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# The shadow nature of positive and twisted quandle cocycle invariants of knots

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## Abstract

Quandle cocycle invariants form a powerful and well developed tool in knot theory. This paper treats their variations — namely, positive and twisted quandle cocycle invariants, and shadow invariants. We interpret the former as particular cases of the latter. As an application, several constructions from the shadow world are extended to the positive and twisted cases. Another application is a sharpening of twisted quandle cocycle invariants for multi-component links.

**Keywords:** quandles; multi-term distributive (co)homology; quandle cocycle invariants; positive quandle cocycle invariants; twisted quandle cocycle invariants; shadow colorings.

## 1 Introduction

A *quandle* is a set  $Q$  endowed with binary operations<sup>1</sup>  $\triangleleft$  and  $\tilde{\triangleleft}$  satisfying

$$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c), \quad (1)$$

$$(a \triangleleft b) \tilde{\triangleleft} b = (a \tilde{\triangleleft} b) \triangleleft b = a, \quad (2)$$

$$a \triangleleft a = a. \quad (3)$$

A group with the conjugation operation  $a \triangleleft b = b^{-1}ab$  is an important example. Quandles were introduced in [Joy82, Mat82] as an efficient tool for constructing knot<sup>2</sup> invariants. Concretely, for a quandle  $Q$  and a knot diagram  $D$ , consider the set  $\mathcal{C}_Q(D)$  of  $Q$ -colorings of  $D$  — that is, assignments  $\mathcal{C}: \text{Arcs}(D) \rightarrow Q$  of elements of  $Q$  to every arc of  $D$ , satisfying the condition on Figure 1  around each crossing.<sup>3</sup> Axioms (1)-(3) force the number  $|\mathcal{C}_Q(D)|$  of such colorings to be stable under Reidemeister moves, and thus to give a knot invariant.

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<sup>1</sup>Operation  $\tilde{\triangleleft}$  is deduced from  $\triangleleft$  using (2); we thus often simply speak about a quandle  $(Q, \triangleleft)$ .

<sup>2</sup>In this paper by *knots* we always mean *oriented knots and links in  $\mathbb{R}^3$* .

<sup>3</sup>Here and afterwards diagrams with unoriented arcs mean that all coherently oriented versions of these diagrams should be considered.



Figure 1: Quandle and shadow colorings

These *quandle invariants* are very powerful, but they do not detect, for instance, the chirality of knots. This flaw was fixed in [CJK<sup>+</sup>03] by enriching colorings with *weights*. Concretely, given an abelian group  $A$ , a map  $\omega: Q \times Q \rightarrow A$  satisfying

$$\omega(a, b) + \omega(a \triangleleft b, c) = \omega(a \triangleleft c, b \triangleleft c) + \omega(a, c), \quad (4)$$

$$\omega(a, a) = 0 \quad (5)$$

is called a *quandle 2-cocycle* of  $Q$ . (The word “cocycle” will be justified below.) The  $\omega$ -*weight* of a  $Q$ -colored knot diagram  $(D, \mathcal{C})$  is defined as a sum

$$\mathcal{W}_\omega(D, \mathcal{C}) = \sum_{\substack{\times \\ a \times b}} \varepsilon(\times) \omega(a, b) \quad (6)$$

over all crossings  $\times$  of  $D$ , where  $\varepsilon(\times)$  is the *sign* of  $\times$  (cf. Figure 2). Axioms (4)-(5) were chosen so that the the multiset of  $\omega$ -weights  $\{\mathcal{W}_\omega(D, \mathcal{C}) \mid \mathcal{C} \in \mathcal{C}_Q(D)\}$  yields a knot invariant, called a *quandle cocycle invariant*.

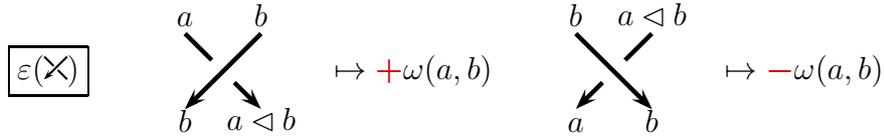


Figure 2: Quandle cocycle invariants

This construction admits several variations. Instead of quandle 2-cocycles, they use some other cohomological data, which we now recall. Together with a quandle  $Q$ , consider a  $Q$ -*module*<sup>4</sup> — that is, a set  $M$  endowed with two maps  $\triangleleft, \tilde{\triangleleft}: M \times Q \rightarrow M$  satisfying Axioms (1)-(2) for all  $a \in M, b, c \in Q$ . The simplest examples are  $Q$  itself, and a one-element set (called a *trivial*  $Q$ -module), with evident module operations. Take also an abelian group  $A$ . Denote by  $C^k(M, Q, A)$  the abelian group of maps from  $M \times Q^{\times k}$  to  $A$ , and put<sup>6</sup>

$$(d_l^k \phi)(m, a_1, \dots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \phi(m \triangleleft a_i, \dots, a_{i-1} \triangleleft a_i, a_{i+1}, \dots, a_{k+1}), \quad (7)$$

$$(d_r^k \phi)(m, a_1, \dots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \phi(m, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{k+1}). \quad (8)$$

Maps  $d_l^k$  and  $d_r^k$  turn out to be anti-commuting differentials. Hence for all  $\alpha_l, \alpha_r$  lying in  $\mathbb{Z}$  (or, more generally, in a commutative ring  $R$  if  $A$  is an  $R$ -module),

<sup>4</sup> $Q$ -modules appeared under the name  $Q$ -(*quandle*-)sets ([FRS95]), and are also known as  $Q$ -*shadows* ([CN11]).

<sup>5</sup>The notation is slightly abusive but convenient.

<sup>6</sup>Subscripts  $l$  &  $r$  refer to an interpretation of these maps as *left* & *right* differentials, cf. [Leb13].

$(C^k(M, Q, A), \alpha_l d_l^k - \alpha_r d_r^k)$  is a cochain complex. Denote by  $C_Q^k(M, Q, A)$  the subgroup of maps vanishing on all tuples  $(m, a_1, \dots, a_k)$  with  $a_i = a_{i+1}$  for some  $i$ . Both  $d_l^k$  and  $d_r^k$  restrict to  $C_Q^k(M, Q, A)$ , and thus so do their linear combinations. The cocycles / coboundaries / cohomology groups of  $(C_Q^k(M, Q, A), \alpha_l d_l^k - \alpha_r d_r^k)$  receive the adjective

- *quandle* if  $\alpha_l = \alpha_r = 1$  ([FRS95, CJK<sup>+</sup>03]);
- *positive quandle* if  $\alpha_l = -\alpha_r = 1$  ([CG14]);
- *(t-)twisted quandle* if  $\alpha_l = 1, \alpha_r = t$ , and  $A$  is a  $\mathbb{Z}[t^{\pm 1}]$ -module ([CES02]);
- *two-term distributive* in the general case ([PS14, Prz11]).

Note that a trivial  $M$  can be safely excluded from consideration (and from our notations), since  $M \times Q^{\times k} \cong Q^{\times k}$ ; we will often talk about  $(M-)$ shadow cocycles / coboundaries / cohomology groups if  $M$  is non-trivial.

*Positive* ([CG14]) and *twisted* ([CES02]) variations of quandle cocycle invariants using positive and, respectively, twisted quandle 2-cocycles with trivial  $M$  are recalled in Section 2. *Shadow invariants* based on shadow quandle 2-cocycles are also reviewed; together with arc colorings, these use region colorings by elements of  $M$ , respecting the rule on Figure 1 (B). In literature, mostly the case  $M = Q$  was considered formally and thoroughly ([RS00, CKS01, Kam02]); the general case seems to be folklore. In this paper, carefully chosen  $Q$ -modules allow us to present twisted (Section 3) and positive (Section 5) quandle cocycle invariants as shadow invariants. This instantly gives a proof of their invariance, avoiding technical verifications. Moreover, all results established for shadow invariants can now be applied in the positive and twisted cases. Such results, discussed in Section 4, include the equality of invariants obtained from cohomologous cocycles; higher-dimensional generalizations; and shadow<sup>7</sup> versions. In Section 6, the shadow ideas lead to a sharpening of twisted quandle cocycle invariants for multi-component links.

## 2 Shadow, positive, and twisted quandle cocycle invariants

In this section we recall three variations of the quandle cocycle invariant construction. From now on, fix a quandle  $(Q, \triangleleft)$  and an abelian group  $A$ .

First, together with the arcs of a knot diagram  $D$ , one can color its *regions* (= the connected components of  $\mathbb{R}^2 \setminus D$ ) by elements of a  $Q$ -module  $M$ . Concretely, define a *shadow*  $(Q, M)$ -*coloring* of  $D$  as a  $Q$ -coloring  $\mathcal{C}$  of the arcs of  $D$  and an assignment  $\mathcal{C}^{sh}: \text{Regions}(D) \rightarrow M$ , compatible in the sense of Figure 1 (B)<sup>8</sup>. The set of such colorings is denoted by  $\mathcal{C}_{Q,M}^{sh}(D)$ . The color of the exterior region  $\Delta_{ex}(D)$  of  $D$  uniquely determines all the other region colors, hence the  $(Q, M)$ -coloring number  $|\mathcal{C}_{Q,M}^{sh}(D)|$  equals  $|\mathcal{C}_Q(D)| \cdot |M|$  and says nothing new about the knot. The situation changes when weights enter the story. Take an  $M$ -*shadow quandle 2-cocycle* of  $Q$

<sup>7</sup>Note the “double-shadow” situation here.

<sup>8</sup>In our diagrams, region colors are underlined for a better readability.

— that is, a map  $\omega: M \times Q \times Q \rightarrow A$  satisfying

$$\begin{aligned} \omega(m, a, b) + \omega(m \triangleleft b, a \triangleleft b, c) + \omega(m, b, c) = \\ \omega(m \triangleleft c, a \triangleleft c, b \triangleleft c) + \omega(m, a, c) + \omega(m \triangleleft a, b, c), \end{aligned} \quad (9)$$

$$\omega(m, a, a) = 0. \quad (10)$$

**Definition 2.1.** The (*shadow*)  $\omega$ -weight of a  $(Q, M)$ -colored knot diagram  $(D, \mathcal{C}, \mathcal{C}^{sh})$  is the sum

$$\mathcal{W}_\omega^{sh}(D, \mathcal{C}, \mathcal{C}^{sh}) = \sum_{\substack{a \times b \\ \underline{m}}} \omega(m, a, b) - \sum_{\substack{a \times b \\ \overline{m}}} \omega(m, a, b) \quad (11)$$

taken over all crossings of  $D$ .

**Theorem 1.** For any  $Q$ -module  $M$ ,  $m \in M$ , and  $M$ -shadow quandle 2-cocycle  $\omega$  of  $Q$ , the multiset

$$\{ \mathcal{W}_\omega^{sh}(D, \mathcal{C}, \mathcal{C}^{sh}) \mid (\mathcal{C}, \mathcal{C}^{sh}) \in \mathcal{C}_{Q, M}^{sh}(D), \mathcal{C}^{sh}(\Delta_{ex}(D)) = m \}$$

depends only on the underlying knot, and not on the choice of its diagram  $D$ .

The (extremely rich) knot invariant thus obtained are called *shadow quandle cocycle invariants*, or simply *shadow invariants*.

Another variation uses a *positive quandle 2-cocycle* of  $Q$  — that is, a map  $\omega: Q \times Q \rightarrow A$  satisfying (5) and

$$\omega(a, c) + \omega(a \triangleleft b, c) = \omega(a \triangleleft c, b \triangleleft c) + \omega(a, b) + 2\omega(b, c). \quad (12)$$

**Definition 2.2.** The (*positive*)  $\omega$ -weight of a  $Q$ -colored knot diagram  $(D, \mathcal{C})$  is the sum

$$\mathcal{W}_\omega^{pos}(D, \mathcal{C}) = \sum_{\substack{a \times b \\ \text{sign}}} \varepsilon^{pos}(\text{sign}) \omega(a, b), \quad (13)$$

where the sign  $\varepsilon^{pos}(\text{sign})$  is defined on Figure 3 (here the checkerboard coloring of the diagram's regions is used, with the exterior region declared white).



Figure 3: Positive quandle cocycle invariants

**Theorem 2** ([CG14]). For any positive quandle 2-cocycle  $\omega$  of  $Q$ , the multiset  $\{ \mathcal{W}_\omega^{pos}(D, \mathcal{C}) \mid \mathcal{C} \in \mathcal{C}_Q(D) \}$  yields a knot invariant.

The knot invariant obtained is referred to as *positive quandle cocycle invariant*.

In order to present the third generalization of quandle cocycle invariants we are interested in, some technical but intuitive definitions are necessary.

**Definition 2.3.** • The *index*<sup>9</sup>  $i(\Delta)$  of a region  $\Delta$  in a knot diagram  $D$  is the algebraic intersection number for  $D$  and a path  $\gamma$  connecting  $\Delta$  to the exterior region  $\Delta_{ex}(D)$ . Sign conventions are indicated in Figure 4(A);  $\gamma$  and  $D$  are supposed to have a finite number of simple transverse intersections.

- The *source region*  $\Delta(\times)$  of a crossing  $\times$  in  $D$  is defined in Figure 4(B)<sup>1011</sup>.
- The *index* of a crossing  $\times$  is set to be  $i(\times) = i(\Delta(\times))$ .

The index of a region/crossing does not depend on the choice of path  $\gamma$  satisfying the description above, and is thus well defined.

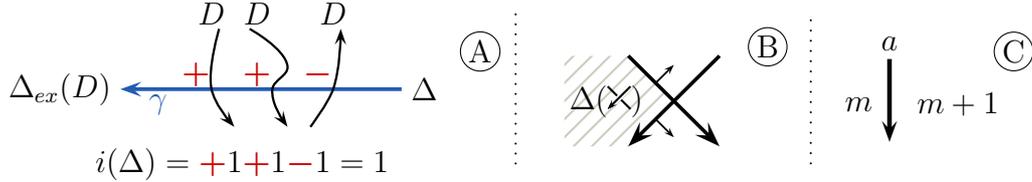


Figure 4: Regions and indices

Until the end of Section 4, suppose  $A$  to be a module over a commutative ring  $R$ , and fix an  $\alpha \in R^*$ . Take an  $\alpha$ -twisted quandle 2-cocycle of  $Q$  — that is, a map  $\omega: Q \times Q \rightarrow A$  satisfying (5) and

$$\omega(a \triangleleft c, b \triangleleft c) - \alpha\omega(a, b) - \omega(a \triangleleft b, c) + \alpha\omega(a, c) + (1 - \alpha)\omega(b, c) = 0. \quad (14)$$

**Definition 2.4.** The (*twisted, or  $\alpha$ -twisted*)  $\omega$ -weight of a  $Q$ -colored knot diagram  $(D, \mathcal{C})$  is the sum

$$\mathcal{W}_\omega^{tw}(D, \mathcal{C}, \alpha) = \sum_{\substack{a \times b}} \varepsilon(\times) \alpha^{-i(\times)} \omega(a, b) \quad (15)$$

taken over all crossings of  $D$ . (Recall that  $\varepsilon(\times)$  is the usual sign of  $\times$ , cf. Figure 2.)

**Theorem 3** ([CES02]). *For any  $\alpha \in R^*$  and any  $\alpha$ -twisted quandle 2-cocycle  $\omega$  of  $Q$ , the multiset  $\{\mathcal{W}_\omega^{tw}(D, \mathcal{C}, \alpha) \mid \mathcal{C} \in \mathcal{C}_Q(D)\}$  yields a knot invariant.*

This knot invariant is called ( $\alpha$ )-twisted quandle cocycle invariant.

To prove Theorems 2 and 3, one could check directly that Reidemeister moves and induced coloring changes preserve the positive/twisted  $\omega$ -weight, as it was done in [CG14, CES02]. Here we prefer a more conceptual solution: in the following sections, the multisets from these theorems are presented as shadow invariants.

### 3 Twisted quandle cocycle invariants via shadows

The first ingredient we need is the  $Q$ -module  $\mathbb{Z}$  with  $m \triangleleft a = m + 1$ . For this  $Q$ -module, the region coloring rule from Figure 1(B) specializes to the one from Figure 4(C). Comparing it with the definition of the index of a region, one proves

<sup>9</sup>The term *Alexander number* is sometimes used instead ([Ale23, CS98, CKS00]).

<sup>10</sup>In our diagrams, a solid crossing stands for a crossing of arbitrary sign.

<sup>11</sup>In other words, it is the region from which all normals to the arcs of  $\times$  point, hence the term.

**Lemma 3.1.** Consider a  $(Q, \mathbb{Z})$ -coloring of a knot diagram  $D$  such that the exterior region receives color 0. Then the color of any region  $\Delta$  coincides with  $i(\Delta)$ .

The region coloring from the lemma will be denoted by  $\mathcal{C}^{ind}: \text{Regions}(D) \rightarrow \mathbb{Z}$ . Remark that it is compatible with any arc coloring.

Next, we show how to transform a twisted quandle 2-cocycle into a (non-twisted) quandle 2-cocycle, at the cost of introducing our non-trivial  $Q$ -module  $\mathbb{Z}$ .

**Lemma 3.2.** A map  $\omega: Q \times Q \rightarrow A$  is an  $\alpha$ -twisted quandle 2-cocycle if and only if the map

$$\begin{aligned} \omega_\alpha: \mathbb{Z} \times Q \times Q &\longrightarrow A, \\ (m, a, b) &\longmapsto \alpha^{-m}\omega(a, b) \end{aligned} \tag{16}$$

is a shadow quandle 2-cocycle.

*Proof.* Conditions (5) for  $\omega$  and (10) for  $\omega_\alpha$  clearly coincide. Further, condition (9) for  $\omega_\alpha$  reads

$$\begin{aligned} \alpha^{-m}\omega(a, b) + \alpha^{-(m+1)}\omega(a \triangleleft b, c) + \alpha^{-m}\omega(b, c) = \\ \alpha^{-(m+1)}\omega(a \triangleleft c, b \triangleleft c) + \alpha^{-m}\omega(a, c) + \alpha^{-(m+1)}\omega(b, c), \end{aligned}$$

which, due to the invertibility of  $\alpha$ , is the same as (14).  $\square$

We further compare  $\alpha$ -twisted  $\omega$ -weights with shadow  $\omega_\alpha$ -weights:

**Lemma 3.3.** For any  $Q$ -colored diagram  $(D, \mathcal{C})$  and any  $\alpha$ -twisted quandle 2-cocycle  $\omega$ , one has

$$\mathcal{W}_\omega^{tw}(D, \mathcal{C}, \alpha) = \mathcal{W}_{\omega_\alpha}^{sh}(D, \mathcal{C}, \mathcal{C}^{ind}).$$

*Proof.* It follows from the definitions and lemmas above.  $\square$

Putting everything together, one gets

**Theorem 4.** For a knot diagram  $D$  and an  $\alpha$ -twisted quandle 2-cocycle  $\omega$ , the multiset  $\{\mathcal{W}_\omega^{tw}(D, \mathcal{C}, \alpha) \mid \mathcal{C} \in \mathcal{C}_Q(D)\}$  of  $\alpha$ -twisted  $\omega$ -weights of  $D$  coincides with the multiset

$$\{\mathcal{W}_{\omega_\alpha}^{sh}(D, \mathcal{C}, \mathcal{C}^{sh}) \mid (\mathcal{C}, \mathcal{C}^{sh}) \in \mathcal{C}_{Q, \mathbb{Z}}^{sh}(D), \mathcal{C}^{sh}(\Delta_{ex}(D)) = 0\}$$

of its  $\mathbb{Z}$ -shadow  $\omega_\alpha$ -weights with the exterior region colored by 0.

*Proof.* According to Lemma 3.3, the first multiset coincides with  $\{\mathcal{W}_{\omega_\alpha}^{sh}(D, \mathcal{C}, \mathcal{C}^{ind}) \mid \mathcal{C} \in \mathcal{C}_Q(D)\}$ . But this is precisely our second multiset, since Lemma 3.1 allows to replace any region  $\mathbb{Z}$ -coloring  $\mathcal{C}^{sh}$  satisfying  $\mathcal{C}^{sh}(\Delta_{ex}(D)) = 0$  with  $\mathcal{C}^{ind}$ .  $\square$

This theorem yields the announced shadow interpretation of twisted quandle cocycle invariants. In particular, combined with Theorem 1, it immediately implies Theorem 3.

## 4 Applications

As applications of our shadow interpretation of twisted quandle cocycle invariants, we now recall some results concerning shadow invariants and discuss their consequences in the twisted setting.

The first such result seems to be folklore. We state it below and give its proof, choosing an argument well adapted for higher-dimensional generalizations.

**Lemma 4.1.** The shadow weights associated to a shadow quandle 2-coboundary are trivial.

*Proof.* Take a  $Q$ -module  $M$ , a map  $\theta: M \times Q \rightarrow A$ , and a  $(Q, M)$ -colored knot diagram  $(D, \mathcal{C}, \mathcal{C}^{sh})$ . Choose a point  $p$  on the knot, and examine the value  $\nu(\gamma) = \theta(\mathcal{C}^{sh}(\Delta(\gamma)), \mathcal{C}(\gamma))$  as you travel along the knot, following its direction; here  $\gamma$  is the arc you are on, and  $\Delta(\gamma)$  is the region to your right. You pass twice through any crossing  $\times$ , and the sum of the increments of  $\nu(\gamma)$  during these two passages is exactly the contribution of  $\times$  to  $\mathcal{W}_{d\theta}^{sh}(D, \mathcal{C})$  (for a positive  $\times$ , it is shown on Figure 5; for a negative one things are similar). Returning to  $p$ , you recover the original value  $\nu(\gamma)$ . Considering such walks along each link component, one concludes that the total increment of  $\nu(\gamma)$  — which is precisely  $\mathcal{W}_{d\theta}^{sh}(D, \mathcal{C})$  — is zero.  $\square$

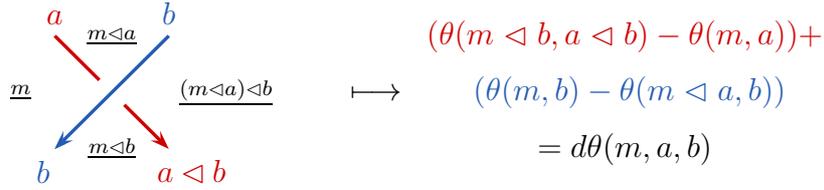


Figure 5: The total cost of passing through a crossing

As a consequence, one obtains

**Proposition 4.2.** Cohomologous twisted quandle 2-cocycles yield the same weights and identical twisted quandle cocycle invariants.

In particular, one can talk about the weight  $\mathcal{W}_{[\omega]}^{tw}$  associated to the twisted quandle 2-cohomology class of  $\omega$ .

*Proof.* To simplify notations, put  $d = d_l^1 - d_r^1$ ,  $d_\alpha = d_l^1 - \alpha d_r^1$ . By the linearity of weights, it suffices to check the triviality of twisted  $d_\alpha \theta$ -weights, with  $\theta: Q \rightarrow A$ . An easy direct verification gives  $(d_\alpha \theta)_\alpha = \alpha d(\theta_\alpha)$  (we use notation (16) and its analogue  $\theta_\alpha(m, a) = \alpha^{-m} \theta(a)$ ). Lemma 3.3 then implies that

$$\mathcal{W}_{d_\alpha \theta}^{tw}(D, \mathcal{C}) = \mathcal{W}_{(d_\alpha \theta)_\alpha}^{sh}(D, \mathcal{C}, \mathcal{C}^{ind}) = \mathcal{W}_{\alpha d \theta_\alpha}^{sh}(D, \mathcal{C}, \mathcal{C}^{ind}) = \alpha \mathcal{W}_{d \theta_\alpha}^{sh}(D, \mathcal{C}, \mathcal{C}^{ind}),$$

which vanishes according to Lemma 4.1.  $\square$

The second application is a shadow enhancement of twisted quandle cocycle invariants. To treat simultaneously the  $Q$ -module  $M$  we want to color regions with and the  $Q$ -module  $\mathbb{Z}$  (with  $m \triangleleft a = m + 1$ ) used in our shadow interpretation of twisted invariants, we use the following elementary observation:

**Lemma 4.3.** The direct product  $M \times M'$  of  $Q$ -modules  $M$  and  $M'$  is also a  $Q$ -module, with the diagonal action  $(m, m') \triangleleft a = (m \triangleleft a, m' \triangleleft a)$ .

Now, repeating verbatim all the arguments from the previous and the beginning of the current sections for the  $Q$ -module  $M \times \mathbb{Z}$  instead of  $\mathbb{Z}$ , one gets

**Theorem 5.** Take a  $Q$ -module  $M$ , a fixed  $m \in M$ , and an  $\alpha$ -twisted  $M$ -shadow 2-cocycle  $\omega$ . The multiset of  $\alpha$ -twisted  $M$ -shadow  $\omega$ -weights

$$\{ \mathcal{W}_\omega^{tw,sh}(D, \mathcal{C}, \mathcal{C}^{sh}, \alpha) \mid (\mathcal{C}, \mathcal{C}^{sh}) \in \mathcal{C}_{Q,M}^{sh}(D), \mathcal{C}^{sh}(\Delta_{ex}(D)) = m \},$$

where 
$$\mathcal{W}_\omega^{tw,sh}(D, \mathcal{C}, \mathcal{C}^{sh}, \alpha) = \sum_{\substack{a \\ \times \\ b}} \varepsilon(\times) \alpha^{-i(\times)} \omega(\mathcal{C}^{sh}(\Delta(\times)), a, b),$$

is a knot invariant<sup>12</sup>. Moreover, cohomologous 2-cocycles yield identical invariants.

The third bonus from our shadow interpretation of twisted quandle cocycle invariants is their higher-dimensional generalization. Indeed, take a  $Q$ -colored  $k - 1$ -dimensional knot diagram  $(D, \mathcal{C})$  in  $\mathbb{R}^k$ , and an  $\alpha$ -twisted quandle  $k$ -cocycle  $\omega$ . Define the  $\alpha$ -twisted  $\omega$ -weight of  $(D, \mathcal{C})$  by a formula analogous to (15), using higher-dimensional versions of indices and signs for crossings of multiplicity  $k$ , and a relevant ordering of adjacent sheets. Lemmas 3.1-3.3 generalize directly. The multiset of  $\alpha$ -twisted  $\omega$ -weights is thus interpreted in terms of higher-dimensional shadow invariants (for more details about the latter, see for instance [FR92, CKS04, PR13]). Moreover, this construction admits a shadow version, similar to that from Theorem 5. Hence any  $\alpha$ -twisted (possibly shadow) quandle  $k$ -cocycle gives rise to a  $k - 1$ -dimensional knot invariant.

## 5 Positive quandle cocycle invariants via shadows

This section is devoted to the case  $\alpha = -1$ . For this  $\alpha$ , the twisted quandle cohomology theory is precisely the positive quandle cohomology theory from [CG14]. In order to show that positive and  $-1$ -twisted quandle cocycle invariants coincide, it remains to compare the weights used in the two constructions:

**Lemma 5.1.** For any  $Q$ -colored diagram  $(D, \mathcal{C})$  and any positive quandle 2-cocycle  $\omega$ , one has  $\mathcal{W}_\omega^{tw}(D, \mathcal{C}, -1) = \mathcal{W}_\omega^{pos}(D, \mathcal{C})$ .

*Proof.* It is sufficient to show that for any crossing  $\times$ , the signs  $\varepsilon(\times)(-1)^{-i(\times)}$  and  $\varepsilon^{pos}(\times)$  coincide. First, observe that, modulo 2, the coloring  $\mathcal{C}^{ind}$  reduces to the checkerboard coloring, under the identification  $0 = \text{white}$ ,  $1 = \text{black}$ . Using this identification, the definition of  $\varepsilon^{pos}(\times)$  (Figure 3) can be restated as on Figure 6 (note that we turned both diagrams so that the over-arc points south-west). Now, if the under-arc points south-east, then  $\varepsilon(\times)$  is 1, and  $i(\times)$  is the color of the region to the west of  $\times$ ; one checks that the latter satisfies  $(-1)^{-i(\times)} = \varepsilon^{pos}(\times)$ , as desired. A change in the orientation of the under-arc changes  $\varepsilon(\times)$  and  $i(\times)$ , but not  $\varepsilon^{pos}(\times)$ , so equality  $\varepsilon(\times)(-1)^{-i(\times)} = \varepsilon^{pos}(\times)$  is preserved.  $\square$

<sup>12</sup>For notations, refer to Definition 2.3.



Figure 6: Sign  $\varepsilon^{pos}$  via region indices

Thus positive quandle cocycle invariants coincide with  $-1$ -twisted ones. Combining this with Theorem 4, one obtains

**Theorem 6.** *For a knot diagram  $D$  and a positive quandle 2-cocycle  $\omega$ , the multiset  $\{ \mathcal{W}_\omega^{pos}(D, \mathcal{C}) \mid \mathcal{C} \in \mathcal{C}_Q(D) \}$  of positive  $\omega$ -weights of  $D$  coincides with the multiset*

$$\{ \mathcal{W}_{\omega_{-1}}^{sh}(D, \mathcal{C}, \mathcal{C}^{sh}) \mid (\mathcal{C}, \mathcal{C}^{sh}) \in \mathcal{C}_{Q, \mathbb{Z}_2}^{sh}(D), \mathcal{C}^{sh}(\Delta_{ex}(D)) = 0 \}$$

of its  $\mathbb{Z}_2$ -shadow  $\omega_{-1}$ -weights with the exterior region colored by 0.

Note that we replaced our  $Q$ -module  $\mathbb{Z}$  with  $\mathbb{Z}_2$ , with the same module operation  $m \triangleleft a = m + 1$ ; this is possible since  $(-1)^m$  is completely determined by  $m \bmod 2$ .

The theorem interprets positive quandle cocycle invariants as shadow ones. This directly shows their invariance, and gives for free their shadow enhancement, answering a question raised in [CG14].

## 6 Twisted quandle cocycle invariants for links

For links with several components, twisted quandle cocycle invariants can be sharpened using a method developed in this section. In what follows, the diagram  $D$  represents a link with  $k$  ordered components, numbered  $1, \dots, k$ .

First, we need a more subtle notion of index.

**Definition 6.1.**

- The  $j$ th index  $i_j(\Delta)$  of a region  $\Delta$  in  $D$  is the index of  $\Delta$  with respect to the  $j$ th component of  $D$ , in the sense of Definition 2.3.
- The  $j$ th index of a crossing  $\bowtie$  is set to be  $i_j(\bowtie) = i_j(\Delta(\bowtie))$ .

For a quandle  $(Q, \triangleleft)$ , consider the equivalence relation induced by  $a \sim a \triangleleft b$ . Corresponding equivalence classes are called *orbits*, their set is denoted by  $Orb(Q)$ , and the orbit of an  $a \in Q$  is denoted by  $\mathcal{O}(a)$ . A  $Q$ -coloring  $\mathcal{C}$  of  $D$  assigns elements of the same orbit to all the arcs belonging to the same component of  $D$ ; for the  $j$ th component, this orbit is denoted by  $\mathcal{C}_*(j)$ .

As before, let  $A$  be a module over a commutative ring  $R$ . Take a collection  $\bar{\alpha} = (\alpha_{\mathcal{O}} \in R^*)_{\mathcal{O} \in Orb(Q)}$ .

**Definition 6.2.**

- An  $\bar{\alpha}$ -twisted quandle 2-cocycle is a map  $\omega: Q \times Q \rightarrow A$  satisfying (5) and

$$\alpha_{\mathcal{O}(c)}^{-1} \omega(a \triangleleft c, b \triangleleft c) - \omega(a, b) - \alpha_{\mathcal{O}(b)}^{-1} \omega(a \triangleleft b, c) + \omega(a, c) + (\alpha_{\mathcal{O}(a)}^{-1} - 1) \omega(b, c) = 0. \quad (17)$$

- The ( $\bar{\alpha}$ -twisted)  $\omega$ -weight of a  $Q$ -colored link diagram  $(D, \mathcal{C})$  is the sum

$$\mathcal{W}_\omega^{tw}(D, \mathcal{C}, \bar{\alpha}) = \sum_{\substack{\bowtie \\ a \bowtie b}} \varepsilon(\bowtie) \alpha_{\mathcal{C}_*(1)}^{-i_1(\bowtie)} \cdots \alpha_{\mathcal{C}_*(k)}^{-i_k(\bowtie)} \omega(a, b). \quad (18)$$

**Theorem 7.** *The multiset  $\{\mathcal{W}_\omega^{tw}(D, \mathcal{C}, \bar{\alpha}) \mid \mathcal{C} \in \mathcal{C}_Q(D)\}$  depends only on the underlying link (and not on the choice of its diagram  $D$ ), and only on the  $\bar{\alpha}$ -twisted quandle cohomology class  $[\omega]$  of  $\omega$ .*

The theorem is proved by interpreting the constructed multisets as shadow invariants. The  $Q$ -module one should use here is the following one:

**Lemma 6.3.** The set  $\bigoplus_{\mathcal{O} \in \text{Orb}(Q)} \mathbb{Z}e_{\mathcal{O}}$  with  $m \triangleleft a = m + e_{\mathcal{O}(a)}$  is a  $Q$ -module.

*Proof.* One has to check that  $e_{\mathcal{O}(b)} + e_{\mathcal{O}(c)} = e_{\mathcal{O}(c)} + e_{\mathcal{O}(b \triangleleft c)}$ , which is obvious since, by definition,  $b$  and  $b \triangleleft c$  belong to the same orbit.  $\square$

As usual, our shadow interpretation gives for free shadow and higher-dimensional generalizations of the theorem.

*Remark 6.4.* Following [Ino13], the multiset from the theorem can be decomposed into multisets  $\{\mathcal{W}_\omega^{tw}(D, \mathcal{C}, \bar{\alpha}) \mid \mathcal{C} \in \mathcal{C}_Q(D), \mathcal{C}_*(1) = \mathcal{O}_1, \dots, \mathcal{C}_*(k) = \mathcal{O}_k\}$ , indexed by  $k$ -tuples of orbits  $\mathcal{O}_1, \dots, \mathcal{O}_k \in \text{Orb}(Q)$ . The invariance of these smaller multisets follows from the fact that Reidemeister moves and induced local coloring changes preserve the orbit of the colors assigned to the arcs of a given link component.

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