A kinetic formulation for multidimensional scalar conservation laws with boundary conditions and applications

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A KINETIC FORMULATION FOR MULTIDIMENSIONAL SCALAR CONSERVATION LAWS WITH BOUNDARY CONDITIONS AND APPLICATIONS

C. IMBERT* AND J. VOVELLE†

Abstract. We state a kinetic formulation of weak entropy solutions of a general multidimensional scalar conservation law with initial and boundary conditions. We first associate with any weak entropy solution a entropy defect measure; the analysis of this measure at the boundary of the domain relies on the study of weak entropy sub and supersolutions and implies the introduction of the notion of sided boundary defect measures. As a first application, we prove that any weak entropy subsolution of the initial-boundary value problem is bounded above by any weak entropy supersolution (Comparison Theorem). We next study a BGK-like kinetic model that approximates the scalar conservation law. We prove that such a model converges by adapting the proof of the Comparison Theorem.

Key words. Conservation law, initial-boundary value problem, boundary defect measures, kinetic traces, weak entropy sub and supersolutions, Comparison Theorem, generalized kinetic solutions, BGK-like kinetic model.

AMS subject classifications. 35L65, 35B50, 35D99, 35F25, 35F30, 35A35

1. Introduction. Let Ω be a strong Lipschitz open subset of $\mathbb{R}^d$. Let $\partial\Omega$ denote its boundary, $n(\overline{x})$ the outward unit normal to Ω at a point $\overline{x} \in \Omega$, $Q = (0, +\infty) \times \Omega$ and $\Sigma = (0, +\infty) \times \partial\Omega$. We consider the following multidimensional scalar conservation law:

$$\partial_t u + \operatorname{div}_x A(u) = 0 \text{ in } Q, \tag{1.1a}$$

with the initial condition:

$$u(0, x) = u_0(x), \forall x \in \Omega, \tag{1.1b}$$

and the boundary condition:

$$u(s, y) = u_b(s, y), \forall (s, y) \in \Sigma. \tag{1.1c}$$

The first step in the understanding of (1.1c) is the work of Bardos, Le Roux and Nédelec [1]: they show that if the initial datum u_0 is BV and the boundary datum is C^2-regular, there exists a unique (weak entropy) solution of (1.1). In particular, they show that an inequality must hold at the boundary. This inequality is known as the BLN condition (see (3.19)). Note that the BLN condition does make sense only if the solution u admits a trace on $\partial\Omega$. In the case of the Cauchy problem with merely essentially bounded (L^∞) data, some notions of generalized solution have been defined. The measure-valued entropy solutions were introduced by DiPerna [9], the entropy process solutions by Eymard, Gallouët and Herbin [11]. These notions of very weak solution are well adapted to the study of the convergence of numerical schemes and error estimates are also available. In the case of the Cauchy-Dirichlet problem with L^∞ data, Otto [25] proposed a notion of *weak entropy solution* $u \in \mathsf{L}^\infty(Q)$, relying on the notion of boundary entropy-flux pairs. An equivalent definition can be given

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by using "Kružkov semi-entropies" (see [8, 30, 34]). An accurate notion of entropy process solution can be given in order to prove the convergence of certain numerical methods [34] but it does not seem possible to get an error estimate with respect to the approximation by vanishing viscosity for example. In order to fill this gap, we follow the ideas developed by Lions, Perthame and Tadmor [18]. Their heuristic idea, which is, in part, a continuation of the works of Brenier [7] and Di Perna [9], is to take into account the decrease of the entropy by introducing an "entropy defect" measure. More precisely, a kinetic function f is associated with the macroscopic function u by setting:

$$f(t,x,\xi) = \begin{cases} 1 & \text{if } 0 < \xi < u(t,x), \\ -1 & \text{if } u(t,x) < \xi < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Such a kinetic function is a so-called equilibrium function. The kinetic formulation of Lions, Perthame and Tadmor states that u is a weak entropy solution of the conservation law if and only if there exists a bounded nonnegative measure m such that:

$$(\partial_t + a.\nabla_x) f = \partial_\xi m \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^d\times\mathbb{R}). \tag{1.3}$$

Next, Perthame [27] showed that these techniques supply a good technical framework to easily prove, for instance, the L^1-contraction property and the error estimate w.r.t. the parabolic approximation, without relying on the dedoubling variable technique.

We start from [27] and we develop analogous techniques for a conservation law with boundary conditions. The main difficulty is to study how the weak entropy solution u and the defect measure m behave at the boundary of the domain. We handle this difficulty by considering the space kinetic trace f^τ of the kinetic function f [32, 33]. As far as the defect measure is concerned, two nonnegative measures $m^b_\pm$ supported by $\Sigma\times\mathbb{R}_\xi$ must therefore be considered. They are characterized by the formula:

$$(-a\cdot n) f^\tau = M f^b + (-a\cdot n)\mathrm{sgn}_\mp + \partial_\xi m^b_\pm \tag{1.4}$$

where the constant M is a Lipschitz constant of the flux A on a compact subset of $\mathbb{R}$ in which the data u_0 and u^b, which are supposed to be measurable essentially bounded functions, take a.e. their values (see Section 2). Relation (1.4) can be understood as a kinetic analogue of the BLN condition[1]. Why do we need two nonnegative measures to describe the behaviour of the entropy defect measure at the boundary ? Because the notion of weak entropy solution is "sided". Let us be more specific. We define weak entropy sub and supersolutions for the initial-boundary value problem and we give a kinetic formulation of them. Hence two different defect measures $m_\pm$ are *a priori* associated with each weak entropy solution. But eventually, we prove they coincide in $Q\times\mathbb{R}_\xi$ and can be different at the boundary. Notion of weak entropy sub and supersolutions for the Cauchy problem were previously considered [2, 15, 3, 4] and comparison principle were established: any weak entropy subsolution of the Cauchy problem is bounded above by any weak entropy supersolution. Such results have also been proved by Terracina [31] for the initial-boundary value problem in the context of BV solutions. We state and prove an analogous result for the initial-boundary

[1] It is a generalization of it in the sense that no strong traces are required; thus merely L^∞ data can be treated

value problem in the context of L^∞ solutions. The L^1-contraction property and the maximum principle follow from it.

We then use our results to study an approximation of the conservation law, namely a kinetic model "à la Bhatnagar-Gross-Krook" (BGK-like kinetic model for short). It was first introduced by Perthame and Tadmor [29] for the Cauchy problem and adapted by Nouri, Omrane and Vila [22, 23] to the initial-boundary value problem. The authors Nouri, Omrane and Vila prove the convergence of the BGK-like kinetic model whenever the data are at equilibrium or not. Here, we restrict our study to the case where the data are at equilibrium and show how, in this framework, the concept of *generalized kinetic solution* can bu used to prove the convergence of the BGK-like kinetic model. Such very weak solutions were introduced by Perthame [28] for the Cauchy problem. They can be viewed as the analogue of the measure-valued solutions of DiPerna [9] or the entropy process solutions of Eymard, Gallouët and Herbin [11]. The definition of a generalized kinetic solution is based on the kinetic formulation: instead of considering an equilibrium function, a solution can be a general kinetic function (See Sections 2 and 5 for precise definitions). The proof of the Comparison Theorem is slightly modified in order to prove that there is at most one generalized kinetic solution of (1.1) and that it is in fact a weak entropy solution. Hence, it permits us to easily pass to the limit in the kinetic model.

To conclude this introduction, let us mention the recent work of Ben Moussa and Szepessy [6] in which is used the concept of measure-valued solution to deal with "very weak solutions" and let us state some other occurence of "kinetic methods" in the study of first order problems with boundary conditions [5, 20], see also [21, 13, 14].

The paper is organized as follows. Section 2 is devoted to notations and assumptions. In Section 3, kinetic formulations of weak entropy solutions (Theorem 3.1) and entropy semisolutions (Proposition 3.3) are stated and proved. In particular, kinetic traces and boundary defect measures are constructed and characterized (Proposition 3.4). In Section 4, the Comparison Theorem (Theorem 4.1) is proved. Section 5 is devoted to the study of the BGK-like kinetic model.

To finish with, let us mention that in a forthcoming paper [10], we study another approximation of the initial-boundary value problem: the parabolic regularization of the conservation law by an artificial viscosity. We get an error estimate between the entropy solution of the conservation law and the regular solution of the parabolic equation. Even if we adapt once again the proof of the Comparison Theorem, additional difficulties arise and the proof is rather long and technical.

2. Preliminaries. We give here some notations, assumptions and basic properties that are used throughout the paper.

The space $\mathbb{R}^d$ is endowed with its usual Euclidian structure. The scalar product is denoted by $x \cdot y$ and the Euclidian norm by $|x|$. For the sake of clarity $\mathbb{R}_t$ and $\mathbb{R}_\xi$ denote the lines of reals respectively related to the t and ξ variables.

DATA. We assume u_0 and u_b to be essentially bounded measurable functions. Let $K > 0$ be a positive constant such that

$$-K \leqslant u_0(x) \leqslant K \text{ for a.e. } x \in \Omega \quad \text{and} \quad -K \leqslant u_b(t,x) \leqslant K \text{ for a.e. } (t,x) \in \Sigma\,.$$

The flux function A is assumed to be locally Lipschitz continuous. Let M be the Lipschitz constant of the function A restricted to $[-K, K]$ and let $a(\xi) = A'(\xi)$.

REMARK 1. *We could as well consider the equation* $\partial_t u + \mathrm{div}_x(A(t,x,u)) = 0$. *All the results presented in this paper remain valid under the assumption that the function*

A is locally Lipschitz continuous with respect to $(t,x) \in [0,T]\times\overline{\Omega}$ uniformly w.r.t. the u variable, while for every $u\in\mathbb{R}$, $(t,x)\mapsto A(t,x,u)$ is in $\mathsf{C}^1([0,T]\times\overline{\Omega})$.

KRUŽKOV SEMI-ENTROPIES. Define

$$\mathrm{sgn}_+(\xi) = \begin{cases} 1 & \text{if} \quad \xi > 0 \\ 0 & \text{if} \quad \xi \leqslant 0 \end{cases} \quad \text{and} \quad \mathrm{sgn}_-(\xi) = \begin{cases} -1 & \text{if} \quad \xi < 0 \\ 0 & \text{if} \quad \xi \geqslant 0 \end{cases}$$

and $\xi^\pm = \mathrm{sgn}_\pm(\xi)\xi$. Let $a\top b$ denote $\max\{a,b\}$ and let $a\bot b$ denote $\min\{a,b\}$. The Kružkov semi-entropies are the convex functions $u\mapsto (u-\kappa)^\pm$ for $\kappa\in\mathbb{R}$. The corresponding entropy fluxes are given by the formula

$$\mathcal{F}^\pm(u,\kappa) = \mathrm{sgn}_\pm(u-\kappa)(A(u)-A(\kappa))\,.$$

KINETIC AND EQUILIBRIUM FUNCTIONS. We previously recalled what an equilibrium function is (see (1.2)). More generally, a kinetic function is a function $f(t,x,\xi)$ such that:

$$\begin{aligned} &0 \leqslant f(t,x,\xi)\mathrm{sgn}(\xi) \leqslant 1, \\ &\partial_\xi f(t,x,\xi) = \delta(\xi) - \nu_{t,x}(\xi) \end{aligned} \tag{2.1}$$

where ν is a Young measure. For an equilibrium function, $\nu_{t,x}(\xi) = \delta(\xi - u(t,x))$. In the following, we also consider two functions associated with any kinetic one:

$$\begin{aligned} f_+(t,x,\xi) &= f(t,x,\xi) - \mathrm{sgn}_-(\xi), \\ f_-(t,x,\xi) &= f(t,x,\xi) - \mathrm{sgn}_+(\xi). \end{aligned}$$

Notice that $\partial_\xi f_\pm = -\nu_{t,x}(\xi)$ and that these functions have no longer a bounded support w.r.t. the kinetic variable ξ. Nevertheless, and it is essential, there exists $\kappa\in\mathbb{R}_\xi$ such that $f_+(t,x,\xi)=0$ if $\xi\geqslant\kappa$ and there exists $\kappa'\in\mathbb{R}_\xi$ such that $f_-(t,x,\xi)=0$ if $\xi\leqslant\kappa'$. We simply say that f_+ vanishes for $\xi\gg 1$ and f_+ vanishes for $\xi\ll -1$. For equilibrium functions, if (t,x) is fixed, then for a.e. $\xi\in\mathbb{R}_\xi$:

$$\begin{aligned} f_+(t,x,\xi) &= \mathrm{sgn}_+(u(t,x)-\xi), \\ f_-(t,x,\xi) &= \mathrm{sgn}_-(u(t,x)-\xi). \end{aligned}$$

LOCALIZATION. The set Ω is assumed to be a strong Lipschitz open subset of $\mathbb{R}^d$, which means that, locally, Ω can be represented as the epigraph of a Lipschitz continuous function. More precisely, there exists a locally finite open cover $\{B_{\lambda_i}\}_{i\in I}$ of $\overline{\Omega}$ and a partition of unity $\{\lambda_i\}_{i\in I}$ of $\overline{\Omega}$ subordinate to $\{B_{\lambda_i}\}_{i\in I}$ such that for any λ :

$$\begin{aligned} \Omega_\lambda &:= \Omega\cap B_\lambda = \{x\in B_\lambda\ ;\ (A_\lambda\, x)_d > h_\lambda(\overline{A_\lambda\, x})\} \\ \partial\Omega_\lambda &:= \partial\Omega\cap B_\lambda = \{x\in B_\lambda\ ;\ (A_\lambda\, x)_d = h_\lambda(\overline{A_\lambda\, x})\} \end{aligned}$$

where $x\mapsto A_\lambda\, x$ is a change of coordinates of $\mathbb{R}^d$ (*i.e.* the composition of a translation and a rotation of $\mathbb{R}^d$) and where $\overline{y}$ stands for $(y_1,\ldots,y_{d-1})$ if $y\in\mathbb{R}^d$. In the following, we also use the notations $Q_\lambda = (0,+\infty)\times\Omega_\lambda$ and $\Sigma_\lambda = (0,+\infty)\times\partial\Omega_\lambda$. When proving the Comparison Theorem and the error estimate, the problem is localized with the help of the functions λ_i. For the sake of clarity, we drop the index i and we suppose that the change of coordinates is trivial: $A = \mathrm{Id}$. The open set $\Pi_\lambda = \{\overline{x}\ ;\ x\in B_\lambda\}\subset\mathbb{R}^{d-1}$ is used to parametrize $\partial\Omega_\lambda$. As a matter of fact, we even identify $\partial\Omega_\lambda$ with the graph of h restricted to Π_λ and Ω_λ with its epigraph. The outward unit normal to Ω_λ at any point $(\overline{x}, h(\overline{x}))$ of $\partial\Omega_\lambda$ is given by

$$n(\overline{x}) := n(\overline{x}, h(\overline{x})) = \frac{1}{\sqrt{1+|\nabla_{\overline{x}} h(\overline{x})|^2}}(\nabla_{\overline{x}} h(\overline{x}), -1).$$

Eventually, in order to make clearer integrations on $\partial\Omega_\lambda$, we use the notation

$$d\overline{\sigma}(\overline{x}) = \sqrt{1+|\nabla_{\overline{x}} h(\overline{x})|^2}\, d\overline{x}\,.$$

REGULARIZATION. Functions that are defined locally, *i.e.* that are defined on Ω_λ and $\partial\Omega_\lambda$, are regularized in the following way. Fix $\delta \in]0,1[$ and consider a smooth function $\theta : \mathbb{R} \to \mathbb{R}^+$ whose support is a subset of $[\delta, 1]$ and such that $\int \theta = 1$. Then define a (right-decentred) regularizing kernel $\theta_\varepsilon := \frac{1}{\varepsilon}\theta(\frac{\cdot}{\varepsilon})$ and set $\gamma_{\alpha,\varepsilon}(t, \overline{x}, x_d) = \theta_\alpha(t) \times \Pi_{i=1}^{d-1}\theta_{\overline{\varepsilon}}(x_i) \times \theta_{\varepsilon_d}(x_d)$. The space regularizing kernel $\Pi_{i=1}^{d-1}\theta_{\overline{\varepsilon}}(x_i) \times \theta_{\varepsilon_d}(x_d)$ is denoted by γ_ε. Consider now a function H defined on Q_λ and a function $\overline{H}$ defined on Σ_λ. Their (local) regularized functions are (both) defined on Q_λ by the following formulae:

$$\begin{cases} H^{\alpha,\varepsilon}(t,x) := (H \times 1_Q) \star \gamma_{\alpha,\varepsilon}(t,x) = \int_Q H(r,z)\gamma_{\alpha,\varepsilon}(t-r, x-z)\, dr\, dz \\ \overline{H}^{\alpha,\varepsilon}(t,x) := (\overline{H} \times 1_\Sigma) \star \gamma_{\alpha,\varepsilon}(t,x) = \int_\Sigma \overline{H}(r,z)\gamma_{\alpha,\varepsilon}(t-r, x-z)\, dr\, d\sigma(z). \end{cases}$$

These two functions equal zero out of Q_λ as soon as $\delta\, \varepsilon_d \geqslant \sqrt{d}\,\mathrm{Lip} h\, \overline{\varepsilon}$, which is always assumed. Of course, if a function ψ is defined both on Q_λ and Σ_λ, then the two means of regularization described above do not lead to the same functions $\psi^{\alpha,\varepsilon}$; nevertheless, there will be no risk of confusion in the forthcoming proofs. Let us also point out the fact that this regularization is local and in fact depends on the map A_λ, even if it is hidden in computations in order to make them more readable.

3. A kinetic formulation of the Cauchy-Dirichlet problem. The main result of the paper is the following kinetic formulation of generalized entropy solutions. For any smooth test function $\phi \in \mathsf{C}_c^\infty(\mathbb{R}^{d+2})$, $\phi^{(t=0)}$ and $\overline{\phi}$ denote respectively the restriction of ϕ to $\{0\} \times \Omega \times \mathbb{R}_\xi$ and to $\Sigma \times \mathbb{R}_\xi$.

THEOREM 3.1. *Consider a bounded function $u \in \mathsf{L}^\infty(Q)$. Let f^0 and f^b be the equilibrium functions associated with u_0 and u_b. Then u is a weak entropy solution of* (1.1) *if and only if there exists a bounded nonnegative measure $m \in \mathcal{M}^+(Q \times \mathbb{R}_\xi)$ and two nonnegative measurable functions $m_+^b, m_-^b \in \mathsf{L}_{\mathrm{loc}}^\infty(\Sigma \times \mathbb{R}_\xi)$ such that the function m_+^b vanishes for $\xi \gg 1$ (resp. the function m_-^b vanishes for $\xi \ll -1$) and such that the equilibrium function f associated with u satisfies for any $\phi \in \mathsf{C}_c^\infty(\mathbb{R}^{d+2})$:*

$$\int_{Q\times\mathbb{R}_\xi} f(\partial_t + a\cdot\nabla_x)\phi + \int_{\Omega\times\mathbb{R}_\xi} f^0\phi^{(t=0)} + \int_{\Sigma\times\mathbb{R}_\xi} (Mf_\pm^b + (-a\cdot n)\mathrm{sgn}_\mp)\overline{\phi} = \int_{Q\times\mathbb{R}_\xi} \partial_\xi\phi dm + \int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi\phi dm_\pm^b \quad (3.1)$$

where M is the Lipschitz constant of the flux function A on $\overline{Q} \times [-K, K]$.

In order to prove and understand this formulation, we define weak entropy sub and supersolutions of the initial-boundary value problem (1.1) and we exhibit a kinetic formulation for these semisolutions.

3.1. Weak entropy sub and supersolutions. Let us define weak entropy sub and supersolutions for the initial-boundary value problem (1.1).

DEFINITION 3.2. *Consider a bounded function* $u \in \mathsf{L}^\infty(Q)$.

1. *The function* u *is a* weak entropy subsolution *(resp.* weak entropy supersolution*) of* (1.1) *if for any* $\kappa \in \mathbb{R}$ *and any* $\phi \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}_t \times \mathbb{R}^d)$, $\phi \geqslant 0$,

$$\int_Q \left[(u(t,x)-\kappa)^\pm \partial_t \phi(t,x) + \mathcal{F}^\pm(u(t,x),\kappa).\nabla_x \phi(t,x)\right] dt\, dx + \int_\Omega (u_0(x)-\kappa)^\pm \phi(0,x) dx + M \int_\Sigma (u_b(s,y)-\kappa)^\pm \phi(s,y) ds\, d\sigma(y) \geqslant 0. \tag{3.2}$$

2. *The function* u *is an* weak entropy solution *of* (1.1) *if it is both a weak entropy subsolution and a supersolution.*

PROPOSITION 3.3. *Let* f^0 *and* f^b *be the equilibrium functions associated with* u_0 *and* u_b. *Consider a bounded function* $u \in \mathsf{L}^\infty(Q)$. *Then* u *is a weak entropy subsolution (resp. weak entropy supersolution) of* (1.1) *if and only if there exists* $m_\pm \in \mathsf{C}(\mathbb{R}_\xi; w-\mathcal{M}^+(\overline{Q}))$ *such that* m_ξ *vanishes for* $\xi \gg 1$ *(resp. for* $\xi \ll -1$*) and such that for any* $\phi \in \mathsf{C}_\mathsf{c}^\infty(\mathbb{R}^{d+2})$,

$$\int_{Q\times\mathbb{R}_\xi} f(\partial_t + a\cdot\nabla_x)\phi + \int_{\Omega\times\mathbb{R}_\xi} f^0 \phi^{(t=0)} + \int_{\Sigma\times\mathbb{R}_\xi} (M f_\pm^b + (-a\cdot n)\mathrm{sgn}_\mp)\overline{\phi} = \int_{\overline{Q}\times\mathbb{R}_\xi} \partial_\xi \phi dm_\pm. \tag{3.3}$$

REMARK 2. *The function* f *satisfies* (3.3) *if and only if the function* $f_\pm$ *satisfies:*

$$\int_{Q\times\mathbb{R}_\xi} f_\pm(\partial_t + a\cdot\nabla_x)\phi + \int_{\Omega\times\mathbb{R}_\xi} f_\pm^0 \phi^{(t=0)} + M\int_{\Sigma\times\mathbb{R}_\xi} f_\pm^b \overline{\phi} = \int_{\overline{Q}\times\mathbb{R}_\xi} \partial_\xi \phi dm_\pm. \tag{3.4}$$

Notice that, here, the expression of the boundary term is simplified. Moreover, (3.4) *is the kinetic equation that appears in the construction* $m_\pm$ *and it is also the one we consider when proving the Comparison Theorem.*

Proof of Proposition 3.3. Consider a weak entropy subsolution (resp. weak entropy supersolution) u of (1.1). Let us fix $\kappa \in \mathbb{R}$ and define a linear form $m_\pm^\kappa$ on $\mathsf{C}_\mathsf{c}^\infty(\overline{Q})$ by:

$$m_\pm^\kappa(\phi) = \int_Q (u-\kappa)^\pm \partial_t\phi + \mathcal{F}^\pm(u,\kappa)\cdot\nabla_x\phi + \int_\Omega (u_0-\kappa)^\pm \phi^{(t=0)} + M\int_\Sigma (u_b-\kappa)^\pm \overline{\phi}. \tag{3.5}$$

Since u is a weak entropy subsolution (resp. weak entropy supersolution), we know that $m_\pm^\kappa(\phi)$ is nonnegative for any κ and any ϕ. We conclude that for any κ, $m_\pm^\kappa$ is a nonnegative measure on $\overline{Q}$ and $m_\pm \in \mathsf{C}(\mathbb{R}_\xi, w-\mathcal{M}^+(\overline{Q}))$. Since $m_\pm \geqslant 0$, we have $||m_\pm|| = m_\pm(1) < +\infty$ by (3.5) and $m_\pm$ is bounded; moreover $m_\pm$ vanishes for $\kappa \gg 1$

(resp. $\kappa \ll 1$). Next, we compute:

$$\int_{\overline{Q}\times\mathbb{R}_\xi} \partial_\xi\phi(t,x,\xi)dm_\pm(t,x,\xi)$$
$$= \int_{Q\times\mathbb{R}_\xi} (u-\xi)^\pm\partial_t\partial_\xi\phi + \mathcal{F}^\pm(u,\xi)\cdot\nabla_x\partial_\xi\phi + \int_{\Omega\times\mathbb{R}_\xi}(u_0-\xi)^\pm\partial_\xi\phi^{(t=0)} + M\int_\Sigma (u_b-\xi)^\pm\overline{\partial_\xi\phi}$$
$$= \int_{Q\times\mathbb{R}_\xi} \mathrm{sgn}_\pm(u-\xi)(\partial_t\phi + a\cdot\nabla_x\phi) + \int_{\Omega\times\mathbb{R}_\xi}\mathrm{sgn}_\pm(u_0-\xi)\phi^{(t=0)} + M\int_\Sigma \mathrm{sgn}_\pm(u_b-\xi)\overline{\phi}$$
$$= \int_{Q\times\mathbb{R}_\xi} f_\pm(\partial_t\phi + a\cdot\nabla_x\phi) + \int_{\Omega\times\mathbb{R}_\xi} f^0_\pm\phi^{(t=0)} + M\int_\Sigma f^b_\pm\overline{\phi}$$
$$= \int_{Q\times\mathbb{R}_\xi} f(\partial_t\phi + a\cdot\nabla_x\phi) + \int_{\Omega\times\mathbb{R}_\xi} f^0\phi^{(t=0)} + \int_\Sigma (Mf^b_\pm + (-a\cdot n)\mathrm{sgn}_\mp)\overline{\phi}.$$

Hence (3.3) is proved.

Conversely, consider $u \in \mathsf{L}^\infty(Q)$ and $g \in \mathsf{C}^\infty_c(\mathbb{R}_t\times\mathbb{R}^d)$. Let $\xi \mapsto E_n(\xi)$ be a smooth approximation of $\xi \mapsto (\xi-\kappa)^\pm$ such that $|E_n'(\xi)| \leqslant 1$ for any positive integer n. Let Ψ be a smooth function with support in $[-2,2]$, values in $[0,1]$ and that equals 1 on $[-1,1]$. Next, define $\Psi_n(\xi) = \Psi(\xi/n)$. Now apply (3.4) to the test function $\phi(t,x,\xi) = g(t,x)\Psi_n(\xi)E_n'(\xi)$:

$$\int_Q \left[\int_{\mathbb{R}_\xi}\Psi_n E_n' f_\pm\right]\partial_t g + \left[\int_{\mathbb{R}_\xi} a\Psi_n E_n' f_\pm\right].\nabla_x g + \int_\Omega\left[\int_{\mathbb{R}_\xi}\Psi_n E_n' f^0_\pm\right] g^{(t=0)}$$
$$+ M\int_\Sigma\left[\int_{\mathbb{R}_\xi}\Psi_n E_n' f^b_\pm\right]\overline{g} = \int_{\overline{Q}\times\mathbb{R}_\xi} g[\Psi_n' E_n' + \Psi_n E_n'']dm_\pm.$$

Letting $n \to +\infty$, we get:

$$\int_Q (u(t,x)-\kappa)^\pm\partial_t g(t,x) + \mathcal{F}^\pm(u(t,x),\kappa)\cdot\nabla_x g(t,x)dtdx + \int_\Omega (u_0(x)-\kappa)^\pm g(0,x)dx$$
$$+ M\int_\Sigma (u_b(s,y)-\kappa)^\pm g(s,y)ds\, d\sigma(y) = \int_Q g(t,x)dm_\pm(t,x,\kappa). \quad (3.6)$$

If moreover g is assumed to be nonnegative, (3.6) yields (3.2). □

3.2. Kinetic traces. In this subsection, we prove the following proposition. See [32, 33] and [19, Lemma 7.34, p. 115].

PROPOSITION 3.4. *Consider a function $f \in L^\infty(Q\times\mathbb{R}_\xi)$ satisfying* (3.3).

1. *There exist two kinetic functions $f^{\tau_0} \in \mathsf{L}^\infty(Q\times\mathbb{R}_\xi)$ and $f^\tau \in \mathsf{L}^\infty(\Sigma\times\mathbb{R}_\xi)$ such that:*

$$\lim_{\alpha\to 0^+}\int_{\Omega\times\mathbb{R}_\xi}\left[\int_0^{+\infty} f(t)\theta_\alpha(t)dt\right]\phi = \int_{\Omega\times\mathbb{R}_\xi} f^{\tau_0}\phi, \quad (3.7)$$
$$\lim_{\varepsilon_d\to 0^+}\int_{[0;+\infty)\times\Pi_\lambda\times\mathbb{R}_\xi}(-a\cdot n)\left[\int_0^{+\infty} f(h(\overline{x})+r)\theta_{\varepsilon_d}(r)\,\lambda(h(\overline{x})+r)dr\right]\psi$$
$$= \int_{[0;+\infty)\times\Pi_\lambda\times\mathbb{R}_\xi}(-a\cdot n)\,f^\tau\,\overline{\lambda}\psi \quad (3.8)$$

for any $\phi \in \mathsf{L}^1(\Omega\times\mathbb{R}_\xi)$ and any $\psi \in \mathsf{L}^1(\Sigma\times\mathbb{R}_\xi)$ and any function λ, element of the partition of unity $\{\lambda_i\}_{i\in I}$.

2. The time kinetic trace f^{τ_0} is bounded above (resp. bounded below) by f^0 and the space kinetic trace f^τ satisfies (1.4) where $m^b_\pm$ denotes the restriction of $m_\pm$ to $\Sigma \times \mathbb{R}_\xi$.

Proof. The proof of the existence of f^{τ_0} and of f^τ such that (3.7), (3.8) hold true can be found in [32, 33]. Let us prove that for any test function $\phi \in \mathsf{C}_c^\infty(\mathbb{R}^{d+2})$:

$$\int_{Q\times\mathbb{R}_\xi} f(\partial_t + a\cdot\nabla_x)\phi + \int_{\Omega\times\mathbb{R}_\xi} f^{\tau_0}\phi^{(t=0)} + \int_{\Sigma\times\mathbb{R}_\xi}(-a\cdot n)f^\tau\overline{\phi} = \int_{Q\times\mathbb{R}_\xi}\partial_\xi\phi dm_\pm. \quad (3.9)$$

Let $\phi \in \mathsf{C}_c^\infty([0;+\infty)\times\Omega\times\mathbb{R}_\xi)$; consider a right-decentred regularizing kernel $\theta_\alpha(r)$; define a cut-off function $w_\alpha(r) = \int_0^r \theta_\alpha(\tau)d\tau$ and apply (3.3) to the test function $w_\alpha(t)\phi(t,x,\xi)$:

$$\int_{Q\times\mathbb{R}_\xi} w_\alpha(t)f(\partial_t + a\cdot\nabla_x)\phi(t,x,\xi)dtdxd\xi + \int_{Q\times\mathbb{R}_\xi}\theta_\alpha(t)f(t,x,\xi)\phi(t,x,\xi)dtdxd\xi$$
$$= \int_{Q\times\mathbb{R}_\xi} w_\alpha(t)\partial_\xi\phi(t,x,\xi)dm(t,x,\xi).$$

Letting $\alpha \to 0+$ and using the Lebesgue dominated convergence theorem and (3.7), we obtain:

$$\int_{Q\times\mathbb{R}_\xi} f(\partial_t + a\cdot\nabla_x)\phi + \int_{\Omega\times\mathbb{R}_\xi} f^{\tau_0}\phi^{(t=0)} = \int_{Q\times\mathbb{R}_\xi}\partial_\xi\phi dm. \quad (3.10)$$

Next, ϕ^λ denotes the function $\phi\,\lambda$ and we define a cut-off function

$$W_{\varepsilon_d}(x) = \int_0^{x_d - h(\overline{x})}\theta_{\varepsilon_d}(s)\,ds\,.$$

We apply (3.10) to the test function $\phi^\lambda W_{\varepsilon_d}$:

$$\int_{Q\times\mathbb{R}_\xi} W_{\varepsilon_d}(x)f(\partial_t + a\cdot\nabla_x)\phi^\lambda(t,x,\xi)dtdxd\xi + \int_{Q\times\mathbb{R}_\xi} f\phi^\lambda a\cdot\nabla_x W_{\varepsilon_d}$$
$$+\int_{\Omega\times\mathbb{R}_\xi} f^{\tau_0}(x,\xi){\phi^\lambda}^{(t=0)}W_{\varepsilon_d}(x)dxd\xi = \int_{Q\times\mathbb{R}_\xi}\partial_\xi\phi^\lambda(t,x,\xi)W_{\varepsilon_d}(x)dm(t,x,\xi). \quad (3.11)$$

In (3.11), we can pass to the limit in each term, except from $\int_{Q\times\mathbb{R}_\xi} f\phi^\lambda a(\xi)\cdot\nabla_x W_{\varepsilon_d}$. Let us study it. Notice that

$$\nabla_x W_{\varepsilon_d}(x) = \theta_{\varepsilon_d}(x_d - h(\overline{x}))(-\nabla_{\overline{x}}h(\overline{x}),1) = -\theta_{\varepsilon_d}(x_d - h(\overline{x}))\sqrt{1+|\nabla_{\overline{x}}h(\overline{x})|^2}\,n(\overline{x}).$$

Hence:

$$\int_{Q\times\mathbb{R}_\xi}\phi^\lambda f a(\xi)\cdot\nabla_x W_{\varepsilon_d}dtdxd\xi$$
$$= \int_{Q\times\mathbb{R}_\xi}\phi^\lambda(t,x,\xi)(-a\cdot n)f(t,x,\xi)\theta_{\varepsilon_d}(x_d - h(\overline{x}))\sqrt{1+|\nabla_{\overline{x}}h(\overline{x})|^2}dtdxd\xi$$
$$= \int_{[0;+\infty)\times\Pi_\lambda\times\mathbb{R}_\xi}(-a\cdot n)\left[\int_{x_d=h(\overline{x})}^{+\infty} f(x_d)\theta_{\varepsilon_d}(x_d - h(\overline{x}))\,\lambda(x_d)dx_d\right]\phi^\lambda\; dtd\overline{\sigma}d\xi.$$

Using (3.8), we get (3.9) with ϕ^λ instead of ϕ as a test-function. Recalling that the function λ is an element of the partition of unit $\{\lambda_i\}_{i\in I}$ and summing this previous inequality over $i \in I$ yields (3.9). We then deduce from (3.2) and (3.9) that (1.4) holds true and that $f^{\tau_0} = f^0 + \partial_\xi m^0_\pm$ where $m^0_\pm$ stands for the restriction of $m_\pm$ to $\{0\} \times \Omega \times \mathbb{R}_\xi$. It follows that:

$$\int f^{\tau_0}_\pm(x,\xi)\mathrm{sgn}_\pm(\xi-\kappa)d\xi \leqslant (u_0(x)-\kappa)^\pm$$

Since f^{τ_0} is a kinetic function and $f^{\tau_0}(\xi) = 0$ for $\xi \gg 1$, we conclude that it can be written under the following form:

$$\begin{aligned} &f^{\tau_0}(x,\xi) = \nu^{\tau_0}_x(\xi,+\infty) + \mathrm{sgn}_- \\ \Big(\text{resp.}\quad &f^{\tau_0}(x,\xi) = \nu^{\tau_0}_x(-\infty;\xi) + \mathrm{sgn}_+\Big). \end{aligned} \tag{3.12}$$

Next, replace κ with $u_0(x)$ and conclude that the support of $\nu^{\tau_0}_x$ lies in $(-\infty, u_0(x)]$ (resp. in $[u_0(x),+\infty)$). Finally, f^{τ_0} satisfies:

$$f^{\tau_0}_+(x,\xi) = \nu^{\tau_0}_x\ (\xi\bot u_0(x), u_0(x)] \leqslant \mathrm{sgn}_+(u_0(x)-\xi) \tag{3.13}$$

$$\Big(\text{resp. } f^{\tau_0}_-(x,\xi) = -\nu^{\tau_0}_x\ [u_0(x), \xi\top u_0(x)) \geqslant \mathrm{sgn}_-(u_0(x)-\xi)\Big). \tag{3.14}$$

This achieves the proof. □

Proof of Theorem 3.1. ¿From Proposition 3.3, we get two measures $m_\pm$. If u is a weak entropy solution of the initial-boundary value problem, then m_+ and m_- coincide in $Q \times \mathbb{R}_\xi$. Indeed, from (3.5), we get:

$$m_\pm(t,x,\kappa) = -\partial_t(u-\kappa)^\pm - \mathrm{div}_x\ \mathcal{F}^\pm(u,\kappa) \text{ in } \mathcal{D}'(Q\times\mathbb{R}_\xi). \tag{3.15}$$

Choosing κ respectively large enough and $-\kappa$ large enough, we obtain that u is a weak solution of (1.1), *i.e.* $\partial_t u + \mathrm{div}_x\ A(u) = 0$ in $\mathcal{D}'(Q)$. Next, we conclude that $m_+ = m_-$ in $Q \times \mathbb{R}_\xi$:

$$m_\pm(t,x,\kappa) = -\frac{1}{2}\partial_t|u-\kappa| - \frac{1}{2}\mathrm{div}_x\ \mathcal{F}(u,\kappa) \text{ in } \mathcal{D}'(Q\times\mathbb{R}_\xi) \tag{3.16}$$

where $\mathcal{F} = \mathcal{F}^+ + \mathcal{F}^-$. Moreover, we proved in Proposition 3.4 that $f^{\tau_0} = f^0 + \partial_\xi m^0_\pm$ and that f^{τ_0} is bounded above and below by f^0. We then conclude that $\partial_\xi m^0_\pm = 0$ hence that $m^0_\pm$ is constant in ξ. Since it equals 0 for large ξ, we conclude that $m^0_\pm = 0$. Eventually, the two measures $m^b_\pm$ are functions: indeed, since they satisfy (1.4) and respectively vanish for $\xi \gg 1$ and $\xi \ll -1$, we have:

$$m^b_+(s,y,\kappa) := M(u_b(s,y)-\kappa)^+ - \int_\kappa^{+\infty}(-a\cdot n)f^\tau_+(s,y,\xi)d\xi \geqslant 0 \tag{3.17}$$

$$m^b_-(s,y,\kappa) := M(u_b(s,y)-\kappa)^- + \int_{-\infty}^{\kappa}(-a\cdot n)f^\tau_-(s,y,\xi)d\xi \geqslant 0\,. \tag{3.18}$$

The proof of Theorem 3.1 is therefore achieved. □

REMARK 3. *Formula* (3.16) *appears in [18, p. 173]. Additional properties of* m *can be derived. See [18].*

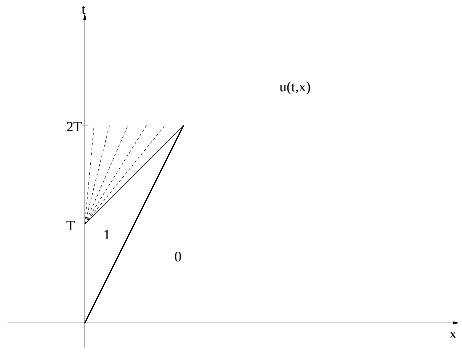

FIG. 3.1. *Weak entropy solution*

LINK WITH THE BLN CONDITION. We detail here the link between the kinetic formulation of weak entropy solutions given in Theorem 3.1 and the BLN condition. Suppose that the function u is a weak entropy solution of Problem (1.1) such that $u \in \mathsf{BV}(Q)$. Let u_τ denote the (strong) trace of the function u on Σ. Obviously, the space kinetic trace is the associated equilibrium function: $f^\tau = \chi_{u_\tau}$ (See Proposition 3.4). Next, remark that:

$$\int_\kappa^{+\infty} a(\xi)\cdot n(y) f_+^\tau(s,y,\xi)d\xi = \mathcal{F}^+(u_\tau(s,y),\kappa)\cdot n(y)$$

and combine with (3.17) in order to get

$$m_+^b(s,y,\kappa) = M(u_b(s,y)-\kappa)^+ + \mathcal{F}^+(u_\tau(s,y),\kappa)\cdot n(y)\,.$$

The fact that the function m_+^b is nonnegative is equivalent to the following condition:

$$\forall\kappa\in[u_b,u_\tau],\quad \mathrm{sgn}_+(u_\tau-u_b)[A(u_\tau)-A(\kappa)]\cdot n \geqslant 0.$$

Similarly $m_-^b \geqslant 0$ if and only if the previous condition holds true replacing sgn_+ with sgn_-. Summing these two inequalities yields the well-know BLN condition [1]:

$$\forall\kappa\in[u_b,u_\tau],\quad \mathrm{sgn}(u_\tau-u_b)[A(u_\tau)-A(\kappa)]\cdot n \geqslant 0. \tag{3.19}$$

3.3. An example. Let us detail the expressions of the entropy defect measure m and the boundary defect measures $m_\pm^b$ for Burgers' equation $\partial_t u + \partial_x(u^2/2) = 0$ considered on the domain $(0,2T)\times(0,+\infty)$ with data $u_0(x)=0$ and

$$u^b(t) = \begin{cases} 1 & \text{if} \quad 0<t<T\,, \\ -1 & \text{if} \quad T<t<2T\,. \end{cases}$$

A shock occurs at the time $t=0$ and a rarefaction wave appears at the time $t=T$. It collides the shock at time $t=2T$. The corresponding weak entropy solution u is represented on Figure 3.1. Then the entropy defect measure is

$$m = \frac{1}{2}\left(\frac{1}{2}[|u-\xi|]_1^0 - [\mathrm{sgn}(u-\xi)(u^2/2-\xi^2/2)]_1^0\right)\delta_L$$

where L is the line $t=2x$ in the (x,t)-plane and where $[G(u)]_1^0 := G(0)-G(1)$. In particular the measure m is concentrated on the line of discontinuity of u and the entropy criterion ensures that it is non-negative. On the other hand, the boundary defect measures are given by

$$m_+^b(t,\xi) = (M(1-\xi)^+ - \operatorname{sgn}^+(1-\xi)(1/2 - \xi^2/2))1_{(0,T)}(t) \\ + (M(1+\xi)^- - \operatorname{sgn}^-(\xi)\xi^2/2)1_{(T,2T)}(t)$$

and

$$m_-^b(t,\xi) = (M(1-\xi)^- - \operatorname{sgn}^-(1-\xi)(1/2 - \xi^2/2))1_{(0,T)}(t) \\ + (M(1+\xi)^+ - \operatorname{sgn}^+(\xi)\xi^2/2)1_{(T,2T)}(t)$$

where M is a constant greater than 1. The identity $a^2 - b^2 = (a+b)(a-b)$ ensures that the two functions are non-negative. The reader would check that the expressions of m and $m_\pm^b$ are consistent with the formula (3.16) and (3.18)-(3.17) respectively.

4. A Comparison Theorem. THEOREM 4.1. *Let $u \in \mathsf{L}^\infty(Q)$ be a weak entropy subsolution of* (1.1) *with data (u_0, u_b) and let $v \in \mathsf{L}^\infty(Q)$ be a weak entropy supersolution of* (1.1) *with data (v_0, v_b). Then:*

$$\frac{1}{T}\int_0^T \int_\Omega (u(t,x) - v(t,x))^+ dx\, dt \leqslant \int_\Omega (u_0(x) - v_0(x))^+ dx \\ + M \int_0^T \int_{\partial\Omega} (u_b(t,x) - v_b(t,x))^+ dt d\sigma . \quad (4.1)$$

In particular, $u \leqslant v$ as soon as $u_0 \leqslant v_0$ and $u_b \leqslant v_b$ (Comparison principle *).* Before proving Theorem 4.1, let us enlight that the L^1-contraction property and the maximum principle follow from it.

COROLLARY 4.2.

1. *Let $u, v \in \mathsf{L}^\infty(Q)$ be two weak entropy solutions of* (1.1). *Then:*

$$\frac{1}{T}\int_0^T \int_\Omega |u(t,x) - v(t,x)| dx\, dt \leqslant \int_\Omega |u_0(x) - v_0(x)| dx \\ + M \int_{(0;T)\times\partial\Omega} |u_b(t,y) - v_b(t,y)| dt d\sigma(y)$$

   (L^1-contraction property *).*
2. *Let u be a weak entropy solution of* (1.1) *and suppose that there exists two constants $U_m, U_M \in \mathbb{R}$ such that:*

$$U_m \leqslant u_0 \leqslant U_M \quad \text{a.e. in } \Omega \quad \text{and} \quad U_m \leqslant u_b \leqslant U_M \quad \text{a.e. in } \Sigma$$

   then $U_m \leqslant u \leqslant U_M$ a.e. in Q (Maximum principle *).*

Proof of Corollary 4.2. The L^1-contraction property is obtained by combining the two equations (4.1) obtained successively with u as a weak entropy subsolution and v as a weak entropy supersolution and with v as a weak entropy subsolution and u as a weak entropy supersolution. In order to prove the maximum principle, one may remark that the constant function U_m is a weak entropy subsolution for data u_0, u_b and that the constant function U_M is a weak entropy supersolution for data u_0, u_b. □

Proof of Theorem 4.1. In order to prove Theorem 4.1, we show that:

$$\int_Q (u-v)^+\partial_t\phi + \mathcal{F}^+(u,v)\cdot\nabla_x\phi + \int_\Omega (u_0-v_0)^+\phi^{(t=0)} + M\int_\Sigma (u_b-v_b)^+\overline{\phi} \geqslant 0, \quad (4.2)$$

holds true for any test function $\phi \in \mathsf{C}_c^\infty(\mathbb{R}_t\times\mathbb{R}^d)$. Passing from (4.2) to (4.1) is classical. Let f, f^0 and f^b (resp. g, g^0 and g^b) denote the equilibrium functions associated with u, u_0 and u_b (resp. with v, v_0 and v_b). The kinetic traces associated with u (resp. with v) are denoted by f^{τ_0} and f^τ (resp. g^{τ_0} and g^τ). Eventually, let m (resp. q) denote the entropy defect measure associated with u (resp. v) and set, for $(s,y,\xi)\in\Sigma\times\mathbb{R}_\xi$,

$$\overline{F_+}(s,y,\xi) = (-a(\xi)\cdot n(y))f_+^\tau(s,y,\xi) \quad \text{and} \quad \overline{G_-}(s,y,\xi) = (-a(\xi)\cdot n(y))g_-^\tau(s,y,\xi)\,.$$

Since u is a weak entropy subsolution of (1.1), the following kinetic equation holds true:

$$\int_{Q\times\mathbb{R}_\xi} f_+(\partial_t + a\cdot\nabla_x)\phi + \int_{\Omega\times\mathbb{R}_\xi} f_+^{\tau_0}\,\phi^{(t=0)} + \int_{\Sigma\times\mathbb{R}_\xi}\overline{F_+}\phi = \int_{Q\times\mathbb{R}_\xi}\partial_\xi\phi dm \qquad (4.3)$$

for any $\phi\in\mathsf{C}_c^\infty(\mathbb{R}^{d+2})$. Let us fix a test function $\phi\in\mathsf{C}_c^\infty(\mathbb{R}^{d+2})$ and apply (4.3) to the test function $\phi^\lambda\star\check{\gamma}_{\alpha,\varepsilon}$, where $\gamma_{\alpha,\varepsilon}$ denotes a right-decentered regularizing kernel and ϕ^λ denotes $\phi\lambda$:

$$\int_{\mathbb{R}^{d+2}} f_+^{\alpha,\varepsilon}(\partial_t + a\cdot\nabla_x)\phi^\lambda + f_+^{\tau_0\,\varepsilon}\theta_\alpha\phi^\lambda + \overline{F_+}^{\alpha,\varepsilon}\phi^\lambda = \int_{\mathbb{R}^{d+2}}\partial_\xi\phi^\lambda\; dm^{\alpha,\varepsilon} \qquad (4.4)$$

where $f_+^{\alpha,\varepsilon} = (f_+\times 1_Q)\star_{t,x}\gamma_{\alpha,\varepsilon}$, $f_+^{\tau_0\,\varepsilon} = (f_+^{\tau_0}\times 1_{\Omega_\lambda})\star_x\gamma_\varepsilon$, $m^{\alpha,\varepsilon} = (m\times 1_Q)\star_{t,x}\gamma_{\alpha,\varepsilon}$ and $\overline{F_+}^{\alpha,\varepsilon} = (\overline{F_+}\times 1_{\Sigma_\lambda})\star_{t,x}\gamma_{\alpha,\varepsilon}$. Now, let us also regularize the kinetic equation satisfied by g, but with different parameters:

$$\int_{\mathbb{R}^{d+2}} g_-^{\beta,\nu}(\partial_t + a\cdot\nabla_x)\phi^\lambda + g_-^{\tau_0\,\nu}\theta_\beta\phi^\lambda + \overline{G_-}^{\beta,\nu}\phi^\lambda = \int_{\mathbb{R}^{d+2}}\partial_\xi\phi^\lambda\; dq^{\beta,\nu} \qquad (4.5)$$

Now apply (4.4) to $-g_-^{\beta,\nu}(t,x,\xi)\phi^\lambda(t,x)$ and (4.5) to $-f_+^{\alpha,\varepsilon}(t,x,\xi)\phi^\lambda(t,x)$ and sum the two equations:

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} -\phi^\lambda(\partial_t + a\cdot\nabla_x)(f_+^{\alpha,\varepsilon}g_-^{\beta,\nu}) + 2\int_{\mathbb{R}^{d+2}}(-f_+^{\alpha,\varepsilon}g_-^{\beta,\nu})(\partial_t + a\cdot\nabla_x)\phi^\lambda\\
&\quad - \int_{\mathbb{R}^{d+2}}\left[f_+^{\tau_0\,\varepsilon}g_-^{\beta,\nu}\theta_\alpha + g_-^{\tau_0\,\nu}f_+^{\alpha,\varepsilon}\theta_\beta\right]\phi^\lambda - \int_{\mathbb{R}^{d+2}}\left[\overline{F_+}^{\alpha,\varepsilon}g_-^{\beta,\nu} + \overline{G_-}^{\beta,\nu}f_+^{\alpha,\varepsilon}\right]\phi^\lambda\\
&\qquad = \int_{\mathbb{R}^{d+2}}\phi^\lambda[\delta_v^{\beta,\nu}\; dm^{\alpha,\varepsilon} + \delta_u^{\alpha,\varepsilon}\; dq^{\beta,\nu}] \qquad (4.6)
\end{aligned}$$

where $\delta_u^{\alpha,\varepsilon} = (\delta(\xi-u(t,x))\times 1_Q)\star\gamma_{\alpha,\varepsilon}$ and $\delta_v^{\beta,\nu} = (\delta(\xi-v(t,x))\times 1_Q)\star\gamma_{\beta,\nu}$. Use the fact that the right-hand side of (4.6) is nonnegative and make an integration by parts in the first line:

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}}(-f_+^{\alpha,\varepsilon}g_-^{\beta,\nu})(\partial_t + a\cdot\nabla_x)\phi^\lambda\\
&\quad - \int_{\mathbb{R}^{d+2}}\left[f_+^{\tau_0\,\varepsilon}g_-^{\beta,\nu}\theta_\alpha + g_-^{\tau_0\,\nu}f_+^{\alpha,\varepsilon}\theta_\beta\right]\phi^\lambda - \int_{\mathbb{R}^{d+2}}\left[\overline{F_+}^{\alpha,\varepsilon}g_-^{\beta,\nu} + \overline{G_-}^{\beta,\nu}f_+^{\alpha,\varepsilon}\right]\phi^\lambda \geqslant 0.
\end{aligned}$$

Now let successively $\beta, \overline{\nu}$ and ν_d go to 0^+ :

$$\int_{Q_\lambda \times \mathbb{R}_\xi} (-f_+^{\alpha,\varepsilon} g_-)(\partial_t + a\cdot\nabla_x)\phi^\lambda - \int_{Q_\lambda \times \mathbb{R}_\xi} f_+^{\tau_0 \varepsilon} g_- \theta_\alpha \phi^\lambda - \int_{Q_\lambda \times \mathbb{R}_\xi} \overline{F_+}^{\alpha,\varepsilon} g_- \, \phi^\lambda \geqslant 0. \quad (4.7)$$

We used the fact that regularized functions equal zero at $t = 0$ and at the boundary. Next, let successively $\alpha, \overline{\varepsilon}$ and ε_d go to 0^+. The first limit is easy to compute:

$$\lim_{\varepsilon_d \to 0^+} \lim_{\overline{\varepsilon} \to 0^+} \lim_{\alpha \to 0^+} \int_{Q_\lambda \times \mathbb{R}_\xi} (-f_+^{\alpha,\varepsilon} g_-)(\partial_t + a \cdot \nabla_x)\phi^\lambda = \int_{Q \times \mathbb{R}_\xi} (-f_+ g_-)(\partial_t + a \cdot \nabla_x)\phi^\lambda$$
$$= \int_Q (u - v)^+ \partial_t \phi^\lambda + \mathcal{F}^+(u, v) \cdot \nabla \phi^\lambda. \quad (4.8)$$

Use (3.7) for g, and (3.13) for f and (3.14) for g :

$$\lim_{\varepsilon_d \to 0^+} \lim_{\overline{\varepsilon} \to 0^+} \lim_{\alpha \to 0^+} - \int_{Q_\lambda \times \mathbb{R}_\xi} f_+^{\tau_0 \varepsilon} g_- \theta_\alpha \phi^\lambda = \lim_{\varepsilon_d \to 0^+} \lim_{\overline{\varepsilon} \to 0^+} - \int_{\Omega_\lambda \times \mathbb{R}_\xi} f_+^{\tau_0 \varepsilon} g_-^{\tau_0} \left(\phi^\lambda\right)^{(t=0)}$$
$$= - \int_{\Omega_\lambda \times \mathbb{R}_\xi} f_+^{\tau_0} g_-^{\tau_0} \left(\phi^\lambda\right)^{(t=0)} \leqslant - \int_{\Omega_\lambda \times \mathbb{R}_\xi} f_+^0 g_-^0 \left(\phi^\lambda\right)^{(t=0)} = \int_\Omega (u_0 - v_0)^+ \left(\phi^\lambda\right)^{(t=0)} \quad (4.9)$$

We proceed analogously with the boundary term.

$$\lim_{\varepsilon_d \to 0^+} \lim_{\overline{\varepsilon} \to 0^+} \lim_{\alpha \to 0^+} - \int_{Q_\lambda \times \mathbb{R}_\xi} \overline{F_+}^{\alpha,\varepsilon} g_- \, \phi^\lambda = \int_{\Sigma \times \mathbb{R}_\xi} (-a \cdot n) f_+^\tau g_-^\tau \overline{\phi^\lambda}$$
$$\leqslant M \int_\Sigma (u_b - v_b)^+ \overline{\phi^\lambda}. \quad (4.10)$$

Let us now justify the inequality in (4.10). In order to do so, we use (1.4) and we represent f^τ and g^τ with their Young measures as in (3.12).

$$\int_{\mathbb{R}_\xi} (-a \cdot n) f_+^\tau g_-^\tau = - \int_{-\infty}^{v_b} \nu^\tau(\xi; +\infty) \partial_\xi q_-^b$$
$$+ \int_{v_b}^{v_b \top u_b} (-a \cdot n) \nu^\tau(\xi; +\infty) \mu^\tau(-\infty; \xi) + \int_{v_b \top u_b}^{+\infty} \mu^\tau(-\infty; \xi) \partial_\xi m_+^b$$
$$\leqslant - \int_{-\infty}^{v_b} q_-^b d\nu^\tau - \left[q_-^b \nu^\tau(\xi; +\infty)\right]_{-\infty}^{v_b} + M(u_b - v_b)^+$$
$$- \int_{v_b \top u_b}^{+\infty} m_+^b d\mu^\tau + \left[m_+^b \mu^\tau(-\infty; \xi)\right]_{v_b \top u_b}^{+\infty} \leqslant M(u_b - v_b)^+$$

Hence, we can pass to the limit in (4.7). By using (4.8), (4.9) and (4.10) and by summing over $i \in I$, (4.1) follows and the proof of Theorem 4.1 is complete. □

5. Convergence of a BGK-like model. In this section, we present the first application of the kinetic formulation we introduced above. Let us consider the following BGK-like model:

$$(\partial_t + a \cdot \nabla_x) f_\varepsilon = \frac{\chi_{u_\varepsilon} - f_\varepsilon}{\varepsilon} \qquad \text{in } Q \times \mathbb{R}_\xi, \quad (5.1a)$$

$$u_\varepsilon(t, x) = \int_\mathbb{R} f_\varepsilon(t, x, \xi) d\xi, \forall (t, x) \in Q, \quad (5.1b)$$

$$f_\varepsilon(0,x,\xi) = f^0(\xi), \forall (x,\xi) \in \Omega \times \mathbb{R}_\xi, \tag{5.1c}$$

$$f_\varepsilon(t,y,\xi) = f^b(y,\xi), \forall (t,y,\xi) \in \Sigma^+, \tag{5.1d}$$

where f^0 and f^b are the equilibrium functions respectively associated with the initial and the boundary data, and where $\Sigma^+ = \{(t,y,\xi) \in \Sigma \times \mathbb{R}_\xi : -a(\xi)\cdot n(y) > 0\}$. The approximation (5.1a)-(5.1b)-(5.1c) for the Cauchy problem (*i.e.* when $\Omega = \mathbb{R}^n$) was first considered by Perthame and Tadmor [29]. They proved that the "hydrodynamic limit" as $\varepsilon \to 0$ is precisely the entropy solution of the initial value problem (1.1a)-(1.1b). Their study relies on the fact that the right-hand side of (5.1a) can be written as the derivative of a measure: $\partial_\xi m_\varepsilon$. This is a consequence of the following observation:

LEMMA 5.1 ([18]). *Let $g \in \mathsf{L}^1(\mathbb{R})$ satisfy $0 \leqslant \operatorname{sgn}(\xi) g(\xi) \leqslant 1$ a.e. Then the function $m_g : \xi \mapsto \int_{-\infty}^{\xi} (\chi_{u_g} - g)(\zeta) d\zeta$ is non-negative.*

As ε goes to 0, the measure m_ε converges to the entropy defect measure m. This kinetic model has been adapted by Nouri, Omrane and Vila [22, 23] to take into account boundary conditions. In [22, 23], data at equilibrium as well as general kinetic ones are considered. The convergence of the kinetic model is proved and, particularly in the non-equilibrium case, the boundary conditions satisfied by the limit such obtained are discussed and compared to the BLN condition. In the present paper, we restrict ourselves to the case of data at equilibirum and we show how the concept of boundary defect measures can help in the understanding of the "hydrodynamic limit"; more precisely, we define approximate boundary defect measures and we prove that they converge to $m^b_\pm$ (See Subsection 3.2). As in [28], we intend to show how a concept of generalized kinetic solution can be used to prove the convergence of the kinetic model associated with (1.1) without "strong" (for instance BV) a priori estimates.

5.1. Solution of the kinetic model. We suppose that Ω is convex. The problem (5.1) admits an integral representation and is therefore solved by a fixed point method. The characteristic of the partial differential operator $\partial_t + a(\xi)\partial_x$ arriving at $(t,x) \in Q$ is the line of equation $X(\tau) = a(\xi)(\tau - t) + x$. If $u_\varepsilon \in \mathsf{C}(0,T;\mathsf{L}^1(\Omega))$, the solution f_ε of the linear equation $\partial_t f_\varepsilon + a(\xi)\cdot\nabla f_\varepsilon + \frac{1}{\varepsilon} f_\varepsilon = \frac{1}{\varepsilon}\chi_{u_\varepsilon}$ satisfies

$$f_\varepsilon(t,x,\xi) = f_\varepsilon(\tau, X(\tau),\xi) e^{\frac{\tau-t}{\varepsilon}} + \int_\tau^t \frac{1}{\varepsilon} \chi_{u_\varepsilon(s,X(s))}(\xi) e^{\frac{s-t}{\varepsilon}} ds \tag{5.2}$$

for any $\tau < t$ such that $X([\tau,t]) \subset \Omega$. Using the boundary condition (5.1d), we see that the computation of the value $f_\varepsilon(t,x,\xi)$ depends on the point of intersection of the characteristic line with the parabolic boundary:

- if $X([0,t]) \subset \Omega$, the characteristic starts from $\{0\} \times \Omega$ at $\tau = 0$ and we put $f_\varepsilon(\tau, X(\tau),\xi) = f^0(x - ta(\xi),\xi)$ in (5.2);
- if there exists $\tau^* \in [0,t]$ such that $X([\tau^*,t]) \subset \Omega$ and $X(\tau^* - 0) \notin \Omega$, the characteristic starts from the boundary Σ at $\tau = \tau^*$ and we put $f_\varepsilon(\tau, X(\tau),\xi) = f^b(\tau^*, X(\tau^*),\xi)$ in (5.2).

Thanks to the integral representation (5.2), it is therefore possible to build an operator T from $\mathsf{C}(0,T;\mathsf{L}^1(\Omega))$ to itself which maps u on $v : (t,x) \mapsto \int_{\mathbb{R}} f_\varepsilon(t,x,\xi) d\xi$. We then show that this operator is a contracting map and the existence and the uniqueness of the solution f_ε of (5.1) follows [29, 22, 28]. This solution satisfies additional properties:

PROPOSITION 5.2 ([29, 22, 28]). *Suppose that Ω is convex. Let $\varepsilon > 0$ and let $f_\varepsilon \in \mathsf{C}(0,T;\mathsf{L}^1(\Omega \times \mathbb{R}_\xi))$ be the solution of (5.1). Under the hypotheses of Section 2, we have:*

1. *f_ε satisfies*
$$0 \leqslant \operatorname{sgn}(\xi) f_\varepsilon(t,x,\xi) \leqslant 1 \text{ for a.e. } (t,x,\xi) \in Q \times \mathbb{R}_\xi\,;$$
2. *there exists a nonnegative function m_ε such that*
$$\frac{\chi_{u_\varepsilon} - f_\varepsilon}{\varepsilon} = \partial_\xi m_\varepsilon; \tag{5.3}$$
3. *for every convex function $\eta \in \mathsf{C}^2(\mathbb{R},\mathbb{R})$ with a bounded derivative η' satisfying $\eta'(0) = 0$:*
$$\int_{Q\times\mathbb{R}_\xi} m_\varepsilon(t,x,\xi)\eta''(\xi)d\xi dx dt \leqslant \int_{\Omega\times\mathbb{R}_\xi} f^0(\xi)\eta'(\xi)d\xi dx + \int_{\Sigma\times\mathbb{R}_\xi} (-a\cdot n)^+(s,y,\xi) f^b(\xi)\eta'(\xi)d\xi dt; \tag{5.4}$$
4. *there exists $\mu \in \mathsf{L}^\infty(\mathbb{R})$, independent of ε and such that $\mu(\xi) = 0$ if $|\xi| \gg 1$ and:*
$$\int_Q m_\varepsilon(t,x,\xi) dx dt \leqslant \mu(\xi); \tag{5.5}$$
5. *for a.e. $(t,x,\xi) \in Q \times \mathbb{R}_\xi$: $f_\varepsilon(t,x,\xi) = 0$ as soon as $|\xi| > K$; consequently*
$$\left| \int_{\mathbb{R}_\xi} f_\varepsilon(t,x,\xi) d\xi \right| \leq K \text{ for a.e. } (t,x) \in Q. \tag{5.6}$$

Sketch of the proof. The fact that f_ε is a kinetic function follows from (5.2). We previously mentioned that (5.3) is a consequence of Lemma 5.1. A rigorous proof of (5.4) relies on the integral representation (5.2). Here is a formal argument: multiply the equation $\partial_t f_\varepsilon + a(\xi)\partial_x f_\varepsilon = \partial_\xi m_\varepsilon$ by $\eta'(\xi)$, integrate the result with respect to (t,x,ξ) and use the fact that $\eta'(\xi) f_\varepsilon(t,x,\xi) \geqslant 0$ (for $\operatorname{sgn}(\eta'(\xi)) = \operatorname{sgn}(\xi)$). Estimate (5.5) is a consequence of (5.4) with $\eta(\xi) = (\xi - \xi_0)^+$ if $\xi_0 > 0$, $\eta(\xi) = (\xi - \xi_0)^-$ if $\xi_0 < 0$. It leads to the expression $\mu = \mu^+ + \mu^-$ with
$$\mu^\pm(\xi) = |\operatorname{sgn}_\pm(\xi)| \ \left(||(u_0 - \xi)^\pm||_{\mathsf{L}^1(\Omega)} + M||(u_b(t,y) - \xi)^\pm||_{\mathsf{L}^1(\Sigma)}\right).$$
Eventually, (5.6) is a consequence of (5.2) and of the first point. □

5.2. Generalized kinetic solutions. In order to prove the convergence of the model, we need to introduce a very weak notion of solution of (1.1).

DEFINITION 5.3. *Consider a kinetic function $f \in \mathsf{L}^\infty(Q \times \mathbb{R}_\xi)$. We say that f is a* generalized kinetic solution *of* (1.1) *if there exists a bounded nonnegative measure $m \in \mathcal{M}^+(Q \times \mathbb{R}_\xi)$ and two nonnegative measurable functions $m_+^b, m_-^b \in \mathsf{L}^\infty_{\mathsf{loc}}(\Sigma \times \mathbb{R}_\xi)$ such that the function m_+^b vanishes for $\xi \gg 1$ (resp. the function m_-^b vanishes for $\xi \ll -1$) and such that* (3.1) *holds true.*

The kinetic formulation can therefore be stated in the following terms: a function u is an entropy solution of (1.1) if and only if its associated equilibrium function is a generalized kinetic solution of (1.1).

THEOREM 5.4. *Any generalized kinetic solution of* (1.1) *is in fact an equilibrium function associated with an entropy solution of the initial-boundary value problem.*

Proof. We just adapt the proof of the Comparison Theorem. Consider a generalized kinetic solution f of the initial-boundary value problem. We can therefore easily prove that for a.e. $t > 0$:

$$\int_{\Omega\times\mathbb{R}_\xi} (-f^+ f^-)(t,x,\xi)dxd\xi \leqslant 0.$$

Now use the fact that f is a kinetic function to get that for a.e. $(t,x) \in Q$:

$$f^-(t,x,\xi) = \nu_{t,x}(-\infty;\xi) \qquad \text{and} \qquad f^+(t,x,\xi) = \nu_{t,x}(\xi;+\infty).$$

Consequently: $\nu_{t,x}(-\infty;\xi) = 0$ or $\nu_{t,x}(\xi;+\infty) = 0$. It follows that $\nu_{t,x}$ is a Dirac mass. The proof is therefore complete. □

5.3. Proof of the convergence. We now state and prove a precise convergence result.

THEOREM 5.5. *Suppose that Ω is convex. Under the hypotheses of Section 2, if f_ε denotes the solution of (5.1), then the sequence of function u_ε defined by $u_\varepsilon(t,x) = \int_{\mathbb{R}} f_\varepsilon(t,x,\xi)d\xi$ converges as $\varepsilon \to 0$ to the entropy solution u of* (1.1) *in any $L^p((0,T)\times\Omega)$, $1 \leqslant p < +\infty$.*

Proof of Theorem 5.5. Let $\overline{f}_\varepsilon$ denote the space kinetic trace of f_ε and consider $\varphi \in \mathsf{C}_c^\infty(\overline{Q}\times\mathbb{R}_\xi)$. By integrating the equation $\partial_t f_\varepsilon + a(\xi)\cdot\partial_x f_\varepsilon = \partial_\xi m_\varepsilon$ against φ we get:

$$\int_{Q\times\mathbb{R}_\xi} f_\varepsilon(\partial_t\varphi + a\cdot\nabla_x\varphi) + \int_{\Omega\times\mathbb{R}_\xi} f^0\varphi^{(t=0)} + \int_{\Sigma\times\mathbb{R}_\xi} (-a\cdot n)\overline{f}_\varepsilon\overline{\varphi} = \int_{Q\times\mathbb{R}_\xi} \partial_\xi\varphi dm_\varepsilon. \quad (5.7)$$

By analogy with (3.17), define the function $m_+^{b,\varepsilon}$ by:

$$m_+^{b,\varepsilon}(t,y,\xi) := M(u_b(t,y)-\xi)^+ - \int_\xi^{+\infty}(-a\cdot n)(\overline{f}_\varepsilon - \mathrm{sgn}_-)(\kappa)d\kappa,$$

and get from (5.7):

$$\int_{Q\times\mathbb{R}_\xi} f_\varepsilon(\partial_t\varphi + a\cdot\nabla_x\varphi) + \int_{\Omega\times\mathbb{R}_\xi} f^0\varphi^{(t=0)} + \int_{\Sigma\times\mathbb{R}_\xi} (Mf_+^b + (-a\cdot n)\mathrm{sgn}_-)\overline{\varphi} = \int_{Q\times\mathbb{R}_\xi} \partial_\xi\varphi dm_\varepsilon + \int_{\Sigma\times\mathbb{R}_\xi} \partial_\xi\overline{\varphi}dm_+^{b,\varepsilon}. \quad (5.8)$$

Let us check that $m_+^{b,\varepsilon}(t,y,\xi)$ is a nonnegative function. Since $\overline{f}_\varepsilon$ is a kinetic function, $\overline{f}_\varepsilon - \mathrm{sgn}_-$ is nonnegative, hence:

$$\begin{aligned} m_+^{b,\varepsilon}(t,y,\xi) &\geqslant M(u_b(t,y)-\xi)^+ - \int_\xi^{+\infty}(-a\cdot n)^+(\overline{f}_\varepsilon(t,y,\kappa) - \mathrm{sgn}_-(\kappa))d\kappa \\ &= M(u_b(t,y)-\xi)^+ - \int_\xi^{+\infty}(-a\cdot n)^+(f^b(t,y,\kappa) - \mathrm{sgn}_-(\kappa))d\kappa \\ &= \int_\xi^{+\infty}(M - (-a\cdot n)^+)(f^b(t,y,\kappa) - \mathrm{sgn}_-(\kappa))d\kappa \\ &\geqslant 0. \end{aligned}$$

Since f_ε is bounded in L^∞-norm and m_ε is bounded in mass by (5.5), we have, up to subsequences,

$$\begin{array}{lll} f_\varepsilon \rightharpoonup f & \text{in} & w-*-L^\infty(Q\times\mathbb{R}_\xi),\\ \overline{f}_\varepsilon \rightharpoonup \widetilde{f} & \text{in} & w-*-L^\infty(\Sigma\times\mathbb{R}_\xi),\\ m_\varepsilon \rightharpoonup m & \text{in} & w-*-\mathcal{M}^+(Q\times\mathbb{R}_\xi) \end{array}$$

where f and $\widetilde{f}$ are, respectively, functions of $L^\infty(Q\times\mathbb{R}_\xi)$ and $L^\infty(\Sigma\times\mathbb{R}_\xi)$ such that (this property is preserved at the $w-*-$limit): $0\leqslant f(\cdot,\xi)\mathrm{sgn}(\xi)\leqslant 1$ and $0\leqslant \widetilde{f}(\cdot,\xi)\mathrm{sgn}(\xi)\leqslant 1$. We first deduce from Proposition 5.2 that:

$$\int_\xi^{+\infty}(-a\cdot n)(\overline{f}_\varepsilon(t,y,\kappa)-\mathrm{sgn}_-(\kappa))d\kappa=\int_\xi^K(-a\cdot n)(\overline{f}_\varepsilon(t,y,\kappa)-\mathrm{sgn}_-(\kappa))d\kappa.$$

It follows that $m_+^{b,\varepsilon}(t,y,\xi)\rightharpoonup m_+^b$ where

$$m_+^b(t,y,\xi):=M(u_b(t,y)-\xi)^+-\int_\xi^K(-a\cdot n)(f^\tau-\mathrm{sgn}_-(\kappa))d\kappa \tag{5.9}$$

so that, at the limit $\varepsilon\to 0$ in (5.8), we have:

$$\int_{Q\times\mathbb{R}_\xi} f(\partial_t\varphi+a\cdot\nabla_x\varphi)+\int_{\Omega\times\mathbb{R}_\xi} f^0\varphi^{(t=0)}+\int_{\Sigma\times\mathbb{R}_\xi}(Mf_+^b+(-a\cdot n)\mathrm{sgn}_-)\overline{\varphi}$$
$$=\int_{Q\times\mathbb{R}_\xi}\partial_\xi\varphi dm+\int_{\Sigma\times\mathbb{R}_\xi}\partial_\xi\overline{\varphi}dm_+^b. \tag{5.10}$$

Besides, it is clear from (5.9) that $m_+^b(t,y,\xi)$ vanishes for $\xi\gg 1$; moreover (5.5) remains true at the limit. Derivating (5.1a) with respect to ξ gives:

$$\partial_\xi f_\varepsilon=\partial_\xi\chi_{u_\varepsilon}+\alpha_\varepsilon=\delta_0(\xi)-\delta_{u_\varepsilon}(\xi)+\alpha_\varepsilon$$

where $\alpha_\varepsilon=\varepsilon\left(\partial_{\xi t}f_\varepsilon+a(\xi)\partial_{\xi x}f_\varepsilon\right)$ tends to zero in $\mathcal{D}'(Q\times\mathbb{R}_\xi)$. We then define a Young measure $\nu_{t,x}(\xi)$ as an adherence value of $\delta(\xi-u_\varepsilon(t,x))$ and we obtain that:

$$\partial_\xi f=\delta_0(\xi)-\nu_{t,x}(\xi)\quad \text{ in }\mathcal{D}'(Q\times\mathbb{R}_\xi).$$

Of course, the same arguments remain valid for m_-^b and consequently f is a generalized kinetic solution of (1.1). In virtue of Theorem 5.4, it is therefore the equilibrium function associated with the unique entropy solution of (1.1). Since f is an equilibrium function, the weak-$*$ convergence of f_ε to f in $L^\infty(Q\times\mathbb{R}_\xi)$ implies the strong convergence of u_ε to u in $L^p(Q)$, $1\leqslant p<+\infty$. The proof is therefore complete. ☐

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