Extremal functions in some interpolation inequalities: Symmetry, symmetry breaking and estimates of the best constants

Jean Dolbeault, Maria J. Esteban

Extremal functions in some interpolation inequalities: Symmetry, symmetry breaking and estimates of the best constants

J. Dolbeault* and M.J. Esteban**

Ceremade (UMR CNRS no. 7534), Université Paris-Dauphine,
Place de Lattre de Tassigny, F-75775 Paris Cédex 16, France
**E-mail: dolbeaul@ceremade.dauphine.fr, **E-mail: esteban@ceremade.dauphine.fr*
http://www.ceremade.dauphine.fr/∼dolbeaul/, http://www.ceremade.dauphine.fr/∼esteban/

This contribution is devoted to a review of some recent results on existence, symmetry and symmetry breaking of optimal functions for Caffarelli-Kohn-Nirenberg (CKN) and weighted logarithmic Hardy (WLH) inequalities. These results have been obtained in a series of papers[1–5] in collaboration with M. del Pino, S. Filippas, M. Loss, G. Tarantello and A. Tertikas and are presented from a new viewpoint.

Keywords: Caffarelli-Kohn-Nirenberg inequality; Gagliardo-Nirenberg inequality; logarithmic Hardy inequality; logarithmic Sobolev inequality; extremal functions; radial symmetry; symmetry breaking; Emden-Fowler transformation; linearization; existence; compactness; optimal constants

1. Two families of interpolation inequalities

Let $d \in \mathbb{N}^*$, $\theta \in [0,1]$, consider the set $\mathcal{D}$ of all smooth functions which are compactly supported in $\mathbb{R}^d \setminus \{0\}$ and define $\vartheta(d,p) := d\,\frac{p-2}{2\,p}$, $a_c := \frac{d-2}{2}$, $\Lambda(a) := (a-a_c)^2$ and $p(a,b) := \frac{2\,d}{d-2+2\,(b-a)}$. We shall also set $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* := \infty$ if $d=1$ or 2. For any $a < a_c$, we consider the two families of interpolation inequalities:

(CKN) *Caffarelli-Kohn-Nirenberg inequalities*[3,4,6] – Let $b \in (a+1/2, a+1]$ and $\theta \in (1/2,1]$ if $d=1$, $b \in (a,a+1]$ if $d=2$ and $b \in [a,a+1]$ if $d \ge 3$. Assume that $p = p(a,b)$, and $\theta \in [\vartheta(d,p),1]$ if $d \ge 2$. There exists a finite positive constant $\mathsf{C}_{\rm CKN}(\theta,p,a)$ such that, for any $u \in \mathcal{D}$,

$$\||x|^{-b}\,u\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \le \mathsf{C}_{\rm CKN}(\theta,p,a)\,\||x|^{-a}\,\nabla u\|^{2\,\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\||x|^{-(a+1)}\,u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{R}^d)}\,.$$

(WLH) *Weighted logarithmic Hardy inequalities*[3,4] – Let $\gamma \ge d/4$ and $\gamma > 1/2$ if $d=2$. There exists a positive constant $\mathsf{C}_{\rm WLH}(\gamma,a)$ such that, for any $u \in \mathcal{D}$, normalized by $\||x|^{-(a+1)}\,u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$,

$$\int_{\mathbb{R}^d} \frac{|u|^2\,\log\left(|x|^{d-2-2\,a}\,|u|^2\right)}{|x|^{2\,(a+1)}}\,dx \le 2\,\gamma\,\log\Big[\mathsf{C}_{\rm WLH}(\gamma,a)\,\|\,|x|^{-a}\,\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\Big]\,.$$

(WLH) appears as a limiting case[3,4] of (CKN) with $\theta = \gamma\,(p-2)$ as $p \to 2_+$. By a standard completion argument, these inequalities can be extended to the set

$\mathcal{D}^{1,2}_a(\mathbb{R}^d) := \{u \in \mathrm{L}^1_{\rm loc}(\mathbb{R}^d) \,:\, |x|^{-a}\,\nabla u \in \mathrm{L}^2(\mathbb{R}^d)$ *and* $|x|^{-(a+1)}\,u \in \mathrm{L}^2(\mathbb{R}^d)\}$. We shall assume that all constants in the inequalities are taken with their optimal values. For brevity, we shall call *extremals* the functions which realize equality in (CKN) or in (WLH).

Let $\mathsf{C}^*_{\rm CKN}(\theta,p,a)$ and $\mathsf{C}^*_{\rm WLH}(\gamma,a)$ denote the optimal constants when admissible functions are restricted to the radial ones. *Radial extremals* are explicit and the values of the constants, $\mathsf{C}^*_{\rm CKN}(\theta,p,a)$ and $\mathsf{C}^*_{\rm WLH}(\gamma,a)$, are known.[3] Moreover, we have

$$\begin{aligned}
&\mathsf{C}_{\rm CKN}(\theta,p,a) \ge \mathsf{C}^*_{\rm CKN}(\theta,p,a) = \mathsf{C}^*_{\rm CKN}(\theta,p,a_c-1)\,\Lambda(a)^{\frac{p-2}{2p}-\theta}\;,\\
&\mathsf{C}_{\rm WLH}(\gamma,a) \ge \mathsf{C}^*_{\rm WLH}(\gamma,a) = \mathsf{C}^*_{\rm WLH}(\gamma,a_c-1)\,\Lambda(a)^{-1+\frac{1}{4\gamma}}\;.
\end{aligned}\tag{1}$$

Radial symmetry for the extremals of (CKN) and (WLH) implies that $\mathsf{C}_{\rm CKN}(\theta,p,a) = \mathsf{C}^*_{\rm CKN}(\theta,p,a)$ and $\mathsf{C}_{\rm WLH}(\gamma,a) = \mathsf{C}^*_{\rm WLH}(\gamma,a)$, while *symmetry breaking* only means that inequalities in (1) are strict.

2. Existence of extremals

Theorem 2.1. *Equality[4] in (CKN) is attained for any $p \in (2,2^*)$ and $\theta \in (\vartheta(p,d),1)$ or $\theta = \vartheta(p,d)$ and $a \in (a_\star^{\rm CKN},a_c)$, for some $a_\star^{\rm CKN} < a_c$. It is not attained if $p=2$, or $a<0$, $p=2^*$, $\theta=1$ and $d\ge3$, or $d=1$ and $\theta=\vartheta(p,1)$.*

Equality[4] in (WLH) is attained if $\gamma\ge1/4$ and $d=1$, or $\gamma>1/2$ if $d=2$, or for $d\ge3$ and either $\gamma>d/4$ or $\gamma=d/4$ and $a\in(a_\star^{\rm WLH},a_c)$, where $a_\star^{\rm WLH} := a_c-\sqrt{\Lambda_\star^{\rm WLH}}$ and $\Lambda_\star^{\rm WLH} := (d-1)\,e\,(2^{d+1}\,\pi)^{-1/(d-1)}\,\Gamma(d/2)^{2/(d-1)}$.

Let us give some hints on how to prove such a result. Consider first Gross' logarithmic Sobolev inequality in Weissler's form[7]

$$\int_{\mathbb{R}^d}|u|^2\,\log|u|^2\,dx \le \frac{d}{2}\,\log\Big(\mathsf{C}_{\rm LS}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\Big)\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)\ \text{s.t.}\ \|u\|_{\mathrm{L}^2(\mathbb{R}^d)}=1\;.$$

The function $u(x) = (2\,\pi)^{-d/4}\,\exp(-|x|^2/4)$ is an extremal for such an inequality. By taking $u_n(x) := u(x+n\,\mathsf{e})$ for some $\mathsf{e}\in\mathbb{S}^{d-1}$ and any $n\in\mathbb{N}$ as test functions for (WLH), and letting $n\to+\infty$, we find that $\mathsf{C}_{\rm LS}\le\mathsf{C}_{\rm WLH}(d/4,a)$. If equality holds, this is a mechanism of loss of compactness for minimizing sequences. On the opposite, if $\mathsf{C}_{\rm LS}<\mathsf{C}_{\rm WLH}(d/4,a)$, which is the case if $a\in(a_\star^{\rm WLH},a_c)$ where $a_\star^{\rm WLH}=a$ is given by the condition $\mathsf{C}_{\rm LS}=\mathsf{C}^*_{\rm WLH}(d/4,a)$, we can establish a compactness result which proves that equality is attained in (WLH) in the critical case $\gamma=d/4$.

A similar analysis for (CKN) shows that $\mathsf{C}_{\rm GN}(p)\le\mathsf{C}_{\rm CKN}(\theta,p,a)$ in the critical case $\theta=\vartheta(p,d)$, where $\mathsf{C}_{\rm GN}(p)$ is the optimal constant in the Gagliardo-Nirenberg-Sobolev interpolation inequalities

$$\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}\le\mathsf{C}_{\rm GN}(p)\,\|\nabla u\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)$$

and $p\in(2,2^*)$ if $d=2$ or $p\in(2,2^*]$ if $d\ge3$. However, extremals are not known explicitly in such inequalities if $d\ge2$, so we cannot get an explicit interval of existence in terms of a, even if we also know that compactness of minimizing sequences for (CKN) holds when $\mathsf{C}_{\rm GN}(p) < \mathsf{C}_{\rm CKN}(\vartheta(p,d),p,a)$. This is the case if $a > a_\star^{\rm CKN}$ where $a = a_\star^{\rm CKN}$ is defined by the condition $\mathsf{C}_{\rm GN}(p) = \mathsf{C}^*_{\rm CKN}(\vartheta(p,d),p,a)$.

It is very convenient to reformulate (CKN) and (WLH) inequalities in cylindrical variables.[8] By means of the Emden-Fowler transformation

$$s = \log|x| \in \mathbb{R}\;, \quad \omega = x/|x| \in \mathbb{S}^{d-1}\;, \quad y = (s,\omega)\;, \quad v(y) = |x|^{a_c-a}\,u(x)\;,$$

(CKN) for u is equivalent to a Gagliardo-Nirenberg-Sobolev inequality on the cylinder $\mathcal{C} := \mathbb{R}\times\mathbb{S}^{d-1}$ for v, namely

$$\|v\|^2_{\mathrm{L}^p(\mathcal{C})} \le \mathsf{C}_{\rm CKN}(\theta,p,a)\,\Big(\,\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}\Big)^\theta\;\|v\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})} \quad \forall\,v\in\mathrm{H}^1(\mathcal{C})$$

with $\Lambda = \Lambda(a)$. Similarly, with $w(y) = |x|^{a_c-a}\,u(x)$, (WLH) is equivalent to

$$\int_{\mathcal{C}} |w|^2\,\log|w|^2\,dy \le 2\,\gamma\,\log\Big[\mathsf{C}_{\rm WLH}(\gamma,a)\,\Big(\|\nabla w\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\Big)\Big]$$

for any $w\in\mathrm{H}^1(\mathcal{C})$ such that $\|w\|_{\mathrm{L}^2(\mathcal{C})} = 1$. Notice that radial symmetry for u means that v and w depend only on s.

Consider a sequence $(v_n)_n$ of functions in $\mathrm{H}^1(\mathcal{C})$, which minimizes the functional

$$\mathcal{E}^p_{\theta,\Lambda}[v] := \Big(\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}\Big)^\theta\;\|v\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})}$$

under the constraint $\|v_n\|_{\mathrm{L}^p(\mathcal{C})} = 1$ for any $n\in\mathbb{N}$. As quickly explained below, if bounded, such a sequence is relatively compact and converges up to translations and the extraction of a subsequence towards a minimizer of $\mathcal{E}^p_{\theta,\Lambda}$.

Assume that $d\ge 3$, let $t := \|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})}/\|v\|^2_{\mathrm{L}^2(\mathcal{C})}$ and $\Lambda = \Lambda(a)$. If v is a minimizer of $\mathcal{E}^p_{\theta,\Lambda}[v]$ such that $\|v\|_{\mathrm{L}^p(\mathcal{C})} = 1$, then we have

$$(t+\Lambda)^\theta = \mathcal{E}^p_{\theta,\Lambda}[v]\,\frac{\|v\|^2_{\mathrm{L}^p(\mathcal{C})}}{\|v\|^2_{\mathrm{L}^2(\mathcal{C})}} = \frac{\|v\|^2_{\mathrm{L}^p(\mathcal{C})}}{\mathsf{C}_{\rm CKN}(\theta,p,a)\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}} \le \frac{\mathsf{S}_d^{\vartheta(d,p)}}{\mathsf{C}_{\rm CKN}(\theta,p,a)}\;\big(t + a_c^2\big)^{\vartheta(d,p)}$$

where $\mathsf{S}_d = \mathsf{C}_{\rm CKN}(1,2^*,0)$ is the optimal Sobolev constant, while we know from (1) that $\lim_{a\to a_c}\mathsf{C}_{\rm CKN}(\theta,p,a) = \infty$ if $d\ge 2$. This provides a bound on t if $\theta > \vartheta(p,d)$. An estimate can be obtained also for v_n, for n large enough, and standard tools of the concentration-compactness method allow to conclude that $(v_n)_n$ converges towards an extremal. A similar approach holds for (CKN) if $d=2$, or for (WLH).

The above variational approach also provides an existence result of extremals for (CKN) in the critical case $\theta = \vartheta(p,d)$, if $a\in(a_1,a_c)$ where $a_1 := a_c - \sqrt{\Lambda_1}$ and $\Lambda_1 = \min\{(\mathsf{C}^*_{\rm CKN}(\theta,p,a_c-1)^{1/\theta}/\,\mathsf{S}_d)^{d/(d-1)}, (a_c^2\,\mathsf{C}^*_{\rm CKN}(\theta,p,a_c-1)^{1/\theta}/\,\mathsf{S}_d)^d$.

If symmetry is known, then there are (radially symmetric) extremals.[3] Anticipating on the results of the next section, we can state the following result which arises as a consequence of Schwarz' symmetrization method (see Theorem 3.2, below).

Proposition 2.1. *Let $d\ge 3$. Then (CKN) with $\theta = \vartheta(p,d)$ admits a radial extremal if[5] $a\in[a_0,a_c)$ where $a_0 := a_c - \sqrt{\Lambda_0}$ and $\Lambda = \Lambda_0$ is defined by the condition $\Lambda^{(d-1)/d} = \vartheta(p,d)\,\mathsf{C}^*_{\rm CKN}(\theta,p,a_c-1)^{1/\vartheta(d,p)}/\,\mathsf{S}_d$.*

A similar estimate also holds if $\theta > \vartheta(d,p)$, with less explicit computations.[5]

3. Symmetry and symmetry breaking

Define

$$\underline{a}(\theta,p) := a_c - \tfrac{2\sqrt{d-1}}{p+2}\sqrt{\tfrac{2\,p\,\theta}{p-2}-1}\,,\quad \tilde{a}(\gamma) := a_c - \tfrac{1}{2}\sqrt{(d-1)(4\,\gamma-1)}\,,$$
$$\Lambda_{\rm SB}(\gamma) := \tfrac{1}{8}\,(4\,\gamma-1)\,e\,\big(\tfrac{\pi^{4\,\gamma-d-1}}{16}\big)^{\frac{1}{4\,\gamma-1}}\,\big(\tfrac{d}{\gamma}\big)^{\frac{4\,\gamma}{4\,\gamma-1}}\,\Gamma\,\big(\tfrac{d}{2}\big)^{\frac{2}{4\,\gamma-1}}\;.$$

Theorem 3.1. *Let $d \geq 2$ and $p \in (2, 2^*)$. Symmetry breaking holds in (CKN) if either*[3,5] *$a < \underline{a}(\theta,p)$ and $\theta \in [\vartheta(p,d),1]$, or*[5] *$a < a_\star^{\rm CKN}$ and $\theta = \vartheta(p,d)$.*

Assume that $\gamma > 1/2$ if $d = 2$ and $\gamma \geq d/4$ if $d \geq 3$. Symmetry breaking holds in (WLH) if[3,5] *$a < \max\{\tilde{a}(\gamma), a_c - \sqrt{\Lambda_{\rm SB}(\gamma)}\}$.*

When $\gamma = d/4$, $d \geq 3$, we observe that $\Lambda_\star^{\rm WLH} = \Lambda_{\rm SB}(d/4) < \Lambda(\tilde{a}(d/4))$ with the notations of Theorem 2.1 and there is symmetry breaking if $a \in (-\infty, a_\star^{\rm WLH})$, in the sense that $\mathsf{C}_{\rm WLH}(d/4,a) > \mathsf{C}^*_{\rm WLH}(d/4,a)$, although we do not know if extremals for (WLH) exist when $\gamma = d/4$.

Results of symmetry breaking for (CKN) with $a < \underline{a}(\theta,p)$ have been established first[1,8,9] when $\theta = 1$ and later[3] extended to $\theta < 1$. The main idea in case of (CKN) is consider the quadratic form associated to the second variation of $\mathcal{E}^p_{\theta,\Lambda}$ around a minimizer among functions depending on s only and observe that the linear operator $\mathcal{L}^p_{\theta,\Lambda}$ associated to the quadratic form has a negative eigenvalue if $a < \underline{a}$. Results[3] for (WLH), $a < \tilde{a}(\gamma)$, are based on the same method.

For any $a < a_\star^{\rm CKN}$, we have $\mathsf{C}^*_{\rm CKN}(\vartheta(p,d),p,a) < \mathsf{C}_{\rm GN}(p) \leq \mathsf{C}_{\rm CKN}(\vartheta(p,d),p,a)$, which proves symmetry breaking. Using well-chosen test functions, it has been proved[5] that $\underline{a}(\vartheta(p,d),p) < a_\star^{\rm CKN}$ for $p-2>0$, small enough, thus also proving symmetry breaking for $a - \underline{a}(\vartheta(p,d),p) > 0$, small, and $\theta - \vartheta(p,d) > 0$, small.

Theorem 3.2. *For all $d \geq 2$, there exists*[2,5] *a continuous function a^* defined on the set $\{(\theta,p) \in (0,1] \times (2,2^*)\,:\,\theta > \vartheta(p,d)\}$ such that $\lim_{p\to 2_+} a^*(\theta,p) = -\infty$ with the property that (CKN) has only radially symmetric extremals if $(a,p) \in (a^*(\theta,p),a_c) \times (2,2^*)$, and none of the extremals is radially symmetric if $(a,p) \in (-\infty, a^*(\theta,p)) \times (2,2^*)$.*

Similarly, for all $d \geq 2$, there exists[5] *a continuous function $a^{**} : (d/4,\infty) \to (-\infty,a_c)$ such that, for any $\gamma > d/4$ and $a \in [a^{**}(\gamma),a_c)$, there is a radially symmetric extremal for (WLH), while for $a < a^{**}(\gamma)$ no extremal is radially symmetric.*

Schwarz' symmetrization allows to characterize[5] a subdomain of $(0,a_c) \times (0,1) \ni (a,\theta)$ in which symmetry holds for extremals of (CKN), when $d \geq 3$. If $\theta = \vartheta(p,d)$ and $p > 2$, there are radially symmetric extremals[5] if $a \in [a_0,a_c)$ where a_0 is given in Propositions 2.1.

Symmetry also holds if $a - a_c$ is small enough, for (CKN) as well as for (WLH), or when $p \to 2_+$ in (CKN), for any $d \geq 2$, as a consequence of the existence of the spectral gap of $\mathcal{L}^p_{\theta,\Lambda}$ when $a > \underline{a}(\theta,p)$.

For given θ and p, there is[2,5] a unique $a^* \in (-\infty,a_c)$ for which there is symmetry breaking in $(-\infty,a^*)$ and for which all extremals are radially symmetric when $a \in$

(a^*, a_c). This follows from the observation that, if $v_\sigma(s,\omega) := v(\sigma\, s,\omega)$ for $\sigma > 0$, then $(\mathcal{E}^p_{\theta,\sigma^2\Lambda}[v_\sigma])^{1/\theta} - \sigma^{(2\,\theta-1+2/p)/\theta^2}\,(\mathcal{E}^p_{\theta,\Lambda}[v])^{1/\theta}$ is equal to 0 if v depends only on s, while it has the sign of $\sigma - 1$ otherwise.

From Theorem 3.1, we can infer that radial and non-radial extremals for (CKN) with $\theta > \vartheta(p,d)$ coexist on the threshold, in some cases.

Numerical results illustrating our results on existence and on symmetry / symmetry breaking have been collected in Fig. 1 below in the critical case for (CKN).

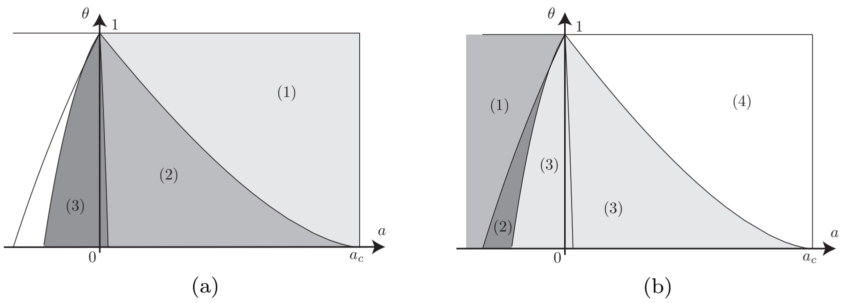

Fig. 1. Critical case for (CKN): $\theta = \vartheta(p,d)$. Here we assume that $d = 5$.
(a) The zones in which existence is known are (1) in which $a \geq a_0$, because extremals are achieved among radial functions, (2) using the *a priori* estimates: $a > a_1$, and (3) by comparison with the Gagliardo-Nirenberg inequality: $a > a_\star^{\rm CKN}$.
(b) The zone of symmetry breaking contains (1) by linearization around radial extremals: $a < \underline{a}(\theta,p)$, and (2) by comparison with the Gagliardo-Nirenberg inequality: $a < a_\star^{\rm CKN}$; in (3) it is not known whether symmetry holds or if there is symmetry breaking, while in (4) symmetry holds by Schwarz' symmetrization: $a_0 \leq a < a_c$.
Numerically, we observe that $\underline{a}$ and $a_\star^{\rm CKN}$ intersect for some $\theta \approx 0.85$.

Acknowledgements. The authors have been supported by the ANR projects CBDif-Fr and EVOL.

References

1. J. Dolbeault, M. J. Esteban and G. Tarantello, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **7**, 313 (2008).
2. J. Dolbeault, M. J. Esteban, M. Loss and G. Tarantello, *Adv. Nonlinear Stud.* **9**, 713 (2009).
3. M. del Pino, J. Dolbeault, S. Filippas and A. Tertikas, *Journal of Functional Analysis* **259**, 2045 (2010).
4. J. Dolbeault and M. J. Esteban, Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities, Preprint, (2010).
5. J. Dolbeault, M. J. Esteban, G. Tarantello and A. Tertikas, Radial symmetry and symmetry breaking for some interpolation inequalities, Preprint, (2010).
6. L. Caffarelli, R. Kohn and L. Nirenberg, *Compositio Math.* **53**, 259 (1984).
7. F. B. Weissler, *Trans. Amer. Math. Soc.* **237**, 255 (1978).
8. F. Catrina and Z.-Q. Wang, *Comm. Pure Appl. Math.* **54**, 229 (2001).
9. V. Felli and M. Schneider, *J. Differential Equations* **191**, 121 (2003).