Simultaneous controllability and discrimination of collections of perturbed bilinear control systems on the Lie group SU(N)

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Abstract

The controllability of bilinear systems is well understood for **finite dimensional** isolated systems where the control can be implemented exactly. However when perturbations are present some interesting theoretical questions are raised. We consider in this paper a control system whose control cannot be implemented exactly but is shifted by a time independent constant in a **discrete** list of possibilities. We prove under general hypothesis that the collection of possible systems (one for each possible perturbation) is simultaneously controllable with a common control. The result is extended to the situations where the perturbations are constant over a common, long enough, time frame. We apply the result to the controllability of quantum systems. **Furthermore, some examples and a convergence result are presented for the situation when an infinite number of perturbations occur.**

Keywords:
quantum control, Lie group controllability, bilinear system, perturbations

1. Introduction

The fundamental importance of addressing the controllability of bilinear systems has long been recognized in engineering control applications (see [1–9]). Among recent applications one may cite the field of quantum control with optical or magnetic external fields (see [5, 9–19]).

Although the controllability is well understood when the system is **of finite dimension, isolated** and the control can be implemented exactly new theoretical and numerical questions are raised when perturbations are present.

The question that is addressed in this paper is related to the simultaneous controllability of bilinear systems. Consider general systems $\frac{dX_k(t)}{dt} = (A_k + u(t)B_k)X_k$ on some finite dimensional Lie group G. Simultaneous controllability is the question of whether all states X_k can be controlled with the same control $u(t)$.

Problems of simultaneous control of **a finite collection of** systems have been addressed recently in applications related to quantum control [20–31]. In such circumstances the system is a collection of molecules or atoms or spin systems and the control is a magnetic field (in NMR) or a laser. The assessment of whether a single control pulse can drive independent (i.e., distinct) quantum systems to their respective target states was addressed theoretically in [20] for general A_k, B_k and applied to the optimal dynamic discrimination of separate quantum systems in [21]. The particular case of identical molecules with $A_k = A$ (constant) and $B_k = \xi_k B, \xi_k \in \mathbb{R}$, was treated in [22, 23] where **it is proved that** all members of an ensemble of randomly oriented molecules subjected to a single ultra-fast laser control pulse can be simultaneously controlled. An independent work [30] treats the circumstance when $A_k = \epsilon_k A, \epsilon_k \in \mathbb{R} \setminus \{-1, 1\}$ and $B_k = B$ (constant) and was used to show controllability for ensembles N-level of quantum systems having different Larmor dispersion. This last result generalizes the findings of [25] for ensembles of spin 1/2 systems.

The infinite dimensional version (an infinite number of systems $A_\epsilon = \epsilon A$ with ϵ taking arbitrary values in an interval $]\epsilon_*, \epsilon^*[$) was treated in [26, 27, 32] for the specific situation of the Bloch equations.

In this paper we extend the result in [30] to the new

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circumstance when $A_k = A + \alpha_k B, \alpha_k \in \mathbb{R}$ and $B_k = B$ (constant) or, equivalently, the simultaneous controllability of systems submitted to time independent perturbations $\frac{dX_k(t)}{dt} = [A + (u(t) + \alpha_k)B]X_k$. As the result in [30] does not apply to this situation, we exploit techniques used previously in [22, 23] and prove positive controllability results.

The perturbation model $A + (u(t) + \alpha_k)B$ was investigated theoretically and numerically in the physical literature independent of any theoretical controllability results. In the quantum computing literature such perturbations are called "fixed systematic errors" (see Section VI.A. equation (40) of [33]) or simply "systematic control error", see [34] where the authors concluded that mitigating such errors may be possible (although at the expense of longer pulse sequences). We give here a theoretical result to sustain this view. We also refer to [35], where the authors design pulse sequences that are generically robust with respect to errors in the amplitude of the control field. In a related recent work the corresponding noise model is called "low frequency noise" (see section IV. C. of [36]): it is defined as the portion of the (control) amplitude noise that has a correlation time that is long (up to 10^3 times) compared to the timescale of the dynamics and as such it can be treated as constant in time. Additional noise models (additive or multiplicative) are presented in [37] in the general quantum control area.

The balance of the paper is as follows: in Section 2 we introduce the general framework and the main notations and in Section 3 we present our main result. In Section 4, we apply our results to the controllability of quantum systems. **The situation of an infinite number of perturbations is discussed in Section 5.** Finally, some conclusions and perspectives of future work are given in Section 6.

2. Problem formulation

Given a matrix M, we denote by $\mathrm{Tr}(M)$ its trace.

Consider the following control systems on $U(N)$

$$\frac{dX(t)}{dt} = (A + u(t)B)X(t), \quad X(0) = Id. \tag{1}$$

Here A and B are skew-Hermitian matrices in $u(N)$. The matrix $X(t)$ evolves in the Lie group of unitary matrices $U(N)$, or, if both the matrices A and B have zero trace, in the Lie group of special unitary matrices $SU(N)$. We will assume without loss of generality (see [7]) that $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$ from now on, i.e., $A, B \in su(N)$ and then $X(t) \in SU(N)$.

The controllability of a system on Lie groups such as (1) is a well-studied problem [4–9]. The literature on the subject of bilinear control relies essentially on the following Theorem (originally due to [38]):

Theorem 1. *Let $A, B \in su(N)$ and denote by $\mathbb{L}_{A,B}$ the Lie subalgebra of $su(N)$ generated by A and B. The system (1) on the Lie group $SU(N)$ is controllable if and only if $\mathbb{L}_{A,B} = su(N)$ or equivalently if $\dim_{\mathbb{R}}\mathbb{L}_{A,B} = N^2 - 1$. Moreover there exists $T_{A,B} > 0$ such that any target can be reached in time $t \geq T_{A,B}$ with controls u such that $|u(s)| \leq 1, \forall s \in [0, t]$.*
Here $\dim_{\mathbb{R}}\mathbb{L}_{A,B}$ stands for the dimension of $\mathbb{L}_{A,B}$ as linear vector space over $\mathbb{R}$.

An important question is what happens if the control $u(t)$ in (1) is submitted to some perturbations in a predefined (discrete) list $\{\alpha_k, k = 1, \cdots, K\}$?

$$\frac{dX_k(t)}{dt} = AX_k(t) + [u(t) + \alpha_k]BX_k(t), \ X_k(0) = Id. \tag{2}$$

Can one still control the systems simultaneously? The real perturbation α_k for a given system is not known beforehand, therefore in order to be certain that the system is controlled, one has to find a control $u(t)$ that simultaneously control all states $X_k(t)$, i.e., find $u(t)$ such that $X_k(T) = V$ for $k = 1, \cdots, K$ (here V is the target state).

Yet a distinct circumstance is when α_k are not arbitrary perturbations but unknown characteristics of the system to be identified. Here the goal is to find $u(t)$ such that, given *distinct* V_k one has $X_k(T) = V_k$. By measuring the state of the system at the final time T, one knows what α_k was effective during $[0, T]$.

In conclusion, our problem can be formalized as follows: let $V_k \in SU(N), k = 1, \cdots, K$ be arbitrary. Is it possible to find $T > 0$ and a measurable $u : [0, T] \to \mathbb{R}$ such that the system given by (2) satisfies $X_k(T) = V_k, \forall \ k = 1, \cdots, K$? If the answer to this question is positive then the system in (2) will be called *simultaneously controllable*.

3. Simultaneous controllability for perturbations

3.1. Tools for simultaneous controllability

In this section we recall some known results on simultaneous controllability. Consider K bilinear systems on $SU(N)$:

$$\frac{dX_k(t)}{dt} = (A_k + u(t)B_k)X_k(t), \ X_k(0) = Id, \tag{3}$$

where $A_k, B_k \in su(N), k = 1, \cdots, K$. We denote by $diag\{M_1, \cdots, M_P\}$ the block diagonal matrix

$\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_P \end{pmatrix}$ obtained by setting the square matrices $M_1, \cdots, M_P$ on its diagonal. This definition allows **to introduce** $\mathcal{A} = diag\{A_1, \cdots, A_K\}$ as a $KN \times KN$ matrix constructed from $A_k, k = 1, \cdots, K$ and $\mathcal{B} = diag\{B_1, ..., B_K\}$. By assembling the K bilinear systems (3), the evolution of this collection of states can be written as a bilinear system (with block diagonal entries) on $(SU(N))^K$:

$$\frac{d\mathbf{X}(t)}{dt} = \mathcal{A}\mathbf{X}(t) + u(t)\mathcal{B}\mathbf{X}(t),\ \mathbf{X}(0) = \mathbf{Id} \in (SU(N))^K. \tag{4}$$

Denote by $\mathbb{L}_{\mathcal{A},\mathcal{B}}$ the Lie algebra generated by the matrices $\mathcal{A}$ and $\mathcal{B}$. Then we have the following result **(see [20], Theorems 1 and 2 p. 277 for the proof and [21], Section III for an application):**

Theorem 2. *The K bilinear systems (3) are simultaneously controllable if and only if* $\mathbb{L}_{\mathcal{A},\mathcal{B}} = (su(N))^K$ *or equivalently*

$$\dim_{\mathbb{R}} \mathbb{L}_{\mathcal{A},\mathcal{B}} = K(N^2 - 1).$$

Here $\dim_{\mathbb{R}} \mathbb{L}_{\mathcal{A},\mathcal{B}}$ *stands for the dimension of* $\mathbb{L}_{\mathcal{A},\mathcal{B}}$ *as linear vector space over* $\mathbb{R}$*. Moreover, there exists* $T_{\mathcal{A},\mathcal{B}} > 0$ *such that any collection of targets* $(V_k)_{k=1}^K \in (SU(N))^K$ *can be reached in time* $t \geq T_{\mathcal{A},\mathcal{B}}$ *with controls* $u(t)$ *such that* $|u(s)| \leq 1, \forall s \in [0, t]$.

A specific result has been proved in [22] **(Section II.B)** for particular choices $A_k = A$ and $B_k = \alpha_k B$.

Theorem 3 ([22]). *Let* $A, B \in su(N)$ *and consider a basis where A is diagonal, denote* $A = (-i)diag\{\lambda_1, ..., \lambda_N\}$, $\lambda_i \in \mathbb{R}$ *being the eigenvalues of iA. In this basis consider the following non-oriented graph* $\mathcal{G}_B = (\mathcal{V}_B, \mathcal{E}_B)$, $\mathcal{V}_B = \{1, 2, \cdots, N\}$, $\mathcal{E}_B = \{(k, l) \mid B_{kl} \neq 0\}$.
Assume that the graph $\mathcal{G}_B$ *is connected and that* $\forall (k_1, l_1), (k_2, l_2) \in \mathcal{E}_B$, $(k_1, l_1) \neq (k_2, l_2) : \lambda_{k_1} - \lambda_{l_1} \neq \lambda_{k_2} - \lambda_{l_2}$. *Consider also* $\alpha_1, \cdots, \alpha_K \in \mathbb{R}$ *such that* $|\alpha_k| \neq |\alpha_l|$, $\forall k \neq l$. *Then the system on* $(SU(N))^K$

$$\frac{dX_k(t)}{dt} = (A + u(t)\alpha_k B)X_k,\ X_k(0) = Id, \tag{5}$$

is simultaneously controllable and $\mathbb{L}_{\mathcal{A},\mathcal{B}} = (su(N))^K$. *Moreover there exists* $T_{A,B,\alpha_1,\cdots,\alpha_K}$ *such that any collection of targets* $(V_k)_{k=1}^K \in (SU(N))^K$ *can be reached in any time t larger than* $T_{A,B,\alpha_1,\cdots,\alpha_K}$ *with controls u such that* $|u(s)| \leq 1, \forall s \in [0, t]$.

3.2. Main result

Using the previous results we can now attack the situation when the control seen by the k-th system is $u(t) + \alpha_k$ and not $u(t)\alpha_k$ as in [22].

Theorem 4. *Consider the bilinear system on* $SU(N)$ *in equation (2), where* $A, B \in su(N)$*. Suppose that* $\mathbb{L}_{[A,B],B} = su(N)$.
Then for any distinct $\alpha_k \in \mathbb{R}$, $k = 1, .., K$, *the collection of systems (2) is simultaneously controllable in the sense described above. Moreover there exists* $T_{A,B,\alpha_1,\cdots,\alpha_K} > 0$ *such that the system is controllable in any time* $t \geq T_{A,B,\alpha_1,\cdots,\alpha_K}$ *with controls u such that* $|u(s)| \leq 1, \forall s \in [0, t]$.

Proof. To assess controllability of (2), we consider it as a system on $(SU(N))^K$ given by matrices $\mathcal{A} = diag\{A + \alpha_1 B, \cdots, A + \alpha_K B\}$ and $\mathcal{B} = diag\{B, \cdots, B\}$. Consider also the Lie algebra $\mathbb{L} = \mathbb{L}_{\mathcal{A},\mathcal{B}}$ spanned by $\mathcal{A}$ and $\mathcal{B}$. Note that $\mathbb{L}_{[A,B],B} = su(N)$ implies $\mathbb{L}_{A,B} = su(N)$. Since $[\mathcal{A}, \mathcal{B}] = diag\{[A, B], \cdots, [A, B]\}$ and since $\mathbb{L}_{[A,B],B} = su(N)$ it follows that $\mathbb{L}$ contains any matrix of the form $diag\{X, \cdots, X\}$, $X \in su(N)$. Thus $\mathbb{L}$ contains $Lie\{\mathcal{A}, diag\{X, \cdots, X\}, X \in su(N)\}$ which contains $diag\{A, \cdots, A\}$ thus contains $\mathcal{A} - diag\{A, \cdots, A\} = diag\{\alpha_1 B, \cdots, \alpha_K B\}$. Consequently $\mathbb{L}$ contains $Lie\{diag\{\alpha_1 B, \cdots, \alpha_K B\}, diag\{X, \cdots, X\}, X \in su(N)\}$. Consider now a particular basis, i.e., the one that diagonalizes the Hermitian matrix $i[A, B]$. Since $\mathbb{L}_{[A,B],B} = su(N)$ the graph $\mathcal{G}_B$ of B (see Theorem 3 for its definition) has to be connected in this basis **(see Section 4 in [9]).** Let us denote by $\bar{X}$ **a matrix such that** $i\bar{X}$ **does not have degenerate transitions (i.e., is "strongly regular" in the terminology of the Definition 2 in [30]). Recall that a matrix** Y **with eigenvalues** λ_1^Y**, ...,** λ_N^Y **has no degenerate transitions if** $\lambda_a^Y - \lambda_b^Y \neq \lambda_i^Y - \lambda_j^Y$ **for all** $(a, b) \neq (i, j)$**.** Then, $\mathbb{L}$ contains $Lie\{diag\{(\alpha_1 + \eta)B, \cdots, (\alpha_K + \eta)B\}, diag\{\bar{X}, \cdots, \bar{X}\}\ \forall\ \eta \in \mathbb{R}\}$. In particular there exists $\bar{\eta} \in \mathbb{R}$ such that $|\alpha_k + \bar{\eta}| \neq |\alpha_j + \bar{\eta}|\ \forall\ j \neq k$. Then by Theorem 3 it follows that $Lie\{diag\{(\alpha_1 + \bar{\eta})B, \cdots, (\alpha_K + \bar{\eta})B\}, diag\{\bar{X}, \cdots, \bar{X}\}\} = (su(N))^K$. Thus the system (2) is controllable. The assertions on $T_{A,B,\alpha_1,\cdots,\alpha_K}$ are consequences of Theorem 3, Q.E.D.

Remark 1. It is important to mention that the condition $\mathbb{L}_{[A,B],B} = su(N)$ is *sufficient but not necessary*. In order to illustrate this remark, we consider $K = 2$ bilinear systems in (2) and choose $A = \frac{1}{i}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$, $B = \frac{1}{i}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\alpha_1 =$

$1, \alpha_2 = -1$. Using the online calculator available at [39], we obtain that $\dim_{\mathbb{R}} \mathbb{L}_{A,B} = 8$, thus $\mathbb{L}_{A,B} = su(3)$ and $\dim_{\mathbb{R}} \mathbb{L}_{[A,B],B} = 4$, thus $\mathbb{L}_{[A,B],B} \neq su(3)$. However $\dim_{\mathbb{R}} \mathbb{L}_{diag\{A+\alpha_1 B, A+\alpha_2 B\}, diag\{B,B\}} = 16$ thus $\mathbb{L}_{diag\{A+\alpha_1 B, A+\alpha_2 B\}, diag\{B,B\}} = (su(3))^2$ and the bilinear systems are simultaneously controllable by the Theorem 2 despite $\mathbb{L}_{[A,B],B} \neq su(3)$.

Remark 2. Having proved the result above for the *bilinear setting*, it is interesting to compare with the analogous result in the *linear case*. For this we consider the following linear systems:

$$\frac{d}{dt}x_1 = Ax_1 + Bu(t), \; x_1(0) = 0,$$

$$\frac{d}{dt}x_2 = Ax_2 + B[u(t) + \alpha], \; x_2(0) = 0.$$

The dynamics of $x_2(t) - x_1(t)$ is not influenced by the control since $\frac{d}{dt}(x_2 - x_1) = A(x_2 - x_1) + B\alpha$, $x_2(0) - x_1(0) = 0$. Hence this collection of systems is not simultaneously controllable.

The result in Theorem 4 can be extended to the situation where the perturbations of the control can depend on time. We will require however that the perturbations be constant on a common, long enough, time interval.

Corollary 5. *Consider the collection of control systems on $SU(N)$ where $A, B \in su(N)$:*

$$\begin{cases} \frac{dY_k(t)}{dt} & = & \Big\{A + (u(t) + \delta_k u(t))B\Big\} Y_k(t), \\ Y_k(0) & = & Y_{k,0} \in SU(N). \end{cases} \tag{6}$$

Suppose that $\mathbb{L}_{[A,B],B} = su(N)$ and there exists $0 < t_1 < t_2 < \infty$ such that $\delta_k u(t) = \alpha_k$ (constant) $\forall t \in [t_1, t_2]$ **and** *$\alpha_k \neq \alpha_\ell$* **for** *$k \neq \ell$. Then there exists $T_{A,B,\alpha_1,\cdots,\alpha_K}$ such that if $t_2 - t_1 \geq T_{A,B,\alpha_1,\cdots,\alpha_K}$ the collection of systems (6) is simultaneously controllable at any time $T \geq t_2$.*

Proof. Let V_k be given targets for the systems (6) at time $T \geq t_2$. Define $u(t)$ to be zero on $[0, t_1] \cup [t_2, T]$ and $V_k^- = Y_k^-(t_1)$ where $Y_k^-(t)$ is the solution of $\frac{dY_k^-(t)}{dt} = (A + \delta_k u(t) B) Y_k^-(t)$, $Y_k^-(0) = Y_{k,0}$ and $V_k^+ = Y_k^+(T)$ where $Y_k^+(t)$ satisfies $\frac{dY_k^+(t)}{dt} = (A + \delta_k u(t) B) Y_k^+(t)$, $Y_k^+(t_2) = Id$. Set targets $W_k = (V_k^+)^{-1} V_k (V_k^-)^{-1}$ for the system (2) on $[0, t_2 - t_1]$ and initial states $X_k(0) = Id$ and let $\tilde{u}(t)$ be the control that drives X_k from $X_k(0) = Id$ to $X_k(t_2 - t_1) = W_k, \forall k = 1, \cdots, K$. Then the control $u(s)$ with $u(s) = 0$, for $s \in [0, t_1[\cup]t_2, T]$ and $u(s) = \tilde{u}(s - t_1)$, for $s \in [t_1, t_2]$ is such that $Y_k(T) = V_k^+ W_k V_k^- = V_k$, Q.E.D.

3.3. Further results on related models

Note that the model in equation (1) implies that the perturbation α_k is present even when the control $u(t)$ is null. In practice, it may sometimes be possible to eliminate the perturbations when the control field is not used and in this situation the controller can switch between a free, unperturbed dynamics and a controlled, perturbed one. This circumstance is modeled as

$$\begin{cases} \frac{dZ_k(t)}{dt} & = & AZ_k(t) + [u(t) + \alpha_k]\xi(t) B Z_k(t), \\ Z_k(0) & = & Z_{k,0} \in SU(N), \end{cases} \tag{7}$$

where the controls are $u(t)$ and $\xi(t)$, but $\xi(t) \in \{0, 1\} \forall t \geq 0$ (ξ being a measurable function). We obtain the following

Theorem 6. *The system (7) is simultaneously controllable if and only if $\mathbb{L}_{A,B} = su(N)$.*

Proof. With $\xi(t)$ as a new control the system (7) is controllable if and only if $\mathbb{L}_{diag\{A,\cdots,A\}, diag\{A+\alpha_1 B,\cdots,A+\alpha_K B\}, diag\{B,\cdots,B\}} = (su(N))^K$ or, equivalently $\mathbb{L}_{diag\{A,\cdots,A\}, diag\{B,\cdots,B\}, diag\{\alpha_1 B,\cdots,\alpha_K B\}} = (su(N))^K$. Denote $\mathbb{L}_1 = \mathbb{L}_{diag\{A,\cdots,A\}, diag\{B,\cdots,B\}, diag\{\alpha_1 B,\cdots,\alpha_K B\}}$. Suppose now $\mathbb{L}_{A,B} = su(N)$. As A, B span the whole $su(N)$ then $\mathbb{L}_1$ contains any matrix of the form $diag\{X, \cdots, X\}$, $X \in su(N)$. From this point the proof is similar as the one of Theorem 4. Of course $\mathbb{L}_{A,B} = su(N)$ is a necessary condition for controllability, which proves the reverse implication, Q.E.D.

Remark 3. For the situation (7) a result analogous to **Corollary 5** can be proved. We leave the proof as an exercise to the reader. **In addition both results remain true when ξ is piecewise constant (with a discrete set of discontinuities).**

4. Application to the control of a quantum system

Consider now a quantum bilinear system (cf. [5, 9, 14]):

$$i\frac{d}{dt}\psi = [H_0 + u(t)\mu]\psi(t), \tag{8}$$

$$H_0 = \begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 2.0 & 0 \\ 0 & 0 & 0 & 0 & 2.15 \end{pmatrix}, \mu = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}. \tag{9}$$

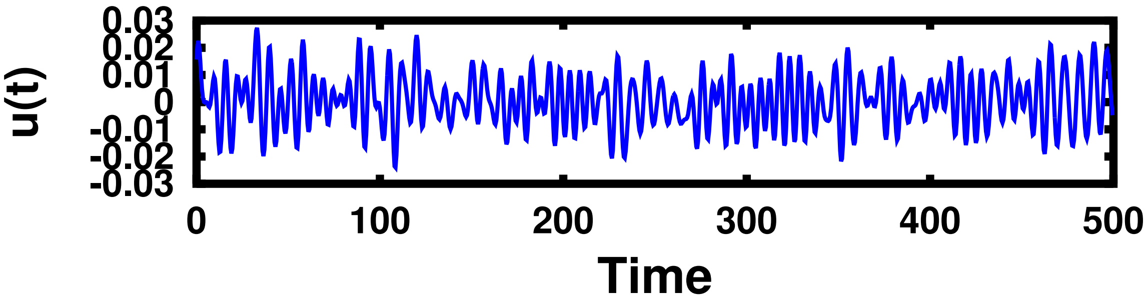

Figure 1: The control that drives ψ_0 to ψ_T (cf. equation (8)) irrespective of the perturbation $\alpha_k \in \{-0.1, 0, 0.1\}$. The quality of the control is over 99% for any perturbation. However the trajectories $\psi(t)$ corresponding to $u(t) - 0.1$, $u(t)$ and $u(t) + 0.1$ are all different.

controlled by the control $u(t)$ and with $\psi(0) = (1/\sqrt{2}, 0, 0, 1/\sqrt{2}, 0)^T$ and target $\psi_T = (0, 1/\sqrt{2}, 0, 0, 1/\sqrt{2})^T$. **This system has been extensively used as a benchmark for testing the controllability of bilinear quantum finite-dimensional systems: controllability criterions, search algorithms to find the controls etc. It has no degenerate transitions but a bi-partite connectivity graph structure: the set of eigenstates** 1 **to** 3 **are not directly connected, same for** 4 **and** 5**. Thus transferring population from eigenstate** 1 **to** 2 **requires a second-order excitation using eigenstates** 4 **or** 5 **as intermediary.** Define $B = \mu/i$ and, for simplicity, $A = [H_0 - 0.2\mathrm{Tr}(H_0).Id]/i$ such that both A and B belong to $su(5)$. Using the tool in [39] we obtain $\dim_{\mathbb{R}} \mathbb{L}_{A,B} = \dim_{\mathbb{R}} \mathbb{L}_{[A,B],B} = 24 = \dim_{\mathbb{R}} su(5)$ and since $\mathbb{L}_{[A,B],B} \subset \mathbb{L}_{A,B} \subset su(5)$ it follows that $\mathbb{L}_{[A,B],B} = \mathbb{L}_{A,B} = su(5)$. Consider the perturbations $\alpha_1 = -0.1, \alpha_2 = 0, \alpha_3 = 0.1$. Therefore **Theorem 4, Corollary 5 and Theorem 6** of the previous section apply. Since $SU(5)$ is transitive on the unit sphere of $\mathbb{C}^N$(cf. [7]) there exists U_T such that $U_T\psi_0 = \psi_T$ and by the Theorem 4 there exists a time T and a control $u : [0, T] \to \mathbb{R}$ such that $u(t), u(t) - 0.1$ and $u(t) + 0.1$ all drive Id to U_T in equation (1) thus all drive the initial state ψ_0 to the final state ψ_T in equation (8). We searched numerically the control $u(t)$ using a so-called monotonic procedure, see [40–45] for details. For $T = 500$, we obtain the control presented in Figure 1. The quality of the control, i.e. the quantity $\frac{|\langle \psi(T), \psi(0) \rangle|}{\|\psi(0)\|}$ is over 99% for all perturbations $\alpha_k, k = 1, 2, 3$. We also tested different pairs of initial and target states (ψ_0, ψ_T) and in all cases high quality controls were found.

5. Extensions to a infinite set of perturbations

We investigate in this section the circumstance when K **(the number of perturbations) is infinite. The controllability of a system consisting of an infinite collection of finite-dimensional systems has been analysed for the situation of the Bloch equation (**$N = 2$**) in [25–27, 32]. To the best of our knowledge no general results are available for generic systems and values of** N**; moreover the counter-example in Theorem 4 in [27] warns that general results may be impossible to obtain.**

We explore two questions: first we give an example that builds on the Bloch equation where a positive controllability result is expected; next we give a procedure for the numerical identification of approximate controls of the system (11).

5.1. An example of perturbed Bloch equation

Recall the notation for the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (10)$$

and consider the Bloch equation with two controls:

$$\begin{aligned} & i\frac{dX_k(t, \omega, \alpha, \beta)}{dt} = \Big\{\omega\sigma_z + [u(t) + \alpha]\sigma_x + \\ & [v(t) + \beta]\sigma_y\Big\} X_k(t, \omega, \alpha, \beta), \\ & X_k(0, \omega, \alpha, \beta) = Id, \\ & \alpha \in]\alpha_*, \alpha^*[, \beta \in]\beta_*, \beta^*[, \omega \in]\omega_*, \omega^*[. \quad (11) \end{aligned}$$

Although a rigorous proof of the controllability would require the tools in [32] and is beyond the scope of this work, we give below the arguments to conclude that this system is controllable (with unbounded controls). Consider the sequence of controls: a $(\pi/2)\sigma_x$ **pulse (for the control** u**), followed**

by free evolution during a unit of time and then a $-(\pi/2)\sigma_x$ pulse (for u) again. This results in the evolution

$$e^{-i(\pi/2)\sigma_x}e^{-i(\omega\sigma_z+\alpha\sigma_x+\beta\sigma_y)}e^{i(\pi/2)\sigma_x} = e^{-i(-\omega\sigma_z+\alpha\sigma_x-\beta\sigma_y)}. \tag{12}$$

Thus the propagator associated to $-\omega\sigma_z+\alpha\sigma_x-\beta\sigma_y$ can be synthesized. A similar computation (now using the control v) allows to construct $-\omega\sigma_z-\alpha\sigma_x+\beta\sigma_y$. Using now infinitesimal times and the formula $e^{U+V} = \lim_{n\to\infty}\left(e^{U/n}e^{V/n}\right)^n$ we have thus at our disposal all propagators $e^{\pm i\omega\sigma_z}$, $e^{\pm i\alpha\sigma_x}$, $e^{\pm i\beta\sigma_y}$. Recall that we also have $e^{\pm i\sigma_x}$, $e^{\pm i\sigma_y}$.

From now on, the argument is similar to that in [27]: commutators of, for instance, $\pm i\omega\sigma_z$ and $\pm i\sigma_x$, produce $\pm i\omega\sigma_y$ and then commutators $\pm i\omega\sigma_z$ and $\pm i\omega\sigma_y$, produce $\pm i\omega^2\sigma_x$, and all other polynomials of ω can be obtain as multiplicative factors before ω_z. Similar arguments allow to further obtain all possible polynomials of three variables ω,α,β. Therefore we obtain (approximate) controllability of the system to any (smooth) target dependent (or not) of the variables ω,α,β.

Remark 4. Not all situations have favorable outcomes. For instance, using same arguments as in Remark page 030302-2 of [26], it is possible to show that for the controlled Hamiltonian $\sigma_z+\alpha\sigma_y+u(t)\sigma_x$ the unknown perturbation $\alpha\in]\alpha_*,\alpha^*[$ cannot always be compensated. Indeed, the attainable propagators are of the form

$$exp\{if_1(\alpha)(\sigma_y-\alpha\sigma_z)+if_2(\alpha)(\sigma_z+\alpha\sigma_y)+if_3(\alpha)\sigma_x\} \tag{13}$$

where f_1, f_2 and f_3 are arbitrary functions of α^2. Thus when for instance $\alpha_* = -\alpha^*$ functions f_1, f_2, f_3 are odd functions which is a restriction for controllability.

5.2. Convergence of the controls for a discrete set of perturbations

We investigate here a numerical algorithm to find the control when the set of perturbations can be a whole (possibly unbounded) interval $I_\alpha\subset\mathbb{R}$. Fix $A,B\in SU(N)$ such that $\mathbb{L}_{[A,B],B} = su(N)$ and let us denote by $X(t,\alpha,u)$ the solution of $\frac{dX(t)}{dt} = (A+(u(t)+\alpha)B)X$ at time t starting from $X(0) = Id$. Fix also a target state $Y(\alpha)\in C^0(I_\alpha;SU(N))$ i.e. continuous with respect to $\alpha\in I_\alpha$.

Consider a sequence of divisions $\mathcal{T}_\ell\subset I_\alpha$: $\alpha_1^\ell < \alpha_1^\ell < ... < \alpha_{K_\ell}^\ell$ of the interval I_α such that $|\mathcal{T}_\ell| := \max_{j=2,K_\ell}|\alpha_j^\ell-\alpha_{j-1}^\ell|$ tends to 0 when ℓ tends to ∞. Fix also a tolerance $\eta\geq 0$. Using the results of the previous sections there exists a time T_ℓ and a control u_ℓ such that $\|X(T_\ell,\alpha_j^\ell,u_\ell)-Y(\alpha_j^\ell)\|\leq\eta$ for all $j=1,...,K_\ell$.

In this section we give a sufficient result that ensures the existence of a control u that steers the initial state Id to the target Y for the whole interval of perturbations I_α up to the tolerance η.

Proposition 7. *Suppose that the sequence T_ℓ is not converging to infinity and $\|u_\ell\|_{L^r([0,T_\ell])}$ are bounded by a common constant for some $1<r<\infty$. Then there exists $T>0$ and $u\in L^r([0,T])$ (independent of α) such that $\|X(T,\alpha,u)-Y(\alpha)\|\leq\eta$ for all $\alpha\in I_\alpha$.*

Proof. Since T_ℓ does not converge to ∞ it has a subsequence converging to some $T\in\mathbb{R}$. Denote again by T_ℓ this subsequence; we can moreover consider that all T_ℓ are either greater or smaller than T, let us say $T_\ell\leq T$ for all ℓ. Extend the domain of definition of u_ℓ on $[0,T]$ with $u_\ell = 0$ on $[T_\ell,T]$; this will not change its L^r norm. Up to extracting another subsequence, there exists $u\in L^r([0,T])$ such that u_ℓ converges weakly in $L^1([0,T])$ to u. Let us prove that u satisfies the required conditions. Fix $\alpha\in I_\alpha$. Since $|\mathcal{T}_\ell|\to 0$ there exists a sequence $\alpha_{k_\ell}^\ell$ such that $\alpha_{k_\ell}^\ell\to\alpha$ when $\ell\to\infty$. We write:

$$\begin{aligned}&\|Y(\alpha)-X(T,\alpha,u)\|\leq\|Y(\alpha)-Y(\alpha_{k_\ell}^\ell)\|\\&+\|Y(\alpha_{k_\ell}^\ell)-X(T_\ell,\alpha_{k_\ell}^\ell,u_\ell)\|\\&+\|X(T_\ell,\alpha_{k_\ell}^\ell,u_\ell)-e^{(T-T_\ell)A}X(T_\ell,\alpha_{k_\ell}^\ell,u_\ell)\|\\&+\|e^{(T-T_\ell)A}X(T_\ell,\alpha_{k_\ell}^\ell,u_\ell)-X(T,\alpha,u)\|.\end{aligned} \tag{14}$$

The first term in the right hand side converges to 0 as $\ell\to\infty$ while the second is bounded by η. The term $\|X(T_\ell,\alpha_{k_\ell}^\ell,u_\ell)-e^{(T-T_\ell)A}X(T_\ell,\alpha_{k_\ell}^\ell,u_\ell)\|$ is equal to $\|Id-e^{(T-T_\ell)A}\|$ and thus converges to 0. The last term can be written as:

$$\begin{aligned}&\|e^{(T-T_\ell)A}X(T_\ell,\alpha_{k_\ell}^\ell,u_\ell)-X(T,\alpha,u)\|\\&=\|e^{(T-T_\ell)A}X(T_\ell,0,\alpha_{k_\ell}^\ell+u_\ell)-X(T,0,\alpha+u)\|\\&=\|X(T,0,\alpha_{k_\ell}^\ell+u_\ell+1_{[T_\ell,T]}\cdot(-\alpha_{k_\ell}^\ell))\\&-X(T,0,\alpha+u)\|.\end{aligned} \tag{15}$$

Since $T_\ell\to T$, it follows that $\alpha_{k_\ell}^\ell+u_\ell+1_{[T_\ell,T]}\cdot(-\alpha_{k_\ell}^\ell)$ converges weakly in $L^1([0,T])$ to $\alpha+u$. From Theorem 3.6 of [46], the weak convergence of $\alpha_{k_\ell}^\ell+u_\ell+1_{[T_\ell,T]}\cdot(-\alpha_{k_\ell}^\ell)$ to $\alpha+u$ ensures that $\lim_{\ell\to\infty}X(T,0,\alpha_{k_\ell}^\ell+u_\ell+1_{[T_\ell,T]}\cdot(-\alpha_{k_\ell}^\ell)) = X(T,0,\alpha+u)$. Combining all estimations we obtain the conclusion:

$$\|X(T,\alpha,u)-Y(\alpha)\|\leq\eta. \tag{16}$$

Remark 5. The Proposition is not a controllability result but can be used numerically to find the control when controllability hold true.

In particular the situation $\eta = 0$ corresponds to exact controllability; however the results in [32] show that approximate controllability is more likely to hold and the controls will be in L^∞_{loc}, thus in all $L^r([0, t])$.

6. Conclusion and perspectives

Using Lie-algebraic methods, sufficient conditions have been derived for the simultaneous controllability of a finite-dimensional system, in the case where the control is submitted to **finite collection of** constant or partially constant perturbations. **Additional arguments have been presented when the number of possible perturbations is infinite.**

This work studied the controllability for possibly large final times. A related question is whether small time local controllability (called STLC) is also true. A further question is whether the result extends to more general, time dependent, perturbations.

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References

[1] E.-D. Sontag, Mathematical control theory: deterministic finite dimensional systems, 2nd Edition, Springer, 1998.

[2] H.-K. Khalil, Nonlinear systems, Macmillan, 1996.

[3] H. Nijmeiher, A. van der Schaft, Nonlinear dynamical control systems, Springer, 1990.

[4] F. S. Leite, P.-E. Crouch, Controllability on classical Lie groups, Mathematics of Control, Signals, and Systems 1 (1988) 31–42.

[5] V. Ramakrishna, M. Salapaka, M. Dahleh, H. Rabitz, A. Peirce, Controllability of molecular-systems, Physical Review A 51 (1995) 960–966.

[6] D. D'Alessandro, Small time controllability of systems on compact Lie groups and spin angular momentum, J. Math. Phys. 42 (9) (2001) 4488–4496.

[7] F. Albertini, D. D'Alessandro, Notions of controllability for quantum mechanical systems, in: 40th IEEE Conference on Decision and Control, Vol. 2, Orlando, FL, 2001, pp. 1589–1594.

[8] C. Altafini, Controllability of quantum mechanical systems by root space decomposition of su(n), Journal of Mathematical Physics 43 (5) (2002) 2051–2062.

[9] G. Turinici, H. Rabitz, Wavefunction controllability in quantum systems, Journal of Physics A: Mathematical and Theoretical 36 (2003) 2565–2576.

[10] S. Rice, M. Zhao, Optical control of quantum dynamics, Wiley, 2000.

[11] G. Huang, T. Tarn, J. Clark, On the controllability of quantum mechanical systems, J. Math. Phys. 24 (1983) 2608–2618.

[12] R. Judson, K. Lehmann, H. Rabitz, W.-S. Warren, Optimal design of external fields for controlling molecular motion-application to rotation, J. Molec. Structure 223 (1990) 425–456.

[13] G. Turinici, On the controllability of bilinear quantum systems, in: M. Defranceschi, C. L. Bris (Eds.), Mathematical models and methods for ab initio Quantum Chemistry, Lecture Notes in Chemistry vol. 74, Springer, 2000, pp. 75–92.

[14] S.-G. Schirmer, H. Fu, A.-I. Solomon, Complete controllability of quantum systems, Phys. Rev. A 63 (2001) 063410.

[15] G. Turinici, H. Rabitz, Quantum Wave Function Controllability, Chemical Physics 267 (1-3) (2001) 1–9.

[16] M.-D. Girardeau, M. Ina, S.-G. Schirmer, T. Gulsrud, Kinematical bounds on evolution and optimization of mixed quantum states, Phys. Rev. A 55 (1997) R1565–R1568.

[17] K. Beauchard, Local controllability of a 1-D Schrödinger equation, Journal de Mathématiques Pures et Appliquées 84 (7) (2005) 851–956. `doi:{10.1016/j.matpur.2005.02.005}`.

[18] K. Beauchard, C. Laurent, Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control, Journal de Mathématiques Pures et Appliquées 94 (5) (2010) 520–554. `doi:{10.1016/j.matpur.2010.04.001}`.

[19] M.-D. Girardeau, S.-G. Schirmer, J.-V. Leahy, R.-M. Koch, Kinematical bounds on optimization of observables for quantum states, Phys. Rev. A 58 (1998) 2684–2689.

[20] G. Turinici, V. Ramakrishna, B. Li, H. Rabitz, Optimal Discrimination of Multiple Quantum Systems: Controllability Analysis, J. Phys. A: Mathematical and General 37 (1) (2003) 273–282.

[21] B. Li, G. Turinici, V. Ramakhrishna, H. Rabitz, Optimal Dynamic Discrimination of Similar Molecules Through Quantum Learning Control, Journal of Physical Chemistry B 106 (33) (2002) 8125–8131.

[22] G. Turinici, H. Rabitz, Optimally controlling the internal dynamics of a randomly oriented ensemble of molecules, Phys. Rev. A 70 (6) (2004) 063412. `doi:10.1103/PhysRevA.70.063412`.

[23] H. Rabitz, G. Turinici, Controlling quantum dynamics regardless of laser beam spatial profile and molecular orientation, Physical review A: Atomic, Molecular and Optical Physics 75 (4) (2007) 043409.

[24] S.-G. Schirmer, I.-C. Pullen, A.-I. Solomon, Controllability of multi-partite quantum systems and selective excitation of quantum dots, Journal of Optics B 7 (2005) S293–S299.

[25] J.-S. Li, N. Khaneja, Ensemble controllability of the Bloch equations, in: 45th IEEE Conference on Decision and Control, San Diego, CA, USA, 2006, pp. 13–15.

[26] J.-S. Li, N. Khaneja, Control of inhomogeneous quantum ensembles, Phys. Rev. A 73 (2006) 030302.

[27] J.-S. Li, N. Khaneja, Ensemble control of Bloch equations, IEEE Trans. Automat. Control 54 (3) (2009) 528–536. `doi:10.1109/TAC.2009.2012983`.
URL `http://dx.doi.org/10.1109/TAC.2009.2012983`

[28] D. Sugny, A. Keller, O. Atabek, D. Daems, C. M. Dion, S. Guerin, H.-R. Jauslin, Control of mixed-state quantum systems by a train of short pulses, Physical Review A (Atomic, Molecular, and Optical Physics) 72 (3) (2005) 032704.

[29] T.-J. Tarn, J. Clark, D. Lucarelli, Controllability of quantum mechanical systems with continuous spectra, in: Decision and Control, 2000. Proceedings of the 39th IEEE Conference on, Vol. 1, 2000, pp. 943–948 vol.1. doi:10.1109/CDC.2000.912894.

[30] C. Altafini, Controllability and simultaneous controllability of isospectral bilinear control systems on complex flag manifolds, Systems & Control Letters 58 (2009) 213–216.

[31] K. Moore, H. Rabitz, Manipulating molecules, Nature Chemistry 4 (2012) 72–73.

[32] K. Beauchard, J.-M. Coron, P. Rouchon, Controllability issues for continuous-spectrum systems and ensemble controllability of Bloch equations, Comm. Math. Phys. 296 (2) (2010) 525–557. doi:10.1007/s00220-010-1008-9.
URL http://dx.doi.org/10.1007/s00220-010-1008-9

[33] K. Khodjasteh, L. Viola, Dynamical quantum error correction of unitary operations with bounded controls, Phys. Rev. A 80 (2009) 032314. doi:10.1103/PhysRevA.80.032314.
URL http://link.aps.org/doi/10.1103/PhysRevA.80.032314

[34] K. Khodjasteh, L. Viola, Dynamically error-corrected gates for universal quantum computation, Phys. Rev. Lett. 102 (2009) 080501. doi:10.1103/PhysRevLett.102.080501.
URL http://link.aps.org/doi/10.1103/PhysRevLett.102.080501

[35] A. M. Souza, G. A. Álvarez, D. Suter, Experimental protection of quantum gates against decoherence and control errors, Phys. Rev. A 86 (2012) 050301. doi:10.1103/PhysRevA.86.050301.
URL http://link.aps.org/doi/10.1103/PhysRevA.86.050301

[36] D. Hocker, C. Brif, M. D. Grace, A. Donovan, T.-S. Ho, K. W. Moore Tibbetts, R. Wu, H. Rabitz, Characterization of control noise effects in optimal quantum unitary dynamics, ArXiv e-printsVersion 1. arXiv:1405.5950.

[37] I. R. Sola, H. Rabitz, The influence of laser field noise on controlled quantum dynamics, The Journal of Chemical Physics 120 (19) (2004) 9009–9016. doi:http://dx.doi.org/10.1063/1.1691803.
URL http://scitation.aip.org/content/aip/journal/jcp/120/19/10.1063/1.1691803

[38] V. Jurdjevic, H. J. Sussmann, Control systems on Lie groups, J. Differ. Equations 12 (1972) 313–329.

[39] Online controllability calculator.
URL https://www.ceremade.dauphine.fr/~turinici/index.php/fr/recherche/calculator.html

[40] M. Belhadj, J. Salomon, G. Turinici, A stable toolkit method in quantum control, Journal of Physics A: Mathematical and Theoretical 41 (36) (2008) 362001–362011.

[41] D. Tannor, V. Kazakov, V. Orlov, Control of photochemical branching: Novel procedures for finding optimal pulses and global upper bounds, in: Time Dependent Quantum Molecular Dynamics ed J Broeckhove and L Lathouwers, New York: Plenum, 1992, pp. 347–360.

[42] W. Zhu, H. Rabitz, A rapid monotonically convergent iteration algorithm for quantum optimal control over the expectation value of a positive definite operator, J. Chem. Phys. 109 (1998) 385–391.

[43] S. G. Schirmer, M. D. Girardeau, J. V. Leahy, Efficient algorithm for optimal control of mixed-state quantum systems, Phys. Rev. A 61 (1999) 012101. doi:10.1103/PhysRevA.61.012101.

[44] L. Baudouin, J. Salomon, Constructive solution of a bilinear optimal control problem for a Schrödinger equation, Systems & Control Letters 57 (6) (2008) 453–464. doi:{10.1016/j.sysconle.2007.11.002}.

[45] Y. Maday, G. Turinici, New formulations of monotonically convergent quantum control algorithms, J. Chem. Phys. 118 (2003) 8191–8196.

[46] J. Ball, J. Marsden, M. Slemrod, Controllability for distributed bilinear systems, SIAM Journal on Control and Optimization 20 (4) (1982) 575–597. arXiv:http://dx.doi.org/10.1137/0320042, doi:10.1137/0320042.
URL http://dx.doi.org/10.1137/0320042