Supplement to "Large sample properties of the matrix exponential spatial specification with an application to FDI" (for reference only; not for publication)

Nicolas Debarsy, Fei Jin, Lung-fei Lee

Note: This supplement includes

(1) QML Estimation of a high order MESS [MESS(p,q)];

(2) Some lemmas;

(3) Proofs of some lemmas;

(4) Proofs of some propositions;

(5) More Monte Carlo results in addition to those in the main paper;

(6) Estimation results of the MESS(1,0) model for the application.

1. QML Estimation of a high order MESS [MESS(p,q)]

Consider the following high order MESS:

$$e^{\boldsymbol{\alpha}\mathbf{W}_n}y_n = X_n\beta + u_n, \quad e^{\boldsymbol{\tau}\mathbf{M}_n}u_n = \epsilon_n, \quad \epsilon_n = (\epsilon_{n1}, \dots, \epsilon_{nn})', \tag{1}$$

where $\boldsymbol{\alpha}\mathbf{W}_n$ denotes $\alpha_1 W_{n1} + \cdots + \alpha_p W_{np}$ for a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_p)'$ and $n \times n$ spatial weights matrices W_{ni}'s, $i = 1, \dots, p$, and $\boldsymbol{\tau}\mathbf{M}_n$ denotes $\tau_1 M_{n1} + \cdots + \tau_q M_{nq}$ for $\boldsymbol{\tau} = (\tau_1, \dots, \tau_q)'$ and $n \times n$ spatial weights matrices M_{nj}'s, $j = 1, \dots, q$. Denote the model as MESS(p,q). We investigate the properties of the QMLE for this model when the disturbances are i.i.d. as assumed in Assumption 5. The quasi log likelihood function of the MESS(p,q), as if the ϵ_{ni}'s were i.i.d. normal, is

$$L_n(\boldsymbol{\theta}) = -\frac{n}{2}\ln(2\pi\sigma^2) + \ln|e^{\boldsymbol{\alpha}\mathbf{W}_n}| + \ln|e^{\boldsymbol{\tau}\mathbf{M}_n}| - \frac{1}{2\sigma^2}(e^{\boldsymbol{\alpha}\mathbf{W}_n}y_n - X_n\beta)'e^{(\boldsymbol{\tau}\mathbf{M}_n)'}e^{\boldsymbol{\tau}\mathbf{M}_n}(e^{\boldsymbol{\alpha}\mathbf{W}_n}y_n - X_n\beta),$$

where $\theta = (\boldsymbol{\gamma}', \sigma^2)'$ with $\boldsymbol{\gamma} = (\boldsymbol{\alpha}, \boldsymbol{\tau}, \beta')'$. Let $\boldsymbol{\theta}_0$ be the true parameter vector. Since $|e^{\boldsymbol{\alpha}\mathbf{W}_n}| = e^{\mathrm{tr}(\boldsymbol{\alpha}\mathbf{W}_n)}$ and $|e^{\boldsymbol{\tau}\mathbf{M}_n}| = e^{\mathrm{tr}(\boldsymbol{\tau}\mathbf{M}_n)}$, as long as W_{ni}'s and M_{nj}'s have zero diagonals, the log Jacobians disappear and the quasi log likelihood function is simplified to

$$L_n(\boldsymbol{\theta}) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(e^{\boldsymbol{\alpha}\mathbf{W}_n}y_n - X_n\beta)'e^{(\boldsymbol{\tau}\mathbf{M}_n)'}e^{\boldsymbol{\tau}\mathbf{M}_n}(e^{\boldsymbol{\alpha}\mathbf{W}_n}y_n - X_n\beta). \tag{2}$$

By contrast, for the high order SARAR model corresponding to (1),

$$(I_n - \boldsymbol{\lambda}\mathbf{W}_n)y_n = X_n\beta + u_n, \quad (I_n - \boldsymbol{\rho}\mathbf{M}_n)u_n = \epsilon_n, \quad \epsilon_n = (\epsilon_{n1}, \dots, \epsilon_{nn})',$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_p)'$ and $\boldsymbol{\rho} = (\rho_1, \dots, \rho_q)'$, the quasi log likelihood function involves the log Jacobian $\ln|(I_n - \boldsymbol{\lambda}\mathbf{W}_n)(I_n - \boldsymbol{\rho}\mathbf{M}_n)| = \ln|I_n - \boldsymbol{\lambda}\mathbf{W}_n| + \ln|I_n - \boldsymbol{\rho}\mathbf{M}_n|$. The stationary regions of $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ can be hard to find and the Jacobian is computationally intensive (Elhorst et al., 2012).

The MESS(p,q) (1) with the notations $\boldsymbol{\alpha}\mathbf{W}_n$ and $\boldsymbol{\tau}\mathbf{M}_n$ resembles the MESS(1,1) presented in (2), thus we have similar expressions for the QMLE. From (2), the QMLE of $\boldsymbol{\gamma}$ is the minimizer of

$$Q_n(\boldsymbol{\gamma}) = (e^{\boldsymbol{\alpha}\mathbf{W}_n} y_n - X_n\beta)' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} (e^{\boldsymbol{\alpha}\mathbf{W}_n} y_n - X_n\beta). \tag{3}$$

For fixed $\boldsymbol{\phi} = (\boldsymbol{\alpha}', \boldsymbol{\tau}')'$, the QMLE of β is

$$\hat{\beta}_n(\boldsymbol{\phi}) = (X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1} X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} y_n. \tag{4}$$

Substituting $\hat{\beta}_n(\boldsymbol{\phi})$ into $Q_n(\boldsymbol{\gamma})$, we obtain a function of only $\boldsymbol{\phi}$:

$$Q_n(\boldsymbol{\phi}) = y_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} y_n, \tag{5}$$

where the projection matrix $H_n(\boldsymbol{\tau}) = I_n - e^{\boldsymbol{\tau}\mathbf{M}_n} X_n (X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1} X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'}$. The QMLE of $\boldsymbol{\phi}$ can be derived by the minimization of $Q_n(\boldsymbol{\phi})$. Corresponding to Assumptions 1, 3 and 4, we make the following assumptions.

Assumption A.1. *Matrices $\{W_{ni}\}$ for $i = 1, \ldots, p$ and $\{M_{nj}\}$ for $j = 1, \ldots, q$ are bounded in both row and column sum norms. The diagonal elements of W_{ni}'s and M_{nj}'s are zero.*

Assumption A.2. *There exists a constant $\delta > 0$ such that $|\alpha_i| \leq \delta$ for $i = 1, \ldots, p$, $|\tau_j| \leq \delta$ for $j = 1, \ldots, q$, and the true $\boldsymbol{\phi}_0$ is in the interior of the parameter space $\boldsymbol{\Phi} = [-\delta, \delta]^{p+q}$.*

Assumption A.3. *The limit $\lim_{n\to\infty} \frac{1}{n} X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n$ exists and is nonsingular for any $\boldsymbol{\tau} \in [-\delta, \delta]^q$, and the sequence of the smallest eigenvalues of $e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n}$ is bounded away from zero uniformly in $\boldsymbol{\tau} \in [-\delta, \delta]^q$.*

To find the identification condition for $\boldsymbol{\phi}$, define

$$\begin{aligned} \bar{Q}_n(\boldsymbol{\phi}) = \min_{\beta} \mathrm{E}\, Q_n(\boldsymbol{\gamma}) &= (X_n\beta_0)' e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} X_n\beta_0 \\ &+ \sigma_0^2 \operatorname{tr}(e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} e^{-\boldsymbol{\tau}_0\mathbf{M}_n}). \end{aligned} \tag{6}$$

The following condition is assumed for the identification uniqueness.

Assumption A.4. *Either (i) $\lim_{n\to\infty} n^{-1} (X_n\beta_0)' e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} X_n\beta_0 \neq 0$ for any $\boldsymbol{\tau}$ and $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}_0$, and $\lim_{n\to\infty} n^{-1} \operatorname{tr}(e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} e^{-\boldsymbol{\tau}_0\mathbf{M}_n}) > 1$ for any $\boldsymbol{\tau} \neq \boldsymbol{\tau}_0$, or (ii) $\lim_{n\to\infty} n^{-1} \operatorname{tr}(e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} e^{-\boldsymbol{\tau}_0\mathbf{M}_n}) > 1$ for any $\boldsymbol{\phi} \neq \boldsymbol{\phi}_0$.*

The consistency of the QMLE follows from the uniform convergence of $[Q_n(\boldsymbol{\phi}) - \bar{Q}_n(\boldsymbol{\phi})]/n$ to zero on the parameter space $\boldsymbol{\Phi}$ and the identification uniqueness. The proof of the following proposition is in 4.

Proposition A.1. *Under Assumptions 2, 5 and A.1–A.4, the QMLE $\hat{\gamma}_n$ of the MESS(p,q) in (1) is consistent.*

For the asymptotic distribution of $\hat{\gamma}_n$, a Taylor expansion of the first-order condition $\frac{\partial Q_n(\hat{\gamma}_n)}{\partial \gamma} = 0$ at the true γ_0 yields

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) = -\Big(\frac{1}{n}\frac{\partial^2 Q_n(\tilde{\gamma}_n)}{\partial \gamma \partial \gamma'}\Big)^{-1} \frac{1}{\sqrt{n}}\frac{\partial Q_n(\gamma_0)}{\partial \gamma}, \tag{7}$$

where $\tilde{\gamma}_n$ is between $\hat{\gamma}_n$ and γ_0. Under regularity conditions, $\frac{1}{n}\frac{\partial^2 Q_n(\tilde{\gamma}_n)}{\partial \gamma \partial \gamma'} = \mathbf{C}_n + o_P(1)$ with $\mathbf{C}_n = \frac{1}{n}\,\mathrm{E}\,\frac{\partial^2 Q_n(\gamma_0)}{\partial \gamma \partial \gamma'}$. We assume that $\mathbf{C}_n$ is nonsingular in the limit.

Assumption A.5. *The limit of $\mathbf{C}_n$ exists and is nonsingular.*

The first-order derivatives of $Q_n(\gamma)$ at γ_0 are

$$\frac{\partial Q_n(\gamma_0)}{\partial \alpha_i} = 2(X_n\beta_0 + e^{-\tau_0 \mathbf{M}_n}\epsilon_n)' e^{-(\alpha_0 \mathbf{W}_n)'} \frac{\partial e^{(\alpha_0 \mathbf{W}_n)'}}{\partial \alpha_i} e^{(\tau_0 \mathbf{M}_n)'}\epsilon_n, \quad ,i = 1,\dots,p, \tag{8}$$

$$\frac{\partial Q_n(\gamma_0)}{\partial \tau_i} = 2\epsilon_n' e^{-(\tau_0 \mathbf{M}_n)'} \frac{\partial e^{(\tau_0 \mathbf{M}_n)'}}{\partial \tau_i}\epsilon_n, \quad i = 1,\dots,q, \tag{9}$$

$$\frac{\partial Q_n(\gamma_0)}{\partial \beta} = -2X_n' e^{(\tau_0 \mathbf{M}_n)'}\epsilon_n, \tag{10}$$

which are linear and quadratic functions of ϵ_n and have mean zero by verification. Thus we may apply the central limit theorem for linear-quadratic forms in Kelejian and Prucha (2001). The proof of the following proposition is in 4.

Proposition A.2. *Under Assumptions 2, 5 and A.1–A.5, $\sqrt{n}(\hat{\gamma}_n - \gamma_0) \xrightarrow{d} N(0, \lim_{n\to\infty} \mathbf{C}_n^{-1}\mathbf{\Omega}_n \mathbf{C}_n^{-1})$, where $\mathbf{C}_n = \frac{1}{n}\,\mathrm{E}\,\frac{\partial^2 Q_n(\gamma_0)}{\partial \gamma \partial \gamma'}$ is a 3×3 symmetric block matrix with the (i,j)th element for $1 \le i,j \le p$ in the $(1,1)$th block being*

$$\frac{2}{n}(X_n\beta_0)' e^{-(\alpha_0 \mathbf{W}_n)'} \frac{\partial e^{(\alpha_0 \mathbf{W}_n)'}}{\partial \alpha_i} e^{(\tau_0 \mathbf{M}_n)'} e^{\tau_0 \mathbf{M}_n} \frac{\partial e^{\alpha_0 \mathbf{W}_n}}{\partial \alpha_j} e^{-\alpha_0 \mathbf{W}_n} X_n\beta_0 + \frac{2\sigma_0^2}{n}\mathrm{tr}\big(e^{-(\alpha_0 \mathbf{W}_n)'} \frac{\partial^2 e^{(\alpha_0 \mathbf{W}_n)'}}{\partial \alpha_i \partial \alpha_j}\big)$$
$$+ \frac{2\sigma_0^2}{n}\mathrm{tr}\big(e^{-(\tau_0 \mathbf{M}_n)'} e^{-(\alpha_0 \mathbf{W}_n)'} \frac{\partial e^{(\alpha_0 \mathbf{W}_n)'}}{\partial \alpha_i} e^{(\tau_0 \mathbf{M}_n)'} e^{\tau_0 \mathbf{M}_n} \frac{\partial e^{\alpha_0 \mathbf{W}_n}}{\partial \alpha_j} e^{-\alpha_0 \mathbf{W}_n} e^{-\tau_0 \mathbf{M}_n}\big),$$

the (i,j)th element for $1 \le i \le p, 1 \le j \le q$ in the $(1,2)$th block being

$$\frac{2\sigma_0^2}{n}\mathrm{tr}\big(e^{-(\tau_0 \mathbf{M}_n)'} e^{-(\alpha_0 \mathbf{W}_n)'} \frac{\partial e^{(\alpha_0 \mathbf{W}_n)'}}{\partial \alpha_i} \frac{\partial e^{(\tau_0 \mathbf{M}_n)'}}{\partial \tau_j}\big) + \frac{2\sigma_0^2}{n}\mathrm{tr}\big(e^{-(\tau_0 \mathbf{M}_n)'} e^{-(\alpha_0 \mathbf{W}_n)'} \frac{\partial e^{(\alpha_0 \mathbf{W}_n)'}}{\partial \alpha_i} e^{(\tau_0 \mathbf{M}_n)'} \frac{\partial e^{\tau_0 \mathbf{M}_n}}{\partial \tau_j} e^{-\tau_0 \mathbf{M}_n}\big),$$

the ith row for $1 \le i \le p$ in the $(1,3)$th block being $-2(X_n\beta_0)' e^{-(\alpha_0 \mathbf{W}_n)'} \frac{\partial e^{(\alpha_0 \mathbf{W}_n)'}}{\partial \alpha_i} e^{(\tau_0 \mathbf{M}_n)'} e^{\tau_0 \mathbf{M}_n} X_n$, the (i,j)th element for $1 \le i,j \le q$ in the $(2,2)$th block being

$$\frac{2\sigma_0^2}{n}\mathrm{tr}\big(e^{-(\tau_0 \mathbf{M}_n)'} \frac{\partial^2 e^{(\tau_0 \mathbf{M}_n)'}}{\partial \tau_i \partial \tau_j}\big) + \frac{2\sigma_0^2}{n}\mathrm{tr}\big(e^{-(\tau_0 \mathbf{M}_n)'} \frac{\partial e^{(\tau_0 \mathbf{M}_n)'}}{\partial \tau_i} \frac{\partial e^{\tau_0 \mathbf{M}_n}}{\partial \tau_j} e^{-\tau_0 \mathbf{M}_n}\big),$$

the elements in the $(2,3)$*th block being all zero, and the* $(3,3)$*th block being* $\frac{2}{n}X_n' e^{(\boldsymbol{\tau}_0 \mathbf{M}_n)'} e^{\boldsymbol{\tau}_0 \mathbf{M}_n} X_n$*, and* $\boldsymbol{\Omega}_n = \mathbf{C}_n + \boldsymbol{\Omega}_{1n}$*, where* $\boldsymbol{\Omega}_{1n}$ *is a symmetric* 3×3 *block matrix with the* (i,j)*th element for* $1 \le i,j \le p$ *in the* $(1,1)$*th block being*

$$
\begin{aligned}
&(\mu_4 - 3\sigma_0^4)\operatorname{vec}_D{}'(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_i} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}) \operatorname{vec}_D(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_j} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}) \\
&+ 2\mu_3 (X_n\beta_0)' e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_i} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'} \operatorname{vec}_D(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_j} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}) \\
&+ 2\mu_3 (X_n\beta_0)' e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_j} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'} \operatorname{vec}_D(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_i} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}),
\end{aligned}
$$

the (i,j)*th element for* $1 \le i \le p$*,* $1 \le j \le q$ *in the* $(1,2)$*th block being*

$$
\begin{aligned}
&(\mu_4 - 3\sigma_0^4)\operatorname{vec}_D{}'(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_i} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}) \operatorname{vec}_D(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} \frac{\partial e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}}{\partial \tau_j}) \\
&+ 2\mu_3 (X_n\beta_0)' e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_i} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'} \operatorname{vec}_D(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} \frac{\partial e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}}{\partial \tau_j}),
\end{aligned}
$$

the i*th row for* $1 \le i \le p$ *in the* $(1,3)$*th block being* $-2\mu_3 \operatorname{vec}_D{}'(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} \frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial \alpha_i} e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}) e^{\boldsymbol{\tau}_0\mathbf{M}_n} X_n$*, the* (i,j)*th element for* $1 \le i,j \le q$ *in the* $(2,2)$*th block being*

$$
(\mu_4 - 3\sigma_0^4)\operatorname{vec}_D{}'(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} \frac{\partial e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}}{\partial \tau_i}) \operatorname{vec}_D(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} \frac{\partial e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}}{\partial \tau_j}),
$$

the i*th row for* $1 \le i \le q$ *in the* $(2,3)$*th block being* $-2\mu_3 \operatorname{vec}_D{}'(2e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} \frac{\partial e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}}{\partial \tau_i}) e^{\boldsymbol{\tau}_0\mathbf{M}_n} X_n$*, and the elements of the* $(3,3)$*th blocks being zero.*

When $\mu_3 = \mu_4 - 3\sigma_0^4 = 0$, $\boldsymbol{\Omega}_{1n} = 0$.

2. Some lemmas

In the following, Lemmas A.1–A.4 can be found in, e.g., Lin and Lee (2010) and Jin and Lee (2012); Lemma A.4, a central limit theorem, is originated in Kelejian and Prucha (2010); and Lemma A.5 is Lemma A.6 in Lee (2007). They are provided here for easy reference. Other lemmas are proved in the subsequent section. Let UB stand for "bounded in both row and column sum norms".

Lemma A.1. *Suppose that* $n \times n$ *matrices* $\{A_n\}$ *are UB. Elements of* $n \times k$ *matrices* $\{X_n\}$ *are uniformly bounded and* $\lim_{n\to\infty} n^{-1}X_n'X_n$ *exists and is nonsingular. Let* $M_n = I_n - X_n(X_n'X_n)^{-1}X_n'$*. Then* $\{M_n\}$ *are UB and* $\operatorname{tr}(M_nA_n) = \operatorname{tr}(A_n) + O(1)$*.*

Lemma A.2. *Suppose that* $A_n = [a_{n,ij}]$ *and* $B_n = [b_{n,ij}]$ *are two* $n \times n$ *matrices and* ϵ_{ni}*'s in* $\epsilon_n = (\epsilon_{n1}, \ldots, \epsilon_{nn})'$ *are independently distributed with mean zero (but may not be i.i.d.). Then,* *(1)* $\mathrm{E}(\epsilon_n \cdot \epsilon_n' A_n \epsilon_n) = (a_{n,11}\,\mathrm{E}(\epsilon_{n1}^3), \ldots, a_{n,nn}\,\mathrm{E}(\epsilon_{nn}^3))'$, *and*

(2) $\mathrm{E}(\epsilon_n' A_n \epsilon_n \cdot \epsilon_n' B_n \epsilon_n) = \sum_{i=1}^n a_{n,ii} b_{n,ii}[\mathrm{E}(\epsilon_{ni}^4) - 3\sigma_{ni}^4] + \mathrm{tr}(\Sigma_n A_n)\,\mathrm{tr}(\Sigma_n B_n) + \mathrm{tr}[\Sigma_n A_n \Sigma_n (B_n + B_n')]$, *where* $\Sigma_n = \mathrm{Diag}(\sigma_{n1}^2, \ldots, \sigma_{nn}^2)$ *with* $\sigma_{ni}^2 = \mathrm{E}(\epsilon_{ni}^2)$, $i = 1, \ldots, n$.

Lemma A.3. *Suppose that* $n \times n$ *matrices* $\{A_n\}$ *are UB, elements of the* $n \times k$ *matrices* $\{C_n\}$ *are uniformly bounded, and* ϵ_{ni}*'s in* $\epsilon_n = (\epsilon_{n1}, \ldots, \epsilon_{nn})'$ *are independent random variables with mean zero and variance* σ_{ni}^2. *The sequence* $\{\mathrm{E}(\epsilon_{ni}^4)\}$ *is bounded. Then* $\epsilon_n' A_n \epsilon_n = O_P(n)$, $\mathrm{E}(\epsilon_n' A_n \epsilon_n) = O(n)$, $n^{-1}[\epsilon_n' A_n \epsilon_n - \mathrm{E}(\epsilon_n' A_n \epsilon_n)] = o_P(1)$ *and* $n^{-1/2} C_n' A_n \epsilon_n = O_P(1)$.

Lemma A.4. *Suppose that* $\{A_n\}$ *is a sequence of symmetric* $n \times n$ *matrices that are UB and* $b_n = (b_{n1}, \ldots, b_{nn})'$ *is an* n*-dimensional column vector such that* $\sup_n n^{-1} \sum_{i=1}^n |b_{ni}|^{2+\eta_1} < \infty$ *for some* $\eta_1 > 0$. *Furthermore, suppose that* $\epsilon_{n1}, \cdots, \epsilon_{nn}$ *are mutually independent with zero means and the moments* $\mathrm{E}(|\epsilon_{ni}|^{4+\eta_2})$ *for some* $\eta_2 > 0$ *exist and are uniformly bounded for all* n *and* i. *Let* $\sigma_{c_n}^2$ *be the variance of* c_n *where* $c_n = \epsilon_n' A_n \epsilon_n + b_n' \epsilon_n - \mathrm{tr}(A_n \Sigma_n)$ *with* Σ_n *being a diagonal matrix with* $\mathrm{E}\,\epsilon_{ni}^2$*'s on its diagonal. Assume that* $n^{-1}\sigma_{c_n}^2$ *is bounded away from zero. Then* $\frac{c_n}{\sigma_{c_n}} \xrightarrow{d} N(0,1)$.

Lemma A.5. *Suppose that* $[Q_n(\gamma) - \bar{Q}_n(\gamma)]$ *converges in probability to zero uniformly in* $\gamma \in \Gamma$ *which is a convex set, and* $\{\bar{Q}_n(\gamma)\}$ *satisfies the identification uniqueness condition at* γ_0. *Let* $\hat{\gamma}_n$ *and* $\hat{\gamma}_n^*$ *be, respectively, the minimizers of* $Q_n(\gamma)$ *and* $Q_n^*(\gamma)$ *in* Γ. *If* $Q_n^*(\gamma) - Q_n(\gamma) = o_P(1)$ *uniformly in* $\gamma \in \Gamma$, *then both* $\hat{\gamma}_n$ *and* $\hat{\gamma}_n^*$ *converge in probability to* γ_0.

In addition, suppose that $\frac{\partial^2 Q_n(\gamma)}{\partial\gamma\partial\gamma'}$ *converges in probability to a well defined limiting matrix, uniformly in* $\gamma \in \Gamma$, *which is nonsingular at* γ_0, *and* $\sqrt{n}\frac{\partial Q_n(\gamma_0)}{\partial\gamma} = O_P(1)$. *If* $\frac{\partial^2 Q_n^*(\gamma)}{\partial\gamma\partial\gamma'} - \frac{\partial^2 Q_n(\gamma)}{\partial\gamma\partial\gamma'} = o_P(1)$ *uniformly in* $\gamma \in \Gamma$ *and* $\sqrt{n}\frac{\partial Q_n^*(\gamma_0)}{\partial\gamma} - \sqrt{n}\frac{\partial Q_n(\gamma_0)}{\partial\gamma} = o_P(1)$, *then* $\sqrt{n}(\hat{\gamma}_n^* - \gamma_0)$ *and* $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$ *have the same limiting distribution.*

Lemmas A.6–A.8 below summarize relevant matrices of the MESS which possess the essential UB property.

Lemma A.6. *Suppose that* $n \times n$ *matrices* $\{M_{n1}\}, \ldots, \{M_{nq}\}$ *are UB. The smallest eigenvalue of* $e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n}$ *is bounded away from zero uniformly over the interval* $[-\delta, \delta]^q$ *for some finite* $\delta > 0$. *Elements of the* $n \times k$ *matrix* X_n *are uniformly bounded. The limit of* $\frac{1}{n} X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n$ *exists and is nonsingular for any* $\boldsymbol{\tau} \in [-\delta, \delta]^q$. *Then* $e^{\boldsymbol{\tau}\mathbf{M}_n}$, $X_n(X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1} X_n'$ *and* $H_n(\boldsymbol{\tau}) = I_n - e^{\boldsymbol{\tau}\mathbf{M}_n} X_n (X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1} X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'}$ *are UB uniformly in* $\boldsymbol{\tau} \in [-\delta, \delta]^q$.

Lemma A.7. *Let* $W_{n1}, \ldots, W_{np}$, $M_{n1}, \ldots, M_{nq}$, A_n *and* B_n *be* $n \times n$ *matrices that are UB,* b_n *be an* n*-dimensional vector with uniformly bounded elements,* X_n *be an* $n \times k$ *matrix with uniformly bounded elements, and* $\epsilon_n = (\epsilon_{n1}, \ldots, \epsilon_{nn})'$ *be a random vector with independent elements that have mean zero and variances* σ_{ni}^2*'s. Assume that* $\lim_{n\to\infty} \frac{1}{n} X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n$ *exists and is nonsingular for any* $\boldsymbol{\tau} \in [-\delta, \delta]^q$

and the sequence $\{\mathrm{E}(\epsilon_{ni}^4)\}$ *is bounded. Then* $\frac{1}{n}b_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \epsilon_n = o_P(1)$ *uniformly on the parameter space* $\boldsymbol{\Phi} = [-\delta, \delta]^{p+q}$, $\frac{1}{n}b_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} B_n e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \epsilon_n = o_P(1)$ *uniformly on* $\boldsymbol{\Phi}$, $\frac{1}{n}[\epsilon_n' A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \epsilon_n - \mathrm{tr}(A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \Sigma_n)] = o_P(1)$ *uniformly on* $\boldsymbol{\Phi}$, $\frac{1}{n}[\epsilon_n' A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} B_n e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \epsilon_n - \mathrm{tr}(A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} B_n e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \Sigma_n)] = o_P(1)$ *uniformly on* $\boldsymbol{\Phi}$, *and* $\frac{1}{n}\mathrm{tr}(A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} (I_n - H_n(\boldsymbol{\tau})) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \Sigma_n) = o(1)$ *uniformly on* $\boldsymbol{\Phi}$, *where* $H_n(\boldsymbol{\tau}) = I_n - e^{\boldsymbol{\tau}\mathbf{M}_n} X_n (X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1} X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'}$ *and* $\Sigma_n = \mathrm{Diag}(\sigma_{n1}^2, \ldots, \sigma_{nn}^2)$.

Lemma A.8. *Let* A_n *be any* $n \times n$ *UB matrix and* $a_n = o_P(1)$. *Then* $||e^{a_n A_n} - I_n||_\infty = o_P(1)$ *and* $||e^{a_n A_n} - I_n||_1 = o_P(1)$.

3. Proofs of Some Lemmas

Proof of Lemma 1. The proof is sketched in (23) and (24) in the main paper. By the mean value theorem,

$$\begin{aligned}&\sqrt{n}[(e_{ni}' e^{-\hat{\alpha}_n W_n} e_{nj} \hat{\beta}_{np} - e_{nr}' e^{-\hat{\alpha}_n W_n} e_{ns} \hat{\beta}_{nq}) - (e_{ni}' e^{-\alpha_0 W_n} e_{nj} \beta_{0p} - e_{nr}' e^{-\alpha_0 W_n} e_{ns} \beta_{0q})] \\ &= \tilde{A}_{1n} \sqrt{n}(\hat{\alpha}_n - \alpha_0, \hat{\beta}_{np} - \beta_{0p}, \hat{\beta}_{nq} - \beta_{0q})',\end{aligned}$$

and

$$\frac{1}{\sqrt{n}}[\mathrm{tr}(e^{-\hat{\alpha}_n W_n})\hat{\beta}_{np} - \mathrm{tr}(e^{-\alpha_0 W_n})\beta_{0p}] = \tilde{A}_{2n}\sqrt{n}[\hat{\alpha}_n - \alpha_0, \hat{\beta}_{np} - \beta_{0p}]',$$

where $\tilde{A}_{1n} = [-e_{ni}' e^{-\tilde{\alpha}_n W_n} W_n e_{nj} \tilde{\beta}_{np} + e_{nr}' e^{-\tilde{\alpha}_n W_n} W_n e_{ns} \tilde{\beta}_{nq}, e_{ni}' e^{-\tilde{\alpha}_n W_n} e_{nj}, -e_{nr}' e^{-\tilde{\alpha}_n W_n} e_{ns}]$ and $\tilde{A}_{2n} = [-\frac{1}{n}\mathrm{tr}(e^{-\tilde{\alpha}_n W_n} W_n)\tilde{\beta}_{np}, \frac{1}{n}\mathrm{tr}(e^{-\tilde{\alpha}_n W_n})]$, with $[\tilde{\alpha}_n, \tilde{\beta}_{np}, \tilde{\beta}_{nq}]$ being between $[\hat{\alpha}_n, \hat{\beta}_{np}, \hat{\beta}_{nq}]$ and $[\alpha_0, \beta_{0p}, \beta_{0q}]$. For the first equalities in (23) and (24) to hold, it is sufficient to show that $\tilde{A}_{1n} - A_{1n} = o_P(1)$ and $\tilde{A}_{2n} - A_{2n} = o_P(1)$. By Lemma A.8, $||e^{(\alpha_0 - \tilde{\alpha}_n)W_n} - I_n||_\infty = o_P(1)$. Then $|e_{ni}'(e^{-\tilde{\alpha}_n W_n} - e^{-\alpha_0 W_n})W_n e_{nj}| = |e_{ni}'(e^{(\alpha_0 - \tilde{\alpha}_n)W_n} - I_n)e^{-\alpha_0 W_n} W_n e_{nj}| \leq ||e^{(\alpha_0 - \tilde{\alpha}_n)W_n} - I_n||_\infty ||e^{-\alpha_0 W_n} W_n||_\infty = o_P(1)$. Similarly, $e_{ni}'(e^{-\tilde{\alpha}_n W_n} - e^{-\alpha_0 W_n})e_{nj} = o_P(1)$. As $\tilde{\beta}_{np} - \beta_{0p} = o_P(1)$, $\tilde{\beta}_{nq} - \beta_{0q} = o_P(1)$ and $\tilde{\gamma}_n - \gamma_0 = o_P(1)$, $\tilde{A}_{1n} - A_{1n} = o_P(1)$. By the mean value theorem, $\frac{1}{n}[\mathrm{tr}(e^{-\tilde{\alpha}_n W_n} W_n) - \mathrm{tr}(e^{-\alpha_0 W_n} W_n)] = o_P(1)$ and $\frac{1}{n}[\mathrm{tr}(e^{-\tilde{\alpha}_n W_n}) - \mathrm{tr}(e^{-\alpha_0 W_n})] = o_P(1)$, then $\tilde{A}_{2n} - A_{2n} = o_P(1)$. The results now follow by Slutsky's lemma. □

Proof of Lemma A.6. As the smallest eigenvalue of $e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n}$ is bounded away from zero uniformly on $[-\delta, \delta]^q$, there exists a constant $\kappa > 0$ such that the smallest eigenvalue of $e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n}$ is greater or equal to κ for any n and $\boldsymbol{\tau} \in [-\delta, \delta]^q$. Write $e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} = \Gamma_n'(\boldsymbol{\tau})\Lambda_n(\boldsymbol{\tau})\Gamma_n(\boldsymbol{\tau})$, where $\Gamma_n(\boldsymbol{\tau})$ is an $n \times n$ orthonormal matrix and $\Lambda_n(\boldsymbol{\tau})$ is a diagonal matrix with the diagonal elements being the eigenvalues of $e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n}$. Then $e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} - \kappa I_n = \Gamma_n'(\boldsymbol{\tau})[\Lambda_n(\boldsymbol{\tau}) - \kappa I_n]\Gamma_n(\boldsymbol{\tau})$ is positive semi-definite, which implies that

$$(\frac{1}{n}\kappa X_n' X_n)^{-1} - (\frac{1}{n} X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1}$$

is also positive semi-definite. Thus, elements of $(\frac{1}{n}X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1}$ and $X_n(\frac{1}{n}X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1} X_n'$ are uniformly bounded in $\boldsymbol{\tau} \in [-\delta, \delta]^q$. It follows that $\frac{1}{n}X_n(\frac{1}{n}X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1} X_n'$ is UB uniformly in $\boldsymbol{\tau} \in [-\delta, \delta]^q$. Let $||.||$ be either the row or column sum norm. Since $||e^{\boldsymbol{\tau}\mathbf{M}_n}|| \leq \sum_{i=0}^{\infty} \frac{||\boldsymbol{\tau}\mathbf{M}_n||^i}{i!} \leq \sum_{i=0}^{\infty} \frac{q^i \delta^i (\max_{1\leq j\leq q} ||M_{nj}||)^i}{i!} = e^{q\delta \max_{1\leq j\leq q} ||M_{nj}||}$, $e^{\boldsymbol{\tau}\mathbf{M}_n}$ is UB uniformly in $\boldsymbol{\tau} \in [-\delta, \delta]^p$ if W_{nj}'s are UB. It follows that $H_n(\boldsymbol{\tau})$ is also UB uniformly in $\boldsymbol{\tau} \in [-\delta, \delta]^q$. □

Proof of Lemma A.7. We only show the results for terms involving $H_n(\boldsymbol{\tau})$, as the results for terms involving B_n can be shown similarly. By Theorem 21.9 on p. 337 of Davidson (1994), the uniform convergence of a sequence of stochastic functions $\{f_n(\boldsymbol{\phi})\}$ on $\boldsymbol{\Phi}$ follows from the pointwise convergence in probability $f_n(\boldsymbol{\phi}) = o_P(1)$ for every $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ and the stochastic equicontinuity of $\{f_n(\boldsymbol{\phi})\}$. For the stochastic equicontinuity, by Theorem 21.10 on p. 339 of Davidson (1994), a sufficient condition is that $|f_n(\boldsymbol{\phi}^*) - f_n(\boldsymbol{\phi})| \leq e_n h(||\boldsymbol{\phi}^* - \boldsymbol{\phi}||)$, for any $\boldsymbol{\phi}^*, \boldsymbol{\phi} \in \boldsymbol{\Phi}$, where $\{e_n\}$ is a stochastically bounded sequence not depending on $\boldsymbol{\phi}$, $h(x)$ is nonstochastic which goes down to 0 as x goes down to 0, and $||\cdot||$ denotes the Euclidean vector norm.

By Lemma A.6, $e^{\boldsymbol{\alpha}\mathbf{W}_n}$ and $H_n(\boldsymbol{\tau})$ are UB uniformly over their respective parameter spaces. Let $T_n(\boldsymbol{\phi}) = e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n}$ and $P_{1n}(\boldsymbol{\tau}) = I_n - H_n(\boldsymbol{\tau})$. Then $\frac{1}{n}b_n' T_n(\boldsymbol{\phi}) A_n \epsilon_n = o_P(1)$ for any $\boldsymbol{\phi} = (\boldsymbol{\alpha}', \boldsymbol{\tau}')'$ in $\boldsymbol{\Phi}$ and $\frac{1}{n}[\epsilon_n' A_n' T_n(\boldsymbol{\phi}) A_n \epsilon_n - \mathrm{tr}(A_n' T_n(\boldsymbol{\phi}) A_n \Sigma_n)] = o_P(1)$ for any $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ by Lemma A.3, and $\frac{1}{n}\mathrm{tr}(A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} P_{1n}(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \Sigma_n) = o(1)$ for any $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ by Lemma A.1. It remains to show the stochastic equicontinuity of the sequences $\{\frac{1}{n}b_n' T_n(\boldsymbol{\phi}) A_n \epsilon_n\}$, $\{\frac{1}{n}[\epsilon_n' A_n' T_n(\boldsymbol{\phi}) A_n \epsilon_n - \mathrm{tr}(A_n' T_n(\boldsymbol{\phi}) A_n \Sigma_n)]\}$ and $\{\frac{1}{n}\mathrm{tr}(A_n e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} P_{1n}(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \Sigma_n)\}$. Let $P_{2n}(\boldsymbol{\tau}) = X_n(X_n' e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} X_n)^{-1} X_n'$. By the mean value theorem,

$$\frac{1}{n}b_n' T_n(\boldsymbol{\phi}^*) A_n \epsilon_n - \frac{1}{n}b_n' T_n(\boldsymbol{\phi}) A_n \epsilon_n = \frac{1}{n}\sum_{i=1}^{p} b_n' \frac{\partial T_n(\tilde{\boldsymbol{\phi}})}{\partial \alpha_i} A_n \epsilon_n (\alpha_i^* - \alpha_i) + \frac{1}{n}\sum_{i=1}^{q} b_n' \frac{\partial T_n(\tilde{\boldsymbol{\phi}})}{\partial \tau_i} A_n \epsilon_n (\tau_i^* - \tau_i),$$

where $\tilde{\boldsymbol{\phi}}$ is between $\boldsymbol{\phi}^*$ and $\boldsymbol{\phi}$, $\frac{\partial T_n(\boldsymbol{\phi})}{\partial \alpha_i} = \big(e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} \frac{\partial e^{\boldsymbol{\alpha}\mathbf{W}_n}}{\partial \alpha_i}\big)^s$, and

$$\frac{\partial T_n(\boldsymbol{\phi})}{\partial \tau_i} = e^{(\boldsymbol{\alpha}\mathbf{W}_n)'}\big[e^{(\boldsymbol{\tau}\mathbf{M}_n)'} \frac{\partial H_n(\boldsymbol{\tau})}{\partial \tau_i} e^{\boldsymbol{\tau}\mathbf{M}_n} + \big(e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) \frac{\partial e^{\boldsymbol{\tau}\mathbf{M}_n}}{\partial \tau_i}\big)^s\big] e^{\boldsymbol{\alpha}\mathbf{W}_n}$$

with $\frac{\partial H_n(\boldsymbol{\tau})}{\partial \tau_i} = e^{\boldsymbol{\tau}\mathbf{M}_n} P_{2n}(\boldsymbol{\tau}) \big(e^{(\boldsymbol{\tau}\mathbf{M}_n)'} \frac{\partial e^{\boldsymbol{\tau}\mathbf{M}_n}}{\partial \tau_i}\big)^s P_{2n}(\boldsymbol{\tau}) e^{(\boldsymbol{\tau}\mathbf{M}_n)'} - \big[\frac{\partial e^{\boldsymbol{\tau}\mathbf{M}_n}}{\partial \tau_i} P_{2n}(\boldsymbol{\tau}) e^{(\boldsymbol{\tau}\mathbf{M}_n)'}\big]^s$. Note that

$$||\frac{\partial (\boldsymbol{\alpha}\mathbf{W}_n)^j}{\partial \alpha_i}||_\infty = ||\sum_{l=0}^{j-1} (\boldsymbol{\alpha}\mathbf{W}_n)^l W_{ni} (\boldsymbol{\alpha}\mathbf{W}_n)^{j-l-1}||_\infty \leq \sum_{l=0}^{j-1} ||\boldsymbol{\alpha}\mathbf{W}_n||_\infty^{j-1} ||W_{ni}||_\infty = j||\boldsymbol{\alpha}\mathbf{W}_n||_\infty^{j-1} ||W_{ni}||_\infty.$$

Then

$$||\frac{\partial e^{\boldsymbol{\alpha}\mathbf{W}_n}}{\partial \alpha_i}||_\infty = ||\sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial (\boldsymbol{\alpha}\mathbf{W}_n)^j}{\partial \alpha_i}||_\infty \leq \sum_{j=1}^{\infty} \frac{||\boldsymbol{\alpha}\mathbf{W}_n||_\infty^{j-1} ||W_{ni}||_\infty}{(j-1)!} = e^{||\boldsymbol{\alpha}\mathbf{W}_n||_\infty} ||W_{ni}||_\infty.$$

Thus, $\frac{\partial e^{\boldsymbol{\alpha}\mathbf{W}_n}}{\partial \alpha_i}$ is bounded in row (similarly, column) sum norm uniformly in $\boldsymbol{\alpha}$. Similarly, $\frac{\partial e^{\boldsymbol{\tau}\mathbf{M}_n}}{\partial \tau_i}$ is bounded in row and sum norms uniformly in $\boldsymbol{\tau}$. Hence, by Lemma A.6, there exists a finite constant c such that all elements of $\frac{1}{n}b_n'\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\alpha_i}A_n$ and $\frac{1}{n}b_n'\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\tau_i}A_n$ in absolute value are bounded by c. Therefore,

$$|\frac{1}{n}b_n'T_n(\boldsymbol\phi^*)A_n\epsilon_n-\frac{1}{n}b_n'T_n(\boldsymbol\phi)A_n\epsilon_n|\leq\frac{(p+q)c}{n}\sum_{i=1}^n|\epsilon_{ni}|\cdot||\boldsymbol\phi^*-\boldsymbol\phi||,$$

where $\frac{1}{n}\sum_{i=1}^n|\epsilon_{ni}|=O_P(1)$ by Markov's inequality. Then $\{\frac{1}{n}b_n'T_n(\boldsymbol\phi)A_n\epsilon_n\}$ is stochastically equicontinuous. For $\{\frac{1}{n}[\epsilon_n'A_n'T_n(\boldsymbol\phi)A_n\epsilon_n-\text{tr}(A_n'T_n(\boldsymbol\phi)A_n\Sigma_n)]\}$, by the mean value theorem,

$$\begin{aligned}
&|\frac{1}{n}[\epsilon_n'A_n'T_n(\boldsymbol\phi^*)A_n\epsilon_n-\text{tr}(A_n'T_n(\boldsymbol\phi^*)A_n\Sigma_n)]-\frac{1}{n}[\epsilon_n'A_n'T_n(\boldsymbol\phi)A_n\epsilon_n-\text{tr}(A_n'T_n(\boldsymbol\phi)A_n\Sigma_n)]|\\
&=|\frac{1}{2n}\sum_{i=1}^p\epsilon_n'A_n'(\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\alpha_i}+\frac{\partial T_n'(\tilde{\boldsymbol\phi})}{\partial\alpha_i})A_n\epsilon_n(\alpha_i^*-\alpha_i)+\frac{1}{2n}\sum_{i=1}^q\epsilon_n'A_n'(\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\tau_i}+\frac{\partial T_n'(\tilde{\boldsymbol\phi})}{\partial\tau_i})A_n\epsilon_n(\tau_i^*-\tau_i)\\
&\quad-\frac{1}{n}\sum_{i=1}^p\text{tr}[A_n'\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\alpha_i}A_n\Sigma_n](\alpha_i^*-\alpha_i)-\frac{1}{n}\sum_{i=1}^q\text{tr}[A_n'\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\tau_i}A_n\Sigma_n](\tau_i^*-\tau_i)|\\
&\leq\Big(\frac{1}{2n}\sum_{i=1}^p|\epsilon_n'A_n'(\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\alpha_i}+\frac{\partial T_n'(\tilde{\boldsymbol\phi})}{\partial\alpha_i})A_n\epsilon_n|+\frac{1}{2n}\sum_{i=1}^q|\epsilon_n'A_n'(\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\tau_i}+\frac{\partial T_n'(\tilde{\boldsymbol\phi})}{\partial\tau_i})A_n\epsilon_n|\\
&\quad+\frac{1}{n}\sum_{i=1}^p|\text{tr}[A_n'\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\alpha_i}A_n\Sigma_n]|+\frac{1}{n}\sum_{i=1}^q|\text{tr}[A_n'\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\tau_i}A_n\Sigma_n]|\Big)||\boldsymbol\phi^*-\boldsymbol\phi||,
\end{aligned}$$

where $\tilde{\boldsymbol\phi}$ lies in between $\boldsymbol\phi^*$ and $\boldsymbol\phi$. As $A_n'(\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\alpha_i}+\frac{\partial T_n'(\tilde{\boldsymbol\phi})}{\partial\alpha_i})A_n$ is symmetric, by the eigenvalue-eigenvector decomposition, there exists othonormal matrix Γ_n and eigenvalue matrix $\Lambda_n=\text{Diag}\{\lambda_{n1},\cdots,\lambda_{nn}\}$ such that

$$\begin{aligned}
\frac{1}{n}|\epsilon_n'A_n'(\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\alpha_i}+\frac{\partial T_n'(\tilde{\boldsymbol\phi})}{\partial\alpha_i})A_n\epsilon_n|&=\frac{1}{n}|\epsilon_n'\Gamma_n\Lambda_n\Gamma_n'\epsilon_n|\leq\frac{1}{n}\max_{i=1,\cdots,n}|\lambda_{ni}|\cdot\epsilon_n'\epsilon_n\\
&\leq\frac{1}{n}||A_n'(\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\alpha_i}+\frac{\partial T_n'(\tilde{\boldsymbol\phi})}{\partial\alpha_i})A_n||_\infty\cdot\epsilon_n'\epsilon_n\leq\frac{c_1}{n}\epsilon_n'\epsilon_n=O_P(1),
\end{aligned}$$

by the spectral radius theorem, for some constant c_1, because $A_n'(\frac{\partial T_n(\boldsymbol\phi)}{\partial\alpha_i}+\frac{\partial T_n'(\boldsymbol\phi)}{\partial\alpha_i})A_n$ is UB uniformly in $\boldsymbol\phi\in\boldsymbol\Phi$. Similarly, $\frac{1}{n}|\epsilon_n'A_n'(\frac{\partial T_n(\tilde{\boldsymbol\phi})}{\partial\tau_i}+\frac{\partial T_n'(\tilde{\boldsymbol\phi})}{\partial\tau_i})A_n\epsilon_n|\leq\frac{c_1}{n}\epsilon_n'\epsilon_n=O_P(1)$. Furthermore, $\frac{1}{n}\text{tr}[A_n'\frac{\partial T_n(\boldsymbol\phi)}{\partial\alpha_i}A_n\Sigma_n]$ and $\frac{1}{n}\text{tr}[A_n'\frac{\partial T_n(\boldsymbol\phi)}{\partial\tau_i}A_n\Sigma_n]$ are bounded uniformly on $\boldsymbol\Phi$. Then $\{\frac{1}{n}[\epsilon_n'A_n'T_n(\boldsymbol\phi)A_n\epsilon_n-\text{tr}(A_n'T_n(\boldsymbol\phi)A_n\Sigma_n)]\}$ is stochastically equicontinuous.

For $\{\frac{1}{n}\text{tr}(A_n'e^{(\boldsymbol\alpha\mathbf{W}_n)'}e^{(\boldsymbol\tau\mathbf{M}_n)'}P_{1n}(\boldsymbol\tau)e^{\boldsymbol\tau\mathbf{M}_n}e^{\boldsymbol\alpha\mathbf{W}_n}A_n\Sigma_n)\}$, as

$$\begin{aligned}
\frac{1}{n}\text{tr}(A_n'e^{(\boldsymbol\alpha\mathbf{W}_n)'}e^{(\boldsymbol\tau\mathbf{M}_n)'}P_{1n}(\boldsymbol\tau)e^{\boldsymbol\tau\mathbf{M}_n}e^{\boldsymbol\alpha\mathbf{W}_n}A_n\Sigma_n)&=\frac{1}{n}\text{tr}(A_n'e^{(\boldsymbol\alpha\mathbf{W}_n)'}e^{(\boldsymbol\tau\mathbf{M}_n)'}e^{\boldsymbol\tau\mathbf{M}_n}e^{\boldsymbol\alpha\mathbf{W}_n}A_n\Sigma_n)\\
&\quad-\frac{1}{n}\text{tr}(A_n'T_n(\boldsymbol\phi)A_n\Sigma_n),
\end{aligned}$$

by the mean value theorem,

$$\begin{aligned}
&\frac{1}{n}\operatorname{tr}(A_n' e^{(\boldsymbol{\alpha}^*\mathbf{W}_n)'} e^{(\boldsymbol{\tau}^*\mathbf{M}_n)'} P_{1n}(\boldsymbol{\tau}^*) e^{\boldsymbol{\tau}^*\mathbf{M}_n} e^{\boldsymbol{\alpha}^*\mathbf{W}_n} A_n \Sigma_n) - \frac{1}{n}\operatorname{tr}(A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} P_{1n}(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \Sigma_n) \\
&= \frac{1}{n}\frac{\partial}{\partial \boldsymbol{\phi}'}\operatorname{tr}(A_n' e^{(\tilde{\boldsymbol{\alpha}}\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\tau}}\mathbf{M}_n)'} P_{1n}(\tilde{\boldsymbol{\tau}}) e^{\tilde{\boldsymbol{\tau}}\mathbf{M}_n} e^{\tilde{\boldsymbol{\alpha}}\mathbf{W}_n} A_n \Sigma_n)\cdot(\tilde{\boldsymbol{\phi}} - \boldsymbol{\phi}) \\
&= \frac{2}{n}\sum_{i=1}^{p}\operatorname{tr}(A_n' e^{(\tilde{\boldsymbol{\alpha}}\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\tau}}\mathbf{M}_n)'} e^{\tilde{\boldsymbol{\tau}}\mathbf{M}_n}\frac{\partial e^{\tilde{\boldsymbol{\alpha}}\mathbf{W}_n}}{\partial \alpha_i} A_n \Sigma_n)(\alpha_i^* - \alpha_i) \\
&\quad + \frac{2}{n}\sum_{i=1}^{q}\operatorname{tr}(A_n' e^{(\tilde{\boldsymbol{\alpha}}\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\tau}}\mathbf{M}_n)'}\frac{\partial e^{\tilde{\boldsymbol{\tau}}\mathbf{M}_n}}{\partial \tau_i} e^{\tilde{\boldsymbol{\alpha}}\mathbf{W}_n} A_n \Sigma_n)(\tau_i^* - \tau_i) \\
&\quad - \frac{1}{n}\sum_{i=1}^{p}\operatorname{tr}\big(A_n'\frac{\partial T_n(\tilde{\boldsymbol{\phi}})}{\partial \alpha_i} A_n \Sigma_n\big)(\alpha_i^* - \alpha_i) - \frac{1}{n}\sum_{i=1}^{q}\operatorname{tr}\big(A_n'\frac{\partial T_n(\tilde{\boldsymbol{\phi}})}{\partial \tau_i} A_n \Sigma_n\big)(\tau_i^* - \tau_i),
\end{aligned}$$

where $\tilde{\boldsymbol{\phi}}$ is between $\boldsymbol{\phi}^*$ and $\boldsymbol{\phi}_0$. By an argument similar to above ones,

$$||\frac{1}{n}\frac{\partial}{\partial \boldsymbol{\phi}'}\operatorname{tr}(A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} P_{1n}(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \Sigma_n)||$$

is bounded by a constant not depending on $\boldsymbol{\phi}$. Thus $\frac{1}{n}\operatorname{tr}(A_n' e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} P_{1n}(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} A_n \Sigma_n)$ is equicontinuous. The results in the lemma then follow from the pointwise convergence and stochastic equicontinuity. □

Proof of Lemma A.8. $||e^{a_n A_n} - I_n||_\infty = ||\sum_{j=1}^{\infty}\frac{1}{j!}a_n^j A_n^j||_\infty \leq \sum_{j=1}^{\infty}\frac{1}{j!}|a_n|^j ||A_n||_\infty^j = e^{|a_n|||A_n||_\infty} - 1 = o_P(1)$. Similarly, $||e^{a_n A_n} - I_n||_1 = o_P(1)$. □

4. Proofs of Some Propositions

Proof of Proposition 1. The consistency of the QMLE $\hat{\gamma}_n$ will follow from the uniform convergence of $[Q_n(\phi) - \bar{Q}_n(\phi)]/n$ to zero and the identification uniqueness condition (White, 1994, Theorem 3.4).

We first show the uniform convergence that $\sup_{\phi\in\Phi}\frac{1}{n}|Q_n(\phi) - \bar{Q}_n(\phi)| = o_P(1)$. As $y_n = e^{-\alpha_0 W_n}(X_n\beta_0 + e^{-\tau_0 M_n}\epsilon_n)$,

$$\begin{aligned}
\frac{1}{n}[Q_n(\phi) - \bar{Q}_n(\phi)] &= \frac{2}{n}(X_n\beta_0)' e^{(\alpha-\alpha_0)W_n'} e^{\tau M_n'} H_n(\tau) e^{\tau M_n} e^{(\alpha-\alpha_0)W_n} e^{-\tau_0 M_n}\epsilon_n \\
&\quad + \frac{1}{n}\epsilon_n' e^{-\tau_0 M_n'} e^{(\alpha-\alpha_0)W_n'} e^{\tau M_n'} H_n(\tau) e^{\tau M_n} e^{(\alpha-\alpha_0)W_n} e^{-\tau_0 M_n}\epsilon_n \\
&\quad - \frac{\sigma_0^2}{n}\operatorname{tr}[e^{-\tau_0 M_n'} e^{(\alpha-\alpha_0)W_n'} e^{\tau M_n'} H_n(\tau) e^{\tau M_n} e^{(\alpha-\alpha_0)W_n} e^{-\tau_0 M_n}] \\
&\quad - \frac{\sigma_0^2}{n}\operatorname{tr}\{e^{-\tau_0 M_n'} e^{(\alpha-\alpha_0)W_n'} e^{\tau M_n'}[I_n - H_n(\tau)] e^{\tau M_n} e^{(\alpha-\alpha_0)W_n} e^{-\tau_0 M_n}\}.
\end{aligned}$$

By Lemma A.7, $\frac{1}{n}[Q_n(\phi) - \bar{Q}_n(\phi)] = o_P(1)$ uniformly on Φ.

We now show that $\frac{1}{n}\bar{Q}_n(\phi)$ is uniformly equicontinuous. By the mean value theorem, for $\phi_1, \phi_2 \in \Phi$,

$$\begin{aligned}\frac{1}{n}[\bar{Q}_n(\phi_1) - \bar{Q}_n(\phi_2)] &= 2(X_n\beta_0)'e^{(\tilde{\alpha}-\alpha_0)W_n'}W_n'e^{\tilde{\tau}M_n'}H_n(\tilde{\tau})e^{\tilde{\tau}M_n}e^{(\tilde{\alpha}-\alpha_0)W_n}X_n\beta_0(\alpha_1-\alpha_2)\\ &+\frac{1}{n}[(X_n\beta_0)'e^{(\tilde{\alpha}-\alpha_0)W_n'}e^{\tilde{\tau}M_n'}(2M_n'H_n(\tilde{\tau})+\frac{\partial H_n(\tilde{\tau})}{\partial\tau})e^{\tilde{\tau}M_n}e^{(\tilde{\alpha}-\alpha_0)W_n}X_n\beta_0](\tau_1-\tau_2)\\ &+\frac{2\sigma_0^2}{n}\operatorname{tr}[e^{-\tau_0M_n'}e^{(\tilde{\alpha}-\alpha_0)W_n'}W_n'e^{\tilde{\tau}M_n'}e^{\tilde{\tau}M_n}e^{(\tilde{\alpha}-\alpha_0)W_n}e^{-\tau_0M_n}](\alpha_1-\alpha_2)\\ &+\frac{2\sigma_0^2}{n}\operatorname{tr}[e^{-\tau_0M_n'}e^{(\tilde{\alpha}-\alpha_0)W_n'}e^{\tilde{\tau}M_n'}M_ne^{\tilde{\tau}M_n}e^{(\tilde{\alpha}-\alpha_0)W_n}e^{-\tau_0M_n}](\tau_1-\tau_2),\end{aligned}$$

where $\tilde{\alpha}$ is between α_1 and α_2, $\tilde{\tau}$ is between τ_1 and τ_2, and $\frac{\partial H_n(\tau)}{\partial\tau} = -M_nP_n(\tau) - P_n(\tau)M_n' + P_n(\tau)(M_n' + M_n)P_n(\tau)$ with $P_n(\tau) = I_n - H_n(\tau)$. Since $P_n(\tau)$ is UB uniformly over the parameter space by Lemma A.6, and so are $e^{\alpha W_n}$ and $e^{\tau M_n}$, there exists some constant c such that

$$\frac{1}{n}|\bar{Q}_n(\phi_1) - \bar{Q}_n(\phi_2)| \le c(|\alpha_1-\alpha_2| + |\tau_1-\tau_2|).$$

Thus $\frac{1}{n}\bar{Q}_n(\phi)$ is uniformly equicontinuous.

Finally, we show that the identification uniqueness condition holds. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $A_n(\phi) = e^{-\tau_0M_n'}e^{(\alpha-\alpha_0)W_n'}e^{\tau M_n'}e^{\tau M_n}e^{(\alpha-\alpha_0)W_n}e^{-\tau_0M_n}$. Since $A_n(\phi)$ is positive definite, λ_i's are all positive. Then by the inequality of arithmetic and geometric means,

$$\begin{aligned}\frac{1}{n}\operatorname{tr}(A_n(\phi)) &= \frac{1}{n}\sum_{i=1}^n\lambda_i \ge (\prod_{i=1}^n\lambda_i)^{1/n} = |A_n(\phi)|^{1/n}\\ &= [e^{-\tau_0\operatorname{tr}(M_n')}e^{(\alpha-\alpha_0)\operatorname{tr}(W_n')}e^{\tau\operatorname{tr}(M_n')}e^{\tau\operatorname{tr}(M_n)}e^{(\alpha-\alpha_0)\operatorname{tr}(W_n)}e^{-\tau_0\operatorname{tr}(M_n)}]^{1/n} = 1,\end{aligned}$$

because $\operatorname{tr}(M_n) = \operatorname{tr}(W_n) = 0$. In addition, $(X_n\beta_0)'e^{(\alpha-\alpha_0)W_n'}W_n'e^{\tau M_n'}H_n(\tau)e^{\tau M_n}e^{(\alpha-\alpha_0)W_n}X_n\beta_0 \ge 0$. Thus, $\frac{1}{n}\bar{Q}_n(\phi) \ge \sigma_0^2$. When $\phi = \phi_0$, $\frac{1}{n}\bar{Q}_n(\phi) = \sigma_0^2$. Assumption 6 implies that whenever $\phi \ne \phi_0$, $\lim_{n\to\infty}\frac{1}{n}\bar{Q}_n(\phi) \ne \sigma_0^2$. Thus, with uniform equicontinuity, the identification uniqueness condition holds.

With the uniform convergence and identification uniqueness condition, the consistency of $\hat{\phi}_n$ follows. Using the formula $\hat{\beta}_n(\phi)$ as a function of ϕ in the main paper, the consistency of $\hat{\beta}_n$ follows by plugging $\hat{\phi}_n$ into the function. □

Proof of Proposition 2. Applying the mean value theorem to the first-order condition $\frac{\partial Q_n(\hat{\gamma}_n)}{\partial\gamma} = 0$ at the true γ_0, we have

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) = -(\frac{1}{n}\frac{\partial^2Q_n(\tilde{\gamma}_n)}{\partial\gamma\partial\gamma'})^{-1}\frac{1}{\sqrt{n}}\frac{\partial Q_n(\gamma_0)}{\partial\gamma}, \tag{11}$$

where $\tilde{\gamma}_n$ is between $\hat{\gamma}_n$ and γ_0. To obtain the asymptotic distribution, we first show that $\frac{1}{n}\frac{\partial^2Q_n(\tilde{\gamma}_n)}{\partial\gamma\partial\gamma'} = \frac{1}{n}\frac{\partial^2Q_n(\gamma_0)}{\partial\gamma\partial\gamma'} + o_P(1) = \frac{1}{n}\operatorname{E}\frac{\partial^2Q_n(\gamma_0)}{\partial\gamma\partial\gamma'} + o_P(1) = C_n + o_P(1)$. The second-order derivatives of $Q_n(\gamma)$ are

$$\begin{aligned}\frac{\partial^2Q_n(\gamma)}{\partial\alpha^2} &= 2y_n'e^{\alpha W_n'}W_n'^2e^{\tau M_n'}e^{\tau M_n}(e^{\alpha W_n}y_n - X_n\beta) + 2y_n'e^{\alpha W_n'}W_n'e^{\tau M_n'}e^{\tau M_n}W_ne^{\alpha W_n}y_n,\\ \frac{\partial^2Q_n(\gamma)}{\partial\alpha\partial\tau} &= 2y_n'e^{\alpha W_n'}W_n'e^{\tau M_n'}(M_n'+M_n)e^{\tau M_n}(e^{\alpha W_n}y_n - X_n\beta),\end{aligned}$$

$$\frac{\partial^2 Q_n(\gamma)}{\partial\alpha\partial\beta} = -2X_n' e^{\tau M_n'} e^{\tau M_n} W_n e^{\alpha W_n} y_n,$$
$$\frac{\partial^2 Q_n(\gamma)}{\partial\tau^2} = 2(e^{\alpha W_n} y_n - X_n\beta)' e^{\tau M_n'}(M_n' M_n + M_n^2) e^{\tau M_n}(e^{\alpha W_n} y_n - X_n\beta),$$
$$\frac{\partial^2 Q_n(\gamma)}{\partial\tau\partial\beta} = -2X_n' e^{\tau M_n'}(M_n + M_n') e^{\tau M_n}(e^{\alpha W_n} y_n - X_n\beta),$$
$$\frac{\partial^2 Q_n(\gamma)}{\partial\beta\partial\beta'} = 2X_n' e^{\tau M_n'} e^{\tau M_n} X_n.$$

We may first write $e^{\tilde{\alpha}_n W_n} = (e^{\tilde{\alpha}_n W_n} - e^{\alpha_0 W_n}) + e^{\alpha_0 W_n}$, $e^{\tilde{\tau}_n M_n} = (e^{\tilde{\tau}_n M_n} - e^{\tau_0 M_n}) + e^{\tau_0 M_n}$ and $\tilde{\beta}_n = (\tilde{\beta}_n - \beta_0) + \beta_0$, and then expand the terms for $\frac{1}{n}\frac{\partial^2 Q_n(\tilde{\gamma}_n)}{\partial\gamma\partial\gamma'}$. By Lemma A.3 and the reduced form of y_n, $\frac{1}{n} y_n' A_n y_n = O_P(1)$ and $\frac{1}{n} X_n' A_n y_n = O_P(1)$, where A_n is an $n \times n$ matrix that is UB. We also note that $||e^{\tilde{\alpha}_n W_n} - e^{\alpha_0 W_n}||_\infty = ||(e^{(\tilde{\alpha}_n - \alpha_0) W_n} - I_n) e^{\alpha_0 W_n}||_\infty \le ||e^{(\tilde{\alpha}_n - \alpha_0) W_n} - I_n||_\infty ||e^{\alpha_0 W_n}||_\infty = o_P(1)$ by Lemma A.8, and similarly $||e^{\tilde{\tau}_n M_n} - e^{\tau_0 M_n}||_\infty = o_P(1)$. Then from the expanded forms of the terms for $\frac{1}{n}\frac{\partial^2 Q_n(\tilde{\gamma}_n)}{\partial\gamma\partial\gamma'}$, we have $\frac{1}{n}\frac{\partial^2 Q_n(\tilde{\gamma}_n)}{\partial\gamma\partial\gamma'} = \frac{1}{n}\frac{\partial^2 Q_n(\gamma_0)}{\partial\gamma\partial\gamma'} + o_P(1)$. The equality $\frac{1}{n}\frac{\partial^2 Q_n(\gamma_0)}{\partial\gamma\partial\gamma'} = \frac{1}{n}\operatorname{E}\frac{\partial^2 Q_n(\theta_0)}{\partial\gamma\partial\gamma'} + o_P(1)$ follows by Lemma A.3 as each element of $\frac{1}{n}\frac{\partial^2 Q_n(\gamma_0)}{\partial\gamma\partial\gamma'} - \frac{1}{n}\operatorname{E}\frac{\partial^2 Q_n(\theta_0)}{\partial\gamma\partial\gamma'}$ is a linear-quadratic function of ϵ_n by the reduced form of y_n.

We now show that $\lim_{n\to\infty} C_n$ is invertible under Assumption 7. Let $\eta = (\eta_1, \eta_2, \eta_3')'$ be a vector whose length is equal to the column dimension of C_n, where η_1 and η_2 are scalars. Consider the linear equation system $\lim_{n\to\infty} C_n\eta = 0$. It is sufficient to show that $\eta = 0$. Since $\lim_{n\to\infty}\frac{1}{n}(e^{\tau_0 M_n} X_n)'(e^{\tau_0 M_n} X_n)$ is nonsingular, from the third row block of $\lim_{n\to\infty} C_n\eta = 0$, we have

$$\eta_3 = \lim_{n\to\infty}[\frac{1}{n}(e^{\tau_0 M_n} X_n)'(e^{\tau_0 M_n} X_n)]^{-1}\frac{1}{n}(e^{\tau_0 M_n} X_n)'\mathbb{W}_n e^{\tau_0 M_n} X_n\beta_0\eta_1.$$

As $\lim_{n\to}\operatorname{tr}(M_n^s M_n^s) \neq 0$, the second row block of $\lim_{n\to\infty} C_n\eta = 0$ implies that

$$\eta_2 = -\eta_1 \lim_{n\to\infty}\operatorname{tr}(\mathbb{W}_n^s M_n^s)/\operatorname{tr}(M_n^s M_n^s).$$

Substituting the expressions for η_2 and η_3 into the first row block of $\lim_{n\to\infty} C_n\eta = 0$, we have

$$\lim_{n\to\infty}[\frac{1}{n}(\mathbb{W}_n e^{\tau_0 M_n} X_n\beta_0)' H_n(\tau_0)(\mathbb{W}_n e^{\tau_0 M_n} X_n\beta_0) + \frac{\sigma_0^2}{2\operatorname{tr}(M_n^s M_n^s)}[\operatorname{tr}(\mathbb{W}_n^s\mathbb{W}_n^s)\operatorname{tr}(M_n^s M_n^s) - \operatorname{tr}^2(\mathbb{W}_n^s M_n^s)]]\eta_1 = 0.$$

Thus Assumption 7 implies that $\eta_1 = 0$. Therefore, $\lim_{n\to\infty} C_n$ is nonsingular.

As $\lim_{n\to\infty} C_n$ is nonsingular, by (11), for large enough n,

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) = -C_n^{-1}\frac{1}{\sqrt{n}}\frac{\partial Q_n(\gamma_0)}{\partial\gamma} + o_P(1),$$

where each element of $\frac{\partial Q_n(\gamma_0)}{\partial\gamma}$ is a linear-quadratic form of ϵ_n as shown in (8) in the main paper. Applying the central limit theorem in Lemma A.4, we have

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) \xrightarrow{d} N(0, \lim_{n\to\infty} C_n^{-1}\Omega_n C_n^{-1}),$$

where $\Omega_n = \frac{1}{n}\,\mathrm{E}\big(\frac{\partial Q_n(\gamma_0)}{\partial \gamma}\frac{\partial Q_n(\gamma_0)}{\partial \gamma'}\big)$, whose explicit expression can be derived by Lemma A.2. When $\epsilon_n \sim N(0, \sigma_0^2 I_n)$, $\tau_0 = 0$ or W_n and M_n can commute, $\Omega_n = 2\sigma_0^2 C_n$, thus the asymptotic VC matrix of $\hat{\gamma}_n$ simplifies to $2\sigma_0^2 \lim_{n\to\infty} C_n^{-1}$. □

Proofs of Proposition 3 and Proposition 4. The proofs resemble those for Propositions 1 and 2 respectively, even though the identification conditions are slightly different, thus they are omitted. □

Proof of Proposition 5. To prove the results in this proposition, it is sufficient to show that (i) $\frac{1}{n}\mathrm{tr}(\hat{\Sigma}_n W_n^s \hat{\Sigma}_n M_n^s) - \frac{1}{n}\mathrm{tr}(\Sigma_n W_n^s \Sigma_n M_n^s) = o_P(1)$, (ii) $\frac{1}{n}\mathrm{tr}(W_n^s M_n \hat{\Sigma}_n) - \frac{1}{n}\mathrm{tr}(W_n^s M_n \Sigma_n) = o_P(1)$,

$$\text{(iii)}\ \frac{1}{n} r_n' e^{\hat{\tau}_n M_n'} e^{\hat{\tau}_n M_n} s_n - \frac{1}{n} r_n' e^{\tau_0 M_n'} e^{\tau_0 M_n} s_n = o_P(1),$$

and (iv) $\frac{1}{n} r_n' e^{\hat{\tau}_n M_n'} \hat{\Sigma}_n e^{\hat{\tau}_n M_n} s_n - \frac{1}{n} r_n' e^{\tau_0 M_n'} \Sigma_n e^{\tau_0 M_n} s_n = o_P(1)$, where the n-dimensional vectors r_n and s_n have uniformly bounded elements.

We first show that (i) holds. Note that $\mathrm{tr}(\Sigma_n W_n^s \Sigma_n M_n^s) = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ni}^2 \sigma_{nj}^2 (w_{n,ij} + w_{n,ji})(m_{n,ij} + m_{n,ji})$, where $w_{n,ij}$ and $m_{n,ij}$ are, respectively, the (i,j)th elements of W_n and M_n. Let H_n be an $n \times n$ symmetric matrix with the (i,j)th element $h_{n,ij}$ being $(w_{n,ij} + w_{n,ji})(m_{n,ij} + m_{n,ji})$. Then H_n is UB, as $\sum_{i=1}^n |h_{n,ij}| \le c\sum_{i=1}^n (|w_{n,ij}| + |w_{n,ji}|)$ for some constant c. To show that (i) holds, we may show that (1) $\frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n (\epsilon_{ni}^2 \epsilon_{nj}^2 - \sigma_{ni}^2 \sigma_{nj}^2) h_{n,ij} = o_P(1)$ and (2) $B_n = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n (\hat{\epsilon}_{ni}^2 \hat{\epsilon}_{nj}^2 - \epsilon_{ni}^2 \epsilon_{nj}^2) h_{n,ij} = o_P(1)$. It has been shown in the proof of Proposition 2 in Lin and Lee (2010) that (1) holds. Thus we only show that (2) holds. As $\hat{\epsilon}_{ni}^2 \hat{\epsilon}_{nj}^2 - \epsilon_{ni}^2 \epsilon_{nj}^2 = (\hat{\epsilon}_{ni}^2 - \epsilon_{ni}^2)\epsilon_{nj}^2 + \epsilon_{ni}^2(\hat{\epsilon}_{nj}^2 - \epsilon_{nj}^2) + (\hat{\epsilon}_{ni}^2 - \epsilon_{ni}^2)(\hat{\epsilon}_{nj}^2 - \epsilon_{nj}^2)$, $B_n = B_{n1} + B_{n2} + B_{n3}$, where $B_{n1} = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n (\hat{\epsilon}_{ni}^2 - \epsilon_{ni}^2)\epsilon_{nj}^2 h_{n,ij}$, $B_{n2} = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n \epsilon_{ni}^2 (\hat{\epsilon}_{nj}^2 - \epsilon_{nj}^2) h_{n,ij}$ and $B_{n3} = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n (\hat{\epsilon}_{ni}^2 - \epsilon_{ni}^2)(\hat{\epsilon}_{nj}^2 - \epsilon_{nj}^2) h_{n,ij}$. We shall show that $B_{ni} = o_P(1)$ for $i = 1, 2, 3$. From the model, we have

$$\begin{aligned}
\hat{\epsilon}_n &= e^{\hat{\tau}_n M_n}(e^{\hat{\alpha}_n W_n} y_n - X_n \hat{\beta}_n) \\
&= [e^{\hat{\tau}_n M_n} e^{(\hat{\alpha}_n - \alpha_0) W_n} e^{-\tau_0 M_n} - I_n]\epsilon_n + e^{\hat{\tau}_n M_n}(e^{(\hat{\alpha}_n - \alpha_0) W_n} - I_n) X_n \beta_0 + e^{\hat{\tau}_n M_n} X_n (\beta_0 - \hat{\beta}_n) + \epsilon_n.
\end{aligned}$$

Then $\hat{\epsilon}_{ni} = a_{ni} + b_{ni} + c_{ni} + \epsilon_{ni}$, where $a_{ni} = e_{ni}[e^{\hat{\tau}_n M_n} e^{(\hat{\alpha}_n - \alpha_0) W_n} e^{-\tau_0 M_n} - I_n]\epsilon_n$, $b_{ni} = e_{ni} e^{\hat{\tau}_n M_n}(e^{(\hat{\alpha}_n - \alpha_0) W_n} - I_n) X_n \beta_0$, and $c_{ni} = e_{ni} e^{\hat{\tau}_n M_n} X_n (\beta_0 - \hat{\beta}_n)$, with e_{ni} being the ith row of the $n \times n$ identity matrix. Thus

$$B_{n3} = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n (a_{ni}^2 + b_{ni}^2 + c_{ni}^2 + 2a_{ni}b_{ni} + 2a_{ni}c_{ni} + 2a_{ni}\epsilon_{ni} + 2b_{ni}c_{ni} + 2b_{ni}\epsilon_{ni} + 2c_{ni}\epsilon_{ni})^2 h_{n,ij}.$$

Since $|b_{ni}| \le ||e^{\hat{\tau}_n M_n}||_\infty ||e^{(\hat{\alpha}_n - \alpha_0) W_n} - I_n||_\infty ||X_n \beta_0||_\infty = o_P(1)$ by Lemma A.8, and $|c_{ni}| \le ||e^{\hat{\tau}_n M_n}||_\infty ||X_n(\beta_0 - \hat{\beta}_n)||_\infty = o_P(1)$, $\frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n d_{ni} h_{n,ij} = o_P(1)$ with d_{ni} being the cross products of b_{ni}^2, c_{ni}^2, $2b_{ni}c_{ni}$, $2b_{ni}\epsilon_{ni}$ and $2c_{ni}\epsilon_{ni}$, by Markov's inequality. We now show that $\frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n a_{ni}^2 \epsilon_{ni}^2 h_{n,ij} = o_P(1)$. By a second order Taylor expansion of a_{ni}, we have $a_{ni} = e_{ni} M_n \epsilon_n (\hat{\tau}_n - \tau_0) + e_{ni} W_n \epsilon_n (\hat{\alpha}_n - \alpha_0) + \frac{1}{2} e_{ni} M_n^2 e^{(\tilde{\tau}_n - \tau_0) M_n} e^{(\tilde{\alpha}_n - \alpha_0) W_n} \epsilon_n (\hat{\tau}_n - \tau_0)^2 + \frac{1}{2} e_{ni} W_n^2 e^{(\tilde{\tau}_n - \tau_0) M_n} e^{(\tilde{\alpha}_n - \alpha_0) W_n} \epsilon_n (\hat{\alpha}_n - \alpha_0)^2 + e_{ni} M_n W_n e^{(\tilde{\tau}_n - \tau_0) M_n} e^{(\tilde{\alpha}_n - \alpha_0) W_n} \epsilon_n (\hat{\tau}_n - \tau_0)(\hat{\alpha}_n - \alpha_0)$, where $\tilde{\tau}_n$ is between $\hat{\tau}_n$ and τ_0, and $\tilde{\alpha}_n$ is between $\hat{\alpha}_n$ and α_0. By the C_r and Cauchy-Schwarz inequalities,

$a_{ni}^2 \leq 5(t_{n1} + t_{n2} + t_{n3} + t_{n4} + t_{n5})$, where $t_{n1} = (e_{ni}M_n\epsilon_n)^2(\hat{\tau}_n - \tau_0)^2$, $t_{n2} = (e_{ni}W_n\epsilon_n)^2(\hat{\alpha}_n - \alpha_0)^2$, $t_{n3} = \frac{1}{4}(e_{ni}M_n^2 e^{(\tilde{\tau}_n-\tau_0)M_n} e^{(\tilde{\alpha}_n-\alpha_0)W_n}\epsilon_n)^2(\hat{\tau}_n-\tau_0)^4$, $t_{n4} = \frac{1}{4}(e_{ni}W_n^2 e^{(\tilde{\tau}_n-\tau_0)M_n} e^{(\tilde{\alpha}_n-\alpha_0)W_n}\epsilon_n)^2(\hat{\alpha}_n-\alpha_0)^4$, and $t_{n5} = (e_{ni}M_nW_n e^{(\tilde{\tau}_n-\tau_0)M_n} e^{(\tilde{\alpha}_n-\alpha_0)W_n}\epsilon_n)^2(\hat{\tau}_n - \tau_0)^2(\hat{\alpha}_n - \alpha_0)^2$. We shall show that $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n t_{nk}\epsilon_{ni}^2 h_{n,ij} = o_P(1)$ for $k = 1, \ldots, 5$. For t_{n1},

$$\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n (e_{ni}M_n\epsilon_n)^2\epsilon_{ni}^2 h_{n,ij} = \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n m_{n,ik}m_{n,il}\epsilon_{nk}\epsilon_{nl}\epsilon_{ni}^2 h_{n,ij}.$$

By the Cauchy-Schwarz inequality, $\mathrm{E}(|\epsilon_{nk}\epsilon_{nl}\epsilon_{nj}^2|) \leq \mathrm{E}^{1/2}(\epsilon_{nk}^2\epsilon_{nl}^2)\,\mathrm{E}^{1/2}(\epsilon_{nj}^4) \leq \mathrm{E}^{1/4}(\epsilon_{nk}^4)\,\mathrm{E}^{1/4}(\epsilon_{nl}^4)\,\mathrm{E}^{1/2}(\epsilon_{nj}^4)$. Then by Markov's inequality, for any $\eta > 0$,

$$\mathrm{P}(|\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n (e_{ni}M_n\epsilon_n)^2 h_{n,ij}| \geq \eta) \leq \frac{c}{n\eta}\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n\sum_{l=1}^n |m_{n,ik}m_{n,il}h_{n,ij}| = O(\frac{1}{\eta})$$

for some constant c, as M_n, W_n and H_n are all UB. Thus $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n (e_{ni}M_n\epsilon_n)^2\epsilon_{ni}^2 h_{n,ij} = O_P(1)$ and $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n t_{n1}\epsilon_{ni}^2 h_{n,ij} = o_P(1)$. Similarly, $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n t_{n2}\epsilon_{ni}^2 h_{n,ij} = o_P(1)$. For t_{n3}, by the Cauchy-Schwarz inequality,

$$t_{n3} \leq \frac{1}{4}e_{ni}M_n^2 e^{(\tilde{\tau}_n-\tau_0)M_n} e^{(\tilde{\alpha}_n-\alpha_0)W_n} e^{(\tilde{\alpha}_n-\alpha_0)W_n'} e^{(\tilde{\tau}_n-\tau_0)M_n'} M_n'^2 e_{ni}'\epsilon_n'\epsilon_n(\hat{\tau}_n - \tau_0)^4 \leq c\epsilon_n'\epsilon_n(\hat{\tau}_n - \tau_0)^4$$

for some constant c. Then $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n t_{n3}\epsilon_{ni}^2|h_{n,ij}| \leq \frac{c(\hat{\tau}_n-\tau_0)^4}{n}\sum_{i=1}^n\sum_{j=1}^n \epsilon_n'\epsilon_n\epsilon_{ni}^2|h_{n,ij}|$, where, by Markov's inequality, $\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \epsilon_n'\epsilon_n\epsilon_{ni}^2|h_{n,ij}| = O_P(1)$. As $\sqrt{n}(\hat{\tau}_n - \tau_0) = O_P(1)$,

$$\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n t_{n3}\epsilon_{ni}^2 h_{n,ij} = o_P(1).$$

Similarly, $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n t_{nk}\epsilon_{ni}^2 h_{n,ij} = o_P(1)$ for $k = 4, 5$. Hence, $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n a_{ni}^2\epsilon_{ni}^2 h_{n,ij} = o_P(1)$. Similarly, we have $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n d_{ni}h_{n,ij} = o_P(1)$ with d_{ni} being any term involving a_{ni} in the expanded form of $(a_{ni}^2 + b_{ni}^2 + c_{ni}^2 + 2a_{ni}b_{ni} + 2a_{ni}c_{ni} + 2a_{ni}\epsilon_{ni} + 2b_{ni}c_{ni} + 2b_{ni}\epsilon_{ni} + 2c_{ni}\epsilon_{ni})^2$. Then $B_{n3} = o_P(1)$. With a similar argument, $B_{n1} = o_P(1)$ and $B_{n2} = o_P(1)$. As a result, (i) holds.

The (ii) follows by a similar argument.

For (iii),

$$\begin{aligned}
&|\frac{1}{n}r_n' e^{\hat{\tau}_n M_n'} e^{\hat{\tau}_n M_n} s_n - \frac{1}{n}r_n' e^{\tau_0 M_n'} e^{\tau_0 M_n} s_n| \\
&= |\frac{1}{n}r_n' e^{\hat{\tau}_n M_n'}(e^{(\hat{\tau}_n-\tau_0)M_n} - I_n)e^{\tau_0 M_n}s_n + \frac{1}{n}r_n' e^{\tau_0 M_n'}(e^{(\hat{\tau}_n-\tau_0)M_n'} - I_n)e^{\tau_0 M_n}s_n| \\
&\leq ||\frac{1}{n}r_n' e^{\hat{\tau}_n M_n'}||_\infty ||e^{(\hat{\tau}_n-\tau_0)M_n} - I_n||_\infty ||e^{\tau_0 M_n}s_n||_\infty + ||\frac{1}{n}r_n' e^{\tau_0 M_n'}||_\infty ||e^{(\hat{\tau}_n-\tau_0)M_n'} - I_n||_\infty ||e^{\tau_0 M_n}s_n||_\infty \\
&= o_P(1),
\end{aligned}$$

by Lemma A.8.

For (iv), write

$$\begin{aligned}
&\frac{1}{n}r_n' e^{\hat\tau_n M_n'}\hat\Sigma_n e^{\hat\tau_n M_n}s_n - \frac{1}{n}r_n' e^{\tau_0 M_n'}\Sigma_n e^{\tau_0 M_n}s_n \\
&= \frac{1}{n}r_n' e^{\tau_0 M_n'}(\hat\Sigma_n - \Sigma_n)e^{\tau_0 M_n}s_n + \frac{1}{n}r_n' e^{\tau_0 M_n'}(\hat\Sigma_n - \Sigma_n)(e^{\hat\tau_n M_n} - e^{\tau_0 M_n})s_n \\
&\quad + \frac{1}{n}r_n'(e^{\hat\tau_n M_n'} - e^{\tau_0 M_n'})(\hat\Sigma_n - \Sigma_n)e^{\hat\tau_n M_n}s_n + \frac{1}{n}r_n'(e^{\hat\tau_n M_n'} - e^{\tau_0 M_n'})\Sigma_n e^{\hat\tau_n M_n}s_n \\
&\quad + \frac{1}{n}r_n' e^{\tau_0 M_n'}\Sigma_n(e^{\hat\tau_n M_n} - e^{\tau_0 M_n})s_n.
\end{aligned}$$

The first term on the r.h.s. of the above equation can be shown to be $o_P(1)$ as the term in (ii). For the second term, note that $||e^{\hat\tau_n M_n} - e^{\tau_0 M_n}||_\infty \leq ||e^{\tau_0 M_n}||_\infty ||e^{(\hat\tau_n - \tau_0)M_n} - I_n||_\infty = o_P(1)$, then $|\frac{1}{n}r_n' e^{\tau_0 M_n'}(\hat\Sigma_n - \Sigma_n)(e^{\hat\tau_n M_n} - e^{\tau_0 M_n})s_n| \leq \frac{c}{n}||e^{\hat\tau_n M_n} - e^{\tau_0 M_n}||_\infty \sum_{i=1}^n |\hat\epsilon_{ni}^2 - \sigma_{ni}^2| \leq \frac{c}{n}||e^{\hat\tau_n M_n} - e^{\tau_0 M_n}||_\infty \sum_{i=1}^n (|\hat\epsilon_{ni}^2 - \epsilon_{ni}^2| + |\epsilon_{ni}^2 - \sigma_{ni}^2|)$ for some constant c, where $\frac{1}{n}\sum_{i=1}^n |\epsilon_{ni}^2 - \sigma_{ni}^2| = O_P(1)$ by Markov's inequality, and $\frac{1}{n}\sum_{i=1}^n |\hat\epsilon_{ni}^2 - \epsilon_{ni}^2| = o_P(1)$ which can be shown as in the proof for (i). Thus the second term is $o_P(1)$. Similarly, the third term is $o_P(1)$. The last two terms are $o_P(1)$ by using the sub-multiplicative property of the row sum matrix norm. Therefore, (iv) holds.

The results in the proposition follow from (i)–(iv). □

Proof of Proposition 6. Using the reduced form for y_n, we have

$$\begin{aligned}
\epsilon_n'(\gamma)P_{ni}\epsilon_n(\gamma) - \mathrm{E}[\epsilon_n'(\gamma)P_{ni}\epsilon_n(\gamma)] &= \epsilon_n' e^{-\tau_0 M_n'} e^{(\alpha-\alpha_0)W_n'} e^{\tau M_n'} P_{ni} e^{\tau M_n} e^{(\alpha-\alpha_0)W_n} e^{-\tau_0 M_n}\epsilon_n \\
&\quad - \sigma_0^2 \operatorname{tr}[e^{-\tau_0 M_n'} e^{(\alpha-\alpha_0)W_n'} e^{\tau M_n'} P_{ni} e^{\tau M_n} e^{(\alpha-\alpha_0)W_n} e^{-\tau_0 M_n}] \\
&\quad + [e^{\tau M_n}(e^{(\alpha-\alpha_0)W_n}X_n\beta_0 - X_n\beta)]' P_{ni}^s e^{\tau M_n} e^{(\alpha-\alpha_0)W_n} e^{-\tau_0 M_n}\epsilon_n, \\
F_n'[\epsilon_n(\gamma) - \mathrm{E}\,\epsilon_n(\gamma)] &= F_n' e^{\tau M_n} e^{(\alpha-\alpha_0)W_n} e^{-\tau_0 M_n}\epsilon_n.
\end{aligned}$$

By Lemma A.7, $\frac{1}{n}\epsilon_n'(\gamma)P_{ni}\epsilon_n(\gamma) - \frac{1}{n}\mathrm{E}[\epsilon_n'(\gamma)P_{ni}\epsilon_n(\gamma)] = o_P(1)$ and $\frac{1}{n}F_n'[\epsilon_n(\gamma) - \mathrm{E}\,\epsilon_n(\gamma)] = o_P(1)$ uniformly on Γ. Thus, $a_n'g_n(\gamma) - \mathrm{E}[a_n'g_n(\gamma)] = o_P(1)$ uniformly on Γ. Similar to the proof for the uniform equicontinuity of $\frac{1}{n}\bar Q_n(\phi)$ in the proof of Proposition 1, there is some constant c such that $\frac{1}{n}|\mathrm{E}[\epsilon_n'(\gamma_1)P_{ni}\epsilon_n(\gamma_1)] - \mathrm{E}[\epsilon_n'(\gamma_2)P_{ni}\epsilon_n(\gamma_2)]| \leq c||\gamma_1 - \gamma_2||$ and $\frac{1}{n}||\mathrm{E}[F_n'\epsilon(\gamma_1)] - \mathrm{E}[F_n'\epsilon(\gamma_2)]|| \leq c||\gamma_1 - \gamma_2||$ for any $\gamma_1, \gamma_2 \in \Gamma$, by the mean value theorem. Then $a_n'\,\mathrm{E}\,g_n(\gamma)$ is uniformly equicontinuous on Γ. The identification condition and the uniform equicontinuity of $a_n'\,\mathrm{E}\,g_n(\gamma)$ imply that the identification uniqueness condition for $\mathrm{E}\,g_n'(\gamma)a_n a_n'\,\mathrm{E}\,g_n(\gamma)$ holds. The consistency of $\hat\gamma_n$ follows from the uniform convergence and the identification uniqueness condition.

By the mean value theorem,

$$\frac{\partial g_n'(\hat\gamma_n)}{\partial\gamma}a_n a_n' g_n(\hat\gamma_n) = 0 = \frac{\partial g_n'(\hat\gamma_n)}{\partial\gamma}a_n a_n'[g_n(\gamma_0) + \frac{\partial g_n(\tilde\gamma_n)}{\partial\gamma'}(\hat\gamma_n - \gamma_0)],$$

where $\tilde{\gamma}_n$ is between $\hat{\gamma}_n$ and γ_0. Thus

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) = -[\frac{\partial g_n'(\hat{\gamma}_n)}{\partial \gamma} a_n a_n' \frac{\partial g_n(\tilde{\gamma}_n)}{\partial \gamma'}]^{-1} \frac{\partial g_n'(\hat{\gamma}_n)}{\partial \gamma} a_n a_n' \sqrt{n} g_n(\gamma_0).$$

We may show that $\frac{\partial g_n(\tilde{\gamma}_n)}{\partial \gamma'} = \frac{\partial g_n(\gamma_0)}{\partial \gamma'} + o_P(1) = \mathrm{E}\, \frac{\partial g_n(\gamma_0)}{\partial \gamma'} + o_P(1)$. The proof is similar to that for $\frac{1}{n} \frac{\partial^2 Q_n(\tilde{\gamma}_n)}{\partial \gamma \partial \gamma'} = \frac{1}{n} \frac{\partial^2 Q_n(\gamma_0)}{\partial \gamma \partial \gamma'} + o_P(1) = \mathrm{E}(\frac{1}{n} \frac{\partial^2 Q_n(\gamma_0)}{\partial \gamma \partial \gamma'}) + o_P(1)$ in the proof of Proposition 2, thus it is omitted. Then

$$\sqrt{n}(\hat{\gamma}_n - \gamma_0) = -[\mathrm{E}\, \frac{\partial g_n'(\gamma_0)}{\partial \gamma} a_n a_n' \,\mathrm{E}\, \frac{\partial g_n(\gamma_0)}{\partial \gamma'}]^{-1} \,\mathrm{E}\, \frac{\partial g_n'(\gamma_0)}{\partial \gamma} a_n a_n' \sqrt{n} g_n(\gamma_0) + o_P(1).$$

The asymptotic distribution of $\hat{\gamma}_n$ now follows from Lemma A.4. The explicit expressions for $\mathrm{E}\, \frac{\partial g_n(\gamma_0)}{\partial \gamma'}$ and $\mathrm{E}[n g_n(\gamma_0) g_n'(\gamma_0)]$ are derived by Lemma A.2. □

Proof of Proposition 7. The generalized Cauchy-Schwarz inequality implies that the optimal weighting matrix $a_n' a_n$ in Proposition 6 is V_n^{-1}. For the consistency, consider

$$g_n'(\gamma) \hat{V}_n^{-1} g_n(\gamma) = g_n'(\gamma) V_n^{-1} g_n(\gamma) + g_n'(\gamma)(\hat{V}_n^{-1} - V_n^{-1}) g_n(\gamma).$$

With $a_n = V_n^{-1/2}$, $a_0 = \lim_{n\to\infty} V_n^{-1/2}$ exists by Assumption 15. The uniform convergence of $a_n' g_n(\gamma) - a_n' \,\mathrm{E}\, g_n(\gamma)$ can be shown similarly as that in the proof of Proposition 6 and the identification condition is given in Assumption 13. Then for the consistency, it only remains to show that $g_n'(\gamma)(\hat{V}_n^{-1} - V_n^{-1}) g_n(\gamma) = o_P(1)$ uniformly on Γ. Let $||\cdot||$ be the Euclidean norm for vectors and matrices. Then,

$$||g_n'(\gamma)(\hat{V}_n^{-1} - V_n^{-1}) g_n(\gamma)|| \leq ||g_n(\gamma)||^2 ||\hat{V}_n^{-1} - V_n^{-1}||.$$

It is shown in the proof of Proposition 6 that $g_n(\gamma) - \mathrm{E}\, g_n(\gamma) = o_P(1)$ uniformly on Γ, and there is some constant c such that $\frac{1}{n} |\,\mathrm{E}[\epsilon_n'(\gamma_1) P_{ni} \epsilon_n(\gamma_1)] - \mathrm{E}[\epsilon_n'(\gamma_2) P_{ni} \epsilon_n(\gamma_2)]| \leq c||\gamma_1 - \gamma_2||$ and $\frac{1}{n} ||\,\mathrm{E}[F_n' \epsilon(\gamma_1)] - \mathrm{E}[F_n' \epsilon(\gamma_2)]|| \leq c||\gamma_1 - \gamma_2||$ for any $\gamma_1, \gamma_2 \in \Gamma$. Then $\mathrm{E}\, g_n(\gamma) = O(1)$ uniformly on Γ and $g_n(\gamma) = O_P(1)$ uniformly on Γ. Since $\hat{V}_n - V_n = o_P(1)$ and V_n is nonsingular, $||\hat{V}_n^{-1} - V_n^{-1}|| = o_P(1)$ by the continuous mapping theorem. Therefore, $g_n'(\gamma)(\hat{V}_n^{-1} - V_n^{-1}) g_n(\gamma) = o_P(1)$ uniformly on Γ.

For the asymptotic distribution, from the proof of Proposition 6, $\frac{\partial g_n(\tilde{\gamma}_{n,o})}{\partial \gamma'} - \mathrm{E}\, \frac{\partial g_n(\gamma_0)}{\partial \gamma'} = o_P(1)$ and

$$\begin{aligned}\sqrt{n}(\hat{\gamma}_{n,o} - \gamma_0) &= -[\frac{\partial g_n'(\hat{\gamma}_{n,o})}{\partial \gamma} \hat{V}_n^{-1} \frac{\partial g_n(\tilde{\gamma}_{n,o})}{\partial \gamma'}]^{-1} \frac{\partial g_n'(\hat{\gamma}_{n,o})}{\partial \gamma} \hat{V}_n^{-1} \sqrt{n} g_n(\gamma_0) \\ &= -[\mathrm{E}\, \frac{\partial g_n'(\gamma_0)}{\partial \gamma} V_n^{-1} \,\mathrm{E}\, \frac{\partial g_n(\gamma_0)}{\partial \gamma'}]^{-1} \,\mathrm{E}\, \frac{\partial g_n'(\gamma_0)}{\partial \gamma} V_n^{-1} \sqrt{n} g_n(\gamma_0) + o_P(1),\end{aligned}$$

where $\tilde{\gamma}_{n,o}$ is between $\hat{\gamma}_{n,o}$ and γ_0. The asymptotic distribution of $\sqrt{n}(\hat{\gamma}_{n,o} - \gamma_0)$ follows from this expansion and Lemma A.4. □

Proof of Proposition 9. Let $Q_n^*(\gamma) = g_n^{*'}(\gamma) V_n^{*-1} g_n^*(\gamma)$ and $\hat{Q}_n^*(\gamma) = \hat{g}_n^{*'}(\gamma) \hat{V}_n^{*-1} \hat{g}_n^*(\gamma)$. We shall verify that $Q_n^*(\gamma)$ and $\hat{Q}_n^*(\gamma)$ satisfy the conditions in Lemma A.5.

First, we show that $||\hat{g}_n^*(\gamma) - g_n^*(\gamma)||_\infty = o_P(1)$ uniformly in γ. To show this result, it is sufficient to show that (a) $\frac{1}{n}\epsilon_n'(\gamma)(\hat{\mathbb{W}}_n - \mathbb{W}_n)\epsilon_n(\gamma) = o_P(1)$ uniformly in γ; (b) $\frac{1}{n}\epsilon_n'(\gamma)\operatorname{Diag}(\hat{\mathbb{W}}_n - \mathbb{W}_n)\epsilon_n(\gamma) = o_P(1)$ uniformly in γ; (c) $\frac{1}{n}\epsilon_n'(\gamma)\operatorname{Diag}(e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0)^{(t)}\epsilon_n(\gamma) = o_P(1)$ uniformly in γ; (d) $\frac{1}{n}\epsilon_n'(\gamma)\operatorname{Diag}(e^{\hat{\tau}_n M_n}X_{nl}^* - e^{\tau_0 M_n}X_{nl}^*)^{(t)}\epsilon_n(\gamma) = o_P(1)$ uniformly in γ; (e) $\frac{1}{n}((e^{\hat{\tau}_n M_n} - e^{\tau_0 M_n})X_{nl}^*)'\epsilon_n(\gamma) = o_P(1)$ uniformly in γ; (f) $\frac{1}{n}(e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0)'\epsilon_n(\gamma) = o_P(1)$ uniformly in γ; and (g) $\frac{1}{n}\operatorname{vec}_{\mathrm{D}}'(\hat{\mathbb{W}}_n - \mathbb{W}_n)\epsilon_n(\gamma)$ uniformly in γ. For (a), by the mean value theorem,

$$\frac{1}{n}\epsilon'(\gamma)(\hat{\mathbb{W}}_n - \mathbb{W}_n)\epsilon_n(\gamma) = \frac{1}{n}\epsilon_n'(\gamma)e^{\tilde{\tau}_n M_n}(M_nW_n - W_nM_n)e^{-\tilde{\tau}_n M_n}\epsilon_n(\gamma)(\hat{\tau}_n - \tau_0),$$

where $\tilde{\tau}$ is between $\hat{\tau}_n$ and τ_0. As in the proof of Lemma A.7, by the spectral radius theorem, $\frac{1}{n}|\epsilon_n'(\gamma)e^{\tilde{\tau}_n M_n}(M_nW_n - W_nM_n)e^{-\tilde{\tau}_n M_n}\epsilon_n(\gamma)| \leq \frac{1}{2n}||[e^{\tilde{\tau}_n M_n}(M_nW_n - W_nM_n)e^{-\tilde{\tau}_n M_n}]^s||_\infty\epsilon_n'(\gamma)\epsilon_n(\gamma) \leq \frac{c}{n}\epsilon_n'(\gamma)\epsilon_n(\gamma)$ for some constant c. The $\frac{1}{n}\epsilon_n'(\gamma)\epsilon_n(\gamma)$ can be expanded by using $\epsilon_n(\gamma) = e^{\tau M_n}e^{(\alpha-\alpha_0)W_n}e^{-\tau_0 M_n}\epsilon_n + e^{\tau M_n}(e^{(\alpha-\alpha_0)W_n}X_n\beta_0 - X_n\beta)$. Let $P_n(\phi) = e^{-\tau_0 M_n'}e^{(\alpha-\alpha_0)W_n'}e^{\tau M_n'}e^{\tau M_n}e^{(\alpha-\alpha_0)W_n}e^{-\tau_0 M_n}$. By Lemma A.7, $\frac{1}{n}\epsilon_n'P_n(\phi)\epsilon_n - \frac{\sigma_0^2}{n}\operatorname{tr}[P_n(\phi)] = o_P(1)$ uniformly in ϕ. As $e^{\alpha A_n}$ is UB uniformly in $\alpha \in [-\delta, \delta]$ for any $n\times n$ UB matrix A_n, $\frac{1}{n}\operatorname{tr}[P_n(\phi)] = O(1)$ uniformly in ϕ. Thus $\frac{1}{n}\epsilon_n'P_n(\phi)\epsilon_n = O_P(1)$ uniformly in ϕ. Other terms in the expanded form of $\frac{1}{n}\epsilon_n'(\gamma)\epsilon_n(\gamma)$ are either $o_P(1)$ uniformly in γ by Lemma A.7 or $O(1)$ uniformly in γ because $e^{\alpha A_n}$ is UB uniformly in $\alpha \in [-\delta, \delta]$ for any $n \times n$ UB matrix A_n. Thus $\frac{1}{n}\epsilon_n'(\gamma)\epsilon_n(\gamma) = O_P(1)$ uniformly in γ. It follows that $\frac{1}{n}\epsilon'(\gamma)(\hat{\mathbb{W}}_n - \mathbb{W}_n)\epsilon_n(\gamma) = o_P(1)$ uniformly in ϕ as $\hat{\tau}_n - \tau_0 = o_P(1)$. For (b), $||\hat{\mathbb{W}}_n - \mathbb{W}_n||_\infty = ||(e^{(\hat{\tau}_n-\tau_0)M_n} - I_n)\mathbb{W}_n(e^{(\tau_0-\hat{\tau}_n)M_n} - I_n) + (e^{(\hat{\tau}_n-\tau_0)M_n} - I_n)\mathbb{W}_n + \mathbb{W}_n(e^{(\tau_0-\hat{\tau}_n)M_n} - I_n)||_\infty \leq ||e^{(\hat{\tau}_n-\tau_0)M_n} - I_n||_\infty||\mathbb{W}_n||_\infty||e^{(\tau_0-\hat{\tau}_n)M_n} - I_n||_\infty + ||e^{(\hat{\tau}_n-\tau_0)M_n} - I_n||_\infty||\mathbb{W}_n||_\infty + ||\mathbb{W}_n||_\infty||e^{(\tau_0-\hat{\tau}_n)M_n} - I_n||_\infty = o_P(1)$, by Lemma A.8. Then $\frac{1}{n}|\epsilon_n'(\gamma)\operatorname{Diag}(\hat{\mathbb{W}}_n - \mathbb{W}_n)\epsilon_n(\gamma)| \leq ||\hat{\mathbb{W}}_n - \mathbb{W}_n||_\infty\frac{1}{n}\epsilon_n'(\gamma)\epsilon_n(\gamma) = o_P(1)$ uniformly in γ. For (c), we have $\frac{1}{n}\epsilon_n'(\gamma)\operatorname{Diag}(e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0)^{(t)}\epsilon_n(\gamma) = \frac{1}{n}\epsilon_n'(\gamma)\operatorname{Diag}(e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0)\epsilon_n(\gamma) - \frac{1}{n}\epsilon_n'(\gamma)\epsilon_n(\gamma)\frac{1}{n}l_n'(e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0)$, where the first term can be shown to be $o_P(1)$ uniformly in γ similarly to (b), and $\frac{1}{n}|l_n'(e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0)| \leq \frac{1}{n}|l_n'|_\infty||e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0||_\infty = o_P(1)$. Thus (c) holds. (d) holds by an argument similar to that for (c). (e) holds since $\frac{1}{n}|((e^{\hat{\tau}_n M_n} - e^{\tau_0 M_n})X_{nl}^*)'\epsilon_n(\gamma)| \leq \sqrt{\frac{1}{n}X_{nl}^{*\prime}(e^{\hat{\tau}_n M_n} - e^{\tau_0 M_n})'(e^{\hat{\tau}_n M_n} - e^{\tau_0 M_n})X_{nl}^*}\sqrt{\frac{1}{n}\epsilon_n'(\gamma)\epsilon_n(\gamma)}$ by the Cauchy-Schwarz inequality. (f) holds by an argument similar to that for (e). For (g), by the cauchy-Schwarz inequality, $\frac{1}{n}|\operatorname{vec}_{\mathrm{D}}'(\hat{\mathbb{W}}_n - \mathbb{W}_n)\epsilon_n(\gamma)| \leq \sqrt{\frac{1}{n}\operatorname{vec}_{\mathrm{D}}'(\hat{\mathbb{W}}_n - \mathbb{W}_n)\operatorname{vec}_{\mathrm{D}}(\hat{\mathbb{W}}_n - \mathbb{W}_n)}\sqrt{\frac{1}{n}\epsilon_n'(\gamma)\epsilon_n(\gamma)}$, where $\frac{1}{n}\operatorname{vec}_{\mathrm{D}}'(\hat{\mathbb{W}}_n - \mathbb{W}_n)\operatorname{vec}_{\mathrm{D}}(\hat{\mathbb{W}}_n - \mathbb{W}_n) \leq ||\hat{\mathbb{W}}_n - \mathbb{W}_n||_\infty^2 = o_P(1)$. Thus (g) holds. Consequently, $||\hat{g}_n^*(\gamma) - g_n^*(\gamma)||_\infty = o_P(1)$ uniformly in γ.

Next we show that $\hat{V}_n^* - V_n^* = o_P(1)$. For this purpose, the following results are helpful: (h) $||\hat{P}_{ni}^* - P_{ni}^*||_\infty = o_P(1)$ and $||\hat{P}_{ni}^* - P_{ni}^*||_1 = o_P(1)$ for $i = 1, \ldots, k^* + 4$ and (i) $||\hat{F}_{ni}^* - F_{ni}^*||_\infty = o_P(1)$ for $i = 1, \ldots, 4$. For (h), we have shown above that $||\hat{P}_{n1}^* - P_{n1}^*||_\infty = ||\hat{\mathbb{W}}_n^* - \mathbb{W}_n^*||_\infty = o_P(1)$, which implies that $||\hat{P}_{n2}^* - P_{n2}^*||_\infty = ||\operatorname{Diag}(\hat{\mathbb{W}}_n^* - \mathbb{W}_n^*)||_\infty \leq ||\hat{\mathbb{W}}_n^* - \mathbb{W}_n^*||_\infty = o_P(1)$. For P_{n3}^*, we have $||\hat{P}_{n3}^* - P_{n3}^*||_\infty = ||\operatorname{Diag}(e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0) - \frac{1}{n}I_nl_n'(e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0)||_\infty \leq (1 + \frac{1}{n}||l'||_\infty)||e^{\hat{\tau}_n M_n}W_nX_n\hat{\beta}_n - e^{\tau_0 M_n}W_nX_n\beta_0||_\infty = o_P(1)$. Similarly, we have $||\hat{P}_{n,l+4}^* - P_{n,l+4}^*||_\infty = o_P(1)$

for $l = 1, \dots, k^*$. The results for the column sum matrix norm are similarly proved. (i) is obvious given the proof of (h). With (h) and (i), we have $\frac{1}{n}|\operatorname{tr}(\hat{P}_{ni}^{*s}\hat{P}_{nj}^{*s} - P_{ni}^{*s}P_{nj}^{*s})| = \frac{1}{n}|\operatorname{tr}[(\hat{P}_{ni}^{*s} - P_{ni}^{s})(\hat{P}_{nj}^{*s} - P_{nj}^{*s}) + P_{ni}^{s}(\hat{P}_{nj}^{*s} - P_{nj}^{*s}) + (\hat{P}_{ni}^{*s} - P_{ni}^{s})P_{nj}^{*s}]| \leq ||\hat{P}_{ni}^{*s} - P_{ni}^{s}||_\infty ||\hat{P}_{nj}^{*s} - P_{nj}^{*s}||_\infty + ||P_{ni}^{s}||_\infty ||\hat{P}_{nj}^{*s} - P_{nj}^{*s}||_\infty + ||\hat{P}_{ni}^{*s} - P_{ni}^{s}||_\infty ||P_{nj}^{*s}||_\infty = o_P(1)$; $\frac{1}{n}|\operatorname{vec}_D'(\hat{P}_{ni}^*)\operatorname{vec}_D'(\hat{P}_{nj}^*) - \operatorname{vec}_D'(P_{ni}^*)\operatorname{vec}_D(P_{nj}^*)| = \frac{1}{n}|\operatorname{vec}_D'(\hat{P}_{ni}^* - P_{ni}^*)\operatorname{vec}_D'(\hat{P}_{nj}^* - P_{nj}^*) + \operatorname{vec}_D'(P_{ni}^*)\operatorname{vec}_D'(\hat{P}_{nj}^* - P_{nj}^*) + \operatorname{vec}_D'(\hat{P}_{ni}^* - P_{ni}^*)\operatorname{vec}_D'(P_{nj}^*)| \leq ||\hat{P}_{ni}^* - P_{ni}||_\infty ||\hat{P}_{nj}^* - P_{nj}||_\infty + ||P_{ni}||_\infty ||\hat{P}_{nj}^* - P_{nj}^*||_\infty + ||\hat{P}_{ni}^* - P_{ni}||_\infty ||P_{nj}^*||_\infty = o_P(1)$; $\frac{1}{n}||\hat{F}_{ni}^{*\prime}\hat{F}_{nj}^* - F_{ni}^{*\prime}F_{nj}^*||_\infty = \frac{1}{n}||(\hat{F}_{ni}^* - F_{ni}^*)'(\hat{F}_{nj}^* - F_{nj}^*) + F_{ni}^{*\prime}(\hat{F}_{nj}^* - F_{nj}^*) + (\hat{F}_{ni}^* - F_{ni}^*)'F_{nj}^*||_\infty \leq ||\hat{F}_{ni}^* - F_{ni}^*||_\infty ||\hat{F}_{nj}^* - F_{nj}^*||_\infty + ||F_{ni}^*||_\infty ||\hat{F}_{nj}^* - F_{nj}^*||_\infty + ||\hat{F}_{ni}^* - F_{ni}^*||_\infty ||F_{nj}^*||_\infty = o_P(1)$; and $\frac{1}{n}||\hat{F}_{ni}^{*\prime}\operatorname{vec}_D(\hat{P}_{nj}^*) - F_{ni}^{*\prime}\operatorname{vec}_D(P_{nj}^*)||_\infty \leq \frac{1}{n}||(\hat{F}_{ni}^* - F_n^*)'\operatorname{vec}_D(\hat{P}_{nj}^* - P_{nj}^*)||_\infty + \frac{1}{n}||F_n^{*\prime}\operatorname{vec}_D(\hat{P}_{nj}^* - P_{nj}^*)||_\infty + \frac{1}{n}||(\hat{F}_{ni}^* - F_n^*)'\operatorname{vec}_D(P_{nj}^*)||_\infty \leq ||\hat{F}_{ni}^* - F_n^*||_\infty ||\hat{P}_{nj}^* - P_{nj}^{*\prime}||_\infty + ||F_n^*||_\infty ||\hat{P}_{nj}^* - P_{nj}^*||_\infty + ||\hat{F}_{ni}^* - F_n^*||_\infty ||P_{nj}^*||_\infty = o_P(1)$. Therefore, $\hat{V}_n^* - V_n^* = o_P(1)$.

As $\hat{V}_n^* - V_n^* = o_P(1)$ and V_n is nonsingular, $\hat{V}_n^{*-1} - V_n^{*-1} = o_P(1)$ by the continuous mapping theorem. As in the proof of Proposition 7, $g_n^*(\gamma) = O_P(1)$ uniformly on Γ. Therefore, $Q_n^*(\gamma) - \hat{Q}_n^*(\gamma) = o_P(1)$ uniformly on Γ. It follows that $\hat{\gamma}_{n,f}^*$ is a consistent estimator of γ_0 as the minimizer of $Q_n^*(\gamma)$ is a consistent estimator of γ_0.

By a similar argument as for $\hat{Q}_n^*(\gamma) - Q_n^*(\gamma) = o_P(1)$ uniformly on Γ, we have $\frac{\partial^2 \hat{Q}_n^*(\gamma)}{\partial\gamma\partial\gamma'} - \frac{\partial^2 Q_n^*(\gamma)}{\partial\gamma\partial\gamma'} = o_P(1)$ uniformly on Γ. Furthermore, $\sqrt{n}\frac{\partial Q_n^*(\gamma_0)}{\partial\gamma} = 2\frac{\partial g_n^{*\prime}(\gamma_0)}{\partial\gamma}V_n^{*-1}\sqrt{n}g_n^*(\gamma_0)$ and $\sqrt{n}\frac{\partial \hat{Q}_n^*(\gamma_0)}{\partial\gamma} = 2\frac{\partial \hat{g}_n^{*\prime}(\gamma_0)}{\partial\gamma}\hat{V}_n^{*-1}\sqrt{n}\hat{g}_n^*(\gamma_0)$. By Lemma A.3, $\sqrt{n}g_n^*(\gamma_0) = O_P(1)$ and $\frac{\partial g_n^{*\prime}(\gamma_0)}{\partial\gamma} = O_P(1)$. By a similar argument as for $g_n^*(\gamma) - \hat{g}_n^*(\gamma) = o_P(1)$ uniformly on Γ, $\frac{\partial \hat{g}_n^{*\prime}(\gamma_0)}{\partial\gamma} - \frac{\partial g_n^{*\prime}(\gamma_0)}{\partial\gamma} = o_P(1)$, and $\sqrt{n}\hat{g}_n^*(\gamma_0) - \sqrt{n}g_n^*(\gamma_0) = o_P(1)$ as $\sqrt{n}(\hat{\tau}_n - \tau_0) = O_P(1)$ and $\sqrt{n}(\hat{\beta}_n - \beta_0) = O_P(1)$. Hence, $\sqrt{n}\frac{\partial Q_n^*(\gamma_0)}{\partial\gamma} = O_P(1)$ and $\sqrt{n}\frac{\partial \hat{Q}_n^*(\gamma_0)}{\partial\gamma} - \sqrt{n}\frac{\partial Q_n^*(\gamma_0)}{\partial\gamma} = o_P(1)$. Therefore, $\hat{\gamma}_{n,f}^*$ has the same limiting distribution as the minimizer of $Q_n^*(\gamma)$. □

Proof of Proposition 10. The proof resembles that for Proposition 6, thus it is omitted. □

Proof of Proposition 11. Note the the (i,j)th element of V_n, for $1 \leq i \leq k_p$ and $1 \leq j \leq k_p$, is $\frac{1}{2n}\operatorname{vec}'(\Sigma_n^{1/2}P_{ni}^s\Sigma_n^{1/2})\operatorname{vec}(\Sigma_n^{1/2}P_{nj}^s\Sigma_n^{1/2}) = \frac{1}{2n}\operatorname{tr}(\Sigma_n^{1/2}P_{ni}^s\Sigma_n^{1/2}\Sigma_n^{1/2}P_{nj}^s\Sigma_n^{1/2}) = \frac{1}{2n}\operatorname{tr}(P_{ni}^s\Sigma_n P_{nj}^s\Sigma_n)$. By a similar argument to that in the proof of Proposition 5, $\hat{V}_n - V_n = o_P(1)$. The rest of the proof resembles that for Proposition 7. □

Proof of Proposition A.1. The consistency of the QMLE $\hat{\gamma}_n$ will follow from the uniform convergence that $\sup_{\boldsymbol{\phi}\in\boldsymbol{\Phi}}\frac{1}{n}|Q_n(\boldsymbol{\phi}) - \bar{Q}_n(\boldsymbol{\phi})| = o_P(1)$ and the identification uniqueness condition.

We first show the uniform convergence. As $y_n = e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}(X_n\beta_0 + e^{-\boldsymbol{\tau}_0\mathbf{M}_n}\epsilon_n)$,

$$\begin{aligned}
\frac{1}{n}[Q_n(\boldsymbol{\phi}) - \bar{Q}_n(\boldsymbol{\phi})] = {} & \frac{2}{n}(X_n\beta_0)'e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'}e^{(\boldsymbol{\alpha}\mathbf{W}_n)'}e^{(\boldsymbol{\tau}\mathbf{M}_n)'}H_n(\boldsymbol{\tau})e^{\boldsymbol{\tau}\mathbf{M}_n}e^{\boldsymbol{\alpha}\mathbf{W}_n}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}e^{-\boldsymbol{\tau}_0\mathbf{M}_n}\epsilon_n \\
& + \frac{1}{n}\epsilon_n'e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'}e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'}e^{(\boldsymbol{\alpha}\mathbf{W}_n)'}e^{(\boldsymbol{\tau}\mathbf{M}_n)'}H_n(\boldsymbol{\tau})e^{\boldsymbol{\tau}\mathbf{M}_n}e^{\boldsymbol{\alpha}\mathbf{W}_n}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}e^{-\boldsymbol{\tau}_0\mathbf{M}_n}\epsilon_n \\
& - \frac{\sigma_0^2}{n}\operatorname{tr}[e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'}e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'}e^{(\boldsymbol{\alpha}\mathbf{W}_n)'}e^{(\boldsymbol{\tau}\mathbf{M}_n)'}H_n(\boldsymbol{\tau})e^{\boldsymbol{\tau}\mathbf{M}_n}e^{\boldsymbol{\alpha}\mathbf{W}_n}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}e^{-\boldsymbol{\tau}_0\mathbf{M}_n}] \\
& - \frac{\sigma_0^2}{n}\operatorname{tr}\{e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'}e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'}e^{(\boldsymbol{\alpha}\mathbf{W}_n)'}e^{(\boldsymbol{\tau}\mathbf{M}_n)'}[I_n - H_n(\boldsymbol{\tau})]e^{\boldsymbol{\tau}\mathbf{M}_n}e^{\boldsymbol{\alpha}\mathbf{W}_n}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}e^{-\boldsymbol{\tau}_0\mathbf{M}_n}\}.
\end{aligned}$$

By Lemma A.7, $\frac{1}{n}[Q_n(\boldsymbol{\phi}) - \bar{Q}_n(\boldsymbol{\phi})] = o_P(1)$ uniformly on $\boldsymbol{\Phi}$.

We now show that $\frac{1}{n}\bar{Q}_n(\boldsymbol{\phi})$ is uniformly equicontinuous. By the mean value theorem, for $\boldsymbol{\phi}, \boldsymbol{\phi}^* \in \boldsymbol{\Phi}$,

$$\begin{aligned}
&\frac{1}{n}[\bar{Q}_n(\boldsymbol{\phi}^*) - \bar{Q}_n(\boldsymbol{\phi})] \\
&= 2\sum_{i=1}^{p}(X_n\beta_0)' e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\alpha}}\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\tau}}\mathbf{M}_n)'} H_n(\tilde{\boldsymbol{\tau}}) e^{\tilde{\boldsymbol{\tau}}\mathbf{M}_n} \frac{\partial e^{\tilde{\boldsymbol{\alpha}}\mathbf{W}_n}}{\partial \alpha_i} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} X_n\beta_0(\alpha_i^* - \alpha_i) \\
&\quad + \frac{2\sigma_0^2}{n}\sum_{i=1}^{p}\mathrm{tr}[e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\alpha}}\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\tau}}\mathbf{M}_n)'} e^{\tilde{\boldsymbol{\tau}}\mathbf{M}_n} \frac{\partial e^{\tilde{\boldsymbol{\alpha}}\mathbf{W}_n}}{\partial \alpha_i} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} e^{-\boldsymbol{\tau}_0\mathbf{M}_n}](\alpha_i^* - \alpha_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{q}(X_n\beta_0)' e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\alpha}}\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\tau}}\mathbf{M}_n)'} (2H_n(\tilde{\boldsymbol{\tau}})\frac{\partial e^{\tilde{\boldsymbol{\tau}}\mathbf{M}_n}}{\partial \tau_i} + \frac{\partial H_n(\tilde{\boldsymbol{\tau}})}{\partial \tau_i} e^{\tilde{\boldsymbol{\tau}}\mathbf{M}_n}) e^{\tilde{\boldsymbol{\alpha}}\mathbf{W}_n} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} X_n\beta_0(\tau_i^* - \tau_i) \\
&\quad + \frac{2\sigma_0^2}{n}\sum_{i=1}^{q}\mathrm{tr}[e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\alpha}}\mathbf{W}_n)'} e^{(\tilde{\boldsymbol{\tau}}\mathbf{M}_n)'} \frac{\partial e^{\tilde{\boldsymbol{\tau}}\mathbf{M}_n}}{\partial \tau_i} e^{\tilde{\boldsymbol{\alpha}}\mathbf{W}_n} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} e^{-\boldsymbol{\tau}_0\mathbf{M}_n}](\tau_i^* - \tau_i),
\end{aligned}$$

where $\tilde{\boldsymbol{\phi}}$ is between $\boldsymbol{\phi}^*$ and $\boldsymbol{\phi}$. By Lemma A.6, $H_n(\boldsymbol{\tau})$, $e^{\boldsymbol{\alpha}\mathbf{W}_n}$ and $e^{\boldsymbol{\tau}\mathbf{M}_n}$ are UB uniformly over their respective parameter spaces. By the proof of Lemma A.7, $\frac{\partial e^{\boldsymbol{\alpha}\mathbf{W}_n}}{\partial \alpha_i}$, $\frac{\partial e^{\boldsymbol{\tau}\mathbf{M}_n}}{\partial \tau_i}$ and $\frac{\partial H_n(\boldsymbol{\tau})}{\partial \tau_i}$ are UB uniformly over their respective parameter spaces. Then there exists some constant c such that

$$\frac{1}{n}|\bar{Q}_n(\boldsymbol{\phi}^*) - \bar{Q}_n(\boldsymbol{\phi})| \leq c(||\boldsymbol{\alpha}^* - \boldsymbol{\alpha}|| + ||\boldsymbol{\tau}^* - \boldsymbol{\tau}||).$$

Thus $\frac{1}{n}\bar{Q}_n(\boldsymbol{\phi})$ is uniformly equicontinuous.

Finally, we show that the identification uniqueness condition holds. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A_n(\boldsymbol{\phi}) = e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'} e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} e^{-\boldsymbol{\tau}_0\mathbf{M}_n}$. Since $A_n(\boldsymbol{\phi})$ is positive definite, λ_i's are all positive. Then by the inequality of arithmetic and geometric means,

$$\begin{aligned}
\frac{1}{n}\mathrm{tr}(A_n(\boldsymbol{\phi})) = \frac{1}{n}\sum_{i=1}^{n}\lambda_i \geq (\prod_{i=1}^{n}\lambda_i)^{1/n} &= |A_n(\boldsymbol{\phi})|^{1/n} \\
&= [e^{-\mathrm{tr}(\boldsymbol{\tau}_0\mathbf{M}_n)} e^{-\mathrm{tr}(\boldsymbol{\alpha}_0\mathbf{W}_n)} e^{\mathrm{tr}(\boldsymbol{\alpha}\mathbf{W}_n)} e^{\mathrm{tr}(\boldsymbol{\tau}\mathbf{M}_n)} e^{\mathrm{tr}(\boldsymbol{\tau}\mathbf{M}_n)} e^{\mathrm{tr}(\boldsymbol{\alpha}\mathbf{W}_n)} e^{-\mathrm{tr}(\boldsymbol{\alpha}_0\mathbf{W}_n)} e^{-\mathrm{tr}(\boldsymbol{\tau}_0\mathbf{M}_n)}]^{1/n} \\
&= 1,
\end{aligned}$$

because $\mathrm{tr}(\boldsymbol{\alpha}\mathbf{M}_n) = \mathrm{tr}(\boldsymbol{\tau}\mathbf{W}_n) = 0$. In addition,

$$(X_n\beta_0)' e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'} e^{(\boldsymbol{\alpha}\mathbf{W}_n)'} e^{(\boldsymbol{\tau}\mathbf{M}_n)'} H_n(\boldsymbol{\tau}) e^{\boldsymbol{\tau}\mathbf{M}_n} e^{\boldsymbol{\alpha}\mathbf{W}_n} e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} X_n\beta_0 \geq 0.$$

Thus, $\frac{1}{n}\bar{Q}_n(\boldsymbol{\phi}) \geq \sigma_0^2$. When $\boldsymbol{\phi} = \boldsymbol{\phi}_0$, $\frac{1}{n}\bar{Q}_n(\boldsymbol{\phi}) = \sigma_0^2$. Assumption 6 implies that whenever $\boldsymbol{\phi} \neq \boldsymbol{\phi}_0$, $\lim_{n\to\infty}\frac{1}{n}\bar{Q}_n(\boldsymbol{\phi}) \neq \sigma_0^2$. Thus the identification uniqueness condition holds.

With the uniform convergence and identification uniqueness condition, the consistency of $\hat{\boldsymbol{\phi}}_n$ follows. The consistency of $\hat{\beta}_n$ follows by plugging $\hat{\boldsymbol{\phi}}_n$ into the function $\hat{\beta}_n(\boldsymbol{\phi})$ in (4). □

Proof of Proposition A.2. To obtain the asymptotic distribution, we use (7) for $\sqrt{n}(\hat{\gamma}_n - \gamma_0)$ derived from a Taylor expansion of the first-order condition. We first show that $\frac{1}{n}\frac{\partial^2 Q_n(\tilde{\gamma}_n)}{\partial\gamma\partial\gamma'} = \frac{1}{n}\frac{\partial^2 Q_n(\gamma_0)}{\partial\gamma\partial\gamma'} + o_P(1) =$

$\frac{1}{n}\,\mathrm{E}\,\frac{\partial^2 Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'} + o_P(1) = \mathbf{C}_n + o_P(1)$. Noting that for any $\boldsymbol{\alpha}^*, \boldsymbol{\alpha} \in [-\delta, \delta]^p$, by expanding $(\boldsymbol{\alpha}^*\mathbf{W}_n)^i = [(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n + \boldsymbol{\alpha}\mathbf{W}_n]^i$, we have

$$\begin{aligned}||(\boldsymbol{\alpha}^*\mathbf{W}_n)^i - (\boldsymbol{\alpha}\mathbf{W}_n)^i||_\infty &\leq \sum_{k=1}^{i}\binom{i}{k}||\boldsymbol{\alpha}\mathbf{W}_n||_\infty^{i-k}||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty^k \\ &= i||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty \sum_{k=1}^{i}\frac{1}{k}\binom{i-1}{k-1}||\boldsymbol{\alpha}\mathbf{W}_n||_\infty^{i-k}||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty^{k-1} \\ &\leq i||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty(||\boldsymbol{\alpha}\mathbf{W}_n||_\infty + ||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty)^{i-1}.\end{aligned}$$

Then $||e^{\boldsymbol{\alpha}^*\mathbf{W}_n} - e^{\boldsymbol{\alpha}\mathbf{W}_n}||_\infty \leq ||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty||\sum_{i=1}^{\infty}\frac{1}{(i-1)!}(||\boldsymbol{\alpha}\mathbf{W}_n||_\infty + ||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty)^{i-1} = ||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty e^{||\boldsymbol{\alpha}\mathbf{W}_n||_\infty + ||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty} \leq ||\boldsymbol{\alpha}^* - \boldsymbol{\alpha}|| \max_{1\leq j\leq p}||W_{nj}||_\infty e^{||\boldsymbol{\alpha}\mathbf{W}_n||_\infty + ||(\boldsymbol{\alpha}^* - \boldsymbol{\alpha})\mathbf{W}_n||_\infty}$. Thus $||e^{\tilde{\boldsymbol{\alpha}}_n\mathbf{W}_n} - e^{\boldsymbol{\alpha}_0\mathbf{W}_n}||_\infty = o_P(1)$. With this result, we can show that $\frac{1}{n}\frac{\partial^2 Q_n(\tilde{\boldsymbol{\gamma}}_n)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'} = \frac{1}{n}\frac{\partial^2 Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'} + o_P(1)$.[1] Furthermore, each element of $\frac{1}{n}\frac{\partial^2 Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'} - \frac{1}{n}\,\mathrm{E}\,\frac{\partial^2 Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'}$ is a linear-quadratic function of ϵ_n, then $\frac{1}{n}\frac{\partial^2 Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'} = \frac{1}{n}\,\mathrm{E}\,\frac{\partial^2 Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'} + o_P(1)$ by Lemma A.3. Hence, $\frac{1}{n}\frac{\partial^2 Q_n(\tilde{\boldsymbol{\gamma}}_n)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'} = \frac{1}{n}\,\mathrm{E}\,\frac{\partial^2 Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'} + o_P(1)$. It follows that

$$\sqrt{n}(\hat{\boldsymbol{\gamma}}_n - \boldsymbol{\gamma}_0) = -\mathbf{C}_n^{-1}\frac{1}{\sqrt{n}}\frac{\partial Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}} + o_P(1),$$

where each element of $\frac{\partial Q_n(\gamma_0)}{\partial\gamma}$ is a linear-quadratic form of ϵ_n as shown in (8)–(10). We now show that $\mathrm{E}\,\frac{\partial Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}} = 0$. As ϵ_{ni}'s are i.i.d. with mean zero, it remains to show that $\mathrm{tr}(e^{-(\boldsymbol{\tau}_0\mathbf{M}_n)'}e^{-(\boldsymbol{\alpha}_0\mathbf{W}_n)'}\frac{\partial e^{(\boldsymbol{\alpha}_0\mathbf{W}_n)'}}{\partial\alpha_i}e^{(\boldsymbol{\tau}_0\mathbf{M}_n)'}) = \mathrm{tr}(\frac{\partial e^{\boldsymbol{\alpha}_0\mathbf{W}_n}}{\partial\alpha_i}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}) = 0$ and $\mathrm{tr}(\frac{\partial e^{\boldsymbol{\tau}_0\mathbf{M}_n}}{\partial\tau_i}e^{-\boldsymbol{\tau}_0\mathbf{M}_n}) = 0$. W.l.o.g., we show that $\mathrm{tr}(\frac{\partial e^{\boldsymbol{\alpha}_0\mathbf{W}_n}}{\partial\alpha_i}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}) = 0$. As $\frac{\partial e^{\boldsymbol{\alpha}_0\mathbf{W}_n}}{\partial\alpha_i} = \sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\frac{1}{j!}(\boldsymbol{\alpha}_0\mathbf{W}_n)^k W_{ni}(\boldsymbol{\alpha}_0\mathbf{W}_n)^{j-1-k}$, $(\boldsymbol{\alpha}_0\mathbf{W}_n)^j e^{-\boldsymbol{\alpha}_0\mathbf{W}_n} = e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}(\boldsymbol{\alpha}_0\mathbf{W}_n)^j$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any two conformable square matrices A and B, we have

$$\begin{aligned}\mathrm{tr}(\frac{\partial e^{\boldsymbol{\alpha}_0\mathbf{W}_n}}{\partial\alpha_i}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}) &= \mathrm{tr}\big(W_{ni}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}\sum_{j=1}^{\infty}\frac{1}{j!}\sum_{k=0}^{j-1}(\boldsymbol{\alpha}_0\mathbf{W}_n)^{j-1}\big) \\ &= \mathrm{tr}\big(W_{ni}e^{-\boldsymbol{\alpha}_0\mathbf{W}_n}\sum_{j=1}^{\infty}\frac{1}{(j-1)!}(\boldsymbol{\alpha}_0\mathbf{W}_n)^{j-1}\big) \\ &= \mathrm{tr}(W_{ni}) = 0.\end{aligned}$$

Applying the central limit theorem in Lemma A.4, we have

$$\sqrt{n}(\hat{\boldsymbol{\gamma}}_n - \boldsymbol{\gamma}_0) \xrightarrow{d} N(0, \lim_{n\to\infty}\mathbf{C}_n^{-1}\boldsymbol{\Omega}_n\mathbf{C}_n^{-1}),$$

where $\boldsymbol{\Omega}_n = \frac{1}{n}\,\mathrm{E}\big(\frac{\partial Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}}\frac{\partial Q_n(\boldsymbol{\gamma}_0)}{\partial\boldsymbol{\gamma}'}\big)$. The explicit expressions for $\mathbf{C}_n$ and $\boldsymbol{\Omega}_n$ can be derived by Lemma A.2.
□

[1]Please see the proof of Proposition 2 for a similar argument.

5. More Monte Carlo Results

Tables 1 – 8 report the simulation results for β_1 and β_2 when $n = 100$ for GMM in the homoskedatic case and for heteroskedastic errors, and all Monte Carlo results for $n = 254$. The results are similar to those for $n = 100$. We observe that bias, standard errors and thus RMSE are lower when the sample size becomes larger.

6. Estimated results for MESS(1,0) for the application

Table 9 contains estimation results of the MESS(1,0). The obtained results are qualitatively the same as those obtained for MESS(1,1).

References

Davidson, J. (1994), *Stochastic Limit Theory: An Introduction for Econometricians*, Oxford University Press, USA.

Elhorst, J. P., Lacombe, D. J. and Piras, G. (2012), 'On model specification and parameter space definitions in higher order spatial econometric models', *Regional Science and Urban Economics* **42**(1), 211–220.

Jin, F. and Lee, L.-F. (2012), 'Approximated likelihood and root estimators for spatial interaction in spatial autoregressive models', *Regional Science and Urban Economics* **42**, 446–458.

Kelejian, H. H. and Prucha, I. R. (2001), 'On the asymptotic distribution of the Moran I test statistic with applications', *Journal of Econometrics* **104**, 219–257.

Kelejian, H. H. and Prucha, I. R. (2010), 'Specification and estimation of spatial autoregressive models with autoregressive and heteroskedastic disturbances', *Journal of Econometrics* **157**, 53–67.

Lee, L.-F. (2007), 'GMM and 2SLS estimation of mixed regressive, spatial autoregressive models', *Journal of Econometrics* **137**, 489–514.

Lin, X. and Lee, L.-F. (2010), 'GMM estimation of spatial autoregressive models with unknown heteroskedasticity', *Journal of Econometrics* **157**, 34–52.

White, H. (1994), *Estimation, inference and specification analysis*, New York: Cambridge University Press.

Table 1: GMM estimation results for $\hat{\beta}_1$ and $\hat{\beta}_2$ for the homoskedastic case with n=100

Results for the $\hat{\beta}_1$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.003	0.006	0.005	0.005	0.006	0.012	0.012	0.012	0.012	0.011	−0.005	0.002	0.002	0.003	0.003	0.002	0.002	0.002	0.002	0.001
$\tau = -2$	*0.100*	*0.102*	*0.101*	*0.102*	*0.101*	*0.084*	*0.084*	*0.090*	*0.089*	*0.090*	*0.095*	*0.099*	*0.099*	*0.099*	*0.099*	*0.079*	*0.080*	*0.079*	*0.078*	*0.080*
	0.100	**0.102**	**0.102**	**0.102**	**0.101**	**0.085**	**0.085**	**0.090**	**0.090**	**0.090**	**0.095**	**0.099**	**0.099**	**0.099**	**0.099**	**0.079**	**0.080**	**0.079**	**0.078**	**0.080**
	−0.004	−0.003	−0.003	−0.004	−0.003	0.006	0.006	0.005	0.006	0.006	−0.005	0.001	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001
$\tau = -1$	*0.083*	*0.084*	*0.084*	*0.084*	*0.084*	*0.066*	*0.066*	*0.067*	*0.066*	*0.066*	*0.096*	*0.095*	*0.095*	*0.095*	*0.095*	*0.079*	*0.079*	*0.079*	*0.079*	*0.079*
	0.083	**0.084**	**0.084**	**0.084**	**0.084**	**0.066**	**0.066**	**0.067**	**0.066**	**0.067**	**0.096**	**0.095**	**0.095**	**0.095**	**0.095**	**0.079**	**0.079**	**0.079**	**0.079**	**0.079**
	−0.003	−0.003	−0.003	−0.003	−0.003	0.002	0.002	0.002	0.002	0.002	−0.003	−0.003	−0.002	−0.003	−0.003	0.000	−0.001	0.000	−0.001	0.000
$\tau = 0$	*0.099*	*0.099*	*0.098*	*0.099*	*0.099*	*0.074*	*0.074*	*0.074*	*0.074*	*0.074*	*0.094*	*0.092*	*0.092*	*0.092*	*0.092*	*0.078*	*0.077*	*0.078*	*0.077*	*0.077*
	0.099	**0.099**	**0.098**	**0.099**	**0.099**	**0.074**	**0.074**	**0.074**	**0.074**	**0.074**	**0.094**	**0.093**	**0.092**	**0.092**	**0.092**	**0.078**	**0.077**	**0.078**	**0.077**	**0.077**
	0.005	−0.001	−0.002	−0.001	−0.001	−0.006	0.001	0.001	0.001	0.001	−0.004	−0.003	−0.003	−0.003	−0.003	−0.002	−0.003	−0.002	−0.003	−0.003
$\tau = 1$	*0.116*	*0.116*	*0.116*	*0.116*	*0.116*	*0.091*	*0.086*	*0.086*	*0.086*	*0.086*	*0.097*	*0.095*	*0.094*	*0.094*	*0.094*	*0.079*	*0.078*	*0.078*	*0.078*	*0.078*
	0.116	**0.116**	**0.116**	**0.116**	**0.116**	**0.092**	**0.086**	**0.086**	**0.086**	**0.086**	**0.097**	**0.095**	**0.094**	**0.094**	**0.094**	**0.079**	**0.078**	**0.078**	**0.078**	**0.078**
	0.004	0.003	0.002	0.001	0.003	0.001	0.001	0.002	0.002	0.002	−0.002	−0.003	−0.003	−0.002	−0.002	−0.005	−0.002	−0.002	−0.002	−0.003
$\tau = 2$	*0.116*	*0.115*	*0.116*	*0.116*	*0.116*	*0.087*	*0.087*	*0.087*	*0.087*	*0.087*	*0.096*	*0.093*	*0.093*	*0.093*	*0.093*	*0.074*	*0.078*	*0.077*	*0.077*	*0.076*
	0.116	**0.115**	**0.116**	**0.116**	**0.116**	**0.087**	**0.087**	**0.087**	**0.087**	**0.087**	**0.096**	**0.093**	**0.093**	**0.093**	**0.093**	**0.074**	**0.078**	**0.077**	**0.077**	**0.076**

Results for the $\hat{\beta}_2$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	−0.001	−0.001	−0.002	−0.003	−0.001	0.018	0.018	0.016	0.019	0.017	−0.011	−0.014	−0.012	−0.012	−0.014	0.001	0.005	0.005	0.001	0.004
$\tau = -2$	*0.221*	*0.221*	*0.220*	*0.219*	*0.220*	*0.190*	*0.192*	*0.194*	*0.192*	*0.192*	*0.274*	*0.273*	*0.274*	*0.276*	*0.273*	*0.235*	*0.232*	*0.232*	*0.231*	*0.233*
	0.221	**0.221**	**0.220**	**0.220**	**0.220**	**0.191**	**0.193**	**0.195**	**0.193**	**0.193**	**0.274**	**0.273**	**0.274**	**0.276**	**0.273**	**0.235**	**0.232**	**0.232**	**0.231**	**0.233**
	−0.013	−0.008	−0.008	−0.010	−0.007	0.012	0.010	0.013	0.014	0.012	−0.016	−0.014	−0.016	−0.013	−0.014	0.001	0.005	0.004	0.005	0.004
$\tau = -1$	*0.272*	*0.272*	*0.273*	*0.271*	*0.271*	*0.230*	*0.230*	*0.230*	*0.230*	*0.231*	*0.295*	*0.298*	*0.296*	*0.295*	*0.296*	*0.255*	*0.253*	*0.254*	*0.253*	*0.254*
	0.272	**0.272**	**0.274**	**0.271**	**0.271**	**0.230**	**0.230**	**0.230**	**0.230**	**0.231**	**0.296**	**0.299**	**0.297**	**0.295**	**0.296**	**0.255**	**0.253**	**0.254**	**0.253**	**0.254**
	−0.014	−0.013	−0.010	−0.013	−0.011	0.002	0.002	0.002	0.006	0.006	−0.022	−0.014	−0.015	−0.015	−0.018	0.003	0.006	0.005	0.006	0.007
$\tau = 0$	*0.300*	*0.301*	*0.301*	*0.301*	*0.302*	*0.256*	*0.257*	*0.257*	*0.256*	*0.256*	*0.312*	*0.312*	*0.310*	*0.309*	*0.310*	*0.267*	*0.269*	*0.268*	*0.269*	*0.268*
	0.300	**0.301**	**0.301**	**0.301**	**0.302**	**0.256**	**0.257**	**0.257**	**0.256**	**0.256**	**0.313**	**0.312**	**0.310**	**0.310**	**0.310**	**0.267**	**0.269**	**0.268**	**0.269**	**0.269**
	−0.022	−0.014	−0.015	−0.014	−0.014	−0.002	−0.003	−0.003	−0.003	−0.003	−0.023	−0.016	−0.018	−0.018	−0.018	0.006	0.006	0.007	0.005	0.006
$\tau = 1$	*0.268*	*0.248*	*0.248*	*0.248*	*0.247*	*0.201*	*0.213*	*0.213*	*0.213*	*0.212*	*0.317*	*0.294*	*0.294*	*0.295*	*0.295*	*0.258*	*0.257*	*0.256*	*0.257*	*0.258*
	0.269	**0.248**	**0.248**	**0.248**	**0.248**	**0.201**	**0.213**	**0.213**	**0.213**	**0.212**	**0.318**	**0.294**	**0.295**	**0.296**	**0.296**	**0.258**	**0.257**	**0.256**	**0.257**	**0.258**
	−0.009	−0.007	−0.007	−0.007	−0.006	−0.003	−0.002	−0.001	−0.002	−0.001	−0.015	−0.015	−0.016	−0.015	−0.014	0.009	−0.002	0.004	0.002	0.003
$\tau = 2$	*0.163*	*0.162*	*0.163*	*0.162*	*0.163*	*0.134*	*0.133*	*0.134*	*0.134*	*0.134*	*0.267*	*0.229*	*0.231*	*0.230*	*0.230*	*0.191*	*0.203*	*0.204*	*0.203*	*0.202*
	0.163	**0.162**	**0.163**	**0.162**	**0.163**	**0.134**	**0.133**	**0.134**	**0.134**	**0.134**	**0.268**	**0.230**	**0.231**	**0.231**	**0.231**	**0.192**	**0.203**	**0.204**	**0.203**	**0.202**

Figures in italics are the standard errors, figures in bold represent RMSE and figures without any layout are the biases.

Table 2: Estimation results for $\hat{\beta}_1$ and $\hat{\beta}_2$ for the heteroskedastic case with n=100

Results for the $\hat{\beta}_1$ estimator

	Results for QML estimation										Results for GMM estimation									
	Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$					Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.003	0.002	0.000	0.003	0.002	0.004	0.003	0.003	0.004	0.004	−0.004	−0.005	−0.006	−0.006	−0.006	−0.010	−0.010	−0.009	−0.010	−0.008
$\tau = -2$	*0.078*	*0.077*	*0.078*	*0.078*	*0.078*	*0.070*	*0.070*	*0.070*	*0.070*	*0.070*	*0.078*	*0.078*	*0.078*	*0.078*	*0.078*	*0.067*	*0.068*	*0.069*	*0.069*	*0.067*
	0.078	**0.077**	**0.078**	**0.078**	**0.078**	**0.070**	**0.070**	**0.070**	**0.070**	**0.070**	**0.079**	**0.079**	**0.078**	**0.078**	**0.078**	**0.068**	**0.069**	**0.070**	**0.069**	**0.068**
	0.003	0.002	0.002	0.002	0.002	0.002	0.003	0.002	0.002	0.003	−0.003	−0.005	−0.005	−0.005	−0.003	−0.007	−0.007	−0.007	−0.007	−0.007
$\tau = -1$	*0.064*	*0.063*	*0.064*	*0.063*	*0.063*	*0.070*	*0.070*	*0.070*	*0.070*	*0.070*	*0.063*	*0.063*	*0.063*	*0.063*	*0.063*	*0.068*	*0.069*	*0.069*	*0.069*	*0.069*
	0.064	**0.063**	**0.064**	**0.063**	**0.063**	**0.070**	**0.070**	**0.070**	**0.070**	**0.070**	**0.063**	**0.063**	**0.063**	**0.063**	**0.063**	**0.068**	**0.069**	**0.069**	**0.069**	**0.069**
	0.003	0.003	0.003	0.004	0.004	0.002	0.001	0.002	0.001	0.002	−0.003	−0.003	−0.003	−0.003	−0.003	−0.006	−0.006	−0.006	−0.006	−0.006
$\tau = 0$	*0.066*	*0.066*	*0.067*	*0.066*	*0.066*	*0.065*	*0.066*	*0.066*	*0.066*	*0.065*	*0.067*	*0.067*	*0.067*	*0.067*	*0.067*	*0.067*	*0.066*	*0.065*	*0.066*	*0.065*
	0.067	**0.066**	**0.067**	**0.066**	**0.066**	**0.065**	**0.066**	**0.066**	**0.066**	**0.065**	**0.067**	**0.067**	**0.067**	**0.067**	**0.067**	**0.067**	**0.066**	**0.066**	**0.066**	**0.066**
	0.003	0.004	0.003	0.004	0.004	0.000	0.000	0.000	0.000	0.000	−0.002	−0.002	−0.002	−0.003	−0.002	−0.005	−0.005	−0.002	−0.006	−0.005
$\tau = 1$	*0.074*	*0.074*	*0.074*	*0.074*	*0.073*	*0.064*	*0.064*	*0.064*	*0.064*	*0.064*	*0.075*	*0.075*	*0.075*	*0.075*	*0.075*	*0.068*	*0.065*	*0.064*	*0.064*	*0.064*
	0.074	**0.074**	**0.074**	**0.074**	**0.073**	**0.064**	**0.064**	**0.064**	**0.064**	**0.064**	**0.075**	**0.075**	**0.075**	**0.075**	**0.075**	**0.069**	**0.065**	**0.064**	**0.065**	**0.064**
	0.003	0.004	0.004	0.004	0.003	−0.001	−0.001	−0.001	−0.001	0.000	−0.002	0.002	−0.002	−0.002	−0.002	−0.004	−0.005	−0.004	−0.007	−0.003
$\tau = 2$	*0.071*	*0.070*	*0.070*	*0.071*	*0.071*	*0.063*	*0.063*	*0.063*	*0.063*	*0.063*	*0.073*	*0.070*	*0.073*	*0.070*	*0.072*	*0.069*	*0.063*	*0.062*	*0.061*	*0.062*
	0.071	**0.070**	**0.070**	**0.071**	**0.071**	**0.063**	**0.063**	**0.063**	**0.063**	**0.063**	**0.073**	**0.070**	**0.073**	**0.070**	**0.072**	**0.069**	**0.063**	**0.062**	**0.062**	**0.062**

Results for the $\hat{\beta}_2$ estimator

	Results for QML estimation										Results for GMM estimation									
	Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$					Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.003	0.003	0.001	0.004	0.002	0.003	0.003	0.003	0.001	0.003	0.002	0.002	0.002	0.002	0.002	0.001	0.004	0.002	0.003	0.002
$\tau = -2$	*0.136*	*0.136*	*0.136*	*0.135*	*0.135*	*0.164*	*0.164*	*0.165*	*0.164*	*0.166*	*0.132*	*0.131*	*0.132*	*0.132*	*0.132*	*0.155*	*0.153*	*0.154*	*0.154*	*0.156*
	0.136	**0.136**	**0.136**	**0.135**	**0.135**	**0.164**	**0.164**	**0.165**	**0.164**	**0.166**	**0.132**	**0.131**	**0.132**	**0.132**	**0.132**	**0.155**	**0.153**	**0.154**	**0.154**	**0.156**
	0.003	0.004	0.004	0.003	0.004	0.003	0.004	0.003	0.003	0.001	0.005	0.006	0.006	0.006	0.005	0.002	0.004	0.004	0.004	0.004
$\tau = -1$	*0.165*	*0.166*	*0.165*	*0.166*	*0.167*	*0.180*	*0.180*	*0.180*	*0.181*	*0.181*	*0.157*	*0.158*	*0.158*	*0.158*	*0.158*	*0.169*	*0.170*	*0.171*	*0.170*	*0.171*
	0.165	**0.166**	**0.165**	**0.166**	**0.167**	**0.180**	**0.180**	**0.180**	**0.181**	**0.181**	**0.158**	**0.158**	**0.158**	**0.158**	**0.158**	**0.169**	**0.170**	**0.171**	**0.171**	**0.171**
	0.000	0.004	0.000	0.004	0.003	0.002	0.001	0.002	0.002	0.000	0.006	0.004	0.004	0.004	0.004	0.002	0.005	0.004	0.003	0.004
$\tau = 0$	*0.189*	*0.191*	*0.190*	*0.190*	*0.190*	*0.190*	*0.190*	*0.190*	*0.191*	*0.190*	*0.179*	*0.179*	*0.179*	*0.179*	*0.179*	*0.183*	*0.180*	*0.179*	*0.179*	*0.179*
	0.189	**0.191**	**0.190**	**0.190**	**0.190**	**0.190**	**0.190**	**0.190**	**0.191**	**0.190**	**0.179**	**0.179**	**0.179**	**0.179**	**0.179**	**0.183**	**0.180**	**0.179**	**0.179**	**0.179**
	0.000	−0.001	0.001	0.003	0.001	−0.001	0.001	−0.001	0.002	0.000	−0.002	−0.004	−0.004	−0.004	−0.004	0.001	0.000	−0.005	0.001	0.000
$\tau = 1$	*0.159*	*0.159*	*0.159*	*0.161*	*0.158*	*0.182*	*0.182*	*0.182*	*0.182*	*0.182*	*0.151*	*0.151*	*0.150*	*0.151*	*0.151*	*0.196*	*0.175*	*0.174*	*0.171*	*0.171*
	0.159	**0.159**	**0.159**	**0.161**	**0.158**	**0.182**	**0.182**	**0.182**	**0.182**	**0.182**	**0.151**	**0.151**	**0.151**	**0.151**	**0.151**	**0.196**	**0.175**	**0.174**	**0.171**	**0.171**
	0.001	0.002	0.001	0.002	0.000	−0.001	−0.001	−0.001	−0.001	−0.002	−0.007	0.002	−0.005	0.001	−0.005	−0.004	−0.001	−0.002	−0.004	−0.003
$\tau = 2$	*0.102*	*0.102*	*0.103*	*0.103*	*0.102*	*0.140*	*0.140*	*0.139*	*0.138*	*0.140*	*0.098*	*0.098*	*0.098*	*0.098*	*0.098*	*0.197*	*0.142*	*0.136*	*0.137*	*0.135*
	0.102	**0.102**	**0.103**	**0.103**	**0.102**	**0.140**	**0.140**	**0.139**	**0.138**	**0.140**	**0.098**	**0.098**	**0.098**	**0.098**	**0.098**	**0.197**	**0.142**	**0.136**	**0.137**	**0.135**

Figures in italics are the standard errors, figures in bold represent RMSE and figures without any layout are the biases.

Table 3: QML estimation results for $\hat{\alpha}$ and $\hat{\tau}$ for the homoskedastic case with n=254

Results for the $\hat{\alpha}$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.006	0.005	0.005	0.004	0.004	0.002	0.001	0.002	0.001	0.002	0.009	0.008	0.009	0.009	0.008	−0.003	−0.002	−0.003	−0.002	−0.002
$\tau = -2$	*0.147*	*0.147*	*0.148*	*0.148*	*0.148*	*0.146*	*0.144*	*0.145*	*0.146*	*0.145*	*0.096*	*0.096*	*0.097*	*0.096*	*0.096*	*0.097*	*0.097*	*0.096*	*0.096*	*0.097*
	0.147	**0.147**	**0.148**	**0.148**	**0.148**	**0.146**	**0.144**	**0.145**	**0.146**	**0.145**	**0.096**	**0.097**	**0.097**	**0.097**	**0.096**	**0.097**	**0.097**	**0.096**	**0.096**	**0.097**
	0.002	0.002	0.002	0.002	0.002	−0.005	−0.005	−0.005	−0.006	−0.005	0.005	0.006	0.005	0.005	0.005	−0.005	−0.005	−0.005	−0.004	−0.005
$\tau = -1$	*0.087*	*0.088*	*0.087*	*0.087*	*0.087*	*0.085*	*0.086*	*0.086*	*0.086*	*0.087*	*0.079*	*0.079*	*0.079*	*0.079*	*0.079*	*0.080*	*0.080*	*0.080*	*0.080*	*0.080*
	0.087	**0.088**	**0.087**	**0.087**	**0.087**	**0.085**	**0.087**	**0.086**	**0.086**	**0.087**	**0.079**	**0.079**	**0.080**	**0.079**	**0.079**	**0.081**	**0.080**	**0.080**	**0.081**	**0.080**
	0.001	0.001	0.002	0.001	0.002	−0.007	−0.006	−0.007	−0.006	−0.007	0.003	0.003	0.003	0.003	0.003	−0.006	−0.005	−0.006	−0.006	−0.005
$\tau = 0$	*0.069*	*0.069*	*0.069*	*0.069*	*0.070*	*0.068*	*0.069*	*0.069*	*0.069*	*0.069*	*0.062*	*0.061*	*0.062*	*0.062*	*0.062*	*0.063*	*0.062*	*0.063*	*0.063*	*0.062*
	0.069	**0.069**	**0.069**	**0.069**	**0.070**	**0.069**	**0.069**	**0.069**	**0.069**	**0.069**	**0.062**	**0.061**	**0.062**	**0.062**	**0.062**	**0.063**	**0.063**	**0.063**	**0.063**	**0.063**
	0.001	0.002	0.002	0.001	0.001	−0.007	−0.007	−0.007	−0.006	−0.007	0.003	0.003	0.003	0.002	0.002	−0.005	−0.005	−0.004	−0.004	−0.004
$\tau = 1$	*0.074*	*0.074*	*0.074*	*0.074*	*0.074*	*0.074*	*0.073*	*0.073*	*0.073*	*0.074*	*0.059*	*0.059*	*0.059*	*0.059*	*0.059*	*0.060*	*0.060*	*0.060*	*0.059*	*0.059*
	0.074	**0.074**	**0.074**	**0.074**	**0.074**	**0.074**	**0.074**	**0.073**	**0.074**	**0.075**	**0.059**	**0.059**	**0.059**	**0.059**	**0.059**	**0.060**	**0.060**	**0.060**	**0.059**	**0.060**
	0.001	0.000	0.001	0.001	0.001	−0.004	−0.003	−0.003	−0.003	−0.004	0.002	0.002	0.002	0.001	0.002	−0.003	−0.003	−0.003	−0.004	−0.003
$\tau = 2$	*0.070*	*0.070*	*0.070*	*0.070*	*0.070*	*0.069*	*0.069*	*0.069*	*0.069*	*0.069*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*
	0.070	**0.070**	**0.070**	**0.070**	**0.070**	**0.069**	**0.069**	**0.069**	**0.069**	**0.069**	**0.055**	**0.055**	**0.056**	**0.055**	**0.055**	**0.055**	**0.055**	**0.055**	**0.055**	**0.055**

Results for the $\hat{\tau}$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.003	0.004	0.004	0.004	0.004	−0.005	−0.004	−0.006	−0.005	−0.006	0.010	0.012	0.013	0.012	0.013	0.015	0.014	0.012	0.013	0.013
$\tau = -2$	*0.183*	*0.183*	*0.184*	*0.183*	*0.183*	*0.185*	*0.184*	*0.185*	*0.185*	*0.185*	*0.230*	*0.230*	*0.231*	*0.231*	*0.230*	*0.212*	*0.212*	*0.209*	*0.209*	*0.210*
	0.183	**0.183**	**0.184**	**0.184**	**0.183**	**0.185**	**0.184**	**0.185**	**0.185**	**0.185**	**0.230**	**0.230**	**0.232**	**0.231**	**0.231**	**0.213**	**0.212**	**0.209**	**0.210**	**0.210**
	0.014	0.014	0.013	0.012	0.015	0.009	0.009	0.007	0.008	0.008	0.035	0.031	0.035	0.035	0.034	0.034	0.034	0.035	0.034	0.036
$\tau = -1$	*0.138*	*0.139*	*0.139*	*0.139*	*0.138*	*0.137*	*0.138*	*0.139*	*0.138*	*0.138*	*0.227*	*0.226*	*0.226*	*0.226*	*0.226*	*0.206*	*0.206*	*0.206*	*0.207*	*0.207*
	0.139	**0.139**	**0.139**	**0.139**	**0.138**	**0.137**	**0.139**	**0.139**	**0.138**	**0.138**	**0.229**	**0.228**	**0.229**	**0.229**	**0.229**	**0.209**	**0.209**	**0.209**	**0.210**	**0.210**
	0.020	0.019	0.019	0.018	0.018	0.014	0.014	0.016	0.014	0.016	0.051	0.053	0.053	0.052	0.050	0.049	0.050	0.051	0.052	0.049
$\tau = 0$	*0.126*	*0.127*	*0.127*	*0.127*	*0.127*	*0.126*	*0.127*	*0.127*	*0.127*	*0.126*	*0.215*	*0.215*	*0.216*	*0.217*	*0.213*	*0.197*	*0.196*	*0.197*	*0.197*	*0.197*
	0.128	**0.128**	**0.128**	**0.128**	**0.129**	**0.127**	**0.128**	**0.128**	**0.128**	**0.127**	**0.221**	**0.222**	**0.222**	**0.223**	**0.219**	**0.203**	**0.202**	**0.203**	**0.204**	**0.203**
	0.026	0.025	0.026	0.025	0.026	0.022	0.023	0.023	0.021	0.022	0.064	0.065	0.066	0.065	0.065	0.061	0.062	0.061	0.061	0.060
$\tau = 1$	*0.130*	*0.129*	*0.129*	*0.129*	*0.129*	*0.129*	*0.129*	*0.129*	*0.129*	*0.130*	*0.217*	*0.217*	*0.217*	*0.219*	*0.218*	*0.198*	*0.198*	*0.197*	*0.195*	*0.196*
	0.132	**0.132**	**0.132**	**0.132**	**0.131**	**0.131**	**0.131**	**0.131**	**0.131**	**0.131**	**0.226**	**0.227**	**0.226**	**0.228**	**0.228**	**0.207**	**0.207**	**0.206**	**0.204**	**0.205**
	0.033	0.034	0.033	0.033	0.033	0.025	0.024	0.024	0.024	0.025	0.083	0.084	0.084	0.084	0.082	0.077	0.079	0.079	0.080	0.079
$\tau = 2$	*0.127*	*0.126*	*0.127*	*0.127*	*0.127*	*0.126*	*0.127*	*0.127*	*0.127*	*0.127*	*0.222*	*0.220*	*0.221*	*0.222*	*0.219*	*0.197*	*0.200*	*0.200*	*0.199*	*0.198*
	0.131	**0.131**	**0.131**	**0.131**	**0.131**	**0.129**	**0.129**	**0.129**	**0.130**	**0.130**	**0.237**	**0.235**	**0.236**	**0.238**	**0.234**	**0.211**	**0.215**	**0.215**	**0.215**	**0.213**

Figures in italics are the standard errors, figures in bold represent RMSE and figures without any layout are the biases.

Table 4: QML estimation results for $\hat{\beta}_1$ and $\hat{\beta}_2$ for the homoskedastic case with n=254

Results for the $\hat{\beta}_1$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
$\tau=-2$	0.003	0.003	0.003	0.003	0.003	−0.002	−0.001	−0.001	−0.001	−0.001	0.004	0.004	0.004	0.004	0.004	−0.002	−0.001	−0.001	−0.001	−0.002
	0.063	*0.063*	*0.063*	*0.064*	*0.063*	*0.065*	*0.064*	*0.064*	*0.065*	*0.065*	*0.058*	*0.058*	*0.057*	*0.058*	*0.058*	*0.060*	*0.059*	*0.059*	*0.059*	*0.059*
	0.063	**0.063**	**0.063**	**0.064**	**0.064**	**0.065**	**0.064**	**0.064**	**0.065**	**0.065**	**0.058**	**0.058**	**0.058**	**0.058**	**0.058**	**0.060**	**0.059**	**0.059**	**0.059**	**0.059**
$\tau=-1$	0.004	0.004	0.004	0.004	0.004	−0.002	−0.001	−0.001	−0.001	−0.001	0.006	0.005	0.005	0.006	0.006	−0.001	−0.002	−0.001	−0.002	−0.002
	0.051	*0.051*	*0.051*	*0.051*	*0.051*	*0.052*	*0.052*	*0.052*	*0.052*	*0.052*	*0.058*	*0.058*	*0.058*	*0.058*	*0.057*	*0.059*	*0.060*	*0.059*	*0.060*	*0.059*
	0.051	**0.051**	**0.051**	**0.051**	**0.051**	**0.052**	**0.052**	**0.052**	**0.052**	**0.052**	**0.058**	**0.058**	**0.058**	**0.058**	**0.058**	**0.059**	**0.060**	**0.059**	**0.060**	**0.059**
$\tau=0$	0.005	0.005	0.005	0.004	0.005	−0.003	−0.002	−0.002	−0.002	−0.003	0.006	0.006	0.006	0.006	0.006	−0.002	−0.001	−0.002	−0.002	−0.002
	0.061	*0.061*	*0.061*	*0.061*	*0.061*	*0.061*	*0.062*	*0.062*	*0.062*	*0.062*	*0.057*	*0.057*	*0.057*	*0.057*	*0.057*	*0.059*	*0.058*	*0.059*	*0.058*	*0.059*
	0.061	**0.061**	**0.061**	**0.061**	**0.061**	**0.061**	**0.062**	**0.062**	**0.062**	**0.062**	**0.057**	**0.057**	**0.057**	**0.057**	**0.057**	**0.059**	**0.058**	**0.059**	**0.058**	**0.059**
$\tau=1$	0.005	0.005	0.005	0.005	0.005	−0.003	−0.003	−0.003	−0.003	−0.003	0.005	0.005	0.005	0.005	0.005	−0.002	−0.003	−0.002	−0.002	−0.002
	0.072	*0.072*	*0.072*	*0.072*	*0.072*	*0.072*	*0.072*	*0.071*	*0.072*	*0.073*	*0.058*	*0.058*	*0.058*	*0.058*	*0.058*	*0.059*	*0.059*	*0.059*	*0.058*	*0.059*
	0.072	**0.072**	**0.072**	**0.072**	**0.072**	**0.072**	**0.072**	**0.071**	**0.072**	**0.073**	**0.058**	**0.058**	**0.058**	**0.058**	**0.058**	**0.059**	**0.059**	**0.059**	**0.058**	**0.059**
$\tau=2$	0.004	0.003	0.003	0.004	0.003	−0.001	0.000	0.000	0.000	−0.001	0.004	0.004	0.004	0.003	0.004	−0.001	−0.002	−0.002	−0.002	−0.002
	0.069	*0.069*	*0.069*	*0.069*	*0.069*	*0.069*	*0.069*	*0.069*	*0.069*	*0.069*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*	*0.055*
	0.069	**0.069**	**0.069**	**0.069**	**0.069**	**0.069**	**0.069**	**0.069**	**0.069**	**0.069**	**0.055**	**0.055**	**0.056**	**0.055**	**0.055**	**0.055**	**0.055**	**0.055**	**0.055**	**0.055**

Results for the $\hat{\beta}_2$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
$\tau=-2$	0.001	0.002	0.001	0.001	0.002	−0.005	−0.004	−0.005	−0.005	−0.004	0.000	−0.001	0.001	0.000	0.001	−0.008	−0.007	−0.007	−0.006	−0.007
	0.140	*0.140*	*0.140*	*0.140*	*0.139*	*0.132*	*0.131*	*0.132*	*0.132*	*0.132*	*0.170*	*0.170*	*0.169*	*0.168*	*0.169*	*0.168*	*0.168*	*0.168*	*0.168*	*0.169*
	0.140	**0.140**	**0.140**	**0.140**	**0.139**	**0.133**	**0.132**	**0.132**	**0.132**	**0.132**	**0.170**	**0.170**	**0.169**	**0.168**	**0.169**	**0.168**	**0.168**	**0.168**	**0.168**	**0.169**
$\tau=-1$	0.001	0.000	0.000	0.001	0.003	−0.005	−0.006	−0.004	−0.006	−0.007	0.001	0.001	0.001	0.002	0.000	−0.007	−0.008	−0.009	−0.008	−0.007
	0.175	*0.175*	*0.174*	*0.174*	*0.174*	*0.168*	*0.169*	*0.167*	*0.168*	*0.168*	*0.187*	*0.187*	*0.186*	*0.186*	*0.187*	*0.183*	*0.185*	*0.183*	*0.185*	*0.184*
	0.175	**0.175**	**0.174**	**0.174**	**0.174**	**0.168**	**0.169**	**0.167**	**0.168**	**0.169**	**0.187**	**0.187**	**0.186**	**0.186**	**0.187**	**0.183**	**0.185**	**0.183**	**0.185**	**0.184**
$\tau=0$	0.000	0.002	−0.001	0.001	0.000	−0.011	−0.012	−0.008	−0.009	−0.010	0.002	0.001	−0.001	0.000	0.002	−0.011	−0.007	−0.010	−0.010	−0.009
	0.195	*0.193*	*0.195*	*0.195*	*0.196*	*0.191*	*0.193*	*0.193*	*0.192*	*0.192*	*0.196*	*0.195*	*0.195*	*0.196*	*0.195*	*0.192*	*0.192*	*0.191*	*0.192*	*0.192*
	0.195	**0.193**	**0.195**	**0.195**	**0.196**	**0.192**	**0.194**	**0.193**	**0.192**	**0.192**	**0.196**	**0.195**	**0.195**	**0.196**	**0.195**	**0.192**	**0.192**	**0.192**	**0.192**	**0.192**
$\tau=1$	−0.002	−0.001	−0.001	0.001	0.001	−0.010	−0.012	−0.011	−0.012	−0.012	0.000	0.001	0.001	0.000	0.001	−0.011	−0.011	−0.010	−0.010	−0.009
	0.159	*0.160*	*0.159*	*0.160*	*0.160*	*0.163*	*0.162*	*0.162*	*0.162*	*0.163*	*0.180*	*0.181*	*0.180*	*0.181*	*0.181*	*0.180*	*0.179*	*0.179*	*0.179*	*0.179*
	0.159	**0.160**	**0.159**	**0.160**	**0.160**	**0.163**	**0.163**	**0.162**	**0.163**	**0.163**	**0.180**	**0.181**	**0.180**	**0.181**	**0.181**	**0.180**	**0.179**	**0.179**	**0.180**	**0.179**
$\tau=2$	0.001	0.000	0.001	0.001	0.001	−0.006	−0.005	−0.005	−0.006	−0.006	0.001	0.002	0.001	0.001	0.002	−0.007	−0.007	−0.007	−0.007	−0.007
	0.101	*0.101*	*0.101*	*0.101*	*0.101*	*0.103*	*0.104*	*0.103*	*0.103*	*0.103*	*0.131*	*0.131*	*0.130*	*0.131*	*0.131*	*0.135*	*0.135*	*0.135*	*0.135*	*0.135*
	0.101	**0.101**	**0.101**	**0.101**	**0.101**	**0.104**	**0.104**	**0.103**	**0.103**	**0.103**	**0.131**	**0.131**	**0.130**	**0.131**	**0.131**	**0.135**	**0.135**	**0.136**	**0.135**	**0.135**

Figures in italics are the standard errors, figures in bold represent RMSE and figures without any layout are the biases.

Table 5: GMM estimation results for $\hat{\alpha}$ and $\hat{\tau}$ for the homoskedastic case with n=254

Results for the $\hat{\alpha}$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	−0.009	−0.009	−0.007	−0.007	−0.006	−0.014	−0.014	−0.008	−0.008	−0.007	0.003	−0.002	−0.003	−0.004	−0.003	−0.004	−0.003	−0.004	−0.005	−0.005
$\tau = -2$	*0.142*	*0.143*	*0.143*	*0.143*	*0.143*	*0.126*	*0.126*	*0.127*	*0.127*	*0.127*	*0.101*	*0.104*	*0.103*	*0.104*	*0.104*	*0.090*	*0.090*	*0.090*	*0.090*	*0.090*
	0.143	**0.143**	**0.143**	**0.144**	**0.143**	**0.127**	**0.127**	**0.127**	**0.127**	**0.127**	**0.101**	**0.104**	**0.103**	**0.104**	**0.104**	**0.090**	**0.090**	**0.090**	**0.090**	**0.090**
	0.001	0.002	−0.001	0.000	0.000	−0.012	−0.012	−0.007	−0.006	−0.006	0.006	0.000	0.001	−0.001	−0.001	−0.005	−0.011	−0.006	−0.005	−0.006
$\tau = -1$	*0.088*	*0.088*	*0.088*	*0.088*	*0.088*	*0.079*	*0.079*	*0.082*	*0.081*	*0.081*	*0.083*	*0.083*	*0.083*	*0.081*	*0.082*	*0.074*	*0.075*	*0.074*	*0.074*	*0.075*
	0.088	**0.088**	**0.088**	**0.088**	**0.088**	**0.080**	**0.080**	**0.082**	**0.081**	**0.081**	**0.083**	**0.083**	**0.083**	**0.081**	**0.082**	**0.074**	**0.076**	**0.074**	**0.074**	**0.076**
	0.002	−0.002	−0.002	−0.002	−0.002	−0.009	−0.009	−0.005	−0.004	−0.004	0.005	0.002	0.002	0.002	0.002	−0.004	−0.004	−0.004	−0.004	−0.004
$\tau = 0$	*0.069*	*0.069*	*0.069*	*0.069*	*0.069*	*0.057*	*0.057*	*0.058*	*0.058*	*0.058*	*0.065*	*0.063*	*0.063*	*0.062*	*0.062*	*0.054*	*0.054*	*0.054*	*0.054*	*0.054*
	0.069	**0.069**	**0.069**	**0.069**	**0.069**	**0.057**	**0.057**	**0.058**	**0.059**	**0.058**	**0.065**	**0.063**	**0.063**	**0.062**	**0.062**	**0.054**	**0.054**	**0.054**	**0.054**	**0.054**
	0.003	−0.001	0.003	−0.001	−0.001	−0.007	−0.004	−0.003	−0.002	−0.002	0.004	0.002	0.000	0.001	0.003	−0.004	−0.003	−0.002	−0.003	−0.003
$\tau = 1$	*0.072*	*0.073*	*0.072*	*0.073*	*0.073*	*0.057*	*0.057*	*0.057*	*0.058*	*0.058*	*0.060*	*0.059*	*0.059*	*0.058*	*0.059*	*0.048*	*0.048*	*0.048*	*0.048*	*0.048*
	0.073	**0.073**	**0.072**	**0.073**	**0.073**	**0.058**	**0.057**	**0.057**	**0.059**	**0.058**	**0.060**	**0.059**	**0.059**	**0.058**	**0.059**	**0.048**	**0.048**	**0.048**	**0.048**	**0.048**
	0.003	0.000	0.003	0.000	0.000	−0.004	−0.002	0.000	0.000	−0.001	0.004	0.003	0.003	0.002	0.002	−0.002	−0.002	−0.002	−0.002	−0.002
$\tau = 2$	*0.068*	*0.069*	*0.068*	*0.069*	*0.068*	*0.054*	*0.054*	*0.055*	*0.055*	*0.055*	*0.056*	*0.055*	*0.054*	*0.055*	*0.055*	*0.046*	*0.046*	*0.045*	*0.045*	*0.045*
	0.068	**0.069**	**0.069**	**0.069**	**0.068**	**0.054**	**0.054**	**0.055**	**0.055**	**0.055**	**0.056**	**0.055**	**0.055**	**0.055**	**0.055**	**0.046**	**0.046**	**0.046**	**0.045**	**0.045**

Results for the $\hat{\tau}$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.017	0.016	0.010	0.009	0.010	0.018	0.018	0.011	0.010	0.010	0.018	0.033	0.033	0.037	0.037	0.017	0.015	0.015	0.016	0.018
$\tau = -2$	*0.179*	*0.178*	*0.186*	*0.185*	*0.186*	*0.169*	*0.169*	*0.170*	*0.168*	*0.168*	*0.250*	*0.260*	*0.260*	*0.259*	*0.257*	*0.209*	*0.209*	*0.209*	*0.209*	*0.209*
	0.180	**0.178**	**0.186**	**0.185**	**0.186**	**0.170**	**0.170**	**0.170**	**0.168**	**0.168**	**0.251**	**0.263**	**0.262**	**0.261**	**0.260**	**0.210**	**0.210**	**0.210**	**0.210**	**0.209**
	0.014	0.014	0.010	0.011	0.010	0.021	0.021	0.014	0.014	0.014	0.029	0.050	0.048	0.050	0.053	0.032	0.035	0.033	0.030	0.031
$\tau = -1$	*0.138*	*0.138*	*0.142*	*0.143*	*0.142*	*0.133*	*0.132*	*0.133*	*0.132*	*0.133*	*0.250*	*0.254*	*0.255*	*0.250*	*0.253*	*0.209*	*0.209*	*0.209*	*0.209*	*0.206*
	0.139	**0.139**	**0.142**	**0.144**	**0.142**	**0.135**	**0.134**	**0.133**	**0.133**	**0.134**	**0.252**	**0.259**	**0.259**	**0.255**	**0.258**	**0.211**	**0.212**	**0.211**	**0.211**	**0.209**
	0.018	0.017	0.017	0.017	0.018	0.021	0.021	0.016	0.016	0.016	0.043	0.058	0.058	0.063	0.059	0.043	0.042	0.042	0.042	0.043
$\tau = 0$	*0.126*	*0.129*	*0.129*	*0.129*	*0.129*	*0.121*	*0.121*	*0.119*	*0.120*	*0.120*	*0.242*	*0.239*	*0.240*	*0.238*	*0.234*	*0.201*	*0.201*	*0.199*	*0.200*	*0.200*
	0.127	**0.130**	**0.130**	**0.130**	**0.131**	**0.123**	**0.122**	**0.120**	**0.121**	**0.121**	**0.246**	**0.246**	**0.247**	**0.246**	**0.242**	**0.205**	**0.205**	**0.204**	**0.204**	**0.204**
	0.023	0.024	0.024	0.024	0.024	0.023	0.019	0.020	0.018	0.018	0.054	0.066	0.060	0.070	0.068	0.043	0.048	0.047	0.047	0.046
$\tau = 1$	*0.128*	*0.129*	*0.129*	*0.129*	*0.130*	*0.121*	*0.118*	*0.121*	*0.121*	*0.121*	*0.240*	*0.240*	*0.242*	*0.232*	*0.236*	*0.200*	*0.204*	*0.204*	*0.203*	*0.203*
	0.130	**0.131**	**0.131**	**0.131**	**0.132**	**0.124**	**0.120**	**0.123**	**0.122**	**0.122**	**0.246**	**0.248**	**0.249**	**0.243**	**0.245**	**0.204**	**0.209**	**0.209**	**0.208**	**0.208**
	0.029	0.028	0.030	0.027	0.028	0.024	0.022	0.020	0.020	0.020	0.067	0.069	0.073	0.075	0.073	0.059	0.058	0.060	0.053	0.059
$\tau = 2$	*0.126*	*0.125*	*0.126*	*0.124*	*0.124*	*0.120*	*0.121*	*0.120*	*0.120*	*0.120*	*0.240*	*0.237*	*0.232*	*0.234*	*0.233*	*0.206*	*0.205*	*0.206*	*0.204*	*0.206*
	0.129	**0.128**	**0.129**	**0.127**	**0.128**	**0.122**	**0.123**	**0.122**	**0.122**	**0.122**	**0.250**	**0.247**	**0.243**	**0.246**	**0.244**	**0.214**	**0.213**	**0.214**	**0.211**	**0.214**

Figures in italics are the standard errors, figures in bold represent RMSE and figures without any layout are the biases.

Table 6: GMM estimation results for $\hat{\beta}_1$ and $\hat{\beta}_2$ for the homoskedastic case with n=254

Results for the $\hat{\beta}_1$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.007	0.007	0.004	0.004	0.004	0.005	0.005	0.003	0.004	0.004	0.004	0.006	0.007	0.006	0.006	0.001	0.001	0.001	0.001	0.001
$\tau=-2$	*0.061*	*0.062*	*0.062*	*0.063*	*0.062*	*0.053*	*0.053*	*0.053*	*0.053*	*0.053*	*0.056*	*0.057*	*0.057*	*0.057*	*0.056*	*0.045*	*0.045*	*0.045*	*0.045*	*0.045*
	0.061	**0.062**	**0.062**	**0.063**	**0.062**	**0.054**	**0.053**	**0.053**	**0.053**	**0.053**	**0.057**	**0.058**	**0.057**	**0.057**	**0.056**	**0.045**	**0.045**	**0.045**	**0.045**	**0.045**
	0.004	0.003	0.000	0.000	0.000	0.002	0.001	0.001	0.002	0.002	0.004	0.006	0.006	0.006	0.005	0.002	0.000	0.002	0.002	0.001
$\tau=-1$	*0.050*	*0.050*	*0.051*	*0.051*	*0.051*	*0.041*	*0.041*	*0.042*	*0.041*	*0.042*	*0.057*	*0.057*	*0.057*	*0.057*	*0.057*	*0.046*	*0.046*	*0.046*	*0.046*	*0.046*
	0.051	**0.050**	**0.051**	**0.051**	**0.051**	**0.041**	**0.041**	**0.042**	**0.042**	**0.042**	**0.057**	**0.057**	**0.057**	**0.058**	**0.057**	**0.046**	**0.046**	**0.046**	**0.046**	**0.046**
	0.005	0.000	0.000	0.000	0.000	−0.002	−0.002	0.000	0.001	0.001	0.005	0.005	0.005	0.005	0.004	0.001	0.001	0.001	0.001	0.001
$\tau=0$	*0.060*	*0.061*	*0.061*	*0.061*	*0.061*	*0.047*	*0.047*	*0.047*	*0.048*	*0.048*	*0.057*	*0.057*	*0.057*	*0.057*	*0.057*	*0.046*	*0.046*	*0.046*	*0.046*	*0.046*
	0.060	**0.061**	**0.061**	**0.061**	**0.061**	**0.047**	**0.047**	**0.047**	**0.048**	**0.048**	**0.057**	**0.057**	**0.057**	**0.057**	**0.057**	**0.046**	**0.046**	**0.046**	**0.046**	**0.046**
	0.006	0.002	0.006	0.002	0.002	−0.003	−0.001	0.000	0.001	0.001	0.006	0.004	0.002	0.004	0.005	−0.001	0.000	0.000	0.000	0.000
$\tau=1$	*0.071*	*0.071*	*0.070*	*0.071*	*0.071*	*0.055*	*0.054*	*0.055*	*0.056*	*0.056*	*0.059*	*0.058*	*0.058*	*0.057*	*0.058*	*0.046*	*0.046*	*0.046*	*0.046*	*0.046*
	0.071	**0.071**	**0.071**	**0.071**	**0.071**	**0.055**	**0.054**	**0.055**	**0.056**	**0.056**	**0.059**	**0.058**	**0.058**	**0.058**	**0.058**	**0.046**	**0.046**	**0.046**	**0.046**	**0.046**
	0.006	0.003	0.006	0.002	0.003	−0.002	0.000	0.002	0.002	0.001	0.006	0.005	0.005	0.004	0.004	−0.001	−0.001	−0.001	−0.001	−0.001
$\tau=2$	*0.068*	*0.068*	*0.068*	*0.068*	*0.068*	*0.053*	*0.053*	*0.054*	*0.054*	*0.054*	*0.056*	*0.055*	*0.055*	*0.055*	*0.055*	*0.045*	*0.045*	*0.045*	*0.045*	*0.045*
	0.068	**0.068**	**0.068**	**0.068**	**0.068**	**0.053**	**0.053**	**0.054**	**0.054**	**0.054**	**0.056**	**0.055**	**0.055**	**0.055**	**0.055**	**0.045**	**0.045**	**0.045**	**0.045**	**0.045**

Results for the $\hat{\beta}_2$ estimator

	Results for $W_n = M_n$										Results for $W_n \neq M_n$									
	Normally distributed error terms					Non-normally distributed error terms					Normally distributed error terms					Non-normally distributed error terms				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.005	0.005	0.005	0.005	0.005	0.001	0.001	0.004	0.005	0.005	0.000	−0.001	0.001	−0.001	0.001	0.000	−0.001	−0.001	−0.001	0.000
$\tau=-2$	*0.139*	*0.138*	*0.134*	*0.133*	*0.134*	*0.108*	*0.108*	*0.108*	*0.106*	*0.106*	*0.168*	*0.168*	*0.167*	*0.168*	*0.170*	*0.135*	*0.136*	*0.136*	*0.136*	*0.135*
	0.139	**0.138**	**0.134**	**0.133**	**0.134**	**0.108**	**0.108**	**0.108**	**0.107**	**0.106**	**0.168**	**0.168**	**0.167**	**0.168**	**0.170**	**0.135**	**0.136**	**0.136**	**0.136**	**0.135**
	0.000	0.000	0.004	0.004	0.003	−0.005	−0.004	0.001	0.002	0.002	−0.003	−0.001	−0.001	−0.002	−0.002	−0.002	−0.011	0.000	−0.001	−0.002
$\tau=-1$	*0.173*	*0.173*	*0.166*	*0.168*	*0.167*	*0.136*	*0.135*	*0.136*	*0.134*	*0.133*	*0.181*	*0.184*	*0.184*	*0.183*	*0.185*	*0.147*	*0.149*	*0.147*	*0.147*	*0.150*
	0.173	**0.173**	**0.166**	**0.168**	**0.167**	**0.136**	**0.135**	**0.136**	**0.134**	**0.133**	**0.181**	**0.184**	**0.184**	**0.183**	**0.185**	**0.147**	**0.149**	**0.147**	**0.147**	**0.150**
	−0.002	0.004	0.004	0.004	0.004	−0.011	−0.011	−0.002	−0.002	−0.002	−0.003	−0.004	−0.001	−0.003	−0.002	−0.001	−0.003	−0.003	−0.004	−0.002
$\tau=0$	*0.193*	*0.188*	*0.188*	*0.188*	*0.188*	*0.156*	*0.155*	*0.156*	*0.152*	*0.152*	*0.193*	*0.193*	*0.194*	*0.193*	*0.190*	*0.153*	*0.153*	*0.153*	*0.154*	*0.153*
	0.193	**0.188**	**0.188**	**0.188**	**0.188**	**0.156**	**0.156**	**0.156**	**0.152**	**0.152**	**0.193**	**0.193**	**0.194**	**0.193**	**0.190**	**0.153**	**0.153**	**0.154**	**0.154**	**0.153**
	−0.003	0.002	−0.003	0.001	0.001	−0.012	−0.006	−0.006	−0.004	−0.005	−0.003	−0.005	0.003	−0.009	−0.005	−0.006	−0.006	−0.006	−0.005	−0.006
$\tau=1$	*0.159*	*0.158*	*0.160*	*0.159*	*0.158*	*0.128*	*0.128*	*0.128*	*0.128*	*0.128*	*0.186*	*0.183*	*0.180*	*0.180*	*0.181*	*0.142*	*0.140*	*0.141*	*0.141*	*0.142*
	0.159	**0.158**	**0.160**	**0.159**	**0.158**	**0.128**	**0.129**	**0.129**	**0.128**	**0.129**	**0.186**	**0.183**	**0.180**	**0.180**	**0.182**	**0.142**	**0.140**	**0.141**	**0.141**	**0.142**
	0.000	0.002	0.001	0.001	0.002	−0.006	−0.004	−0.002	−0.002	−0.002	0.003	0.000	−0.004	−0.004	−0.001	−0.007	−0.007	−0.008	−0.004	−0.006
$\tau=2$	*0.100*	*0.100*	*0.101*	*0.100*	*0.099*	*0.079*	*0.081*	*0.082*	*0.082*	*0.082*	*0.138*	*0.133*	*0.134*	*0.133*	*0.133*	*0.104*	*0.104*	*0.104*	*0.102*	*0.104*
	0.100	**0.100**	**0.101**	**0.100**	**0.099**	**0.079**	**0.081**	**0.082**	**0.082**	**0.082**	**0.138**	**0.133**	**0.134**	**0.133**	**0.133**	**0.105**	**0.105**	**0.104**	**0.102**	**0.104**

Figures in italics are the standard errors, figures in bold represent RMSE and figures without any layout are the biases.

Table 7: Estimation results for $\hat{\alpha}$ and $\hat{\tau}$ for the heteroskedastic case with n=254

Results for the $\hat{\alpha}$ estimator

	Results for QML estimation										Results for GMM estimation									
	Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$					Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.000	0.001	0.000	0.001	0.001	−0.001	0.000	−0.001	0.000	0.000	0.000	0.000	0.000	−0.001	−0.002	0.001	−0.002	−0.002	−0.001	−0.002
$\tau = -2$	*0.059*	*0.059*	*0.059*	*0.059*	*0.059*	*0.046*	*0.046*	*0.046*	*0.046*	*0.046*	*0.059*	*0.059*	*0.059*	*0.059*	*0.059*	*0.046*	*0.045*	*0.045*	*0.045*	*0.045*
	0.059	**0.059**	**0.059**	**0.059**	**0.059**	**0.046**	**0.046**	**0.046**	**0.046**	**0.046**	**0.059**	**0.059**	**0.059**	**0.059**	**0.059**	**0.046**	**0.045**	**0.045**	**0.045**	**0.045**
	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	0.000	0.000	0.000	0.001	0.000	−0.001	−0.001	−0.002	−0.002	−0.001
$\tau = -1$	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.026*	*0.025*	*0.025*	*0.025*	*0.025*	*0.030*	*0.025*	*0.024*	*0.024*	*0.024*
	0.025	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.026**	**0.025**	**0.025**	**0.025**	**0.025**	**0.030**	**0.025**	**0.025**	**0.024**	**0.024**
	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	0.001	0.000	0.000	0.000	0.000	0.000	−0.001	−0.001	−0.001	−0.001
$\tau = 0$	*0.018*	*0.018*	*0.018*	*0.018*	*0.018*	*0.018*	*0.018*	*0.018*	*0.018*	*0.018*	*0.020*	*0.018*	*0.018*	*0.018*	*0.018*	*0.023*	*0.019*	*0.018*	*0.018*	*0.018*
	0.018	**0.018**	**0.018**	**0.018**	**0.018**	**0.018**	**0.018**	**0.018**	**0.018**	**0.018**	**0.020**	**0.018**	**0.018**	**0.018**	**0.018**	**0.023**	**0.019**	**0.018**	**0.018**	**0.018**
	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	−0.001	0.000	−0.001	0.001	0.001	0.001	0.001	0.001	0.000	−0.001	0.000	−0.001	0.000
$\tau = 1$	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.022*	*0.022*	*0.022*	*0.022*	*0.022*	*0.026*	*0.024*	*0.024*	*0.024*	*0.024*	*0.024*	*0.022*	*0.022*	*0.022*	*0.022*
	0.025	**0.025**	**0.025**	**0.025**	**0.025**	**0.022**	**0.022**	**0.022**	**0.022**	**0.022**	**0.026**	**0.024**	**0.024**	**0.024**	**0.024**	**0.024**	**0.022**	**0.022**	**0.022**	**0.022**
	0.000	0.000	0.000	0.000	0.000	0.000	−0.001	−0.001	0.000	−0.001	0.001	0.001	0.001	0.001	0.001	0.000	−0.001	0.000	0.000	−0.001
$\tau = 2$	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.022*	*0.023*	*0.022*	*0.022*	*0.022*	*0.026*	*0.025*	*0.025*	*0.025*	*0.026*	*0.027*	*0.024*	*0.023*	*0.023*	*0.023*
	0.025	**0.025**	**0.025**	**0.025**	**0.025**	**0.022**	**0.023**	**0.022**	**0.022**	**0.022**	**0.026**	**0.025**	**0.026**	**0.026**	**0.026**	**0.027**	**0.024**	**0.023**	**0.023**	**0.023**

Results for the $\hat{\tau}$ estimator

	Results for QML estimation										Results for GMM estimation									
	Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$					Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.000	0.001	0.001	0.000	0.002	0.026	0.027	0.029	0.026	0.027	0.006	0.003	0.003	0.003	0.005	0.029	0.033	0.032	0.031	0.031
$\tau = -2$	*0.121*	*0.121*	*0.120*	*0.121*	*0.120*	*0.194*	*0.194*	*0.194*	*0.195*	*0.193*	*0.123*	*0.124*	*0.124*	*0.124*	*0.118*	*0.192*	*0.197*	*0.196*	*0.196*	*0.196*
	0.121	**0.121**	**0.120**	**0.121**	**0.120**	**0.196**	**0.196**	**0.197**	**0.196**	**0.195**	**0.123**	**0.124**	**0.124**	**0.124**	**0.118**	**0.194**	**0.199**	**0.198**	**0.198**	**0.198**
	0.007	0.007	0.007	0.006	0.008	0.045	0.045	0.043	0.044	0.043	0.006	0.008	0.007	0.007	0.007	0.029	0.049	0.052	0.053	0.053
$\tau = -1$	*0.107*	*0.106*	*0.107*	*0.107*	*0.105*	*0.190*	*0.190*	*0.191*	*0.189*	*0.191*	*0.110*	*0.110*	*0.110*	*0.110*	*0.110*	*0.189*	*0.187*	*0.190*	*0.190*	*0.191*
	0.107	**0.107**	**0.107**	**0.107**	**0.105**	**0.196**	**0.195**	**0.196**	**0.195**	**0.195**	**0.110**	**0.110**	**0.110**	**0.110**	**0.110**	**0.192**	**0.193**	**0.197**	**0.198**	**0.198**
	0.012	0.014	0.012	0.013	0.012	0.052	0.053	0.051	0.054	0.052	0.006	0.011	0.011	0.011	0.011	0.008	0.045	0.060	0.060	0.061
$\tau = 0$	*0.106*	*0.104*	*0.105*	*0.104*	*0.105*	*0.189*	*0.188*	*0.189*	*0.187*	*0.188*	*0.108*	*0.108*	*0.108*	*0.108*	*0.108*	*0.191*	*0.187*	*0.188*	*0.188*	*0.188*
	0.106	**0.104**	**0.106**	**0.105**	**0.106**	**0.196**	**0.195**	**0.196**	**0.195**	**0.195**	**0.108**	**0.108**	**0.108**	**0.108**	**0.108**	**0.192**	**0.192**	**0.197**	**0.198**	**0.198**
	0.017	0.019	0.019	0.019	0.019	0.063	0.063	0.063	0.064	0.065	0.012	0.017	0.018	0.018	0.017	−0.014	0.040	0.068	0.070	0.069
$\tau = 1$	*0.108*	*0.107*	*0.107*	*0.106*	*0.107*	*0.193*	*0.192*	*0.193*	*0.192*	*0.192*	*0.110*	*0.109*	*0.109*	*0.109*	*0.109*	*0.200*	*0.192*	*0.190*	*0.192*	*0.193*
	0.110	**0.108**	**0.109**	**0.108**	**0.109**	**0.203**	**0.202**	**0.203**	**0.203**	**0.202**	**0.110**	**0.111**	**0.110**	**0.110**	**0.110**	**0.201**	**0.196**	**0.202**	**0.204**	**0.205**
	0.025	0.025	0.025	0.026	0.024	0.079	0.080	0.082	0.079	0.080	0.021	0.022	0.023	0.022	0.022	−0.056	0.033	0.073	0.073	0.072
$\tau = 2$	*0.107*	*0.108*	*0.107*	*0.107*	*0.107*	*0.196*	*0.197*	*0.197*	*0.196*	*0.197*	*0.112*	*0.110*	*0.110*	*0.109*	*0.110*	*0.216*	*0.207*	*0.197*	*0.198*	*0.199*
	0.110	**0.110**	**0.110**	**0.110**	**0.110**	**0.211**	**0.212**	**0.213**	**0.212**	**0.213**	**0.114**	**0.112**	**0.112**	**0.112**	**0.112**	**0.223**	**0.209**	**0.210**	**0.212**	**0.212**

Figures in italics are the standard errors, figures in bold represent RMSE and figures without any layout are the biases.

Table 8: Estimation results for $\hat{\beta}_1$ and $\hat{\beta}_2$ for the heteroskedastic case with n=254

Results for the $\hat{\beta}_1$ estimator

	Results for QML estimation										Results for GMM estimation									
	Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$					Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
$\tau = -2$	*0.031*	*0.031*	*0.031*	*0.031*	*0.031*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.032*	*0.031*	*0.031*	*0.031*	*0.032*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*
	0.031	**0.031**	**0.031**	**0.031**	**0.031**	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**	**0.032**	**0.031**	**0.031**	**0.031**	**0.032**	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**
	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
$\tau = -1$	*0.025*	*0.025*	*0.024*	*0.025*	*0.025*	*0.027*	*0.027*	*0.027*	*0.027*	*0.027*	*0.024*	*0.025*	*0.025*	*0.025*	*0.024*	*0.026*	*0.027*	*0.027*	*0.027*	*0.027*
	0.025	**0.025**	**0.025**	**0.025**	**0.025**	**0.027**	**0.027**	**0.027**	**0.027**	**0.027**	**0.024**	**0.025**	**0.025**	**0.025**	**0.025**	**0.026**	**0.027**	**0.027**	**0.027**	**0.027**
	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
$\tau = 0$	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.025*	*0.025*	*0.025*	*0.025*	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*
	0.026	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**	**0.025**	**0.025**	**0.025**	**0.025**	**0.026**	**0.026**	**0.026**	**0.026**	**0.026**
	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
$\tau = 1$	*0.028*	*0.028*	*0.028*	*0.028*	*0.028*	*0.025*	*0.025*	*0.025*	*0.025*	*0.025*	*0.029*	*0.028*	*0.028*	*0.028*	*0.028*	*0.027*	*0.026*	*0.025*	*0.025*	*0.025*
	0.028	**0.028**	**0.028**	**0.028**	**0.028**	**0.025**	**0.025**	**0.025**	**0.025**	**0.025**	**0.029**	**0.028**	**0.028**	**0.028**	**0.028**	**0.027**	**0.026**	**0.025**	**0.025**	**0.025**
	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.002	0.002	0.002	0.001	0.000	−0.001	0.000	0.000	0.000
$\tau = 2$	*0.026*	*0.026*	*0.026*	*0.026*	*0.026*	*0.024*	*0.024*	*0.024*	*0.024*	*0.024*	*0.027*	*0.027*	*0.027*	*0.027*	*0.027*	*0.029*	*0.025*	*0.024*	*0.024*	*0.025*
	0.026	**0.026**	**0.026**	**0.026**	**0.026**	**0.024**	**0.024**	**0.024**	**0.024**	**0.024**	**0.027**	**0.027**	**0.027**	**0.027**	**0.027**	**0.029**	**0.025**	**0.024**	**0.024**	**0.025**

Results for the $\hat{\beta}_2$ estimator

	Results for QML estimation										Results for GMM estimation									
	Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$					Heteroskedastic with $W_n = M_n$					Heteroskedastic with $W_n \neq M_n$				
	α					α					α					α				
	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
	−0.003	−0.003	−0.003	−0.003	−0.003	−0.005	−0.005	−0.005	−0.005	−0.005	−0.002	−0.001	−0.001	−0.001	−0.002	−0.004	−0.003	−0.004	−0.003	−0.004
$\tau = -2$	*0.049*	*0.049*	*0.049*	*0.049*	*0.049*	*0.061*	*0.061*	*0.061*	*0.061*	*0.061*	*0.050*	*0.049*	*0.049*	*0.049*	*0.050*	*0.061*	*0.060*	*0.061*	*0.061*	*0.061*
	0.049	**0.049**	**0.049**	**0.049**	**0.049**	**0.061**	**0.061**	**0.061**	**0.061**	**0.061**	**0.050**	**0.049**	**0.049**	**0.049**	**0.050**	**0.061**	**0.061**	**0.061**	**0.061**	**0.061**
	−0.005	−0.005	−0.005	−0.005	−0.005	−0.006	−0.006	−0.006	−0.006	−0.006	−0.003	−0.003	−0.003	−0.003	−0.002	−0.004	−0.004	−0.004	−0.004	−0.004
$\tau = -1$	*0.062*	*0.061*	*0.061*	*0.061*	*0.062*	*0.067*	*0.066*	*0.066*	*0.067*	*0.067*	*0.061*	*0.061*	*0.061*	*0.061*	*0.061*	*0.067*	*0.066*	*0.066*	*0.066*	*0.066*
	0.062	**0.061**	**0.062**	**0.062**	**0.062**	**0.067**	**0.067**	**0.067**	**0.067**	**0.067**	**0.061**	**0.061**	**0.061**	**0.061**	**0.061**	**0.067**	**0.066**	**0.067**	**0.067**	**0.067**
	−0.006	−0.006	−0.006	−0.006	−0.006	−0.006	−0.006	−0.005	−0.006	−0.006	−0.004	−0.004	−0.004	−0.004	−0.004	−0.004	−0.004	−0.004	−0.004	−0.004
$\tau = 0$	*0.070*	*0.070*	*0.070*	*0.070*	*0.070*	*0.069*	*0.070*	*0.069*	*0.070*	*0.070*	*0.074*	*0.070*	*0.070*	*0.070*	*0.070*	*0.074*	*0.071*	*0.070*	*0.070*	*0.070*
	0.070	**0.070**	**0.070**	**0.070**	**0.070**	**0.070**	**0.070**	**0.070**	**0.070**	**0.070**	**0.074**	**0.070**	**0.070**	**0.070**	**0.070**	**0.074**	**0.071**	**0.070**	**0.070**	**0.070**
	−0.005	−0.004	−0.005	−0.005	−0.005	−0.005	−0.006	−0.005	−0.005	−0.005	−0.001	−0.003	−0.003	−0.003	−0.003	−0.005	−0.004	−0.004	−0.004	−0.004
$\tau = 1$	*0.058*	*0.058*	*0.057*	*0.058*	*0.058*	*0.065*	*0.065*	*0.065*	*0.066*	*0.066*	*0.063*	*0.059*	*0.059*	*0.059*	*0.059*	*0.080*	*0.072*	*0.065*	*0.066*	*0.065*
	0.058	**0.058**	**0.058**	**0.058**	**0.058**	**0.066**	**0.066**	**0.066**	**0.066**	**0.066**	**0.063**	**0.059**	**0.059**	**0.059**	**0.059**	**0.080**	**0.072**	**0.065**	**0.066**	**0.065**
	−0.002	−0.002	−0.002	−0.002	−0.001	−0.003	−0.002	−0.002	−0.003	−0.003	0.000	−0.001	−0.001	−0.001	−0.001	−0.005	0.000	−0.002	−0.002	−0.004
$\tau = 2$	*0.036*	*0.037*	*0.037*	*0.037*	*0.037*	*0.049*	*0.049*	*0.049*	*0.049*	*0.049*	*0.038*	*0.038*	*0.037*	*0.037*	*0.038*	*0.082*	*0.063*	*0.049*	*0.050*	*0.049*
	0.036	**0.037**	**0.037**	**0.037**	**0.037**	**0.049**	**0.049**	**0.049**	**0.049**	**0.049**	**0.038**	**0.038**	**0.037**	**0.037**	**0.038**	**0.082**	**0.063**	**0.049**	**0.050**	**0.049**

Figures in italics are the standard errors, figures in bold represent RMSE and figures without any layout are the biases.

Table 9: Estimation results for different specifications

	(1)	(2)	(3)	(4)	(5)	(6)
Cons.	-2.048	-3.013	-4.618	-2.566	-3.013	-3.093
	(4.628)	(4.660)	(4.474)	(4.093)	(4.280)	(3.330)
$LGDP$	1.107^{***}	1.114^{***}	1.285^{***}	1.106^{***}	1.114^{***}	1.117^{***}
	(0.243)	(0.246)	(0.236)	(0.231)	(0.240)	(0.196)
$LPOP$	-0.584^{**}	-0.585^{**}	-0.763^{***}	-0.580^{**}	-0.585^{**}	-0.589^{***}
	(0.243)	(0.246)	(0.236)	(0.246)	(0.253)	(0.205)
$OECD$	1.065^{*}	1.037^{*}	0.962^{*}	1.079^{*}	1.037^{*}	1.059^{*}
	(0.545)	(0.551)	(0.531)	(0.597)	(0.608)	(0.584)
$LDIS$	-1.244^{***}	-1.200^{***}	-1.216^{***}	-1.241^{***}	-1.200^{***}	-1.200^{***}
	(0.222)	(0.220)	(0.211)	(0.201)	(0.200)	(0.150)
$TARIFFS$	0.107	0.106	0.149	0.106	0.106	0.107
	(0.112)	(0.113)	(0.109)	(0.084)	(0.084)	(0.075)
MP	1.280	1.211	1.433	1.312	1.212	1.301
	(1.052)	(1.105)	(1.068)	(1.122)	(1.152)	(1.092)
Spat auto	-0.331^{**}	0.265^{**}	0.280^{***}	-0.334^{**}	0.265^{**}	0.275^{**}
in y	(0.147)	(0.109)	(0.104)	(0.164)	(0.117)	(0.111)
n	35	35	35	35	35	35

Standard errors between brackets; (1) is homoskedastic SARAR, (2) is homo. MESS(1,1) by QML, (3) is homo. MESS(1,1) by GMM, (4) is heteroskedastic SARAR, (5) is hetero. MESS(1,1) by QML and (6) is hetero. MESS(1,1) by GMM; *, ** and *** correspond to significance at the 10%, 5% and 1% respectively.