SELF-SIMILARITY FOR BALLISTIC AGGREGATION EQUATION

Miguel Escobedo, Stéphane Mischler

SELF-SIMILARITY FOR BALLISTIC AGGREGATION EQUATION

November 1, 2009

Miguel Escobedo[1], Stéphane Mischler[2]

Abstract

We consider ballistic aggregation equation for gases in which each particle is identified either by its mass and impulsion or by its sole impulsion. For the constant aggregation rate we prove existence of self-similar solutions as well as convergence to the self-similarity for generic solutions. For some classes of mass and/or impulsion dependent rates we are also able to estimate the large time decay of some moments of generic solutions or to build some new classes of self-similar solutions.

1 Introduction

In the present work, we are concerned with ballistic aggregation Smoluchowski like models for which we establish quantitative information on the qualitative behavior of solutions. By *ballistic aggregation*, also (improperly) called *kinetic coalescence* in previous works [2, 7], we mean that we consider a system of particles identified by their mass and impulsion which undergo an aggregation mechanism. That differs from the simplest aggregation mechanism introduced by Smoluchowski [15] in which model the particles are identified by their sole mass.

Let us be more precise. We denote by $P = P_y$ with $y = (m, p)$ a particle of mass $m > 0$ and impulsion $p \in \mathbb{R}^d$. The space of particles states is then $Y = \mathbb{R}_+ \times \mathbb{R}^d$ and the velocity of the particle P_y is $v = p/m$. We assume that at a microscopic level (the level of particles) the rate of collision of two particles $P = P_y$ and $P' = P_{y'}$ is a given nonnegative function $a = a(y, y')$ and when these two particles collide they join to form one aggregated particle $P'' = P_{y''}$ in such a way that the mechanism conserves total mass and total impulsion. In other words, the microscopic mechanism reads

$$P_y + P_{y'} \overset{a(y,y')}{\longrightarrow} P_{y''},$$

[1]Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, E–48080 Bilbao, Spain. E-mail : `mtpesmam@lg.ehu.es`

[2]Ceremade - UMR 7534, Université Paris - Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France. E-mail : `mischler@ceremade.dauphine.fr`

with $y'' = (m'', p'')$ given by

$$m'' = m + m', \qquad p'' = p + p'.$$

It is worth mentioning that the above reaction dissipates kinetic energy since that, denoting by $\mathcal{E}^\sharp = m^\sharp\,|v^\sharp|^2/2$ the kinetic energy of particle $P^\sharp$, we have

$$\begin{aligned}\mathcal{E}^{**} - \mathcal{E} - \mathcal{E}^* &= \frac{1}{2}\frac{|p+p^*|^2}{m+m_*} - \frac{1}{2}\frac{|p|^2}{m} - \frac{1}{2}\frac{|p^*|^2}{m_*} \\ &= -\frac{1}{2}\frac{m\,m_*}{m+m_*}\,|v-v_*|^2.\end{aligned}$$

At the mesoscopic (or statistical or mean field) level, the system is described at time $t \ge 0$ by the density function $f(t,y) \ge 0$ of particles with state $y \in Y$. For a given initial distribution f_{in}, the evolution of the density f is described by the Smoluchoswki/Boltzmann like equation:

$$\begin{aligned}\partial_t f &= Q(f) \quad \text{in} \quad (0,+\infty)\times Y, && (1.1)\\ f(0) &= f^{in} \quad \text{in} \quad Y. && (1.2)\end{aligned}$$

The collision operator $Q(f)$ is given by $Q(f) = Q_1(f) - Q_2(f)$, where

$$\begin{aligned}Q_1(f)(y) &= \frac{1}{2}\int_{\mathbb{R}^d}\int_0^m a(y', y-y')\; f(y')\; f(y-y')\; dm'dp', && (1.3)\\ Q_2(f)(y) &= \int_{\mathbb{R}^d}\int_0^\infty a(y,y')\; f(y)\; f(y')\; dm'dp'. && (1.4)\end{aligned}$$

The two following examples of functions a have been considered in relation with models in astrophysics [17, 8]:

$$a(y,y') = a_{HS}(y,y') := (m^{1/3} + m'^{1/3})^2\,|v - v'|, \qquad (1.5)$$

$$a(y,y') = a_{NP}(y,y') := \frac{m+m'}{mm'}\,\frac{1}{|v-v'|^2}. \qquad (1.6)$$

This model is seen as a simple test case or elementary analog of more realistic situations in fluid mechanics or astrophysics [1, 9]. We refer to the introduction of [14, 2, 7] for an elementary introduction to physics motivation of such a model. We also refer to [1, 9, 16, 17] and to the references quoted in [14, 2, 7] for a more detailed discussion about physics of aggregation.

In the context described above it is very natural to impose on the initial data f_{in} to have finite number of particles and momentum. This condition reads:

$$0 \le f^{in} \in L^1\left(Y, (1+m+|p|)\,dydp\right). \qquad (1.7)$$

Existence of solutions under that condition has been proved in [14, 2, 7]. It has also been proved that

$$f(t,\cdot) \to 0 \quad \text{in} \quad L^1(Y), \;\; \text{as } t \to +\infty, \qquad (1.8)$$

that is that the total number of particles tends to 0.

A more detailed description of the asymptotic behaviour of the solutions may be obtained by considering scaling invariance properties of the equations. This may be done for example by studying the so-called self similar solutions as it is possible to do for the Smoluchowski equation, see [4, 6] and the references therein for recent results in that direction for that model. A first difficulty to this end is to determine the relevant scalings of the equation (1.1), (1.3), (1.4). We are very far from being able to treat the general case when the aggregation kernel $a(y, y')$ actually depends on both mass and momentum of the two colliding particles and even when the aggregation kernel $a(y, y')$ only depends on the momentum of the two colliding particles. We then may be less ambitious and just ask for whether a more accurate version than (1.8) for some rate of aggregation a is available? We may imagine to answer that question in several ways listed below by order of accuracy, and indeed depending of the case we will establish any of such a kind of information.

- **Answer 1.** Upper bound on moment: $\exists\,\bar\alpha$, $\exists\,\nu$, $C \in (0,\infty)$ such that

$$M_{\bar\alpha}(f(t,.)) \le \frac{C}{t^\nu} \qquad \forall\, t \ge 1.$$

- **Answer 2.** Upper and lower bound on moments: $\exists\,\bar\alpha$, $\exists\,\nu_i = \nu_i(\bar\alpha)$, $C_i = C_i(\bar\alpha) \in (0,\infty)$, such that

$$\frac{C_1}{t^{\nu_1}} \le M_{\bar\alpha}(f(t,.)) \le \frac{C_2}{t^{\nu_2}} \qquad \forall\, t \ge 1.$$

- **Answer 3.** Existence of self-similar solution: there exists some profile function $\varphi_\infty : Y \to \mathbb{R}_+$, some exponents λ, μ, $\nu \in \mathbb{R}$ such such that the function

$$\varphi(t,m,p) := t^\lambda\, \varphi_\infty(t^\mu\, m, t^\nu\, p)$$

is a solution to equation (1.1), (1.3), (1.4).

- **Answer 4.** Convergence to self-similarity: for any given solution f there exists a self-similar solution φ such that $f \sim \varphi$ as $t \to \infty$, in a sense to be specified.

Here depending of the model, we define the moment of order $\bar\alpha$ of f in the following way:

- when $f = f(y)$ with $y = m \in Y = (0,\infty)$ or $y = p \in Y = \mathbb{R}^d$, then $\bar\alpha = \alpha \in \mathbb{R}$ and

$$M_{\bar\alpha}(f) = M_\alpha(f) = \int_Y |y|^\alpha\, f\, dy; \tag{1.9}$$

- when $f = f(y)$ with $y = (m,p) \in Y = (0,\infty) \times \mathbb{R}^d$, then $\bar\alpha = (\alpha,\beta) \in \mathbb{R}^2$ and

$$M_{\bar\alpha}(f) = M_{\alpha,\beta}(f) = \int_Y m^\alpha\, |p|^\beta\, f\, dy. \tag{1.10}$$

The results obtained in this work are very partials and may be classified as follows.

In Section 2 we consider the case of the kernel $a_{HS}(y,y')$ (which depends on both mass and momentum) and the only result we are able to prove is a upper estimate on some moments (that is a result of type "Answer 1").

In the remainder of the paper, we focus our attention on some toy models in which the aggregation rate a depends upon the only impulsion or upon the only masses, namely $a(y,y') = a(p,p')$, $a(y,y') = a(m,m')$ or even $a(y,y') \equiv 1$. The relation with the initial problem is not clear, and in particular it seems that a velocity depending aggregation rate $a(y,y') = a(v,v')$ should be more natural that an impulsion depending aggregation rate $a(y,y') = a(p,p')$. Anyway, on the one hand such kind of aggregation rates has been considered by physicists, see [1, 9, 16, 17], and on the other hand our results and methods can give some ideas in order to tackle the so much more difficult models where the aggregation rate depends on both mass and momentum.

Then, in Section 3 we consider a class of kernels which only depend on the momentum p and p', we establish some moment estimates of type "Answer 2", from which we deduce the rather strange conclusion that solutions do not enjoin a self-similar property (nor self-similar solution exists). That result sow doubt about the fact that in the case of the mass and impulsion hard spheres kernel, solutions develop self-similar behavior.

We treat in Section 4 the case where the kernel depends only on the masses m m' of the colliding particles and we exhibit a new class of self-similar solution (that is "Answer 3"). Lastly, in Section 5 the case of constant kernel is treated, for which results of type "Answer 3" and "Answer 4" are established.

We end that introduction by some remarks and open questions. A common feature of these equations is that

$$M_{1,0}(t) \equiv M_{1,0}(0) \qquad \text{and} \qquad M_{0,0}(t) \to 0 \quad \text{as} \quad t \to \infty,$$

and when the cross-section a is homogeneous of order $\bar{\gamma}$ (which belongs to $\mathbb{R}$ or $\mathbb{R}^2$) it is likely that

$$M_{\bar{\gamma}}(t) \equiv \frac{1}{t} \quad \text{as} \quad t \to \infty, \tag{1.11}$$

a result which is also known to be true for the coagulation equation (see [6, 5, 4]) and for the inelastic Boltzmann equation (see [11] and the references therein). The equivalence (1.11) is established for the the impulsion depending and the mass depending aggregation rate, but only one side of that equivalence is proved in the case of the true mass and impulsion depending hard spheres aggregation rate. We ask then.

Open question 1. Is it true that the assymptotic equivalence behavior (1.11) holds for some true mass and impulsion depending aggregation rate?

An other interesting question should be to establish some asymptotic behavior of typical velocity or impulsion depending quantity. A way to express that in mathematical terms is the following:

Open question 2. Is it possible to exhibit some moment $M_{\bar{\alpha}}$ for which we may determinate the long time behavior of $M_{\bar{\alpha}}/M_0$ (even just saying that it converges to something)?

2 Mass & impulsion dependence case: a remark on the hard spheres model.

Let us recall the following result

Theorem 2.1 *[7, Theorem 2.6, Theorem 2.8 and Lemma 3.3] Assume that a satisfies*

$$
\begin{aligned}
&0 \le a(y,y') = a(y',y) \le k_S(y)\,k_S(y'), \quad \forall\, y,y' \in Y,\\
&a(m,-p,m',-p') = a(m,p,m',p') \quad \forall\,(m,p),(m',p') \in Y,\\
&a(m,p,m',p') \le a(m,p,m',-p') \quad \forall\,(m,p),(m',p') \in Y \text{ s.t. } \langle p,p'\rangle > 0,
\end{aligned}
$$

with $k_S(y) := 1 + m + |p| + |v|$. *For any even (in the p variable) initial condition* $0 \le f_{in} \in L^1(Y; k_S^2(y)\,dy)$, *there exists a unique solution* $f \in C([0,T); L^1(Y; k_S(y)\,dy)) \cap L^\infty(0,T; L^1(Y; k_S^2(y)\,dy))\ \forall T > 0$, *which furthermore satisfies*

$$\int_Y f(t,.)\,m\,dy \equiv \ Cst, \tag{2.1}$$

$$f(t,.) \text{ is even, so that } \int_Y f(t,.)\,p\,dy \equiv \ 0, \tag{2.2}$$

$$\int_Y f(t,.)\,|v|^k\,dy \le \int_Y f_{in}\,|v|^k\,dy, \quad \forall\, k > 0, \tag{2.3}$$

$$\int_Y f(t,.)\,|p|^2\,dy \le \int_Y f_{in}\,|p|^2\,dy, \tag{2.4}$$

$$\int_Y f\,m^\alpha\,dy \ \to\ 0 \quad \text{when} \quad t \to \infty, \quad \forall\, \alpha < 1. \tag{2.5}$$

Remark 2.2 *(i) It is worth mentioning that the hard spheres collision rate* a_{HS} *does satisfy the assumption of Theorem 2.1, but not the Manev rate* a_{NP}.

(ii) As a consequence of (2.1), (2.3), (2.4) and (2.5) we deduce that

$$M_{\alpha,\beta}(t) := \int_Y f(t,.)\,m^\alpha\,|p|^\beta\,dy \ \to\ 0 \quad \text{as} \quad t \to \infty \tag{2.6}$$

whenever (α,β) *belongs to the region*

$$\{\beta \in [0,2],\ \alpha < 1 - \beta/2\} \cup \{\beta \ge 2,\ \alpha < 2 - \beta\}.$$

In the case of the hard spheres model we are able to quantify the rate of decay of one of the moment functions of the solution. More precisely, we have the following result.

Lemma 2.3 *Assume that* $a = a_{HS}$. *With the assumption of Theorem 2.1 there holds* $A^{-1} := M_{-1/3,1}(0) < \infty$ *and*

$$M_{-1/3,1}(t) \le \frac{1}{A + t/4} \qquad \forall\, t \ge 0. \tag{2.7}$$

Proof of Lemma 2.3. First we have $M_{-1/3,1}(0) < \infty$ because

$$m^{-1/3}\,|p| = m^{2/3}\,|v| \le m^{4/3} + |v|^2 \le 2\,k_S^2.$$

Now, from the expression (1.1)-(1.2) of the collision kernel we have

$$\int_Y Q(f,f)\,m^{-1/3}\,|p|\,dy = \frac{1}{2}\int_Y\int_Y \Delta_{-1/3,\,1}\,f\,f'\,dydy',$$

with

$$\Delta_{-1/3,\,1} = [(m+m')^{-1/3}\,|p+p'| - m^{-1/3}\,|p| - (m')^{-1/3}\,|p'|]\,[r+r']^2\,|v-v'|.$$

On one hand $-\Delta_{-1/3,\,1} \geq 0$ because

$$(m+m')^{1/3}\left(\frac{|p|}{m^{1/3}} + \frac{|p'|}{(m')^{1/3}}\right) \geq |p| + |p'| \geq |p+p'|.$$

On the other hand, if we only take into account the values of v and v' where $v \cdot v' < 0$ and suppose that, for example, $|p| = \min(|p|, |p'|)$ we have

$$\begin{aligned} -\Delta_{-1/3,\,1} &\geq \left(\frac{|p|}{m^{1/3}} + \left(\frac{|p'|}{(m')^{1/3}} - \frac{|p'|}{(m+m')^{1/3}}\right)\right)[r^2 + (r')^2]\,[|v| + |v'|] \\ &\geq \left(\frac{|p|}{m^{1/3}}\right)[(r')^2]\,[|v'|] = \frac{|p|}{m^{1/3}}\,\frac{|p'|}{(m')^{1/3}}. \end{aligned}$$

Whence, by evenness of f

$$\begin{aligned} \frac{d}{dt}\int_Y f\,\frac{|p|}{m^{1/3}}\,dy &\leq -\frac{1}{2}\int_{Y^2,\,v\cdot v'<0} \frac{|p|}{m^{1/3}}\,\frac{|p'|}{(m')^{1/3}} f\,f'\,dydy' \\ &\leq -\frac{1}{4}\left(\int_Y f\,\frac{|p|}{m^{1/3}}\,dy\right)^2, \end{aligned}$$

from which (2.7) straightforwardly follows. □

3 The impulsion dependence case $a = a(p, p_*)$

We consider now the equation (1.1), (1.3), (1.4) with a collision kernel a independent of the mass of the colliding particles. We may then integrate the equation with respect to the mass and obtain that the function of t and p, $\int_0^\infty f(t,m,p)\,dm$, that we shall still denote f, satisfies the equation:

$$\partial_t f = Q(f,f) \quad \text{in} \quad (0,+\infty)\times\mathbb{R}^d, \tag{3.1}$$

$$f(0) = f_{in} \quad \text{in} \quad \mathbb{R}^d, \tag{3.2}$$

the collision operator $Q(f)$ is given by $Q(f,f) = Q_1(f,f) - Q_2(f,f)$, where

$$Q_1(f,f)(y) = \frac{1}{2}\int_{\mathbb{R}^d} a(p', p-p')\;f(p')\;f(p-p')\,dp', \tag{3.3}$$

$$Q_2(f,f)(p) = \int_{\mathbb{R}^d} a(p,p')\;f(p)\;f(p')\,dp'. \tag{3.4}$$

We focus on the cases

$$a(p,p') = |p-p'|^\gamma, \qquad \gamma \in [0,2], \quad d \in \mathbb{N}^*. \tag{3.5}$$

Before stating our main result we need some definitions and notations. We say that a function f on $\mathbb{R}^d$ is even if

$$f(-p) = f(p) \qquad \forall\, p \in \mathbb{R}^d,$$

it is radially symmetric if

$$f(Rp) = f(p) \qquad \forall\, p \in \mathbb{R}^d,\ R \in SO(d)$$

where $SO(d)$ stands for the rotation group on $\mathbb{R}^d$. For any weight function $k : \mathbb{R}^d \to \mathbb{R}_+$ we define the "moment of order k" of the non negative density measure$f \in M^1_{loc}(\mathbb{R}^d)$ by

$$M_k(f) := \int_{\mathbb{R}^d} k(p)\, f(dp),$$

and we define M^1_k as the set of Radon measures μ such that $M_k(|\mu|) < \infty$. For any $\alpha \in \mathbb{R}_+$ we use the shorthand notation

$$M_\alpha := \int_{\mathbb{R}} f(p)\, |p|^\alpha\, dp,$$

that is $M_\alpha = M_k(f)$ for $k(p) = |p|^\alpha$ and the shorthand notation $M^1_\alpha = M^1_\ell$ for $\ell(p) = 1 + |p|^\alpha$.

Theorem 3.1 *Consider the aggregation rate (3.5).*

(i) For any even initial datum $f_{in} \in M^1_{2\alpha}$, $\alpha \in \mathbb{N}\backslash\{0,1\}$, *there exists a unique even solution* $f \in C([0,T]; M^1(\mathbb{R}^d) - weak) \cap L^\infty(0,T; M^1_{2\alpha}(\mathbb{R}^d))$ *to equation (3.1)–(3.4). For any* $\alpha \in [0,1]$ *the function* $t \mapsto M_\alpha(t)$ *is decreasing and* $f(t,.)$ *is radially symmetric for any* $t \geq 0$ *if furthermore* f_{in} *is radially symmetric.*

(ii) Moreover, the solution $f(t,.)$ *satisfies*

$$\frac{1}{M_\gamma(0)^{-1} + k_1\, t} \leq M_\gamma(t) \leq \frac{1}{M_\gamma(0)^{-1} + k_2\, t} \qquad \forall\, t \geq 0, \tag{3.6}$$

for some constants $k_i = k_i(\gamma, d) \in (0, \infty)$.

One of the main tools in order to establish that result is to consider moment equations. As it is classical for the coagulation equation, but here using one more change of variable $p' \to -p'$, any even solution f to equation (3.1)–(3.4) satisfies (at least formally) the fundamental moment equation

$$\begin{aligned} \frac{d}{dt} M_\alpha &= \frac{1}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\, f'\, a(p,p')\, [|p+p'|^\alpha - |p|^\alpha - |p'|^\alpha]\, dpdp' \\ &= \frac{1}{4} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\, f'\, \big\{ a(p,p')\, \big[|p+p'|^\alpha - |p|^\alpha - |p'|^\alpha\big] \\ &\qquad\qquad + a(p,-p')\, \big[|p-p'|^\alpha - |p|^\alpha - |p'|^\alpha\big] \,\big\}\, dpdp'. \end{aligned} \tag{3.7}$$

More precisely, we consider in this Section the case $\gamma \in (0,2)$ and $d \in \mathbb{N}^*$, the case $\gamma = 1$ and $d = 1$ and the case $\gamma = 2$ and $d \in \mathbb{N}^*$. The case $\gamma = 0$ and $d = 1$ is treated in Section 5. We shall use the following notation for the moments of order $\alpha \in \mathbb{N}$:

3.1 Proof of the existence and uniqueness part in Theorem 3.1.

We prove in this subsection an uniqueness and existence result for a general class of aggregation rates by adapting some arguments from [10, 7], see also [13]. We then deduce the existence and uniqueness part in Theorem 3.1.

Lemma 3.2 *We consider a continuous aggregation rate* $a : \mathbb{R}^{2d} \to \mathbb{R}_+$ *which satisfies*

$$\begin{aligned} a(-p,-p') &= a(p,p') \qquad \forall\, p,p' \in \mathbb{R}^d, & (3.8)\\ a(p,p') &\le a(-p,p') \qquad \forall\, p,p' \in \mathbb{R}^d,\ p\cdot p' > 0, & (3.9) \end{aligned}$$

a even weight function $k : \mathbb{R}^d \to \mathbb{R}_+$ *and we define*

$$\Delta_k(p,p') := a(p,p')\,[k(p'') + k(p') - k(p)], \quad \tilde{\Delta}_k(p,p') = A(p,p') + A(-p,p').$$

We assume that

$$a(p,p') \le C\,k(p)\,k(p') \quad \text{and} \quad \tilde{A}(p,p') \le C\,k(p)\,k(p')^2. \tag{3.10}$$

For any given even initial datum $f_{in} \in M^1_k(\mathbb{R}^d)$ *there exists no more than one even solution* $f \in C([0,T); M^1_k(\mathbb{R}^d)) \cap L^\infty(0,T; M^1_{k^2}(\mathbb{R}^d))$ *to equations (3.1)–(3.4).*

Remark 3.3 *(i) The same result holds without the evenness assumption on the density function when the second condition in (3.10) is replaced by*

$$A(p,p') \le C\,k(p)\,k(p')^2.$$

We refer to [10, 7] where such kind of result is proved in a L^1 *framework. The same result also holds for radially symmetric solutions when we assume that*

$$a(Rp,Rp') = a(p,p') \quad \forall\, p,p' \in \mathbb{R}^d,\ R \in SO(d), \tag{3.11}$$

and the second condition in (3.10) is replaced by

$$\int_{R\in SO(d)} A(p,R\,p')dR \le C\,k(p)\,k(p')^2.$$

(ii) The same kind of result holds for aggregation rate defined on Y^2 *with* $Y = (0,\infty)\times\mathbb{R}^d$ *as it is the case when particles are identified by their mass and impulsion, see [7].*

Proof of Lemma 3.2. *Step 1.* We claim that for $g \in C([0,T); M^1_k - weak)$, $G \in L^1(0,T; M^1_k)$ and $b \in C((0,T)\times\mathbb{R}^d;\mathbb{R}_+)$ such that

$$\partial_t g = G - b\,g \quad \text{in the sense of } \mathcal{D}'([0,T)\times\mathbb{R}^d), \tag{3.12}$$

the differential inequality

$$\frac{d}{dt}\|g\,k\|_{M^1} \le \|G\,k\|_{M^1} - \|b\,g\,k\|_{M^1} \tag{3.13}$$

holds in the sense of $\mathcal{D}'([0,T))$. First, it is clear using a classical duality argument that equation (3.12) has at most one solution. Indeed, given two solutions $g_1, g_2 \in$

$C([0,T]; M^1_k - weak)$, we have for any $t \in (0,T)$, $\varphi_t \in C_{comp}(\mathbb{R}^d)$ and denoting by $\varphi \in C_{comp}([0,t] \times \mathbb{R}^d)$ the solution to the dual homogeneous equation $\partial_t \varphi = b\,\varphi$

$$\int_{\mathbb{R}^d} (g_2 - g_1)(t)\, \varphi_t \, dp = \int_0^t \int_{\mathbb{R}^d} \{(\partial_s g_2 - \partial_s g_1)\, \varphi + (g_2 - g_1)\, \partial_s \varphi\}\, ds dp = 0.$$

Now, for any $g_\varepsilon(0) \in C_{K_\varepsilon} := \{u \in C(\mathbb{R}^d);\ \operatorname{supp} u \subset K_\varepsilon\}$, with $K_\varepsilon \subset \mathbb{R}^d$ a compact, and any $G_\varepsilon \in L^1(0,T; C_{K_\varepsilon})$ there exists a (unique) solution $g_\varepsilon \in C([0,T); C_{K_\varepsilon})$ to equation (3.12) which furthermore satisfies

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |g_\varepsilon|\, k \, dy &= \int_{\mathbb{R}^d} (G_\varepsilon - b\, g_\varepsilon)\, \mathrm{sign} g_\varepsilon \, k\, dy \\ &\leq \int_{\mathbb{R}^d} |G_\varepsilon|\, k\, dy - \int_{\mathbb{R}^d} |g_\varepsilon|\, a\, k\, dy. \end{aligned} \tag{3.14}$$

Here, $\mathrm{sign} g_\varepsilon = 1$ if $g_\varepsilon > 0$, $\mathrm{sign} g_\varepsilon = 0$ if $g_\varepsilon = 0$, $\mathrm{sign} g_\varepsilon = -1$ if $g_\varepsilon < 0$. Finally, we can build (by a standard truncation and regularization by convolution process) the sequences (G_ε) and $g_\varepsilon(0)$ such that furthermore $G_\varepsilon \rightharpoonup G$, $g_\varepsilon(0) \rightharpoonup g(0)$ in the weak sense of measures in M^1_k, $\|G_\varepsilon(s)\|_{M^1_k} \leq \|G(s)\|_{M^1_k}$ for a.e. $s \in (0,T)$, $\|g_\varepsilon(0)\|_{M^1_k} \leq \|g(0)\|_{M^1_k}$. By the previous uniqueness argument we have $g_\varepsilon \rightharpoonup g$ in the weak sense of measure and we get (3.13) by passing to the limit in (3.14).

Step 2. Let us consider two solutions $f_1, f_2 \in C([0,T); M^1_k(\mathbb{R}^d)) \cap L^\infty(0,T; M^1_{k^2}(\mathbb{R}^d))$ which are evens and let us denote $D = f_2 - f_1$, $S = f_1 + f_2$. By a standard algebraic computation D satisfies the following equation

$$\begin{aligned} \partial_t D &= \hat{Q}(f_2, f_2) - \hat{Q}(f_1, f_1) = \hat{Q}(D, S) \\ &= \hat{Q}_1(D, S) - S\, L(D) - L(S)\, D, \end{aligned}$$

where

$$\hat{Q}_i(\varphi, \psi) = \frac{1}{2}\, (Q_i(\varphi, \psi) + Q_i(\psi, \varphi)), \qquad L(\varphi) := \int_{\mathbb{R}^d} a(p, p')\, \varphi(p')\, dp'.$$

Because of the assumption made on a and f we have $D \in C([0,T]; M^1_k - weak)$, $G := \hat{Q}_1(D, S) - S\, L(D) \in L^\infty(0,T; M^1_k)$ and $0 \leq b := L(S) \in C([0,T] \times \mathbb{R}^d)$ so that the first step implies

$$\begin{aligned} \frac{d}{dt} \|D\|_{M^1_k} &\leq \|(\hat{Q}_1(D,S) - S\, L(D))\, k\|_{M^1} - \|D\, k\, L(S)\|_{M^1} \\ &\leq \frac{1}{2} \int\!\!\int a\, [k'' + k']\, |D(dp)|\, S(dp') - \frac{1}{2} \int\!\!\int a\, k\, |D(dp)|\, S(dp') \\ &\leq \frac{1}{4} \int\!\!\int \tilde{A}\, |D(dp)|\, S(dp') \leq \frac{C}{4}\, \|S\|_{M^1_{k^2}}\, \|D\|_{M^1_k}. \end{aligned}$$

Uniqueness follows by using the Gronwall lemma. $\square$

Lemma 3.4 *Consider a continuous aggregation rate $a : \mathbb{R}^{2d} \to \mathbb{R}_+$ which satisfies (3.8) (resp. (3.11)), (3.9) as well as*

$$a(p, p') \leq C\, (k + k') \qquad \forall\, p,\, p' \in \mathbb{R}^d, \tag{3.15}$$

for the weight function $k(p) = 1 + |p|^2$ and some constant $C \in (0,\infty)$. For any given even (resp. radially symmetric) initial datum $f_{in} \in M^1_{2\alpha}(\mathbb{R}^d)$ there exists at least one even (resp. radially symmetric) solution $f \in C([0,T); M^1(\mathbb{R}^d) - weak) \cap L^\infty(0,T; M^1_{2\alpha}(\mathbb{R}^d))$ to equation (3.1)–(3.4), and this one furthermore satisfies $t \mapsto M_\beta(t)$ is decreasing for any $\beta \in [0,1]$.

Remark 3.5 *It is likely that by adapting some arguments introduced in [12], see also [3, 10], for any even (resp. radially symmetric) initial datum $f_{in} \in L^1_{2\alpha}(\mathbb{R}^d)$ the approximating solution $f_n(t,.)$ built in the proof below is a Cauchy sequence in $C([0,T; L^1(\mathbb{R}^d))$ so that we may conclude $f \in C([0,T); L^1(\mathbb{R}^d)) \cap L^\infty(0,T; L^1_{2\alpha}(\mathbb{R}^d))$.*

Proof of Lemma 3.4. We define the sequence of bounded aggregation rates $a_n := a \wedge n$, for which classically fixed point argument (see for instance [7] which deals with some similar situation) implies the existence of a unique even (resp. radially symmetric) solution $f_n \in C([0,T); L^1_{2\alpha}(\mathbb{R}^d))$ to equation (3.1)–(3.4) associated with a_n for any initial datum $f_{in,n} \in L^1_{2\alpha+2}(\mathbb{R}^d)$, $\alpha \in \mathbb{N}$, $\alpha \geq 2$. Then, we have for any $\beta \in \mathbb{N}^*$, $\beta \leq \alpha$

$$\begin{aligned}\frac{d}{dt}\int f_n\,(1+|p|^{2\beta}) &= \frac{1}{2}\int f_n\,f_n'\,a_n\,\Big[(|p|^2 + 2\,p\cdot p' + |p'|^2)^\beta - |p|^{2\beta} - |p'|^{2\beta} - 1\Big] \\ &= \int f_n\,f_n'\,a_n\,\Big[2\,\beta\,p\cdot p'\,|p|^{2(\beta-1)} - 1/2\Big] \\ &\quad + \sum \mu_{\beta_1,\beta_2,\beta_2}\int f_n\,f_n'\,a_n\,(p\cdot p')^{\beta_1}\,|p|^{2\,\beta_2}\,\,|p|^{2\,\beta_3},\end{aligned}$$

where in the last sum we have $\beta_1 + \beta_2 + \beta_3 = \beta$ and ($\beta_1 \geq 2$ or ($\beta_2 \geq 1$ and $\beta_3 \geq 1$)) or, in other words, $|p\cdot p'|^{\beta_1}\,|p|^{2\,\beta_2}\,\,|p|^{2\,\beta_3} \leq |p|^{2\,\beta'}\,|p'|^{2\,(\beta-\beta')}$ with $1 \leq \beta' \leq \beta - 1$. Since we also have

$$\begin{aligned}&\int f_n\,f_n'\,a_n\,p\cdot p'\,|p|^{2(\beta-1)} = \\ &\quad = \int_{p\cdot p'>0} f_n\,f_n'\,(a(p,p')\wedge n - a(-p,p')\wedge n)\,p\cdot p'\,|p|^{2(\beta-1)} \leq 0,\end{aligned}$$

we conclude with

$$\frac{d}{dt}\int f_n\,(1+|p|^{2\beta}) \leq \sum_{1\leq\beta'\leq\beta-1}\mu_{\beta'}\int f_n\,f_n'\,a\,|p|^{2\,\beta'}\,|p'|^{2\,(\beta-\beta')}. \tag{3.16}$$

When $\beta = 1$ the set of admissible values of β' is empty, and we recover a result from [4]

$$\frac{d}{dt}\int f_n\,(1+|p|^2) \leq 0,$$

so that

$$\sup_{[0,T]}\|f_n\|_{L^1_k} \leq \|f_{in,n}\|_{L^1_k}. \tag{3.17}$$

When $\beta \geq 2$, gathering (3.15), (3.16) and (3.17), we easily conclude by a iterative argument that

$$\sup_{[0,T]}\|f_n\|_{L^1_{k^\beta}} \leq C_T(\beta, \|f_{in,n}\|_{L^1_{k^\beta}}). \tag{3.18}$$

Considering a sequence $(f_{in,n})$ such that $f_{in,n} \rightharpoonup f_{in}$ in the weak sense of measure and $\|f_{in,n}\|_{L^1_{k^\beta}}$ remains bounded, we easily pass to the limit in the equation satisfied by f_n thanks to (3.18). The fact that $t \mapsto M_\beta(t)$ is decreasing comes from the fact that $p \mapsto |p|^\beta$ is a sub-additive function when $\beta \in [0,1]$, so that $\Delta_\beta \le 0$ and then $d/dt M_\beta(t) \le 0$. $\square$

Proof of the existence and uniqueness part in Theorem 3.1. It is clear that $a(p,p') = |p-p'|^\gamma$ satisfies (3.8), (3.9), the first inequality in (3.10) and (3.15). Moreover, the second inequality in (3.10) holds since we have

$$\begin{aligned}\tilde\Delta_2 &= |p-p_*|^\gamma\,(|p+p_*|^2+|p_*|^2-|p|^2+1) + |p+p_*|^\gamma\,(|p-p_*|^2+|p_*|^2-|p|^2+1)\\ &= 2\,(|p-p_*|^\gamma - |p+p_*|^\gamma)\,p\cdot p' + (|p-p_*|^\gamma + |p+p_*|^\gamma)\,(2\,|p_*|^2+1),\end{aligned}$$

where the first term in non positive and the second term is bounded by say $8\,(k')^2\,k$, using that $|p \pm p_*|^\gamma \le 2\,(|p|^\gamma + |p'|^\gamma)$. We conclude by using Lemma 3.2 and Lemma 3.4. $\square$

3.2 Proof of the rate decay part in Theorem 3.1 when $\gamma < 2$.

For an even initial datum $f_{in} \in M^1_4(\mathbb{R}^d)$ we consider the unique even solution $f \in C([0,T]; M^1 - weak) \cap L^\infty(0,T;M^1_4)$, $\forall T$, given by Theorem 3.1(i). This one satisfies the moment equation

$$\frac{d}{dt} M_\gamma = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,\Delta_\gamma\,dpdp' = \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,\tilde\Delta_\gamma\,dpdp', \tag{3.19}$$

with

$$-\Delta_\gamma = |p-p'|^\gamma\,[|p+p'|^\gamma - |p|^\gamma - |p'|^\gamma]$$

and

$$-\,\tilde\Delta_\gamma = |p-p'|^\gamma\,\left[|p+p'|^\gamma - |p|^\gamma - |p'|^\gamma\right] + |p+p'|^\gamma\,\left[|p-p'|^\gamma - |p|^\gamma - |p'|^\gamma\right]. \tag{3.20}$$

We split the proof of Theorem 3.1(ii) in several steps.

Step 1. One the one hand, for any given $A > 0$ and any $p, p' \in \mathbb{R}^d$ such that $A^{-1}\,|p'| \le |p| \le A\,|p|$ we easily get

$$\begin{aligned}|\Delta_\gamma| &\le (|p|+|p'|)^\gamma\,\max\left[(|p|+|p'|)^\gamma, |p|^\gamma + |p'|^\gamma\right]\\ &\le 2^4\,\max(|p|,|p'|)^{2\gamma} \le 2^4\,A^\gamma\,(|p|\,|p'|)^\gamma.\end{aligned} \tag{3.21}$$

On the other hand, we define $M := \max(|p|,|p'|)$, $m := \min(|p|,|p'|)$, $x := m/M \in [0,1]$, $\varepsilon := \hat p \cdot \hat p' \in [-1,1]$ and we compute (in the first line we have assumed that $|p| = M$ which is not a restriction to the generality because of the symmetry of $\tilde\Delta_\gamma$)

$$\begin{aligned}-\,\tilde\Delta_\gamma &= M^{2\gamma}\,\left\{|\hat p - x\,\hat p'|^\gamma\,\left[1 + x^\gamma - |\hat p + x\,\hat p'|^\gamma\right] + |\hat p + x\,\hat p'|^\gamma\,\left[1 + x^\gamma - |\hat p - x\,\hat p'|^\gamma\right]\right\}\\ &= M^{2\gamma}\,\Big\{(1+x^\gamma)\,[(1+2\,\varepsilon\,x + x^2)^{\gamma/2} + (1-2\,\varepsilon\,x+x^2)^{\gamma/2}]\\ &\qquad -2\,(1+2\,\varepsilon\,x+x^2)^{\gamma/2}\,(1-2\,\varepsilon\,x+x^2)^{\gamma/2}\Big\}\\ &= M^{2\gamma}\,\left\{2\,x^\gamma + \mathcal{O}(x^2)\right\} \le 3\,M^{2\gamma}\,x^\gamma = 3\,(|p|\,|p'|)^\gamma\end{aligned} \tag{3.22}$$

uniformly on $\varepsilon \in [-1,1]$ and $x \le A_0^{-1}$ for $A_0 \ge 1$ large enough.

Gathering (3.21) and (3.22) we obtain

$$\frac{1}{4}\,\tilde{\Delta}_\gamma \geq -k_1\,|p|^\gamma\,|p'|^\gamma \qquad \forall\, p, p' \in \mathbb{R}^d,$$

with $k_1 := \max(3/4, 2^3\,A_0^\gamma)/4$, and equation (3.19) then implies

$$\frac{d}{dt} M_\gamma \geq -k_1\,M_\gamma^2.$$

We straightforwardly obtain the first inequality in (3.6) by integrating this differential equation.

Step 2. First, together with the variables M, x and ε introduced in Step 1, we define $r > 0$ and $u \in [0,1]$ by setting $r^2 := |p|^2 + |p'|^2$ and $u := 2\,p \cdot p'/r^2$, so that $|p \pm p'|^2 = r^2\,(1 \pm u)$. Splitting the positive and the negative terms in identity (3.20), we have

$$\begin{aligned} -\tilde{\Delta}_\gamma &= (|p|^\gamma + |p'|^\gamma)\,(|p-p'|^\gamma + |p+p'|^\gamma) - 2\,|p-p'|^\gamma\,|p+p'|^\gamma \\ &= r^{2\gamma}\left\{ \frac{(|p|^2)^{\gamma/2} + (|p'|^2)^{\gamma/2}}{(|p|^2+|p'|^2)^{\gamma/2}}\,\Big[(1+u)^{\gamma/2} + (1-u)^{\gamma/2}\Big] - 2\,(1+u)^{\gamma/2}\,(1-u)^{\gamma/2} \right\}. \end{aligned}$$

Since $\gamma/2 \in [0,1]$, the map $x \mapsto x^{\gamma/2}$ is sub-additive, and we obtain

$$\begin{aligned} -\tilde{\Delta}_\gamma &\geq r^{2\gamma}\left\{ \Big[(1+u)^{\gamma/2} + (1-u)^{\gamma/2}\Big] - 2\,(1+u)^{\gamma/2}\,(1-u)^{\gamma/2} \right\} \\ &\geq M^{2\gamma}\,(1+u)^{\gamma/2}\,(1-u)^{\gamma/2}\,\phi(u), \quad \phi(u) := \Big[(1-u)^{-\gamma/2} + (1+u)^{-\gamma/2}\Big] - 2. \end{aligned}$$

We easily verify that ϕ is increasing on $[0,1]$ so that $\phi(u) > \phi(0) = 0$ for any $u \in [-1,1]$, $u \neq 0$. Coming back to the variables M, x and ε, that is $\phi(u) > 0$ for any $p, p' \in \mathbb{R}^d$ such that the associated variables M, x and ε satisfy $M > 0$, $x > 0$ and $\varepsilon \neq 0$. Moreover, when $\varepsilon = 0$ (p and p' are orthogonal vectors) we also have

$$\begin{aligned} -\tilde{\Delta}_\gamma &= 2(|p|^2 + |p'|^2)^{\gamma/2}\,\Big[|p|^\gamma + |p'|^\gamma - (|p|^2 + |p'|^2)^{\gamma/2}\Big] \\ &\geq 2\,M^{2\gamma}\,\Big[1 + x^\gamma - (1+x^2)^{\gamma/2}\Big] > 0 \end{aligned}$$

for any $p, p' \in \mathbb{R}^d$ such that the associated variables M and x satisfy $M > 0$, $x > 0$, because the function $z \mapsto z^{\gamma/2}$ is strictly sub-additive on $\mathbb{R}_+$, that is $(z+z')^{\gamma/2} < z^{\gamma/2} + (z')^{\gamma/2}$ for any $z, z' > 0$. Gathering these two lower bounds on $-\tilde{\Delta}_\gamma$, it yields

$$-\,\tilde{\Delta}_\gamma \geq M^{2\,\gamma}\,\psi(x,\varepsilon) \tag{3.23}$$

with $\psi(x,\varepsilon) > 0$ for any $x > 0$ and $\varepsilon \in [-1,1]$.

Next, coming back to (3.22), we also deduce

$$-\,\tilde{\Delta}_\gamma = M^{2\gamma}\,\left\{2\,x^\gamma + \mathcal{O}(x^2)\right\} \geq M^{2\gamma}\,x^\gamma \tag{3.24}$$

uniformly on $\varepsilon \in [-1,1]$ and $x \leq A_0^{-1}$ for $A_0 \geq 1$ large enough. Gathering (3.23) with (3.24) we deduce that for some constant $k_2 > 0$ we have

$$\forall\, p, p' \in \mathbb{R}^d \qquad -\,\frac{1}{4}\tilde{\Delta}_\gamma \geq k_2\,M^{2\gamma}\,x^\gamma = k_2\,(|p|\,|p'|)^\gamma,$$

and equation (3.19) then implies

$$\frac{d}{dt} M_\gamma \leq -k_2\,M_\gamma^2.$$

The second inequality in (3.6) is again obtained by integrating this differential equation.

3.3 The case $a(y,y') = |p - p'|$, $d = 1$.

In the particular case under consideration $d = 1$ and $\gamma = 1$, we can establish a more accurate version of the decay estimate on the moment M_1 together with additional moment estimates.

Lemma 3.6 *Assume $a(y,y') = |p - p'|$ and $d = 1$. For any even initial datum $f_{in} \in M^1_3(\mathbb{R})$ the unique solution $f \in C([0,T]; M^1(\mathbb{R})) \cap L^\infty(0,T; M^1_3(\mathbb{R}))$ of (3.1)-(3.4) given by Theorem 3.1 satisfies for any $t \geq 0$*

$$\max\left(\frac{M_0(0)}{(1+M_1(0)\,t/2)^2}, \frac{2^{3/2}\,M_0(0)}{(2+3M_3^{1/3}(0)\,t)^{3/2}}\right) \leq M_0(t) \leq \frac{M_0(0)}{(1+M_1(0)t)^{1/2}} \tag{3.25}$$

$$\frac{1}{M_1(0)^{-1}+t} \leq M_1(t) \leq \frac{1}{M_1(0)^{-1}+t/2} \tag{3.26}$$

$$\frac{M_2(0)}{(1+M_1(0)\,t/2)^2} \leq M_2(t) \leq M_2(0) \tag{3.27}$$

$$\frac{M_3(0)}{(1+M_1(0)\,t/2)^2} \leq M_3(t) \leq M_3(0). \tag{3.28}$$

Remark 3.7 *The above estimates on the behaviour of $M_1(t)$ for t large are quite good. That is not the case for the estimates on M_α, $\alpha = 0,2,3$ which seem to be rather partial. Worst, with these bounds we can not even know what is the limit of any of the quotients of moments $M_\alpha(t)/M_1(t)$ for $\alpha = 0,2,3$ as $t \to \infty$. The value of such a limit would indicate whether the solution $f(t)$ has a tendency to concentrate or to spread as t increases (see also below the discussion concerning the case $\gamma = 2$).*

Proof of Lemma 3.6. Introducing the notations $M = \max(|p|,|p'|)$, $m = \min(|p|,|p'|)$, we systematically exploit the differential equation

$$\frac{d}{dt}M_\alpha = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,\Delta_\alpha\,dpdp' \tag{3.29}$$

with

$$\Delta_\alpha := [M-m]\,[(M+m)^\alpha - M^\alpha - m^\alpha] + [M+m]\,[(M-m)^\alpha - M^\alpha - m^\alpha].$$

Step 1. $\alpha = 1$. We have

$$\Delta_1 = -2\,(M+m)\,m,$$

from which we deduce

$$\frac{d}{dt}M_1(t) = -\frac{M_1^2(t)}{2} - \frac{B_1(t)}{2}, \qquad B_1(t) := \int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,\{\min(|p|,|p'|)\}^2\,dpdp'.$$

Since $0 \leq \{\min(|p|,|p'|)\}^2 \leq |p|\,|p'|$, we have $0 \leq B_1(t) \leq M_1^2(t)$ and we obtain the two closed differential inequalities

$$-M_1^2(t) \leq \frac{d}{dt}M_1(t) \leq -\frac{M_1^2(t)}{2}, \tag{3.30}$$

from which we deduce (3.26).

Step 2. $\alpha = 0$. We have

$$\Delta_0 = -2\,M,$$

from which we deduce

$$\frac{d}{dt} M_0(t) = -\frac{B_0(t)}{2}, \qquad B_0(t) := \int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,\max(|p|,|p'|)\,dpdp'. \tag{3.31}$$

Since $|p| \le \max(|p|,|p'|) \le |p| + |p'|$, we have $M_0\,M_1 \le B_0 \le 2\,M_0\,M_1$ and then

$$-\,M_0\,M_1 \le \frac{d}{dt} M_0 \le -\frac{1}{2}\,M_0\,M_1. \tag{3.32}$$

Using the previous estimate (3.26) on $M_1(t)$ we get

$$-\frac{M_0(t)}{M_1^{-1}(0) + t/2} \le \frac{d}{dt} M_0(t) \le -\frac{M_0(t)}{2(M_1^{-1}(0) + t)},$$

from which we deduce the first lower estimate as well as the upper bound in (3.25).

Step 3. $\alpha = 2$. We have

$$\Delta_2 = -4\,m\,M^2$$

from which we deduce

$$\frac{d}{dt} M_2(t) = -B_2(t), \qquad B_2(t) := \int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,\min(|p|,|p'|)\,|p|\,|p'|\,dpdp'.$$

Using that $0 \le \min(|p|,|p'|)\,|p|\,|p'| \le |p|^2\,|p'|$ together with (3.26), we obtain

$$-M_2\,\frac{1}{M_1(0)^{-1} + t/2} \le -M_2\,M_1 \le \frac{d}{dt} M_2(t) \le 0,$$

which implies (3.27).

Step 4. $\alpha = 3$. We have

$$0 \ge \Delta_3 = -2\,M\,m^3 - 2\,m^4 \ge -4\,M\,m^3 \ge -4\,|p|^3\,|p'|,$$

from what we deduce

$$0 \ge \frac{d}{dt} M_3(t) \ge -M_1\,M_3,$$

which again implies (3.28).

Step 5. $\alpha = 0$ *again.* Coming back to the moment M_0, we write for any $\varepsilon > 0$

$$\begin{aligned}
\frac{d}{dt} M_0 &= -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,|p' - p|\,dpdp' \\
&\ge -\frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,(\varepsilon + \frac{1}{\varepsilon}\,|p - p'|^2)\,dpdp' \\
&\ge -\frac{\varepsilon}{4}\,M_0^2 - \frac{2}{\varepsilon}\,M_0\,M_2.
\end{aligned}$$

By interpolation we have $M_2(t) \le M_0^{1/3}(t)\, M_3^{2/3}(t)$. Since, by (3.28), $M_3(t) \le M_3(0)$ for all $t > 0$ we deduce $M_2(t) \le M_0^{1/3}(t)\, M_3^{2/3}(0)$. Therefore

$$\frac{d}{dt} M_0(t) \;\ge\; -\frac{\varepsilon}{4}\, M_0^2 - \frac{2}{\varepsilon}\, M_0^{4/3}\, M_3^{2/3}(0)$$

We now chose $\varepsilon \equiv \varepsilon(t) > 0$ such that $\varepsilon\, M_0^2 = \frac{1}{\varepsilon}\, M_0^{4/3}\, M_3^{2/3}(0)$, or equivalently $\varepsilon = M_0^{-1/3}\, M_3^{1/3}(0)$. With that choice of $\varepsilon(t)$ the moment equation reads

$$\frac{d}{dt} M_0(t) \ge -\frac{9}{4}\, M_3^{1/3}(0)\, M_0^{5/3},$$

from which we deduce the second lower estimate in (3.25). □

Remark 3.8 *In the last step, we may also argue as follows. Gathering the estimate* $\max(|p|, |p'|) \ge (|p|\,|p'|)^{1/2}$*, the differential equation (3.31) and the interpolation estimate* $M_1^{5/2} \le M_{1/2}^2\, M_3^{1/2}$ *we obtain thanks to (3.28)*

$$\frac{d}{dt} M_0 \le -\frac{1}{dt}\, M_3^{1/2}(0)\, M_1^{5/2}(t).$$

Together with (3.26) we recover the second lower estimate in (3.25).

3.4 The case $a = |p - p_*|^2$

In the particular case under consideration $\gamma = 2$ and $d \in \mathbb{N}^*$, we can close the family of moment equations for any moments $M_{2\alpha}$, $\alpha \in \mathbb{N}$. In the following lemma we give the expression of moments up to order 4, showing a (unexpected?) non self-similar behavior of solutions.

Lemma 3.9 *Assume* $a(y, y') = |p - p'|^2$ *and* $d \in \mathbb{N}^*$*. There exists a numerical constant* $k_d \in (0, \infty)$*,* $k_1 := 2$*, such that for any radially symmetric initial datum* $f_{in} \in M_6^1(\mathbb{R})$ *the unique radially symmetric solution* $f \in C([0,T]; M^1(\mathbb{R})) \cap L^\infty(0,T; M_6^1(\mathbb{R}))$ *of (3.1)-(3.4) given by Theorem 3.1 satisfies for any* $t \ge 0$

$$M_0(t) = \frac{M_0(0)}{(M_2(0)^{-1} + 2\, k_d\, t)^{1/(2k_d)}} \tag{3.33}$$

$$M_2(t) = \frac{1}{M_2(0)^{-1} + 2\, k_d\, t} \tag{3.34}$$

$$M_4(t) = M_4(0)\,(M_2(0)^{-1} + 2\, k_d\, t)^{1/k_d - 2}. \tag{3.35}$$

Proof of Lemma 3.9. We proceed in several steps.

Step 1. $\alpha = 2$. Using the fact that f is radially symmetric (so that the odd moments of f vanish) and the notations $p = r\,\sigma$, $r = |p|$, $p' = r'\,\sigma'$, $r' = |p'|$, the fundamental moment identity (3.7) implies

$$\begin{aligned}
\frac{d}{dt} M_2 &= \frac{1}{2} \int_{\mathbb{R}^d}\!\int_{\mathbb{R}^d} f\, f'\, [|p|^2 - 2\, p\cdot p' + |p'|^2]\,(2\, p\cdot p')\, dp dp' \\
&= -2 \int_{\mathbb{R}^d}\!\int_{\mathbb{R}^d} f\, f'\, [p\cdot p']^2\, dp dp' \\
&= -2 \int_0^\infty\!\int_0^\infty f(r)\, f(r')\, r^{d+1}\,(r')^{d+1}\, dr dr' \times \int_{S^{d-1}}\!\int_{S^{d-1}} [\sigma\cdot\sigma']^2\, d\sigma d\sigma' \\
&= -2\, k_d\, M_2^2,
\end{aligned}$$

with

$$
\begin{aligned}
k_d &:= \left(\int_{S^{d-1}} \int_{S^{d-1}} [\sigma \cdot \sigma']^2 \, d\sigma d\sigma' \right) \times \text{meas}(S^{d-1})^{-2} \\
&= \text{meas}(S^{d-1})^{-1} \int_{S^{d-1}} \sigma_1^2 \, d\sigma.
\end{aligned}
$$

We compute $k_1 = 1$, $k_2 = 1/2$. The expression (3.34) immediately follows by integrating that ODE.

Step 2. $\alpha = 0$. When $\alpha = 0$, the fundamental moment identity (3.7) and the fact that f is radially symmetric imply

$$
\begin{aligned}
\frac{d}{dt} M_0 &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \, f' \, [|p|^2 - 2 \, p \cdot p' + |p'|^2] \, (-1) \, dp dp' \\
&= -M_2 \, M_0.
\end{aligned}
$$

Integrating that ODE with the help of (3.34) we get (3.33).

Step 3. $\alpha = 4$. When $\alpha = 4$, the fundamental moment identity (3.7) and the fact that f is radially symmetric imply

$$
\begin{aligned}
\frac{d}{dt} M_4 &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \, f' \, [|p|^2 - 2 \, p \cdot p' + |p'|^2] \, [4 \, (p \cdot p')^2 + 8 \, |p|^2 \, (p \cdot p') + 2 \, |p|^2 \, |p'|^2] \, dp dp' \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \, f' \, \left\{ [2 \, |p|^2] \, [4 \, (p \cdot p')^2 + 2 \, |p|^2 \, |p'|^2] - 16 \, |p|^2 \, (p \cdot p')^2 \right\} \, dp dp' \\
&= 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f \, f' \, \left\{ |p|^4 \, |p'|^2 - 2 \, |p|^2 \, (p \cdot p')^2 \right\} \, dp dp' \\
&= (2 - 4 \, k_d) \, M_2 \, M_4.
\end{aligned}
$$

Integrating that ODE with the help of (3.34) we get (3.35). □

Remark 3.10 *(i) On the one hand, the moment $M_\alpha(g(t,.))$ of a self-similar function g of the form $g(t,p) = t^\mu \, G(t^\nu \, p)$ satisfies*

$$
M_\alpha(g(t,.)) = C_\alpha \, t^{\mu - (d+\alpha) \, \nu}.
$$

On the other hand, when $d = 1$ we have $k_1 = 1$ so that the solution f of equation (3.1)-(3.4) satisfies

$$
M_0(f(t,.)) \sim C_0' \, t^{-1/2}, \quad M_2(f(t,.)) \sim C_2' \, t^{-1}, \quad M_4(f(t,.)) \sim C_4' \, t^{-1}.
$$

Since the long time behavior of these functions are incompatibles, there does not exist any self-similar solution with self-similar profile $G \in M_6^1(\mathbb{R})$.

(ii) When $d = 1$, to make the ideas simpler, the moment $M_{2\alpha}$ satisfies the edo

$$
\frac{d}{dt} M_{2\alpha} = \sum_{\beta=1}^{\alpha-1} \binom{2\alpha}{2\beta} \, M_{2\beta} \, M_{2(\alpha+1-\beta)} - \sum_{\beta=0}^{\alpha-1} \binom{2\alpha}{2\beta+1} \, M_{2\beta+2} \, M_{2(\alpha-\beta)}.
$$

In particular, we find

$$
\frac{d}{dt} M_6 = 3 \, M_2 \, M_6 - 5 \, M_4^2.
$$

When $M_2(0) = 1/2$ (for the sake of simplification again), the solution is

$$M_6(t) = \left(M_6(0) - 2\, M_4(0)^2 + \frac{2\, M_4(0)^2}{(1+t)^{5/4}} \right) (1+t)^{3/2} \qquad \forall\, t \geq 0,$$

with $M_6(0) - 2\, M_4(0)^2 > 0$ (Holder inequality). The solutions of equation (3.1)-(3.4) have a rather strange behavior since that

$$M_0 \sim \kappa_0\, t^{-1/2}, \quad \frac{M_2}{M_0} \sim \kappa_1\, t^{-1/2}, \quad \frac{M_6}{M_0} \sim \kappa_2\, t^{3/2},$$

In some sense, the behavior is in part comparable with the solutions of the inelastic Boltzmann equation which energy (here the M_2 moment) dissipates and in part comparable with the solutions to Smoluchoski equation which high moments rapidly increase. It is worth mentioning that here the "mean second moment" (that is M_2/M_0) tends to 0 in the large time assymptotic. The opposite feature occurs for some models dealt in section 4.

4 The mass dependence case $a = a(m, m_*)$

Consider now the problem (1.1)-(1.4) where the kernel $a(y, y')$ only depends on the masses of the particles, namely

$$a(y, y') = a(m, m'), \tag{4.1}$$

and introduce the associated Smoluchowski equation

$$\begin{aligned} \frac{\partial F}{\partial t}(t, m) = \frac{1}{2} \int_0^m & F(t, m - m') F(t, m')\, a(m - m', m')\, dm' \\ & - \int_0^\infty F(t, m) F(t, m')\, a(m, m')\, dm'. \end{aligned} \tag{4.2}$$

For any function $\psi \in L^1(\mathbb{R}^3)$ we define the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ by

$$\hat{\psi}(\eta) = (\mathcal{F}\psi)(\eta) = \int_{\mathbb{R}^3} \psi(p)\, e^{-i\, p \cdot \eta}\, dp, \qquad (\mathcal{F}^{-1}\psi)(p) = (2\,\pi)^{-3} \int_{\mathbb{R}^3} \psi(\eta)\, e^{i\, p \cdot \eta}\, dp.$$

Theorem 4.1 *For any smooth function a on $\mathbb{R}^3$ homogeneous of degree θ^{-1}, $\theta \in (0, \infty)$, and such that $\varphi := \mathcal{F}^{-1}(e^{-a(\cdot)}) \geq 0$, and for any solution $F \equiv F(t, m)$ to the coagulation equation (4.2) with coagulation kernel $a(m, m')$, the function $f(t, m, p)$ defined by*

$$f(t, m, p) = m^{-3\,\theta}\, F(t, m)\, \varphi\left(\frac{p}{m^\theta} \right), \tag{4.3}$$

is a solution of the equation (1.1), (1.3), (1.4) for the same aggregation kernel.

Remark 4.2 *Theorem 4.1 is not a general existence result of solutions to (1.1), (1.3), (1.4). Notice indeed that the initial data satisfied corresponding to these solutions are all of the form $f(0, m, p) = m^{-3\theta} F_{in}(m)\, \varphi\left(p/m^\theta\right)$. An example of admissible function a is $a(p) := |p|^2$, so that that $\theta = 1/2$.*

Proof of Theorem 4.1. We have to check that the function $f(t,m,p)$ defined by (4.3) solves (1.1), (1.3), (1.4). We start with writing

$$\begin{aligned}\frac{\partial f}{\partial t} &= m^{-3\theta}\varphi\left(\frac{p}{m^\theta}\right)\frac{\partial F}{\partial t}\\ &= m^{-3\theta}\varphi\left(\frac{p}{m^\theta}\right)\left[\frac{1}{2}\int_0^m F(t,m-m')F(t,m')\,a(m-m',m')\,dm'\right.\\ &\qquad\left.-\int_0^\infty F(t,m)F(t,m')\,a(m,m')\,dm'\right].\end{aligned} \tag{4.4}$$

On the one hand, using that

$$\int_{\mathbb{R}^3}\varphi(p)\,dp = \mathcal{F}(\varphi)(0) = e^{-a(0)} = 1,$$

the last term in (4.4) gives

$$\begin{aligned}& m^{-3\theta}\varphi\left(\frac{p}{m^\theta}\right)\int_0^\infty F(t,m)F(t,m')\,a(m,m')\,dm' =\\ &= m^{-3\theta}\varphi\left(\frac{p}{m^\theta}\right)F(t,m)\int_0^\infty a(m,m')\,F(t,m')\int_{\mathbb{R}^3}(m')^{-3\theta}\varphi(\frac{p'}{m'^\theta})\,dp'\\ &= f(t,m,p)\int_0^\infty\int_{\mathbb{R}^3} a(m,m')\,f(t,m',p')\,dp'.\end{aligned} \tag{4.5}$$

On the other hand, let us define the function

$$g(m,p) = m^{-3\,\theta}\varphi(p/m^\theta).$$

Using the definition of φ and the homogeneity of a, it satisfies for any $0 < m' < m$

$$\begin{aligned}\hat{g}(m,\eta) &= \hat{\varphi}(m^\theta\,\eta) = \exp(-a(m^\theta\,\eta)) = \exp(-m\,a(\eta))\\ &= \exp(-m'\,a(\eta))\,\exp(-(m-m')\,a(\eta))\\ &= \hat{g}(m',\eta)\,\hat{g}(m-m',\eta),\end{aligned}$$

or coming back to the origin function

$$g(m,p) = \int_{\mathbb{R}^3} g(m',p')\,g(m-m',p-p')\,dp'.$$

Using that identity in the first (gain) term in (4.4), we get

$$\begin{aligned}& m^{-3\theta}\varphi\left(\frac{p}{m^\theta}\right)\int_0^m F(t,m-m')F(t,m')\,a(m-m',m')\,dm' =\\ &= g(m,p)\int_0^m F(t,m-m')F(t,m')\,a(m-m',m')\,dm'\\ &= \int_{\mathbb{R}^3}\int_0^m F(t,m-m')g(m-m',p-p')\,F(t,m')g(m',p')\,a(m-m',m')\,dm'dp'\\ &= \int_{\mathbb{R}^3}\int_0^m f(t,m-m',p-p')\,f(t,m',p')\,a(m-m',m')\,dm'dp'.\end{aligned} \tag{4.6}$$

We conclude that f satisfies 1.1), (1.3), (1.4) by gathering (4.4), (4.5) and (4.6). $\square$

The previous Theorem is useful in order to prove the existence of self similar solutions for some kernels $a(m,m')$ as it is seen in the following corollary.

Corollary 4.3 *Suppose that a and θ are as in Theorem 4.1. Assume further that F is a self similar solution of the coagulation equation with coagulation kernel $a(m,m')$. Then the function f defined by (4.3) is a self similar solution of (1.1), (1.3), (1.4).*

Proof of Corollary 4.3. The hypothesis on F means that for some functions Φ, $\nu(t)$ and $\mu(t)$ it may be written as:

$$F(t,m)=\nu(t)\Phi(\mu(t)\,m).$$

Therefore f is a self-similar function since it may be written as

$$\begin{aligned} f(t,m,p) &= m^{-3\theta}\,\nu(t)\Phi(\mu(t)\,m)\,\varphi\left(\frac{p}{m^\theta}\right) \\ &= \nu(t)\mu(t)^{3\theta}\,(\mu(t)\,m)^{-3\theta}\;\Phi(\mu(t)\,m)\varphi\left(\frac{\mu(t)^\theta\,p}{(\mu(t)m)^\theta}\right) \\ &= \nu(t)\mu(t)^{3\theta}\,\Psi\left(\mu(t)\,m,\mu(t)^\theta\,p\right) \end{aligned}$$

with $\Psi(M,P)=M^{-3\theta}\,\Phi(M)\,\varphi\left(P/M^\theta\right)$. □

Remark 4.4 *(i) Self similar solutions for equation (1.1), (1.3), (1.4) had already been obtained in [7]. They correspond to the case $\theta=1/2$ of the above Corollary.*

(ii) Self similar solutions of the coagulation equation are well known to exist for the cases $a(m,m')=1$, $a(m,m')=m+m'$ and $a(m,m')=m\,m'$. Their existence for several other kernels with homogenetity $\lambda<1$ have been proved in [5] and [6]. In that last case, these self similar solutions are of the form:

$$F(t,m)=t^{-\frac{2}{1-\lambda}}\Phi\left(\frac{m}{t^{\frac{1}{1-\lambda}}}\right). \tag{4.7}$$

We deduce under the assumption of the above Corollary that

$$f(t,m,p)=t^{-\frac{2}{1-\lambda}}m^{-3\theta}\,\Phi\left(\frac{m}{t^{\frac{1}{1-\lambda}}}\right)\,\varphi\left(\frac{p}{m^\theta}\right). \tag{4.8}$$

is a self similar solutions to equation (1.1), (1.3), (1.4) for the same kernel $a(m,m')$. A straightforward calculation yields

$$P_k(t)=\int_{\mathbb{R}^d}\int_0^\infty |p|^k\,f(t,m,p)\,dm\,dp=t^{-\frac{1-k\theta}{1-\lambda}}\int_{\mathbb{R}^d}|P|^k\varphi(P)dP\int_0^\infty M^{k\theta}\,\Phi(M)\,dM. \tag{4.9}$$

As a consequence, we have $P_0\to 0$, $P_1\to 0$ and more generally $P_k\to 0$ whenever $k<\theta^{-1}$ but $P_k/P_0\to\infty$ for any $k>0$ and $P_k\to\infty$ whenever $k<\theta^{-1}$. The rough physics interpretation is that the total number of particle decreases, the total impulsion of the gas also decreases, but for instance the mean second moment P_2/P_0 tends to infinity in the large time asymptotic, which is the opposite behavior with respect to the one discussed in Remark 3.10. Here, the behavior is quite similar with the bahavior of the solutions to Smoluchoski equation since the mean impulsion moment $P_k/P_0\to\infty$ for any $k>0$. That makes again a difference with the model discussed in Remark 3.10.

5 The constant case $a = 1$

For the sake of simplicity, we restrict our study to the case $d = 1$. It is likely that it extends to higher dimension $d \in \mathbb{N}^*$.

Theorem 5.1 *Suppose that the initial data f_{in} is even, regular and good decreasing properties. Then (1.1), (1.2), (1.3) (1.4) has a solution given by:*

$$f(t,m,p) = \mathcal{F}^{-1}\left(\mathcal{L}^{-1}F\right)(t,m,p) \tag{5.1}$$

$$F(t,\zeta,\xi) = \frac{H_0^2}{(H_0 + (t/2))^2\left(\frac{1}{F(0,\zeta,\xi)} - \frac{H_0\, t/2}{H_0+(t/2)}\right)}, \tag{5.2}$$

with $H_0 := M_{0,0}(f_{in})^{-1}$ as defined in (2.6). Furthermore, f satisfies

$$t^{5/2}\, f(t, t\, m, \sqrt{t}\, p) \ \rightharpoonup \varphi_\infty(m,p) := \frac{4\, H_0^2}{\sqrt{2\,\pi\,\mathcal{A}\,\mathcal{B}}}\frac{e^{-\frac{2\,H_0^2\,m}{\mathcal{A}}}\, e^{-\frac{\mathcal{A}\,p^2}{2\,\mathcal{B}\,m}}}{\sqrt{m}} \tag{5.3}$$

in the weak sense of measure $\sigma(M^1(Y), C_c(Y))$ as $t \to +\infty$, where

$$\mathcal{A} = H^2(0)\int_0^\infty \int_{\mathbb{R}} m\, f(0,m,p) dp dm, \tag{5.4}$$

$$\mathcal{B} = \frac{H^2(0)}{2}\int_0^\infty \int_{\mathbb{R}} p^2\, f(0,m,p) dp dm. \tag{5.5}$$

Proof of Theorem 5.1 We first notice that the equation (1.1), (1.3) (1.4) is now:

$$\partial_t f(t,m,p) = \frac{1}{2}\int_{\mathbb{R}^d}\int_0^m f(t, m-m', p-p')\, f(t,m',p') dm'\, dp' \\ - f(t,m,p)\int_{\mathbb{R}^d}\int_0^\infty f(t,m',p')\, dm'\, dp'. \tag{5.6}$$

This equation may be explicitly solved using Fourier transform with respect to $p \in \mathbb{R}$ and Laplace transform with respect to $m > 0$. Of course this needs the transform F of the function f to be defined. This has then to be checked once the expression of f is obtained. We thus define

$$F(t,\zeta,\xi) = \int_0^\infty \int_{\mathbb{R}^d} e^{-m\,\zeta}\, e^{-i\,p\,\xi} f(t,m,p)\, dp\, dm. \tag{5.7}$$

We then take formally Fourier and Laplace transforms in (5.6) to obtain the Bernouilli equation:

$$\partial_t F(t,\zeta,\xi) = \frac{1}{2}F^2(t,\zeta,\xi) - M_0(t)\, F(t,\zeta,\xi) \tag{5.8}$$

$$M_0(t) = F(t,0,0). \tag{5.9}$$

We first notice, taking $\zeta = \xi = 0$ in (5.8), that $M_0(t)$ satisfies $\frac{d}{dt}M_0(t) = -\frac{1}{2}M_0^2(t)$ from where

$$M_0(t) = \frac{1}{H_0 + t/2}. \tag{5.10}$$

Classical ODE integration methods lead that the solution of (5.8) is the function $F(t,\zeta,\xi)$ given by (5.2). On the one hand, the function $t \mapsto (H_0\, t/2)/(H_0 + t/2)$ is strictly increasing with limit in infinity equal to H_0, so that for any $\delta \in (0,1)$ there exists $T \in (0,\infty)$

$$\forall\, t \in [0,T] \qquad \frac{H_0\, t/2}{H_0 + t/2} \le H_0\,(1-\delta), \tag{5.11}$$

and on the other hand

$$|F(0,\zeta,\xi)| \le \int_0^\infty \int_{\mathbb{R}} f(0,m,p)\, dmdp = H_0^{-1}. \tag{5.12}$$

Gathering (5.11) and (5.12) the fraction in the right hand side of (5.2) is well defined for all $t > 0$. More precisely for any $t \in [0,T]$

$$\begin{aligned} \left| \frac{1}{F(0,\zeta,\xi)} - \frac{H_0\, t/2}{H_0 + (t/2)} \right| \quad &\ge \quad \left| \frac{1}{F(0,\zeta,\xi)} \right| - \frac{H_0\, t/2}{H_0 + (t/2)} \\ &\ge \quad |F(0,\zeta,\xi)|^{-1} - H_0\,(1-\delta) \ge \delta\; |F(0,\zeta,\xi)|^{-1}, \end{aligned}$$

which implies

$$|F(t,\zeta,\xi)| \le \frac{H_0^2}{\delta\,(H_0 + t/2)^2} |F(0,\zeta,\xi)|. \tag{5.13}$$

As a consequence, any "good" decay and regularity properties of the initial data f_{in} ensure "good" decay and regularity properties of $F(0,\zeta,\xi)$. It is then possible to take the inverse Fourier and Laplace transforms of $F(t,\zeta,\xi)$ to define the function $f(t,m,p)$.
If one is interested in the behaviour of $f(t,m,p)$ as $t \to \infty$ it is a classical argument to consider the rescaled functions φ associated to f by the relation

$$\varphi(t,M,P) := t^{5/2}\, f(t, t\,M, \sqrt{t}\,P), \tag{5.14}$$

so that

$$f(t,m,p) = t^{-5/2}\,\varphi\left(t, \frac{m}{t}, \frac{p}{\sqrt{t}}\right). \tag{5.15}$$

Taking the Fourier and Laplace transform in both side yields

$$F(t,\zeta,\xi) \quad = \quad t^{-1}\Phi(t, t\,\zeta, \sqrt{t}\,\xi) \tag{5.16}$$

with

$$\Phi(t,\zeta,\xi) \quad = \quad \frac{tH_0^2}{(H_0 + (t/2))^2 \left(\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})} - \frac{H_0\, t/2}{H_0 + (t/2)} \right)}. \tag{5.17}$$

Since we are interested in the long time behaviour of $\Phi(\cdot,\zeta,\xi)$ for all ζ and ξ fixed , we may write:

$$\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})} - \frac{H_0\, t/2}{H_0 + (t/2)} \quad = \quad \frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})} - H_0 + \frac{H_0^2}{H_0 + (t/2)}$$

and consider the auxiliary function

$$
\begin{aligned}
\Psi(t,\zeta,\xi) &= \frac{tH_0^2}{((t/2))^2\left(\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})}-H_0+\frac{H_0^2}{(t/2)}\right)} \\
&= \frac{4H_0^2}{t\left(\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})}-H_0+\frac{2\,H_0^2}{t}\right)}. \qquad (5.18)
\end{aligned}
$$

We perform the following expansion up to the order $o(1/t)$:

$$
\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})}-H_0=\frac{\zeta}{t}\frac{\partial F^{-1}}{\partial\zeta}(0,0,0)+\frac{\xi}{\sqrt{t}}\frac{\partial F^{-1}}{\partial\xi}(0,0,0)+\frac{1}{2}\frac{\xi^2}{t}\frac{\partial^2F^{-1}}{\partial\xi^2}(0,0,0)+o\left(\frac{1}{t}\right). \qquad (5.19)
$$

Since by hypothesis f is even with respect to p, we have

$$
\frac{\partial F}{\partial\xi}(0,0,0)=-i\int_0^\infty\int_{\mathbb{R}}f_{in}(m,p)\,p\,dp\,dm=0
$$

and then:

$$
\frac{\partial F^{-1}}{\partial\xi}(0,0,0)=-\frac{1}{F(0,0,0)^2}\frac{\partial F}{\partial\xi}(0,0,0)=0. \qquad (5.20)
$$

We also have

$$
\frac{\partial^2F}{\partial\xi^2}(0,0,0)=-\int_0^\infty\int_{\mathbb{R}}p^2\,f(0,m,p)dpdm,
$$

which with the help of (5.20) implies

$$
\begin{aligned}
\frac{\partial^2F^{-1}}{\partial\xi^2}(0,0,0) &= -F^{-2}(0,0,0)\frac{\partial^2F}{\partial\xi^2}(0,0,0)+F^{-3}(0,0,0)\left(\frac{\partial F}{\partial\xi}(0,0,0)\right)^2 \\
&= H^2(0)\int_0^\infty\int_{\mathbb{R}}p^2\,f(0,m,p)dpdm \quad =2\,\mathcal{B}. \qquad (5.21)
\end{aligned}
$$

Similarly, we compute

$$
\frac{\partial F}{\partial\zeta}(0,0,0) \quad = \quad -\int_0^\infty\int_{\mathbb{R}}m\,f(0,m,p)dpdm,
$$

which implies

$$
\frac{\partial F^{-1}}{\partial\zeta}(0,0,0)=-\frac{1}{F^2(0,0,0)}\frac{\partial F}{\partial\zeta}(0,0,0)=\mathcal{A}. \qquad (5.22)
$$

Thanks to (5.19), (5.20), (5.21) and (5.22), we deduce that (5.18) reads now:

$$
\Psi(t,\zeta,\xi)=\frac{4H^2(0)}{(\zeta\,\mathcal{A}+\xi^2\,\mathcal{B}+2H^2(0)+o(1))}
$$

from where

$$
\lim_{t\to+\infty}\Phi(t,\zeta,\xi)=\lim_{t\to+\infty}\Psi(t,\zeta,\xi)=\frac{4H^2(0)}{\mathcal{A}\,\zeta+\mathcal{B}\,\xi^2+2H^2(0)}=:\Psi_\infty(\zeta,\xi). \qquad (5.23)
$$

In order to come back to the original variables, we recall that from standard integral calculus for any $\mathcal{C}, \mathcal{D} > 0$

$$\frac{1}{(2\pi)^{1/2}} \int_0^\infty \int_{\mathbb{R}} e^{-m\,\zeta}\, e^{-i\,p\,\xi}\, \frac{e^{-\mathcal{C}\,m} e^{-\frac{|p|^2}{2\,\mathcal{D}\,m}}}{\sqrt{\mathcal{D}\,m}}\, dp\, dm = \frac{1}{\zeta + \mathcal{D}\,\xi^2 + \mathcal{C}},$$

from where choosing $\mathcal{C} := 2\,H_0^2/\mathcal{A}$ and $\mathcal{D} := \mathcal{B}/\mathcal{A}$, we obtain

$$\left(\mathcal{F}^{-1}\mathcal{L}^{-1}\right)(\Psi_\infty) = \frac{4\,H_0^2}{(2\pi)^{1/2}\,\mathcal{A}}\, \frac{e^{-\mathcal{C}\,m} e^{-\frac{|p|^2}{2\,\mathcal{D}\,m}}}{\sqrt{\mathcal{D}\,m}} = \varphi_\infty(m,p)$$

as defined in (5.3). Finally, (5.24) implies that $\varphi(t,.) \rightharpoonup \varphi_\infty$ in the weak sens of measure, which is nothing but (5.3). $\square$

The previous Theorem shows the convergence of some of the solutions of (1.1)-(1.4) to a function which is a self similar solution of the equation (1.1), (1.3), (1.4), i.e. a solution of the form

$$f(t,m,p) = t^{-\alpha}\varphi(t^{-1}\,m, t^{-\beta}\,p) \tag{5.24}$$

for some function φ. The numbers α and β define the scaling of the self similar solutions. In the Theorem 5.1 we have $\alpha = 5/2$ and $\beta = 1/2$. It turns out that equation (1.1), (1.3), (1.4) has more than one self similar solution with the same scaling as it is shown in the next Theorem.

Theorem 5.2 *Let $\Phi \in C^1(\mathbb{R}^d)$ is such that*

$$g(y,x) = \mathcal{F}_\xi^{-1}\mathcal{L}_\zeta^{-1}\left(\frac{2}{2\,\zeta\Phi\left(\frac{\xi^2}{\zeta}\right)+1}\right)$$

is well defined for $x \in \mathbb{R}^d$ and $y > 0$. Then

$$t^{-\frac{5}{2}} g\left(\frac{m}{t}, \frac{p}{\sqrt{t}}\right). \tag{5.25}$$

is a self similar solution to (1.1), (1.3), (1.4).

Proof of Theorem 5.2. We look after self similar solutions of the form (5.25). The function g must then solve:

$$-\frac{5}{2}\,g - y\partial_y g - \frac{1}{2}x\partial_x g = \frac{1}{2}\int_{\mathbb{R}}\int_0^y g(y-y', x-x')\, g(y',y')dy'\,dx' - \\ -g\int_{\mathbb{R}}\int_0^\infty g(y',x')dy'\,dx'. \tag{5.26}$$

We integrate this equation with respect to x and y and obtain

$$\int_{\mathbb{R}}\int_0^\infty g(y',x')dy'\,dx' = 2. \tag{5.27}$$

We now Fourier transform with respect to x and Laplace transform with respect to y:

$$\zeta\partial_\zeta\widehat{g}+\frac{1}{2}\xi\partial_\xi\widehat{g}=\frac{1}{2}\widehat{g}^2-\widehat{g}.$$

We divide by $\widehat{g}^2$ and define $G=1/\widehat{g}$:

$$\zeta\partial_\zeta G+\xi\partial_\xi G=G-\frac{1}{2}.$$

The function G may then be any function of the form:

$$G(\zeta,\xi)=\zeta\Phi\left(\frac{\xi^2}{\zeta}\right)+\frac{1}{2}$$

for any arbitrary derivable function Φ. Therefore

$$\widehat{g}(\zeta,\xi)=\frac{2}{2\,\zeta\Phi\left(\frac{\xi^2}{\zeta}\right)+1},\tag{5.28}$$

with, due to (5.27):

$$\lim_{\zeta\to0,\,\xi\to0}\frac{2}{2\,\zeta\Phi\left(\frac{\xi^2}{\zeta}\right)+1}=2\Longleftrightarrow\lim_{\zeta\to0,\,\xi\to0}\zeta\Phi\left(\frac{\xi^2}{\zeta}\right)=0.$$

If we want to define the function g from (5.28) the function Φ must be such that $\widehat{g}$ has an inverse Fourier and Laplace transform. $\square$

Remark 5.3 *If $\Phi(z)=z+1$,*

$$\begin{aligned}\hat{g}(\zeta,\xi)=\frac{2}{2\,\zeta\left(\frac{\xi^2}{\zeta}+1\right)+1}&=\frac{2}{2\,(\xi^2+\zeta)+1}\\&=\mathcal{L}\left(\mathcal{F}\left(\frac{e^{-\frac{y}{2}}\,e^{-\frac{x^2}{4y}}}{\sqrt{2}\sqrt{y}}\right)\right).\end{aligned}$$

This is the profile of the self similar solution which appears in Theorem 5.1. It is easy to obtain particular solutions g, some of them are explicit others are not. If, for example, $\Phi\equiv1$ then $g(y,x)=e^{-y^2}\delta_{x=0}$. Another explicit example is for $\Phi(z)=z$ which gives $g(x,y)=\sqrt{\pi}\delta_{y=0}e^{-\frac{|x|}{\sqrt{2}}}$. On the other hand, if we take $\Phi(z)=\sqrt{z}$, the inverse Laplace transform, let us call it $h(y,\xi)$, is still explicit:

$$h(y,\xi)=\mathcal{L}_\zeta^{-1}\left(\frac{2}{2\sqrt{\zeta\,\xi^2}}+1\right)=\frac{\frac{\sqrt{\xi^2}}{\sqrt{\pi}\,\sqrt{y}}-e^{\frac{y}{\xi^2}}Erfc\left(\sqrt{\frac{y}{\xi^2}}\right)}{4\xi^2}.\tag{5.29}$$

It remains to check that $h(y,\cdot)$ has an inverse Fourier transform with respect to the variable ξ. It is easily checked that, for all $y>0$ fixed:

$$\begin{aligned}h(y,\xi)&=\mathcal{O}\left(\frac{\xi}{y^{3/2}}\right),\ \text{ as }\xi\to0\\h(y,\xi)&=\frac{\frac{1}{\sqrt{\pi}}\sqrt{\frac{\xi^2}{y}}-1}{4\xi^2}+\mathcal{O}\left(\frac{y}{\xi^2}\right)\ \text{ as }|\xi|\to+\infty.\end{aligned}$$

This function is then in $L^2(\mathbb{R})$ with respect to the ξ variable and has then an inverse Fourier transform with respect to ξ which is $g(y,x)$:

$$g(y,x) = \mathcal{F}_\xi^{-1}(h(y,\cdot))(x).$$

Moreover, for all $y > 0$, $g(y,\cdot) \in L^2(\mathbb{R})$ and the convolution of $g(y,\cdot)$ with itself is well defined

$$\mathcal{F}\left(g(y-y',\cdot) * g(y',\cdot)\right)(\xi) = h(y-y',\xi)\, h(y',\xi)$$

and

$$\int_0^y \mathcal{F}\left(g(y-y',\cdot) * g(y',\cdot)\right)(\xi) dy = \int_0^y h(y-y',\xi)\, h(y',\xi)\, dy.$$

Therefore,

$$\int_{\mathbb{R}} \left|\mathcal{F}\left(g(y-y',\cdot) * g(y',\cdot)\right)(\xi)\right| d\xi \quad \leq \quad \int_0^y \int_{\mathbb{R}} \left|h(y-y',\xi)\, h(y',\xi)\right| d\xi\, dy = \sum_{k=1}^{6} I_k,$$

with

$$\begin{aligned}
I_1 &:= \int_0^{y/2} \int_{|\xi| \leq y'^{1/2} \leq (y-y')^{1/2}} \left|h(y-y',\xi)\, h(y',\xi)\right| d\xi\, dy', \\
I_2 &:= \int_0^{y/2} \int_{y'^{1/2} \leq |\xi| \leq (y-y')^{1/2}} \left|h(y-y',\xi)\, h(y',\xi)\right| d\xi\, dy', \\
I_3 &:= \int_0^{y/2} \int_{y'^{1/2} \leq (y-y')^{1/2} \leq |\xi|} \left|h(y-y',\xi)\, h(y',\xi)\right| d\xi\, dy', \\
I_4 &:= \int_{y/2}^{y} \int_{|\xi| \leq (y-y')^{1/2} \leq y'^{1/2}} \left|h(y-y',\xi)\, h(y',\xi)\right| d\xi\, dy', \\
I_5 &:= \int_{y/2}^{y} \int_{(y-y')^{1/2} \leq |\xi| \leq y'^{1/2}} \left|h(y-y',\xi)\, h(y',\xi)\right| d\xi\, dy', \\
I_6 &:= \int_{y/2}^{y} \int_{(y-y')^{1/2} \leq y'^{1/2} \leq |\xi|} \left|h(y-y',\xi)\, h(y',\xi)\right| d\xi\, dy'.
\end{aligned}$$

We must verify that each term is finite. Indeed, we have

$$\begin{aligned}
I_1 &\leq C \int_0^{y/2} \int_{|\xi| \leq y'^{1/2} \leq (y-y')^{1/2}} \frac{\xi^2}{y'^{3/2}(y-y')^{3/2}} d\xi\, dy' \\
&\leq C \int_0^{y/2} \frac{min\{y'^{3/2},\, (y-y')^{3/2}\}}{y'^{3/2}(y-y')^{3/2}} dy' < \infty; \\
I_2 &\leq C \int_0^{y/2} \int_{y'^{1/2} \leq |\xi| \leq (y-y')^{1/2}} \frac{|\xi|}{(y-y')^{3/2}} \left(\frac{1}{\sqrt{y'}\,|\xi|} + \frac{1}{\xi^2} + \mathcal{O}\left(\frac{y'}{\xi^2}\right)\right) d\xi\, dy' \\
&\leq \frac{C}{y^{3/2}} \int_0^{y/2} \int_{y'^{1/2} \leq |\xi| \leq (y-y')^{1/2}} \left(\frac{1}{\sqrt{y'}} + \frac{1}{|\xi|} + 1\right) d\xi dy \\
&\leq \frac{C}{y^{3/2}} \int_0^{y/2} \int_{y'^{1/2} \leq |\xi| \leq (y-y')^{1/2}} \left(\frac{2}{\sqrt{y'}} + 1\right) d\xi dy < \infty;
\end{aligned}$$

$$
\begin{aligned}
I_3 &\le C\int_0^{y/2}\int_{y'^{1/2}\le (y-y')^{1/2}\le|\xi|}\left(\frac{1}{\sqrt{y'}\,|\xi|}+\frac{1}{\xi^2}+\mathcal{O}\left(\frac{y'}{\xi^2}\right)\right)\times\\
&\quad\times\left(\frac{1}{\sqrt{y-y'}\,|\xi|}+\frac{1}{\xi^2}+\mathcal{O}\left(\frac{y-y'}{\xi^2}\right)\right)d\xi\,dy'\\
&\le C\int_0^{y/2}\int_{y'^{1/2}\le (y-y')^{1/2}\le|\xi|}\left(\frac{1}{\sqrt{y'}\sqrt{y-y'}\,|\xi|^2}+\frac{1}{|\xi|^3}\left(\frac{1}{\sqrt{y-y'}}+\frac{1}{\sqrt{y'}}\right)+\right.\\
&\quad\left.+\frac{1}{\xi^4}+\mathcal{O}\left(\frac{y'}{\sqrt{y-y'}|\xi|^3}\right)+\mathcal{O}\left(\frac{y-y'}{\sqrt{y'}|\xi|^3}\right)+\mathcal{O}\left(\frac{y+y^2}{|\xi|^4}\right)\right)d\xi dy\\
&\le \frac{C}{\sqrt{y}}\int_0^{y/2}\frac{1}{\sqrt{y'}}\int_{y'^{1/2}\le (y-y')^{1/2}\le|\xi|}\frac{d\xi}{|\xi|^2}\,dy+\\
&\quad +C\int_0^{y/2}\left(\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{y'}}\right)\int_{y'^{1/2}\le (y-y')^{1/2}\le|\xi|}\frac{d\xi}{|\xi|^3}\,dy+\\
&\quad +C\int_0^{y/2}\int_{y'^{1/2}\le (y-y')^{1/2}\le|\xi|}\frac{d\xi}{|\xi|^4}\,dy\\
&\le \frac{C}{\sqrt{y}}\int_0^{y/2}\frac{dy}{\sqrt{y'}\,\sqrt{(y-y')}}+C\int_0^{y/2}\left(\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{y'}}\right)\frac{dy}{y-y'}+C\int_0^{y/2}\frac{dy}{(y-y')^2}.
\end{aligned}
$$

Similar estimates show that the integrals I_4, I_5 and I_6 converge. The function $\int_0^y h(y-y',\xi)\,h(y',\xi)\,dy$ is then in $L^1(\mathbb{R})$ and has then an inverse Fourier transform which is

$$
\int_0^y (g(y-y',\cdot)*g(y',\cdot))(\xi)\,dy.
$$

References

[1] G. F. Carnevale, Y. Pomeau, W. R. Young, *Statistics of Ballistic Agglomeration*, Phys. Rev. Lett., **64**, (1990), 2913-2916.

[2] M. Escobedo, Ph. Laurençot, S. Mischler, *On a kinetic equation for coalescing particles,* Comm. in Maths. Phys. **246** (2004); 237–267.

[3] M. Escobedo, S. Mischler, *On a Quantum Boltzmann equation for a gas of photons*, J. Math. Pures Appl. **80** No 5 (2001), 471-515

[4] M. Escobedo, S. Mischler, *On self-similarity for the coagulation equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2006), no. 3, 331–362

[5] M. Escobedo, S. Mischler, M. R. Ricard, *On self-similarity and stationary problem for fragmentation and coagulation models*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005); 99-125.

[6] N. Fournier, Ph. Laurençot, *Existence of self-similar solutions to Smoluchowski's coagulation equation.* Comm. Math. Phys., **256** (3) (2005) 589-609.

[7] N. Fournier, S. Mischler, *A Boltzmann equation for elastic, inelastic, and coalescing collisions.* J. Math. Pures Appl., **84** (9) (2005): 1173-1234.

[8] A.V. Bobylev and R. Illner, *Collision integrals for attractive potentials*, J. Statist. Phys. **95** (1999), 633–649.

[9] Y. Jiang, F. Leyvraz, *Scaling theory for ballistic aggregation.*, J. Phys. A: Math. Gen., **26**, (1993), L179-L186.

[10] Ph. Laurençot, S. Mischler, *On coalescence equations and related models*, Modeling and computational methods for kinetic equations, 321–356, Model. Simul. Sci. Eng. Technol., Birkhuser Boston, Boston, MA, 2004.

[11] S. Mischler, C. Mouhot, *Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres*, Comm. Math. Phys., (3) 287, (2009)

[12] S. Mischler, B. Wennberg, *On the Spatially Homogeneous Boltzmann equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **16** No 4 (1999), 467-501

[13] J.R. Norris, *Smoluchowski's coagulation equation: uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent*, Ann. Appl. Probab. 9 (1999), no. 1, 78–109.

[14] J.M. Roquejoffre and Ph. Villedieu, *A kinetic model for droplet coalescence in dense sprays*, Math. Models Methods Appl. Sci. **11** (2001), 867–882.

[15] M. Smoluchowski, *Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen*, Zeitschrift f. physik. Chemie **92** (1917), 129–168.

[16] E. Trizac, P.L. Krapivsky, *Correlations in ballistic processes*, Phys. Rev. Lett., **91**, n. 21, (2003).

[17] G. Wetherill, *The Formation and Evolution of Planetary Systems* (Cambridge: Cambridge University Pres), 1988.