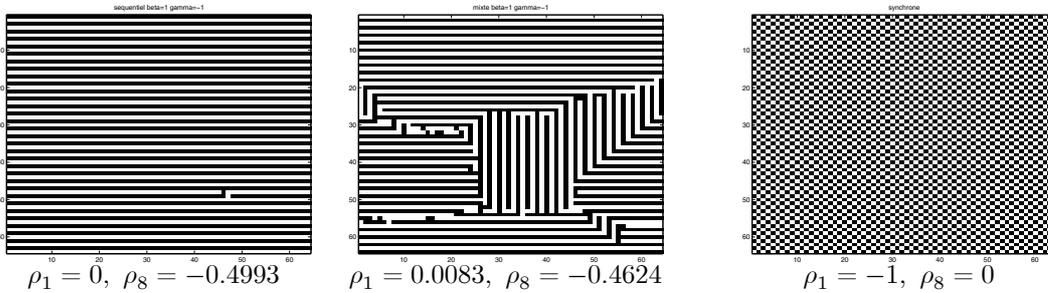
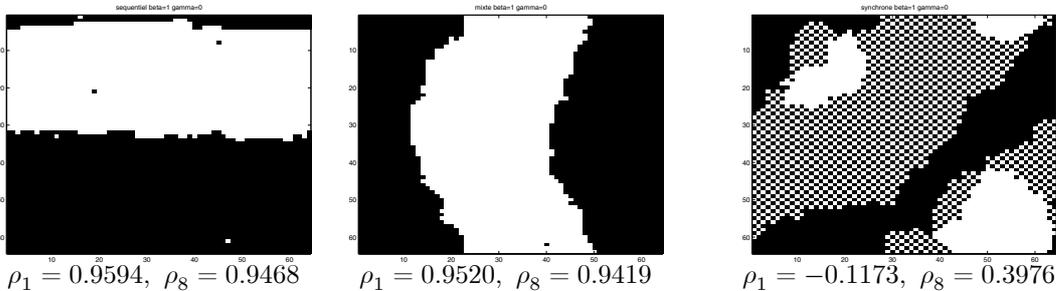
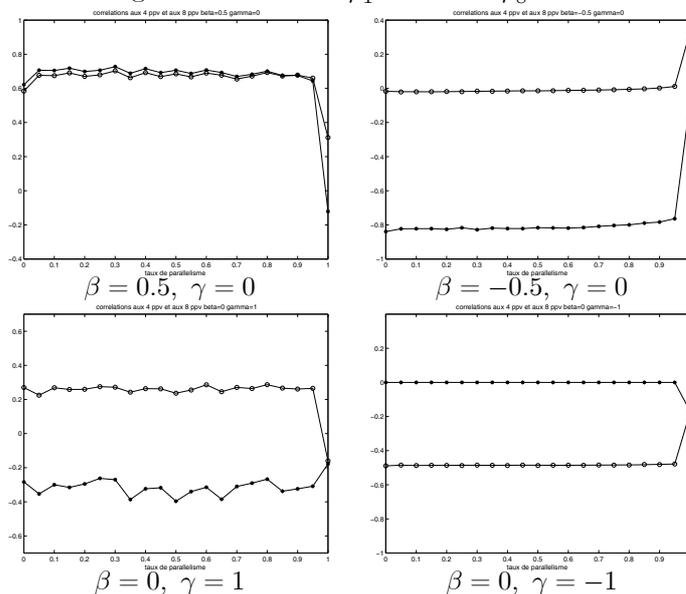


Fig. 2: Simulations of fields with parameters  $\beta = 1$ ,  $\gamma = 0$  for different dynamics  
Sequential,  $\tau = 0$       Partially synchronous  $\tau = 50\%$       Synchronous,  $\tau = 1$



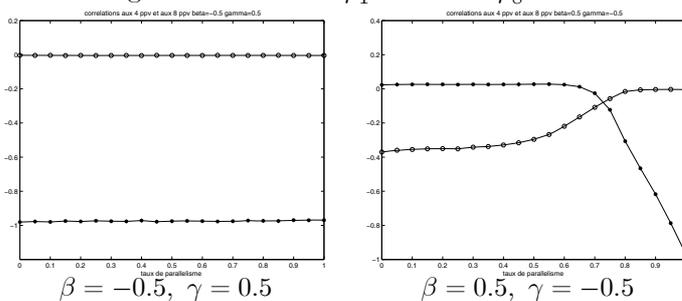
Then we consider only the partially synchronous dynamics. We repeat hundred times each simulation and compute the mean of the empirical correlations  $\rho_1$  and  $\rho_8$ . We draw the evolution of those two correlations when  $\tau$  increases from 0 to 1 by steps of 5%. We know that for large positive values of  $\beta$ , the stationary distribution doesn't depend of  $\tau$ ,  $0 < \tau < 1$ ; in fact, we observe that the correlations are constant from 5% to 95% in the cases  $\beta > 0$  ( $\beta \geq 0.5$  is large enough) and  $\gamma = 0$ , but also if  $\beta < 0$  ( $\beta \leq -0.5$ ) and  $\gamma = 0$ . In both cases we observe constancy or a light gap between the sequential dynamics with  $\tau = 0$  and the partial dynamics with  $\tau = 5\%$ ; most of all, we observe discontinuity between  $\tau = 0.95$  and  $\tau = 1$ . The same thing occurs if we permute parameters  $\beta$  and  $\gamma$  (for instance  $\beta = 0$ ,  $\gamma = 1$  or  $\beta = 0$ ,  $\gamma = -1$ ), see Figure 3.

Fig. 3: Correlations  $\rho_1$  —\*— and  $\rho_8$  —o—



Finally, if the parameters have different signs, we observe different behaviours; we observe constant correlations in the case  $\beta = -0.5, \gamma = 0.5$ ; in fact the observed images at different parallelisms rates are all similar to the sequential and synchronous dynamics cases, leading to chessboard like configurations (see Figure 4a). On the other hand, if  $\beta > 0$ , and  $\gamma < 0$ , the behaviour of the correlations is quite different; the correlations are quite equal for small parallelism rate and then slowly meet the total synchronous correlations values. It seems there is a threshold rate from which the configurations change, from the correlations point of view (see Figure 4b)

Fig. 4: Correlations  $\rho_1$  —\*— and  $\rho_8$  —o—



In conclusion, our simulations allow us to illustrate the theoretical results; we have shown that the choice of the dynamics is very important in standards adoption context, as well as in other application fields. Moreover, we point out that except in specific examples, the limit ergodic distributions remain unknown for the synchronous and partially (general) synchronous choices; we may suppose that they coincide or strongly differ for specific values of the dynamics parameters, as presented in the previous examples.

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