Global sensitivity analysis for the boundary control of an open channel

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Abstract— **The goal of this paper is to solve the global sensitivity analysis for a particular control problem. More precisely, the boundary control problem of an open-water channel is considered, where the boundary conditions are defined by the position of a downstream overflow gate and an upperstream underflow gate. The dynamics of the water depth and of the water velocity are described by the Shallow Water equations, by taking into account the bottom and friction slopes. Since some physical parameters are unknown, a stabilizing boundary control is first computed for their nominal values, and then a sensitivity analysis is performed to measure the impact of the uncertainty in the parameters on a given** *to-be-controlled output*. **The unknown physical parameters are described by some probability distribution functions. Numerical simulations are performed to measure the first-order and total sensitivity indices.**

I. Introduction

In this work we consider the boundary stabilization of an open channel. The model we consider is described by the Shallow-Water equations, which are conservation laws perturbed by non-homogeneous terms due to the effects of the bottom slope, the slope's friction, and the lateral supply. The boundary actions are defined as the position of both spillways located at the extremities of the reach. In [5], [6], the authors designed stabilizing boundary output feedback controllers, with an exponential convergence to the equilibrium of water level and water flow. Our interest is motivated by the following remark: in a given real open channel, many of the involved parameters are uncertain, e.g., because of measurement uncertainties (bottom slope, friction slope, ...). Our aim is then to study the sensitivity of the efficiency of the control of the open channel with respect to the uncertainties in these parameters.

Previous papers on this topic include those considering the insensitizing problem, for e.g., the wave equation (see [1] among other references). They address a local sensitivity problem, that is they study the derivative of the quantity of interest around a given value of one parameter. The work we propose here presents a double originality. Firstly we will address simultaneously the sensitivity to all uncertain parameters, and not just one. And secondly, we propose to investigate the global sensitivity, when the parameters vary around their mean values, following prescribed probability distributions (Gaussian or uniform). Therefore, the present approach can be seen as a global analysis of the control. This will be addressed using statistical techniques.

Sensitivity analysis aims to find the most sensitive parameters, i.e. parameters whose variation have the largest impact on the output quantity. Local sensitivity analysis essentially compute the derivative of the output with respect to the parameter, at a given value of the parameter. Global (stochastic) sensitivity analysis (see e.g. [19] for a review) assume that the parameters can vary widely, either in a given range, or around a given value. In this framework, the parameters are assumed to follow suitable probability distributions. One way of measuring sensitivities is to compute sensitivity indices, such as Sobol indices [21], which quantify the contribution of a given parameter or set of parameters to the output variance: the larger the index value, the greater the sensitivity.

These indices are in general impossible to compute exactly, and must be estimated. Classical approaches of effective computation use Monte-Carlo type methods, see the survey [11]. The Monte-Carlo approach requires a large number of model runs. As in general the model is complex and requires large computing time, it is beneficial to replace the full model by a metamodel, that is an approximate but fast model. In this work we used the reduced basis method [9], [10], [17], [23], [14]. The reduced basis method consists in solving the discrete model PDE in a smaller dimension space, i.e. to look for a solution in a space spanned by a reduced basis instead of a large generic basis.

Due to space limitation, the proofs are omitted.

This paper is organized as follows. Section II presents the model and states the problem. Section III presents the sensitivity analysis in our context, and describes the numerical computation of indices using Monte Carlo approach and reduced basis metamodelling. Section IV presents numerical results. We conclude and give outlooks in Section V.

II. Boundary control of an open channel and problem statement

A. *Quasilinear equation*

Let us consider the classical Shallow Water equations describing the flow dynamics inside of an open-channel. For an introduction of such model and related control problems see e.g. [4], [2], [6]. This model describes the space and time-evolution of the water depth $H = H(x,t)$ and horizontal water velocity $V = V(x,t)$ and is written as follows, for all $(x,t) \in [0,L] \times \mathbb{R}_+$,

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \partial_x \begin{pmatrix} HV \\ \frac{1}{2}V^2 + gH \end{pmatrix} + \begin{pmatrix} 0 \\ g(S_f - S_b) \end{pmatrix} = 0 \,, \tag{1}$$

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where L stands for the length of the pool, g is the gravity constant, S_b is the bottom slope and S_f is the friction slope. The bottom slope S_b may depend on x in the literature, but here we consider S_b constant.

Let us denote the water flow by Q. It is given by $Q(x,t) = BH(x,t)V(x,t)$ where B is the channel width. In the present work, we suppose there are two gates, one at $x = 0$ and one at $x = L$, which are respectively

- a submerged underflow gate:

$$Q(0,t) \quad = \quad U_0 B \mu_0 \sqrt{2g(z_{up} - H(0,t))}, \qquad (2)$$

where z_{up} is the water level before the gate, μ_0 the water flow coefficient and U_0 the position of the spillway (see Fig. 1),

- a submerged overflow gate:

$$H(L,t) \quad = \quad \left(\frac{Q^2(L,t)}{2gB^2\mu_L^2}\right)^{1/3} + h_s + U_L, \qquad (3)$$

where h_s is the height of the fixed part of the overflow gate, μ_L the water flow coefficient at this gate and U_L the position of the spillway (see Fig. 1).

The controls are the positions U_0 and U_L of both spillways located at the extremities of the pool and related to the state variables H and Q.

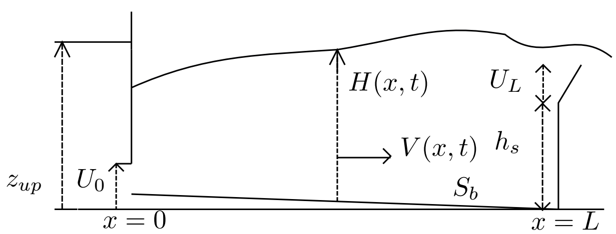

Fig. 1: Sketch of a channel: one reach with a downstream overflow gate and an upperstream underflow gate.

Other kinds of boundary conditions (BC) can be considered e.g., two submerged underflow gates (or two submerged overflow gates) at $x = 0$ and at $x = L$.

There are sufficient stability conditions written in terms of the boundary conditions for the stability of (1) with the boundary conditions (2)-(3). These sufficient conditions exploit Lyapunov functions (see e.g. [5]), or analysis of the characteristic curves (see [18], [15]).

There are various empirical models that are available in the literature for the friction slope (see e.g. [3], [6] which are considering two different models). Let us pick the model of [3] for S_f, that is

$$S_f = C\frac{V^2}{H}, \qquad (4)$$

where C is a constant friction coefficient.

B. Linearized model and stabilizing control laws

A steady-state solution or *equilibrium* of (1) is a time-independent solution of this equation. Let us consider a space-independent equilibrium, denoted $H^\star, V^\star$ and defined by:

$$V^\star = \left(\frac{S_b Q^\star}{BC}\right)^{1/3}, \quad H^\star = \frac{Q^\star}{BV^\star}. \qquad (5)$$

From (1) and (4), it implies $S_f = S_b$ and $S_b H^\star = C(V^\star)^2$. The linearized Shallow Water equations around such an equilibrium are computed in [3]. Denoting the deviations of the states with respect to such an equilibrium by $h(x,t) = H(x,t) - H^\star$ and $v(x,t) = V(x,t) - V^\star$, we can write as follows:

$$\begin{array}{c}\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} V^\star & H^\star \\ g & V^\star \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} \\ + \begin{pmatrix} 0 & 0 \\ -\frac{gC(V^\star)^2}{(H^\star)^2} & \frac{2gCV^\star}{H^\star} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0 \; . \end{array} \qquad (6)$$

Now, introducing the classical characteristic coordinates, that are defined by, for all $(x,t) \in [0,L] \times \mathbb{R}_+$,

$$\begin{array}{l} \xi_1(x,t) = v(x,t) + h(x,t)\sqrt{\frac{g}{H^\star}} \\ \xi_2(x,t) = v(x,t) - h(x,t)\sqrt{\frac{g}{H^\star}} \end{array} \qquad (7)$$

and the characteristic velocities:

$$\lambda_1 = V^\star + \sqrt{gH^\star} \; , \; -\lambda_2 = V^\star - \sqrt{gH^\star} \; . \qquad (8)$$

Assume that the flow is fluvial, that is $gH^\star > V^{\star 2}$. Under this condition, the characteristic velocities have opposite signs, that is $\lambda_1 > 0$ and $-\lambda_2 < 0$.

The linearized Shallow Water equations (6) may be rewritten as

$$\begin{array}{c}\partial_t \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \partial_x \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \\ + \begin{pmatrix} \gamma & \delta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \end{array} \qquad (9)$$

with the parameters

$$\begin{array}{l} \gamma = gC\frac{(V^\star)^2}{H^\star}\left(\frac{1}{V^\star} - \frac{1}{2\sqrt{gH^\star}}\right) \; , \\ \delta = gC\frac{(V^\star)^2}{H^\star}\left(\frac{1}{V^\star} + \frac{1}{2\sqrt{gH^\star}}\right) \; . \end{array} \qquad (10)$$

With direct computation, the following proposition could be obtained.

Proposition 2.1: *Given any constant values k_0 and k_L, defining the controls U_0 and U_L by, for all $t \geq 0$,*

$$U_0(t) = \frac{H(0,t)\left(V^\star - \frac{1+k_0}{1-k_0}(H(0,t) - H^\star)\sqrt{\frac{g}{H^\star}}\right)}{\mu_0\sqrt{2g(z_{up} - H(0,t))}} \qquad (11)$$

$$U_L(t) = -\left(\frac{H(L,t)\left(V^\star + \frac{1+k_L}{1-k_l}(H(L,t) - H^\star)\sqrt{\frac{g}{H^\star}}\right)}{\sqrt{2g}\mu_L}\right)^{\frac{2}{3}} + H(L,t) - h_s \; , \qquad (12)$$

the boundary conditions (2) and (3) may be rewritten as

$$\begin{pmatrix} \xi_1(0,t) \\ \xi_2(L,t) \end{pmatrix} = \begin{pmatrix} 0 & k_0 \\ k_L & 0 \end{pmatrix} \begin{pmatrix} \xi_1(L,t) \\ \xi_2(0,t) \end{pmatrix} \; . \qquad (13)$$

Note that by defining the to-be-measured outputs as the water heights at both extremities of the channel, namely $H(0,t)$, and $H(L,t)$, the previous result defines output feedback controllers U_0 and U_L. Moreover, note that this is an exact expression.

It is possible to combine the previous with Lyapunov techniques to compute stabilizing controllers. This is one of the contributions of [3] which is recalled here:

Proposition 2.2: *([3]) For any $(k_0, k_L) \in \mathbb{R}$ such that*

$$\max\left\{|k_0|\sqrt{\frac{\lambda_1\gamma}{\lambda_2\delta}}, |k_L|\sqrt{\frac{\lambda_2\delta}{\lambda_1\gamma}}\right\} < 1 \ , \tag{14}$$

where γ and δ are defined in (10), defining U_0 and U_L with Proposition 2.1, the system (9) with the boundary conditions (13) is exponentially stable (in L^2-norm). More precisely, there exist $\nu > 0$ and $M > 0$ such that, for every initial condition $(\xi_1^0, \xi_2^0) \in L^2((0, L); \mathbb{R}^2)$, the solution to the Cauchy problem (9) with the boundary conditions (13) and the initial condition

$$(\xi_1(x, 0), \xi_2(x, 0)) = (\xi_1^0(x), \xi_2^0(x)) \ , \ \forall x \in (0, L) \tag{15}$$

satisfies

$$\|(\xi_1(\cdot, t), \xi_2(\cdot, t))\|_{L^2((0,L);\mathbb{R}^2)} \leq Me^{-\nu t}\|(\xi_1^0, \xi_2^0)\|_{L^2((0,L);\mathbb{R}^2)}.$$

The proof of the previous result is given in [3] by noting that (14) is equivalent to [3, Condition (9)].

C. Problem statement

Assume now that we want to use this study to compute a stabilizing control for a real-life channel. In this case, a legitimate question to ask is whether the controls computed using the theoretical model are accurate enough to stabilize the real-life channel according to Proposition 2.2. More specifically, most of the physical parameters in this problem are measured quantities, potentially endowed with measurement errors. Thus the theoretical channel may differ from the real-life one. And as the control has been designed on the theoretical channel, it may lead to a difference in the quality of the stabilization of the real-life channel. In this paper, we want to find which physical parameters have the largest influence on this quality of stabilization.

Indeed, if there are measurement errors, the real-life model is still governed by (9), but the true values for the parameters are unknown, *a priori* different from the nominal values. However, the only way to design the control is to use for each parameter its nominal value. It leads to non-usual boundary conditions, which we linearize in order to keep the resolution simple. Let us denote with nom in subscript the nominal value of a parameter. For instance, $z_{up,\text{nom}}$ is the nominal value of z_{up}. The quantities $V^\star_{\text{nom}}$ and $H^\star_{\text{nom}}$ are defined by (5), with all parameters fixed to their nominal values. We obtain the following

Proposition 2.3: *Given any constant values k_0 and k_L, defining the controls U_0 and U_L by, for all $t \geq 0$,*

$$U_0(t) = \frac{H(0,t)\left(V^\star_{\text{nom}} - \frac{1+k_0}{1-k_0}(H(0,t) - H^\star_{\text{nom}})\sqrt{\frac{g}{H^\star_{\text{nom}}}}\right)}{\mu_0\sqrt{2g(z_{up,\text{nom}} - H(0,t))}} \tag{16}$$

$$U_L(t) = -\left(\frac{H(L,t)\left(V^\star_{\text{nom}} + \frac{1+k_L}{1-k_L}(H(L,t) - H^\star_{\text{nom}})\sqrt{\frac{g}{H^\star_{\text{nom}}}}\right)}{\sqrt{2g}\mu_L}\right)^{\frac{2}{3}} + H(L,t) - h_{s,\text{nom}} \ , \tag{17}$$

the boundary conditions (2) and (3) for the real-life model are linearized as

$$\left(1 - \frac{\mathcal{B} + \sqrt{g/H^\star}}{2\sqrt{g/H^\star}}\right)\xi_1(0,t) + \frac{\mathcal{B} + \sqrt{g/H^\star}}{2\sqrt{g/H^\star}}\xi_2(0,t) = \mathcal{A} \tag{18}$$

$$-\frac{\mathcal{D} - \sqrt{g/H^\star}}{2\sqrt{g/H^\star}}\xi_1(L,t) + \left(1 + \frac{\mathcal{D} - \sqrt{g/H^\star}}{2\sqrt{g/H^\star}}\right)\xi_2(L,t) = \mathcal{C} \tag{19}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are suitable values.

Remark 2.4: *We remark that if the nominal parameters coincide with the true ones, Proposition 2.1 and Proposition 2.3 provide the same boundary conditions.*

Let us define by $\mu = (h_s, B, S_b, C, z_{up}, \xi_1^0, \xi_2^0)$ the vector of uncertain parameters. The uncertainty on these parameters is modeled by random variables. Parameters h_s, B, S_b, z_{up} are modeled by Gaussian distributions whose means are the nominal values of parameters (their measure) and whose standard deviations reflect the uncertainties on these measures. Initial conditions (ξ_1^0, ξ_2^0) and parameter C are modeled by uniform distributions. We refer to Table I for more details on these distributions.

Name	Nominal value	Comment
h_s	$4m$	uncertainty: $\mathcal{N}(4, 0.03)$
B	$80m$	uncertainty $\mathcal{N}(80, 1.03)$
S_b	0.0002	uncertainty $\mathcal{N}(2 \times 10^{-4}, 2.5 \times 10^{-6})$
C	0.001	uncertainty $\mathcal{U}([9 \times 10^{-4}, 0.0011])$
z_{up}	$10m$	uncertainty $\mathcal{N}(10, 0.13)$
ξ_1^0	0	initial value, uncertainty $\mathcal{U}([-0.01, 0.01])$
ξ_2^0	0	initial value, uncertainty $\mathcal{U}([-0.01, 0.01])$
k_0	0.6	known
k_L	0.7	known
μ_0	0.6	known
μ_L	0.73	known
$Q^\star$	50	known
g	9.81	acceleration of gravity, known

TABLE I: Uncertainty on input parameters. $\mathcal{N}(m, \sigma)$ is a normal distribution of mean value m and standard deviation σ, and $\mathcal{U}([a, b])$ is the uniform distribution on $[a, b]$.

The stability of the system is then measured by the so-called *to-be-controlled output*:

$$f(\mu) = \sqrt{\int_{t=0}^{T^\star}\int_{x=0}^{L}(Q^\star - Q(x,t))^2 \, \mathrm{d}x \, \mathrm{d}t} \,, \tag{20}$$

where $T^\star$ is a given time horizon. In our study, the parameters (k_0, k_L) will be fixed by the controller. Recall that ξ is governed by Equations (9) with boundary conditions given by Proposition 2.3. The sensitivity of the *to-be-controlled output* to the input parameters μ will be derived by performing a global sensitivity analysis whose basements are recalled in the next section. The closed-loop system is sketched out in Figure 2 where the uncertainties appear.

III. Sensitivity analysis

In the following, one wants to measure the sensitivity of the *to-be-controlled output* $f(\mu)$ with respect to the uncertainties on the parameter vector μ.

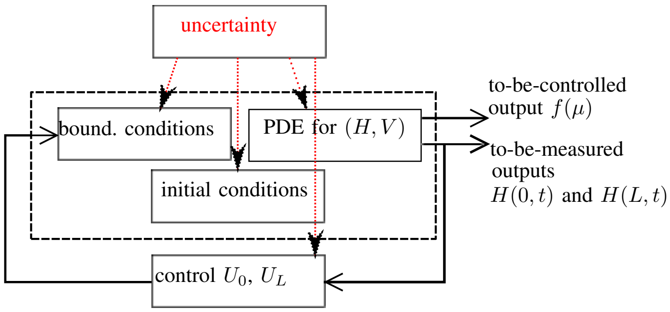

Fig. 2: Control loop in case of uncertainties

A. Global sensitivity: a variance-based approach

We adopt a stochastic framework. Input parameters μ_1, …, μ_p (here $p = 7$) are assumed to be independent and are thus modeled by one-dimensional distributions, as made precise in Table I. The *to-be-controlled output* can then be considered as a scalar random variable $Y = f(\mu)$. The conditional expectation $\mathrm{E}(Y|\mu_i)$ is a random variable which gives the mean of Y over the distributions of the μ_j ($j \neq i$), when μ_i is fixed. It is the best approximation in the mean square sense of Y which depends on μ_i only. Its variance quantifies the influence of μ_i on the dispersion of Y. The Sobol' sensitivity indices are obtained by normalizing this variance. Thus, the Sobol' first order sensitivity index of input parameter μ_i is defined by

$$\mathrm{S}_{\{i\}} = \frac{\mathrm{Var}\big(\mathrm{E}(Y|\mu_i)\big)}{\mathrm{Var}(Y)}.$$

It belongs to the interval $[0,1]$. More generally, one can define sensitivity indices of any order $r \in \{1,\dots,p\}$, starting from the functional ANOVA decomposition (see [7]). Let us first introduce some notation. We assume that f is a real square integrable function, $\mathfrak{u}$ is a subset of $\{1,\dots,p\}$, $\mathfrak{u}^c$ stands for its complement, its cardinal is denoted $r = |\mathfrak{u}|$, and $\mu_\mathfrak{u}$ represents the random vector with components μ_i, $i \in \mathfrak{u}$. The ANOVA decomposition then states that $Y = f(\mu)$ can be uniquely decomposed into summands of increasing dimensions

$$f(\mu) = \sum_{\mathfrak{u}\subseteq\{1,\dots,p\}} f_\mathfrak{u}(\mu_\mathfrak{u}) \tag{21}$$

where $f_\emptyset = \mathbb{E}[Y]$ and the other components have zero mean value and are mutually uncorrelated. The Sobol' index [21] of order $r = |\mathfrak{u}|$ with respect to the combination of all the variables in $\mathfrak{u} \subseteq \{1,\dots,p\}$ is then defined as

$$\mathrm{S}_\mathfrak{u} = \frac{\sigma_\mathfrak{u}^2}{\sigma^2} = \frac{\mathrm{Var}\big[f_\mathfrak{u}(\mu_\mathfrak{u})\big]}{\mathrm{Var}[Y]}\,. \tag{22}$$

The main effect of the i^{th} factor is measured by $\mathrm{S}_{\{i\}}$; then, for $i \neq j$, the interaction effect[1] due to the i^{th} and the j^{th} factors, that cannot be explained by the sum of the individual effects of μ_i and μ_j, is measured by $\mathrm{S}_{\{i,j\}}$, and so on (see [19]). For any $i \in \{1,\dots,p\}$, we also define a total sensitivity index $\mathrm{S}^T_{\{i\}}$ to express the overall output sensitivity to an input μ_i by

$$\mathrm{S}^T_{\{i\}} = \sum_{\mathfrak{v}\subset\{1,\dots p\} \text{ such that } i\in\mathfrak{v}} \mathrm{S}_\mathfrak{v}\,. \tag{23}$$

B. An estimator for Sobol' indices

In our context, no analytical formula is available for the Sobol' indices, which we thus need to estimate. In this subsection, we introduce the classical Monte Carlo estimator of Sobol' indices first introduced in [21].

We first need some notation. Let $\mathfrak{u}$ be a non-empty subset of $\{1,\dots p\}$. For any $i \in \{1,\dots,p\}$, let $\mu_i^{j,1}$ and $\mu_i^{j,2}$, $j = 1,\dots,n$ be two *independent and identically distributed* (i.i.d.) samples of size n of the parameter μ_i. Recall that the p parameters μ_1, …, μ_p are distributed according to distributions given in Table I. We now define

$$\begin{aligned}\mu_\mathfrak{u}^j &= (\mu_i^{j,1}\,,\ i\in\mathfrak{u})\\ \mu_{\mathfrak{u}^c}^{j,1} &= (\mu_i^{j,1}\,,\ i\in\mathfrak{u}^c)\\ \mu_{\mathfrak{u}^c}^{j,2} &= (\mu_i^{j,2}\,,\ i\in\mathfrak{u}^c)\,.\end{aligned}$$

Finally, for $k = 1$ and 2, consider

$$Y_\mathfrak{u}^{j,k} = f(\mu_\mathfrak{u}^j, \mu_{\mathfrak{u}^c}^{j,k})\ . \tag{24}$$

For practical purposes, we need to define the closed Sobol' index of order $r = |\mathfrak{u}|$ with respect to the combination of all the variables in $\mathfrak{u} \subseteq \{1,\dots,p\}$ as

$$\mathrm{S}_\mathfrak{u}^{\text{closed}} = \frac{\mathrm{Var}\big(\mathrm{E}(Y|\mu_\mathfrak{u})\big)}{\mathrm{Var}(Y)}. \tag{25}$$

Let us remark that for first-order indices, $\mathrm{S}_{\{i\}}^{\text{closed}}$ and $\mathrm{S}_{\{i\}}$ coincide.

We define the estimator $\hat{\mathrm{S}}_{\mathfrak{u},n}^{\text{closed}}$ of $\mathrm{S}_\mathfrak{u}^{\text{closed}}$ as

$$\hat{\mathrm{S}}_{\mathfrak{u},n}^{\text{closed}} = \frac{\dfrac{1}{n}\displaystyle\sum_{j=1}^n Y_\mathfrak{u}^{j,1}Y_\mathfrak{u}^{j,2} - \left(\dfrac{1}{n}\displaystyle\sum_{j=1}^n \dfrac{Y_\mathfrak{u}^{j,1}+Y_\mathfrak{u}^{j,2}}{2}\right)^2}{\dfrac{1}{n}\displaystyle\sum_{j=1}^n \dfrac{(Y_\mathfrak{u}^{j,1})^2+(Y_\mathfrak{u}^{j,2})^2}{2} - \left(\dfrac{1}{n}\displaystyle\sum_{j=1}^n \dfrac{Y_\mathfrak{u}^{j,1}+Y_\mathfrak{u}^{j,2}}{2}\right)^2} \tag{26}$$

with $\mathfrak{u} = \{i\}$ for first-order indices and $\mathfrak{u} = \{i,j\}$ for closed second-order indices. This estimator was first introduced in [16]. The asymptotic properties of $\hat{\mathrm{S}}_{\mathfrak{u},n}^{\text{closed}}$ are stated in [12, Propositions 2.2 and 2.5]. This estimator requires a large number n of model evaluations. Therefore, to reduce the cost, we opt for a metamodel approach, replacing equations (9) by a reduction of a linear version of (9). We refer to Section III-C for more details on the reduction procedure. Let us now define the respective estimators of $\mathrm{S}_{\{i\}}$, $\mathrm{S}^T_{\{i\}}$ and $\mathrm{S}_{\{i,j\}}$ as

$$\hat{\mathrm{S}}_{\{i\},n} = \hat{\mathrm{S}}_{\{i\},n}^{\text{closed}}\,,\ \hat{\mathrm{S}}^T_{\{i\},n} = 1 - \hat{\mathrm{S}}_{\{i\}^c,n}^{\text{closed}}$$

and

$$\hat{\mathrm{S}}_{\{i,j\},n} = \hat{\mathrm{S}}_{\{i,j\},n}^{\text{closed}} - \hat{\mathrm{S}}_{\{i\},n}^{\text{closed}} - \hat{\mathrm{S}}_{\{j\},n}^{\text{closed}}\,.$$

[1]Following the terminology of sensitivity analysis (see e.g. [19]), interaction effect is the combination effect of several input parameters on the *to-be-controlled output*.

Theorem 3.1: *Assume that* $\mathbb{E}(Y^4) < \infty$. *Let* $\alpha \in (0,1)$ *(typically* $\alpha = 0.05$ *or* 0.10*). Then an asymptotic confidence interval (CI) of level* $1-\alpha$ *for* $\mathrm{S}_{\{i\}}$ *is given by*

$$\left[\hat{\mathrm{S}}^{\text{closed}}_{\{i\},n} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \hat{\mathrm{S}}^{\text{closed}}_{\{i\},n} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right],$$

with $z_{1-\frac{\alpha}{2}}$ *is the* $1-\frac{\alpha}{2}$ *quantile of the* $\mathcal{N}(0,1)$ *distribution and where* $\sigma = \sigma_{\{i\}}$. *The analogous of this result is also true for* $\mathrm{S}^T_{\{i\}}$ *and* $\mathrm{S}_{\{i,j\}}$.

Proof of Theorem 3.1 (sketch). Combining the vectorial central limit theorem stated in [8, Theorem 3.1] for vectors of closed Sobol' indices with the delta method (as in [22]) applied to the particular case of linear transformations, we get the result of Theorem 3.1. □

C. Discretized and reduced model

1) Discretization of the model: We use an implicit upwind scheme to discretize (9). We denote by k the discrete time index, by $i = 1, \dots, Nx$ the space index (where Nx is the number of discretization points in space), and by $\xi_j^{i,k}$ an approximation of ξ_j ($j = 1,2$) at the ith point of the uniform space grid on $[0, L]$ with Nx points and at the kth timestep. We denote by $\Delta x = L/Nx$ and $\Delta t = T^\star/Nt$ the space and time steps, respectively.

Using a classical upwind scheme, we get the following approximations:

$$\partial_t\xi_1 + \lambda_1\partial_x\xi_1 \approx \left(\frac{\xi_1^{i,k+1} - \xi_1^{i,k}}{\Delta t} + \lambda_1\frac{\xi_1^{i,k+1} - \xi_1^{i-1,k+1}}{\Delta x}\right)_{i,k}$$

and:

$$\partial_t\xi_2 - \lambda_2\partial_x\xi_2 \approx \left(\frac{\xi_2^{i,k+1} - \xi_2^{i,k}}{\Delta t} - \lambda_2\frac{\xi_2^{i+1,k+1} - \xi_2^{i,k+1}}{\Delta x}\right)_{i,k},$$

which give, when combined to (9), the following implicit recurrence (in k) relations:

$$\left(\frac{1}{\Delta t} + \frac{\lambda_1}{\Delta x} + \gamma\right)\xi_1^{i,k+1} - \frac{\lambda_1}{\Delta x}\xi_1^{i-1,k+1} + \delta\xi_2^{i,k+1} = \frac{\xi_1^{i,k}}{\Delta t}$$

$$\left(\frac{1}{\Delta t} - \frac{\lambda_2}{\Delta x} + \delta\right)\xi_2^{i,k+1} + \frac{\lambda_2}{\Delta x}\xi_2^{i,k+1} + \gamma\xi_1^{i,k+1} = \frac{\xi_2^{i,k}}{\Delta t}.$$

The boundary conditions (18) and (19) can readily be incorporated so as to write, at each time step, a linear system of equations that has to be solved so as to find ξ_1^{k+1} and ξ_2^{k+1} from ξ_1^k and ξ_2^k. An approximation of the *to-be-controlled output* can then be obtained by discretizing the double integral in (20).

2) Model reduction: As the experimental model will have to be numerically solved for a large number of parameter values, we use the *reduced basis* technique (see e.g., [17], and [13] for space-time reduction) so as to accelerate the resolutions of the above mentioned systems. We use a space-time reduced basis approach, which can be summed up as follows: we introduce the vector $\xi = (\xi_1^0, \xi_2^0, \xi_1^1, \xi_2^1, \dots, \xi_1^N t, \xi_2^N t)$, which is the solution of some large (dimension $2 \times (Nt+1) \times Nx$) sparse linear system of equations $A(\mu)\xi = b(\mu)$, where $A(\mu)$ and $b(\mu)$ are appropriate functions of the true parameter values μ. One can check that $A(\mu)$ and $b(\mu)$ satisfy the so-called affine decomposition hypothesis (as introduced in [17]), hence the classical reduced basis algorithms (op. cit.) can be readily applied to this system.

IV. Numerical results

A. Parameters

For the numerical implementations, we have chosen the channel length $L = 250$ m, and the time horizon $T^\star = 75$ s. The parameters k_0 and k_L have been fixed to 0.6 and 0.7. The discretization parameters were settled to $\Delta t = 5$ s, $\Delta x = 5$ m, and for the reduction, a reduced basis obtained from proper orthogonal decomposition [20] of size 11, obtained from a snapshot of size 100. For the estimation of Sobol' indices, the Monte-Carlo sample size was fixed equal to $50\,000$. We provide in the next sub-section asymptotic CI of level 0.95 provided in Theorem 3.1.

Note that the length of a CI is a direct measure of the estimation precision.

B. Indices estimations

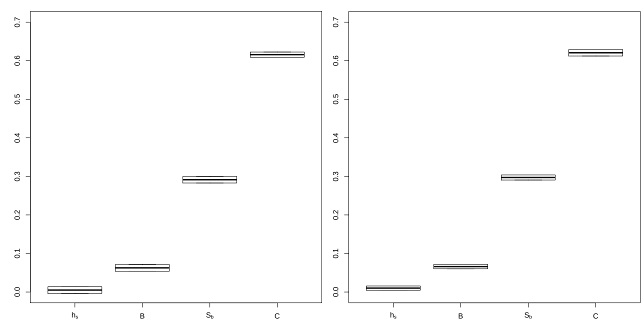

Fig. 3: First-order (left) and total (right) Sobol' indices with 95% CI for parameters h_s, B, S_b and C

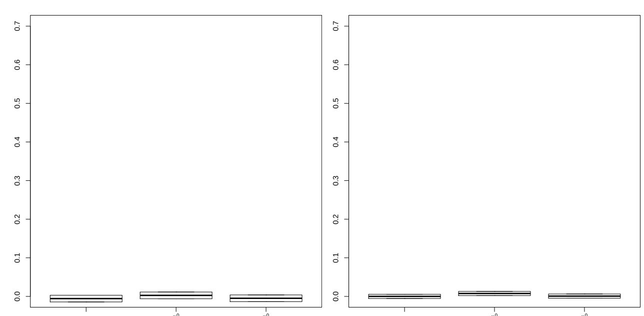

Fig. 4: First-order (left) and total (right) Sobol' indices with 95% CI for parameters z_{up}, ξ_1^0, and ξ_2^0

On Figures 3 and 4 the results read as follows. For each uncertain parameter, the bold centered line on the left part of the figure (respectively on the right part) gives, on the vertical axis, the estimation of the first-order (resp. total) Sobol' index. The upper and lower thin lines define the upper and lower bounds of the 95% CI. Note that, by definition, the first-order index is smaller than the corresponding total index. We conclude with a confidence level of 95% that the influence of the parameters h_s and z_{up} are not significant for this *to-be-controlled output*. As expected from the choice of $T^\star$ and the exponential control rate, the initial conditions (ξ_1^0, ξ_2^0) are not significant as well. The parameter C is the parameter having the most significant effect, parameters S_b and B are influent as well but in a less extent.

As for each parameter, as the difference between the total and the first-order indices is not significant, it means that the interactions are negligible. Thus the *to-be-controlled output* is additive in the parameters.

V. Conclusion

In this paper, the global sensitivity analysis has been performed when considering probabilistic distribution functions standing for the uncertainty in some unknown physical parameters. It allows us to describe the impact of the parameters on a *to-be-controlled output* in a boundary control problem. This boundary control is motivated by an application for the flow control in an open channel. This work lets many research lines open. In particular, it could be interesting to vary parameters μ_0 and μ_L, as they are in general ill-estimated.

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