Matrix Representation of Bounded Linear Operators By Bessel Sequences, Frames and Riesz Sequence

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Abstract:

In this work we will investigate how to find a matrix representation of operators on a Hilbert space $\mathcal{H}$ with Bessel sequences, frames and Riesz bases as an extension of the known method of matrix representation by ONBs. We will give basic definitions of the functions connecting infinite matrices defining bounded operators on l^2 and operators on $\mathcal{H}$. We will show some structural results and give some examples. Furthermore in the case of Riesz bases we prove that those functions are isomorphisms. We are going to apply this idea to the connection of Hilbert-Schmidt operators and Frobenius matrices. Finally we will use this concept to show that every bounded operator is a generalized frame multiplier.

1. Introduction

From practical experience it became apparent that the concept of an orthonormal basis is not always useful. This led to the concept of frames, which was introduced by Duffin and Schaefer [12] and today it is one of the most important foundations of sampling theory [1].
The standard matrix description [8] of operators O using an ONB (e_k) is by constructing an matrix M with the entries $M_{j,k} = \langle Oe_k, e_j \rangle$. In [6] a concept was presented, where an operator R is described by the matrix $\left(\left\langle R\phi_j, \tilde{\phi}_i \right\rangle \right)_{i,j}$ with (ϕ_i) being a frame and $(\tilde{\phi}_i)$ its canonical dual. Such a kind of representation is used for the description of operators in [15] using Gabor frames and [19] using linear independent Gabor systems. In this work we are presenting the main ideas for Bessel sequences, frames and Riesz sequences and also look at the dual function which assigns an operator to a matrix. For proofs and details we refer to [3].

2. Notation and Preliminaries

2.1 Hilbert spaces and Operators

Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all linear and bounded operators from the Hilbert space $\mathcal{H}_1$ to $\mathcal{H}_2$. Furthermore we will denote the range of an operator A by $ran(O)$ and its kernel by $ker(A)$.
Let X, Y, Z be sets, $f : X \to Z$, $g : Y \to Z$ be arbitrary functions. The *Kronecker product* $\otimes_o : X \times Y \to Z$ is defined by $(f \otimes_o g)(x, y) = f(x) \cdot g(y)$. Let $f \in \mathcal{H}_1$, $g \in \mathcal{H}_2$ then define the *inner tensor product* as an operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ by $(f \otimes_i \overline{g})(h) = \langle h, g \rangle f$ for $h \in \mathcal{H}_2$.

2.1.1 Hilbert Schmidt Operators

A bounded operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is called a *Hilbert-Schmidt* ($\mathcal{HS}$) [18] operator if there exists an ONB $(e_n) \subseteq \mathcal{H}_1$ such that $\|T\|_{\mathcal{HS}} := \sqrt{\sum\limits_{n=1}^{\infty} \|Te_n\|_{\mathcal{H}_2}^2} < \infty$. Let $\mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2)$ denote the space of Hilbert Schmidt operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

2.2 Frames

A sequence $\Psi = (\psi_k | k \in K)$ is called a *frame* [5, 7] for the Hilbert space $\mathcal{H}$, if constants $A, B > 0$ exist, such that

$$A \cdot \|f\|_{\mathcal{H}}^2 \leq \sum_k |\langle f, \psi_k \rangle|^2 \leq B \cdot \|f\|_{\mathcal{H}}^2 \ \forall\, f \in \mathcal{H} \quad (1)$$

A sequence $\Psi = (\psi_k)$ is called a *Bessel sequence* with Bessel bound B if it fulfills the right inequality above. The index set will be omitted in the following, if no distinction is necessary.
A complete sequence (ψ_k) in $\mathcal{H}$ is called a *Riesz basis* if there exist constants A, $B > 0$ such that the inequalities

$$A \|c\|_2^2 \leq \left\| \sum_{k \in K} c_k \psi_k \right\|_{\mathcal{H}}^2 \leq B \|c\|_2^2$$

hold for all finite sequences (c_k).

3. Representing Operators with Frames

Let (ψ_k) be a frame in $\mathcal{H}_1$. An existing operator $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is uniquely determined by its images of the frame elements. For $f = \sum\limits_k c_k \psi_k$

$$U(f) = U(\sum_k c_k \psi_k) = \sum_k c_k U(\psi_k).$$

On the other hand, contrary to the case for ONBs, we cannot just choose a Bessel sequence (η_k) and define an operator just by choosing $V(\psi_k) := \eta_k$ and setting $V(\sum\limits_k c_k \psi_k) = \sum\limits_k c_k \eta_k$. This is in general not well-defined. Only if

$$\sum_k c_k \psi_k = \sum_k d_k \psi_k \Longrightarrow \sum_k c_k \eta_k = \sum_k d_k \eta_k$$

this definition is non-ambiguous, i.e. if $ker\left(D_{\psi_k}\right) \subseteq ker\left(D_{\eta_k}\right)$. This condition is certainly fulfilled, if D_{ψ_k} is injective, i.e. for Riesz bases.
This problem can be avoided by using the following definition

$$V(f) := \sum_k \left\langle f, \tilde{\psi}_k \right\rangle \eta_k. \tag{2}$$

As (η_k) forms a Bessel sequence, the right hand side of Eq. (2) is well-defined. It is clearly linear, and it is bounded. The Bessel condition is necessary in the case of ONBs to get a bounded operator, too [8]. But contrary to the ONB case, here, in general, $V(\psi_k) \neq \eta_k$. So this option does not seem very useful. Instead of changing the sequence with which the coefficients are resynthezised, an operator can also be described by changing the coefficients, as presented in the following sections.

4. Matrix Representation

4.1 Motivation: Solving Operator Equalities

Given an operator equality $O \cdot f = g$ it is natural to discretize it to find a solution. Let $\Phi = (\phi_k)$ be a frame. Let us suppose that for a given g with coefficients $d = (d_k) = (\langle g, \phi_k \rangle)$ and a matrix representation M of O there is an algorithm to find the least square solution of

$$M \cdot c = d \tag{3}$$

for example using the pseudoinverse [7]. Still, if using frames, we can not expect to find a true solution for the operator equality just by applying $D_{\tilde{\Phi}}$ on c as in general c is not in $ran(C_\Phi)$ even if d is. But we see the following:

$$Of = g \Longleftrightarrow \sum_k \langle f, \phi_k \rangle O\tilde{\phi}_k = g \Longleftrightarrow$$

$$\Longleftrightarrow \sum_k \langle f, \phi_k \rangle \left\langle O\tilde{\phi}_k, \phi_k \right\rangle = \langle g, \phi_k \rangle$$

$$\Longleftrightarrow \mathcal{M}^{(\Phi,\tilde{\Phi})}(O) \cdot C_\Phi f = C_\Phi g.$$

It can be easily seen that this is equivalent to projecting c on $ran(C)$, solving $MC_\Phi D_{\tilde{\Phi}} c = d$, which is a common idea found in many algorithms, for example for a recent one see [20].

This gives us an algorithm for finding an approximative solution to the inverse operator problem $Of = g$.

1. Set $M = \mathcal{M}^{(\Phi,\tilde{\Phi})}(O)$.

2. Find a good finite dimensional approximation M_N of M by using the finite section method [14, 16] and

3. then apply an algorithm like e.g. the QR factorization [21] to find a solution for the operator equation.

4. and synthezise with the dual frame $\tilde{\Phi}$.

4.2 Bessel sequences

Theorem 4.2.1 *Let $\Psi = (\psi_k)$ be a Bessel sequence in $\mathcal{H}_1$ with bound B, $\Phi = (\phi_k)$ in $\mathcal{H}_2$ with B'.*

1. *Let $O : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded, linear operator. Then the infinite matrix*

$$\left(\mathcal{M}^{(\Phi,\Psi)}(O)\right)_{m,n} = \langle O\psi_n, \phi_m \rangle$$

defines a bounded operator from l^2 to l^2 with $\|\mathcal{M}\|_{l^2 \to l^2} \leq \sqrt{B \cdot B'} \cdot \|O\|_{\mathcal{H}_1 \to \mathcal{H}_2}$. As an operator $l^2 \to l^2$

$$\mathcal{M}^{(\Phi,\Psi)}(O) = C_\Phi \circ O \circ D_\Psi$$

This means the function $\mathcal{M}^{(\Phi,\Psi)} : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{B}(l^2, l^2)$ is a well-defined bounded operator.

2. *On the other hand let M be an infinite matrix defining a bounded operator from l^2 to l^2, $(Mc)_i = \sum_k M_{i,k} c_k$. Then the operator $\mathcal{O}^{(\Phi,\Psi)}$ defined by*

$$\left(\mathcal{O}^{(\Phi,\Psi)}(M)\right) h = \sum_k \left(\sum_j M_{k,j} \langle h, \psi_j \rangle \right) \phi_k,$$

$$\left\| \mathcal{O}^{(\Phi,\Psi)}(M) \right\|_{\mathcal{H}_1 \to \mathcal{H}_2} \leq \sqrt{B \cdot B'} \, \|M\|_{l^2 \to l^2}.$$

$$\mathcal{O}^{(\Phi,\Psi)}(M) = D_\Phi \circ M \circ C_\Psi = \sum_k \sum_j M_{k,j} \cdot \phi_k \otimes_i \overline{\psi}_j$$

This means the function $\mathcal{O}^{(\Phi,\Psi)} : \mathcal{B}(l^2, l^2) \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a well-defined bounded operator.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{H}_1 \arrow[r, "\mathcal{O}^{(\Phi,\Psi)}(M)", "O"'] \arrow[d, shift left, "C_\Psi"] & \mathcal{H}_2 \arrow[d, shift left, "C_\Phi"] \\
l^2 \arrow[r, "\mathcal{M}^{(\Psi,\Phi)}(O)", "\mathrm{M}"'] \arrow[u, shift left, "D_\Psi"] & l^2 \arrow[u, shift left, "D_\Phi"]
\end{tikzcd}$$

Figure 1: The operator induced by a matrix M and the matrix induced by an operator O.

If we do not want to stress the dependency on the frames and there is no change of confusion, the notation $\mathcal{M}(O)$ and $\mathcal{O}(M)$ will be used.
In the above theorem we have avoided the issue, when an infinite matrix defines a bounded operator from l^2 to l^2. A criterion has been proved in [9]:

Theorem 4.2.2 *An infinite matrix M defines a bounded operator from l^2 to l^2, if and only if $(M^*M)^n$ is defined for all $n = 1, 2, 3, \dots$ and $\sup_n \sup_l \left| \left[(M^*M)^n_{l,l} \right]^{1/n} \right| < \infty$.*

For similar conditions see [17].

4.3 Frames

Proposition 4.3.1 *Let $\Psi = (\psi_k)$ be a frame in $\mathcal{H}_1$ with bounds A, B, $\Phi = (\phi_k)$ in $\mathcal{H}_2$ with A', B'. Then*

1. $\left(\mathcal{O}^{(\Phi,\Psi)} \circ M^{(\tilde{\Phi},\tilde{\Psi})}\right) = Id = \left(\mathcal{O}^{(\tilde{\Phi},\tilde{\Psi})} \circ M^{(\Phi,\Psi)}\right)$. *And therefore for all $O \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$:*
$$O = \sum_{k,j} \left\langle O\tilde{\psi}_j, \tilde{\phi}_k \right\rangle \phi_k \otimes_i \overline{\psi}_j$$
2. *$\mathcal{M}^{(\Phi,\Psi)}$ is injective and $\mathcal{O}^{(\Phi,\Psi)}$ is surjective.*
3. *Let $\mathcal{H}_1 = \mathcal{H}_2$, then $\mathcal{O}^{(\Psi,\tilde{\Psi})}(Id_{l^2}) = Id_{\mathcal{H}_1}$*
4. *Let $\Xi = (\xi_k)$ be any frame in $\mathcal{H}_3$, and $O : \mathcal{H}_3 \to \mathcal{H}_2$ and $P : \mathcal{H}_1 \to \mathcal{H}_3$. Then*
$$\mathcal{M}^{(\Phi,\Psi)}(O \circ P) = \left(\mathcal{M}^{(\Phi,\Xi)}(O) \cdot \mathcal{M}^{(\tilde{\Xi},\Psi)}(P)\right)$$

As a direct consequence we get the following corollary:

Corollary 4.3.2 *For the frame $\Phi = (\phi_k)$ the function $\mathcal{M}^{(\Phi,\tilde{\Phi})}$ is a Banach-algebra monomorphism between the algebra of bounded operators $(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1), \circ)$ and the infinite matrices of $\left(\mathcal{B}(l^2, l^2), \cdot\right)$.*

Lemma 4.3.3 *Let $O : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear and bounded operator, let $\Psi = (\psi_k)$ and $\Phi = (\phi_k)$ be frames in $\mathcal{H}_1$ resp. $\mathcal{H}_2$. Then $\mathcal{M}^{(\Phi,\tilde{\Psi})}(O)$ maps $ran\,(C_\Psi)$ into $ran\,(C_\Phi)$ with*
$$\left(\langle f, \psi_k \rangle\right)_k \mapsto \left(\langle Of, \phi_k \rangle\right)_k .$$
If O is surjective, then $\mathcal{M}^{(\Phi,\tilde{\Psi})}(O)$ maps $ran\,(C_\Psi)$ onto $ran\,(C_\Phi)$. If O is injective, $\mathcal{M}^{(\Phi,\tilde{\Psi})}(O)$ is also injective.

The other function $\mathcal{O}$ is in general not so "well-behaved". It is, if the dual frames are biorthogonal. In this case these functions are isomorphisms, see the next section.

4.4 Riesz sequences

Theorem 4.4.1 *Let $\Phi = (\phi_k)$ be a Riesz basis for $\mathcal{H}_1$, $\Psi = (\psi_k)$ one for $\mathcal{H}_2$. The functions $\mathcal{M}^{(\Phi,\Psi)}$ and $\mathcal{O}^{(\tilde{\Phi},\tilde{\Psi})}$ between $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and the infinite matrices in $\mathcal{B}(l^2, l^2)$ are bijective. $\mathcal{M}^{(\Phi,\Psi)}$ and $\mathcal{O}^{(\tilde{\Phi},\tilde{\Psi})}$ are inverse to each other. For $\mathcal{H}_1 = \mathcal{H}_2$ the identity is mapped on the identity by $\mathcal{M}^{(\Phi,\Psi)}$ and $\mathcal{O}^{(\tilde{\Phi},\tilde{\Psi})}$. If furthermore $\Psi = \Phi$ then $\mathcal{M}^{(\Phi,\tilde{\Phi})}$ and $\mathcal{O}^{(\Phi,\tilde{\Phi})}$ are Banach algebra isomorphisms, respecting the identities id_{l^2} and $id_{\mathcal{H}}$.*

5. Matrix Representation of $\mathcal{HS}$ Operators

We now have the adequate tools to state that $\mathcal{HS}$ operators correspond exactly to the Frobenius matrices, as expected. Let A be an m by n matrix, then $\|A\|_{fro} = \sqrt{\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} |a_{i,j}|^2}$ is the *Frobenius norm*. Let us denote the set of all matrices with finite Frobenius norm by $l^{(2,2)}$, the set of *Frobenius matrices*.

Proposition 5.0.2 *Let $\Psi = (\psi_k)$ be a Bessel sequence in $\mathcal{H}_1$ with bound B, $\Phi = (\phi_k)$ in $\mathcal{H}_2$ with B'. Let M be a matrix in $l^{(2,2)}$. Then $\mathcal{O}^{(\Phi,\Psi)}(M) \in \mathcal{HS}(\mathcal{H}_1, \mathcal{H}_2)$, the Hilbert Schmidt class of operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, with $\|\mathcal{O}(M)\|_{\mathcal{HS}} \le \sqrt{BB'}\,\|M\|_{fro}$.*
Let $O \in \mathcal{HS}$, then $\mathcal{M}^{(\Phi,\Psi)}(O) \in l^{(2,2)}$ with $\|\mathcal{M}(O)\|_{fro} \le \sqrt{BB'}\,\|O\|_{\mathcal{HS}}$.

5.1 Matrices and the Kernel Theorems

For $L^2(\mathbb{R}^d)$ the $\mathcal{HS}$ operators are exactly those integral operators with kernels in $L^2\left(\mathbb{R}^{2d}\right)$ [18]. This means that there exists a $\kappa_O \in L^2(\mathbb{R}^{2d})$ such an operator can be described as
$$(Of)(x) = \int \kappa_O(x,y) f(y) dy$$
Or in weak formulation
$$\langle Of, g \rangle = \int\int \kappa_O(x,y) f(y) \overline{g}(x) dy dx = \langle \kappa_O, f \otimes_o \overline{g} \rangle . \tag{4}$$
From 4.2.1 we know that
$$O = \sum_{j,k} \left\langle O\tilde{\psi}_j, \tilde{\phi}_k \right\rangle \phi_k \otimes_i \overline{\psi}_j$$
and so

Corollary 5.1.1 *Let $O \in \mathcal{HS}\left(L^2\left(\mathbb{R}^d\right)\right)$. Let $\Psi = (\psi_j)$ and $\Phi = (\phi_k)$ be frames in $L^2\left(\mathbb{R}^d\right)$. Then the kernel of O is given as:*
$$\kappa_O = \sum_{j,k} \mathcal{M}^{(\tilde{\Psi},\tilde{\Phi})}(O)_{k,j} \cdot \phi_k \otimes_o \overline{\psi}_j$$
This directly leads to the next concept.

6. Generalized Bessel Multipliers

Let m be a sequence and $\mathfrak{diag}(m)$ the matrix that has this sequence as diagonal. Then define
$$\mathbf{M}_{m,\Phi,\Psi} := \mathcal{O}^{(\Phi,\Psi)}\left(\mathfrak{diag}(m)\right) = \sum_k m_k \cdot \phi_k \otimes \psi_k$$
This means we have arrived quite naturally at the definition of frame multipliers as introduced in [2].
It is a very natural idea to extend this definition to include more side-diagonals:

Definition 6.0.2 *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert-spaces, let $(\psi_k)_{k\in L} \subseteq \mathcal{H}_1$ and $(\phi_k)_{k\in K} \subseteq \mathcal{H}_2$ be Bessel sequences. Let M be a $(K \times L)$-matrix that defines a bounded operator from l^2 to l^2. Define the operator $\mathbf{M}_{M,(\phi_k),(\psi_k)} : \mathcal{H}_1 \to \mathcal{H}_2$, the* generalized Bessel multiplier *for the Bessel sequences (ψ_k) and (ϕ_k), as the operator*

$$\mathbf{M}_{m,(\phi_k),(\psi_k)}(f) = \sum_l \sum_k M_{l,k} \langle f, \psi_k \rangle \phi_l.$$

The sequence m is called the symbol *of $\mathbf{M}$. If the sequence is a frame, we call the operator a 'generalized frame multiplier'.*

For Gabor frames, this is a particular case of the 'generalized Gabor multipliers' as found in [10] or [11] in this volume.
Using the results above we can write

Proposition 6.0.3 *For two frames $(\psi_k) \subseteq \mathcal{H}_1$ and $(\phi_k\} \subseteq \mathcal{H}_2$ every operator $O : \mathcal{H}_1 \to \mathcal{H}_2$ can be written as a generalized frame multiplier with the symbol $M_{l,k} = \left\langle O\tilde{\psi}_k, \tilde{\phi}_l \right\rangle$.*

Further results as the following are easy to prove:

Theorem 6.0.4 *Let $\mathbf{M} = \mathbf{M}_{m,\phi_k,\psi_k}$ be a Bessel multiplier for the Bessel sequences $(\psi_k) \subseteq \mathcal{H}_1$ and $(\phi_k\} \subseteq \mathcal{H}_2$ with the bounds B and B'. Then*

1. *If $M, M^* \in l^{1,\infty}$ with $\|M\|_{1,\infty} = K_1$ and $\|M^*\|_{1,\infty} = K_2$ then $\mathbf{M}$ is a well defined bounded operator with $\|\mathbf{M}\|_{Op} \le \sqrt{B'BK_1K_2}$.*
2. *If $\sup\limits_n \left\|M^{(n)}\right\|_{Op} = K < \infty$ then $\mathbf{M}$ is a well defined bounded operator with $\|\mathbf{M}\|_{Op} \le \sqrt{B'B}K$.*
3. *If $(M^*M)^n$ is defined for $n = 1, 2, \dots$ and $\sup\limits_n \sup\limits_i \left[\langle M^*M)^n_{i,i}\right]^{1/n} = K < \infty$ then $\|\mathbf{M}\|_{Op} \le \sqrt{B'B}K$.*
4. *If $\phi_k = \psi_k$ and $M \in \mathcal{B}(l^2)$ is a positive matrix, $\mathbf{M}$ is positive.*
5. *Let $M \in \mathcal{B}(l^2)$, then $\left(\mathbf{M}_{M,(\phi_k),(\psi_k)}\right)^* = \mathbf{M}_{M^*,(\psi_k),(\phi_k)}$. Therefore if M is self-adjoint and $\phi_k = \psi_k$, $\mathbf{M}$ is self-adjoint.*
6. *Let $M \in \mathcal{B}(l^2)$ be a matrix such that $\lim\limits_n \left\|M^{(n)} - M\right\|_{Op} = 0$, then $\mathbf{M}$ is compact.*
7. *If $M \in l^{2,2}$, $\mathbf{M}$ is a Hilbert Schmidt operator with $\|M\|_{\mathcal{HS}} \le \sqrt{B'}\sqrt{B}\ \|M\|_{2,2}$.*

Here for an operator A we denote $A^{(n)} = P_nAP_n$, where $P_n(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots, x_{n-1}, 0, 0, \dots)$, see [14] (finite sections).

7. Perspectives

In this work we have investigated the basic idea of matrix representations using frames. An interesting question, as discussed in Section 4.1, is how to find a good finite approximation matrix. For first ideas in the Gabor case see [13, 10, 11, 22, 4].

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