Regularity of the American put option in the Black-Scholes model with general discrete dividends

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Regularity of the American put option in the Black-Scholes model with general discrete dividends

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Abstract

We analyze the regularity of the value function and of the optimal exercise boundary of the American Put option when the underlying asset pays a discrete dividend at known times during the lifetime of the option. The ex-dividend asset price process is assumed to follow the Black-Scholes dynamics and the dividend amount is a deterministic function of the ex-dividend asset price just before the dividend date. This function is assumed to be non-negative, non-decreasing and with growth rate not greater than 1. We prove that the exercise boundary is continuous and that the smooth contact property holds for the value function at any time but the dividend dates. We thus extend and generalize the results obtained in [JV11] when the dividend function is also positive and concave. Lastly, we give conditions on the dividend function ensuring that the exercise boundary is locally monotonic in a neighborhood of the corresponding dividend date.

Introduction

We consider the American put option with maturity T and strike K written on an underlying stock S. Like in [JV11], we assume that the stochastic dynamics of the ex-dividend price process of this stock can be modelled by the Black Scholes model and that this stock is paying discrete dividends at deterministic times $0 \leq t_d^I < t_d^{I-1} < \cdots < t_d^i < \cdots < t_d^1 < T$. At each dividend time t_d^i, the value of the stock becomes $S_{t_d^i} = S_{t_d^i-} - D_i\left(S_{t_d^i-}\right)$ where $D_i(S_{t_d^i-})$ is the value of the dividend payment (see Figure 1). We suppose that each dividend function $D_i : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing, non-negative and such that $x \mapsto x - D_i(x)$ is also non-decreasing and non-negative.

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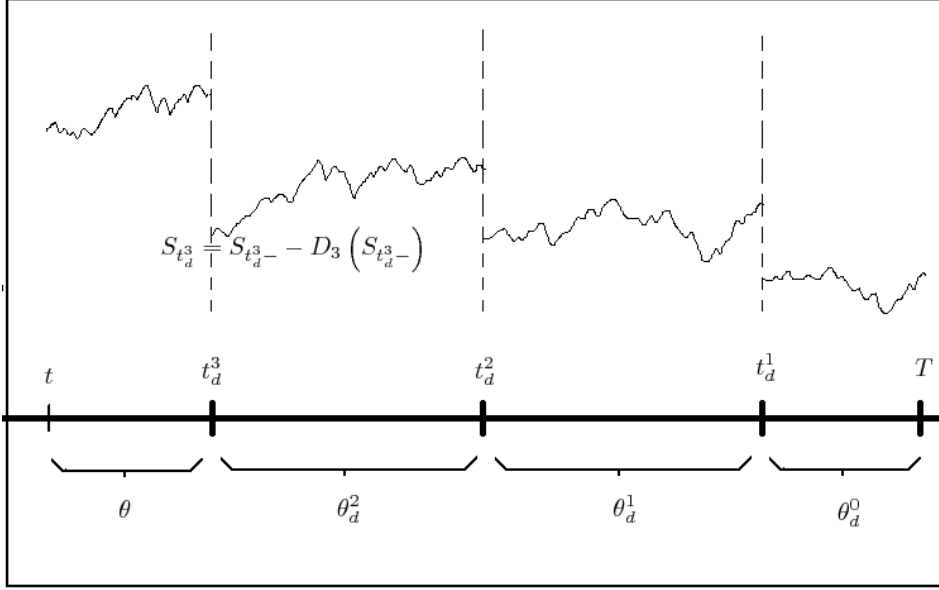
$$S_{t_d^3} = S_{t_d^3-} - D_3\left(S_{t_d^3-}\right)$$

Figure 1: A trajectory of the stock price process

We are interested in the value of the American Put option with strike K and maturity T. Since we are in a Markovian framework, the price can be characterized in terms of a value function depending of the time t and the stock price at time t. For the sake of consistency, we will denote this value function by u_0 for the case without dividends.

The case without dividend was studied by McKean [McK65] and Van Moerbeke [VM76]. McKean first linked this optimal stopping time problem to a free-boundary problem involving both the pricing function u_0 and the exercise boundary denoted by c_0. As it is proved in a more general framework in [EK81], a stopping time solving this optimal stopping time problem is given by the first time the price process crosses this boundary. Van Moerbeke derived an integral equation which involves both c_0 and its derivative, but in later work by Kim [Kim90], Jacka [Jac93] and Carr, Jarrow and Myneni [CJM92] an integral equation was derived which only involves c_0 itself. The regularity and uniqueness of solutions to this equation was left as an open problem in those papers. Uniqueness was proven by Peskir [Pes05]. Convexity was proved in [CCJZ08] and in [Eks04]. Infinite regularity of c_0 at all points prior to the maturity was formally proved by Chen and Chadam [CC06]. Then Bayraktar and Xing [BX09] proved that this remains true if the underlying asset pays continuous dividends at a fixed rate. In practice, continuous dividends are not a satisfying model since dividends are paid once a year

or quarterly. That is why we are interested in dividends that are paid at a number of discrete points in time.

When we assume discrete dividend payments, in general, the value function of the Put option will no longer be convex in the stock price variable, even if convexity is preserved for linear dividend functions. Moreover, the optimal exercise boundary will become discontinuous at the dividend dates and before the dividend dates it may not be monotone. Integral formulas for the exercise boundary which are similar to the ones in [CJM92] have been derived under the assumption that the boundary is Lipschitz continuous (see Göttsche and Vellekoop [GV11]) or locally monotonic (Vellekoop & Nieuwenhuis [VNar]). In this paper we continue the study, undertaken in [JV11], of conditions under which such regularity properties of the optimal exercise boundary under discrete dividend payments can be proven.

We prove that the exercise boundary is continuous at any time which is not a dividend date and that the smooth contact property holds for the value function of the option. We considerably extend the results obtained in [JV11], where the continuity of the exercise boundary and the smooth contact property were only obtained in a left-hand neighborhood of the first dividend date when the corresponding dividend function was assumed to be globally concave and linear with a positive slope in a neighbohood of the origin. Under the much more restrictive assumption of global linearity of all the dividend functions, the smooth contact property and the right-continuity (resp. continuity) of the exercise boundary was proved to hold globally (resp. in a left-hand neighborhood of each dividend date). We also extend the result obtained in [JV11] on the decrease of the exercise boundary in a left-hand neighborhood of the first (resp of each) dividend date when the corresponding dividend function was assumed to be positive and concave (resp. when all dividend functions were supposed to be linear) : we give more general sufficient conditions on each dividend function for the exercise boundary to be either non-decreasing or non-increasing in a left-hand neighborhood of the corresponding dividend date.

In the first section, we introduce our notations and assumptions. In the second section, we recall the existence results for the value function and the exercise boundary stated in [JV11]. The third section is devoted to the smooth-fit property and relies on a viscosity solution approach combined with an estimation of the derivative of the value function with respect to the time variable. In the fourth section, we prove the continuity result for the exercise boundary, which is known to be upper-semicontinuous by continuity of the value function. The right-continuity is obtained by comparison with the optimal boundary of the Put option in the Black-Scholes model without dividend. The left-continuity follows from the characterization of the continuation region as the set of points where the spatial derivative of the value function is greater than -1. In the fifth section, we are interested in the local

behaviour of the exercise boundary in a neighborhood of the dividend date. To be able to analyse this behaviour, we have to assume that the stock level at which the dividend function becomes positive lies in the post-dividend exercise region. When the dividend function has a positive slope at this point, we obtain a first order expansion for the exercise boundary at the dividend date. We also provide sufficient conditions for the exercise boundary to be locally monotonic.

1 Notations and assumptions

1.1 Notations

- $\left(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P}\right)$ is a probability space with a right continuous filtration, $(B_s)_{s\geq 0}$ a $(\mathcal{F}_s)$-brownian motion under $\mathbb{P}$, and $\mathbb{Q}$ is the probability measure defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\frac{\sigma^2}{2}t + \sigma B_t}.$$

- $\bar{S}_t^x$ is a geometric brownian motion satisfying : $d\bar{S}_t^x = r\bar{S}_t^x dt + \sigma\bar{S}_t^x dB_t$ and $\bar{S}_0^x = x$. Its density at time t is denoted $\mathrm{p}(t,x,y) = \frac{\mathbf{1}_{\{y>0\}}}{\sigma y\sqrt{2\pi t}} \exp\left(-\frac{1}{2\sigma^2 t}\left(\ln\left(\frac{y}{x}\right) - (r - \frac{\sigma^2}{2})t\right)^2\right)$,

- $\mathcal{A}$ is the Black-Scholes operator defined for any $\mathcal{C}^2$ function f by $\mathcal{A}f(x) = -rf(x) + rxf'(x) + \frac{\sigma^2 x^2}{2}f''(x)$,

- the set of all the stopping times of $(\mathcal{F}_s)_{s\leq\theta}$ is abusively denoted by $\{\tau \in [0,\theta]\}$.

Recursive construction Let $(\theta_d^i = t_d^i - t_d^{i+1})_{0\leq i\leq I-1}$ with the convention $t_d^0 = T$ denote the durations between the dividend dates. For non-negative values of θ and x, we define by induction

- $u_0(\theta, x) = \sup_{\tau\in[0,\theta]} \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}_\tau^x\right)^+\right]$ the price of the American put option in the Black-Scholes model without dividends when the time to maturity is θ and the spot level x. The corresponding exercise boundary is $c_0(\theta)$ such that $\{x : u(0,x) > (K - x)^+\} = (c_0(\theta), +\infty)$. Let $v(\theta, x)$ be the value function of the American put option with normalized strike 1 in the Black Scholes model without dividends and $\bar{c}(\theta)$ the associated exercise boundary. One has :

$$u_0(\theta, x) = \sup_{\tau\in[0,\theta]} \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}_\tau^x\right)^+\right] = K\sup_{\tau\in[0,\theta]} \mathbb{E}\left[e^{-r\tau}\left(1 - \bar{S}_\tau^{x/K}\right)^+\right] = Kv\left(\theta, \frac{x}{K}\right)$$

and consequently $c_0(\theta) = \sup\{x|u_0(\theta, x) = (K - x)^+\} = K\bar{c}(\theta)$.

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- $\forall i \in \{1, \ldots, I\}$,

$$u_i(\theta, x) = \sup_{\tau \in [0, \theta]} \mathbb{E}\left[e^{-r\tau} \left(K - \bar{S}_\tau^x \right)^+ \mathbf{1}_{\{\tau < \theta\}} + e^{-r\theta} u_{i-1}(\theta_d^{i-1}, \bar{S}_\theta^x - D_i(\bar{S}_\theta^x)) \mathbf{1}_{\{\tau = \theta\}} \right].$$

Note that $u_i(0, x) = u_{i-1}(\theta_d^{i-1}, x - D_i(x))$.

- Any stopping time τ such that $u_i(\theta, x) = \mathbb{E}\left[e^{-r\tau} \left(K - \bar{S}_\tau^x \right)^+ \mathbf{1}_{\{\tau < \theta\}} + e^{-r\theta} u_i(0, \bar{S}_\theta^x) \mathbf{1}_{\{\tau = \theta\}} \right]$
will be abusively called **an optimal stopping time for** $u_i(\theta, x)$.

1.2 Assumptions

In all what follows, we assume that

$$(A) \; \forall i \in \{1, \ldots, I\}, \begin{cases} \text{(a) } D_i \text{ is non-decreasing and non-negative,} \\ \text{(b) } \rho_i : x \mapsto x - D_i(x) \text{ is non-decreasing and non-negative.} \end{cases}$$

2 Previous results

Under (A), we can reformulate Proposition 1.5 [JV11] with our notations,

Proposition 2.1. *Suppose that $t < t_d^i < t_d^{i-1} < \cdots < t_d^1 < T$ and set $\theta = t_d^i - t$, $\theta_d^0 = T - t_d^1$, and for $j = 1 \ldots i - 1$, $\theta_d^j = t_d^j - t_d^{j+1}$, then the value at time t when the spot price of the stock is equal to x of the American put option with strike K and maturity T is given by $u_i(\theta, x)$.*

With these notations, at time $t = t_d^i$, if the spot price of the stock is x, the price of the put option is $u_{i-1}(\theta_d^{i-1}, x)$. When $D_i(x)$ is positive, it differs from $u_i(0, x) = u_{i-1}(\theta_d^{i-1}, x - D_i(x))$. The next Lemma follows from Lemma 1.3 [JV11].

Lemma 2.2. *For each $\theta \geq 0$, the mapping $x \mapsto u_i(\theta, x)$ is non-increasing and $x \mapsto x + u_i(\theta, x)$ is non-decreasing.*

Like in Lemma 1.3 [JV11], one easily deduces the existence of the exercise boundary.

Corollary 2.3 (Exercise boundary). *For any $\theta \geq 0$, it exists $c_i(\theta) \in [0, K)$ such that :* $u_i(\theta, x) > (K - x)^+ \Leftrightarrow x > c_i(\theta)$

By Proposition 2.1, the exercise boundary of the Put option in our model with discrete dividends is

$$t \in [0, T) \mapsto \sum_{i=0}^{I} c_i(t_d^i - t) \mathbf{1}_{\{t_d^{i+1} \leq t < t_d^i\}} \text{ with convention } t_d^0 = T.$$

With a slight abuse of terminology, we also call exercise boundaries the functions c_i. Notice that because of time-reversal, left-continuity of the c_i implies right-continuity of the true exercise boundary and that right-continuity of the c_i implies left-continuity of the true boundary on $[0, t_d^I) \cup (t_d^I, t_d^{I-1}) \cup \cdots \cup (t_d^1, T)$ with existence of left-hand limits at the dividend dates.

According to Lemma 1.4 [JV11], one has

Proposition 2.4 (Regularity result). *The value function* $(\theta, x) \mapsto u_i(\theta, x)$ *is continuous on* $\mathbb{R}_+ \times \mathbb{R}_+$. *On the continuation region defined as* $\{(\theta, x) | \theta > 0, \ x > c_i(\theta)\}$, *this function is* $\mathcal{C}^{1,2}$ *and satisfies :*

$$-\partial_\theta u_i(\theta, x) - r u_i(\theta, x) + r x \partial_x u_i(\theta, x) + \frac{\sigma^2}{2} x^2 \partial_{xx} u_i(\theta, x) = 0.$$

Moreover, the left-hand derivative $\partial_{xx-} u_i(\theta, x)$ *of* $\partial_x u_i(\theta, \bullet)$ *is well defined.*

The upper-semi continuity of $c_i(\bullet)$ is a consequence of the continuity of u_i.

Corollary 2.5. *For any* $\theta \geq 0$, $\limsup_{\theta' \to \theta} c_i(\theta') \leq c_i(\theta)$.

Remark 2.6. *Since the dividend function* D_i *is non-negative,* $u_i(\theta, x) \geq u_{i-1}(\theta + \theta_d^{i-1}, x)$ *and therefore* $u_i(\theta, x) \geq u_0 \left(\theta + \sum_{j=1}^i \theta_d^{i-1}, x\right)$. *We deduce that* $c_i(\theta) \leq K \bar{c} \left(\theta + \sum_{j=1}^i \theta_d^{i-1}\right)$.

3 Smooth-fit property

In this section, we are going to prove the smooth-fit property :

Proposition 3.1 (Smooth-fit). *For all* $\theta > 0$, $u_i(\theta, \bullet)$ *is* $\mathcal{C}^1$.

The proof is based on the viscosity super-solution property of u_i and estimations of the time derivative of this function stated in the two next lemmas.

Lemma 3.2. $(\theta, x) \mapsto u(\theta, x)$ *is a viscosity supersolution of*
$\min(\partial_\theta u_i(\theta, x) - \mathcal{A} u_i(\theta, \bullet)(x), u_i(\theta, x) - (K - x)^+) = 0$ *with* $u_i(0, x) = u_{i-1}(\theta_d^{i-1}, \rho_i(x))$

Proof. It comes from the definition of u_i that $u_i(\theta, x) \geq (K - x)^+$.
Let $\phi(t, x)$ be a test function such that $0 = (u_i - \phi)(\theta, x) = \min_V (u_i - \phi)$ where $V = (\theta - \eta, \theta] \times (x - \eta, x + \eta)$ for a certain $\eta > 0$. Let τ be the first exit time of $\bar{S}^x$ outside the ball centered at x with radius η and let $0 < \epsilon < \eta$. Because of the minimum property of (θ, x), one has

$$\mathbb{E}\left[e^{-r(\tau \wedge \epsilon)}(u_i(\theta - (\tau \wedge \epsilon), \bar{S}^x_{\tau \wedge \epsilon}) - \phi(\theta - (\tau \wedge \epsilon), \bar{S}^x_{\tau \wedge \epsilon}))\right] \geq u_i(\theta, x) - \phi(\theta, x).$$

Applying Itô formula to $e^{-rt}\phi(\theta - t, \bar{S}_t^x)$ between $t = 0$ and $\tau \wedge \epsilon$, we deduce that

$$\mathbb{E}\left[\int_0^{\tau \wedge \epsilon} e^{-rt}(\partial_\theta \phi(\theta - t, \bar{S}_t^x) - \mathcal{A}\phi(\theta - t, \bullet)(\bar{S}_t^x))dt\right] \geq \mathbb{E}\left[\left(u_i(\theta, x) - e^{-r(\tau \wedge \epsilon)}u_i(\theta - (\tau \wedge \epsilon), \bar{S}_{\tau \wedge \epsilon}^x)\right)\right].$$

Since, by the dynamic programming principle, for any stopping time $\eta \leq \theta$, one has $u_i(\theta, x) \geq \mathbb{E}\left[e^{-r\eta}u_i(\theta - \eta, \bar{S}_\eta^x)\right]$, the right-hand-side is non-negative. We deduce that

$$\mathbb{E}\left[\frac{1}{\epsilon}\int_0^{\tau \wedge \epsilon} e^{-rt}(\partial_\theta \phi(\theta - t, \bar{S}_t^x) - \mathcal{A}\phi(\theta - t, \bullet)(\bar{S}_t^x))dt\right] \geq 0.$$

By sending ϵ to zero, we obtain the supersolution inequality from Lebesgue's theorem :

$$\partial_\theta \phi(\theta, x) - \mathcal{A}\phi(\theta, \bullet)(x) \geq 0.$$

$\square$

Lemma 3.3. *For any $i \geq 0$, $\theta > 0$ and $x \geq 0$ one has*

$$\limsup_{\theta' \to \theta}\left|\frac{u_i(\theta', x) - u_i(\theta, x)}{\theta' - \theta}\right| \leq r(K + x) + x\left(r\left(2\mathcal{N}\left(\frac{2r}{\sigma}\sqrt{\theta}\right) - 1\right) + \sigma\frac{e^{-2\frac{r^2}{\sigma^2}\theta}}{\sqrt{2\pi\theta}}\right),$$

$$|\partial_{xx-}u_i(\theta, x)| \leq \mathbf{1}_{\{x \geq c_i(\theta)\}}\frac{2}{\sigma^2 c_i^2(\theta)}\left(2rK + \left(3r + \frac{\sigma}{\sqrt{2\pi\theta}}\right)c_i(\theta)\right).$$

Moreover $\partial_x u_i(\theta, x)$ admits a right-hand limit at $c_i(\theta)$ denoted by $\partial_x u_i(\theta, c_i(\theta)^+)$ and $\partial_x u_i(\theta, c_i(\theta)^+) \in [-1, 0]$.

The proof of these estimates, which relies on the scaling property of the Brownian motion and Lemma 2.2, is postponed in Appendix. We are now able to prove Proposition 3.1.

Proof. Let $c = c_i(\theta)$. By Lemma 3.3, the limit $\partial_x u_i(\theta, c+) = \lim_{y \downarrow c}\partial_x u_i(\theta, y)$ exists. We adapt a viscosity solution argument given in [Pha09] : supposing that $\partial_x u(\theta, c+) > -1$, we are going to obtain a contradiction. For $\epsilon > 0$, let $\phi_\epsilon(x) = (K - c)^+ + \alpha(x - c) + \frac{1}{2\epsilon}(x - c)^2$ where $-1 = \partial_x u_i(\theta, c-) < \alpha < \partial_x u_i(\theta, c+)$. Since $c < K$, it exists an open interval $(x_\epsilon, y_\epsilon) \subset [0, K]$ containing c such that $\min_{x \in (x_\epsilon, y_\epsilon)}(u_i(\theta, x) - \phi_\epsilon(x)) = u_i(\theta, c) - \phi_\epsilon(c) = 0$.
We set

$$\beta = 2(3r + \frac{\sigma}{\sqrt{\pi\theta}})K \quad \text{and} \quad \phi(\theta - t, x) = \phi_\epsilon(x) - \beta t.$$

By Lemma 3.3, for any $(t, x) \in [0, \frac{\theta}{2}] \times [0, K]$, one has $u_i(\theta - t, x) - u_i(\theta, x) \geq -\frac{\beta}{2}t$. Therefore $0 = (u_i - \phi)(\theta, c) = \min_{(t,x) \in (\frac{\theta}{2}, \theta] \times (x_\epsilon, y_\epsilon)}(u_i - \phi)(t, x)$. By the supersolution property of u_i stated in Lemma 3.2, we deduce that

$$0 \leq \partial_\theta \phi(\theta, c) - \mathcal{A}\phi(\theta, \bullet)(c) = \beta + r(K - c) - rc\alpha - \frac{\sigma^2 c^2}{2\epsilon}.$$

By sending ϵ to zero, we get the desired contradiction. $\square$

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4 Continuity of the exercise boundary

Proposition 4.1. *Under (A), for any $i \in \{0, \ldots, I\}$, the function $\theta \mapsto c_i(\theta)$ is continuous on $[0, +\infty)$.*

The right continuity will be proved in Section 4.1 whereas the left continuity will be proved in Section 4.2.

Remark 4.2. *In particular, we deduce from this result the behaviour of the exercise boundary at the dividend time.*

Since $c_i(0) = \sup \left\{ x \geq 0 | u_{i-1}(\theta_d^{i-1}, x - D_i(x)) = K - x \right\}$ and for $y \in [0, c_{i-1}(\theta_d^{i-1})) \; u_{i-1}(\theta_d^{i-1}, y) = K - y$, one has $c_i(0) = c_{i-1}(\theta_d^{i-1}) \wedge \inf \{x \geq 0 | D_i(x) > 0\}$ and

Corollary 4.3. *Under (A), for any $i \in \{1, \ldots, I\}$, $\lim_{t \to 0_+} c_i(t) = c_{i-1}(\theta_d^{i-1}) \wedge \inf \{x \geq 0 | D_i(x) > 0\}$.*

As $c_i(0) = 0$ when $\forall x > 0$, $D_i(x) > 0$, this result generalizes Lemma 2.1 [JV11].

4.1 Right continuity

The right continuity of the exercise boundary is based on a comparaison result with the exercise boundary $\bar{c}$ of the classical American put option with strike 1 in the Black-Scholes model without dividends.

Lemma 4.4. *For $\theta \geq 0$ and $t \geq 0$, one has : $c_i(\theta + t) \geq \left(K \left(1 - e^{-rt} \right) + c_i(\theta) e^{-rt} \right) \bar{c}(t)$*

Proof. Let $\tau = \tilde{\tau} \wedge t$ where $\tilde{\tau}$ is an optimal stopping time for $u_i(\theta + t, x)$. By the dynamic programming principle, one has

$$u_i(\theta + t, x) = \mathbb{E} \left[e^{-r\tau} (K - \bar{S}_\tau^x)^+ \mathbf{1}_{\{\tau < t\}} + \mathbf{1}_{\{\tau = t\}} e^{-rt} u_i(\theta, \bar{S}_t^x) \right].$$

Since $x \mapsto u_i(\theta, x)$ is non-increasing and using the fact for any $0 \leq \alpha \leq K$, $(K - x)^+ \leq$

$(K - (\alpha \wedge x))^+ = (K - \alpha) + (\alpha - x)^+$, one deduces

$$u_i(\theta + t, x) \le \mathbb{E}\left[e^{-r\tau}(K - \bar{S}_\tau^x)^+ \mathbf{1}_{\{\tau < t\}} + \mathbf{1}_{\{\tau = t\}} e^{-rt}\left(K - c_i(\theta) \wedge \bar{S}_t^x\right)^+ \right]$$

$$\le \mathbb{E}\left[\begin{array}{l} e^{-r\tau}\left(K - \left\{c_i(\theta) + (K - c_i(\theta))\left(1 - e^{-r(t-\tau)}\right)\right\} \wedge \bar{S}_\tau^x\right)^+ \mathbf{1}_{\{\tau < t\}} \\ + \mathbf{1}_{\{\tau = t\}} e^{-rt}\left(K - c_i(\theta) \wedge \bar{S}_t^x\right)^+ \end{array} \right]$$

$$\le \mathbb{E}\left[e^{-r\tau}\left(K - \left\{c_i(\theta) + (K - c_i(\theta))\left(1 - e^{-r(t-\tau)}\right)\right\} \wedge \bar{S}_\tau^x\right)^+ \right]$$

$$\le \mathbb{E}\left[e^{-r\tau}\left(K - \left\{c_i(\theta) + (K - c_i(\theta))\left(1 - e^{-r(t-\tau)}\right)\right\}\right)\right)\right]$$

$$\quad + \mathbb{E}\left[e^{-r\tau}\left(c_i(\theta) + (K - c_i(\theta))\left(1 - e^{-r(t-\tau)}\right) - \bar{S}_\tau^x\right)^+ \right]$$

$$\le (K - c_i(\theta))e^{-rt} + \mathbb{E}\left[e^{-r\tau}\left(K\left(1 - e^{-rt}\right) + c_i(\theta)e^{-rt} - \bar{S}_\tau^x\right)^+ \right]$$

where we used $(K - c_i(\theta))(1 - e^{-r(t-\tau)}) \le (K - c_i(\theta))(1 - e^{-rt})$ for the last inequality.

Since τ is a stopping-time not greater then t, for $x \le \left(K\left(1 - e^{-rt}\right) + c_i(\theta)e^{-rt}\right)\bar{c}(t)$, the second term of the right-hand side is not greater than $\left(K\left(1 - e^{-rt}\right) + c_i(\theta)e^{-rt} - x\right)$. Therefore, one has $u_i(\theta + t, x) \le (K - x)^+$ and $c_i(\theta + t) \ge x$. $\qquad \square$

Corollary 4.5. *The function $\theta \mapsto c_i(\theta)$ is right continuous.*

Proof. Because $\lim_{t \to 0} \bar{c}(t) = 1$ (cf [KS91] p.71-80), Lemma 4.4 implies that $\liminf_{\theta' \downarrow \theta} c_i(\theta') \ge c_i(\theta)$. We conclude with the upper-semicontinuity property stated in Corollary 2.5. $\qquad \square$

We recall (cf [KS91]) that $\bar{c}(\infty) \stackrel{\text{def}}{=} \lim_{\theta \to +\infty} \bar{c}(\theta)$ exists and is equal to $\frac{2r}{2r + \sigma^2}$.

Corollary 4.6. *One has $\lim_{\theta \to +\infty} c_i(\theta) = K\bar{c}(\infty)$. Moreover, when $r > 0$, $\forall \theta > 0$, $c_i(\theta) > 0$.*

Proof. If $r = 0$ then $\bar{c} \equiv 0$ and the statement clearly holds.
Let us now assume that $r > 0$. Since $u_i(t, x) \ge u_0(t, x)$, we have $c_i(t) \le K\bar{c}(t)$. Writing Lemma 4.4 for $\theta = 0$, we deduce that

$$\forall t \ge 0, \ -(K - c_i(0))e^{-rt}\bar{c}(t) \le c_i(t) - K\bar{c}(t) \le 0.$$

We obtain the first statement by taking the limit $t \to \infty$ in this inequality.
For $\theta = 0$, Lemma 4.4 also implies $c_i(t) \ge K(1 - e^{-rt})\bar{c}(t)$. Since $\bar{c}$ is non-increasing with positive limit at infinity, we deduce that $c_i(t) > 0$ as soon as $t > 0$. $\qquad \square$

4.2 Left continuity

The left continuity is based on the characterization of the continuation region in terms of the spatial derivative of u_i stated in the next proposition.

Proposition 4.7. *Under* (A), *the property*

(P_i) : *For any* $\theta > 0$ *and* $x \geq 0$ *one has* $x > c_i(\theta) \iff 1 + \partial_x u_i(\theta, x) > 0$

holds for any $i \in \{0, \cdots, I\}$.

The proof of Proposition 4.7 will be done by induction on i. The main tools to deduce the induction hypothesis at rank i from the one at rank $i-1$ are in the following lemmas, the proofs of which are postponed to the Appendix.

Lemma 4.8. *Let* $\theta > 0$, $x > c_i(\theta)$ *and* τ *denote the smallest optimal stopping time for* $u_i(\theta, x)$. *Then* $y \mapsto \mathbb{P}\left(\tau = \theta | \bar{S}^x_\theta = y\right)$ *is non-decreasing and is positive on* $(K, +\infty)$.

The function $u_i(0, x)$ being Lipschitz continuous by Lemma 2.2, it is absolutely continuous and therefore dx a.e. differentiable. We denote by $\partial_x u_i(0, x)$ its a.e. derivative.

Lemma 4.9. *Let* $\theta > 0$, $x \geq 0$ *and* τ *be an optimal stopping time for* $u_i(\theta, x)$. *Then one has*

$$1 + \partial_x u_i(\theta, x) \geq \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau = \theta\}}\left(1 + \partial_x u_i(0, \bar{S}^x_\theta)\right)\right].$$

Moreover, $\bar{\tau} \stackrel{\text{def}}{=} \lim_{\epsilon \to 0_+} \inf\left\{t \geq 0 | \bar{S}^{x+\epsilon}_t \leq c_i(\theta - t)\right\}$ *is an optimal stopping time and satisfies*

$$1 + \partial_x u_i(\theta, x) = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\bar{\tau} = \theta\}}\left(1 + \partial_x u_i(0, \bar{S}^x_\theta)\right)\right].$$

We are now proving Proposition 4.7.

Proof. First, for $i = 0$, due to [KS91], $x \mapsto u_i(\theta, x)$ is convex and so (P_0) is true.
Let us suppose that (P_{i-1}) holds for $i \in \{1, \cdots, I-1\}$.
By (A), $\kappa_i \stackrel{\text{def}}{=} \sup\left\{x \geq 0 | x - D_i(x) \leq c_{i-1}(\theta^{i-1}_d)\right\}$ is such that

$$\forall x \geq 0, \ x - D_i(x) \leq c_{i-1}(\theta^{i-1}_d) \Leftrightarrow x \leq \kappa_i.$$

Moreover, D_i is differentiable dx a.e. and equal to the integral of its a.e. derivative which takes its values in $[0, 1]$. We denote this a.e. derivative by D'_i. Since $u_i(0, x) = u_{i-1}(\theta^{i-1}_d, x - D_i(x))$ where $u_{i-1}(\theta^{i-1}_d, x)$ is C^1 by Proposition 3.1, one easily checks that

$$dx \text{ a.e., } \partial_x u_i(0, x) = (1 - D'_i(x))\partial_y u_{i-1}(\theta^{i-1}_d, y)|_{y=x-D_i(x)} \tag{1}$$

where the second term of the right-hand-side belongs to $[-1, 0]$ by Lemma 2.2. There are two possibilities :

- either $\kappa_i < \infty$ and then for $x > \kappa_i$, $1 + \partial_y u_{i-1}(\theta^{i-1}_d, y)|_{y=x-D_i(x)} > 0$ by (P_{i-1}) so that $1 + \partial_x u_i(0, x) > 0$ a.e. by Equation (1),

- or $\kappa_i = +\infty$ and then $D_i(x) = \int_0^x D_i'(y)dy \sim x$ as $x \to \infty$. Therefore there exists a borel set $\mathbf{C} \subset (K, +\infty)$ with infinite Lebesgue measure, on which D_i' takes values in $\left[\frac{1}{2}, 1\right]$. By Equation (1), for almost every $x \in \mathbf{C}$, $1 + \partial_x u_i(0, x) \geq \frac{1}{2}$.

So there exists of a borel set $A \subset (K, +\infty)$ which is non neglictible for the Lebesgue measure and such that for every $x \in A$, $1 + \partial_x u_i(0, x) > 0$.

Using the first statement of Lemma 4.9 then $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_\theta} = \frac{e^{-r\theta} \bar{S}_\theta^x}{x}$, one obtains

$$
\begin{aligned}
1 + \partial_x u_i(\theta, x) &\geq \quad \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{\tau = \theta\}} \left(1 + \partial_x u_i(0, \bar{S}_\theta^x) \right) \right] \\
&= \quad e^{-r\theta} \int_0^{+\infty} \frac{y}{x} \left(1 + \partial_x u_i(0, y) \right) \mathbb{P} \left(\tau = \theta | \bar{S}_\theta^x = y \right) \mathrm{p}(\theta, x, y) dy \\
&\geq \quad e^{-r\theta} \int_A \frac{y}{x} \left(1 + \partial_x u_i(0, y) \right) \mathbb{P} \left(\tau = \theta | \bar{S}_\theta^x = y \right) \mathrm{p}(\theta, x, y) dy.
\end{aligned}
$$

By Lemma 4.8, the last quantity is positive and the assertion is proved. $\qquad\square$

Proposition 4.10. $\theta \mapsto c_i(\theta)$ *is left continuous.*

Proof. By Corollary 2.5, we just need to prove that it does not exist $\theta > 0$ such that $\liminf_{t \to 0_+} c_i(\theta - t) < c_i(\theta)$.

Let us suppose that it exists such a $\theta > 0$ and obtain a contradiction. Let $c_- \stackrel{\text{def}}{=} \liminf_{t \to 0_+} c_i(\theta - t)$ and $(t_n)_n$ be a decreasing sequence in $(0, \theta)$ tending to zero and such that $c_i(\theta - t_n)$ tend to c_-. Then, by Lemma 4.4 written with $(s - t_n, \theta - s)$ replacing (t, θ), we obtain that for $s \in (t_n, \theta)$, $c_i(\theta - s) \leq c_i(\theta - t_n) \frac{e^{r(s - t_n)}}{\bar{c}(s - t_n)}$. So $\lim_{t \to 0_+} c_i(\theta - t) = c_-$. Then it exists $\eta \in (0, c_i(\theta))$, $\delta_0 \in (0, \theta/2)$, such that $\forall t \in (0, 2\delta_0)$ $c_i(\theta - t) < c_i(\theta) - \eta$. Let $x < y$ be such that $c_i(\theta) - \eta < x < y \leq c_i(\theta)$. One has

$$
y - x + u_i(\theta, y) - u_i(\theta, x) = 0. \tag{2}
$$

Let us define $\tau = \inf \left\{ t \geq 0 \ \middle| \ t + \left| \bar{S}_t^1 - 1 \right| \geq \delta_0 \wedge \frac{x - c_i(\theta) + \eta}{x} \right\}$. For $\theta' \in (\theta, \theta - \delta_0)$ and $z \geq x$, one has $\forall t \in [0, \tau]$, $\bar{S}_t^z \geq \bar{S}_t^x \geq c_i(\theta) - \eta > c_i(\theta' - t)$ and by Proposition 2.4, $u_i(\theta', z) = \mathbb{E} \left[e^{-r\tau} u_i(\theta' - \tau, \bar{S}_\tau^z) \right]$. Since u_i is continuous and bounded by K, letting θ' tend to θ, we get by dominated convergence $u_i(\theta, z) = \mathbb{E} \left[e^{-r\tau} u_i(\theta - \tau, \bar{S}_\tau^z) \right]$. We deduce

$$
\begin{aligned}
y - x + u_i(\theta, y) - u_i(\theta, x) &= \mathbb{E} \left[e^{-r\tau} \left(\bar{S}_\tau^y - \bar{S}_\tau^x + u_i(\theta - \tau, \bar{S}_\tau^y) - u_i(\theta - \tau, \bar{S}_\tau^x) \right) \right] \\
&= \mathbb{E}^{\mathbb{Q}} \left[\int_x^y \left(1 + \partial_x u_i(\theta - \tau, \bar{S}_\tau^z) \right) dz \right].
\end{aligned}
$$

But since $\mathbb{Q} \left(\tau > 0 \text{ and } \forall z \geq x, \ \bar{S}_\tau^z > c_i(\theta - \tau) \right) = 1$, the right-hand side is positive by Proposition 4.7, which contradicts Equation (2). $\qquad\square$

On Figure 2, we represent two different exercise boundaries computed through a binomial tree method following [VN06]. In both cases, $c_1(0) = \kappa_1 = 20$. In case (a), the boundary appears to be smooth whereas in case (b), it seems to be merely continuous (at time 0.04, even continuity is not so clear from the figure).
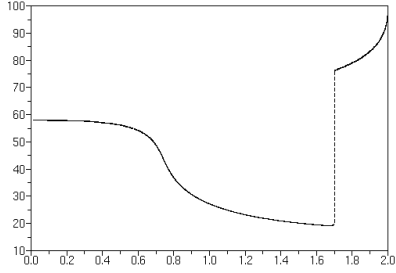
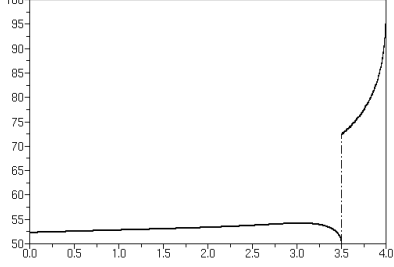(a) Maturity is 2 with one dividend time at 1.7; $D_1(x) = \frac{1}{5}\left((x-20)^+ - (x-30)^+\right)$
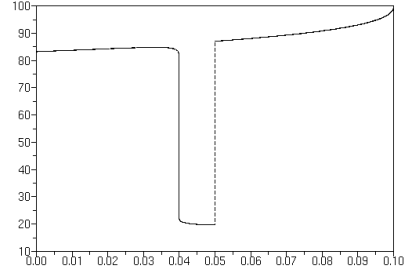
(b) Maturity is 0.1 with one dividend time at 0.05; $D_1(x) = \min\left(\frac{9}{8}, \frac{2}{9}\left((x-20)^+\right)^2\right)$

Figure 2: Exercise boundaries of an American put option with different maturities for different dividend functions. Strike is 100, diffusion parameters are $r = 0.04$ and $\sigma = 0.3$.

5 Local behaviour of the exercise boundary near the dividend dates

In this section, we are going to show how the behaviour of the exercise boundary is driven by the shape of the function $u_i(0, .)$.

We recall that $c_i(0) = \min\left(c_{i-1}(\theta_d^{i-1}), \inf\{x \geq 0 | D_i(x) > 0\}\right)$. We are able to precise the local behaviour of the exercise boundary near the dividend dates only when $c_i(0) < c_{i-1}(\theta_d^{i-1})$. Notice that by Lemma 4.4, this condition is satisfied as soon as $\inf\{x \geq 0 | D_i(x) > 0\} < \left(K(1 - e^{-r\theta_d^{i-1}}) + e^{-r\theta_d^{i-1}}c_{i-1}(0)\right)\bar{c}(\theta_d^{i-1})$. On Figure 3 are represented two different exercise boundaries computed through a binomial tree method following [VN06]. Notice that in each case, a dividend is paid if the stock price is over 50. On the left (resp. right) one, $c_1(.)$ seems to be locally increasing (resp. decreasing) on $[0, \epsilon)$ for ϵ small enough. In Proposition 5.3 and 5.6, we give sufficient conditions on the dividend functions for these local monotonicity properties to hold.

(a) $D_1(x) = \min\left(\dfrac{1125}{32}, \dfrac{8}{1125}\left((x-50)^+\right)^2\right)$
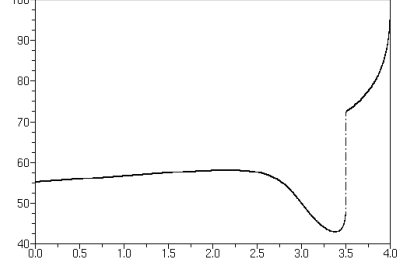
(b) $D_1(x) = 0.05(x-50)^+$

Figure 3: Exercise boundaries of an American put option of maturity 4 with one dividend time at 3.5 for different dividend functions. Strike is 100, diffusion parameters are $r = 0.04$ and $\sigma = 0.3$.

5.1 Equivalent of the exercise boundary for dividend functions with positive slope at $c_i(0)_+$

Proposition 5.1. *If $c_i(0) > 0$ and $\liminf_{x \to c_i(0)_+} \frac{D_i(x)}{x - c_i(0)} > 0$, then $c_i(\theta) - c_i(0) \sim_{\theta \to 0_+} -\sigma c_i(0)\sqrt{\theta |\ln \theta|}$.*

Notice that the second hypothesis implies that $c_i(0) = \inf\{x \geq 0 | D_i(x) > 0\}$ and therefore that $\inf\{x \geq 0 | D_i(x) > 0\} \leq c_{i-1}(\theta_d^{i-1})$ with possible equality. In order to prove Proposition 5.1, we need the following lemma, the proof of which is postponed in Appendix.

Lemma 5.2. *Suppose that $c_i(0) > 0$ and that it exists $\alpha > 0$, $\beta \in [1, 2)$ and an open set $V \subset \mathbb{R}_+^\star$ containing $c_i(0)$ such that :*

$$\forall x \in V, \ u_i(0, x) - (K - x)^+ \geq \alpha \left|(x - c_i(0))^+\right|^\beta. \tag{3}$$

Then $\forall \delta > 1$, $\exists \Theta_\delta > 0$, $\forall \theta \in [0, \Theta_\delta]$, $c_i(\theta) \leq c_i(0) \exp\left\{-\sigma\sqrt{\theta\left((2 - \beta)|\ln \theta| - (\beta + \delta)\ln|\ln \theta|\right)}\right\}$.

We are now able to prove Proposition 5.1.

Proof. Since $c_i(0) \leq c_{i-1}(\theta_d^{i-1}) < K$ and for $x \in [0, K]$, $u_{i-1}(0, x) \geq K - x + D_i(x)$, the positivity of $\liminf_{x \to c_i(0)_+} \frac{D_i(x)}{x - c_i(0)}$ implies that the second hypothesis of Lemma 5.2 is satisfied with $\beta = 1$. Hence, for θ small enough, $c_i(\theta) \leq c_i(0)e^{-\sigma\sqrt{\theta(|\ln \theta| - 3\ln|\ln \theta|)}}$. By Lemma 4.4, we know that $c_i(\theta) \geq c_i(0)\bar{c}(\theta) + (1 - e^{-r\theta})(K - c_i(0))\bar{c}(\theta)$, where, according to [Lam95], $\bar{c}(\theta) - 1 \sim_{\theta \downarrow 0} -\sigma\sqrt{\theta |\ln \theta|}$. Since $\sqrt{\theta(|\ln \theta| - 3\ln|\ln \theta|)} \sim_{\theta \downarrow 0} \sqrt{\theta |\ln \theta|}$, we easily conclude. $\square$

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5.2 Monotonicity of the value function

The monotonicity of the value function around the i-th dividend time is closely related to the sign, on a right-hand neighborhood of $c_i(0)$, of the Black-Scholes operator applied to $u_i(0,.) = u_{i-1}(\theta_d^{i-1}, \rho_i(.))$ where $\rho_i(x) = x - D_i(x)$. In the previous sections, the derivative of D_i was thought in the sense of distributions. From now on, we assume that D_i is the difference of two convex functions in order to apply the Itô-Tanaka formula. So the derivative of D_i (resp. ρ_i) is considered as the left-hand derivative.

5.2.1 Exercise boundary locally non-decreasing

To obtain this property, we need negativity of the Black-Scholes operator applied to $u_i(0,.)$ in a right-hand neighborhood of $c_i(0)$.

Proposition 5.3. *Assume that $c_i(0) < c_{i-1}(\theta_d^{i-1})$, that D_i is the difference of two convex functions, and that the positive part of the Jordan-Hahn decomposition of the measure D_i'' is absolutely continuous with respect to the Lebesgue measure. Assume moreover that, if g_i denotes the density of the absolutely continuous part of D_i'', it exists $\varepsilon \in (0, c_{i-1}(\theta_d^{i-1}) - c_i(0))$ and $C_1 \in [0, +\infty)$ such that*

$$\forall x \leq c_i(0) + \varepsilon, \; -rD_i(x) + rxD_i'(x) + \frac{\sigma^2 x^2}{2} g_i(x) \leq rK - \varepsilon$$

$$\forall x > c_i(0) + \varepsilon, \; g_i(x) \leq C_1 x^{C_1}.$$

Then it exists a neighborhood of $(0, c_i(0))$ in $\mathbb{R}_+ \times \mathbb{R}_+$ such that u_i is non-increasing w.r.t θ in this neighborhood. Moreover, the exercise boundary c_i is non-decreasing in a neighborhood of 0.

Remark 5.4. *This result is a generalization of Proposition 2.2 in [JV11] which states the same local monotonicity property of the value function at the first dividend date when $c_1(0) = 0$ and D_1 is a non-zero concave function satisfying assumption (A). Indeed concavity implies that $g_1(x) \leq 0$ and $D_1(x) - rxD_1'(x) \geq D_1(0)$ where $D_1(0) = 0$ by (A). When $r > 0$ and $c_i(0) = 0$, generalizing the proofs of Lemma 2.1 and Corollary 2.3 [JV11], one may check that $c_i(\theta) \leq rK\theta \limsup_{x\to 0^+} \frac{x}{D_i(x)} + o(\theta)$ as $\theta \to 0$ and that, under the assumptions of Proposition 5.3, if $\frac{x}{D_i(x)}$ admits a finite right-hand limit at $x = 0$, $c_i(\theta) \sim_{\theta \to 0^+} rK\theta \lim_{x\to 0^+} \frac{x}{D_i(x)}$.*

The function $D_i(x) = \min\left(\alpha, \frac{(r-\eta)K}{\sigma^2 c_i^2(0)}\left((x - c_i(0))^+\right)^2\right)$ satisfies (A) and the assumptions of Proposition 5.3 when $c_i(0) > 0$, for $\eta \in (0, r)$ and $\alpha \in (0, \frac{\sigma^2 c_i^2(0)}{4(r-\eta)K}]$.

To prove the proposition, we need the following lemma, the proof of which is postponed in appendix.

Lemma 5.5. *Let $p \geq 0$ and for $t_1 \geq 0$, $\tau_{t_1} = \inf\left\{w \geq 0 | \bar{S}_w^x \geq c_i(t_1 - w)\mathbf{1}_{\{w<t_1\}} + c_i(0)\mathbf{1}_{\{w \geq t_1\}}\right\}$ with the convention $\inf \emptyset = +\infty$.*

$$\forall \alpha > 0, \ \exists \eta > 0, \ \lim_{v \to 0+} \ \sup_{t_1 \leq \eta, x \leq c_i(0)+\alpha} \frac{\mathbb{E}\left[\left(1 + \left(\bar{S}_v^x\right)^p\right)\mathbf{1}_{\left\{\tau_{t_1} \geq v, \ \bar{S}_v^x \geq c_i(0)+2\alpha\right\}}\right]}{\mathbb{P}(\tau_{t_1} \geq v)} = 0.$$

We are now able to prove Proposition 5.3.

Proof. Let $0 \leq s < t$, $x > c_i(t)$ and τ be the smallest optimal stopping time for (t, x). Since $\tau \wedge s$ is a stopping time not greater than s, $u_i(s, x) \geq \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}_\tau^x\right)\mathbf{1}_{\{\tau<s\}} + e^{-rs}u_i(0, \bar{S}_s^x)\right]$. Using $(K - x)^+ \leq u_i(0, x)$, we deduce

$$u_i(t, x) - u_i(s, x) \leq \mathbb{E}\left[\mathbf{1}_{\{\tau \geq s\}}\left(e^{-r\tau}u_i(0, \bar{S}_\tau^x) - e^{-rs}u_i(0, \bar{S}_s^x)\right)\right].$$

By Lemma 6.1, on $\tau > s$,

$$e^{-r\tau}u_i(0, \bar{S}_\tau^x) - e^{-rs}u_i(0, \bar{S}_s^x) = \int_s^\tau e^{-rv}\left\{\begin{array}{l} -ru_i(0, \bar{S}_v^x) + r\bar{S}_v^x\partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x))\rho_i'(\bar{S}_v^x) \\ +\frac{\sigma^2}{2}\left(\bar{S}_v^x\rho_i'\left(\bar{S}_v^x\right)\right)^2\partial_{xx}u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x)) \end{array}\right\} dv$$

$$+ \frac{1}{2}\int_s^\tau \int_{\mathbb{R}} e^{-rv}\partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(a))\rho_i''(da)dL_v^a(\bar{S}^x)$$

$$+ M_\tau - M_s \tag{4}$$

where $M_t = \int_0^t \sigma e^{-rv}\bar{S}_v^x\partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x))\rho_i'(\bar{S}_v^x)dB_v$. As $\mathbb{E}\left[\langle M\rangle_t\right] \leq \sigma^2 tx^2 e^{\sigma^2 t}$, M_t is a true martingale and

$$\mathbb{E}\left[\mathbf{1}_{\{\tau \geq s\}}(M_\tau - M_s)\right] = \mathbb{E}\left[\mathbf{1}_{\{\tau \geq s\}}(\mathbb{E}\left[M_\tau | \mathcal{F}_s\right] - M_s)\right] = 0. \tag{5}$$

The function $y \mapsto \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(y))$ belongs to $[-1, 0]$ by Lemma 2.2 and is equal to -1 on $[0, c_i(0) + \varepsilon]$ since then $\rho_i(y) \leq y \leq c_i(0) + \varepsilon < c_{i-1}(\theta_{i-1}^d)$. Since for any $a \geq 0$, $t \mapsto L_t^a$ is a non-decreasing process and $\rho_i'' = -D_i''$, using the growth assumption on g_i, we deduce that $\mathbb{P}$-almost surely

$$\int_s^\tau \int_{\mathbb{R}} e^{-rv}\partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(a))\rho_i''(da)dL_v^a(\bar{S}^x)$$

$$\leq \int_s^\tau \int_{\mathbb{R}} e^{-rv}\left(\mathbf{1}_{\{a \leq c_i(0)+\varepsilon\}}g_i(a) + \mathbf{1}_{\{a > c_i(0)+\varepsilon\}}C_1 a^{C_1}\right)dadL_v^a(\bar{S}^x)$$

Using Exercise 1.15 p.232 [RY91], we deduce that

$$\int_s^\tau \int_{\mathbb{R}} e^{-rv}\partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(a))\rho_i''(da)dL_v^a(\bar{S}^x)$$

$$\leq \int_s^\tau \sigma^2 e^{-rv}\left(\bar{S}_v^x\right)^2 (\mathbf{1}_{\left\{\bar{S}_x^x \leq c_i(0)+\varepsilon\right\}}g_i(\bar{S}_v^x) + C_1\mathbf{1}_{\left\{\bar{S}_v^x > c_i(0)+\varepsilon\right\}}(\bar{S}_v^x)^{C_1})dv. \tag{6}$$

By Lemma 3.3 and since $c_i(0) + \varepsilon < c_{i-1}(\theta_d^{i-1})$, it exists a finite constant C_2 not depending on s and t such that

$$\int_s^\tau e^{-rv} \left(\bar{S}_v^x \rho_i'\left(\bar{S}_v^x \right) \right)^2 \partial_{xx} u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x)) dv \leq C_2 \int_s^\tau e^{-rv} \left(\bar{S}_v^x \right)^2 \mathbf{1}_{\{\bar{S}_v^x > c_i(0) + \varepsilon\}} dv. \quad (7)$$

For $y \leq c_i(0) + \varepsilon$, $u_{i-1}(\theta_d^{i-1}, \rho_i(y)) = K - \rho_i(y)$ and

$$-r u_i(0, y) + r y \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(y)) \rho_i'(y) = -rK - rD_i(y) + ry D_i'(y)$$

where D_i is equal to 0 on $[0, c_i(0)]$. Hence the assumptions ensure that

$$-r u_i(0, y) + r y \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(y)) \rho_i'(y) + \frac{\sigma^2 y^2}{2} g_i(y) \leq \begin{cases} -rK & \text{if } y \leq c_i(0) \\ -\varepsilon & \text{if } y \in (c_i(0), c_i(0) + \varepsilon] \end{cases} \quad (8)$$

When $y > c_i(0) + \varepsilon$, since $\partial_x u_{i-1} \leq 0$ and $\rho_i' \geq 0$, $-r u_i(0, y) + r y \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(y)) \rho_i'(y)$ is non-positive.

Taking expectations in Equation (4) and using Equation (5), Equation (6), Equation (7), Equation (8), we deduce that it exists a constant $M > 0$ such that

$$u_i(t, x) - u_i(s, x) \leq \int_s^t e^{-rv} \mathbb{P}(\tau \geq v) \left\{ \begin{array}{l} -(rK \wedge \varepsilon) \\ +M \dfrac{\mathbb{E}\left[\mathbf{1}_{\{\tau \geq v, \bar{S}_v^x > c_i(0) + \varepsilon\}} \left(1 + \left(\bar{S}_v^x \right)^{2+C_1} \right) \right]}{\mathbb{P}(\tau \geq v)} \end{array} \right\} dv \quad (9)$$

Applying Lemma 5.5 (with $p = 2 + C_1$, $t_1 = t$ and $\alpha = \frac{\varepsilon}{2}$), we obtain that for t small enough, uniformly in $x \leq c_i(0) + \frac{\varepsilon}{2}$, the right-hand-side of Equation (9) is non-positive.

With Proposition 4.1, we deduce the existence of $\eta > 0$ such that $\sup_{w \in [0,\eta]} c_i(w) \leq c_i(0) + \frac{\epsilon}{2}$ and

$$\forall 0 \leq s < t < \eta, \ \forall x \in (c_i(t), c_i(0) + \frac{\epsilon}{2}], \ u_i(t, x) \leq u_i(s, x).$$

This inequality is still true for $x \leq c_i(t)$ since then $u_i(t, x) = (K - x)^+ \leq u_i(s, x)$. For $0 \leq s < t < \eta$, we conclude that $u_i(t, c_i(s)) \leq u_i(s, c_i(s)) = K - c_i(s)$, which implies that $c_i(s) \leq c_i(t)$. $\qquad \square$

5.2.2 Exercise boundary locally non-increasing

To obtain this property, we need positivity of the Black-Scholes operator applied to $u_i(0, .)$ in a right-hand neighborhood of $c_i(0)$.

Proposition 5.6. *Assume that $c_i(0) < c_{i-1}(\theta_d^{i-1})$, that D_i is the difference of two convex functions, and that the negative part of the Jordan-Hahn decomposition of the measure D_i'' is absolutely continuous with respect to the Lebesgue measure.*

Assume moreover that, if g_i denotes the density of the absolutely continuous part of the measure D_i'', it exists $\varepsilon \in (0, c_{i-1}(\theta_d^{i-1}) - c_i(0))$ and $C_1 \in [0, +\infty)$ such that

on $(c_i(0), c_i(0) + \varepsilon]$, D_i is C^2 and such that $-rD_i(x) + rxD_i'(x) + \dfrac{\sigma^2 x^2}{2} g_i(x) \geq rK + \varepsilon$,

$\forall x > c_i(0) + \varepsilon,\ g_i(x) \leq -C_1 x^{C_1}.$

Then it exists a neighborhood of $(0, c_i(0))$ in $\mathbb{R}_+ \times \mathbb{R}_+$ such that u_i is non-decreasing w.r.t θ in this neighborhood. Moreover the exercise boundary c_i is non-increasing in a neighborhood of 0.

Remark 5.7. *When $c_i(0) = 0$, there is no non-negative function D_i satisfying the differential inequality on a neighborhood of $(c_i(0), c_i(0) + \varepsilon)$.*
For $c_i(0) > 0$ and $\alpha \in (0,1)$, the function

$$D_i(x) = \alpha(x - c_i(0))^+ + \left(\frac{1}{\sigma c_i(0)}\right)^2 (r(K - \alpha c_i(0)) + \eta)\left((x - c_i(0))^+\right)^2 e^{-\frac{x^2}{\eta}}$$

satisfies (A) and the assumptions of Proposition 5.6 when $\eta > 0$ is small enough.

Proof. Let $0 \leq s < t$, $x > c_i(s)$ and τ be the smallest optimal stopping time for (s, x). We set $\bar{\tau} = \tau \mathbf{1}_{\{\tau < s\}} + \mathbf{1}_{\{\tau = s\}} \left(\inf\left\{v \geq s | \bar{S}_v^x \leq c_i(0)\right\} \wedge t\right)$. We have

$$u_i(t, x) - u_i(s, x) \geq \mathbb{E}\left[\mathbf{1}_{\{\tau = s\}}\left(e^{-r\bar{\tau}} u_i(t - \bar{\tau}, \bar{S}_{\bar{\tau}}^x) - e^{-rs} u_i(0, \bar{S}_s^x)\right)\right].$$

Since on $\{\tau = s\}$, $\bar{S}_s^x \geq c_i(0)$, on $\{\tau = s, \bar{\tau} < t\}$, $\bar{S}_{\bar{\tau}}^x = c_i(0)$, and $u_i(t - \bar{\tau}, c_i(0)) \geq (K - c_i(0)) = u_i(0, c_i(0))$. We then deduce that

$$u_i(t, x) - u_i(s, x) \geq \mathbb{E}\left[\mathbf{1}_{\{\bar{\tau} \geq s\}}\left(e^{-r\bar{\tau}} u_i(0, \bar{S}_{\bar{\tau}}^x) - e^{-rs} u_i(0, \bar{S}_s^x)\right)\right].$$

Applying Lemma 6.1, arguing like in the proof of Proposition 5.3 about the local martingale part and using that dv a.e. on $[s, t]$, $\bar{\tau} \geq v$ implies $\bar{S}_v^x > c_i(0)$, we get

$$u_i(t, x) - u_i(s, x) \geq \mathbb{E}\left[\int_s^t \mathbf{1}_{\{\bar{\tau} \geq v, \bar{S}_v^x > c_i(0)\}} e^{-rv} \left\{ \begin{array}{l} -ru_i(0, \bar{S}_v^x) + r\bar{S}_v^x \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x))\rho_i'(\bar{S}_v^x) \\ + \frac{\sigma^2}{2}\left(\bar{S}_v^x \rho_i'\left(\bar{S}_v^x\right)\right)^2 \partial_{xx} u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_v^x)) \end{array} \right\} dv\right]$$
$$+ \frac{1}{2}\mathbb{E}\left[\int_s^t \int_{\mathbb{R}} \mathbf{1}_{\{\bar{\tau} \geq v\}} e^{-rv} \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(a))\rho_i''(da) dL_v^a(\bar{S}^x)\right].$$

Like in the proof of Proposition 5.3, one checks that

$$\forall y \in (c_i(0), c_i(0) + \varepsilon],\ -ru_i(0, y) + ry\partial_x u_i(\theta_d^{i-1}, \rho_i(y))\rho_i'(y) + \frac{\sigma^2 y^2}{2} g_i(y) \geq \varepsilon$$
$$\forall y > c_i(0) + \varepsilon,\ -ru_i(0, y) + ry\partial_x u_i(\theta_d^{i-1}, \rho_i(y))\rho_i'(y) \geq -r(K + y),$$

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$$\int_s^t \mathbf{1}_{\{\bar\tau \geq v\}} e^{-rv} \left(\bar S_v^x \rho_i'\left(\bar S_v^x\right)\right)^2 \partial_{xx} u_{i-1}(\theta_d^{i-1}, \rho_i(\bar S_v^x)) dv \geq -C_2 \int_s^t \mathbf{1}_{\{\bar\tau \geq v, \bar S_v^x > c_i(0)+\varepsilon\}} e^{-rv} \left(\bar S_v^x\right)^2 dv,$$

and that

$$\int_s^t \int_{\mathbb{R}} \mathbf{1}_{\{\bar\tau \geq v\}} e^{-rv} \partial_x u_{i-1}(\theta_d^{i-1}, \rho_i(a)) \rho_i''(da) dL_v^a(\bar S^x)$$
$$\geq \int_s^t \mathbf{1}_{\{\bar\tau \geq v\}} e^{-rv} \sigma^2 \left(\bar S_v^x\right)^2 \left[g_i(\bar S_v^x) \mathbf{1}_{\{\bar S_v^x \leq c_i(0)+\varepsilon\}} - C_1 (\bar S_v^x)^{C_1} \mathbf{1}_{\{\bar S_v^x > c_i(0)+\varepsilon\}} \right] dv.$$

Gathering all the inequalities, we get that it exists a finite constant $M \geq 0$ such that :

$$u_i(t,x) - u_i(s,x) \geq \int_s^t \left\{ \mathbb{P}\left(\bar\tau \geq v\right) e^{-rv} \varepsilon - \mathbb{E}\left[\mathbf{1}_{\{\bar\tau \geq v, \ \bar S_v^x > c_i(0)+\varepsilon\}} M \left(1 + \left(\bar S_v^x\right)^{2+C_1}\right)\right] \right\} dv. \tag{10}$$

Applying Lemma 5.5 (with $p = 2 + C_1$, $t_1 = s$ and $\alpha = \frac{\varepsilon}{2}$), we obtain that for t small enough, uniformly for $x \leq c_i(0) + \frac{\varepsilon}{2}$, the right-hand-side of Equation (10) is non-negative.

With Proposition 4.1, we deduce the existence of $\eta > 0$ such that $\sup_{w \in [0,\eta]} c_i(w) \leq c_i(0) + \frac{\varepsilon}{2}$ and that

$$\forall 0 \leq s < t < \eta, \ \forall x \in (c_i(s), c_i(0) + \frac{\varepsilon}{2}), \ u_i(s,x) \leq u_i(t,x).$$

This inequality is still true for $x \leq c_i(s)$ since then $u_i(s,x) = (K-x)^+ \leq u_i(t,x)$.

Then, as soon as $0 \leq s < t < \eta$, $u_i(s, c_i(t)) \leq u_i(t, c_i(t)) = K - c_i(t)$ which implies that $c_i(t) \leq c_i(s)$. $\qquad \square$

Conclusion and further research

The continuity of the exercise boundary as well as the smooth contact property are likely to be generalized in a model with discrete dividends where the underlying asset price has a local volatility dynamics between the dividend dates with a positive local volatility function. We plan to investigate this extension in a future work. Assuming that the underlying stock price evolves as the exponential of some Lvy process between the dividend dates provides another natural generalization of the Black-Scholes model that could be considered (see [LM08] for the case without discrete dividends).

6 Appendix

6.1 Proof of Lemma 3.3

Proof. The existence of the right-hand limit at $c_i(\theta)$ for $\partial_x u_i(\theta, x)$ is an easy consequence of the second estimation. Since for $x < c_i(\theta)$, $\partial_{xx} u_i(\theta, x) = 0$ and for $x > c_i(\theta)$, by Proposition 2.4 and Lemma 2.2,

$$
\begin{aligned}
|\partial_{xx} u_i(\theta, x)| &= \left| \frac{2}{\sigma^2 x^2} \left(\partial_\theta u_i(\theta, x) + r u_i(\theta, x) - r x \partial_x u_i(\theta, x) \right) \right| \\
&\leq \frac{2}{\sigma^2 x^2} |\partial_\theta u_i(\theta, x)| + \frac{2r}{\sigma^2} \left(\frac{K}{x^2} + \frac{1}{x} \right),
\end{aligned}
$$

the second estimation is easily deduced from the first one. To prove the first estimation, we set

$$
V_i : (\gamma, \nu, x) \mapsto \sup_{\tau \in [0,1]} \mathbb{E} \left[e^{-\gamma \frac{\nu^2}{2} \tau} \left(K - x e^{\frac{\nu^2}{2}(\gamma-1)\tau + \nu B_\tau} \right)^+ \mathbf{1}_{\{\tau < 1\}} + e^{-\gamma \frac{\nu^2}{2}} u_i(0, x e^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1}) \mathbf{1}_{\{\tau=1\}} \right]
$$

Because of the scaling property of the brownian motion, for any positive $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $\theta \in \mathbb{R}^+$, $\sup_{\tau \in [0,1]} \mathbb{E} \left[f(\theta \tau, \sqrt{\theta} B_\tau) \right] = \sup_{\tau \in [0,\theta]} \mathbb{E} \left[f(\tau, B_\tau) \right]$.

We deduce that $V_i \left(\frac{2r}{\sigma^2}, \sigma \sqrt{\theta}, x \right) = u_i(\theta, x)$ and

$$
\limsup_{\theta' \to \theta} \left| \frac{u_i(\theta', x) - u_i(\theta, x)}{\theta' - \theta} \right| = \frac{\sigma}{2\sqrt{\theta}} \limsup_{\nu' \to \sigma\sqrt{\theta}} \left| \frac{V_i(\frac{2r}{\sigma^2}, \nu', x) - V_i(\frac{2r}{\sigma^2}, \sigma\sqrt{\theta}, x)}{\nu' - \sigma\sqrt{\theta}} \right|.
$$

Therefore it is enough to check that

$$
\forall x, \nu \geq 0, \ \limsup_{\nu' \to \nu} \left| \frac{V_i(\gamma, \nu', x) - V_i(\gamma, \nu, x)}{\nu' - \nu} \right| \leq \nu \gamma (K + x) + x \left(\gamma \nu (2\mathcal{N}(\gamma \nu) - 1) + 2 \frac{e^{-\gamma^2 \frac{\nu^2}{2}}}{\sqrt{2\pi}} \right). \tag{11}
$$

Setting $(\gamma, \nu) = (\frac{2r}{\sigma^2}, \sigma\sqrt{\theta})$, the optimality of $\tau = \inf \left\{ t \geq 0 | u_i(\theta - t, \bar{S}_t^x) + \bar{S}_t^x \leq K \right\} \wedge \theta$ for $u_i(\theta, x)$ translates into the optimality of

$$
\tau^\star \stackrel{\text{def}}{=} \inf \left\{ t \geq 0 | V_i(\gamma, \nu\sqrt{1-t}, x e^{\frac{\nu^2}{2}(\gamma-1)t + \nu B_t}) + x e^{\frac{\nu^2}{2}(\gamma-1)t + \nu B_t} \leq K \right\} \wedge 1
$$

for $V_i(\gamma, \nu, x)$. This implies that

$$
V_i(\gamma, \nu, x) + x = K \mathbb{E} \left[e^{-\frac{\nu^2}{2} \gamma \tau^\star} \right] + \mathbb{E} \left[\mathbf{1}_{\{\tau^\star = 1\}} e^{-\frac{\nu^2}{2} \gamma} \left(u_i(0, x e^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1}) + x e^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1} - K \right) \right].
$$

For any $\nu' \geq 0$, by definition of V_i,

$$
V_i(\gamma, \nu', x) \geq K \mathbb{E} \left[e^{-\frac{\nu'^2}{2} \gamma \tau^\star} \right] + \mathbb{E} \left[\mathbf{1}_{\{\tau^\star = 1\}} e^{-\frac{\nu'^2}{2} \gamma} \left(u_i(0, x e^{\frac{\nu'^2}{2}(\gamma-1)+\nu' B_1}) + x e^{\frac{\nu'^2}{2}(\gamma-1)+\nu' B_1} - K \right) \right].
$$

Using that $x \mapsto x + u_i(0, x)$ is 1-lipschitz and non-decreasing by Lemma 2.2, then $u_i(0, .) \leq K$ and $(1 - e^x)^+ \leq (-x)^+ \leq |x|$, one deduces

$$
\begin{aligned}
V_i(\gamma, \nu', x) - V_i(\gamma, \nu, x) \geq & K \mathbb{E}\left[\left\{e^{-\frac{\nu'^2}{2}\gamma\tau^\star} - e^{-\frac{\nu^2}{2}\gamma\tau^\star}\right\} \mathbf{1}_{\{\tau^\star < 1\}}\right] \\
& + \left[e^{-\frac{\nu'^2}{2}\gamma} - e^{-\frac{\nu^2}{2}\gamma}\right] \mathbb{E}\left[\mathbf{1}_{\{\tau^\star = 1\}}\left(u_i(0, xe^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1}) + xe^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1}\right)\right] \\
& - e^{-\frac{\nu'^2}{2}\gamma}\mathbb{E}\left[\mathbf{1}_{\{\tau^\star = 1\}} xe^{\frac{\nu^2}{2}(\gamma-1)+\nu B_1}\left(1 - e^{(\nu'-\nu)\left((\gamma-1)\frac{\nu+\nu'}{2}+B_1\right)}\right)^+\right]. \\
\geq & -K(e^{-\frac{\nu^2}{2}\gamma} - e^{-\frac{\nu'^2}{2}\gamma})^+(\mathbb{P}(\tau^\star < 1) + \mathbb{P}(\tau^\star = 1)) \\
& - x\left(1 - e^{\frac{\nu^2-\nu'^2}{2}\gamma}\right)^+ \mathbb{E}\left[\mathbf{1}_{\{\tau^\star = 1\}}e^{-\frac{\nu^2}{2}+\nu B_1}\right] \\
& - e^{\frac{\nu^2-\nu'^2}{2}\gamma}|\nu' - \nu|\mathbb{E}\left[\mathbf{1}_{\{\tau^\star = 1\}}xe^{-\frac{\nu^2}{2}+\nu B_1}\left|(\gamma-1)\frac{\nu+\nu'}{2} + B_1\right|\right] \\
\geq & -(K + x)\gamma|\nu - \nu'|\frac{\nu+\nu'}{2} - e^{\frac{|\nu^2-\nu'^2|}{2}\gamma}|\nu' - \nu|x\mathbb{E}\left[\left|(\gamma-1)\frac{\nu+\nu'}{2} + \nu + B_1\right|\right].
\end{aligned}
$$

Remarking that for $y \in \mathbb{R}$, $\mathbb{E}|y + B_1| = y(2\mathcal{N}(y) - 1) + \frac{2e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$ and combining the resulting inequality with the one deduced by exchanging ν and ν', we conclude that Equation (11) holds. $\qquad\square$

6.2 Proofs of the auxiliary results of Section 4.2

6.2.1 Proof of Lemma 4.8

Proof. Let $\theta > 0$ and $x > c_i(\theta)$. For $a, b \in \mathbb{R}$ and $t \in [0, \theta]$, we define $Y_t^{a,b} = a + \frac{t}{\theta}(b - a) + \Xi_t$ where $(\Xi_s)_{s \in [0,\theta]}$ is a brownian bridge on $[0, \theta]$ starting and ending at 0. Then $\left(Y_t^{a,b}\right)_{t \in [0,\theta]}$ is a brownian bridge on $[0, \theta]$ starting at a and ending at b. For $y \geq 0$,

$$
\begin{aligned}
\mathbb{P}\left(\tau = \theta | \bar{S}_\theta^x = y\right) &= \mathbb{P}\left(\forall t \in [0, \theta], Y_t^{0, \frac{1}{\sigma}\left(\ln\frac{y}{x} - \left(r - \frac{\sigma^2}{2}\right)\theta\right)} > \frac{1}{\sigma}\left(\ln\frac{c_i(\theta - t)}{x} - \left(r - \frac{\sigma^2}{2}\right)t\right)\right) \\
&= \mathbb{P}\left(\forall t \in [0, \theta], \quad \Xi_t > \frac{1}{\sigma}\left(\ln\frac{c_i(\theta - t)}{x} - \frac{t}{\theta}\ln\frac{y}{x}\right)\right)
\end{aligned}
$$

and the monotonicity of $y \mapsto \mathbb{P}\left(\tau = \theta | \bar{S}_\theta^x = y\right)$ easily follows. For $y > K$, this implies

$$
\frac{\mathbb{P}(\tau = \theta, \bar{S}_\theta^x \in (K, y))}{\mathbb{P}(\bar{S}_\theta^x \in (K, y))} \leq \mathbb{P}(\tau = \theta | \bar{S}_\theta^x = y).
$$

Therefore, to prove the second assertion, we only need to check $\mathbb{P}(\tau = \theta, \bar{S}_\theta^x \in (K, y)) > 0$. Let $\eta = \inf\left\{t \geq 0 | \bar{S}_t^x = \frac{y+K}{2}\right\}$. As $\sup_{t \geq 0} c_i(t) \leq K$, one has

$$\left\{\tau > \eta, \eta < \theta, \forall v \in [0, \theta - \eta] \bar{S}_{\eta+v}^x \in (K, y)\right\} \subset \left\{\tau = \theta, \bar{S}_\theta^x \in (K, y)\right\}.$$

By the strong Markov property and the continuity of the Black-Scholes model, one deduces

$$\mathbb{P}(\tau = \theta, \bar{S}_\theta^x \in (K, y)) \geq \mathbb{E}\left[\mathbf{1}_{\{\tau > \eta, \eta < \theta\}} \mathbb{P}(\forall v \in [0, t], \bar{S}_v^{\frac{y+K}{2}} \in (K, y))\Big|_{t=\theta-\eta}\right]$$

$$\geq \mathbb{P}(\tau > \eta, \eta < \theta) \mathbb{P}(\forall v \in [0, \theta], \bar{S}_v^{\frac{y+K}{2}} \in (K, y))$$

$$\geq \mathbb{P}\left(\tau = \theta, \bar{S}_\theta^x \geq y\right) \mathbb{P}(\forall v \in [0, \theta], \bar{S}_v^{\frac{y+K}{2}} \in (K, y)).$$

The last factor in the right-hand-side is positive. By comonotony,

$$\mathbb{P}\left(\tau = \theta, \bar{S}_\theta^x \geq y\right) = \mathbb{E}\left[\mathbb{P}(\tau = \theta | \bar{S}_\theta^x) \mathbf{1}_{\{\bar{S}_\theta^x \geq y\}}\right] \geq \mathbb{P}(\tau = \theta) \mathbb{P}\left(\bar{S}_\theta^x \geq y\right).$$

One concludes by remarking that

$$K\mathbb{E}\left[e^{-r\tau}\right] - x + \mathbb{E}\left[e^{-r\theta} \mathbf{1}_{\{\tau=\theta\}}\left(u_i(0, \bar{S}_\theta^x) + \bar{S}_\theta^x - K\right)\right] = u_i(\theta, x) > K - x \geq K\mathbb{E}\left[e^{-r\tau}\right] - x$$

implies positivity of $\mathbb{P}(\tau = \theta)$. $\qquad \square$

6.2.2 Proof of Lemma 4.9

Proof. Let $\theta, \epsilon > 0$, $x \geq 0$ and τ be an optimal stopping time for $u_i(\theta, x)$. Since

$$u_i(\theta, x + \epsilon) \geq \mathbb{E}\left[e^{-r\tau}\left(K - \bar{S}_\tau^{x+\epsilon}\right)^+ \mathbf{1}_{\{\tau < \theta\}} + e^{-r\theta} u_i(0, \bar{S}_\theta^{x+\epsilon}) \mathbf{1}_{\{\tau=\theta\}}\right]$$

and $(K - \bar{S}_\tau^{x+\epsilon})^+ - (K - \bar{S}_\tau^x)^+ \geq \bar{S}_\tau^x - \bar{S}_\tau^{x+\epsilon}$, we have

$$
\begin{aligned}
\frac{u_i(\theta, x+\epsilon) - u_i(\theta, x)}{\epsilon} &\geq \frac{1}{\epsilon}\mathbb{E}\left[e^{-r\tau}\left(\bar{S}_\tau^x - \bar{S}_\tau^{x+\epsilon}\right)\mathbf{1}_{\{\tau<\theta\}} + e^{-r\theta}\left(u_i(0, \bar{S}_\theta^{x+\epsilon}) - u_i(0, \bar{S}_\theta^x)\right)\mathbf{1}_{\{\tau=\theta\}}\right] \\
&= -\mathbb{E}\left[e^{-r\tau}\bar{S}_\tau^1 \mathbf{1}_{\{\tau<\theta\}}\right] + \mathbb{E}\left[e^{-r\theta}\bar{S}_\theta^1 \frac{u_i(0, \bar{S}_\theta^{x+\epsilon}) - u_i(0, \bar{S}_\theta^x)}{\bar{S}_\theta^{x+\epsilon} - \bar{S}_\theta^x}\mathbf{1}_{\{\tau=\theta\}}\right] \\
&= -\mathbb{Q}\left(\tau < \theta\right) + \mathbb{E}^\mathbb{Q}\left[\frac{u_i(0, \bar{S}_\theta^{x+\epsilon}) - u_i(0, \bar{S}_\theta^x)}{\bar{S}_\theta^{x+\epsilon} - \bar{S}_\theta^x}\mathbf{1}_{\{\tau=\theta\}}\right] \\
&= -1 + \mathbb{E}^\mathbb{Q}\left[\left(1 + \frac{u_i(0, \bar{S}_\theta^{x+\epsilon}) - u_i(0, \bar{S}_\theta^x)}{\bar{S}_\theta^{x+\epsilon} - \bar{S}_\theta^x}\right)\mathbf{1}_{\{\tau=\theta\}}\right]
\end{aligned}
$$

where we used $\bar{S}_\theta^x = x\bar{S}_\theta^1$ for the first equality and $\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_\theta} = e^{-r\theta}\bar{S}_\theta^1$ for the second one. The first assertion is deduced by dominated convergence using that, according to Lemma 2.2,

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$x \mapsto u_i(0, x)$ is 1-Lipschitz and therefore almost surely differentiable,.

The smallest optimal stopping time for $u_i(\theta, x + \epsilon)$ is $\tau^\epsilon = \theta \wedge \inf \left\{ t \in [0, \theta] | \bar{S}_t^{x+\epsilon} \leq c_i(\theta - t) \right\}$. Clearly, $\mathbb{P}$-almost surely, for any $\epsilon > \epsilon'$, $\tau^\epsilon \geq \tau^{\epsilon'}$ and one may define $\bar{\tau}$ as $\lim_{\epsilon \to 0_+} \tau^\epsilon$. Moreover, $\bar{\tau} \geq \tau^\star$ where $\tau^\star$ is the smallest optimal stopping time for $u_i(\theta, x)$. As $(\mathcal{F}_t)_t$ is a right-continuous filtration, $\bar{\tau}$ is a stopping time (cf (4.17) p.46 of [RY91]). By optimality of τ^ϵ,

$$u_i(\theta, x + \epsilon) = \mathbb{E}\left[e^{-r\tau^\epsilon} \right] K - (x + \epsilon) + \mathbb{E}\left[e^{-r\theta} \mathbf{1}_{\{\tau^\epsilon = \theta\}} \left(u_i\left(0, \bar{S}_\theta^{x+\epsilon}\right) + \bar{S}_\theta^{x+\epsilon} - K \right) \right].$$

Since $x \mapsto x + u_i(0, x)$ is 1-Lipschitz, one may take the limit $\epsilon \to 0$ in this equality and obtain

$$u_i(\theta, x) = \mathbb{E}\left[e^{-r\bar{\tau}} \right] K - (x + \epsilon) + \mathbb{E}\left[e^{-r\theta} \mathbf{1}_{\{\bar{\tau} = \theta\}} \left(u_i\left(0, \bar{S}_\theta^{x}\right) + \bar{S}_\theta^{x} - K \right) \right],$$

which implies that $\bar{\tau}$ is also an optimal stopping time for $u_i(\theta, x)$.

When $\tau^\epsilon < \theta$, $\bar{S}_{\tau^\epsilon}^x \leq \bar{S}_{\tau^\epsilon}^{x+\epsilon} \leq K$. Therefore

$$
\begin{aligned}
\frac{u_i(\theta, x+\epsilon) - u_i(\theta, x)}{\epsilon} &\leq \frac{1}{\epsilon} \mathbb{E}\left[e^{-r\tau^\epsilon} \left(\bar{S}_{\tau^\epsilon}^x - \bar{S}_{\tau^\epsilon}^{x+\epsilon} \right) \mathbf{1}_{\{\tau^\epsilon < \theta\}} + e^{-r\theta} \left(u_i(0, \bar{S}_\theta^{x+\epsilon}) - u_i(0, \bar{S}_\theta^{x}) \right) \mathbf{1}_{\{\tau^\epsilon = \theta\}} \right] \\
&= -1 + \mathbb{E}^{\mathbb{Q}}\left[\left(1 + \frac{u_i(0, \bar{S}_\theta^{x+\epsilon}) - u_i(0, \bar{S}_\theta^{x})}{\bar{S}_\theta^{x+\epsilon} - \bar{S}_\theta^{x}} \right) \mathbf{1}_{\{\tau^\epsilon = \theta\}} \right].
\end{aligned}
$$

Letting $\epsilon \to 0$ in this inequality, we obtain by dominated convergence $\partial_x u_i(\theta, x) + 1 \leq \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\{\bar{\tau} = \theta\}} \left(1 + \partial_x u_i(0, \bar{S}_\theta^{x}) \right) \right]$, which concludes the proof. $\square$

6.3 Proofs of the auxiliary results of Section 5

6.3.1 Proof of Lemma 5.2

Proof. Let $\theta > 0$. Using the definition of u_i, Equation (3), and the Cauchy Schwarz inequality, we get

$$
\begin{aligned}
u_i(\theta, x) &\geq Ke^{-r\theta} - x + e^{-r\theta} \mathbb{E}\left[u_i(0, \bar{S}_\theta^x) + \bar{S}_\theta^x - K \right] \\
&\geq Ke^{-r\theta} - x + e^{-r\theta} \mathbb{E}\left[\alpha \left| (\bar{S}_\theta^x - c_i(0))^+ \right|^\beta \right] \\
&\quad + e^{-r\theta} \mathbb{E}\left[\mathbf{1}_{\{\bar{S}_\theta^x \notin V\}} \left(u_i(0, \bar{S}_\theta^x) + \bar{S}_\theta^x - K - \alpha \left| \left(\bar{S}_\theta^x - c_i(0) \right)^+ \right|^\beta \right) \right] \\
&\geq Ke^{-r\theta} - x + e^{-r\theta} \mathbb{E}\left[\alpha \left| (\bar{S}_\theta^x - c_i(0))^+ \right|^\beta \right] - e^{-r\theta} \mathbb{E}\left[\mathbf{1}_{\{\bar{S}_\theta^x \notin V\}} \alpha \left| \bar{S}_\theta^x \right|^\beta \right] \\
&\geq Ke^{-r\theta} - x + e^{-r\theta} \mathbb{E}\left[\alpha \left| (\bar{S}_\theta^x - c_i(0))^+ \right|^\beta \right] - \alpha e^{-r\theta} x^\beta e^{\beta\left(r - \frac{\sigma^2}{2} \right)\theta + \beta^2 \sigma^2 \theta} \sqrt{\mathbb{P}(\bar{S}_\theta^x \notin V)}.
\end{aligned}
$$

Let $\epsilon > 0$ be such that $(c_i(0) - 2\epsilon, c_i(0) + 2\epsilon) \subset V$. For $x \in (c_i(0) - \epsilon, c_i(0) + \epsilon)$,

$$\mathbb{P}\left(\bar{S}_\theta^x \notin V \right) \leq \mathbb{P}\left(\bar{S}_\theta^x \notin (x - \epsilon, x + \epsilon) \right) \leq 2\mathcal{N}\left\{ \frac{1}{\sigma\sqrt{\theta}} \left(\left(r + \frac{\sigma^2}{2} \right)\theta + \ln\max\left(\frac{x - \epsilon}{x}, \frac{x}{x + \epsilon} \right) \right) \right\}.$$

We deduce that

$$u_i(\theta, x) \geq Ke^{-r\theta} - x + e^{-r\theta}\mathbb{E}\left[\alpha\left|\left(\bar{S}_\theta^x - c_i(0)\right)^+\right|^\beta\right] + o(\theta), \tag{12}$$

where the term $o(\theta)$ is uniform for $x \in (c_i(0) - \epsilon, c_i(0) + \epsilon)$. In order to bound the third term of the right-hand-side from below, we first deal with $\phi(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[\left|\left(\bar{S}_\theta^1 - 1\right)^+\right|^\beta\right]$. Using the change of variables $z = \sigma\sqrt{\theta}u$ for the second equality, we have

$$
\begin{aligned}
\phi(\theta) &= \int_0^{+\infty} z^\beta e^{-\frac{1}{2\sigma^2\theta}\left(\ln(1+z) - \left[r - \frac{\sigma^2}{2}\right]\theta\right)^2} \frac{dz}{\sqrt{2\pi\theta}\sigma(1+z)} \\
&\geq e^{-\left[\frac{r}{\sigma} - \frac{\sigma}{2}\right]^2\theta} \int_0^{+\infty} z^\beta e^{-\frac{1}{\sigma^2\theta}\ln^2(1+z)} \frac{dz}{\sqrt{2\pi\theta}\sigma(1+z)} \\
&\geq e^{-\left[\frac{r}{\sigma} - \frac{\sigma}{2}\right]^2\theta} \int_0^{+\infty} z^\beta e^{-\frac{z^2}{\sigma^2\theta}} \frac{dz}{\sqrt{2\pi\theta}\sigma(1+z)} = e^{-\left[\frac{r}{\sigma} - \frac{\sigma}{2}\right]^2\theta}\sigma^\beta\theta^{\frac{\beta}{2}} \int_0^{+\infty} \frac{u^\beta e^{-u^2}du}{\sqrt{2\pi}(1 + u\sigma\sqrt{\theta})} \\
&\geq e^{-\left[\frac{r}{\sigma} - \frac{\sigma}{2}\right]^2\theta}\sigma^\beta\theta^{\frac{\beta}{2}} \int_0^{+\infty} \frac{u^\beta e^{-u^2}}{\sqrt{2\pi}}\left(1 - u\sigma\sqrt{\theta}\right)du \\
&= e^{-\left[\frac{r}{\sigma} - \frac{\sigma}{2}\right]^2\theta}\sigma^\beta\theta^{\frac{\beta}{2}} \frac{1}{\sqrt{8\pi}}\left[\Gamma\left(\frac{1+\beta}{2}\right) - \sigma\sqrt{\theta}\Gamma\left(\frac{3+\beta}{2}\right)\right] \\
&= e^{-\left[\frac{r}{\sigma} - \frac{\sigma}{2}\right]^2\theta}\sigma^\beta\theta^{\frac{\beta}{2}} \frac{1}{\sqrt{8\pi}}\Gamma\left(\frac{1+\beta}{2}\right)\left[1 - \sigma\sqrt{\theta}\frac{1+\beta}{2}\right].
\end{aligned}
$$

Thus, for $\theta < \frac{1}{\sigma^2(1+\beta)^2}$ and $C = \frac{1}{2}e^{-\frac{\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2}{\sigma^2(1+\beta)^2}}\frac{\sigma^\beta}{\sqrt{8\pi}}\Gamma\left(\frac{1+\beta}{2}\right)$, one has $\phi(\theta) \geq C\theta^{\frac{\beta}{2}}$.

Let $x < c_i(0)$ and $\tau = \inf\left\{t \geq 0 | \bar{S}_t^x \geq c_i(0)\right\}$. For $\theta < \frac{1}{\sigma^2(1+\beta)^2}$, using the strong Markov property then Formula 2.0.2 p.223 [BS96], one has

$$
\begin{aligned}
\mathbb{E}\left[\left|\left(\bar{S}_\theta^x - c_i(0)\right)^+\right|^\beta\right] &= |c_i(0)|^\beta\,\mathbb{E}\left[\mathbb{E}\left[\left|\left(\bar{S}_{\theta-\tau}^1 - 1\right)^+\right|^\beta \middle| \mathcal{F}_\tau\right]\mathbf{1}_{\{\tau<\theta\}}\right] \\
&= |c_i(0)|^\beta\,\mathbb{E}\left[\phi\left(\theta - \tau\right)\mathbf{1}_{\{\tau<\theta\}}\right] \\
&\geq |c_i(0)|^\beta\,C\theta^{\frac{\beta}{2}}\mathbb{E}\left[\left(1 - \frac{\tau}{\theta}\right)^{\frac{\beta}{2}}\mathbf{1}_{\{\tau<\theta\}}\right] \\
&\geq |c_i(0)|^\beta\,C\theta^{\frac{\beta}{2}}\frac{1}{\sigma}\ln\frac{c_i(0)}{x}\int_0^\theta\left(1 - \frac{t}{\theta}\right)^{\frac{\beta}{2}}\frac{1}{\sqrt{2\pi t^3}}e^{-\frac{1}{2\sigma^2 t}\left(\left(\frac{\sigma^2}{2} - r\right)t + \ln\frac{c_i(0)}{x}\right)^2}dt \\
&\geq |c_i(0)|^\beta\,e^{\frac{1}{2\sigma^2}\left[2\left(\frac{\sigma^2}{2} - r\right)\ln\frac{x}{c_i(0)} - \left(\frac{\sigma^2}{2} - r\right)^2\theta\right]}C\theta^{\frac{\beta}{2}} \\
&\qquad \times \underbrace{\frac{1}{\sigma\sqrt{2\pi\theta}}\ln\frac{c_i(0)}{x}\int_0^1(1-u)^{\frac{\beta}{2}}\frac{1}{\sqrt{u^3}}e^{-\frac{1}{2\sigma^2\theta u}\ln^2\frac{c_i(0)}{x}}du}_{:=\psi(\theta,x)}.
\end{aligned}
$$

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Hence

$$\exists M, \eta > 0, \ \forall (\theta, x) \in (0, \eta) \times (c_i(0) e^{-\sigma \theta^{\frac{1}{3}}}, c_i(0)), \ \mathbb{E}\left[\left|\left(\bar{S}_\theta^x - c_i(0)\right)^+\right|^\beta\right] \geq M \theta^{\frac{\beta}{2}} \psi(\theta, x). \quad (13)$$

Setting $\gamma(x) = \frac{1}{\sigma\sqrt{\theta}} \ln \frac{c_i(0)}{x}$, we have $\psi(\theta, x) = \frac{\gamma(x)}{\sqrt{2\pi}} \int_0^1 (1-u)^{\frac{\beta}{2}} u^{-\frac{3}{2}} e^{-\frac{\gamma^2(x)}{2u}} du$. With the change of variables $t = \frac{1}{u} - 1$, we deduce that $\psi(\theta, x) = \frac{\gamma(x)}{\sqrt{2\pi}} e^{-\frac{\gamma^2(x)}{2}} \Gamma\left(\frac{\beta}{2} + 1\right) \mathrm{U}\left(\frac{\beta}{2} + 1; \frac{3}{2}; \frac{\gamma^2(x)}{2}\right)$ where $\mathrm{U}(a, b, z) = \frac{1}{\Gamma(a)} \int_0^{+\infty} e^{-tz} t^{a-1} (1+t)^{b-a-1} dt$ is the confluent hypergeometric function of the second kind. By 13.5.2 p.504 [AS72],

$$\text{for } z \to +\infty, \ \mathrm{U}\left(\frac{\beta}{2} + 1; \frac{3}{2}; z\right) = z^{-(\frac{\beta}{2}+1)}(1 + O(1/z)).$$

Then we choose θ small enough to ensure that $x(\theta) = c_i(0) e^{-\sigma\sqrt{\theta((2-\beta)|\ln \theta| - (\delta+\beta) \ln |\ln \theta|)}}$ is well defined. Since $\gamma(x(\theta)) = \sqrt{(2-\beta)|\ln \theta| - (\delta + \beta) \ln |\ln \theta|}$ tends to $+\infty$ as $\theta \to 0$, we deduce

$$\psi(\theta, x(\theta)) = \frac{\Gamma\left(\frac{\beta}{2} + 1\right) 2^{1+\frac{\beta}{2}}}{((2-\beta)|\ln \theta| - (\delta+\beta) \ln |\ln \theta|)^{\frac{\beta+1}{2}} \sqrt{2\pi}} \theta^{1-\frac{\beta}{2}} |\ln \theta|^{\frac{\delta+\beta}{2}} \left(1 + O\left(\frac{1}{|\ln \theta|}\right)\right)$$

$$= \frac{\Gamma\left(\frac{\beta}{2} + 1\right) 2^{1+\frac{\beta}{2}}}{\sqrt{2\pi}(2-\beta)^{\frac{\beta+1}{2}}} \theta^{1-\frac{\beta}{2}} |\ln \theta|^{\frac{\delta-1}{2}} \left(1 + O\left(\frac{\ln |\ln \theta|}{|\ln \theta|}\right)\right).$$

Plugging this into Equation (13), we conclude that it exists a constant $\kappa > 0$ such that as $\theta \to 0$,

$$\mathbb{E}\left[\left|\left(\bar{S}_\theta^{x(\theta)} - c_i(0)\right)^+\right|^\beta\right] \geq \kappa \theta |\ln \theta|^{\frac{\delta-1}{2}} \left(1 + O\left(\frac{\ln |\ln \theta|}{|\ln \theta|}\right)\right).$$

With Equation (12), this implies that

$$u_i(\theta, x(\theta)) \geq K - x(\theta) + \theta \left(\kappa |\ln \theta|^{\frac{\delta-1}{2}} - rK\right) + o(\theta)$$

and the conclusion follows by positivity of the factor $\kappa |\ln \theta|^{\frac{\delta-1}{2}} - rK$ for θ small enough. $\qquad \square$

6.3.2 Proof of Lemma 5.5.

Proof. Ideas are similar to those of the proof of Proposition 2.2 of [JV11]. For $\alpha > 0$, according to Proposition 4.1, there exists $\eta > 0$ such that $\sup_{w \in [0,\eta]} c_i(w) \leq c_i(0) + \frac{\alpha}{2}$. Let us suppose that $t_1 \in [0, \eta]$. Let $x \leq c_i(0) + \alpha$ and $v \geq 0$.
Setting $\tilde{\tau} = \inf\left\{w \geq 0 | \bar{S}_w^x \geq c_i(0) + \alpha\right\}$, we have

$$\mathbf{1}_{\{\tau \geq v\}} \geq \mathbf{1}_{\left\{\tau \geq \tilde{\tau}, \ \tilde{\tau} \leq v, \ \forall w \in [\tilde{\tau}, v], \ \bar{S}_w^x > c_i(0) + \alpha\right\}}$$

Using the strong Markov property, we deduce that

$$\mathbb{P}\left(\tau \geq v\right) \geq \mathbb{P}\left(\tau \geq \tilde{\tau}, \ \tilde{\tau} \leq v\right) \mathbb{P}\left(\inf_{w \in [0,v]} \bar{S}_w^1 > \frac{c_i(0) + \frac{\alpha}{2}}{\hat{c}_i(0) + \alpha}\right). \tag{14}$$

Whereas, by continuity of the trajectories of $\bar{S}^x$ and since $x \leq c_i(0) + \alpha$,

$$\mathbf{1}_{\left\{\tau \geq v, \ \bar{S}_v^x \geq c_i(0) + 2\alpha\right\}} \leq \mathbf{1}_{\left\{\tau \geq \tilde{\tau}, \tilde{\tau} \leq v, \ \bar{S}_v^x \geq c_i(0) + 2\alpha\right\}}.$$

Again by the strong Markov property, we deduce that

$$\mathbb{E}\left[\left(\bar{S}_v^x\right)^p \mathbf{1}_{\left\{\tau \geq v, \ \bar{S}_v^x \geq c_i(0) + 2\alpha\right\}}\right] \leq \mathbb{E}\left[\mathbf{1}_{\{\tau \geq \tilde{\tau}, \tilde{\tau} \leq v\}}\left(c_i(0) + \alpha\right)^p \mathbb{E}\left[\left(\bar{S}_w^1\right)^p \mathbf{1}_{\left\{\bar{S}_w^1 \geq \frac{c_i(0) + 2\alpha}{c_i(0) + \alpha}\right\}}\right]_{w = v - \tilde{\tau}}\right]. \tag{15}$$

Then by defining $\tilde{\mathbb{P}}$ as $\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{p\sigma B_t - \frac{p^2 \sigma^2 t}{2}}$, we get

$$\mathbb{E}\left[\left(\bar{S}_v^x\right)^p \mathbf{1}_{\left\{\tau \geq v, \ \bar{S}_v^x \geq c_i(0) + 2\alpha\right\}}\right] \leq \mathbb{P}\left(\tau \geq \tilde{\tau}, \ \tilde{\tau} \leq v\right)\left(c_i(0) + \alpha\right)^p e^{\left(pr + \sigma^2 \frac{p(p-1)}{2}\right)v} \sup_{0 \leq w \leq v} \tilde{\mathbb{P}}\left(\bar{S}_w^1 \geq \frac{c_i(0) + 2\alpha}{c_i(0) + \alpha}\right). \tag{16}$$

Notice that for any $t, x, y \geq 0$, $\mathbb{P}(\bar{S}_t^x \geq y) \leq \tilde{\mathbb{P}}(\bar{S}_t^x \geq y)$. So, we deduce that

$$\frac{\mathbb{E}\left[\left(1 + \left(\bar{S}_v^x\right)^p\right) \mathbf{1}_{\left\{\tau \geq v, \ \bar{S}_v^x \geq c_i(0) + 2\alpha\right\}}\right]}{\mathbb{P}(\tau \geq v)} \leq \frac{\left(1 + (c_i(0) + \alpha)^p e^{\left(pr + \sigma^2 \frac{p(p-1)}{2}\right)v}\right) \sup_{0 \leq w \leq v} \tilde{\mathbb{P}}\left(\bar{S}_w^1 \geq \frac{c_i(0) + 2\alpha}{c_i(0) + \alpha}\right)}{\mathbb{P}\left(\inf_{w \in [0,v]} \bar{S}_w^1 > \frac{c_i(0) + \frac{\alpha}{2}}{\hat{c}_i(0) + \alpha}\right)}. \tag{17}$$

This concludes the proof since when v tends to 0, the numerator tends to 0 whereas the denominator tends to 1. $\qquad\square$

6.3.3 Itô tanaka formula

Lemma 6.1. *For $i \geq 1$, assume that D_i is difference of two convex functions. Then*

$$du_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_t^x)) = \partial_x u_i(\theta_d^{i-1}, \rho_i(\bar{S}_t^x))\rho_i'(\bar{S}_t^x)d\bar{S}_t^x + \frac{1}{2}\int_{\mathbb{R}} \partial_x u_i(\theta_d^{i-1}, \rho_i(a))dL_t^a(\bar{S}^x)\rho_i''(da)$$

$$+ \frac{1}{2}\partial_{xx} u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}_t^x))\left(\rho_i'(\bar{S}_t^x)\right)^2 d\left\langle \bar{S}^x \right\rangle_t$$

Proof. By the Itô-Tanaka formula,

$$d\rho_i(\bar{S}_t^x) = \rho_i'(\bar{S}_t^x)d\bar{S}_t^x + \frac{1}{2}\int_{\mathbb{R}} dL_t^a(\bar{S}^x)\rho_i''(da).$$

Hence $X_t = \rho_i(\bar{S}^x_t)$ is a continuous semi-martingale with bracket $\langle X \rangle_t = \int_0^t \left(\rho'_i(\bar{S}^x_s) \right)^2 d \left\langle \bar{S}^x \right\rangle_s$. By Lemma 3.3, since $\theta_d^{i-1} > 0$, the function $f(x) = \partial_{xx} u_{i-1} \left(\theta_d^{i-1}, \bullet \right)$ is bounded. The next lemma ensures that

$$du_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}^x_t)) = \partial_x u_i(\theta_d^{i-1}, \rho_i(\bar{S}^x_t)) \left(\rho'_i(\bar{S}^x_t) d\bar{S}^x_t + \frac{1}{2} \int_{\mathbb{R}} \rho''_i(da) dL^a_t(\bar{S}^x) \right)$$
$$+ \frac{1}{2} \partial_{xx} u_{i-1}(\theta_d^{i-1}, \rho_i(\bar{S}^x_t)) \left(\rho'_i(\bar{S}^x_t) \right)^2 d \left\langle \bar{S}^x \right\rangle_t.$$

One concludes since, by Proposition 1.3 p.222 [RY91], $\mathbb{P} \otimes |\rho''_i|(da)$ a.e., the measure $dL^a_t(\bar{S}^x)$ is supported by $\{ t : \bar{S}^x_t = a \}$.

$\square$

Lemma 6.2. *Let X be a continuous semi-martingale and f a $\mathcal{C}^1$ function, $\mathcal{C}^2$ on $[0, x^\star)$ and $(x^\star, +\infty)$, such that either $\inf_{x \in \mathbb{R}} f''(x)$ or $\sup_{x \in \mathbb{R}} f''(x)$ is finite. Then, almost surely,*

$$\int_0^t \mathbf{1}_{\{X_s = x^\star\}} d \langle X \rangle_s = 0 \ \text{and} \ f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d \langle X \rangle_s.$$

Proof. The first assertion is a consequence of the occupation times formula and ensures that differentiability of f' at $x^\star$ is not needed for the right-hand-side of the second equality to be well defined. By hypothesis, it exists $0 \leq M < \infty$ such that either $x \mapsto f(x) + Mx^2$ or $x \mapsto f(x) - Mx^2$ is convex and consequently f is the difference of two convex functions. So we can apply the Itô-Tanaka formula and conclude by the occupation times formula. $\square$

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