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Abstract. We study the uniformly elliptic fully nonlinear PDE $F(D^2u, Du, u, x) = f(x)$ in Ω, where F is a convex positively 1-homogeneous operator and $\Omega \subset \mathbb{R}^N$ is a regular bounded domain. We prove non-existence and multiplicity results for the Dirichlet problem, when the two principal eigenvalues of F are of different sign. Our results extend to more general cases, for instance, when F is not convex, and explain in a new light the classical results of Ambrosetti-Prodi type in elliptic PDE.

1 Introduction and Main Results

This paper is devoted to the study of the existence and the uniqueness of solutions of the Dirichlet boundary value problem

$$\left\{\begin{array}{rcll} H(D^2u, Du, u, x) & = & f(x) & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$-boundary, $f \in L^\infty(\Omega)$, and $H(M, p, u, x)$ is an uniformly elliptic fully nonlinear operator, globally Lipschitz in (M, p) and locally Lipschitz in u. A particular type of operators to which our results apply are Isaacs and Hamilton-Jacobi-Bellman operators. Boundary value problems of this type have been very extensively studied in the framework of classical, strong and viscosity solutions, see for example [35], [24], [28], [19], [14], [17]. Most work on fully nonlinear problems concerns *proper* operators, that is, the case when H is nonincreasing in u. Recently nonproper problems of type (1) have been studied in [36] and [37], see also the references in these papers. The present work continues a study started in [37].

For all $M \in \mathcal{S}_N(\mathbb{R}), p \in \mathbb{R}^N$, define the extremal operators $\mathcal{L}^-, \mathcal{L}^+$ by

$$\mathcal{L}^-(M, p) = \mathcal{M}^-_{\lambda,\Lambda}(M) - \gamma\,|p|, \qquad \mathcal{L}^+(M, p) = \mathcal{M}^+_{\lambda,\Lambda}(M) + \gamma\,|p|,$$

for some positive constants λ, Λ, γ. Here $\mathcal{M}^+, \mathcal{M}^-$ denote the Pucci operators $\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}} \mathrm{tr}(AM)$, $\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A\in\mathcal{A}} \mathrm{tr}(AM)$, where $\mathcal{A} \subset \mathcal{S}_N$ denotes the set of matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$.

We suppose that H in (1) satisfies the following hypothesis : for all $M \in \mathcal{S}_N(\mathbb{R}), p \in \mathbb{R}^N, u \in \mathbb{R}, x \in \Omega$, and for some constants A_0, c, δ,

$$F(M,p,u,x) - A_0 \le H(M,p,u,x) \le \mathcal{L}^+(M,p) + c|u| + A_0, \tag{2}$$

where $F(M,p,u,x)$ is some nonlinear operator, such that

$$\left\{\begin{array}{c} \mathcal{L}^-(M,p) - \delta|u| \le F(M,p,u,x) \le \mathcal{L}^+(M,p) + \delta|u| \\ \\ F(tM,tp,tu,x) = tF(M,p,u,x) \;\text{ for } t \ge 0 \\ \\ F \text{ is convex in } (M,p,u), \qquad F(M,0,0,x) \in C(\mathcal{S}_N(\mathbb{R}) \times \overline{\Omega}, \mathbb{R}). \end{array}\right. \tag{3}$$

We assume that H is Lipschitz continuous and uniformly elliptic, in the following sense : for each $R \in \mathbb{R}$ there exists $c_R \in \mathbb{R}$ such that for all $M, N \in \mathcal{S}_N(\mathbb{R})$, $p, q \in \mathbb{R}^N$, $x \in \Omega$, $u, v \in [-R, R]$,

$$\left\{\begin{array}{lcl} H(M,p,u,x) - H(N,q,v,x) & \ge & \mathcal{L}^-(M-N, p-q) - c_R|u-v| \\ H(M,p,u,x) - H(N,q,v,x) & \le & \mathcal{L}^+(M-N, p-q) + c_R|u-v|. \end{array}\right. \tag{4}$$

Note that (3) implies (4) with $H = F$ and $c_R = \delta$, see [37] (or inequalities (7) below). Our final standing assumption is that the proper operator

$$H_v[u] := H(D^2u, Du, v(x), x) - u \tag{5}$$

satisfies the comparison principle for each $v \in C(\overline{\Omega})$, in the sense that if $H_v[u_1] \ge f \ge H_v[u_2]$ in Ω, and $v_1 = v_2 = 0$ on $\partial\Omega$ then $v_1 \le v_2$ in Ω. This is satisfied for instance when H is Hölder continuous in x with a sufficiently large Hölder constant or when H is convex in M and $H(M,0,0,x)$ is uniformly continuous. Many other conditions which ensure uniqueness for proper equations can be found in [19], [31], [14], [32].

For instance, F can be a Hamilton-Jacobi-Bellman (HJB) operator, that is, a supremum of linear second order operators with bounded coefficients and continuous second order coefficients – see [37] for examples and discussions. HJB operators are basic in control theory. On the other hand, H can be an Isaacs operator, that is, a sup-inf of linear operators (these operators are essential in game theory). The Dirichlet problem for such operators has been widely studied in the proper case, and still many open question subsist, see the references above. Of course H can be a semilinear or quasilinear operator satisfying the hypotheses we made.

It was shown in [34], [9], [37] (see also [11], [5] for related results) that under hypothesis (3) F has two principal eigenvalues $\lambda_1^+(F,\Omega) \le \lambda_1^-(F,\Omega)$, which correspond to a positive and a negative eigenfunction, such that (1) with $H = F$ has a unique solution for all f if $\lambda_1^+ > 0$, while if $\lambda_1^- > 0 \ge \lambda_1^+$ then (1) has a solution for $f \ge 0$ but (1) does not have solutions for $f \le 0$, $f \not\equiv 0$. The question of uniqueness in the last case was left open, since $\lambda_1^- > 0$ alone does not imply a comparison principle. It is this question that we address in the present article. We will show that uniqueness fails when only one of the two eigenvalues is positive.

We will use the following decomposition of the right-hand side $f(x)$ in (1)

$$f(x) = -t\phi(x) + h(x),$$

where $t \in \mathbb{R}$, $\phi = \varphi_1^+(F_0,\Omega)$ is the first positive eigenfunction of the operator $F_0(M,p,x) = F(M,p,0,x)$, normalized so that $\max_\Omega \phi = 1$. The existence of $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, $p < \infty$, $\phi > 0$ in Ω, $F_0(D^2\phi, D\phi, x) = -\lambda_0^+\phi$ in Ω was shown in [37]. Since F_0 is proper, we have $\lambda_0^+ = \lambda_1^+(F_0,\Omega) > 0$, see [37].

Whenever we speak of a solution of (1) we shall mean a function in $C(\overline{\Omega})$ which satisfies (1) in the L^N-viscosity sense. See [14] for definitions and properties of these solutions. Note that $u \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ satisfies (1) almost everywhere in Ω if and only if it is a L^N-viscosity solution of (1).

Here is our main result.

Theorem 1 *Suppose F and H verify (2)–(5), and*

$$\lambda_1^+(F,\Omega) < 0 < \lambda_1^-(F,\Omega). \tag{6}$$

Then for each $h \in L^\infty(\Omega)$ there exists a number $t^*(h) \in \mathbb{R}$ such that:
(1) if $t < t^*(h)$ then (1) has at least two solutions ;
(2) if $t = t^*(h)$ then (1) has at least one solution ;
(3) if $t > t^*(h)$ then (1) has no solutions.
The map $h \to t^*(h)$ is continuous from $L^\infty(\Omega)$ to $\mathbb{R}$.

Remark 1. If $H(M,p,u,x)$ is convex in M then the solutions obtained in Theorem 1 belong to $W^{2,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, for all $p < \infty$.

The acknowledged reader may have noticed that the conclusion in Theorem 1 is similar to results obtained in the framework of the so-called Ambrosetti-Prodi problem, classical in the theory of semilinear elliptic PDE's. We shall quote here the original work [3], as well as the subsequent developments [10], [33], [20], [29], [23], [38], [16], [22]. Quasilinear operators were recently considered in [4], [7]. Here is the most typical Ambrosetti-Prodi type result : given the operator $H_L(M,p,u,x) = \mathrm{tr}(A(x)M) + b(x).p + g(x,u)$, if g is a Lipschitz function such that $g(x,u) \geq c_1 u^+ - c_2 u^- - c_0$, and if $c_1 > \lambda_1 > c_2$, where λ_1 is the usual first eigenvalue of the linear operator $L(M,p,x) = \mathrm{tr}(A(x)M) + b(x).p$, then the same conclusion as in Theorem 1 holds for (1) with $H = H_L$. Actually, this statement is nothing but Theorem 1 applied to $H = H_L$ and $F = F_L$, where

$$F_L(M,p,u,x) = \mathrm{tr}(A(x)M) + b(x).p + c_1 u^+ - c_2 u^-.$$

Here $u^+ = \max\{u,0\}, u^- = -\min\{u,0\}$, $A \in C(\overline{\Omega})$ is a positive definite matrix, b is a bounded vector, and $c_1 > c_2$. Then $\lambda_1^+(F_L,\Omega)$ (resp. $\lambda_1^-(F_L,\Omega)$) is obviously equal to $\lambda_1 - c_1$ (resp. $\lambda_1 - c_2$).

In other words, and this is the second main conclusion of the paper, the Ambrosetti-Prodi phenomenon turns out to be due to nonuniqueness of solutions of the Dirichlet problem for a convex nonlinear operator with one positive and one negative principal eigenvalue.

Remark 2. Many of the quoted papers on the Ambrosetti-Prodi problem contain results also for systems of equations or for the case when $g(x,u)$ in H_L does not have a linear but rather a power growth in u. Such extensions for fully nonlinear equations and systems of type (1) might be the subject of a future work.

Remark 3. It is only a matter of technicalities to show the results extend to the case when $h(x)$ in (1) and A_0 in (2) belong to $L^p(\Omega)$, $p > N$.

The next section contains the proof of Theorem 1. Its overall scheme (that is, the statements of the steps of the proof) is similar to the classical one used to prove the Ambrosetti-Prodi type results quoted above. It combines Perron's method with a priori bounds and degree theory, see the next section for more details. Of course, the proofs of some steps are rather different, and require a specific nonlinear approach. We find it quite remarkable how naturally the theory of viscosity solutions and eigenvalues for fully nonlinear operators permit to carry out these proofs. We begin the next section by an overview.

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Remark. The results of this paper first appeared as a preprint in 2008. This preprint was later used a basis for the more detailed study of resonance problems in [26], as well as for the result of the same type in [21] on more general fully nonlinear operators modeled on $|\nabla u|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2 u)$. Other related results can be found in [6], [27].

2 Proof of Theorem 1

From now on $h \in L^\infty(\Omega)$ will be fixed and we shall refer to (1) as problem $(\mathcal{P}_t)$ or $(\mathcal{P}_{t,h})$, when we need to stress the dependence on t or h.

We first give the plan of the proof of Theorem 1.

1. prove an a priori upper bound on t, such that $(\mathcal{P}_t)$ has a solution ;
2. prove an a priori bound on u, for $t \geq -C$;
3. prove subsolutions of $(\mathcal{P}_t)$ exist for all t, supersolutions exist for sufficiently small t, deduce by Perron's method that solutions of $(\mathcal{P}_t)$ exist for $t \in (-\infty, t^*)$;
4. prove for each $t \in (-\infty, t^*)$ there exists a subsolution of $(\mathcal{P}_t)$ which is smaller than all solutions of $(\mathcal{P}_t)$;
5. use fixed point theorems and degree theory to conclude ;

Let us review the main points and the difficulties in the proofs. Steps 1 and 2 above are rather classical for operators in divergence form, that is, for cases when (1) has an equivalent formulation in terms of integrals. Then one can prove Step 1 by testing the equation with the first eigenfunction of F_0 and after that carry out a contradiction (blow-up) argument to obtain the statement in Step 2. This is not possible for operators in non-divergence form. Recently a different method was developed in [22], for the semilinear operators F_L, H_L, which gives a simultaneous proof of Steps 1 and 2, and which applies to operators with power growth in u. The proof in [22] depends on the linearity of $L = F_0$. We will show here that it is actually the nonlinear structure of F and H, as described in our hypotheses, which provides for such a method to be applicable.

Further, Step 3 above is proved with the help of an one-sided Alexandrov-Bakelman-Pucci (ABP) inequality combined with an existence result, both obtained in [37], for operators with only one positive principal eigenvalue, which we recall below.

Another important difference with the semilinear case appears in proving Step 4. If $F = F_L$ then it is automatic that the restriction of F_L to the cone $\{(M, p, u, x) : u \leq 0\}$ satisfies a comparison principle in this cone (since F_L is linear and coercive there). In the nonlinear case this is not clear ; however we manage to prove that subsolutions can be chosen to satisfy properties which permit to us to use a more restrictive comparison result, which we establish, based on the fraction rather than the difference between the two functions that we compare – see Lemma 2.5 and the comments there.

Finally, the multiplicity result (Step 5) relies on an argument which uses the properties of the Leray-Schauder degree of compact maps.

We next list several preliminary results, mostly from [37]. It was shown in [37] that hypothesis (3) implies

$$\left\{\begin{array}{lcl} F(M-N,p-q,u-v,x) & \geq & F(M,p,u,x)-F(N,q,v,x) \\ F(M+N,p+q,u+v,x) & \leq & F(M,p,u,x)+F(N,q,v,x), \end{array}\right. \tag{7}$$

for all $M,N\in\mathcal{S}_N(\mathbb{R}),p,q\in\mathbb{R}^N,u,v\in\mathbb{R},x\in\Omega$.

We recall that the principal eigenvalues of F are defined by

$$\begin{array}{lcl} \lambda_1^+(F,\Omega) & = & \sup\{\lambda\in\mathbb{R}\mid \exists\psi>0 \text{ in } \Omega,\ F(D^2\psi,D\psi,\psi,x)+\lambda\psi\leq 0 \text{ in } \Omega\}, \\ \lambda_1^-(F,\Omega) & = & \sup\{\lambda\in\mathbb{R}\mid \exists\psi<0 \text{ in } \Omega,\ F(D^2\psi,D\psi,\psi,x)+\lambda\psi\geq 0 \text{ in } \Omega\}. \end{array}$$

In the sequel we shall need the following *one-sided* ABP estimate, obtained in [37]. A complete version of the Alexandrov-Bakelman-Pucci inequality for proper operators can be found in [14] (an ABP inequality for the Pucci operator was first proved in [13]). We recall that λ_1^+,λ_1^- are bounded above and below by constants which depend only on $N,\lambda,\Lambda,\gamma,\delta,\Omega$, and that both principal eigenvalues of any proper operator are positive, see [37].

Theorem 2 ([37]) *Suppose the operator F satisfies (3).*

I. *If $\lambda_1^-(F,\Omega)>0$ then for any $u\in C(\overline{\Omega})$, $f\in L^N(\Omega)$, the inequality $F(D^2u,Du,u,x)\leq f$ implies*

$$\sup_\Omega u^- \leq C(\sup_{\partial\Omega} u^- + \|f^+\|_{L^N(\Omega)}),$$

where C depends on Ω, $N,\lambda,\Lambda,\gamma,\delta$.

II. *In addition, if $\lambda_1^+(F,\Omega)>0$ then $F(D^2u,Du,u,x)\geq f$ implies*

$$\sup_\Omega u \leq C(\sup_{\partial\Omega} u^+ + \|f^-\|_{L^N(\Omega)}).$$

Set $E_p=W^{2,p}_{\mathrm{loc}}(\Omega)\cap C(\overline{\Omega})$, $p\geq N$. We shall use the following existence result.

Theorem 3 ([37]) *Suppose the operator F satisfies (3).*

I. *If $\lambda_1^-(F,\Omega)>0$ then for any $f\in L^p(\Omega),p\geq N$, such that $f\geq 0$ in Ω, there exists a solution $u\in E_p$ of $F(D^2u,Du,u,x)=f$ in Ω, $u=0$ on $\partial\Omega$, such that $u\leq 0$ in Ω.*

II. *In addition, if $\lambda_1^+(F,\Omega)>0$ then for any $f\in L^p(\Omega),p\geq N$, there exists a unique solution $u\in E_p$ of $F(D^2u,Du,u,x)=f$ in Ω, $u=0$ on $\partial\Omega$.*

We will actually need to apply parts II in the above two theorems only to the proper operator $F_0(M,p,x) = F(M,p,0,x)$.

We now move to the proof of Theorem 1. First we will show that solutions of $(\mathcal{P}_t)$ admit an a priori bound, which is uniform in $t \in (m,\infty)$, for each $m \in \mathbb{R}$. In the sequel C will denote a constant which may change from line to line and which depends on $N, \lambda, \Lambda, \gamma, \delta$, A_0, c, Ω, and $\|h\|_{L^\infty(\Omega)}$.

The next proposition realizes Steps 1 and 2 (see the beginning of this section) of the proof of Theorem 1.

Proposition 2.1 *For each $m_0 \in \mathbb{R}_+$ there exists a constant C such that for any $t \geq -m_0$ and any solution u of $(\mathcal{P}_t)$ we have*

$$\|u\|_{L^\infty(\Omega)} \leq C \qquad \textit{and} \qquad t \leq C.$$

In particular, there do not exist solutions of $(\mathcal{P}_t)$ for large t.

Proof. We divide the proof in three steps.

Claim 1. For each $m_0 \in \mathbb{R}_+$ there exists a constant $C = C(m_0)$ such that for any $t \geq -m_0$ and any solution u of $(\mathcal{P}_t)$ with this t we have

$$\|u^-\|_{L^\infty(\Omega)} \leq C.$$

Proof. This is an immediate consequence of (2),(6), and Theorem 2 I with f replaced by $m_0\phi + h$. □

Claim 2. There exists a constant C such that for solution u of $(\mathcal{P}_t)$ we have

$$t \leq C(1 + \|u\|_{L^\infty(\Omega)}).$$

Proof. By (2) and the definition of ϕ we have

$$F(D^2u, Du, u, x) - \frac{t}{\lambda_0^+} F(D^2\phi, D\phi, 0, x) \leq h(x) + A_0. \tag{8}$$

By (7) and (3) we have (recall we have set $F_0(M,p,x) = F(M,p,0,x)$)

$$\begin{aligned} F(M,p,u,x) &\geq F(M,p,0,x) - F(0,0,-u,x) \\ &\geq F_0(M,p,x) - \delta|u|. \end{aligned}$$

Hence, by (7), (8), and the homogeneity of F

$$-F_0\left(D^2\left(-u + \frac{t}{\lambda_0^+}\phi\right), D\left(-u + \frac{t}{\lambda_0^+}\phi\right), x\right) \leq h(x) + A_0 + \delta|u|.$$

Then the second part of Theorem 2 implies that for all $x \in \Omega$

$$-u(x) + \frac{t}{\lambda_0^+}\phi(x) \le C\|h(x) + A_0 + \delta|u|\|_{L^\infty(\Omega)}.$$

Taking x such that $\phi(x) = \max_\Omega \phi = 1$ finishes the proof of Claim 2. $\square$

Conclusion. Suppose the a priori bound on u in the statement of Proposition 2.1 is false, that is, there exist sequences $\{t_n\}, \{u_n\}$ such that $t_n \ge -m_0$, $\|u_n\|_{L^\infty(\Omega)} \to \infty$, and

$$H(D^2u_n, Du_n, u_n, x) = -t_n\phi + h.$$

By (2), (3) and Claim 2 we have

$$\begin{array}{rcl} \mathcal{L}^-(D^2u_n, Du_n) & \le & \delta\|u_n\|_{L^\infty(\Omega)} + m_0 + A_0 + h \\ \mathcal{L}^+(D^2u_n, Du_n) & \ge & -C(1 + \|u_n\|_{L^\infty(\Omega)}) + h. \end{array}$$

Hence, setting $v_n = u_n/\|u_n\|$ (so that $\|v_n\|_{L^\infty(\Omega)} = 1$),

$$\mathcal{L}^-(D^2v_n, Dv_n) \le C \quad \text{and} \quad \mathcal{L}^+(D^2v_n, Dv_n) \ge -C.$$

We now use the following result from the general theory of viscosity solutions of fully nonlinear PDE (it is a particular case, for instance, of Proposition 4.2 in [17]).

Proposition 2.2 *For any given $M \in \mathbb{R}$ the set of functions $u \in C(\overline{\Omega})$ such that*

$$\|u\|_{L^\infty(\Omega)} \le M, \qquad \mathcal{L}^-(D^2u, Du) \le M, \qquad \text{and} \qquad \mathcal{L}^+(D^2u, Du) \ge -M$$

is precompact in $C(\overline{\Omega})$.

Hence a subsequence of $\{v_n\}$ converges uniformly to a function v in $\overline{\Omega}$. Note that $v \ge 0$ in Ω, by Claim 1, and $\|v\|_{L^\infty(\Omega)} = 1$.

Again by (2) $F(D^2u_n, Du_n, u_n, x) \le m_0 + A_0 + h$, so, by the homogeneity of F

$$F(D^2v_n, Dv_n, v_n, x) \le o(1).$$

By viscosity solutions theory (see Theorem 3.8 in [14]) we can pass to the limit in this inequality, obtaining

$$F(D^2v, Dv, v, x) \le 0. \tag{9}$$

We recall the following strong maximum principle (Hopf lemma), a consequence from the results in [8].

Proposition 2.3 ([8]) *Let $\mathcal{O} \subset \mathbb{R}^N$ be a regular domain and let $\gamma \in \mathbb{R}$, $\delta \le 0$. Suppose $w \in C(\overline{\mathcal{O}})$ is a viscosity solution of*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 w) - \gamma|Dw| - \delta w \le 0 \text{ in } \mathcal{O},$$

and $w \ge 0$ in $\mathcal{O}$. Then either $w \equiv 0$ in $\mathcal{O}$ or $w > 0$ in $\mathcal{O}$ and at any point $x_0 \in \partial\mathcal{O}$ at which $w(x_0) = 0$ we have

$$\liminf_{t \searrow 0} \frac{w(x_0 + t\nu) - w(x_0)}{t} > 0,$$

where ν is the interior normal to $\partial\mathcal{O}$ at x_0.

Therefore $v > 0$ in Ω. Using (9), the existence of such function contradicts the definition of $\lambda_1^+(F, \Omega)$ and the hypothesis $\lambda_1^+(F, \Omega) < 0$.

Hence $\|u\|_{L^\infty(\Omega)}$ is bounded, and, by Claim 2, t is bounded as well. □

We turn to existence of subsolutions and supersolutions of $(\mathcal{P}_t)$. We shall need the following boundary Lipschitz estimate for fully nonlinear equations (for a proof see Proposition 4.9 in [37]).

Proposition 2.4 *Suppose H satisfies (4) and Ω satisfies an uniform exterior sphere condition. Suppose $u \in C(\overline{\Omega})$ satisfies $H(D^2u, Du, u, x) = h$, $u = 0$ on $\partial\Omega$, where $h \in L^\infty(\Omega)$. Then there exists a constant k depending on $N, \lambda, \Lambda, \gamma, \delta$, diam$(\Omega)$, $\|u\|_{L^\infty(\Omega)}$, $\|h\|_{L^\infty(\Omega)}$, and the radius of the exterior spheres, such that for each $x_0 \in \partial\Omega$*

$$|u(x)| \le k|x - x_0| \qquad \text{for each } \; x \in \Omega.$$

First we deal with the existence of supersolutions.

Lemma 2.1 *There exists $t_0 \in \mathbb{R}$, depending on the constants in (2)-(4) and on $\|h\|_{L^\infty(\Omega)}$, such that for each $t \le t_0$ there exists a supersolution $\overline{u}$ of $(\mathcal{P}_t)$, such that $\overline{u} \ge 0$ in Ω, $\overline{u} \in E_p$, $p < \infty$.*

Proof. Let $\overline{u}$ be the unique solution of the Dirichlet problem (see Theorem 3 above, or Corollary 3.10 in [14])

$$\left\{ \begin{array}{rcll} \mathcal{L}^+(D^2\overline{u}, D\overline{u}) & = & -h^-(x) & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{10}$$

The ABP inequality shows that $\overline{u} \ge 0$ in Ω, $\|\overline{u}\|_{L^\infty(\Omega)} \le C$ and $\overline{u}$ satisfies the boundary inequality in Proposition 2.4. On the other hand, the Hopf lemma and the inequality $F_0(D^2\phi, D\phi, x) \leq 0$ imply that there exists a constant $\alpha > 0$ such that for all $x_0 \in \partial\Omega$

$$\liminf_{t\searrow 0} \frac{\phi(x_0 + t\nu) - \phi(x_0)}{t} \geq \alpha,$$

where ν is the inner normal to $\partial\Omega$. Therefore there exists $t_0 < 0$ such that $-t_0\phi \geq \delta\overline{u}$ in Ω. Hence by (2) we have $H(D^2\overline{u}, D\overline{u}, \overline{u}, x) \leq -t\phi + h$, for all $t \leq t_0$, which is the required result. $\square$

The next lemma concerns the existence of subsolutions.

Lemma 2.2 *For any $t \in \mathbb{R}$ there exists a subsolution $\underline{u} \leq 0$ in Ω, $\underline{u} \in E_p$, $p < \infty$, of $(\mathcal{P}_t)$. In addition, given a compact interval $I \subset \mathbb{R}$, $\underline{u}$ can be chosen so that $\underline{u} \leq u$ in Ω, for all solutions u of $(\mathcal{P}_t)$, $t \in I$.*

The difficulty in Lemma 2.2 is in the second statement. As a step in its proof, we will obtain the following uniform boundary Hopf Lemma, which is of independent interest.

Lemma 2.3 *Assume Ω satisfies an uniform interior sphere condition. Suppose F satisfies (3), $\lambda_1^-(F, \Omega) > 0$, and $f \not\equiv 0, 0 \leq f \leq M$ in Ω. Then there exists $\alpha_0 > 0$ depending only on $\lambda, \Lambda, \nu, \delta$, Ω, and M, such that for any solution of $F(D^2u, Du, u, x) = f$ in Ω, $u \leq 0$ in Ω, $u = 0$ on $\partial\Omega$, and all $x_0 \in \partial\Omega$ we have*

$$V_{x_0}(u) := \liminf_{t\searrow 0} \frac{u(x_0) - u(x_0 + t\nu)}{t} \geq \alpha_0.$$

Proof. Suppose the lemma is false, that is, there is a sequence of solutions $u_n \leq 0$ in Ω and points $x_n \in \partial\Omega$ (we can suppose $x_n \to x \in \partial\Omega$) such that $V_{x_n}(u_n) \to 0$. Note that $\|u_n\|_{L^\infty(\Omega)} \leq C$, by Theorem 2. From (3) we have

$$\mathcal{L}^-(D^2u_n, Du_n) \leq C \quad \text{and} \quad \mathcal{L}^+(D^2u_n, Du_n) \geq -C.$$

By Proposition 2.2 a subsequence of $\{u_n\}$ converges uniformly to a function u in $\overline{\Omega}$, and $F(D^2u, Du, u, x) = f$ in Ω. Note that, by the strong maximum principle, $u_n < 0$ and $u < 0$ in Ω (since $f \not\equiv 0$ excludes $u_n \equiv 0$ or $u \equiv 0$).

By (3) and properties of Pucci operators ($\mathcal{M}^-(M) = -\mathcal{M}^+(-M)$), the positive functions $v_n = -u_n$ satisfy

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2v_n) - \nu|Dv_n| - \delta v_n \leq 0 \tag{11}$$

in Ω. Let ρ be the radius of the interior spheres. Fix $p \in \partial\Omega$ and let $B_\rho \subset \Omega$ be a ball tangent to $\partial\Omega$ at p. Introduce the (standard) barrier function, defined in B_ρ,

$$z(r) = e^{-\beta r^2} - e^{-\beta \rho^2},$$

where r is the distance to the center of B_ρ and β is a positive constant yet to be chosen. We recall the following fact.

Lemma 2.4 *Suppose $u \in C^2(B)$ is a radial function, defined on a ball B, say $u(x) = g(|x|)$. Then the matrix $D^2u(x)$ has $g''(|x|)$ as a simple eigenvalue, and $|x|^{-1}g'(|x|)$ as an eigenvalue of multiplicity $N-1$.*

Using this lemma and the fact that

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{\{e_i>0\}} e_i + \Lambda \sum_{\{e_i<0\}} e_i, \quad \mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{\{e_i>0\}} e_i + \lambda \sum_{\{e_i<0\}} e_i,$$

where e_i denote the eigenvalues of M, an elementary computation shows that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 z) - \nu|Dz| - \delta z \geq 0 \tag{12}$$

in the annulus $B_\rho \setminus B_{\rho/2}$, if $\beta = \beta(\rho)$ is chosen sufficiently large. Let the point $q_n \in \partial B_{\rho/2}$ be such that $v_n(q_n) = \min_{\partial B_{\rho/2}} v_n$ and set

$$\sigma_n = \frac{v_n(q_n)}{e^{-\beta(\rho/2)^2} - e^{-\beta\rho^2}}.$$

Then $\sigma_n z \leq v_n$ on $\partial(B_\rho \setminus B_{\rho/2})$ and, by the comparison principle for proper operators (see [14] or [37], note that the operator which appears in (11),(12) is proper), $\sigma_n z \leq v_n$ in $B_\rho \setminus B_{\rho/2}$. Hence

$$\sigma_n \frac{\partial z}{\partial \nu}(p) \leq -V_p(v_n) = V_p(u_n),$$

which implies

$$\min_{\partial B_{\rho/2}} v_n \leq a_0 V_p(u_n)$$

for some $a_0 > 0$, which depends on the appropriate quantities, and for all $p \in \partial\Omega$. Therefore, there exists a sequence of points $y_n \in \Omega$ such that $\mathrm{dist}(y_n, \partial\Omega) \geq \rho/2$ and $v_n(y_n) \to 0$. Hence there exists a point $y \in \Omega$ such that $v(y) = 0$, a contradiction. $\square$

Proof of Lemma 2.2. Set $M = A_0 + \sup_{t\in I} \|-t\phi + h\|_{L^\infty(\Omega)}$. By Theorem 3, there exists a solution $\underline{u} < 0$ in Ω of $F(D^2\underline{u}, D\underline{u}, \underline{u}, x) = M$ in Ω, $\underline{u} = 0$ on $\partial\Omega$. Hence $\underline{u}$ is a subsolution of $(\mathcal{P}_t)$ for $t \in I$, by (2).

Next, note that if u is a solution of $(\mathcal{P}_t)$ for some $t \in I$, then both functions $\psi = u$ and $\psi = 0$ are solutions of the inequality

$$F(D^2\psi, D\psi, \psi, x) \leq F(D^2\underline{u}, D\underline{u}, \underline{u}, x).$$

Since $-u^- = \min\{u, 0\}$ and the minimum of two viscosity supersolutions is a viscosity supersolution, we have

$$F(D^2(-u^-), D(-u^-), -u^-, x) \leq F(D^2\underline{u}, D\underline{u}, \underline{u}, x).$$

Observe we cannot directly infer from this inequality that $\underline{u} \leq -u^- \leq u$ since F does not satisfy a comparison principle ($\lambda_1^+(F, \Omega) < 0$). However, as we will show now, we can gain enough information on these functions in order to prove the inequality by considering their quotient instead of their difference.

By Proposition 2.4 and Lemma 2.3 we can fix k sufficiently large so that for any solution u of $(\mathcal{P}_t)$, $t \in I$, and any $x_0 \in \partial\Omega$ we have

$$\limsup_{t \searrow 0} \frac{-u^-(x_0 + t\nu)}{k\underline{u}(x_0 + t\nu)} \leq \frac{1}{4}$$

Note that $k\underline{u}$ is a subsolution of $(\mathcal{P}_t)$ for $k \geq 1$ and $t \in I$, by (2) and (3).

Fix a solution u of $(\mathcal{P}_t)$, $t \in I$. Then there exists $d > 0$ sufficiently small, so that, setting $\Omega_d = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) > d\}$, we have

$$0 < w := \frac{-u^-}{k\underline{u}} \leq \frac{1}{2} \qquad \text{in } \Omega \setminus \Omega_d.$$

The proof of Lemma 2.2 is finished with the help of the following comparison result.

Lemma 2.5 *Suppose v_1, v_2 are such that $v_1 \leq 0, v_2 < 0$ in Ω, $v_2 \in E_p$, $p < \infty$,*

$$\begin{aligned} F(D^2v_1, Dv_1, v_1, x) &\leq F(D^2v_2, Dv_2, v_2, x) \qquad \text{in } \Omega, \qquad (13)\\ 0 &< F(D^2v_2, Dv_2, v_2, x) \qquad \text{in } \Omega, \qquad (14)\end{aligned}$$

and, for some $d > 0$, $w := \dfrac{v_1}{v_2} < \dfrac{1}{2}$ in $\Omega \setminus \Omega_d$. Then $v_1 > v_2$ in Ω.

Remark. Considering the quotient rather than the difference of two functions can often be a successful technique when proving comparison results for a nonlinear operator. For fully nonlinear equations this has been used, in a different setting, for instance in [11].

Proof of Lemma 2.5. For any two vectors $p, q \in \mathbb{R}^N$ we denote the symmetric tensorial product by $p \otimes q = \frac{1}{2}(p_i q_j + p_j q_i)_{i,j=1}^N \in \mathcal{S}_N$. By replacing v_1 by wv_2 in (13) and by using (7) and the homogeneity of F we get

$$\begin{aligned} wF(Dv_2, Dv_2, v_2, x) \quad &+ \quad v_2 F(D^2 w + 2\frac{Dv_2}{v_2} \otimes Dw, Dw, 0, x) \qquad (15) \\ = F(wDv_2, wDv_2, wv_2, x) \quad &- \quad F(-v_2 D^2 w - 2Dv_2 \otimes Dw, -v_2 Dw, 0, x) \\ &\leq \quad F(D^2 v_1, Dv_1, v_1, x) \leq F(D^2 v_2, Dv_2, v_2, x), \end{aligned}$$

where we have used the equality

$$D^2(u_1 u_2) = u_1 Du_2 + 2Du_1 \otimes Du_2 + u_2 Du_1,$$

valid for $u_1, u_2 \in E_p$. In case u_1 is only continuous, we use test functions in E_p to prove (15) - this is very standard, so we shall omit it.

We obtain from (15)

$$\tilde{F}(D^2(w-1), D(w-1), x) + c(x)(w-1) \geq 0 \quad \text{ in } \Omega_{d/2}, \qquad (16)$$

where we have set

$$\tilde{F}(M, p, x) = F(M + 2b(x) \otimes p, p, 0, x),$$

$$b(x) = \frac{Dv_2(x)}{v_2(x)} \in L^\infty(\Omega_{d/2}),$$

$$c(x) = \frac{F(D^2 v_2(x), Dv_2(x), v_2(x), x)}{v_2(x)} < 0,$$

by (14). Note that $w-1 < 0$ in a neighbourhood of $\partial\Omega_{d/2}$. Then the existence of a point in $\Omega_{d/2}$ at which $w-1$ attains a positive maximum would contradict (16) - just test (16) with a constant function. So $w-1 \leq 0$. Finally, $w-1 < 0$ is a consequence of the strong maximum principle. □

The following existence result is an easy consequence from the previous lemmas.

Proposition 2.5 *There exists a number t^* such that problem $(\mathcal{P}_t)$ has a solution for $t \leq t^*$ and does not have a solution for $t > t^*$.*

Proof. We use the following lemma which is based on the Perron's method. This type of result in the viscosity setting goes back at least to Ishii [30] – recall we assumed in the introduction that the operator $H_v[u]$ satisfies the comparison principle.

Lemma 2.6 *Suppose $u_0 \in E_N$ is a subsolution and $v_0 \in E_N$ is a supersolution of $H(D^2u, Du, u, x) = f$, where $f \in L^\infty(\Omega)$, H satisfies (4). Suppose in addition that $u_0 \le v_0$ in Ω, $u_0 \le 0$ on $\partial\Omega$, and $v_0 \ge 0$ on $\partial\Omega$. Then there exists a solution u of*

$$\begin{cases} H(D^2u, Du, u, x) &= f \quad \textit{in} \quad \Omega \\ u &= 0 \quad \textit{on} \quad \partial\Omega. \end{cases}$$

For a proof of this lemma see for example Lemma 4.3 in [37].

Next, set

$$t^* = \sup\{t \in \mathbb{R} \,:\, (\mathcal{P}_t) \text{ has a supersolution }\}.$$

It follows from Lemmas 2.6 and 2.2 that if for some t problem $(\mathcal{P}_t)$ has supersolution then it has a solution. It is obvious that if u is a supersolution for $(\mathcal{P}_{t_0})$ then it is also a supersolution for all $(\mathcal{P}_t)$, $t < t_0$. By Lemma 2.1 t^* is well defined and by Proposition 2.1 t^* is finite. The existence of solution for $t = t^*$ follows from a passage to the limit $t_n \to t^*$, thanks to Proposition 2.2 and Theorem 3.8 in [14]. □

Now we can move to the realization of Step 5 of the proof of Theorem 1. The argument which follows is inspired by a classical reasoning of Amann [1], [2]. We refer for instance to [15] for a systematic treatment of existence results based on degree theory.

In what follows we shall use the following global $C^{1,\alpha}$-estimate, proved in [40], [39], [41].

Theorem 4 *Suppose H satisfies (4), Ω is a $C^{1,1}$-domain and u is a solution of (1). Then there exists $\alpha, C_0 > 0$ depending on $N, \lambda, \Lambda, \gamma, \delta, \Omega$, such that $u \in C^{1,\alpha}(\Omega)$, and*

$$\|u\|_{C^{1,\alpha}(\Omega)} \le C_0 \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\right).$$

Let t_1 be such that there exists a solution $\overline{u}$ for $(\mathcal{P}_{t_1})$. Fix $t < t_1$. Then $\overline{u}$ is a strict supersolution of $(\mathcal{P}_t)$. By Lemma 2.2 there is a subsolution $\underline{u}$ of $(\mathcal{P}_t)$ such that $\underline{u} < \overline{u}$ in Ω. By the choice of $\underline{u}, \overline{u}$ and Hopf's lemma, we can also ensure that $\frac{\partial \underline{u}}{\partial\nu} < \frac{\partial \overline{u}}{\partial\nu}$ on $\partial\Omega$.

Let c_{R_0} is the constant from hypothesis (4), with

$$R_0 = \max\{\|\underline{u}\|_{L^\infty(\Omega)}, \|\overline{u}\|_{L^\infty(\Omega)}\}.$$

For any $v \in C(\overline{\Omega})$ we define $H_v(M, p, x) = H(M, p, v(x), x)$. For each $v \in C(\overline{\Omega})$ we denote with $u = K_t(v)$ the solution of the Dirichlet problem

$$\begin{cases} H_v(D^2u, Du, x) - c_{R_0}u &= f(x) - c_{R_0}v \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{17}$$

This problem has a unique solution, by hypothesis (5), Theorem 3, and Perron's method. By the ABP inequality K_t maps bounded sets in $C(\overline{\Omega})$ into bounded sets in $C(\overline{\Omega})$. Hence, by Proposition 2.2 and Theorem 4 (recall $C^{1,\alpha}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ is compact) the map K_t sends bounded sets in $C^1(\overline{\Omega})$ into precompact sets in $C^1(\overline{\Omega})$, that is, $K_t : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ is a compact map. Note that solutions of (1) are fixed points of K_t and vice versa.

Define

$$\mathcal{O} = \{v \in C^1(\overline{\Omega}) \ : \ \underline{u} < v < \overline{u} \text{ in } \Omega \ \text{ and } \ \frac{\partial \underline{u}}{\partial \nu} < \frac{\partial v}{\partial \nu} < \frac{\partial \overline{u}}{\partial \nu} \text{ on } \partial\Omega\}.$$

Note the set $\mathcal{O}$ is open in $C^1(\overline{\Omega})$.

Claim. $K_t(\overline{\mathcal{O}}) \subset \mathcal{O}$. In particular, $K_t(\overline{\mathcal{O}}) \cap \partial\mathcal{O} = \emptyset$.

To prove this claim it is sufficient to show that if $\underline{u} \le v \le \overline{u}$ in Ω then $\underline{u} < K_t(v) < \overline{u}$ in Ω and $\frac{\partial \underline{u}}{\partial \nu} < \frac{\partial K_t(v)}{\partial \nu} < \frac{\partial \overline{u}}{\partial \nu}$ on $\partial\Omega$.

So let $v \in C(\overline{\Omega})$ be such that $\underline{u} \le v \le \overline{u}$ and set $u = K_t(v)$. Then we have, by (4),

$$\begin{aligned} H(D^2u, Du, \overline{u}(x), x) &= H(D^2u, Du, \overline{u}(x), x) + c_R\overline{u} - c_R\overline{u} \\ &\ge H(D^2u, Du, v(x), x) + c_R v - c_R\overline{u} \\ &= f(x) + c_R u - c_R\overline{u} \\ &> H(D^2\overline{u}, D\overline{u}, \overline{u}(x), x) + c_R(u - \overline{u}). \end{aligned}$$

This implies, again by (4),

$$\mathcal{L}^+(D^2(u-\overline{u}), D(u-\overline{u})) - c_R(u-\overline{u}) > 0$$

in Ω, and $u - \overline{u} = 0$ on $\partial\Omega$. It follows from the maximum principle for proper operators (or from Theorem 2) and from the strong maximum principle that $u < \overline{u}$ in Ω and $\frac{\partial u}{\partial \nu} < \frac{\partial \overline{u}}{\partial \nu}$ on $\partial\Omega$. In the same way we obtain the inequality for $\underline{u}$. □

To finish the proof of our main theorem we shall use the following lemma, concerning the Leray-Schauder degree of the compact map $I - K_t$. It is well-known how to prove this type of result, we give a proof for completeness.

Lemma 2.7 *For any $t_0 \in (-\infty, t^*)$ there exist $R_1, R_2 \in \mathbb{R}$ such that $R_1 < R_2$ and*

$$\deg(I - K_{t_0}, \mathcal{O} \cap \mathcal{B}_{R_1}, 0) = 1 \quad \text{and} \quad \deg(I - K_{t_0}, \mathcal{B}_{R_2}, 0) = 0, \tag{18}$$

where $\mathcal{B}_R = \{u \in C^1(\overline{\Omega}) \ : \ \|u\|_{C^1(\overline{\Omega})} < R\}$.

Proof. Let $\bar{R}$ be an upper bound (given by Theorem 4) for $C^1(\overline{\Omega})$-norms of solutions of (17) with $\|v\|_{L^\infty(\Omega)} \le R_0$. Set $R_1 = \max\{\bar{R}, \|\underline{u}\|_{C^1(\overline{\Omega})}, \|\overline{u}\|_{C^1(\overline{\Omega})}\}+1$. To prove the first equality in (18), fix $w \in \mathcal{O}\cap\mathcal{B}_{R_1}$ and consider the compact homotopy $H(s,v) = H_s(v) = sK_{t_0}(v) + (1-s)w$, for $s \in [0,1]$, $v \in C(\overline{\Omega})$. By the choice of R_1 and the claim above we have $(I - H_s)(u) \neq 0$ for all $u \in \partial(\mathcal{O}\cap\mathcal{B}_{R_1})$ and all $s \in [0,1]$. Hence

$$\deg(I - H_1, \mathcal{O}\cap\mathcal{B}_{R_1}, 0) = \deg(I - H_0, \mathcal{O}\cap\mathcal{B}_{R_1}, 0) = 1,$$

since H_0 is a constant mapping.

By combining Proposition 2.1 with Theorem 4 we see that for each m_0 there exists an uniform bound $\tilde{C}(m_0)$ for the $C^1(\overline{\Omega})$-norms of the solutions of $(\mathcal{P}_t)$ with $t \ge m_0$. Then we take $R_2 = \max\{\tilde{C}+1, R_1+1\}$, where $\tilde{C} = \tilde{C}(t_0)$. Set $t_1 = t^* + 1$. Clearly the mapping $K(t,u) = K_t(u), t \in [t_0,t_1]$, is a compact homotopy linking K_{t_0} to K_{t_1}. Further, we have $(I - K_t)(u) \neq 0$ for all $u \in \partial\mathcal{B}_{R_2}$ and all $t \in [t_0,t_1]$, by Proposition 2.1 and the choice of R_2. Hence

$$\deg(I - K_{t_0}, \mathcal{B}_{R_2}, 0) = \deg(I - K_{t_1}, \mathcal{B}_{R_2}, 0).$$

But the last degree is zero, since K_{t_1} has no fixed points at all, by Proposition 2.5. This proves the second equality in (18). $\square$

So, to complete the proof of the multiplicity result in Theorem 1 we can use the excision property of the degree together with Lemma 2.7, which leads to $\deg(I - K_{t_0}, \mathcal{B}_{R_2} \setminus (\overline{\mathcal{O}\cap\mathcal{B}_{R_1}}), 0) = -1$, hence problem (1) (i.e. problem $(\mathcal{P}_{t_0})$) has a second solution in $\mathcal{B}_{R_2} \setminus (\mathcal{O}\cap\mathcal{B}_{R_1})$, apart from the solution in $\mathcal{O}\cap\mathcal{B}_{R_1}$, given by Proposition 2.5.

Finally, let us show the mapping $h \to t^*(h)$ is continuous. Suppose that $h_n \rightrightarrows h$ in $\overline{\Omega}$. Set $t_n^* = t^*(h_n), t^* = t^*(h)$. Note t_n^* is bounded above, by Proposition 2.1. Furthermore, we have $t_n^* \ge t^*(-\|h\|_{L^\infty(\Omega)} - 1)$ for large n, since any solution of (1) with h replaced by $-\|h\|_{L^\infty(\Omega)} - 1$ is a supersolution of $(\mathcal{P}_{t_n^*,h_n})$. So t_n^* is bounded. Take a subsequence of t_n^* and let a be the limit of some subsequence of this subsequence (which we denote by t_n^* again). Let u_n be a solution of $(\mathcal{P}_{t_n^*,h_n})$ (we already know such a solution exists). By Proposition 2.1 $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Hence, by the equation satisfied by u_n, (4) and Proposition 2.2, some subsequence of u_n converges to a solution of $(\mathcal{P}_{a,h})$. Hence $a \le t^*$.

Suppose $a < a+3\varepsilon < t^*$, for some $\varepsilon > 0$. Let $\overline{u}$ be a positive supersolution of $(\mathcal{P}_{a+3\varepsilon,h})$ - we already know such supersolutions exist. Let w_n be the solution of the Dirichlet problem

$$\left\{\begin{array}{rcll} \mathcal{L}^+(D^2 w_n, Dw_n) &=& h_n - h & \text{in} \quad \Omega \\ w_n &=& 0 & \text{on} \quad \partial\Omega. \end{array}\right.$$

By the ABP inequality and the boundary estimate (Theorem 2 and Proposition 2.4), we have $w_n \rightrightarrows 0$ and $c_R|w_n| \le \varepsilon\phi$ in Ω for large n, where c_R is the constant from (4), with $R = \|\overline{u}\|_{L^\infty(\Omega)} + 1$.

Set $v_n = \overline{u} + w_n$. Then, by (4), if n is sufficiently large,

$$
\begin{aligned}
H(D^2v_n, Dv_n, v_n, x) &\le H(D^2v_n, Dv_n, v_n, x) - H(D^2\overline{u}, D\overline{u}, \overline{u}, x) \\
&\quad -(a+3\varepsilon)\phi + h \\
&\le \mathcal{L}^+(D^2w_n, Dw_n) + c_R w_n - (t_n^* + 2\varepsilon)\phi + h \\
&\le -(t_n^* + \varepsilon)\phi + h_n,
\end{aligned}
$$

Hence v_n is a positive supersolution of $(\mathcal{P}_{t_n^*+\varepsilon, h_n})$ which implies that this problem has a solution as well (we know subsolutions always exist). This is a contradiction with the definition of t_n^*.

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