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Abstract

This manuscript deals with the stabilization of a class of nonlinear discrete-time systems under control saturations including time-varying parameter dependency. The studied control law consists of the gain scheduled feedback of the measured output and of the nonlinearity present in the dynamics of the controlled system. Furthermore the saturations are taken into account by modeling the nonlinear saturated system through a deadzone nonlinearity satisfying a *modified sector condition*. Thus, as for precisely known systems, LMI stabilization conditions are proposed for such a generic system. These conditions can be cast into convex programming problems for design purposes. An illustrative example stresses out the efficiency of the main result.

Key words: Nonlinear systems, Saturations, Parameter dependent Lyapunov functions, LMIs, Robustness, Gain-scheduled output control.

1 Introduction

The control design of systems subject to nonlinearities, including saturation, has attracted considerable attention in control theory literature for many years [1]. It has been motivated by the large variety of practical applications and the occurrence in industrial processes. Besides the classical issue of stability analysis and stabilization of systems with saturated input [2–4], several approaches have been proposed to cope with controller performances, like disturbance attenuation [5–7] or using a representation as the feedback interconnection of a linear system with a sector bounded nonlinearity. This last option has been received attention with de proposal of new design tools [4,8–10] and, in particular, by considering the control saturations (see for instance [11–16]).

The case of nonlinear systems subject to actuator amplitude limitations and for which the dynamics can be decomposed into the feedback interconnection described above, is treated in [17,14] for continuous and discrete-time precisely known systems, respectively. The framework of Time-Varying Parameter (TVP) dependent and nonlinear systems in discrete-time is considered in [18], by assuming that the TVP was unknown. Then the provided approach consists in considering a Parameter Dependent Lyapunov Function (PDLF) associated with a parameter independent state feedback control law and of the nonlinearity that models the nonlinear part of the open-loop system dynamics.

In practice, the current state is not usually available in its entirety and the considered control law class should involve a measured output. The prob-

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lem of the output feedback is a problem in automatic control which is still widely open (see [19–21] and references therein). For the LTI systems, full order dynamic output controllers could be designed via mainly two technics: a judicious pyramidal change of variables [22] and the projection lemma [10]. They have been both applied for systems including saturations [6,23] and [24]. Nevertheless these techniques are not well adapted to systems including a TVP. The matrices of the output dynamic should be independent of the TVP (that is robust controller) or only quasi-TVP dependent (only some matrices are TVP-dependent, see [6]). Such a framework is helpful and adapted to continuous-time system governed via a network [25], by including non-uniformaly sampling [26], quantification of the state and input saturation [27]. This framework also allows to cope with the fuzzy control problem of nonlinear systems using Takagi-Sugeno fuzzy models [28] where, particularly, the membership functions are the (known) TVPs.

In this paper, we propose to consider the class of control laws formulated as gain-scheduled feedbacks of the measured output and of the nonlinearity. It is assumed that the current TVP is available (by estimation or measurement). Sufficient conditions are provided here by using a modified sector condition for taking the saturation nonlinearity into acount, to design a Gain-Scheduled control law. Thus Linear Matrix Inequalities (LMI) conditions for local stabilization allow to cast this control design problem into a convex programming problem. To cope with control law based on measured output, we add to LMIs a set of Linear Matrix Equalities. Thus the obtained result will be less conservative than ones available in the literature, with a moderate increasing of LMI dimensions.

This work is organized as follows: In section 2, the framework of the control design problem is presented. Section 3 provides some preliminary results and definitions, which allow to obtain the main result, formalized as a convex programming problem in Section 4. The paper ends with a numerical example, issued from the litterature of network control systems and concluding remarks respectively in Sections 5 and 6.

Notations. Relative to a matrix $A \in \mathbb{R}^{m\times n}$, A' denotes its transpose, and $A_{(i)}$, $i = 1, \dots m$, denotes its ith row. If $A = A' \in \mathbb{R}^{n\times n}$, then $A < 0$ ($A \leq 0$) means that A is negative (semi-)definite. The components of any vector $x \in \mathbb{R}^n$ are denoted $x_{(i)}$, $\forall i = 1, \dots n$. Inequalities between vectors are component-wise: $x \leq 0$ means that $x_{(i)} \leq 0$ and $x \leq y$ means that $x_{(i)} - y_{(i)} \leq 0$. I_n denotes the $n \times n$ identity matrix. The symbol $\star$ stands for symmetric blocks in matrices. For a symmetric and positive-definite matrix $M \in \mathbb{R}^{n\times n}$, the ellipsoidal set $\mathcal{E}(M)$ associated with M is given by $\{x \in \mathbb{R}^n; x'Mx \leq 1\}$.

2 Problem presentation

Consider a nonlinear discrete-time TVP-dependent system represented by:

$$
\begin{aligned}
x_{k+1} &= A(\xi_k)\, x_k + G(\xi_k)\, \varphi(z_k) + B(\xi_k)\, \mathrm{sat}(u_k), && (1)\\
z_k &= L(\xi_k)\, x_k, && (2)\\
y_k &= C x_k. && (3)
\end{aligned}
$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $z_k \in \mathbb{R}^p$ and $y_k \in \mathbb{R}^q$ are at time k the state, the input, the output of the system and the measured output, respectively. $\xi_k \in \mathbb{R}^N$ is the unknown TVP at time k belonging to the unit simplex Ξ:

$$
\Xi = \{\xi \in \mathbb{R}^N \ ; \ \sum_{i=1}^{N} \xi_{(i)} = 1 \ , \quad \xi_{(i)} \geq 0 \ , \quad i = 1, \dots, N\}. \qquad (4)
$$

ξ_k can be viewed as a model uncertainty. The structure of the system matrices are assumed to be a N-vertices polytope of the form:

$$\left[\, A\left(\xi_k\right)\; B\left(\xi_k\right)\; G\left(\xi_k\right)\; L\left(\xi_k\right) \right] = \sum_{i=1}^{N} \xi_{k(i)} \left[\, A_i\; B_i\; G_i\; L_i \right]. \tag{5}$$

The matrix C involved in (3) is assumed to be full row rank and independent with respect to ξ_k. The nonlinearity $\varphi(.) : \mathbb{R}^p \to \mathbb{R}^p$ verifies the cone bounded sector condition $\varphi(.) \in [0_p, \Omega]$, [29,3], that is $\varphi(0) = 0$ and there exists a symmetric positive definite matrix $\Omega = \Omega' \in \mathbb{R}^{p\times p}$ such that, $\forall z_k \in \mathbb{R}^p$, all components of $\varphi_{(l)}(\cdot)$ independently verify the cone bounded sector condition

$$\varphi_{(l)}(z_k)[\varphi(z_k) - \Omega z_k]_{(l)} \leq 0\ ,\ \ \forall l = 1, \cdots, p. \tag{6}$$

By summing these inequalities with weigthing them by p arbitrary positive scalars $(\delta_l(\xi_k))^{-1} > 0$, we obtain

$$\sum_{l=1}^{p}(\delta_l(\xi_k))^{-1}\varphi_{(l)}(z_k)[\varphi(z_k) - \Omega z_k]_{(l)} = \varphi'(z_k)(\Delta(\xi_k))^{-1}[\varphi(z_k) - \Omega z_k] \leq 0, \tag{7}$$

where $\Delta(\xi_k) \triangleq \mathrm{diag}\{\delta_l(\xi_k)\} \in \mathbb{R}^{p\times p}$ is diagonal and positive.

By definition (6), which is assumed in the sequel, the nonlinearity $\varphi(\cdot)$ globally verifies the sector condition (7), for any diagonal and positive matrix $\Delta(\xi_k)$. Thus, $\Delta(\xi_k)$ represents a degree-of-freedom and can be an optimization variable. Notice, however, that in a more general case where there may exist dependencies among different components of $\varphi(\cdot)$, it could be possible to consider only the sector condition provided in [29,3]: $\varphi'(z_k)[\varphi(z_k) - \Omega z_k] \leq 0$, by restricting the degree-of-freedom $\Delta(\xi_k)$ to $\Delta(\xi_k) = \delta(\xi_k)I_p$. Ω, which is independent on the parameter ξ_k, is given by the designer and assumed to be known in the following.

The control inputs are bounded in amplitude, and the standard saturation function is considered:

$$\mathrm{sat}(u(t))_{(l)} = \mathrm{sat}(u_{(l)}(t)) = \mathrm{sign}(u_{(l)}(t)) \min(\rho_{(l)}, |u_{(l)}(t)|), \tag{8}$$

$\forall l = 1, \dots, m$, where $\rho_{(l)} > 0$ denotes the symmetric amplitude bound relative to the l-th control.

Throughout this work, we assume that the vector ρ is fixed and predefined and in addition that the current parameter vector, ξ_k, is available in real time (measured or estimated). Thus, by extending the kind of control law provided in [14,18], the following gain scheduled control law is considered:

$$u_k = K(\xi_k)y_k + \Gamma(\xi_k)\varphi(z_k) = K(\xi_k)Cx_k + \Gamma(\xi_k)\varphi(z_k) \tag{9}$$

where the $m \times q$-matrix $K(\xi_k)$ is a gain scheduled output feedback and the $m \times p$-matrix $\Gamma(\xi_k)$ is a gain scheduled feedback associated to the nonlinearity $\varphi(.)$. Thus, with non trivial $\Gamma(\xi_k)$, this feedback control law requires either the knowledge of $\varphi(.)$ or its availability as a signal [8]. The corresponding closed-loop system reads:

$$x_{k+1} = A(\xi_k)x_k + G(\xi_k)\varphi(z_k) + B(\xi_k)\mathrm{sat}\left(K(\xi_k)y_k + \Gamma(\xi_k)\varphi(z_k)\right). \tag{10}$$

In the sequel the following problem is considered:

Problem 1 (Robust stabilization under saturating actuators)

Determine gain scheduled feedback matrices $K(\xi_k)$ and $\Gamma(\xi_k)$ and a region $\mathcal{S}_0 \subseteq \mathbb{R}^n$, as large as possible, such that for any initial condition $x_0 \in \mathcal{S}_0$ the origin of the corresponding TVP-dependent closed-loop system (10) is uniformly asymptotically stable for any $\varphi(.)$ verifying the sector condition (7) and *for any sequency* $\{\xi_k\}_{k\in\mathbb{N}}$.

Before presenting the main result associated with Problem 1, some technical lemmas and definitions are pointed out in the following section.

3 Preliminaries

Consider the generic dead-zone nonlinearity $\Psi(u_k) = u_k - \mathrm{sat}(u_k)$. By considering u_k given by (9), we can rewrite the closed-loop system (10) under the form

$$x_{k+1} = A_{cl}(\xi_k)x_k + G_{cl}(\xi_k)\varphi(z_k) - B(\xi_k)\Psi(u_k), \tag{11}$$

where $A_{cl}(\xi_k) = A(\xi_k) + B(\xi_k)K(\xi_k)C$ and $G_{cl}(\xi_k) = G(\xi_k) + B(\xi_k)\Gamma(\xi_k)$.

The following Lemma will be used to consider the dead-zone as a nonlinearity belonging to a generalized sector condition. For given ξ_k-dependent matrices $H(\cdot) \in \mathbb{R}^{m\times n}$ and $F(\cdot) \in \mathbb{R}^{m\times p}$, consider the set $\mathcal{S}(H(\cdot), F(\cdot), \rho)$ defined by

$$\Big\{x \in \mathbb{R}^n \mid -\rho \leq \Big(H(\xi)x - F(\xi)\varphi\Big(L(\xi)x_k\Big)\Big) \leq \rho\ ;\ \forall\xi \in \Xi\Big\}. \tag{12}$$

Lemma 2 *Consider ξ_k-dependent $m \times q$-matrix $K(\cdot)$, $m \times n$-matrix $E_1(\cdot)$ and $m \times p$-matrices $\Gamma(\cdot), E_2(\cdot)$ and note $H(\cdot) = K(\cdot)C - E_1(\cdot)$ and $F(\cdot) = \Gamma(\cdot) - E_2(\cdot)$. If x_k is an element of $\mathcal{S}(H(\cdot), F(\cdot), \rho)$, then by noting $u_k = K(\xi_k)y_k + \Gamma(\xi_k)\varphi(z_k)$, the nonlinearity $\Psi(u_k)$ satisfies the following inequality*

$$\Psi(u_k)'\Big(T(\xi_k)\Big)^{-1}\Big[\Psi(u_k) - E_1(\xi_k)x_k - E_2(\xi_k)\varphi(z_k)\Big] \leq 0 \tag{13}$$

for any diagonal positive definite matrix $T(\xi_k) \in \mathbb{R}^{m\times m}$.

Proof: It follows the same lines as the one of Lemma 1 in [11] (see also [30]). □

Let us consider a Parameter Dependent Lyapunov Function (PDLF) defined by

$$V : \begin{cases} \mathbb{R}^n \times \Xi \longrightarrow \mathbb{R}^+, \\ \\ (x_k, \xi_k) \longmapsto V(x_k, \xi_k). \end{cases} \tag{14}$$

The Parameter Dependent Level Set (PDLS) associated to V is given by

$$L_V \triangleq \Big\{ x_k \in \mathbb{R}^n \mid V(x_k, \xi_k) \le 1 \ , \ \forall \xi_k \in \Xi \Big\} \tag{15}$$

The notion of contractive sets is basic for determining regions of asymptotic stability for the saturating closed-loop system (11). The following definition of contractivity is adapted to consider both the parameter uncertainties and the sector bounded characterization of nonlinearity $\varphi(.)$.

Definition 3 *The PDLS L_V is* robustly absolutely contractive *with respect to the trajectories of system (11), if $\forall x_k \in L_V$, $\forall \xi_k \in \Xi$ and $\forall \varphi(.) \in [0_p, \Omega]$,*

$$V(x_{k+1}, \xi_{k+1}) - V(x_k, \xi_k) < 0. \tag{16}$$

To provide the desired (local) stabilization conditions, we consider in the sequel the class of PDLF of the form

$$V(x_k, \xi_k) = x_k' Q^{-1}(\xi_k) x_k, \quad \text{with } Q(\xi_k) = \sum_{i=1}^{N} \xi_{k(i)} Q_i, \ Q_i = Q_i' > 0. \tag{17}$$

It is noteworthy that the PDLF is not linear with respect to the uncertainty ξ_k as usual in dedicated literature.

Lemma 4 *The PDLS L_V (15), associated with the considered PDLF class*

(17), verifies:

$$L_V = \bigcap_{\xi_k \in \Xi} \mathcal{E}(Q^{-1}(\xi_k)) = \bigcap_{i \in \{1,\cdots,N\}} \mathcal{E}(Q_i^{-1}) \tag{18}$$

Proof: $x \in L_V \Leftrightarrow \forall \xi \in \Xi, V(x,\xi) < 1 \Leftrightarrow x \in \bigcap_{\xi_k \in \Xi} \mathcal{E}(Q^{-1}(\xi_k))$. In addition, $\bigcap_{\xi_k \in \Xi} \mathcal{E}(Q^{-1}(\xi_k)) \subset \bigcap_{i \in \{1,\cdots,N\}} \mathcal{E}(Q_i^{-1})$. On the other hand, to prove that $\bigcap_{i \in \{1,\cdots,N\}} \mathcal{E}(Q_i^{-1}) \subset \bigcap_{\xi_k \in \Xi} \mathcal{E}(Q^{-1}(\xi_k))$, consider $x \in \bigcap_{i \in \{1,\cdots,N\}} \mathcal{E}(Q_i^{-1})$, then $\forall i = 1, \cdots, N$, $x' Q_i^{-1} x < 1$, which is equivalent by Schur complement to $\begin{bmatrix} 1 & x' \\ x & Q_i \end{bmatrix} > 0$. Thus $\forall \xi \in \Xi$, $\begin{bmatrix} 1 & x' \\ x & Q(\xi) \end{bmatrix} > 0$. This implies that $x \in \mathcal{E}(Q^{-1}(\xi))$, $\forall \xi \in \Xi$, or $x \in \bigcap_{\xi \in \Xi} \mathcal{E}(Q^{-1}(\xi))$. $\square$

The problem and the definitions of the used tools being set, two technical lemmas allowing to obtain the main theorem in Section 4, are presented.

Lemma 5 *Consider, for $i = 1, \cdots, N$, the existence of symmetric positive definite matrices $Q_i \in \mathbb{R}^{n \times n}$, positive diagonal matrices $\Delta_i \in \mathbb{R}^{p \times p}$ and $T_i \in \mathbb{R}^{m \times m}$, matrices $U_i \in \mathbb{R}^{n \times n}$, $Y_{1,i} \in \mathbb{R}^{m \times q}$, $Z_{1,i} \in \mathbb{R}^{m \times n}$, $W_i \in \mathbb{R}^{q \times q}$ and $Y_{2,i}, Z_{2,i} \in \mathbb{R}^{m \times p}$ such that $\forall i, j = 1, \cdots, N$, in one hand:*

$$\mathcal{M}_{ij} = \begin{bmatrix} -Q_i & \mathcal{M}_{jj}^1 & \mathcal{M}_{jj}^2 & -B_j T_j \\ \star & \mathcal{M}_j^3 & U_j' L_j' \Omega & Z_{1,j}' \\ \star & \star & -2\Delta_j & Z_{2,j}' \\ \star & \star & \star & -2T_j \end{bmatrix} < 0 \tag{19}$$

and $\forall i = 1, \cdots, N$, $\forall j = 1, \cdots, N-1$ *and* $\forall h = j+1, \cdots, N$

$$\mathcal{M}_{ijh} = \begin{bmatrix} -2Q_i & \mathcal{M}^1_{jh} + \mathcal{M}^1_{hj} & \mathcal{M}^2_{jh} + \mathcal{M}^2_{hj} & -B_j T_h - B_h T_j \\ \star & \mathcal{M}^3_j + \mathcal{M}^3_h & (U'_j L'_h + U'_h L'_j)\Omega & Z'_{1,j} + Z'_{1,h} \\ \star & \star & -2(\Delta_j + \Delta_h) & Z'_{2,j} + Z'_{2,h} \\ \star & \star & \star & -2(T_j + T_h) \end{bmatrix} < 0, \quad (20)$$

where $\mathcal{M}^1_{jh} = A_j U_h + B_j Y_{1,h} C$; $\mathcal{M}^2_{jh} = G_j \Delta_h + B_j Y_{2,h}$ *and* $\mathcal{M}^3_j = Q_j - U_j - U'_j$. *And, in the other hand:*

$$W_i C = C U_i, \quad \forall i = 1, \cdots, N. \quad (21)$$

Furthermore, by assuming that x_k *is such that (7) and (13) are both verified, then the gain scheduled output control (* a priori *not linear with respect to* ξ_k*)*

$$u_k = K(\xi_k) y_k + \Gamma(\xi_k)\varphi(z_k) = Y_1(\xi_k)(W(\xi_k))^{-1} y_k + Y_2(\xi_k)(\Delta(\xi_k))^{-1}\varphi(z_k), \quad (22)$$

with

$$\begin{bmatrix} Y_1(\xi_k) & Y_2(\xi_k) & U(\xi_k) & \Delta(\xi_k) & Z_1(\xi_k) & Z_2(\xi_k) & W(\xi_k) \end{bmatrix} = \sum_{i=1}^{N} \xi_{k(i)} \begin{bmatrix} Y_{1,i} & Y_{2,i} & U_i & \Delta_i & Z_{1,i} & Z_{i,2} & W_i \end{bmatrix} \quad (23)$$

implies the inequality (16).

Proof: As C is full row rank and Q_i positive definite, it follows from (21) that W_i is full rank for all $i = 1, \cdots, N$. This yields

$$B(\xi_k)Y(\xi_k)C = B(\xi_k)Y(\xi_k)W^{-1}(\xi_k)CU(\xi_k) = B(\xi_k)K(\xi_k)CU(\xi_k).$$

By summing

$$\mathcal{M}(\xi_k,\xi_{k+1}) = \sum_{i,j=1}^{N} \xi_{k+1,(i)}\xi_{k,(j)}^2\mathcal{M}_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N-1}\sum_{h=j+1}^{N} \xi_{k+1,(i)}\xi_{k,(j)}\xi_{k,(h)}\mathcal{M}_{ijh}, \quad (24)$$

by using definitions (23), and by introducing $K(\xi_k)$ and $\Gamma(\xi_k)$ as defined by (22) and the changes of variables of gain-scheduled matrices

$$E_1(\xi_k) = Z_1(\xi_k)(U(\xi_k))^{-1}; \quad E_2(\xi_k) = Z_2(\xi_k)(\Delta(\xi_k))^{-1}, \quad (25)$$

we obtain

$$\mathcal{M}(\xi_k,\xi_{k+1}) = \begin{bmatrix} -Q(\xi_{k+1}) & A_{cl}(\xi_k)U(\xi_k) & G_{cl}(\xi_k)\Delta(\xi_k) & -B(\xi_k)T(\xi_k) \\ \star & Q(\xi_k)-U(\xi_k)-U'(\xi_k) & U'(\xi_k)L'(\xi_k)\Omega & Z_1'(\xi_k) \\ \star & \star & -2\Delta(\xi_k) & Z_2'(\xi_k) \\ \star & \star & \star & -2T(\xi_k) \end{bmatrix} < 0. \quad (26)$$

From this last inequality, following [31], we deduce that

$$U'(\xi_k)Q^{-1}(\xi_k)U(\xi_k) \geq U(\xi_k) + U'(\xi_k) - Q(\xi_k). \quad (27)$$

Using Inequality (27) and the change of basis $\text{diag}[I; U^{-1}(\xi_k); \Delta^{-1}(\xi_k); T^{-1}(\xi_k)]$ leads to a matrix inequality which can reformulated by Schur complement by

$$\begin{bmatrix} A_{cl}'(\xi_k) \\ G_{cl}'(\xi_k) \\ -B'(\xi_k) \end{bmatrix} Q^{-1}(\xi_{k+1}) \begin{bmatrix} A_{cl}'(\xi_k) \\ G_{cl}'(\xi_k) \\ -B'(\xi_k) \end{bmatrix}' - \begin{bmatrix} Q^{-1}(\xi_k) & L'(\xi_k)\Omega\Delta^{-1}(\xi_k) & E_1'(\xi_k)T^{-1}(\xi_k) \\ \star & -2\Delta^{-1}(\xi_k) & E_2'(\xi_k)T^{-1}(\xi_k) \\ \star & \star & -2T^{-1}(\xi_k) \end{bmatrix} < 0. \quad (28)$$

By multiplying this last inequality at left by $\begin{bmatrix} x_k' & \varphi'(z_k) & \psi'(u_k) \end{bmatrix}$ and at right by its transpose, one has

$$V(x_{k+1},\xi_{k+1}) - V(x_k;\xi_k) - 2\varphi'(z_k)(\Delta(\xi_k))^{-1}[\varphi(z_k) - \Omega z_k] \\ - 2\Psi(u_k)'\Big(T(\xi_k)\Big)^{-1}\Big[\Psi(u_k) - E_1(\xi_k)x_k - E_2(\xi_k)\varphi(x_k)\Big] < 0. \quad (29)$$

The assumption (13) and the sector condition (7) imply the relation (16). □

Lemma 6 *Consider, for $i = 1,\cdots,N$, that there exists symmetric positive definite matrices $Q_i \in \mathbb{R}^{n\times n}$, positive diagonal matrices $\Delta_i \in \mathbb{R}^{p\times p}$, matrices $U_i \in \mathbb{R}^{n\times n}$, $Y_{1,i} \in \mathbb{R}^{m\times q}$, $W_i \in \mathbb{R}^{q\times q}$, $Z_{1,i} \in \mathbb{R}^{m\times n}$ and $Y_{2,i}, Z_{2,i} \in \mathbb{R}^{m\times p}$such that $\forall i = 1,\cdots,N$ and $\forall l = 1,\cdots,m$ equalities (21) hold and inequalities:*

$$\mathcal{N}_{i,l} = \begin{bmatrix} -Q_i + U_i + U_i' & \star & \star \\ -\Omega L_i U_i & 2\Delta_i & \star \\ (Y_{1,i}C - Z_{1,i})_{(l)} & (Y_{2,i} - Z_{2,i})_{(l)} & \rho^2_{(l)} \end{bmatrix} > 0 \quad (30)$$

and $\forall i = 1,\cdots,N-1$, $\forall j = i+1,\cdots N$ and $\forall l = 1,\cdots,m$:

$$\mathcal{N}_{ij,l} = \begin{bmatrix} -Q_i - Q_j + U_i + U_j + U_i' + U_j' & \star & \star \\ -\Omega(L_i U_j + L_j U_i) & 2(\Delta_i + \Delta_j) & \star \\ (Y_{1,i}C - Z_{1,i} + Y_{1,j}C - Z_{1,j})_{(l)} & (Y_{2,i} - Z_{2,i} + Y_{2,j} - Z_{2,j})_{(l)} & 2\rho^2_{(l)} \end{bmatrix} > 0. \quad (31)$$

Then

$$L_V \subset \mathcal{S}(H(\cdot), F(\cdot), \rho), \quad (32)$$

where $H(\xi) = K(\xi)C - E_1(\xi)$ and $F(\xi) = \Gamma(\xi) - E_2(\xi)$ with definitions (22) and (25).

Proof: By summing

$$\mathcal{N}_l(\xi_k) = \sum_{i=1}^{N} \xi_{k,(i)}^2 \mathcal{N}_{i,l} + \sum_{i=1}^{N-1} \sum_{i=p+1}^{N} \xi_{k,(i)} \xi_{k,(p)} \mathcal{N}_{ip,l}, \tag{33}$$

and due to inequality (27), the change of base $\mathrm{diag}[U^{-1}(\xi_k); \Delta^{-1}(\xi_k); 1]$ leads to a matrix inequality, which writes by Schur complement

$$\begin{bmatrix} -Q^{-1}(\xi_k) & \star \\ -\Delta^{-1}(\xi_k)\Omega L(\xi_k) & 2\Delta^{-1}(\xi_k) \end{bmatrix} - \frac{1}{\rho_{(l)}^2} \begin{bmatrix} H'_{(l)}(\xi_k) \\ F'_{(l)}(\xi_k) \end{bmatrix} \begin{bmatrix} H_{(l)}(\xi_k) & F_{(l)}(\xi_k) \end{bmatrix} > 0. \tag{34}$$

By multiplying this last inequality at left by $[\, x'_k \;\; \varphi'(z_k) \,]$ and at right by its transpose, one has

$$\begin{aligned} V(x_k, \xi_k) + 2\varphi'(z_k)(\Delta(\xi_k))^{-1}[\varphi(z_k) - \Omega L(\xi_k) x_k] \\ - \frac{1}{\rho_{(l)}^2} \left\| H_{(l)}(\xi_k) x_k - F_{(l)}(\xi_k)\varphi(z_k) \right\|^2 \geq 0. \end{aligned} \tag{35}$$

Hence, by considering the sector condition (7), it follows that:

$$V(x_k, \xi_k) \geq \frac{1}{\rho_{(l)}^2} \left\| H_{(l)}(\xi_k) x_k - F_{(l)}(\xi_k)\varphi(z_k) \right\|^2. \tag{36}$$

For all $x_k \in L_V$, $V(x_k, \xi_k) < 1$, which implies that $x_k \in \mathcal{S}(H(\cdot), F(\cdot), \rho)$. We obtain the relation (32). □

The PDLS L_V is then the set $\mathcal{S}_0$ of initial condition x_0 of uncertain closed-loop system which is uniformly asymptotically stable for any non-linearity $\varphi(\cdot)$ verifying the sector condition (7). This set L_V is convex, because this is the intersection of convex sets. However, contrary as mentioned in [16], L_V cannot be formulated in the general case as a convex hull of different ellipsoidal sets.

4 Control design via convex programming

The optimization problem consists in determining a control defined by (22), with the largest set $L_V = \mathcal{S}_0$, under the constraints (19), (20), (30) and (31). For obtaining the largest set L_V, we consider a $\sqrt{\alpha}$-radius ball included into L_V:

$$\mathcal{E}\left(\frac{1}{\alpha}I\right) = \{x \in \mathbb{R}^n; x'x \leq \alpha\} \subset L_V = \mathcal{S}_0. \tag{37}$$

This inclusion is equivalent to $\mathcal{E}\left(\frac{1}{\alpha}I\right) \subset \mathcal{E}(Q_i^{-1})$, $\forall i = 1, \cdots, N$, or (by noting $\mu = \dfrac{1}{\alpha}$; see [10]) to:

$$\begin{bmatrix} \mu I_n & I_n \\ I_n & Q_i \end{bmatrix} > 0, \quad \forall i = 1, \cdots, N. \tag{38}$$

Thus, the following convex programming problem is proposed in the main Theorem:

Theorem 7 *By considering symmetric positive definite matrices $Q_i \in \mathbb{R}^{n\times n}$, positive diagonal matrices $\Delta_i \in \mathbb{R}^{p\times p}$, matrices $U_i \in \mathbb{R}^{n\times n}$, $Y_{1,i} \in \mathbb{R}^{m\times q}$, $W_i \in \mathbb{R}^{q\times q}$, $Z_{1,i} \in \mathbb{R}^{m\times n}$ and $Y_{2,i}, Z_{2,i} \in \mathbb{R}^{m\times p}$, for $i = 1, \cdots, N$ and a scalar $\mu \in \mathbb{R}$, the convex optimization problem*

$$\min_{W_i, Q_i, U_i, \Delta_i, T_i, Z_{1,i}, Z_{2,i}, Y_{1,i}, Y_{2,i}} \mu$$

subject to LMIs (19), (20), (30), (31) *and* (38) *and to equalities* (21).

leads to a gain-scheduled control law represented by (22), solution of Problem 1.

Proof: The proof is straightforward by using Lemmas 5 and 6 and Inclusion (37), with $\mu = \dfrac{1}{\alpha}$. $\square$

Notice that a gain scheduled $m \times n$ state-feedback $K(\xi_k)$ is recovered in (9) by considering $q = n$ and $C = I_n$, *i.e.* $y_k = x_k$. This implies that $W_i = U_i$, due to equalities (21) and thus that equalities (21) and variables W_i cannot be considered in Theorem 7.

Remark 8 *The framework proposed to find a solution to the stabilization problem can be extended in different ways to cope with other control problems, as for instance ones related to $\mathcal{L}_2$-gain and λ-contractivity [7]. Notice also that, at least conceptually, it can be possible to develop a similar gain scheduling solution to the stabilization problem by using the polytopic representation of the saturation nonlinearity [1,6]. This potential solution would imply more complex conditions, numerically or even for implementation [17].*

5 Numerical example

Consider the following data for system (2)-(7), with $N = 2$:

$$A_1 = \begin{bmatrix} -1.1 & 0.4 \\ -0.2 & 1.1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.2 & 0.7 \\ 0.6 & 1.3 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1.3 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}, G_2 = \begin{bmatrix} 1.6 \\ 0 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 1.1 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0.9 \end{bmatrix}, \; \rho = \tfrac{1}{2}.$$

The nonlinearity φ verifying the sector condition with $\Omega = 0.7$ is given by $\varphi(z) = 0.3z(1 + \sin(z))$. Both cases $C = [1 \; 1]$ and $C = I_2$ are considered and we obtain respectively $\mu = 2.60$ and $\mu = 1.60$.

In Figure 1, the sets $\mathcal{E}(Q_1^{-1})$ and $\mathcal{E}(Q_2^{-1})$ corresponding to the above syn-

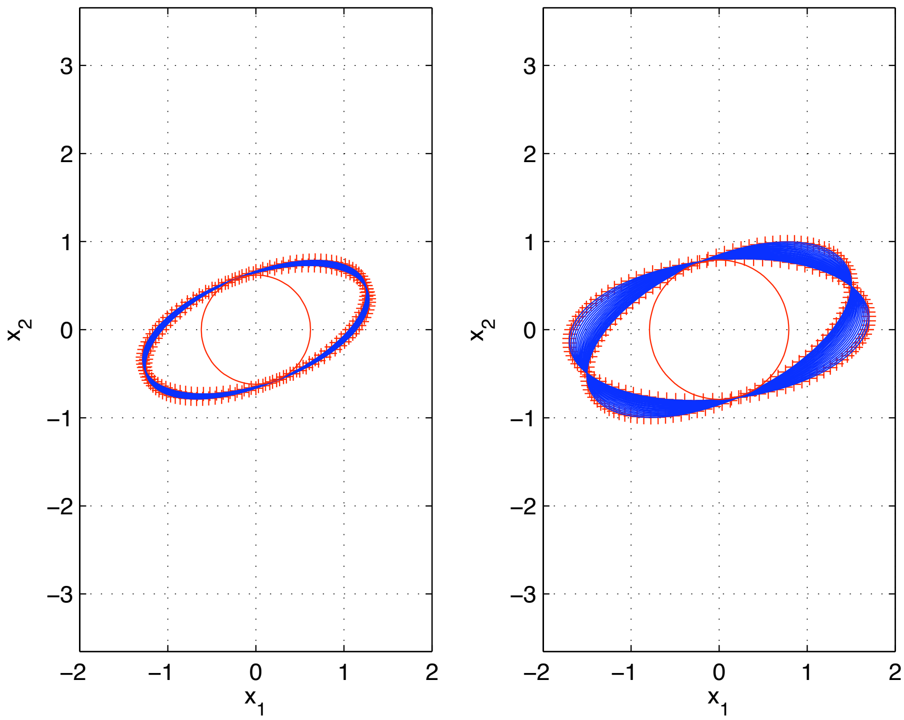

Fig. 1. Sets $\mathcal{E}(Q_1^{-1})$ and $\mathcal{E}(Q_2^{-1})$ obtained from the optimization problem, and largest sphere included in L_V for $C = [2\ 3]$ (left) and $C = I_2$ (right).

thesis result are plotted with symbol '+'. The solid lines denotes several sets $\mathcal{E}(Q^{-1}(\xi))$, for several parameters $\xi \in \Xi$, uniformely distributed. Finally the solid sphere denotes the largest sphere $\mathcal{E}(\mu I)$, included in the intersection of $\mathcal{E}(Q_1^{-1})$ and $\mathcal{E}(Q_2^{-1})$. The set $\mathcal{S}_0 = L_V$ is depicted by the intersection of both ellipsoids $\mathcal{E}(Q_1^{-1})$ and $\mathcal{E}(Q_2^{-1})$.

For the case $C = [1\ 1]$, we consider an initial state belonging to $L_V/\mathcal{E}(\mu I_2)$. The obtained trajectory is plotted with the control u_k and its saturation $\mathrm{sat}(u_k)$ on Figure 2, with respect to the discrete time, for an arbitrary sequence of TVP ξ_k. The saturation $\mathrm{sat}(u_k)$ is emphasized at the first sampled times.

Finally, in order to emphasize the efficient improvement of our method with respect to the literature, the conservative method provided in [18] leads to

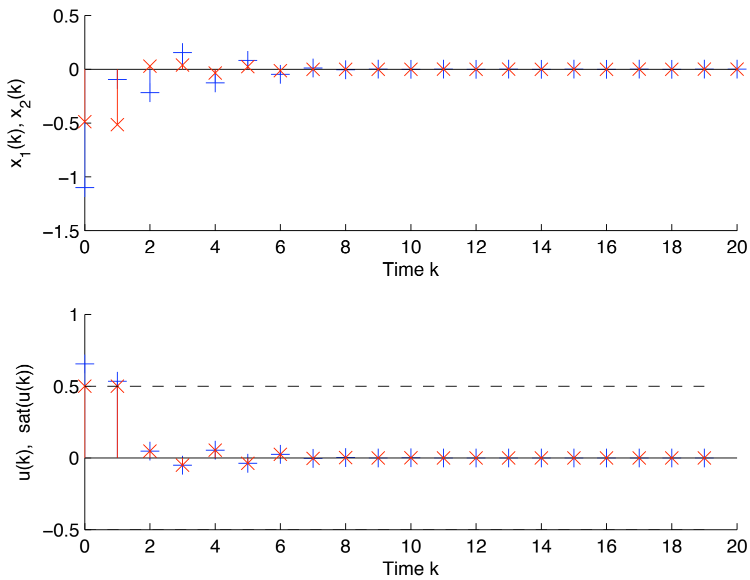

Fig. 2. A particular trajectory: a) state components (x_1 with '+'; x_2 with '×'); b) control u_k with '+' and saturated control $\mathrm{sat}(u_k)$ with '×'.

$\mu = 2.09$, that is 30% greater than μ obtained using our proposed approach for $C = I_2$.

6 Conclusion

This paper provides a gain-scheduled output control design for systems coping with nonlinear time-varying parameter dependent systems subject to saturated actuators. The nonlinearity is taken into account by a cone bounded sector. The proposed LMI conditions are based on the use of a parameter dependent Lyapunov function and a modified sector condition for representing the saturation nonlinearity. The control design problem is formulated as an optimization problem under LMI conditions and Linear Matrix Equalities. Its resolution leads to a solution less conservative than the ones available in the literature.

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