ON SYMMETRICALLY GROWING BODIES

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ABSTRACT. Two continuum theories of growing bodies are presented. In the first, the growing body consists of elements called the growing body points assumed to be identifiable in all stages of growth. The growth of the body is modeled by mapping the growing body into a material manifold containing the material points. The second theory is obtained from the first by taking the quotient under the action of the group of diffeomorphisms of the growing body and the various growth stages are represented by shapes of the growing body. Kinematics and stress theory are formulated by constructing an infinite dimensional fiber bundle structure for the configuration space.

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1. INTRODUCTION

This work presents a setting for the formulation of the mechanics of growing bodies. By the mechanics of growing bodies we mean a theory in which the material structure of the body does not remain fixed. Material points may be added or removed from the body.

We consider two general kinds of growing body theories. The first, reviewed in Section 2 and discussed in [9], considers growing bodies whose parts are identifiable throughout the various growth stages. Such growing bodies are intended to model systems such as the human body where it is quite natural to assume that we can identify the parts and points in the body although the material points (or cells) that they contain change during growth. The growing body is defined as the set containing these identifiable points. Our ability to identify the various growing body points can be motivated by an assumption that they have different properties. The material points are assumed to be elements of a material manifold—a Euclidean space for simplicity. A growth stage is specified by means of an embedding of the growing body into the material manifold and we refer to such an embedding as a *content*. The image of the content contains the material points that constitute the body at the corresponding growth stage.

The second theory, pertaining to *symmetrically growing bodies*, abandons the assumption that the various growing body points are identifiable. As such, it is intended to model phenomena like solidification or crystal growth. The term "symmetry" is used because this theory is obtained from the former when one considers the action of the group of diffeomorphisms of the growing body manifold on the various contents of the growing body. Specifically, we now identify two contents if they differ by a diffeomorphism of the growing body. Such an equivalence class of contents is traditionally referred to as a *shape* in the global analysis literature.

The basic framework used in the formulation of the theories is the construction of an infinite dimensional manifold structure for the configuration space of a mechanical system (as in [8]), and then defining generalized velocities and generalized forces as elements of the tangent and cotangent bundles, respectively. In both theories, the configuration spaces have structures of fiber bundles. The base manifold in the case of growing bodies is the collection of contents. For symmetrically growing bodies, the base manifold is the collection of shapes of the growing body and its manifold structure is given by [5], [6], [1], [2] and [7]. The fiber over a particular content or a shape is the collection of the configurations of the image of the content or the shape, respectively, in the physical space. In both cases, material velocity fields provide connections on the respective fiber bundles.

This basic framework implies that forces are Schwartz distributions. Stress theory is obtained by requiring these distributions to be of order one and representing them by measures—the stress measures. Densities representing these measures, if they exist, are the stress fields. Forces associated with the growth of the body are analogous to Gurtin's *configurational forces* (see [4]). The stresses associated with the growth of the body are analogous to the Eshelby tensor, [3].

2. Review of Growing Bodies

The notion of a growing body is introduced in order to model a situation in which the material structure of a body is not fixed so material points are added and removed from the body. While the material structure is allowed to vary, a growing body has additional structure that allows one to identify its elements—the growing body points—throughout the growth of the body. Thus we make the following definitions.

Definition 2.1. A *growing body*, $\mathcal{B}$, is a compact three dimensional submanifold with a boundary of a three dimensional Euclidean space.

Definition 2.2. The *material manifold* is a three dimensional Euclidean space M with tangent space $\mathbf{V}$.

Definition 2.3. A *content* of the growing body $\mathcal{B}$ is a smooth embedding $c\colon \mathcal{B} \to M$.

The material manifold represents the collection of material points, and as such, the image of the growing body under a content is a *simple body* of continuum mechanics. The collection of all contents of the growing body, the *content space*, is $\mathrm{Emb}(\mathcal{B}, M)$. In the following, the physical space will be modeled by $\Re^3$.

Definition 2.4. A *configuration* κ of the growing body is a smooth embedding $\kappa\colon c\{\mathcal{B}\} \to \Re^3$, for some content c.

Since $c\{\mathcal{B}\}$ is a simple body, $Q_{c\{\mathcal{B}\}} = \mathrm{Emb}(c\{\mathcal{B}\}, \Re^3)$ is the collection of its configurations in space. The *configuration space of the growing body* is therefore

$$Q_{\mathcal{B}} = \bigcup_c Q_{c\{\mathcal{B}\}}, \qquad c \in \mathrm{Emb}(\mathcal{B}, M).$$

The mapping $\pi\colon Q_{\mathcal{B}} \to \mathrm{Emb}(\mathcal{B}, M)$ such that $\pi(\kappa) = c$ if $\kappa \in Q_{c\{\mathcal{B}\}}$ will be referred to as the *configuration space projection*. The following proposition is an application of a standard result (e.g., [7]) on spaces of embeddings.

Proposition 2.1. The content space is an open subset of $C^\infty(\mathcal{B}, M)$, and as such, it is a Frechet manifold whose tangent space at any content may be identified with $C^\infty(\mathcal{B}, \mathbf{V})$. Similarly, $Q_{c\{\mathcal{B}\}}$ is open in $C^\infty(c\{\mathcal{B}\}, \Re^3)$ so $Q_{c\{\mathcal{B}\}}$ is a Frechet manifold whose tangent space at any configuration can be identified with $C^\infty(c\{\mathcal{B}\}, \Re^3)$.

Proposition 2.2. The configuration space of the growing body has the structure of a trivializable fiber bundle whose typical fiber is $\mathrm{Emb}(\mathcal{B}, \Re^3)$.

Proof. A natural global fiber bundle chart

$$\varPhi \colon Q_{\mathcal{B}} \to \mathrm{Emb}(\mathcal{B}, M) \times \mathrm{Emb}(\mathcal{B}, \Re^3)$$

is defined on $Q_{\mathcal{B}}$ by $\varPhi(\kappa) = (\pi(\kappa), \kappa \circ \pi(\kappa))$. $\square$

For a configuration κ of the growing body, the second component $e = \kappa \circ \pi(\kappa)$ of $\varPhi(\kappa)$ will be referred to as the *extent* corresponding to κ.

Definition 2.5. A *generalized velocity* is an element $\dot{\kappa}$ of the tangent bundle $TQ_{\mathcal{B}}$.

Using the global chart $\varPhi$, a generalized velocity may be represented by $(\dot{c}, \dot{e}) \in C^\infty(\mathcal{B}, \mathbf{V}) \times C^\infty(\mathcal{B}, \Re^3)$ to which we will refer as the *growth rate* and *extent rate*, respectively.

Definition 2.6. A *material velocity field* $\mathbf{v}$ is an element of the vertical subbundle $VQ_{\mathcal{B}} \subset TQ_{\mathcal{B}}$, i.e., $\mathbf{v} \in T_\kappa Q_{c\{\mathcal{B}\}}$, $c = \pi(\kappa)$ for some $\kappa \in Q_{\mathcal{B}}$.

The term "material velocity field" is used because an element of $T_\kappa Q_{c\{\mathcal{B}\}}$, represents a generalized velocity of the simple body $c\{\mathcal{B}\}$ at its configuration κ in space.

Proposition 2.3. There is a natural connection on $TQ_{\mathcal{B}}$ such that the vertical component associated with a generalized velocity $\dot{\kappa} \in T_\kappa Q_{\mathcal{B}}$ is given by

$$\mathbf{v}(X) = \dot{e} \circ c^{-1}(X) - \mathrm{D}\kappa(X)\big(\dot{c} \circ c^{-1}(X)\big),$$

where, $(\dot{c}, \dot{e})$ are the representatives of $\dot{\kappa}$ in the global chart $\varPhi$ and D is the differentiation operator.

Proof. Consider the following diagram.

$$\begin{tikzcd}
T_\kappa Q_{c\{\mathcal{B}\}} \arrow[r, shift left, "\text{Inclusion}"] & T_\kappa Q_{\mathcal{B}} \arrow[l, shift left, "\Delta"] \arrow[r, shift left, "T\pi"] \arrow[d, shift right, "T_\kappa\varPhi"'] & T_c\mathrm{Emb}(\mathcal{B}, M) \arrow[l, shift left, "\varGamma"] \arrow[dl, shift left, "\hat{\varGamma}"] \\
& T_c\mathrm{Emb}(\mathcal{B}, M) \times T_e\mathrm{Emb}(\mathcal{B}, \Re^3) \arrow[ul, "\hat{\Delta}"] \arrow[u, shift right, "T_\kappa\varPhi^{-1}"'] \arrow[ur, shift left, "\mathrm{pr}_1"] &
\end{tikzcd}$$

Here,

$$\hat{\varGamma} \colon T_c\mathrm{Emb}(\mathcal{B}, M) = C^\infty(\mathcal{B}, \mathbf{V}) \to C^\infty(\mathcal{B}, \mathbf{V} \times \Re^3) = T_\kappa\varPhi\{T_\kappa Q_{\mathcal{B}}\}$$

is defined by $\hat{\varGamma}(\dot{c}) = (\mathit{1}, \mathrm{D}\kappa \circ c)(\dot{c})$, i.e., $\hat{\varGamma}(\dot{c})(\zeta) = (\dot{c}(\zeta), \mathrm{D}\kappa_{c(\zeta)}(\dot{c}(\zeta)))$ and

$$\hat{\Delta} \colon T_c\mathrm{Emb}(\mathcal{B}, M) \times T_e\mathrm{Emb}(\mathcal{B}, \Re^3) \to T_e\mathrm{Emb}(\mathcal{B}, \Re^3)$$

is defined by $\hat{\Delta} = c^{-1*} \circ \mathrm{pr}_2 \circ (\mathit{1} - \hat{\varGamma} \circ \mathrm{pr}_1)$, where asterisks denote pullbacks, e.g., $c^*(\mathbf{v}) := \mathbf{v} \circ c$ and $c^{-1*}(\dot{c}) := \dot{c} \circ c^{-1*}$. Hence, specifically,

$$\hat{\Delta}(\dot{c}, \dot{e}) = \dot{e} \circ c^{-1} - \mathrm{D}\kappa(\dot{c} \circ c^{-1}).$$

From the definitions it follows that $\mathrm{pr}_1 \circ \hat{\Gamma} = \mathit{1}$, and $(\mathit{1} - \hat{\Gamma} \circ \mathrm{pr}_1)(\dot{c}, \dot{e}) = (0, \dot{e} - (\mathrm{D}\kappa \circ c)(\dot{c}))$. The mappings Γ and Δ are defined so as to make the diagram commutative.

□

Generalized forces are elements of the cotangent bundle of the corresponding configuration space. In the sequel we denote dual spaces and cotangent bundles by asterisks. In particular, we have the following definitions.

Definition 2.7. A *content force* f_c at the content c of the growing body is an element of $T_c^* \mathrm{Emb}(\mathcal{B}, M) = C^\infty(\mathcal{B}, \mathbf{V})^*$. An *extent force* f_e at the extent e is an element of $T_e^* \mathrm{Emb}(\mathcal{B}, \Re^3) = C^\infty(\mathcal{B}, \Re^3)^*$. A *simple body force* on the simple body B at the configuration $\kappa \colon B \to \Re^3$ is an element f_m of $T^* \mathrm{Emb}(B, \Re^3) = C^\infty(B, \Re^3)^*$. A *growing body force* $f_\mathcal{B}$ at the configuration κ is an element of $T_\kappa^* Q_\mathcal{B}$.

We can use the global chart Φ in order to represent growing body forces by means of content and extent forces. Thus,

$$T_\kappa^* \Phi \colon T_c^* \mathrm{Emb}(\mathcal{B}, M) \times T_e^* \mathrm{Emb}(\mathcal{B}, \Re^3) \to T_\kappa^* Q_\mathcal{B}, \quad e = \kappa \circ c$$

gives the representation

$$f_\mathcal{B}(\dot{\kappa}) = T_\kappa^* \Phi(f_c, f_e)(\dot{\kappa}) = f_c(\dot{c}) + f_e(\dot{e}),$$

where $(\dot{c}, \dot{e})$ are the representatives of the generalized velocity $\dot{\kappa}$ under the chart Φ.

Similarly, the decomposition $(T\pi, \Delta)$, provided by the connection, induces the mapping

$$(T\pi, \Delta)^* \colon (T_{\pi(\kappa)} \mathrm{Emb}(\mathcal{B}, M) \times T_\kappa Q_{c\{\mathcal{B}\}})^* \to T_\kappa^* Q_\mathcal{B}$$

that gives a representation of $f_\mathcal{B}$ by means of a force $f_a \in T_c^* \mathrm{Emb}(\mathcal{B}, M)$ and a simple body force f_m in the form

$$f_\mathcal{B}(\dot{\kappa}) = [(T\pi, \Delta)^*(f_a, f_e)](\dot{\kappa}) = f_a(\dot{c}) + f_m(\mathbf{v}), \quad \dot{c} = T\pi(\dot{\kappa}), \ \mathbf{v} = \Delta(\dot{\kappa}).$$

The situation is illustrated in the following diagram.

$$\begin{tikzcd}
T_\kappa^* Q_{c\{\mathcal{B}\}} \arrow[r, shift right, "\Delta^*"'] \arrow[dr, "\hat{\Delta}^*"'] & T_\kappa^* Q_\mathcal{B} \arrow[l, shift right, "\mathrm{Inclusion}^*"'] \arrow[r, shift right, "\Gamma^*"'] \arrow[d, shift left, "T_\kappa^* \Phi^{-1}"] & T_c^* \mathrm{Emb}(\mathcal{B}, M) \arrow[l, shift right, "T^* \pi"'] \arrow[dl, shift right, "{\mathrm{pr}_1}^*"'] \\
& T_c^* \mathrm{Emb}(\mathcal{B}, M) \times T_e^* \mathrm{Emb}(\mathcal{B}, \Re^3) \arrow[u, shift left, "T_\kappa^* \Phi"] \arrow[ur, shift right, "\hat{\Gamma}^*"'] &
\end{tikzcd}$$

The relation between the various components representing a growing body force are given by

$$\begin{aligned} f_c &= f_a - f_m \circ (\mathrm{D}\kappa \circ c^{-1}) \circ c^{-1*} \\ f_e &= f_m \circ c^{-1*}, \end{aligned}$$

whose inverse relations are

$$\begin{aligned} f_a &= f_c + f_e \circ (\mathrm{D}\kappa \circ c) \\ f_m &= f_e \circ c^*. \end{aligned}$$

First order stress theory is obtained if one assumes, as we do for the rest of this section, that forces are distributions of order one. Such forces are represented by *stress measures* (see [8] and [10]). For the sake of simplicity we assume that the various stress measures are given in terms of smooth densities with respect to the volume measures on the various regions. (See [9] for a general presentation.) The representation using the stress densities is of the form

$$f_{\mathcal{B}}(\dot{\kappa}) = \int_{\mathcal{B}} \dot{c} \cdot s_c \, dV_{\mathcal{B}} + \int_{\mathcal{B}} \mathrm{D}\dot{c} \cdot S_c \, dV_{\mathcal{B}} + \int_{\mathcal{B}} \dot{e} \cdot s_e \, dV_{\mathcal{B}} + \int_{\mathcal{B}} \mathrm{D}\dot{e} \cdot S_e \, dV_{\mathcal{B}}.$$

Here, the first two terms on the right represent f_c, the last two represent f_e, the vector fields s_c and s_e are the *ambient force fields* and the tensor fields S_c and S_e are *stress tensor fields*. Using the Gauss theorem one can rewrite the last equation in terms of body force fields b_c, b_e and surface force fields t_c, t_e, satisfying,

$$\begin{aligned} t_c &= S_c(n), & t_e &= S_e(n), && \text{on } \partial\mathcal{B}, \\ b_c &= s_c - \operatorname{Div} S_c, & b_e &= s_e - \operatorname{Div} S_e, && \text{on } \mathcal{B}, \end{aligned}$$

in the form

$$f_{\mathcal{B}}(\dot{\kappa}) = \int_{\partial\mathcal{B}} (\dot{c} \cdot t_c + \dot{e} \cdot t_e) \, dA_{\mathcal{B}} + \int_{\mathcal{B}} (\dot{c} \cdot b_c + \dot{e} \cdot b_e) \, dV_{\mathcal{B}}.$$

Similarly, it is possible to use stress fields for the representation of the components f_a and f_m to obtain

$$f_{\mathcal{B}}(\dot{\kappa}) = \int_{B} (\dot{c} \circ c^{-1}) \cdot s_a \, dV + \int_{B} \mathrm{D}(\dot{c} \circ c^{-1}) \cdot S_a \, dV \\ + \int_{B} \mathbf{v} \cdot s_m \, dV + \int_{B} \mathrm{D}\mathbf{v} \cdot S_m \, dV.$$

Here, $B = c\{\mathcal{B}\}$ and V is the volume on B. The representation by force fields is of the form

$$f_{\mathcal{B}}(\dot{\kappa}) = \int_{\partial B} (\dot{c} \cdot t_a + \mathbf{v} \cdot t_m) \, dA + \int_{B} (\dot{c} \cdot b_a + \mathbf{v} \cdot b_m) \, dV,$$

where,

$$\begin{aligned} t_a &= S_a(n), & t_m &= S_m(n), && \text{on } \partial B, \\ b_a &= s_a - \operatorname{Div} S_a, & b_m &= s_m - \operatorname{Div} S_m, && \text{on } B, \end{aligned}$$

and A is the area measure on ∂B.

3. Kinematics of Symmetrically Growing Bodies

The foregoing discussion assumed that points in the growing body kept their identities though the various content mappings. The physical motivation behind this assumption is that the various points in the growing body have properties that vary smoothly over the body. These measurable (in the physical sense) properties are used in order to identify these points.

The present section abandons this assumption and we consider a growing body that is homogeneous in the sense that it is not possible to distinguish between the growing body and its image under a diffeomorphism. Consider the action Ψ of the group of diffeomorphisms of $\mathcal{B}$, $\mathrm{Diff}(\mathcal{B})$, on the content space $\mathrm{Emb}(\mathcal{B}, M)$, given by

$$\Psi : \mathrm{Emb}(\mathcal{B}, M) \times \mathrm{Diff}(\mathcal{B}) \to \mathrm{Emb}(\mathcal{B}, M), \quad \Psi(c, g) = c \circ g, \quad g \in \mathrm{Diff}(\mathcal{B}).$$

If $c_2 = c_1 \circ g$ for some diffeomorphism g then their images, the simple bodies $c_2\{\mathcal{B}\}$ and $c_1\{\mathcal{B}\}$, are identical. Hence, $Q_{c_2\{\mathcal{B}\}} = Q_{c_1\{\mathcal{B}\}}$, i.e., the fibers of $Q_{\mathcal{B}}$ over them are identical. If the body is homogeneous so that the distribution of the properties in the growing body is not affected by diffeomorphisms of $\mathcal{B}$, the physical motivation for our ability to identify the elements of the growing body is no longer valid and one has to identify two contents $c_1,\ c_2$ satisfying $c_2 = c_1 \circ g$. Thus, we have an equivalence relation ϱ on $\mathrm{Emb}(\mathcal{B}, M)$.

Definition 3.1. The *space of shapes* of a symmetrically growing body is the quotient set $\mathcal{S} = \mathrm{Emb}(\mathcal{B}, M)/\varrho$. An element $\chi \in \mathcal{S}$ will be referred to as a *shape* (see [5], [7], [1] and [2]).

Thus, for a symmetrically growing body the role of the contents is assumed by shapes. As we have seen above, an equivalence class $\chi \in \mathcal{S}$ can be identified with the image of any of its members—a simple body. In the following we will use this identification and, for example, we write Q_χ for the collection of configurations in space of the simple body χ. It seems to us that the notion of a shape of a symmetrically growing body is similar to the notion of a *control volume* used by Gurtin [4].

In order to formulate mechanics in this situation, one has to provide $\mathcal{S}$ with a differentiable structure. This has been done in a general situation by [1], [2], [5], [6] and [7].

Proposition 3.1. The collection $\mathcal{S}$ is a Frechet manifold. A chart on $\mathcal{S}$ is constructed as follows. Let χ_0 be a shape and let $\eta\colon \partial\chi_0 \times \Re \to W \subset M$ be a tubular neighborhood of $\partial\chi_0$ in M. There is a neighborhood $U \subset C^\infty(\partial\chi_0)$ ($C^\infty(\partial\chi_0)$ is the space of smooth functions on $\partial\chi_0$) such that $\varPhi : U \to \mathcal{S}, \varPhi(u) = \{\eta\big(X, u(X)\big) \mid X \in \partial\chi_0\}$ is a chart on $\mathcal{S}$.

Since M has a Euclidean metric, we can use tubular neighborhoods that are normal to the boundary $\partial\chi$. The tangent space $T_\chi\mathcal{S}$ can be identified with the space of smooth sections of the normal bundle $\nu : N \to \partial\chi$ to the boundary of χ. In other words, just as we have taken equivalence classes of embeddings under the action of the group of diffeomorphisms to construct $\mathcal{S}$ from $\mathrm{Emb}(\mathcal{B}, M)$ the tangent space $T_\chi\mathcal{S}$ can be obtained by taking equivalence classes in $C^\infty(\mathcal{B}, \mathbf{V})$, the tangent space to $\mathrm{Emb}(\mathcal{B}, M)$, under the action of infinitesimal diffeomorphisms—vector fields on $\mathcal{B}$ that are tangent to its boundary. The resulting equivalence classes are the sections of the normal bundle. In the sequel we will identify an element $\dot\chi \in T_\chi\mathcal{S}$ with the single component, a C^∞ function, that the corresponding normal vector field has with respect to the outward unit normal n to $\partial\chi$.

Definition 3.2. The *configuration space* of a symmetrically growing body is

$$Q = \bigcup_{\chi \in \mathcal{S}} Q_\chi.$$

Charts on Q are constructed by extending artificially normal vector fields on $\partial\chi$ to χ in order to generate diffeomorphisms of χ with the images of neighboring shapes. This is done using the "dragging of the domain" of [5] defined as follows.

Definition 3.3. Let $\eta : \partial\chi_0 \times \Re \to W \subset M$ be a tubular neighborhood of $\partial\chi_0$ in M. A *dragging of the domain* χ_0 along η is a differentiable mapping $\delta : U \to \mathrm{Diff}^\infty(M)$, with $U \subset C^\infty(\partial\chi_0)$ a domain of a chart $\varPhi : U \to \mathcal{S}, \varPhi(u) = \{\eta\big(X, u(X)\big) \mid X \in \partial\chi_0\}$ containing $\partial\chi_0$, that satisfies:

(i) $\delta(u)(X) = \eta\big(X, u(X)\big)$ for all $X \in \partial\chi_0$,
(ii) $\delta(0) = 1_M$,
(iii) $\delta(u)(X) \neq X$ only in a neighborhood of $\partial\chi_0$ contained in W.

It is possible to show (see [5]) that such a dragging of the domain can always be constructed. Once a dragging of the domain is given, one can construct a local trivialization in a neighborhood of χ_0. If $\chi = \Phi(u) = \{\eta\big(X, u(X)\big) \mid X \in \partial\chi_0\}$, $u \in U$ the diffeomorphism $Q_{\chi_0} \to Q_\chi$ is given by $\kappa_0 \mapsto \kappa_0 \circ \delta(u)^{-1}$.

Proposition 3.2. The configuration space Q is a bundle $\pi_{\mathcal{S}} : Q \to \mathcal{S}$ whose fiber at χ is $Q_\chi = \mathrm{Emb}(\chi, \Re^3)$. The fibers are isomorphic to $\mathrm{Emb}(\mathcal{B}, \Re^3)$—the space of extents.

Unlike the case where the symmetry requirement was not imposed, there is no natural identification of Q_χ with $\mathrm{Emb}(\mathcal{B}, \Re^3)$ because χ is not a unique embedding but an equivalence class.

We recall that the tangent space $T_\kappa Q_{\mathcal{B}}$ was identified with

$$C^\infty(\mathcal{B}, T(M \times \Re^3)) = C^\infty(\mathcal{B}, \mathbf{V} \times \Re^3).$$

Using charts, an element $\dot\kappa \in T_\kappa Q$ can be identified with a pair

$$(\dot\chi, \dot e) \in C^\infty(\nu) \times C^\infty(\mathcal{B}, \Re^3),$$

containing a section of the normal bundle and an extent rate.

Consider an

$$X \in \mathrm{Interior}(\chi) \subset M$$

and a motion $\kappa(t)$ in Q such that $\kappa(0) \in Q_\chi$. Then, X is in the interior of the images of $\chi(t) = \pi_s(\kappa(t))$ for t in a neighborhood of zero. Thus, the value $\kappa(t)(X)$ is well defined for all t in that neighborhood of the zero and the derivative, the material velocity field,

$$\mathbf{v}(X) = \frac{d}{dt}\kappa(t)(X)\Big|_{t=0},$$

can be calculated. It can be shown that $\mathbf{v}$ is well defined even on the boundary of χ and in fact $\mathbf{v}$ is C^{k-1} if κ is C^k (see [11]). As in Section 2 we use Δ to denote the mapping $\dot\kappa \mapsto \mathbf{v}$.

Proposition 3.3. The mapping Δ is the vertical projection of a connection on Q.

4. Forces on Symmetrically Growing Bodies

In accordance with the general setting we make the following definition.

Definition 4.1. A *force* on a symmetrically growing body at the configuration $\kappa \in Q$ is an element of $T^*_\kappa Q$.

The decomposition $(T\pi, \Delta) \colon T_\kappa Q \to T_{\pi(\kappa)}\mathcal{S} \times C^\infty(\chi, \Re^3)$ generates a decomposition of forces. Thus, a force on a symmetrically growing body may be represented in the form

$$f(\dot\kappa) = f_{\mathcal{S}}(T\pi(\dot\kappa)) + f_m(\Delta(\dot\kappa)) = f_{\mathcal{S}}(\dot\chi) + f_m(\mathbf{v}).$$

The force f_m and its representation by stresses were considered in Section 2. Note that since shapes are invariant with respect to diffeomorphisms of $\mathcal{B}$, one cannot use a shape to pull back fields defined on χ onto $\mathcal{B}$ as was done in section 2. For this reason we must use the representation by fields defined on the identifiable χ, i.e., by f_m as in section 2. Forces in $T^*_\chi\mathcal{S}$ have been considered in [12].

Definition 4.2. A *shape force* $f_{\mathcal{S}}$ at the shape χ is an element of $T^*_\chi\mathcal{S}$ that is continuous with respect to the C^1 norm when $T_\chi\mathcal{S}$ is identified with the space of C^∞ sections of the normal bundle.

Proposition 4.1. A force $f_{\mathcal{S}}$ can be represented in the form

$$f_{\mathcal{S}}(\dot{\chi}) = \int\limits_{\partial\chi} \dot{\chi}\, d\sigma_{\mathcal{S}} + \int\limits_{\partial\chi} \mathrm{D}\dot{\chi}\cdot d\Sigma_{\mathcal{S}},$$

where $\sigma_{\mathcal{S}}$ is a real valued measure over $\partial\chi$ and $\Sigma_{\mathcal{S}}$ is a measure over $\partial\chi$ valued in $T\partial\chi$.

The measures representing the shape force (accretive force in the terminology of [12]) are the corresponding stress measures. If stress measures that represent the shape force are given, the force may be restricted to a two dimensional submanifold P of $\partial\chi$ by

$$f_{\mathcal{S}\,P}(\dot{\chi}) = \int\limits_{P} \dot{\chi}\, d\sigma_{\mathcal{S}} + \int\limits_{P} \mathrm{D}\dot{\chi}\cdot d\Sigma_{\mathcal{S}}$$

Since $\partial\chi$ has no boundary, the stress measures can be approximated by smooth densities with respect to the area measure on $\partial\chi$. Thus we have the following.

Proposition 4.2. Every shape force $f_{\mathcal{S}}$ may be approximated with arbitrary accuracy by the smooth real function $s_{\mathcal{S}}$ over $\partial\chi$ and a smooth vector field tangent to $\partial\chi$ in the form

$$f_{\mathcal{S}}(\dot{\chi}) \approx \int\limits_{\partial\chi} (\dot{\chi}\, s_{\mathcal{S}} + \mathrm{D}\dot{\chi}\cdot S_{\mathcal{S}})\, dA.$$

If $s_{\mathcal{S}}$ and $S_{\mathcal{S}}$ are given, then, the force $f_{\mathcal{S}\,P}$ for a two dimensional submanifold with boundary P of $\partial\chi$ is given by

$$f_{\mathcal{S}}(\dot{\chi}) = \int\limits_{P} \dot{\chi}\, b_{\mathcal{S}}\, dA + \int\limits_{\partial P} \dot{\chi}\cdot t_{\mathcal{S}}\, dL,$$

where $\mathrm{Div}\, S_{\mathcal{S}} + b_{\mathcal{S}} = s_{\mathcal{S}}$ on P and $S_{\mathcal{S}}\cdot\lambda = t_{\mathcal{S}}$ on ∂P. Here, λ is the unit normal on $\partial\chi$ to ∂P. The field $b_{\mathcal{S}}$ is the *shape body force* and the field $t_{\mathcal{S}}$ is the *shape surface force*.

Clearly, when we consider the whole of χ, $\partial(\partial\chi) = 0$ and the second integral vanishes.

The representation of a symmetrically growing force by smooth stress fields is therefore in the form

$$f(\dot{\kappa}) = \int\limits_{\partial\chi} (\dot{\chi}\, s_{\mathcal{S}} + \mathrm{D}\dot{\chi}\cdot S_{\mathcal{S}})\, dA + \int\limits_{\chi} (\mathbf{v}\cdot s_m + \mathrm{D}\mathbf{v}\cdot S_m)\, dV,$$

and the representation in terms of body forces and surface forces is

$$f(\dot{\kappa}) = \int\limits_{\partial\chi} \dot{\chi}\, b_{\mathcal{S}}\, dA + \int\limits_{\partial\chi} \mathbf{v}\cdot t_m\, dA + \int\limits_{\chi} \mathbf{v}\cdot b_m\, dV.$$

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