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THE GAUSS-BONNET OPERATOR OF AN INFINITE GRAPH

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Abstract. — We propose a general condition, to ensure essential self-adjointness for the Gauß-Bonnet operator $D = d + \delta$, based on a notion of completeness as Chernoff. This gives essential self-adjointness of the Laplace operator both for functions and 1-forms on infinite graphs. This is used to extend Flanders result concerning solutions of Kirchhoff's laws.

Résumé. Nous proposons une condition générale qui assure le caractère essentiellement auto-adjoint de l'opérateur de Gauss-Bonnet $D = d + \delta$, basée sur une notion de complétude comme Chernoff. Comme conséquence, l'opérateur de Laplace agissant sur les fonctions et les 1-formes de graphes infinis est essentiellement auto-adjoint. Nous utilisons ce cadre pour étendre le résultat de Flanders à propos des solutions des lois de Kirchhoff.

1. Introduction

Operators on infinite graphs are of large interest and a lot of recent works deals with this subject. One approach can be to study how techniques of spectral geometry can be extended on graphs regarded as one-dimensional simplicial complexes. We refer to Dodziuk [D84, DK87] for general presentation of this approach and to [CdV98, CTT11] for the geometric point of view, and also [CdV91] for the relation between Kirchhoff's laws and Hodge theory.

We consider here only connected locally finite infinite graphs and we study Kirchhoff's laws. Flanders has first studied this question on infinite graphs seen as infinite electric networks, see [F71]. Several authors have clarified and extended Flanders work on electric networks, see for instance Thomassen [T90], Soardi [S94], Doyle & Snell [DS99], Zemanian [Z08], Georgakopoulos [G10], Carmesin [Cm12] and also the book of Jorgensen & Pearse [JP14] for a general approach.
Flanders main result is that there exists a unique current flow in an infinite network with a finite number of sources which is the limit of flows with finite support.

In our paper, this question is approached by the study of a Dirac type operator: *the Gauß-Bonnet operator* $D = d + \delta$, introduced on an infinite graph considered as a one-dimentional simplicial complex. Indeed, this operator is a generalisation of the Dirac operator studied on $\mathbb{Z}$ by Golenia & Haugomat in [GH12]. We give a general condition on the graph by defining the notion of $\chi-$*completeness*, see Section 3.2. One of the main results is essential self-adjointeness of the Gauß-Bonnet operator, when the graph is $\chi-$complete (or *complete homogeneous*). This condition

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covers the situations of [M09], and [T10] (or [T12]), it is satisfied by locally finite graphs which are complete for some intrinsic pseudo metric, as defined in [HKMW13] (although the results of [HKMW13] are valid in a more general context of graphs not necessarily locally finite), and it is a discrete version of a result of Chernoff, see [Ch73], in the case of manifolds. One of the applications in his paper concludes that, on a complete manifold, every power of the Dirac operator $d+\delta$ is essentially self-adjoint. In particular, for every power of the Laplace-Beltrami operator, essential self-adjointness is true.

In Section 4.3, we define the property of *positivity at infinity* for Dirac type operators. And by adding this assumption on our Gauß-Bonnet operator, we prove that its range is closed and consequently the Hodge property holds, in a similar result as Anghel's for compact Riemannian manifold, see [A93]. This situation permits us to enlarge the conditions on the current source and the voltage source in the Flanders problem. In Section 5, we give new examples of infinite graphs where it applies.

2. Preliminaries

2.1. **Definitions on Graphs.** (cf. [LP14]) A graph K is a simplicial complex of dimension one. We denote by $\mathcal{V}$ the set of vertices and $\mathcal{E}$ the set of *oriented edges*, considered as a subset of $\mathcal{V}\times\mathcal{V}$. We assume that $\mathcal{E}$ is symmetric without loops:

$$v\in\mathcal{V}\Rightarrow (v,v)\notin\mathcal{E},\quad (v_1,v_2)\in\mathcal{E}\Rightarrow (v_2,v_1)\in\mathcal{E}.$$

Choosing an orientation of the graph consists of defining a partition of $\mathcal{E}$:

$$\begin{aligned}&\mathcal{E}^+\sqcup\mathcal{E}^- = \ \mathcal{E}\\ &(v_1,v_2)\in\mathcal{E}^+ \Longleftrightarrow (v_2,v_1)\in\mathcal{E}^-.\end{aligned}$$

For $e=(v_1,v_2)\in\mathcal{E}$, let's set

$$e^+=v_2,\ e^-=v_1,\ -e=(v_2,v_1).$$

e^+ and e^- are called boundary points of the edge e.

2.1.1. A *path* between two vertices x,y in $\mathcal{V}$ is a finite set of edges $e_1,\dots,e_n, n\geq 1$ such that

$$e_1^-=x,\ e_n^+=y\ \text{ and, if }\ n\geq 2,\ \forall j,\ \ 1\leq j\leq (n-1)\Rightarrow e_j^+=e_{j+1}^-.$$

Notice that each path has a beginning and an end, and that an edge is a path.
Let us denote Γ_{xy} the set of the paths from the vertex x to the vertex y.

2.1.2. The graph is *connected* if two vertices are always related by a path, *ie.* if Γ_{xy} is non empty for all x,y in $\mathcal{V}$.

2.1.3. The graph is *locally finite* if each vertex belongs to a finite number of edges. The *degree* or *valence* of a vertex $x\in\mathcal{V}$ is the cardinal of the set $\{e\in\mathcal{E};\ e^+=x\}$.

2.1.4. A *subgraph* of a graph K is a graph $K_0 = (\mathcal{V}_0, \mathcal{E}_0)$ such that $\mathcal{V}_0 \subset \mathcal{V}$ and $\mathcal{E}_0 \subset \mathcal{E}$.

Remark 1. *All the graphs we shall consider on the sequel will be connected, locally finite, so with countably many vertices.*

2.2. **Functions and forms.** The $0-$cochains are just scalar functions on $\mathcal{V}$, we denote the set by $C^0(K)$.
The $1-$cochains or forms are odd scalar functions on $\mathcal{E}$ we denote the set by $C^1(K)$. Thus we have

$$C^0(K) = \mathbb{C}^{\mathcal{V}},$$
$$C^1(K) = \{\varphi : \mathcal{E} \to \mathbb{C}, \varphi(-e) = -\varphi(e)\}.$$

The sets of cochains with finite support are denoted by $C_0^0(K),\ C_0^1(K)$.
To obtain Hilbert spaces we need weights, let us give

$$c : \mathcal{V} \to \mathbb{R}_+^*,$$

and

$$r : \mathcal{E} \to \mathbb{R}_+^* \text{ even}$$

so $r(-e) = r(e)$.
They define scalar products:

$$\forall\ f, g \in C_0^0(K)\,;\ \langle f, g\rangle = \sum_{v\in\mathcal{V}} c(v) f(v) \bar{g}(v)$$

$$\forall\ \varphi, \psi \in C_0^1(K)\,;\ \langle \varphi, \psi\rangle = \frac{1}{2}\sum_{e\in\mathcal{E}} r(e)\varphi(e)\bar{\psi}(e) \tag{1}$$

Remark 2. *As the products $r(e)\varphi(e)\bar{\psi}(e),\ e \in \mathcal{E}$ in (1) are even, the term $\dfrac{1}{2}$ allows to recover the usual definition.*

Remark 3. *In the context of electric networks, our weight on edges would play the role of the conductance, the intensity would be on $e \in \mathcal{E} : I(e) = r(e)\varphi(e)$ and the energy $\|\varphi\|^2 = \frac{1}{2}\sum_{e\in\mathcal{E}} \frac{1}{r(e)} I(e)^2$. So, indeed, $\frac{1}{r(e)}$ is the resistance of the edge e!*

Let us finally define the Hilbert spaces $L_2(\mathcal{V})$ and $L_2(\mathcal{E})$ as the sets of cochains with finite norm, we have

$$L_2(\mathcal{V}) = \overline{C_0^0(K)},$$
$$L_2(\mathcal{E}) = \overline{C_0^1(K)}.$$

and put

$$\mathcal{H} = L_2(\mathcal{V}) \oplus L_2(\mathcal{E}), \forall F = (f, \varphi) \in \mathcal{H},\ \|F\|^2 = \|f\|^2 + \|\varphi\|^2. \tag{2}$$

Comment. $L_2(\mathcal{V})$ and $L_2(\mathcal{E})$ can be considered as subspaces of $\mathcal{H}$, this justifies that all the L_2-norms have the same notation.

2.3. **Operators.**

2.3.1. *The difference operator.* It is the operator
$$\mathrm{d} : C_0^0(K) \to C_0^1(K),$$
given by
$$\mathrm{d}(f)(e) = f(e^+) - f(e^-), \tag{3}$$
for $f \in C_0^0(K), e \in \mathcal{E}$.

2.3.2. *The coboundary operator.* It is δ the formal adjoint of d.
Thus it satisfies
$$\langle \mathrm{d}f, \varphi\rangle = \langle f, \delta\varphi\rangle \tag{4}$$
for all $f \in C_0^0(K)$ and $\varphi \in C_0^1(K)$.

Lemma 4. *The coboundary operator* $\delta : C_0^1(K) \to C_0^0(K)$, *acts as*
$$\delta(\varphi)(x) = \frac{1}{c(x)} \sum_{e, e^+ = x} r(e)\varphi(e). \tag{5}$$

Proof. — Using the equation (4), we have
$$\frac{1}{2} \sum_{e\in\mathcal{E}} r(e) \left(f(e^+) - f(e^-)\right) \bar{\varphi}(e) = \frac{1}{2} \sum_{x\in\mathcal{V}} f(x) \overline{\Big(\sum_{e^+ = x} r(e)\varphi(e) - \sum_{e^- = x} r(e)\varphi(e)\Big)}$$
But $r\varphi$ is odd and $\mathcal{E}$ symmetric, so
$$\sum_{e^- = x} r(e)\varphi(e) = -\sum_{e^+ = x} r(e)\varphi(e).$$
We remark that the sum entering in the formula (5) of δ is finite due to the hypothesis that the graph is locally finite. □

Remark 5. *The operator* d *is defined by (3) on* $C^0(K)$*, but to define* δ *on* $C^1(K)$*, we need an hypothesis on* K*: we suppose that the graph is locally finite. This hypothesis could be weakened by assuming that the edge weights* $r(e)$, $e \in \mathcal{E}$ *are summable around each vertex as considered in* [KL12].

With these two operators we can define the following two operators.

2.3.3. *The Gauß-Bonnet operator.* It is the endomorphism
$$D = \mathrm{d} + \delta : C_0^0(K) \oplus C_0^1(K) \to C_0^0(K) \oplus C_0^1(K)$$
given by
$$D(f, \varphi) = (\delta\varphi, \mathrm{d}f)$$
for all $f \in C_0^0(K)$ and $\varphi \in C_0^1(K)$.

Remark 6. *On a locally finite graph, the operator* D *extends to* $C^0(K) \oplus C^1(K)$ *and we still denote it* D *if there is no confusion.*

The domain $C_0^0(K) \oplus C_0^1(K)$ of D is dense in the Hilbert space $\mathcal{H}$ (defined in (2)). This operator is symmetric and of Dirac type, *i.e.* D^2 is of Laplace type.

2.3.4. *Laplacian.* By definition, it is

$$\Delta = D^2 : C_0^0(K) \oplus C_0^1(K) \circlearrowleft .$$

This operator preserves the direct sum $C_0^0(K) \oplus C_0^1(K)$, so we can write

$$\Delta = \Delta_0 \oplus \Delta_1.$$

2.4. **Metrics.** By analogy to Riemannian geometry, we call *metric* an even function

$$a : \mathcal{E} \to \mathbb{R}_+^*.$$

It defines a distance on the graph K in the following way.
One first defines the *length of a path*: for $\gamma = (e_1, \dots, e_n)$

$$l_a(\gamma) = \sum_{j=1}^{n} \sqrt{a(e_j)}.$$

Then the *metric distance* between two vertices x, y is given by

$$d_a(x,y) = \inf_{\gamma \in \Gamma_{xy}} l_a(\gamma).$$

3. Closability and Self-adjointness

3.1. **Closability.**

Lemma 7. *If the graph K is connected and locally finite the operators* d *and* δ *are closable.*

Proof. — Let us suppose that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_0^0(K)$ such that $\|f_n\| \to 0$ and $(\mathrm{d}(f_n))_n$ converges. Let us denote by φ this limit.
We have to show that $\varphi = 0$. If

$$\|f_n\| + \|\mathrm{d}(f_n) - \varphi\| \to 0,$$

then for each vertex v, $f_n(v)$ converges to 0 and for each edge e, $\mathrm{d}(f_n)(e)$ converges to $\varphi(e)$. But by the first statement and the expression of d, for each edge e, $\mathrm{d}(f_n)(e)$ converges to 0.

The same can be done for δ: convergence in norm to 0 of a sequence $(\varphi_n)_n$ implies pointwise convergence to 0 which implies pointwise convergence of $\delta(\varphi_n)$ to 0, because of local finiteness of the graph ; if $\delta(\varphi_n)$ converges in norm, it must be to 0. □

Thus, we can consider different extensions of these operators in the framework of Hilbert spaces (see [RS80]).

The smallest extension is the closure, denoted $\bar{d} = \mathrm{d}_{min}$ (resp. $\bar{\delta} = \delta_{min}$ and $\bar{D} = D_{min}$) has the domain

$$\mathrm{Dom}(\mathrm{d}_{min}) = \{ f \in L_2(\mathcal{V});\ \exists (f_n)_{n\in\mathbb{N}},\ f_n \in C_0^0(K),\ L_2 - \lim_{n\to\infty} f_n = f, \\ L_2 - \lim_{n\to\infty} \mathrm{d}(f_n) \text{ exists } \} \quad (6)$$

for such an f, one puts
$$\mathrm{d}_{min}(f) = \lim_{n\to\infty} \mathrm{d}(f_n).$$

The largest is $\mathrm{d}_{max} = \delta^*$, the adjoint operator of δ_{min}, (resp. $\delta_{max} = \mathrm{d}^*$, the adjoint operator of d_{min}.)

3.2. **A sufficient condition for self-adjointness of D.**

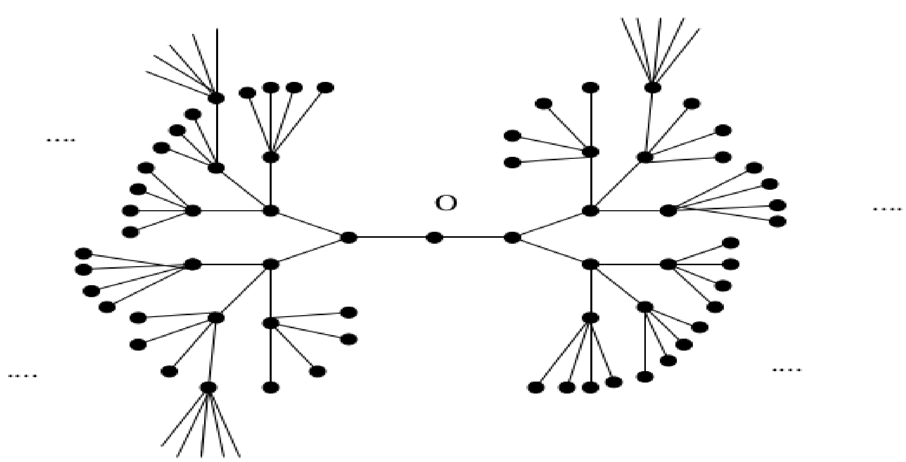
3.2.1. *Geometric hypothesis for the graph K.*

Definition 8. *The graph K is* $\chi-$complete *if there exists a increasing sequence of finite sets $(B_n)_{n\in\mathbb{N}}$ such that $\mathcal{V} = \Cup B_n$ and there exist related functions χ_n satisfying the following three conditions:*

(i) $\chi_n \in C_0^0(K)$, $0 \le \chi_n \le 1$
(ii) $v \in B_n \;\Rightarrow\; \chi_n(v) = 1$
(iii) $\exists C > 0, \forall n \in \mathbb{N},\, x \in \mathcal{V},\; \dfrac{1}{c(x)} \displaystyle\sum_{e, e^{\pm}=x} r(e)\mathrm{d}\chi_n(e)^2 \le C.$

For this type of graphs one has
$$\forall p \in \mathbb{N},\, \exists n_p,\, n \ge n_p \Rightarrow\; \forall e \in \mathcal{E},\text{ such that } e^+ \text{ or } e^- \in B_p,\; \mathrm{d}\chi_n(e) = 0 \tag{7}$$
$$\mathcal{E} = \Cup \mathcal{E}_n \text{ if } \mathcal{E}_n = \{e \in \mathcal{E}, e^+ \in B_n \text{ or } e^- \in B_n\} \tag{8}$$
$$\forall f \in L_2(\mathcal{V}),\; \lim_{n\to\infty} < \chi_n f, f >= \|f\|^2 \tag{9}$$
$$\forall \varphi \in L_2(\mathcal{E}), \|\varphi\|^2 = \lim_{n\to\infty} \frac{1}{2}\sum_{e\in\mathcal{E}} r(e)\chi_n(e^+)|\varphi(e)|^2 \tag{10}$$
$$\text{and } \lim_{n\to\infty} \sum_{e\in \mathrm{supp}(\mathrm{d}\chi_n)} r(e)|\varphi(e)|^2 = 0. \tag{11}$$

Example 9. Let us consider an infinite tree with increasing valence:



Taking constant weights on vertices and edges, this graph is $\chi-$complete. Indeed, one can define *generations* of vertices on such a graph: the considered origin vertex O is of generation 0 and valence 2, it is related to two vertices which are of generation 1 and valence 3, and more generally there are $2n!$ vertices of generation n and valence $(n + 2)$.

One defines $B_n, n \in \mathbb{N}$, as the set of vertices of generation less than n^2 and χ_n constant on each generation of vertices:

$$x \text{ of generation } p \Rightarrow \chi_n(x) = \Big(\frac{(n+1)^2 - p}{2n+1} \wedge 1\Big) \vee 0.$$

So, $p \leq n^2 \Rightarrow \chi_n(x) = 1$ and $p \geq (n+1)^2 \Rightarrow \chi_n(x) = 0$ while $|d\chi_n(e)| \leq 1/(2n+1)$ is in fact supported on edges between generations larger than n^2 and less than $(n+1)^2$. To verify the condition (iii), one has to calculate for these generations, $(p+2)/(2n+1)^2 \leq ((n+1)^2+2)/(2n+1)^2$ which is bounded independently on n.

Remark 10. *The condition of $\chi-$completeness covers many situations that have been already studied. Particularily it is satisfied by locally finite graphs which are complete for some intrinsic pseudo metric, as defined in* [HKMW13].

Lemma 11. *If the graph admits an* intrinsic path metric *d such that $(\mathcal{V}, d)$ is complete, then the graph is $\chi-$complete.*

Proof. — The hypothesis means that our infinite, connected, locally finite, weighted graph admits a metric a as defined in Section 2.4 such that

$$\forall x \in \mathcal{V},\, \frac{1}{c(x)} \sum_{e \in \mathcal{E}, e^+ = x} r(e)a(e) \leq 1$$

(the relation between our notations and those of [HKMW13] is: $\sigma^2 = a$). We suppose also that the metric distance d_a defines $(\mathcal{V}, d_a)$ as a complete metric space.

We then define the functions χ_n as follows. Fix O a vertex in $\mathcal{V}$ and put

$$\forall n \in \mathbb{N},\, B_n = \{x \in \mathcal{V};\, d_a(O,x) \leq n\},\, \chi_n(x) = \sup\{(1 - d_a(x, B_n)), 0\} \tag{12}$$

As pointed in [HKMW13] completeness of $(\mathcal{V}, d_a)$ gives that the B_n are finite. We verify that

(i) The support of χ_n is finite: it is included in $\{x;\ d_a(x, B_n) \leq 1\} \subset B_{n+1}$.
(ii) $x \in B_n \Rightarrow d_a(x, B_n) = 0 \Rightarrow \chi_n(x) = 1$
(iii) finally, by the triangle inequality, $d\chi_n(e)^2 \leq a(e)$; then the condition of *intrinsic metric* gives:

$$\forall x \in \mathcal{V},\, \frac{1}{c(x)} \sum_{e \in \mathcal{E}, e^+ = x} r(e)d\chi_n(e)^2 \leq 1.$$

$\square$

Remark 12. *If we consider the metric already introduced in* [CTT11] *(but to study non complete situations)*

$$a(e) = \frac{\min(c(e^+), c(e^-))}{r(e)} \tag{13}$$

and with bounded valence:

$$\exists A > 0,\, \forall v \in \mathcal{V}, \quad \sharp\{e \in \mathcal{E},\, e^+ = v\} \leq A.$$

then, if the graph is complete for this metric, χ-completeness is also satisfied. Indeed, $\frac{a}{A}$ is an intrinsic metric, because:

$$\forall x \in \mathcal{V},\ \frac{1}{c(x)} \sum_{e \in \mathcal{E}, e^+ = x} r(e) a(e) \leq A.$$

It is also the case in the situation of [M09] *where the hypothesis taken give that*

$$\sup_{x \in B_n} \frac{1}{c(x)} \sum_{e, e^{\pm} = x} r(e) \mathrm{d}\chi_n(e)^2 = o(1)$$

for some χ_n satisfying $\mathrm{d}\chi_n(e)^2 = O(n^{-2})$.

Theorem 1. *Let K be a connected, locally finite graph. If K is χ-complete, then the operator D is essentially self-adjoint.*

Proof. — First note that

(a) If $\mathrm{d}_{min} = \mathrm{d}_{max}$ *and* $\delta_{min} = \delta_{max}$ *then D is essentially self-adjoint.* Indeed, D is a direct sum and if $F = (f, \varphi) \in \mathrm{Dom}(D^*)$ then $\varphi \in \mathrm{Dom}(d^*)$ and $f \in \mathrm{Dom}(\delta^*)$ and then, by hypothesis, $\varphi \in \mathrm{Dom}(\delta_{min})$ and $f \in \mathrm{Dom}(d_{min})$, thus $F \in \mathrm{Dom}(\bar{D})$.

For the following we need some formulas taken in [M09]. First we set, for each $f \in C^0(K)$

$$\tilde{f}(e) = \frac{1}{2}(f(e^+) + f(e^-)). \tag{14}$$

The function $\tilde{f}$ is even on the edges. We have

$$\forall f, g \in C^0(K),\ \forall e \in \mathcal{E}, \quad \mathrm{d}(fg)(e) = f(e^+)\mathrm{d}g(e) + \mathrm{d}f(e)g(e^-) \tag{15}$$
$$= \tilde{f}(e)\mathrm{d}g(e) + \tilde{g}(e)\mathrm{d}f(e)$$

$$\forall f \in C^0(K), \varphi \in C^1(K),\ \forall v \in \mathcal{V}, \quad \delta(\tilde{f}\varphi)(v) = f(v)\delta\varphi(v) - \frac{1}{2c(v)} \sum_{e^+ = v} r(e)\mathrm{d}f(e)\varphi(e). \tag{16}$$

We prove now these two equalities.

(b) If $f \in$ $\mathrm{Dom}(\mathrm{d}_{max})$ *then* $\|(f - \chi_n f)\| + \|\mathrm{d}(f - \chi_n f)\| \to 0$ *when $n \to \infty$.* This will show that $\mathrm{d}_{min} = \mathrm{d}_{max}$.

Let $f \in \mathrm{Dom}(\mathrm{d}_{max})$, we can then calculate

$$\|(f - \chi_n f)\|^2 \leq \sum_{v \notin B_n} c(v)|f(v)|^2 \overset{n\to\infty}{\longrightarrow} 0$$

because $f \in L_2(\mathcal{V})$. For the second term, the relation (15) gives

$$\mathrm{d}(f - \chi_n f)(e) = (1 - \chi_n)(e^+)\mathrm{d}(f)(e) - f(e^+)\mathrm{d}(\chi_n)(e).$$

Because of (10) (and with an abuse of notation),

$$\lim_{n\to\infty} \|(1 - \chi_n)(e^+)\mathrm{d}(f)(e)\| = 0$$

On the other hand,

$$\begin{aligned}
\|f(e^+)\mathrm{d}(\chi_n)(e)\|^2 &= \sum_{e\in\mathcal{E}} r(e)|f(e^+)|^2|\mathrm{d}(\chi_n)(e)|^2 \\
&= \sum_{x\in\mathcal{V}} |f(x)|^2 \sum_{e^+=x} r(e)|\mathrm{d}(\chi_n)(e)|^2 \\
&\leq \sum_{x\in\mathcal{V},\exists e\in \mathrm{supp}(d\chi_n), e^+=x} Cc(x)|f(x)|^2
\end{aligned}$$

by the hypothesis (iii). The property (8) permits to conclude that this term tends to 0 as $n\to\infty$.

(c) If $\varphi \in \mathrm{Dom}(\delta_{max})$ then $\|(\varphi-\tilde{\chi}_n\varphi)\| + \|\delta(\varphi-\tilde{\chi}_n\varphi)\| \to 0$ when $n\to\infty$. This will show that $\delta_{min}=\delta_{max}$.

Let $\varphi\in\mathrm{Dom}(\delta_{max})$, by the properties (7) and (8) we know that

$$\forall p\in\mathbb{N},\ \forall n\geq n_p,\quad \|\varphi-\tilde{\chi}_n\varphi\|^2 \leq \sum_{e\in\mathcal{E}_p^c} r(e)|\varphi(e)|^2$$

so $\lim\limits_{n\to\infty}\|\varphi-\tilde{\chi}_n\varphi\|=0$.

On the other hand, by (16)

$$\begin{aligned}
\delta(\varphi-\tilde{\chi}_n\varphi)(v) &= \delta\left(\widetilde{(1-\chi_n)}\varphi\right)(v) \\
&= (1-\chi_n)(v)\delta\varphi(v) + \frac{1}{2c(v)}\sum_{e^+=v} r(e)\mathrm{d}\chi_n(e)\varphi(e)
\end{aligned}$$

Clearly

$$\lim_{n\to\infty}\|(1-\chi_n)\delta\varphi\|=0$$

because $\delta\varphi\in L_2(\mathcal{V})$. For the second term, we use (iii) and the Cauchy-Schwarz inequality:

$$\begin{aligned}
\forall v\in\mathcal{V},\ \Big|\sum_{e^+=v} r(e)\mathrm{d}\chi_n(e)\varphi(e)\Big|^2 &\leq \sum_{e^+=v} r(e)|\mathrm{d}\chi_n(e)|^2 \sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n),e^+=v} r(e)|\varphi(e)|^2 \\
&\leq Cc(v)\sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n),e^+=v} r(e)|\varphi(e)|^2
\end{aligned}$$

$$\begin{aligned}
\text{so, } \sum_{v\in\mathcal{V}} c(v)\Big|\frac{1}{2c(v)}\sum_{e^+=v} r(e)\mathrm{d}\chi_n(e)\varphi(e)\Big|^2 &\leq C\sum_{v\in\mathcal{V}}\sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n),e^+=v} r(e)|\varphi(e)|^2 \\
&\leq C\sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n)} r(e)|\varphi(e)|^2.
\end{aligned}$$

This term tends to 0 by properties (7) and (8). □

Proposition 13. *Let K be a connected, locally finite graph. The operator D is essentially self-adjoint if and only if the operator Δ is essentially self-adjoint.*

Proof. — If D is essentially self-adjoint, then $\mathrm{Im}(D \pm i)$ are dense and $(\bar{D} \pm i)$ are invertible. This is a result for essentially self-adjoint operators (Corollary of Theorem VIII.3 in [RS80]). By the second property we know that

$$\exists C_2 > 0, \forall F \in \mathrm{Dom}(\bar{D}),\ \|F\|_{L_2} \leq C_2 \|(\bar{D} \pm i)(F)\|_{L_2}. \tag{17}$$

Note also that

$$D(C_0^0(K) \oplus C_0^1(K)) \subset C_0^0(K) \oplus C_0^1(K).$$

Now, by the theorem of von Neumann, $(\bar{D})^2 = D^* \bar{D}$ is self-adjoint when $D^* = \bar{D}$ and it is an extension of Δ. As a consequence, the domain of $(\bar{D})^2$ contains the domain of $\bar{\Delta}$, the closure of Δ. But

$$\mathrm{Dom}(\bar{\Delta}) \subset \mathrm{Dom}((\bar{D})^2) \Rightarrow \mathrm{Dom}((\bar{D})^2) \subset \mathrm{Dom}(\Delta^*).$$

In fact, we have also $\mathrm{Dom}(\Delta^*) \subset \mathrm{Dom}((\bar{D})^2)$: let $\Psi \in \mathrm{Dom}(\Delta^*)$, then

$$\exists C_1 > 0,\ \forall F \in C_0^0(K) \oplus C_0^1(K),\quad |\langle (\Delta + 1)(F), \Psi \rangle| \leq C_1 \|F\|_{L_2}.$$

We now consider the linear form defined on $C_0^0(K) \oplus C_0^1(K)$, by

$$G \longmapsto \langle (D - i)G, \Psi \rangle$$

For all $G \in \mathrm{Im}(D + i)$, $\exists F \in C_0^0(K) \oplus C_0^1(K)$, such that $G = (D + i)(F)$ so $G \in C_0^0(K) \oplus C_0^1(K)$ and, using (17)

$$|\langle (D - i)G, \Psi \rangle| = |\langle (\Delta + 1)F, \Psi \rangle| \leq C_1 \|F\|_{L_2} \leq C_1 C_2 \|G\|_{L_2}$$

Hence

$$\exists C > 0,\ \forall G \in \mathrm{Im}(D + i),\ |\langle (D - i)G, \Psi \rangle| \leq C \|G\|_{L_2}. \tag{18}$$

But $\mathrm{Im}(D + i)$ is dense, it means that the considered linear form extends continuously on L_2 or that $(D + i)\Psi \in L_2$. Thus $\Psi \in \mathrm{Dom}(\bar{D})$ because $\bar{D}$ is self-adjoint. It is then clear that $D(\Psi) \in \mathrm{Dom}(\bar{D})$:

$$\forall F \in C_0^0(K) \oplus C_0^1(K),\quad |\langle D(F), D(\Psi) \rangle| = |\langle \Delta(F), \Psi \rangle| \leq (C_1 + \|\Psi\|_{L_2}) \|F\|_{L_2}.$$

So, we have proved

$$\mathrm{Dom}(\Delta^*) \subset \mathrm{Dom}((\bar{D})^2) \Rightarrow \mathrm{Dom}((\bar{D})^2) \subset \mathrm{Dom}(\bar{\Delta})$$

because $\Delta^{**} = \bar{\Delta}$, and finally

$$\mathrm{Dom}(\bar{\Delta}) = \mathrm{Dom}((\bar{D})^2)$$

and then $\bar{\Delta} = (\bar{D})^2$ is self-adjoint.

Let us now look at the converse: if Δ is essentially self-adjoint then, by the Corollary of Theorem VIII.3 in [RS80], $\mathrm{Im}(\Delta + 1)$ is dense but

$$\Delta + 1 = (D + i)(D - i) = (D - i)(D + i) \Rightarrow \mathrm{Im}(\Delta + 1) \subset \mathrm{Im}(D \pm i).$$

Thus $\mathrm{Im}(D \pm i)$ are both dense and D is essentially self-adjoint. □

This proof essentially follows [Ch73], it uses in a very significant way the fact that D maps elements of finite support into themselves.

Corollary 14. *Let K be a connected, locally finite graph. If K is χ-complete, then the operator Δ is essentially self-adjoint.*

Remark 15. — *The case studied in* [T10], *namely a complete graph for the metric*

$$a(e) = \frac{\sqrt{c(e^+)c(e^-)}}{r(e)}$$

and with a valence bounded by A can be handled with the same kind of calculus, although it is not clear that this metric is intrinsic. Indeed, with the same χ_n (defined by (12)), the bound now satisfied is

$$\exists C > 0\ ,\ \forall e \in \mathcal{E},\ n \in \mathbb{N}\ , \quad r(e)\mathrm{d}\chi_n(e)^2 \leq C\sqrt{c(e^+)c(e^-)}.$$

We write

$$\begin{aligned}\sum_{e\in\mathcal{E}} r(e)f(e^-)\mathrm{d}\chi_n(e)\bar{\varphi}(e) &= \frac{1}{2}\sum_{e\in\mathcal{E}} r(e)(f(e^+)+f(e^-))\mathrm{d}\chi_n(e)\bar{\varphi}(e)\\ &\leq \frac{1}{2}\sqrt{\sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n)} r(e)|\varphi(e)|^2}\sqrt{\sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n)} r(e)|f(e^+)+f(e^-)|^2\mathrm{d}\chi_n(e)^2}\end{aligned}$$

$$\begin{aligned}\text{and} \sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n)} & r(e)|f(e^+)+f(e^-)|^2\mathrm{d}\chi_n(e)^2\\ &= \sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n)} r(e)\Big[|(f(e^+)-f(e^-)|^2 + 4\,\mathrm{Re}\Big(f(e^+)\bar{f}(e^-)\Big)\Big]\mathrm{d}\chi_n(e)^2\\ &= \sum_{e\in\mathrm{supp}(\mathrm{d}\chi_n)} r(e)|\mathrm{d}(f)(e)|^2 + 4\,\mathrm{Re}\Big(\sum_{x\in\mathcal{V}} f(x)\sum_{e^+=x} r(e)\bar{f}(e^-)\mathrm{d}\chi_n(e)^2\Big)\end{aligned}$$

the first term tends to 0 *by completeness and the second is bounded as follows*

$$\begin{aligned}\mathrm{Re}\Big(\sum_{x\in\mathcal{V}}|f(x)|\sum_{e^+=x} r(e)|f(e^-)|\mathrm{d}\chi_n(e)^2\Big)&\\ \leq C\sum_{x\in\mathcal{V}}|f(x)| \sum_{e\in\mathrm{supp}\,\mathrm{d}\chi_n, e^+=x} |f(e^-)|\sqrt{c(e^+)c(e^-)}&\\ \leq AC \sum_{x\in\mathcal{V},\exists e\in\mathrm{supp}\,\mathrm{d}\chi_n, e^+=x} c(x)|f(x)|^2&\end{aligned}$$

because, as $\mathcal{E}$ is symmetric, one has

$$\sum_{x\in\mathcal{V},\exists e\in\mathrm{supp}\,\mathrm{d}\chi_n, e^+=x} c(x)|f(x)|^2 = \sum_{x\in\mathcal{V},\exists e\in\mathrm{supp}\,\mathrm{d}\chi_n, e^-=x} c(x)|f(x)|^2$$

So the second term also tends to 0, because of completeness and bounded valence.

4. Flanders Theorem

4.1. **Flanders problem.** In 1971, Flanders published a very nice result [F71] concerning resistive networks. The problem is the following (keeping the notations of Flanders): Let i be a finite current source, *i.e.* an element of $C^0_0(K)$, and E' a finite voltage source, *i.e.* an element of $C^1_0(K)$,

is there a resulting current flow, and is it unique?

i.e. find L_2-solutions I of the problem (Kirchhoff's laws):

$$\begin{cases} \text{(Kirchhoff's current law)} & \delta(I) + i = 0, \\ \text{(Kirchhoff's voltage law)} & \forall Z,\, \partial Z = 0, \quad \displaystyle\int_Z E' = \int_Z I, \end{cases} \tag{19}$$

Here Z is a cycle, *i.e.* a 1-chain (a formal finite sum of oriented edges) with no boundary.

Geometrically, if we write $Z = \sum_{e \in \mathcal{E}^+} z_e e$, $z_e \in \mathbb{Z}$, the boundary ∂ of a 1-chain is an operator defined on the edges by $\partial(e) = e^+ - e^-$.

On the sequel we will prefer a skew-symmetric notation:

$$\begin{aligned} Z &= \frac{1}{2} \sum_{e \in \mathcal{E}} z_e e,\ z_e \in \mathbb{Z}, \text{ with } z_e = -z_{-e} \\ \partial Z &= \sum_{x \in \mathcal{V}} \Big(\sum_{e^+ = x} z_e \Big) x. \end{aligned} \tag{20}$$

The integral in (19) has to be understood in the simplicial framework:

$$\int_Z I = \frac{1}{2} \sum_{e \in \mathcal{E}} z_e I(e) \tag{21}$$

Flanders studies this problem for an infinite graph with weight $c = 1$ on vertices (Remark that our weight on edges is in fact the inverse of the resistances r introduced by Flanders, so our unknown I corresponds to $r.I$ in the notations of Flanders). He shows that this problem has a unique L_2-solution which is the limit of finite flows (*ie.* solutions on an increasing sequence of finite subgraphs) if i has zero mean value $\sum_{v \in \mathcal{V}} i(v) = 0$.

4.2. **Flanders type Theorem.** In the framework we have introduced in Section 2, this question is related to the question of the Hodge decomposition. Indeed, the second condition tells us that the periods of I are given by those of E', this determine the harmonic component of I, *i.e.* the orthogonal projection of I on $\mathrm{Ker}(\delta)$, while the complementary component must be sent by δ on $-i$. So we have to look for $I = E_0 + I_0$ such that E_0 is the harmonic component of E' and I_0 satisfies $-i = \delta(I_0)$ and $\int_Z I_0 = 0$ on cycles.

Remark 16. *Indeed, this question is related with the* uniqueness problem *studied very carefully in a lot of works, we refer to* [LP14] *for a precise presentation. It appears that, at least with finite source current and no voltage current, the situation mostly studied, (i.e. i has finite support and $E' = 0$), the two general solutions are the* free current *which is the solution proposed by Flanders, and the* wired current *which is the solution of* minimal energy *(i.e. of minimal L_2-norm). It is clear, with the previous decomposition $I = E_0 + I_0$, that there exists at most one I_0 and the solution of minimal energy is given by the choice of E_0 with minimal norm. When $E' = 0$ this solution is $E_0 = 0$. The* uniqueness problem *is to find conditions where there is no choice although for E_0. We will not study this question here but focus on the existence question which concerns in fact I_0.*

Definition 17. *Any cycle Z defines a unique 1-cochain $E_Z \in C_0^1(K)$ such that $\forall E \in L_2(\mathcal{E})$*

$$\int_Z E = \langle E, E_Z \rangle$$

and we have the formula

$$Z = \sum_{e \in \mathcal{E}^+} z_e e,\ z_e \in \mathbb{Z} \quad \Rightarrow E_Z = \sum_{e \in \mathcal{E}^+} \frac{z_e}{r_e} e^*.$$

where the cochain e^* is defined by $e^*(e) = 1$ and $e^*(e') = 0$ if $e' \neq \pm e$.

An L_2-cycle Z is an (infinite) cycle such that $E_Z \in L_2(\mathcal{E})$.

Lemma 18. *For any L_2-cycle Z the 1-cochain E_Z satisfies formally that $\delta E_Z = 0$. Hence*

$$E_Z \in \operatorname{Ker} \delta_{\max}.$$

This result is a simple consequence of (20).

Lemma 19. *For any $E \in L_2(\mathcal{E})$ orthogonal to* $\operatorname{Ker} \delta_{\max}$ *and any L_2-cycle Z*

$$\int_Z E = 0.$$

Proof. — Indeed, for any L_2-cycle Z, $E_Z \in \operatorname{Ker} \delta_{\max} \subset L_2(\mathcal{E})$ and

$$\int_Z E = \langle E, E_Z \rangle = 0.$$

$\square$

Remark 20. *The uniqueness problem is then related to sufficient conditions for* $\operatorname{Ker} \delta_{\max}$ *to be generated by the E_Z, a priori we could consider finite cycles, or L_2-cycles. If $\varphi \in \operatorname{Ker} \delta_{\max} \subset L_2(\mathcal{E})$ is orthogonal to any E_Z, Z finite cycle, then there exists a function $f \in C^0(K)$ such that $\varphi = \mathrm{d}f$, so there exists a harmonic function with finite energy, but not necessarily L_2.*

Theorem 2. *Let K be a connected, locally finite graph. We suppose that it is $\chi-$complete such that the operator D defined on $C_0^0(K) \oplus C_0^1(K)$ is essentially self-adjoint. Then for any $i \in C_0^0(K)$ satisfying $\sum_{v \in \mathcal{V}} c(v) i(v) = 0$ and for any $E' \in L_2(\mathcal{E})$ there exists a unique solution of minimal energy $I \in \operatorname{Dom}(\bar{\delta})$ of the problem:*

$$\delta(I) + i = 0,\ \text{and } \forall Z,\ L_2-\text{cycle} \quad \int_Z E' = \int_Z I. \tag{22}$$

Proof. — By hypothesis, $\bar{\delta} = \delta_{\max}$. The space $\operatorname{Ker} \bar{\delta}$ is closed in $L_2(\mathcal{E})$, so any element $I \in L_2(\mathcal{E})$ can be writen $I = E_0 + I_0$ with $E_0 \in \operatorname{Ker} \bar{\delta}$ and I_0 in its orthogonal complement. By Lemma 19, and Remark 16, if we look for a solution of minimal energy, E_0 must be the orthogonal projection of E' on $\mathcal{Z} \subset \operatorname{Ker} \bar{\delta}$, the closure of the vector space generated by all the E_Z, Z L_2-cycle and then, by definition of the E_Z,

$$\forall Z\ L_2\text{-cycle}\ ,\ \int_Z E' = \int_Z E_0.$$

Now, the existence of I_0 is related to the property of $-i$ to be in the range of $\bar{\Delta}$. In the case where i has finite support, we can do as follows: let K_0 be a finite connected subgraph of K (see 2.1.4, vertices of K_0 are vertices of K and edges of K_0 are edges of K). We suppose that the support of i is included in K_0. Denote by d_0 the difference operator of K_0. The Laplacian Δ_0 of K_0 is self-adjoint and $\operatorname{Im}\Delta_0 = \operatorname{Ker}\Delta_0^{\perp}$. Thus, as $\operatorname{Ker}\Delta_0 = \mathbb{R}$ consists of constant functions

$$\langle i, 1\rangle = 0 \Rightarrow \exists f \in C^0(K_0),\ -i = \Delta_0(f).$$

Let $\varphi \in C_0^1(K)$ be the extension of $\mathrm{d}_0 f$ by 0 on the edges which don't belong to K_0. This form is certainly different from $\mathrm{d}f$ but $\delta\varphi = -i$.

We define now I_0 as the orthogonal projection of φ on the orthogonal complement of $\operatorname{Ker}\bar{\delta}$, it means that I_0 differs from φ by an element of $\operatorname{Ker}\bar{\delta}$ and that $I_0 \in \operatorname{Ker}\bar{\delta}^{\perp}$. Using Lemma 19, we conclude that:

$$\delta I_0 = -i \text{ and } \forall Z\ L_2\text{-cycle}\ ,\ \int_Z I_0 = 0.$$

Thus $I_0 + E_0$ is the solution of the problem with minimal energy. □

Remark 21. — *In the original theorem of Flanders, E' has finite support, and we only take care of finite cycles, but the proof extends easily to $E' \in L_2(\mathcal{E})$ if we consider only L_2-cycles. The question is how to extend it to more general i or can we characterize* $\operatorname{Im}(\bar{\delta})$? *If we can prove that* $\operatorname{Im}(\bar{\delta})$ *is closed, then the Hodge decomposition applies (see (24)) and the answer will be quite simple; that is what we explore below.*

4.3. **Anghel's hypothesis.** In [A93], N. Anghel shows that a Dirac type operator D defined on a complete manifold is Fredholm if and only if D^2 is *positive at infinity.*

Let us define the complementary of a subgraph of a graph.

Definition 22. *For a subgraph K_0 of a graph K, we define the* complementary graph $K_0^c = (\mathcal{V}^c, \mathcal{E}^c)$ *as follows*

$$\mathcal{V}^c = \mathcal{V} \smallsetminus \mathcal{V}_0, \quad \mathcal{E}^c = \{e \in \mathcal{E} \smallsetminus \mathcal{E}_0,\ \partial(e) \subset \mathcal{V}^c\}.$$

Remark 23. (1) *In particular boundary points of edges in $\mathcal{E}_0$ belong to $\mathcal{V}_0$.*
(2) *As a consequence of the definition, $\mathcal{E}^c$ avoids the edges with one end in $\mathcal{V}^c$ and one in $\mathcal{V}_0$.*

Following [KL10], we define the *boundary* of a subgraph K_0 to be its edge boundary :

$$\partial(K_0) = \mathcal{E} \smallsetminus (\mathcal{E}_0 \cup \mathcal{E}^c).$$

Definition 24. *We say that a closed Dirac type operator D is* positive at infinity *if there exists a finite connected subgraph $K_0 = (\mathcal{V}_0, \mathcal{E}_0)$ of K such that*

$$\exists C > 0, \quad \forall (f,\varphi) \in L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c) \cap \operatorname{Dom}(D), \quad \|(f,\varphi)\| \le C\|D(f,\varphi)\| \tag{23}$$

where $D(f,\varphi)$ is in fact D applied to the prolongations by 0 of (f,φ).

(Remark that this definition gives rather positivity of Δ.)

Theorem 3. *If the graph (connected and locally finite) is $\chi-$complete and if its Gauß-Bonnet operator*

$$D = \mathrm{d} + \delta$$

(which is essentially self-adjoint) satisfies that $\bar{D}$ is positive at infinity, then $\mathrm{Im}(\bar{D})$ *is closed and, as a consequence, the Hodge property holds :*

$$L_2(\mathcal{E}) = \mathrm{Ker}\,\bar{\delta} \oplus \mathrm{Im}(\bar{\mathrm{d}}), \quad L_2(\mathcal{V}) = \mathrm{Ker}\,\bar{\mathrm{d}} \oplus \mathrm{Im}(\bar{\delta}). \tag{24}$$

Proof. — The condition (23) implies that the closed restriction operator D^c of $\bar{D}$ on K_0^c :

$$D^c : \mathrm{Dom}(D^c) \subset L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c) \to L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c)$$

is continuous (for the graph norm on $\mathrm{Dom}(D^c)$), injective and with closed image. By the inversion theorem, there exists

$$P : L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c) \to \mathrm{Dom}(D^c)$$

such that $P \circ D^c = \mathbb{I}$, and $\mathbb{I} - D^c \circ P$ is the orthogonal projector on the subspace $\mathrm{Im}(D^c)^\perp$.

Let now $\psi \in \overline{\mathrm{Im}(\bar{D})}$. It means:

$$\exists \text{ a sequence } (\sigma_n)_{n\in\mathbb{N}} \text{ in } \mathrm{Dom}(\bar{D}), \quad \sigma_n \in \mathrm{Ker}(\bar{D})^\perp, \text{ and } \lim_{n\to\infty} \bar{D}(\sigma_n) = \psi.$$

The sequence (σ_n) is bounded. If not, (σ_n) admits a subsequence whose norm tends to $+\infty$, denoting this subsequence (σ_n) again, we construct

$$\varphi_n = \frac{\sigma_n}{\|\sigma_n\|}.$$

It satisfies

$$\|\varphi_n\| = 1, \lim_{n\to\infty} \bar{D}(\varphi_n) = 0.$$

Then the restriction of $\bar{D}(\varphi_n)$ to K_0^c also converge to 0 in $L_2(\mathcal{V}^c) \times L_2(\mathcal{E}^c)$.
But the set of vertices not in $\mathcal{V}^c$ and the set of edges not in $\mathcal{E}^c$ are finite. As φ_n is bounded, by passing to a subsequence, we can suppose that all their values in these finite sets converge, and by the same argument we can suppose that the value of φ_n on the vertices which are boundary points of edges in $\partial(K_0)$ converge. By local finiteness we conclude that $\bar{D}(\varphi_{n|K_0^c})$ converges.

By (23), then also $\varphi_{n|K_0^c}$ converges, thus finally φ_n converges, let φ be the limit, it satisfies

$$\|\varphi\| = 1, \ \varphi \in \mathrm{Ker}(\bar{D})^\perp, \ \bar{D}(\varphi) = 0.$$

Thre is a contradiction.

So we can suppose that (σ_n) is bounded, then by the same kind of reasoning, we show that $(\sigma_n)_n$ admits a subsequence which converges, let σ be this limit. As $\bar{D}$ is closed and $\bar{D}(\sigma_n)$ converges, then $\sigma \in \mathrm{Dom}(\bar{D})$ and $\bar{D}(\sigma) = \psi$. □

We see that the reasoning is separated for $0-$forms and $1-$forms. This gives:

Corollary 25. *Let K be a graph (connected and locally finite) $\chi-$complete so its Gauß-Bonnet operator $D = \mathrm{d} + \delta$ is essentially self-adjoint. If* d *satisfies the condition*

$$\exists C > 0, \quad \forall f \in L_2(\mathcal{V}^c) \cap \mathrm{Dom}(\bar{\mathrm{d}}), \quad \|f\| \leq C\|\bar{\mathrm{d}} f\| \tag{25}$$

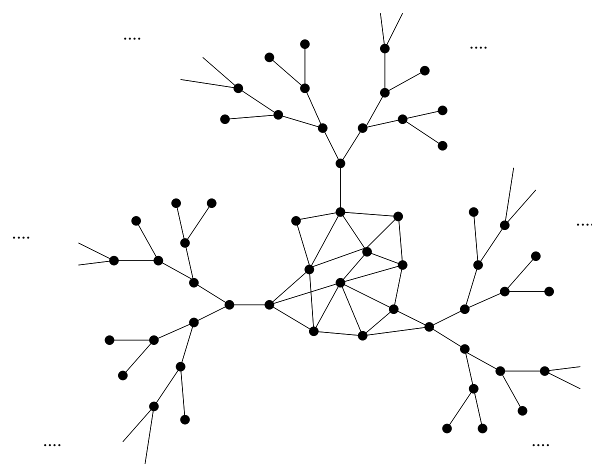
for the complementary of some finite graph, then $\mathrm{Im}\,\bar{\mathrm{d}}$ *is closed and*

$$L_2(\mathcal{E}) = \mathrm{Ker}\,\bar{\delta} \oplus \mathrm{Im}(\bar{\mathrm{d}}).$$

And there exists a similar statement for δ.

5. Examples

It is clear that if K possesses infinitely many cycles (as infinite ladders, or infinite grids), the condition of positivity at infinity will not be satisfied because there will be elements in $\mathrm{Ker}\,\delta$ with support as far as we want. A family of examples could be a graph with finite geometry: there exists a finite subgraph K_0 such that K_0^c is a disconnected (finite) union of branches.



Proposition 26. *If the connected graph K admits a finite subgraph such that its complementary is a finite union of trees with constant valence larger than 3, then, considered with the weights constant equal to* 1 *on vertex and edges, it is $\chi-$complete and* $\mathrm{Im}\,\bar{\mathrm{d}}$ *is closed.*

Proof. — We will prove that d is positive at infinity, *ie.* on each tree. Let U be a tree with a base point and valence $p+1$, $p \geq 2$. We apply Corollary 17 of [KL10], taking the notations of this paper (in particular $\sharp$ denotes the cardinality): in our case $D_U = p+1$ is finite, so it suffices to show that the isoperimetric constant α_U is positive.
Recall that

$$\alpha_U = \inf_{W \subset U, \text{finite}} \frac{\sharp(\partial W)}{\sharp W}. \tag{26}$$

For a tree, one has a notion of *height*: the base point is of height 0, and for another point its height is the necessary number of edges to join it to the base point.

Let W be a finite set of vertices of U, we shall show by reccurence on $\sharp W$ that

$$\sharp(\partial W) \geq \sharp W.$$

If $\sharp W = 1$, then $\sharp(\partial W) = p+1$. If $\sharp W = n \geq 1$, let $x \in W$ be a point of highest height in W and y is the point just below. Then define $W' = W - \{x\}$ so $\sharp W' = \sharp W - 1$ and

$$y \in W \Rightarrow \sharp(\partial W) = p - 1 + \sharp(\partial W')$$
$$y \notin W \Rightarrow \sharp(\partial W) = p + 1 + \sharp(\partial W')$$

In all cases, applying the reccurence hypothesis, we get:

$$\sharp(\partial W) \geq p - 1 + \sharp(\partial W') \geq p - 1 + \sharp W - 1 \geq \sharp W.$$

□

Corollary 27. *Such a graph (as in the proposition 26) satisfies also that* $\operatorname{Im} \bar{\delta}$ *is closed and* $\operatorname{Ker} \bar{\mathrm{d}} = \{0\}$ *(because constants are not in* L_2*), so* $\bar{\delta}$ *is surjective.*
As a consequence, for such a graph Flanders problem (19) has always a unique solution with minimal energy.

Proof. — Indeed, if (25) is satisfied, then

$$\forall f \in \operatorname{Dom}(\Delta^c) \subset L_2(\mathcal{V}^c), \quad \|f\| \leq C^2 \|\Delta(f)\|. \tag{27}$$

Thus, by the same reasoning as before the range of $\bar{\Delta}$ acting on functions is closed. Now if $(\varphi_n)_n$ is a sequence of 1-forms such that $\delta(\varphi_n)$ converges, we can apply the Hodge decomposition (24) at φ_n, because of the Proposition 26:

$$\exists f_n \in \operatorname{Dom}(\bar{\mathrm{d}}) \text{ such that } \delta \circ \mathrm{d}(f_n) \in L_2(\mathcal{V}) \text{ and converges.}$$

But we can extract a subsequence of $(f_n)_n$ which converges, because of (27). □

Proposition 28. *If the connected graph* K *admits a finite subgraph such that its complementary is a finite union of trees with valencelarger than 3, then, considered with the weights equal to the valence on vertices and constant equal to* 1 *on edges, it is* $\chi-$*complete and* $\operatorname{Im} \bar{\mathrm{d}}$ *is closed.*

Proof. — It is clear that such a graph satisfies the condition of $\chi-$completeness. The fact that d is positive at infinity is again a consequence of the results of [KL10]. Indeed, by hypothesis we have $\forall v \in \mathcal{V}, m(v) = \sharp\{e \in \mathcal{E}, e^+ = v\}$ at least on the "tree-part", thus is it equal to the function n introduced in [KL10] and their d is constant equal to 1. By their Proposition 15, the quadratic form on a part U is bounded from below by $1 - \sqrt{1 - \alpha_U^2}$ if α_U is the isoperimetric constant introduced in (26) but now with the volumes $|.|$ defined by the weights:

$$\alpha_U = \inf_{W \subset U, \text{finite}} \frac{|\partial W|}{|W|}.$$

Let W be a finite part of a tree. Its number of (oriented) edges is $\sum_{v \in W} m(v) = |W|$. But, because it is in a tree the number of interior edges is at most $2.\sharp(W)$. Thus

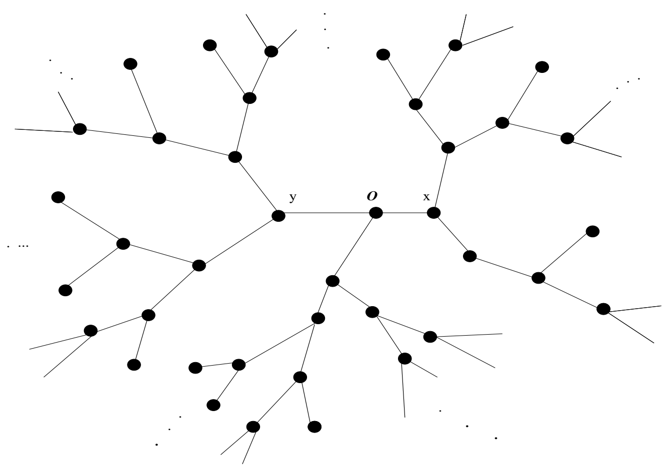
$$\frac{|\partial W|}{|W|} \geq \frac{\sum_{v\in W}(m(v)-2)}{\sum_{v\in W} m(v)} \geq \frac{1}{3}$$

because $m(v) \geq 3$. □

The same Corollary as before holds, for the same reasons.

Corollary 29. *Such a graph (as in the Proposition 28) satisfies also that* $\operatorname{Im}\bar{\delta}$ *is closed and* $\operatorname{Ker}\bar{\mathrm{d}} = \{0\}$ *(because constants are not in* L_2*), so* $\bar{\delta}$ *is surjective.*
As a consequence, for such a graph, Flanders problem (19) has always a unique solution with minimal energy.

Remark 30. *Take care to the fact that in these situations* $\operatorname{Ker}\bar{\delta}$ *can be non trivial : on a tree of valence 3, with all the weights equal to 1, fix a point* O*, it has at least two edges which go to infinity:* (x, O) *and* (y, O)*.*



Let φ *be the form such that*

$$\varphi(x, O) = 1,\ \varphi(y, O) = -1$$

at the n-level on the branch emanating from x *we put the value of* φ *to be* $\frac{1}{2^n}$*, and at the n-level on the branch emanating from* y *we put the value of* φ *to be* $\frac{-1}{2^n}$*.*
Elsewhere, we put $\varphi(e) = 0$*.*
It is easy to verify that such a φ *is in* L_2 *and satisfies* $\delta(\varphi) = 0$*, see also* [Ay13].

Remark 31. *In these two last cases the Laplacian is bounded, and the non zero spectrum is bounded from below because the isoperimetric constant* α_U *admits a bound independent on* U*.*

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