An Efficient Algorithm for the Computation of Optimum Paths in Weighted Graphs

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Abstract. The goal of this paper is to identify a more efficient algorithm for the computation of the path of minimum ratio (i.e. the quotient of the weight divided by the length) in a weighted graph. The main application of this technique is to improve the efficiency of reachability analysis for flat counter automata with Difference Bound Matrix constraints on transitions. A previous result showed that these paths could be defined using Presburger arithmetic. However, this method is costly, as the complexity of deciding satisfiability of Presburger formulae has a double exponential lower bound. Our solution avoids the use of Presburger arithmetic, by computing the function between any $n \in \mathbb{N}$ and the weight of the minimal path of length n. The function is computed iteratively, by computing the minimal fixed point of a system of set constraints, involving semilinear sets. This requires a min operator on linear sets, which is implemented using rewrite rules.

1 Introduction

Flat counter automata [6], [8], [3] have received a lot of attention recently, as they represent an important class of infinite-state systems, for which the reachability and termination problems are decidable. These results have been used in a number of successful verification tools, e.g. FAST [2], LASH [9] or TREX [1].

Comon and Jurski show in [6] that the reachability problem for a flat counter automaton can be expressed in Presburger arithmetic, given that the automata have transition guards that are conjunctions of relations of the form $x - y \leq c$, where x and y denote either the current or the future (primed) values of the counters, and c is an integer constant. These constraints are known as Difference Bound Matrix (DBM). To our knowledge, their result concerns the most general class of flat counter automata, considered so far.

Recent work by Bozga, Iosif and Lakhnech [4] addresses the same class of counter automata as the one in [6], but presents a simplified proof of decidability, based on the translation of a control loop with a DBM constraint into a weighted automaton, such that, for each pair of counters x and y of the original loop, we have $x^0 - y^n \leq w$, where w is the weight of the minimum weight path of length n between given states of the weighted automaton. Moreover, it can be shown that the minimal weight is a linear function of the length n of the path. Computing this weight, as a function of the length, i.e. $w = \alpha \cdot n + \beta$, for some $\alpha, \beta \in \mathbb{Z}$ is crucial in order to express the global relation between the input and output values of the counters. However, for the purposes of the proof in [4], full Presburger arithmetic was used in order to define this function.

The goal of this paper is therefore to present a more efficient method for the computation of the minimum weight function in a given weighted graph. We introduce a new approach to this problem, which overpasses the method based on Presburger arithmetic. Our method is based on the computation of the least fixed point of a system of set constraints on semilinear sets. This computation is shown to terminate if an entire elementary cycle is considered at the time, rather than just single edges. Since the number of elementary cycles is exponential in the number of nodes, we consider only the elementary cycles of minimum ratio (= the weight of the cycle divided by its length). It can be shown that the number of optimal cycles is polynomial in the number of graph nodes. Moreover, a polynomial-time algorithm for finding these elementary cycles has been reported in [5]. In order for the solution of the system of set constraints to be a function, i.e. a set of pairs $\langle n, w \rangle$ that is a functional relation, we use a $\min(X)$ operator that returns, for any semilinear set X the set of minimal points. For efficiency reasons, this operator has been implemented using a set of rewriting rules, that work on semilinear sets.

The techniques described in this paper have been implemented, the result being a tool for the analysis of flat counter systems, which is to be released soon.

2 Notation and Definitions

For the rest of the paper, let us consider a fixed weighted graph $G = \langle V, E, w \rangle$, where:

- V is a finite set of nodes,
- $E \subseteq V \times V$ the set of edges.
- $w : E \to \mathbb{Z}$ is the weight function.

A path π in a graph is a sequence of nodes $v_0, v_1, ..., v_n$ such that $\pi = (v_0, v_1, ..., v_n)$, where $(v_i, v_{i+1}) \in E, \forall i = 1, ..., n$. The length $|\pi|$ is equal to the number of edges of the path. The weight $w(\pi)$ is equal to the sum of the weights on the path. A path between two nodes s and t is called *optimal* of length m if there is no other path of length m and with smaller weight.

A cycle c is a path such that the start node and the end node are the same. The cycle mean $\rho(c)$ is defined as $\frac{w(c)}{|c|}$. A cycle is called elementary if there is a sequence of nodes in which there is never twice the same node. An elementary cycle is called *critical* if it has the minimum cycle mean amongst all the elementary cycles of the graph. A node is critical if it belongs to a critical cycle.

A linear set $a_0 + a_1\mathbb{N} + ... + a_n\mathbb{N}$ where $a_0, a_1, ..., a_n \in \mathbb{Z}^k$ denote the set of all points $x \in \mathbb{Z}^k$ of the form $x = a_0 + \sum_{i=1}^{n} x_i a_i$, where $x_i \in \mathbb{N}$. A semilinear set is a finite union of linear sets.

Given two sets $A, B \subseteq \mathbb{Z}^k$, we define $A \oplus B = \{a + b | a \in A, b \in B\}$. It could be shown that if A and B are semilinear sets, then $A \oplus B$ does so.

3 The Problem

Let $u, v \in V$ be two nodes in the graph, and P_{uv} be the relation between lengths and weights of paths from u to v: $P_{uv} = \{(m, w) \mid \exists \pi$ from u to v such that $|\pi| = m, w(\pi) = w\}$. This set is semilinear, for any nodes u and v. This could be proved either directly, or by using the Presburger formula defined in [4]. Now, the set of optimal paths from u to v is:

$\min P_{uv} = \{(m, w) \mid (m, w) \in P_{uv} \text{ and } \forall (m, w') \in P_{uv} \Rightarrow w \leq w'\}$

This set is also semilinear, given that P_{uv} is semilinear.

Our aim is to effectively compute $\min P_{st}$ for some given s and t by overpassing the construction in [4] which produces a Presburger formula that is too complex to be useful in practice.

4 Our Solution

4.1 The Initial Approach

Let $s \in V$ be a fixed node in the graph.

The sets P_{sv} are the least fixpoint solution with $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in P_{ss}$ of the following system of equations, for all v:

$$P_{sv} = \bigcup_{e=(u,v)\in E} P_{su} \oplus \begin{pmatrix} 1 \\ w(e) \end{pmatrix}$$

These equations give a direct way to compute the sets P_{sv} as the limit of the sequences $P_{sv}^{(k)}$ defined as follows:

$$P_{sv}^{(k+1)} = \bigcup_{e=(u,v)\in E} P_{su}^{(k)} \oplus \begin{pmatrix} 1 \\ w(e) \end{pmatrix}$$

started with $P_{ss}^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $P_{sv}^{(0)} = \emptyset$ for all $v \neq s$. Unfortunately, the computation above cannot be carried out iteratively because the limit cannot be reached in a finite number of steps, if there are cycles in the graph.

Nevertheless, we can extend the equations above to take into account the elementary cycles(EC), as follows:

$$P_{sv} = \bigcup_{e=(u,v)\in E} P_{su} \oplus \begin{pmatrix} 1 \\ w(e) \end{pmatrix} \oplus \bigoplus_{c\in EC(v)} \begin{pmatrix} |c| \\ w(e) \end{pmatrix} \mathbb{N}$$

Now, it can be shown that by iterating these equations, we always reach the limit in a finite number of steps (cf. Theorem 1). The second step is to extract the set of minimal points out of a semilinear set (cf. Section 4.3).

4.2 The Improved Approach

A problem with the previous approach is that it requires to compute and use *all* the elementary cycles in the graph. Or, it is well known that their number is at worst exponential.

Since we are interested in computing the optimal paths, we restrict ourselves to elementary critical cycles(ECC) only and optimal paths. That is, we attempt to compute sets X_{sv} as the least fixpoint solution with $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in X_{ss}$ to the system of equations, for all v:

$$X_{sv} = \min \left(\bigcup_{e=(u,v)\in E} X_{su} \oplus \begin{pmatrix} 1 \\ w(e) \end{pmatrix} \oplus \bigoplus_{c \in ECC(v)} \begin{pmatrix} |c| \\ w(c) \end{pmatrix} \mathbb{N} \right)$$

As before, we compute the sets X_{sv} as the limit of sequences $X_{sv}^{(k)}$ defined as follows:

$$X_{sv}^{(k+1)} = \min \left(\bigcup_{e=(u,v)\in E} X_{su}^{(k)} \oplus \begin{pmatrix} 1 \\ w(e) \end{pmatrix} \oplus \bigoplus_{c \in ECC(v)} \begin{pmatrix} |c| \\ w(c) \end{pmatrix} \mathbb{N} \right)$$

started with $X_{ss}^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $X_{sv}^{(0)} = \emptyset$ for all $v \neq s$.

The following theorem states the correctness of the improved approach:

Theorem 1. *Given a weighted graph $G = \langle V, E, w \rangle$, for all pairs of nodes $u, v \in V$ we have:*

- *[correctness] the limit set X_{uv} is semilinear, and $X_{uv} = \min P_{uv}$,*
- *[termination] the sequence $X_{uv}^{(k)}$ converges in a finite number of steps.*

To calculate the critical cycles, we use the Howard's algorithm [7], which is known to perform in a remarkable *almost linear* average execution time.

4.3 Implementing the min Operator

The previous approaches requires the min operator to be implemented on semilinear sets X. The formal definition of this operator is: $\min X = \{(m, w) \mid (m, w) \in X \text{ and } \forall (m, w') \in X \Rightarrow w \leq w'\}$

This operator is effectively implemented by the set of rewriting rules given in table 1.

The rule (1) applies to linear sets having more than one generator and reduces their number by one. In this transformation, the optimal generator is kept (the

$$(1)\qquad Y + \binom{m_1}{w_1}\mathbb{N} + \binom{m_2}{w_2}\mathbb{N} \;\rightsquigarrow\; \bigcup_{k=0}^{M/m_2-1} Y + \binom{m_1}{w_1}\mathbb{N} + \binom{km_2}{kw_2}$$

$[w_1/m_1 \leq w_2/m_2, \quad M = lcm(m_1, m_2)]$

$$(2)\qquad \binom{m_0}{w_0} + \binom{m_1}{w_1}\mathbb{N} \cup \binom{m_0'}{w_0'} + \binom{m_2}{w_2}\mathbb{N} \;\rightsquigarrow\; \bigcup_{k=0}^{M/m_1-1} \binom{m_0+km_1}{w_0+kw_1} + \binom{M}{w_1M/m_1}\mathbb{N} \;\cup\; \bigcup_{k=0}^{M/m_2-1} \binom{m_0'+km_2}{w_0'+kw_2} + \binom{M}{w_2M/m_2}\mathbb{N}$$

$[m_1 \neq m_2, \quad gcd(m_1, m_2)|(m_0 - m_0'), \quad M = lcm(m_1, m_2)]$

$$(3)\qquad \binom{m_0}{w_0} + \binom{m}{w_1}\mathbb{N} \cup \binom{m_0'}{w_0'} + \binom{m}{w_2}\mathbb{N} \;\rightsquigarrow\; \bigcup_{k=0}^{n_1} \binom{m_0+km}{w_0+kw_1} \cup \bigcup_{k=0}^{n_2} \binom{m_0'+km}{w_0'+kw_2} \cup \binom{m_0+(n_1+1)m}{w_0+(n_1+1)w_1} + \binom{m}{w_1}\mathbb{N}$$

$[m|(m_0 - m_0'), \quad w_1 < w_2, \quad n_1 = max(0, [((w_0 - w_0') + w_2(m_0' - m_0)/M)/(w_2 - w_1)])]$
$[\quad n_2 = max(0, [((w_0 - w_0') + w_1(m_0' - m_0)/M)/(w_2 - w_1)])]$

$$(4)\qquad \binom{m_0}{w_0} + \binom{m}{w}\mathbb{N} \cup \binom{m_0'}{w_0'} + \binom{m}{w}\mathbb{N} \;\rightsquigarrow\; \bigcup_{k=0}^{n-1} \binom{m_0+km}{w_0+kw} \cup \binom{m_0'}{w'} + \binom{m}{w}\mathbb{N}$$

$[m|(m_0 - m_0'), \quad m_0' \geq m_0, \quad n = (m_0' - m_0)/m, \quad w' = min(w_0', w_0 + nw)]$

$$(5)\qquad \binom{m_0}{w_0} + \binom{m}{w}\mathbb{N} \cup \binom{m_0'}{w_0'} \;\rightsquigarrow\; \binom{m_0}{w_0} + \binom{m}{w}\mathbb{N}$$

$[m|(m_0 - m_0'), \quad m_0' \geq m_0, \quad w_0' \geq w_0 + w(m_0' - m_0)/m]$

$$(6)\qquad \binom{m_0}{w_0} + \binom{m}{w}\mathbb{N} \cup \binom{m_0'}{w_0'} \;\rightsquigarrow\; \bigcup_{k=0}^{n-1} \binom{m_0+km}{w_0+kw} \cup \binom{m_0'}{w_0'} \cup \binom{m_0+(n+1)m}{w_0+(n+1)w} + \binom{m}{w}\mathbb{N}$$

$[m|(m_0 - m_0'), \quad m_0' \geq m_0, \quad w_0' < w_0 + w(m_0' - m_0)/m, \quad n = (m_0' - m_0)/m]$

$$(7)\qquad \binom{m_0}{w_0} \cup \binom{m_0}{w_0'} \;\rightsquigarrow\; \binom{m_0}{w_0}$$

$[w_0 \leq w_0']$

Table 1. Rewriting rules for the min operator

one having the best ration w/m) while the others are taken only a finite number of times.

The rules (2-4) apply to a semilinear sets containing precisely two overlapping linear sets with one generator each (i.e., sequences). The rule (2) handles the case where the generators have distinct periods (m values) and simply 'split' them in order to obtain a common period (the least common multiple of both periods). Then, rules (3-4) select the minimum amongst two overlapping linear sets having the same period. Notice that rule (3) applies for the case where the generator weights are different, and rule (4) applies for the case where the generator weights are the same.

Rules (5-6) apply to semilinear sets containing precisely two overlapping linear sets, one with no generator (i.e, an isolated point) and one with some generator (a sequence). Rule (5) handles the case where the isolated point has a bigger value than the one provided by the sequence, and rule (6) the converse. Finally, rule (7) applies for two isolated points choosing the minimal one.

The following theorem states the corectness of this method:

Theorem 2. *Let X be a semilinear set in two dimensions (m, w):*

1. *[soundness] If X rewrites to X' using one of the rules above then $\min X = \min X'$.*
2. *[completness] If X cannot be rewritten by any of the rules then $\min X = X$;*
3. *[termination] Any derivation of X using the rules above eventually terminates in a finite number of steps.*

5 An Example

Consider the graph G given in figure 1. We are interested to obtain optimal paths from the node 1 to all other nodes in the graph.

The system of equations is the following:

$$
\begin{aligned}
X_{11} &= \min\left\{X_{12} \oplus \begin{pmatrix}1\\3\end{pmatrix}\right\} \oplus \begin{pmatrix}2\\4\end{pmatrix}\mathbb{N}\\
X_{12} &= \min\left\{X_{11} \oplus \begin{pmatrix}1\\1\end{pmatrix}\right\} \oplus \begin{pmatrix}2\\4\end{pmatrix}\mathbb{N}\\
X_{13} &= \min\left\{\left(X_{11} \oplus \begin{pmatrix}1\\2\end{pmatrix}\right) \bigcup \left(X_{13} \oplus \begin{pmatrix}1\\2\end{pmatrix}\right) \bigcup \left(X_{14} \oplus \begin{pmatrix}1\\2\end{pmatrix}\right)\right\} \oplus \begin{pmatrix}2\\3\end{pmatrix}\mathbb{N}\\
X_{14} &= \min\left\{X_{13} \oplus \begin{pmatrix}1\\1\end{pmatrix}\right\} \oplus \begin{pmatrix}2\\3\end{pmatrix}\mathbb{N}\\
X_{15} &= \min\left\{X_{14} \oplus \begin{pmatrix}1\\2\end{pmatrix}\right\}
\end{aligned}
$$

After solving the above system, the minimal solution is the following:

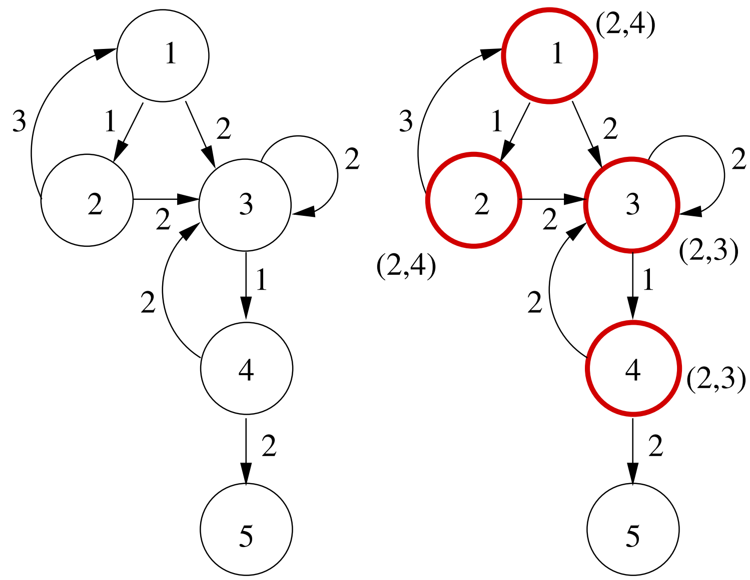

Fig. 1. (a) an example graph (b) the graph annotated with critical nodes and critical cycles

$$X_{11} = \left\{ \binom{2}{4} \mathbb{N} \right\}$$
$$X_{12} = \left\{ \binom{1}{1} + \binom{2}{4} \mathbb{N} \right\}$$
$$X_{13} = \left\{ \binom{1}{2} + \binom{2}{3} \mathbb{N}, \binom{2}{3} + \binom{2}{3} \mathbb{N} \right\}$$
$$X_{14} = \left\{ \binom{2}{3} + \binom{2}{3} \mathbb{N}, \binom{3}{4} + \binom{2}{3} \mathbb{N} \right\}$$
$$X_{15} = \left\{ \binom{3}{5} + \binom{2}{3} \mathbb{N}, \binom{4}{6} + \binom{2}{3} \mathbb{N} \right\}$$

6 Conclusions

We have addressed the problem of finding minimal ratio paths in a weighted graph. This problem is of interest to the verification community, as it provides an effective method for the analysis of flat counter automata with DBM constraints, which is one of the most general class of infinite state systems, known to be decidable. Our solution relies on computing the least fixed point of a system of set constraints. This computation is accelerated by adding one elementary cycle at the time, and using a performant algorithm to select the elementary cycles of minimal ratio. Then computing the minimum function is done by applying exhaustively a set of rewriting rules. The techniques presented have been implemented in a tool for the analysis of flat counter automata which is to be released soon.

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