Control of interbank contagion under partial information

Hamed Amin, Andreea Minca, Agnès Sulem

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Abstract

We consider a stylized core-periphery financial network in which links lead to the creation of projects in the outside economy but make banks prone to contagion risk. The controller seeks to maximize, under budget constraints, the value of the financial system defined as the total amount of external projects. Under partial information on interbank links, revealed in conjunction with the spread of contagion, the optimal control problem is shown to become a Markov decision problem. We find the optimal intervention policy using dynamic programming.

Our numerical results show that the value of the system depends on the connectivity in a non-monotonous way: it first increases with connectivity and then decreases with connectivity. The maximum value attained depends critically on the budget of the controller and the availability of an adapted intervention strategy. Moreover, we show that for highly connected systems, it is optimal to increase the rate of intervention in the peripheral banks rather than in core banks.

Keywords: Systemic risk, Optimal control, Financial networks.

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1 Introduction

In this paper we propose a stylized hierarchical model for a financial network with two classes of banks: core and peripheral.

All banks have a certain amount of projects funded by deposits, independently of the other banks. In addition, there are peripheral banks that have access to available (local) projects but do not hold sufficient deposits and an external creditor that can lend to the core banks. Core banks can intermediate credit between the external creditor and the peripheral banks: A core bank can borrow from another core bank or from an external creditor and can lend to another core bank or to a peripheral bank that has available projects that need funding. The peripheral banks can only borrow from the core banks. The total number of credit lines from the external creditor to the core banks is equal to the total number of credit lines extended by core banks to peripheral banks. [1]

We define the value of the financial system as the total amount of external projects funded directly by deposits or by credit intermediated through the core banks. Since we consider the aggregate value of the financial system, it is beyond the scope of our model to consider the interest rate on the loans which only has the effect of redistributing the value among banks. The connectivity of the core banks plays a dual role: more connectivity allows core banks to intermediate more credit from the external creditor to the peripheral banks. This increases the value of the system as more available projects get funded. On the other hand, more connectivity of the core banks creates risk of contagion. A core bank may fail to maintain credit to the other banks.[2] When a core bank fails, we assume that it withdraws all credit from all debtor banks and liquidates its external project entirely.

We study the optimal intervention policy of a lender of last resort that injects liquidity into the system either in the core banks or in the peripheral banks after the failure of an initial core bank. The criterion of the controller is the total value of the financial system at the end of the contagion.

The flow of information available to the controller is modeled as a link-revealing filtration. This filtration is monotonous in space: at each step, the failure cluster, i.e., the set of failed banks and their revealed links, increases and the controller learns of a new link to a failed bank. For tractability reasons, we learn the lenders of a failed bank one by one, according to some arrival times of this information (which can be arbitrarily close, but distinct). At each of these arrival times, the link-revealing filtration increases. Given the observation of the failure cluster, the conditional law of the unobserved part of the network depends only on the current failure cluster and not its history. In this sense the system is Markovian.

Our main result (Theorem 5) states that the optimal control policy is Markovian and depends only on the observed failure cluster. Moreover, we prove a state space collapse result that the Markovian system can be described by a Markov chain with a lower dimension, that gives the cross-distribution of the distance to failure. We show that the optimal control is feedback on the state of the lower-dimensional Markov chain. This provides us with a tractable framework for the optimal control problem, solvable by the dynamic programming approach.

In the numerical results we apply this framework to answer the following questions: How does the

[1] There is vast empirical evidence for a tiered structure of interbank networks for given countries, see e.g Boss et al. (2004) for Austria, Cont et al. (2012) for Brazil, Craig and Von Peter (2014) for Germany, etc.

[2] This failure can be triggered by a bank run initially, and then spread through contagion.

value of the system and the optimal control depend on the connectivity of the core banks? Should intervention be targeted towards core banks or towards peripheral banks?

To ensure that our results are driven by connectivity and not size, we introduce a numeraire in our model represented by the value of one link, and normalized to 1: one link can fund one unit of available project and also intervention to stop contagion through one link costs 1. The difference in scale from core to peripheral banks is captured by the difference in their connectivity. Between a pair of banks, there can be multiple links.

We vary the connectivity of the core banks, where zero connectivity means a financial system without any core banks that act as intermediaries, i.e., all projects are funded locally. We find that up to a certain level of connectivity, the value of the system increases with connectivity and afterwards it decreases with connectivity. Our numerical results point to the existence an of "optimal" level of connectivity depends on the initial number of failures in the system and the budget of the controller. At this optimal level of connectivity the system depends, in case there are defaults, on the controller, and in particular on the availability of a strategy adapted to the flow of information generated by the spread of contagion. For highly connected systems with initial failures, even in presence of intervention the value of the system may fall below the value of the disconnected system.

We then analyze the intervention policy. In the low connectivity regime, as we increase connectivity, the rate of intervention in core banks increases. On the contrary, for highly connected systems, as we increase connectivity the optimal rate of intervention is higher for peripheral banks than for core banks.

We interpret these results as follows. In the low connectivity regime, there is little scope for contagion among core banks since non-failed core banks can absorb the failure of a counterparty without any intervention. So the controller intervenes on the periphery, to stop contagion from the core to periphery. In the medium connectivity regime, intervention occurs on core banks: in this regime, the number of linkages among core banks increases and contagion to non-failed core banks is possible since their initial distance to failure is small compared to the connectivity of a failed counterparty. In the high connectivity regime, the number of cycles in the subnetwork of core banks is too large. For a fixed budget, there is a substantial risk that even if there is an intervention on core banks, contagion will spread due to these cycles among core banks and eventually reach all banks anyway. Therefore it may be preferable in this case to inject in the peripheral banks directly and save them with certainty.

Relation to previous literature Our work is in the area of systemic risk, see Amini and Minca (2013), Fouque and Langsam (2013) and the references therein. This paper is related to strand of works on various types of control of contagion in financial networks (see e.g. Rogers and Veraart (2013)).

The random network model was studied in the context of systemic risk in Amini et al. (2013), who give asymptotic results on the scope of contagion in heterogenous networks, where core-periphery networks are a particular case of the heterogenous networks. Our focus here is not on the scope of contagion *per se*, but on the tradeoff between financial contagion, which is the price to pay for connectivity, and the creation of new projects, which is the benefit of connectivity. This tradeoff appears in the optimization criterion of the controller.

The idea of cost-benefit of the links is a key component in the financial network formation literature, see e.g. Blume et al. (2013) for a setting based on random graphs. Their focus is different, as they study the stability of a network formed by strategic banks that take into account the benefits and risks of connectivity.

The control of contagion in an interbank network is studied in Minca and Sulem (2014) under full observation of the interbank network in a model that extends Eisenberg and Noe (2001) to account for the interplay between insolvency and funding liquidity risk. The problem leads to a convex combinatorial optimization problem, which is tractable only if the controller intervenes only on the set of core banks. Contrary to this limitation, in the current model, under partial information revealed step by step, we prove a state space collapse theorem which allows for tractability.

In most previous works on contagion in directed financial networks, the sense of contagion is in the opposite sense of lending and the cascades are interpreted as cascades of insolvencies. In our paper, we consider similarly to the simulation studiies Gai and Kapadia (2010), Gai et al. (2011), Müller (2006) that contagion goes in the same direction as the credit: the lines of credit are withdrawn when the lender fails.

Our work is also related to recent literature on spatial risk measures, see Chen et al. (2013), Föllmer (2014). Similarly to these works, the complexity of the model is spatial and not temporal, as our controller faces a problem in space as contagion may spread from the initially failed bank to the system, leading if no intervention to a larger number of failures.

Several previous works point to the link between connectivity and financial stability, e.g. Battiston et al. (2012), Amini et al. (2013), Acemoglu et al. (2013), Garnier et al. (2013). Our work complements this literature, by adding explicitly in the model the creation of projects driven by the connectivity of the core banks which act as intermediaries.

Within a reduced form approach to systemic risk, Giesecke and Kim (2011) propose estimators for the hazard rate of a systemic event; Fouque and Ichiba (2013), Carmona et al. (2013) model the evolution of the bank reserves using coupled diffusion processes: the liquidity provided by the central bank is captured by using a rate of borrowing and lending. The critical difference is that in our model banks do not only lend to each other but also fund external projects. Consequently, while Carmona et al. (2013) find that adding liquidity in the system does not affect systemic risk, in our model, the control affects the value of the financial network which depends both on systemic risk and the external projects.

2 The model

2.1 General description

We fix a probability space under which the financial network is uniformly chosen among all networks in which core banks have a fixed, non-random, connectivity and peripheral banks have connectivity at most one. We consider the stylized case in which all core banks have the same connectivity. While the connectivity is fixed, the counterparties (lenders and borrowers) of the core banks are randomly chosen. Note that it is sufficient to establish the counterparties of the core banks, since the peripheral banks can only connect to core banks, and the external creditor to core banks only.

Initially, banks are endowed with a certain resilience in front of the withdrawal of their credit, called distance to failure. Each time a credit line is withdrawn from a bank, its distance to failure decreases by 1. When it reaches 0, the bank fails and will liquidate its project and withdraw any credit from counterparties, so contagion occurs. We consider a worse case where the project is liquidated entirely and all credit lines are withdrawn, so we do not allow for partial liquidations or withdrawals. This is a tractability assumption. We call the failure cluster the set of failed banks and the credit lines that were withdrawn by them.

At each step, there is the observed failure cluster and an unobserved part of the financial network. Information arrival is conditioned on the spread of contagion: as long as there are failures in the system we learn the counterparties that have credit lines from the banks that are failed. When contagion stops, i.e., there are no more such links to be revealed, the system is no longer observed. As information arrives and a new link to a failed bank is revealed, the corresponding counterparty will decrease its distance to failure. When a link is revealed, in absence of intervention, the distance to failure of the counterparty decreases by 1 in case the bank was not failed and stays at 0 in case the counterparty already failed. In case there is an intervention, the banks remains at the same distance to failure. The controller can have a maximum number of interventions.

2.2 The system before the shock

We consider a financial system with c *core* banks and p *peripheral* banks with the total number of banks

$$n = c + p.$$

Each core bank holds y_c in external projects and each peripheral bank holds y_p in external projects, funded through their own deposits independently of all the other banks. There is also an external creditor, formally represented as node 0. In addition, there exists a financial network that intermediates credit lines from the external creditor to the peripheral banks. This is shown in Figure 1, in the particular case with one core bank and one peripheral bank.

In general, the chain of intermediate core banks is larger than one, and multiple core banks intermediate the credit from the external creditor to a peripheral bank. This has opposite network effects: more peripheral banks have the chance to obtain credit lines, but contagion is possible among the intermediary core banks.

We now describe the network model. Core banks have connectivity λ: they have λ borrowers, that can be other core banks or peripheral banks, and λ lenders, which can be other core banks or the external creditor. Peripheral banks have at most connectivity 1 and can only borrow from core banks. We consider networks which allow for *multiple* links between two nodes, i.e., multi-networks. This corresponds to the case where there is more than one unit of a loan between the pair of banks.

We represent the financial system as a directed unweighted multi-network $([n] \cup \{0\}, \mathcal{E})$, where $[n] := \{1, \ldots, n\}$ represents the set of banks and $\mathcal{E}$ the set of links. For a link $(i, j) \in \mathcal{E}$, i is the lender and j is the borrower. Links are unweighted, that is, there is a standardized value of a loan

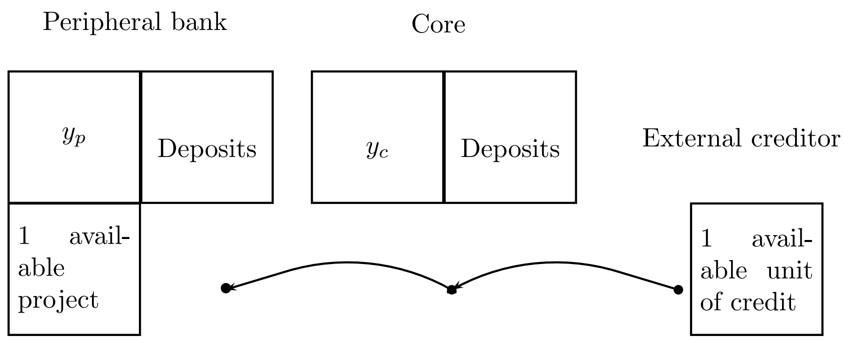


Figure 1: Credit intermediation by one core bank: one unit of credit from the external creditor is intermediated by the core bank and extended to a peripheral bank, that invests this unit of loan into one unit of additional project. In general, there may be multiple intermediary core banks.

normalized to 1 and which plays in our model the role of a numéraire. We allow for multiple links between two core banks. The number of in-coming links of core banks, equal to the number of out-going links of core banks, is

$$m := c\lambda.$$

If we adjust for double counting of the links among core banks, we obtain the total number of links in the network

$$2m - L_C,$$

with the number of links among core banks given by

$$L_C := \#\{(i,j) \in \mathcal{E}, i,j \in [c]\}.$$

We can now describe the benefit of the financial network. Peripheral banks that receive one credit line from core banks fund one new unit of external project.

This modeling choice can be interpreted in the following way: core banks have access to credit lines from the external creditor and they can choose to fund directly their own projects or to intermediate the credit. The quantity y_c can be thought as being chosen optimally by core banks: there is a cost associated to the search of projects, so after the amount y_c of funded projects they prefer to lend to other banks. It is beyond the scope of this paper to model the optimal choice of y_c by the core banks.

The number of additional projects extended by the peripheral banks using the credit provided by core banks is given by

$$N := \#\{(i,j) \in \mathcal{E}, i \in [c], j \in [p]\}. \tag{1}$$

We also have the following *balance sheet equation*, that the total number of links exiting the core

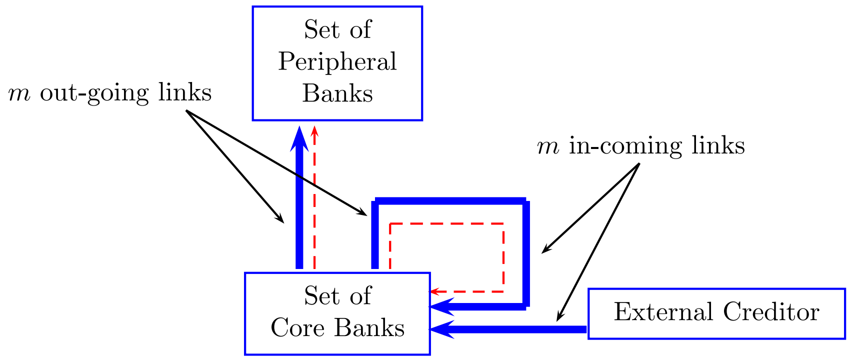


Figure 2: Direction of credit lines. More connectivity of the core banks amounts to more credit lines towards peripheral banks but also more cycles among core banks. Contagion happens in the direction of credit lines: when a bank fails it withdraws all credit lines form its borrowers.

banks is equal to the total number of links entering the core banks, i.e.,

$$\#\{(i,j) \in \mathcal{E}, i \in [c], j \in [c] \cup [p]\} = \{(i,j) \in \mathcal{E}, i \in [c] \cup \{0\}, j \in [c]\}, \tag{2}$$

which in turn requires that

$$N = \{(i,j) \in \mathcal{E}, i \in \{0\}, j \in [c]\}.$$

In total, the number of units of project in absence of an initial shock is given by

$$\bar{J} := cy_c + py_p + N \times 1, \tag{3}$$

where 1 corresponds to a unit of project. The first terms are independent of the network since they are projects funded by each bank through their own deposits. As connectivity increases in the network, the third term may increase. However, as shown in Figure 2, contagion occurs in the same direction as direction of the links and therefore an increase of connectivity increases contagion risk. When a core bank fails, it withdraws its credit lines and liquidates its projects (y_c units). When a peripheral bank fails, it liquidates its projects (y_p units).

In the sequel we fix a probability space $(G_{n+1.m}, \mathbb{P})$, where $G_{n+1.m}$ denotes the set of networks with $n+1$ nodes (n banks and the external creditor) and at most m links. The financial network $\mathcal{E}$ and any partial observation of this network live on the space $G_{n+1,m}$.

Under the probability measure $\mathbb{P}$, the law of $\mathcal{E}$ is given as follows. Each core bank is assigned λ in-coming half links and λ out-going half links, corresponding to the number of lenders and borrowers. The identity of these lenders and borrowers is random and drawn as follows. We first draw the borrowers of core banks. Each peripheral bank is assigned one in-coming half-link. In total we have so far $m = c\lambda$ out-going half links and $m + p$ in-coming half links. An out-going half link represents an offer of a credit line. An in-coming half link represents a candidate to borrow one credit line. Only when these half-links are matched, this corresponds to an effective loan. Moreover, only when the chosen candidate is a peripheral bank, a new unit of project is created. When the chosen candidate is a core bank, then this does not correspond to a new unit of project, but to an additional intermediary.

We draw uniformly and independently, without replacement, m in-coming half-links out of the total $m + p$ in-coming half-links. We match the m out-going half-links with the drawn m in-coming links. When an out-going half-link is matched with an in-coming half-link, a link is established.

This draw establishes the borrowers of all core banks. Importantly, this draw also establishes all links among core banks. Any unmatched in-coming half edge of a core bank represents a candidate to borrow which is not matched by an offer from another core bank. Therefore it is matched to the external creditor node 0, to ensure the balance equation (2).

An unmatched in-coming half edge of a peripheral bank is removed, i.e., peripheral banks do not have access to credit lines from the external creditor.

Remark 1. *Note that, given the observation of any subset of matched links, the unobserved matched links also have the same law of uniform draw without replacement, and this conditional law depends on the observed set of matched links only through the number of matched links going out of core banks and not the history of the past matching.*

It follows that the random variable N in (1) has the following law

$$N \sim Hyp(m + p, p, m), \tag{4}$$

where $Hyp(\alpha, s, d)$ denotes the hypergeometric distribution, i.e., the distribution of the probability of successes in d draws without replacement from a finite population of size α containing exactly s successes.

2.3 Dynamics of Contagion and Information

Each bank is endowed with a random variable denoted $\theta_0(i), i \in [n]$ which represents how many credit lines can be withdrawn from bank i, before bank i fails. We call this variable "initial distance to failure" and it takes values in $\{0, \ldots, \vartheta_{max}\}$, where distance 0 marks a failed bank. We assume that when a bank fails, it withdraws all credit lines from its debtors and also liquidated entirely its projects. Our model of contagion is a linear threshold model, with the initial distance to failure the threshold. The threshold model for cascades in random graphs was introduced by Watts (2002). However, despite its simplicity, rigorous mathematics results on the equivalence to the spread of contagion in financial networks were proven only recently in Amini et al. (2013), in the case without a controller and without external projects.

We now endow the probability space $(G_{n+1.m}, \mathbb{P})$ with a filtration $(\mathcal{G}_k)_{k \leq m}$ that models at the same time the financial contagion and the flow of information: at each step an out-going link of a failed bank is revealed. This corresponds to a credit line that is being withdrawn and consequently to a decrease in the distance to failure of the borrower at the end of the link. Rather than temporal, information has here a spatial dimension. We call this filtration the "link-revealing" filtration.

The failure cluster started by the failure of a core bank, is given below by the set of failed banks and the set of revealed links. It is defined as a random graph that evolves in the space $G_{n+1.m}$. At any point during the cascade dynamics, there are at most $n+1$ failed nodes and at most m out-going links of failed nodes. The cascade of failures and the flow of information is determined as follows.

Definition 2. *Cascade of contagion and information*

Initial condition.
Let the set of initially failed banks $F_0 := \{i \in [n] \mid \theta_0(i) = 0\}$. *Let the set of revealed links exiting initially failed nodes* $\mathcal{R}_0 := \emptyset$. *Let the set of hidden links exiting initially failed banks* $\mathcal{H}_0 := \{(i,j) \in \mathcal{E}, i \in F_0\}$. *Let the sigma-algebra representing the information available initially* $\mathcal{G}_0 := \sigma((\theta_0(i), i \in [n]))$.

Dynamics.
Let $k \in [1, m]$. *If* $\mathcal{H}_{k-1} = \emptyset$, *Stop.*

Otherwise: Let $(i_k, j_k) \in \mathcal{H}_{k-1}$ *a hidden link, chosen uniformly over the set* $\mathcal{H}_{k-1}$.[3]

Let $\mathcal{R}_k := \mathcal{R}_{k-1} \cup \{(i_k, j_k)\}$.

Let $\mathcal{G}_k = \mathcal{G}_{k-1} \vee \sigma(\{i_k, j_k\})$.

If $\theta_{k-1}(j_k) > 0$, *set* $\theta_k(j_k) := \theta_{k-1}(j_k) - 1$; *Otherwise* $\theta_k(j_k) := \theta_{k-1}(j_k)$.

For $i \neq j_k$, $\theta_k(i) = \theta_{k-1}(i)$.

If $\theta_k(j_k) = 0$ *and* $\theta_{k-1}(j_k) = 1$, *then* $F_k := F_{k-1} \cup \{j_k\}$; *Set*

$$\mathcal{H}_k := \mathcal{H}_{k-1} \setminus \{(i_k, j_k)\} \cup \{(j_k, j) \in \mathcal{E}\}. \tag{5}$$

Otherwise, $F_k := F_{k-1}$;[4] *Set*

$$\mathcal{H}_k := \mathcal{H}_{k-1} \setminus \{(i_k, j_k)\}. \tag{6}$$

At any step of the cascade, the links exiting failed banks are partitioned in two categories: revealed and hidden. For tractability we need that only one of these links is revealed at one time, therefore we choose them randomly one by one. During the cascade, new information arrives in the system when a new hidden link is revealed. Since the hidden link is exiting a failed node, the counterparty will have a credit line withdrawn. We model this by reducing the distance to failure of the (revealed) counterparty by one. With each link revealed, the number of hidden links decreases by one (see (6)). If the counterparty fails, i.e., its distance to failure reaches zero, then the set of

[3]Note that when the hidden link (i_k, j_k) is revealed at time k, all links exiting the failed bank i_k, i.e. $\{(i_k, j) \in \mathcal{E}\}$, are either revealed or in the set of hidden links. If they are hidden, they will remain in the set of hidden links, until revealed one by one.

[4]Note that at any step $k \in [0, m]$, we have $F_k = \{i, \theta_k(i) = 0\}$.

failures increases. If this failed bank is a peripheral bank, it does not have any additional links. If it is a core bank, all its links are not revealed immediately but they become hidden links. Thus the number of hidden links jumps up (see (5)).

In sum, the number of hidden links decreases by one at each step of the algorithm but jumps up when a core bank fails. Note also that the stopping time T at which we reveal all hidden links is always smaller than $m = c\lambda$, the total number of out-going links of core banks.

The set of hidden links $\mathcal{H}_k$ is not adapted to the filtration generated by the revealed links $\mathcal{G}_k$. This is what justifies the notion of "partial information" in our model. The hidden link (j_k, j), added to the set of hidden links in (5) represents a lending relation between a failed node j_k and one of its debtors. This link will belong to failure cluster, but is not revealed at step k when j_k fails. It will be revealed later. In other words, the controller knows when a node j_k fails that someone will have a credit line withdrawn by j_k but she does not yet know who. In this sense, the distance to failure $\theta_k(i)$ is only an observed distance to failure, since the hidden link will decrease the distance to failure of i later. Likewise, F_k is only an observed set of failures.

3 Optimal intervention problem

We now consider the previous setting, but we assume that there exists a controller that intervenes a maximum number of times M (which represents the budget constraint) at steps $1 \le k \le m$. When the controller intervenes on a bank, its distance to failure increases by 1. We interpret the controller as a lender of last resort: in absence of intervention a bank will see its credit line withdrawn by a failed bank. With intervention, this credit will be replaced by a line of credit from the lender of last resort.

3.1 Contagion and information under intervention

We consider the case when the controller may intervene at each step only on one bank, which we understand is the end of the revealed link. For this type of control, we introduce the control set

$$\mathcal{U}_M := \{(u_k \in \{0,1\}, 1 \le k \le m), \sum_{1 \le k \le m} u_k \le M\}.$$

We will show later (Proposition 7) that for the optimization criterion that we consider, this space is sufficient, i.e., it would not be optimal for the controller to intervene on multiple banks at the same time or on banks which are not the end of the revealed link.

We now define the controlled cascade of information and contagion, where the control process takes values in $\mathcal{U}_M$ and is adapted to the link-revealing filtration.

Definition 3. *Controlled cascade of information and contagion*

Initial condition.
Let $\theta_0(i), i \in [n]$ given. Let the set of initially failed banks $F_0 := \{i \in [n] \mid \theta_0(i) = 0\}$. Let the set of revealed links exiting initially failed nodes $\mathcal{R}_0 = \emptyset$. Let the set of hidden links exiting initially

failed banks $\mathcal{H}_0 := \{(i,j) \in \mathcal{E}, i \in F_0\}$. *Let the sigma-algebra representing the information available initially* $\mathcal{G}_0^u = \sigma((\theta_0(i), i \in [n]))$. *Set* $u_0(i) = 0, i \in [n]$ *and* $v_0(i) = 0, i \in [n]$.

Dynamics.
Let $k \in [1, m]$. *If* $\mathcal{H}_{k-1}^u = \emptyset$, *Stop.*

Otherwise, let $(i_k^u, j_k^u) \in \mathcal{H}_{k-1}^u$ *chosen uniformly over the set* $\mathcal{H}_{k-1}^u$.

Let $\mathcal{R}_k^u \leftarrow \mathcal{R}_{k-1}^u \cup \{(i_k^u, j_k^u)\}$.

Set $\mathcal{G}_k^u = \mathcal{G}_{k-1}^u \vee \sigma(\{i_k^u, j_k^u\})$.

Choose u_k $\mathcal{G}_k^u$*-measurable, with values in* $\{0,1\}$, *such that* $\sum_{i\in[n]} v_k^u(i) \le M$, *where*

$$v_k^u(i) := v_{k-1}^u(i) + u_k \mathbb{1}_{i=j_k^u}, i \in [n].$$

If $\theta_{k-1}^u(j_k^u) > 0$, $\theta_k^u(j_k^u) := \theta_{k-1}^u(j_k^u) - 1 + u_k$; *Otherwise* $\theta_k^u(j_k^u) = \theta_{k-1}^u(j_k^u)$.

For $i \neq j_k^u$, *set* $\theta_k^u(i) = \theta_{k-1}^u(i)$.

If $\theta_k^u(j_k^u) = 0$ *and* $\theta_{k-1}^u(j_k^u) = 1$,
then $F_k^u := F_{k-1}^u \cup \{j_k^u\}$; $\mathcal{H}_k^u := \mathcal{H}_{k-1}^u \setminus \{(i_k^u, j_k^u)\} \cup \{(j_k^u, j) \in \mathcal{E}\}$.

Otherwise, $F_k^u = F_{k-1}^u$; $\mathcal{H}_k^u := \mathcal{H}_{k-1}^u \setminus \{(i_k^u, j_k^u)\}$.

Note that indeed the filtration defined above is the link-revealing filtration:

$$\mathcal{G}_k^u = \sigma(\mathcal{R}_l^u, 0 \le l \le k, (\theta_0(i))_{i\in[n]})$$

It is immediate to see from the definition above that $(\theta_k^u(i))_{i\in[n]}$ is $\mathcal{G}_k^u$-adapted: for all $k \le T^u$

$$\theta_k^u(i) = \theta_0^u(i) - \#\{(j,i) \in \mathcal{R}_k^u\} + v_k(i),$$

with $v_k(i)$ a function of $(\mathcal{R}_{l_1}, l_1 \le l)$ and $(\theta_0(j))_{j\in[n]}$.

For $k = 1, \ldots, T^u$, we refer to $(v_{k-1}^u(i), i \in [n])$, $(\theta_{k-1}^u(i), i \in [n])$ as the state of the system at $k-1$ and to the end of the revealed link j_k^u, as the "new observation" at time k. Given the state of the system at time $k-1$, the new observation at time k, and an adapted control u, it is immediate to see from Definition of the cascade, that the state at time k is completely determined.

3.2 Optimal control problem

We consider as optimization criterion the value of the financial system at the end of the cascade process. This is defined similarly to the case without an initial shock in (3)

$$J_{T^u}^u = \#\{i \in [c], \theta_{T^u}^u(i) > 0\} y_c + \#\{i \in [p], \theta_{T^u}^u(i) > 0\} y_p + \mathbb{E}[N_{T^u} \mid \mathcal{G}_{T^u}^u], \tag{7}$$

where

$$N_{T^u}^u := \#\{(i,j) \in \mathcal{E}, i \in [c]\, j \in [p], \theta^u(j) > 0)\}. \tag{8}$$

The last term in (7) represents the expectation of the number of loans renewed by core banks to the peripheral banks, i.e., the external projects that were funded by peripheral banks by borrowing from core banks. Note that in the first two terms of (7), we count in only the projects of non-failed banks, since we considered that the projects of failed banks are liquidated.[5]

We can define now the optimal control problem.

Problem 4 (Optimal control under partial information)**.**

$$\Phi_0 := \max_{u \in \mathcal{U}^a} \mathbb{E}\big(J^u_{T^u})|\mathcal{G}^u_0\big)$$

where $\mathcal{U}^a$ denotes the set of $(\mathcal{G}^u_k)_{1 \le k \le m}$-adapted processes with values in $\mathcal{U}_M$.

Since the control space $\mathcal{U}^a$ is finite, there exists an optimal control. However, *a priori* this control depends on the whole history of the system, which would make the problem intractable. In the next section, we prove the main result of the paper, namely a state space collapse theorem: contagion and the optimal control can be determined using some aggregates of the previous state of the system. Moreover we show that these aggregates represent a controlled Markov chain.

3.3 State space collapse theorem

We now prove the main result of the paper, namely that the control problem is a Markov decision problem of a low-dimensional chain. Intuitively, from the point of view of the controller, nodes that have the same connectivity and distance to failure are identical. Therefore, we only need to keep track of their number during the cascade, rather than their individual state, which is a significant reduction in the state of the system. Since we assumed that all core banks have the same connectivity, we only need to keep track of the number of core banks at each distance to failure. For simplicity of the exposition, we also assume that peripheral banks have maximum distance to failure 1, so they can only have failed or be at distance to failure 1.

Starting from the variables defined in the previous section $((v^u_k(i), i \in [n]), (\theta^u_k(i), i \in [n])$, we thus define the following aggregates.

$$A^u_k := \big((C^u_k(\vartheta), \vartheta = 1 \dots \vartheta_{max}), P^u_k, I^{c,u}_k, I^{p,u}_k\big),\ k = 0, \dots, m, \tag{9}$$

with

- $C^u_k(\vartheta), \vartheta = 1 \dots \vartheta_{max}$: the number of core banks having distance to failure equal to ϑ at step k:
$$C^u_k(\vartheta) := \#\{i \in [c], \theta^u(i) = \vartheta\};$$
- P^u_k: the number of peripheral banks that have not failed up to step k (and consequently at distance to failure 1 at step k):
$$P^u_k := \#\{i \in [p], \theta^u(i) = 1\};$$

[5]This above definition of the value of the financial system is in the case of zero recovery rates for the external project. We could introduce a recovery rate, but this would not change our results which are essentially qualitative.

- $I_k^{c,u}(\vartheta) := \sum_{i\in[c],\theta(i)=\vartheta} v_k^u(i)$: the number of interventions up to step k on core banks having distance to failure equal to ϑ;
- $I_k^{p,u} := \sum_{i\in[p]} v_k^u(i)$: the number of interventions up to step k on peripheral banks.

For all $k = 1, \ldots, m$, A_k^u takes values in

$$\mathcal{A} := \{(x_c, x_p, i_c, i_p) \in [0,c]^{\vartheta_{max}} \times [0,p] \times [0,M]^{\vartheta_{max}+1} \mid \sum_{\vartheta=1}^{\vartheta_{max}} x_c(\vartheta) \leq c, \sum_{\vartheta=1}^{\vartheta_{max}} i_c + i_p \leq M\}. \quad (10)$$

In the sequel we will use the following aggregate of the new observation and will refer to this also as "the new observation" at time k:

$$Y_k := (\mathbb{1}_{j_k^u \in [c]}, \theta_k(j_k^u))$$

with values in

$$\mathcal{O} := \{(\ell, \vartheta), \ell \in \{0,1\}, \vartheta \leq \vartheta_{max}\}.$$

We have the following main theorem.

Theorem 5. *(State space collapse)*
Let $\mathcal{U}^{Feedback}$ be the set of feedback controls $u = (u_k)_{k\in[1,m]} \in \mathcal{U}^a$ of the form

$$u_k = U_k(A_{k-1}^u, Y_k), \text{ for } 1 \leq k \leq m,$$

with U_k a deterministic function. For any feedback control $u \in \mathcal{U}^{Feedback}$, A^u is a Markov chain, with respect to its own filtration $\big(\sigma((A_l^u)_{l\leq k})\big)_{k\leq m}$.

Moreover, there exists an optimal control $u^* \in \mathcal{U}^a$ for Problem 4 and we have that $u^* \in \mathcal{U}^{Feedback}$.

Proof. The proof of the theorem necessitates several steps. We show for any adapted control u with values in $\mathcal{U}_M$ that:

(i) The terminal criterion can be written using only the aggregate variables at the stopping time T^u, $A_{T^u}^u$:

First, note that the stopping time representing the end of the cascade T^u can be written, using the aggregate state, as the following exit time

$$T^u = \inf\{k \mid \big(c - \sum_{\vartheta=1}^{\vartheta_{max}} C_k^u(\vartheta)\big)\lambda - k = 0\}. \quad (11)$$

This represents the time there are no more hidden links in Definition 3 of the cascading failures since the number of hidden links that exit failed core banks is given by $\#\mathcal{H}_k = \big(c - \sum_{\vartheta=1}^{\vartheta_{max}} C_k^u(\vartheta)\big)\lambda - k$, i.e. the total number of links that exit failed core banks minus the number of revealed links k.

Second, the terminal criterion in (7) can be written as

$$J^u_{T^u} := \sum_{\vartheta=1}^{\vartheta_{max}} C^u_{T^u} y_c + P^u_{T^u} y_p + \mathbb{E}(N^u_{T^u} \mid \mathcal{G}^u_{T^u}). \tag{12}$$

The conditional law of the random variable N_{T^u} on the $\mathcal{G}^u_{T^u}$ is the law of a hypergeometric random variable

$$N_{T^u} \sim Hyp(m + p - T^u, \#\{i \in [p], \theta^u(i) > 0\} - I^{p,u}_{T^u}, m - T^u).$$

This represents a Hypergeometric random variable whose conditional distribution has the following parameters: number of draws $m - T^u$(remaining unobserved out-going links of core banks), the size of the population $m + p - T^u$ (remaining unobserved in-coming links of core and peripheral banks). The number of remaining unobserved in-coming links of peripheral banks is given by the peripheral banks that were never matched, which are the peripheral banks that stayed at their initial distance to failure without any intervention, i.e.,

$$\#\{i \in [p], \theta^u(i) > 0\} - I^{p,u}_{T^u}.$$

It is easy to see that N_{T^u} has the above conditional distribution, from the conditional uniform (without replacement) law of the matching of banks, see Remark 1.

Then, we have

$$\mathbb{E}[N_{T^u} \mid \mathcal{G}^u_{T^u}] = \frac{(m - T^u)\big(\#\{i \in [p], \theta^u(i) > 0\} - I^{p,u}_{T^u}\big)}{m + p - T^u}.$$

We thus conclude this step of the proof.

(ii) The transition probabilities of the aggregate variables A^u_{k+1} depend only on the previous state A^u_k, the new (aggregate) observation $Y_{k+1} = (\mathbb{1}_{j^u_{k+1} \in [c]}, \theta(j^u_{k+1}))$ and the control at time $k+1$, u_{k+1}:

For $k \in [T^u, m]$:

$$A^u_{k+1} = A^u_k.$$

For $k \in [0, T^u)$, we have the following transition probabilities for the aggregate state A^u_k (where we only specify the changes in the aggregate state variables).

If $u_{k+1} = 0$,

$$\begin{cases} C^u_{k+1}(\vartheta) = C^u_k(\vartheta) - 1, C^u_{k+1}(\vartheta - 1) = C^u_k(\vartheta - 1) + 1 & \text{for } ((\mathbb{1}_{j^u_{k+1} \in [c]}, \theta(j^u_{k+1})) = (1, \vartheta), \vartheta > 1 \\ C^u_{k+1}(1) = C^u_k(1) - 1 & \text{for } ((\mathbb{1}_{j^u_{k+1} \in [c]}, \theta(j^u_{k+1})) = (1, 1) \\ P^u_{k+1} = P^u_k - 1 & \text{for } ((\mathbb{1}_{j^u_{k+1} \in [c]}, \theta(j^u_{k+1})) = (0, 1). \end{cases}$$

The second line above represents the case when there is a new failure.

If $u_{k+1} = 1$,

$$\begin{cases} I^{c,u}_{k+1}(\vartheta) = I^{c,u}_{k}(\vartheta) + 1 & \text{for } (\mathbb{1}_{j^u_{k+1}\in[c]}, \theta(j^u_{k+1})) = (1, \vartheta), \vartheta \geq 1 \\ I^{p,u}_{k+1} = I^{p,u}_{k} + 1 & \text{for } ((\mathbb{1}_{j^u_{k+1}\in[c]}, \theta(j^u_{k+1})) = (0, 1). \end{cases}$$

(iii) The conditional law of the new observation $Y_{k+1} = (\mathbb{1}_{j^u_{k+1}\in[c]}, \theta(j^u_{k+1}))$ on the sigma-algebra $\mathcal{G}^u_k$ depends only on the aggregate variable A^u_k. We have the following lemma.

Lemma 6. *Let $k < T^u_k$. The node j^u_{k+1}, representing the end node of the link (i^u_{k+1}, j^u_{k+1}) chosen uniformly over the set $\mathcal{H}^u_k \neq \emptyset$ has the following law conditional on the sigma-algebra $\mathcal{G}^u_k$.*

$$\mathbb{P}[j^u_{k+1} = i \mid \mathcal{G}^u_k] = \begin{cases} \frac{(\lambda - (\vartheta_{max} - \theta^u_k(i)) - v^u_k(i)}{m+p-k} & i \in [c] \\ \frac{1 - v^u_k(i)}{m+p-k} & i \in [p] \end{cases} \tag{13}$$

Moreover we have for $\vartheta = 1, \ldots, \vartheta_{max}$,

$$\begin{aligned} p_k(1, \vartheta) &:= \mathbb{P}[(\mathbb{1}_{j^u_{k+1}\in[c]}, \theta(j^u_{k+1})) = (1, \vartheta) \mid \mathcal{G}^u_k] = \frac{C^u_k(\vartheta)\big(\lambda - (\vartheta_{max} - \vartheta) - I^{c,u}_k(\vartheta)\big)}{m+p-k}, \\ p_k(0, 1) &:= \mathbb{P}[(\mathbb{1}_{j^u_{k+1}\in[c]}, \theta(j^u_{k+1})) = (0, 1) \mid \mathcal{G}^u_k] = \frac{P^u_k(\vartheta) - I^{p,u}_k}{m+p-k}. \end{aligned} \tag{14}$$

Proof. Remark that $\mathcal{H}^u_k \neq \emptyset$ for $k < T^u$. The denominator represents in each of the cases above the number of available in-coming half links of a node. At step k one credit line will be withdrawn uniformly over the set of hidden links. Therefore the conditional law of the identity of the end link is given by the uniform matching to the remaining half-links. Equations 14 result by summing over all core banks at distance to failure $\vartheta \in [1, \vartheta_{max}]$ and respectively over all peripheral banks. ☐

It is now clear from the above that we can write

$$A^u_{k+1} = f(A^u_k, Y_{k+1}, u_{k+1}), \tag{15}$$

with the $\mathcal{G}^u_k$- conditional law of the new observation Y_{k+1} depending only on A^u_k. Therefore A^u_k is a Markov chain (adapted to its own filtration).

By the standard theory of control of Markov chains, see e.g. Bertsekas and Shreve (1978), the last statement of the theorem now follows. ☐

The following proposition allow us to further reduce the dimension of the problem, under criterion (7). Regarding the interventions on the core banks, we will only need to keep track of their total number. It will be understood that interventions on core banks always occur at distance to failure 1 and there are no interventions on core banks at distance $\vartheta > 1$:

$$I_k^{c,u}(\vartheta) = 0,\ k = 0, \dots, m,\ \vartheta > 1.$$

This will not be necessarily the case if we consider other optimality criteria.

The proposition also states that we lose nothing by restricting the control to the set $\mathcal{U}_M$ rather than the larger control set which allows intervention on multiple banks at the same time denoted by

$$\bar{\mathcal{U}}_M := \{(u_k \in \{0,1\}^n, 1 \le k \le m), \sum_{1 \le k \le m} \sum_{i \in [n]} u_k(i) \le M\},$$

with $\mathcal{U}_M \subseteq \bar{\mathcal{U}}_M$.

Proposition 7. *Let the optimization criterion $J^u_{T^u}$ given by (7) and consider the Problem 4 with the control set $\bar{\mathcal{U}}_M$. Then, for $1 \le k \le m$, $u_k(i) = 0$ if $i \neq j_k^u$ or if $i = j_k^u$ and $\theta(j_k^u) > 1$.*

Proof. The dependence of the criterion $J^u_{T^u}$ on the set of core banks $\{i \in [c] \mid \theta_{T^u}(i) > 0\}$ is only through its cardinal. Any given core bank will modify the state of the other banks only after the time it fails. (The set of hidden links $\mathcal{H}_k$ that drives the contagion grows only when there is a new failure of a core bank). Moreover, each core bank that fails is chosen randomly from the set of core banks at distance to failure 1 before they are chosen. Before this time, each of these banks' distance to failure decreased by 1 each time it was chosen as an end node of a hidden link. Let $\tau(i) := \inf\{k, j_k^u = i, \theta_k^u(i) = 1\}$, possibly ∞. Then for $k < \tau(i)$, $u_k(i) = 0$. $\square$

4 Numerical analysis

4.1 Dynamic Programming method

Using the state space collapse Theorem 5 and Proposition 7, the problem is now tractable and can be solved using dynamic programming. We have the following result.

Proposition 8. *The value function for Problem 4 is given by a function ϕ_0 as $\Phi_0 = \phi_0(A_0)$*

where the function ϕ_0 is defined backward recursively by the Bellman equation:
For $x = (x_c, x_p, i_c, i_p) \in \mathcal{A}$

$$\phi_k(x) = \begin{cases} \max_{u_{k+1} \in \mathcal{U}^{Feedback}_{k+1,x}} \mathbb{E}\big(\phi_{k+1}(A^u_{k+1}) \mid A^u_k = x\big), & \big(c - \sum_{\vartheta=1}^{\vartheta_{max}} x_c(\vartheta)\big)\lambda > k \\ J\Big(\big(c - \sum_{\vartheta=1}^{\vartheta_{max}} x_c(\vartheta)\big)\lambda, x\Big) & \textit{otherwise}, \end{cases} \tag{16}$$

where $U^{Feedback}_{k+1,x}$ is the set of strategies of the form $U_{k+1}(x, y)$ (for $x \in \mathcal{A}, y \in \mathcal{O}$) with values in $\{0, 1 \wedge (M - i_c - i_p)\}$ and with A^u_{k+1} given from (15) by

$$A^u_{k+1} = f\big(A^u_k, Y_{k+1}, U_{k+1}(A^u_k, Y_{k+1})\big)$$

Number of core banks	$c = 5$
Number of peripheral banks	$p = 40$
Number of projects of core banks	$y_c = 1$
Number of projects of peripheral banks	$y_p = 1$
Connectivity	$\lambda \in [1, 20]$
Maximum distance to failure	$\vartheta_{max} = 3$
Intervention budget	$M = 4$

Table 1: Parameters of the stylized model.

and the expectation in (16) is understood as the conditional law of Y_{k+1} on the event $\{A_k^u = x\}$. Note that if we start the backward recursion at step m, we are guaranteed to be in the second case above. We can further write, using the above proposition stating that control only occurs at distance to failure 1

$$\begin{aligned}\phi_k(x) = &\sum_{\vartheta>1} p_k(1,\vartheta)\phi_{k+1}(x' = x_c - e_\vartheta + e_{\vartheta-1}, x_p, i_c, i_p) \\ &+ p_k(1,1)\max\{\phi_{k+1}(x' = x_c - e_\vartheta, x_p, i_c, i_p), \phi_{k+1}(x' = x_c, x_p, i_c+1, i_p)\mathbb{1}_{i_c+1, i_p<M}\} \\ &+ p_k(0,1)\max\{\phi_{k+1}(x' = x_c, x_p - 1, i_c, i_p), \phi_{k+1}(x' = x_c, x_p, i_c, i_p+1)\mathbb{1}_{i_c+1, i_p<M}\} \\ &+ \big(1 - \sum_{\vartheta\geq 1} p_k(1,\vartheta) - p_k(0,1)\big)\phi_{k+1}(x) \end{aligned} \tag{17}$$

where e_ϑ is the ϑ-th unit vector in $\mathbb{R}^{\theta_{max}}$ and the transition probabilities are given in (14).

The first line corresponds to the case when a core bank at distance to failure > 1 is chosen. The second line corresponds to the case when a core bank at distance to failure 1 is chosen. The third line corresponds to the case when a peripheral bank at distance to failure 1 is chosen. The last line corresponds to the case when the chosen bank already failed.

4.2 Numerical experiments

We solve Problem 4 by implementing the dynamic programming equation (17) with the numerical values of the parameters given in Table 1, unless otherwise stated.

Connectivity and value of the financial system We first compare the value of the financial system under different connectivities and different intervention budgets. We understand by value of the financial system the value of the optimal control Problem 4 at step 0 under criterion (7).

Figure 3 shows that the relation between value and connectivity is non-monotonous. When the initial state of the system is such that there are no failed banks ($\sum_{\vartheta=1}^{3} C_0(\vartheta) = c$), then the value of the system always increases with connectivity. When there is at least an initially failed bank in the system, $\sum_{\vartheta=1}^{3} C_0(\vartheta) < c$, then the expected value of the system first increases with connectivity and then decreases with connectivity. This holds for the case without intervention and for various intervention budgets. A priori, connectivity increases the value of the financial system. However, when there are initially failed banks in the system, contagion also increases and the value of the system is given by these two opposite effects. This means that after a certain level of connectivity, which depends on the initial state of the system, the negative effect of contagion becomes predominant and the value of the system decreases with connectivity. Therefore, there is an optimal level of connectivity for each initial state for a given budget, and this optimum is increasing in the budget of the controller.

Another important insight is that up to a certain level of connectivity, the value of the financial system is higher than the value in absence of the network (connectivity $\lambda = 0$). Connectivity, up to a point, allows for an increase in the value of the system, but this increase is significant when there is a controller that mitigates contagion when there are initial failures in the system. Indeed, as Figure 4 shows, optimal control ensures that the benefit from the connectivity is not surpassed by the contagion cost, up to a certain connectivity. Note that the preserved value (difference in value with intervention and without intervention) surpasses the intervention budget. This conclusion depends on our assumption on the complete liquidations and we expect that it may no longer hold under partial liquidations.

We now describe the policy of the controller. We compare the number of interventions on core and peripheral banks for varying connectivity. Figure 5 plots the difference in the expected number of interventions on the core and peripheral banks, under two cases $y_c = 1,\ y_p = 1$ (the core and peripheral banks have the same number of projects, so the difference between core and peripheral banks is driven by connectivity alone) and $y_c = 10,\ y_p = 1$ (the number of projects of core banks is much larger than the number of projects of peripheral banks). When there are no failed core banks initially in the system, we verify that there are also no interventions as there is no contagion in the system.

When there is at least one initially failed core bank in the system, we distinguish the following pattern. In the low connectivity regime, the number of interventions in peripheral banks increases with connectivity. After a certain level of connectivity (medium connectivity regime) it is optimal to increase the number of interventions on core banks. Then, in the high connectivity regime, it is again optimal to intervene towards the peripheral banks. This can be interpreted as follows. In the low connectivity regime, cores are mainly connected to peripheral banks and thus the contagion is direct from core to peripheral. In this case it is better to intervene directly on the peripheral banks. In the medium connectivity regime, core banks get connected to other core banks and there is contagion among core banks. It is then optimal to intervene on core banks in order to prevent this type of contagion. Then in the high connectivity regime, the number of cycles among core banks is large enough that contagion amount them cannot be stopped. It is then again optimal to intervene on the peripheral banks to stop as least the contagion from core to peripheral. This effect holds even if the number of projects of core banks is much larger than the number of projects of peripheral banks, so it is a connectivity-driven effect.

The value of adapted intervention In this section, we compare the optimal intervention adapted to the filtration $\mathcal{G}_k^u$ with the case of open-loop control, i.e., when the corresponding budget is used only at time 0 to increase the initial distance to failure of certain banks so as to maximize the value in the system

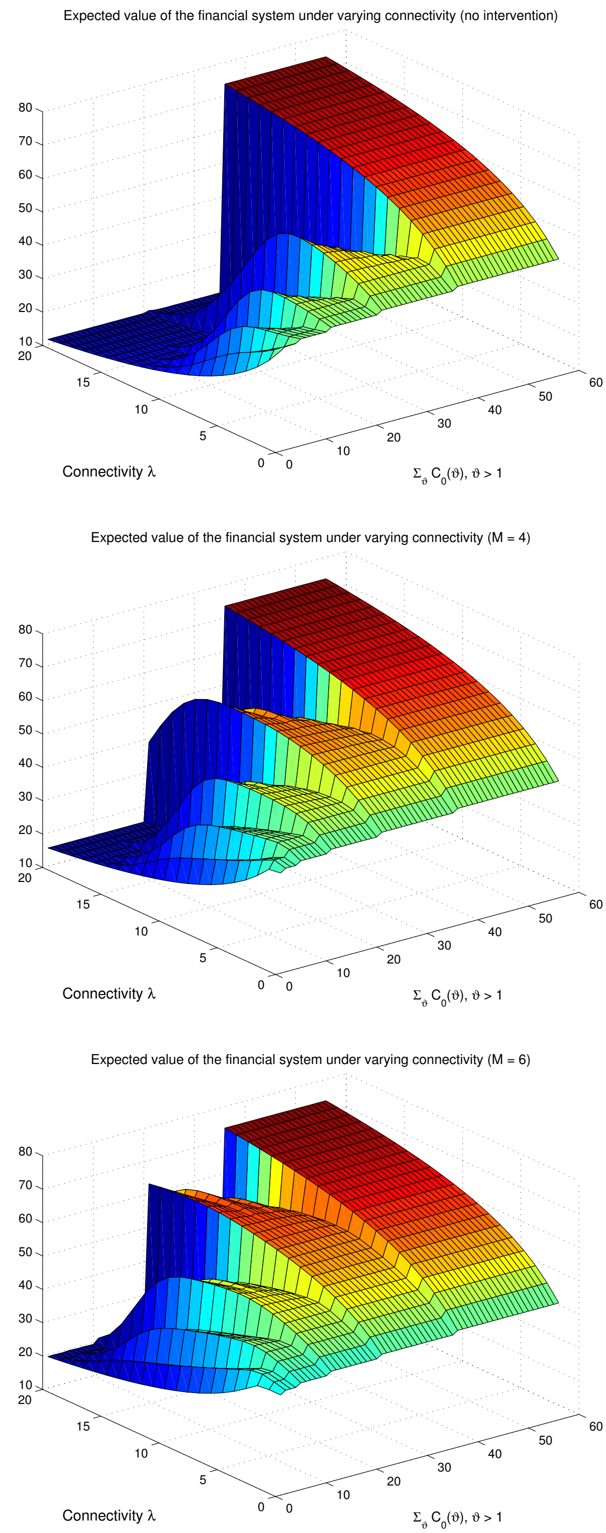


Figure 3: Value of the financial system (Φ_0) under varying connectivity and different intervention budgets. The horizontal axis represents the triplet giving the initial state of the core banks $(C_0(1), C_0(2), C_0(3))$ in increasing order of $\sum_{\vartheta=1}^{3} C_0(\vartheta)$ (the number of non-failed core banks). $P_0 = p$. The different slices correspond to different values of $\sum_{\vartheta=1}^{3} C_0(\vartheta) \in [0, c]$. For connectivity 0, the network is absent.

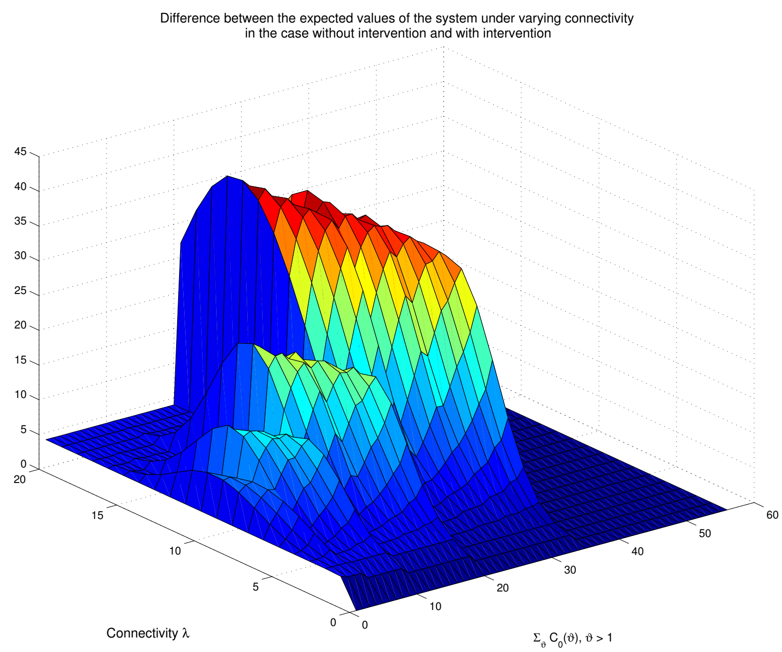
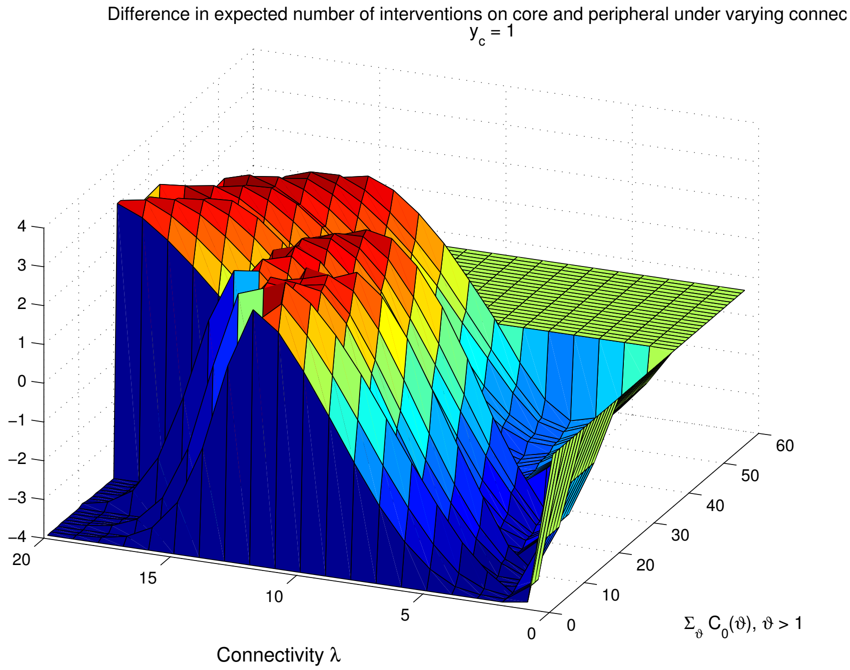


Figure 4: Difference between the values of the system under varying connectivity in the case without intervention and with intervention ($M = 4$). The horizontal axis represents the triplet giving the initial state of the core banks $(C_0(1), C_0(2), C_0(3))$ in increasing order of $\sum_{\vartheta=1}^{3} C_0(\vartheta)$ (the number of non-failed core banks). $P_0 = p$. The different slices correspond to different values of $\sum_{\vartheta=1}^{3} C_0(\vartheta) \in [0, c]$. For connectivity 0, the network is absent.



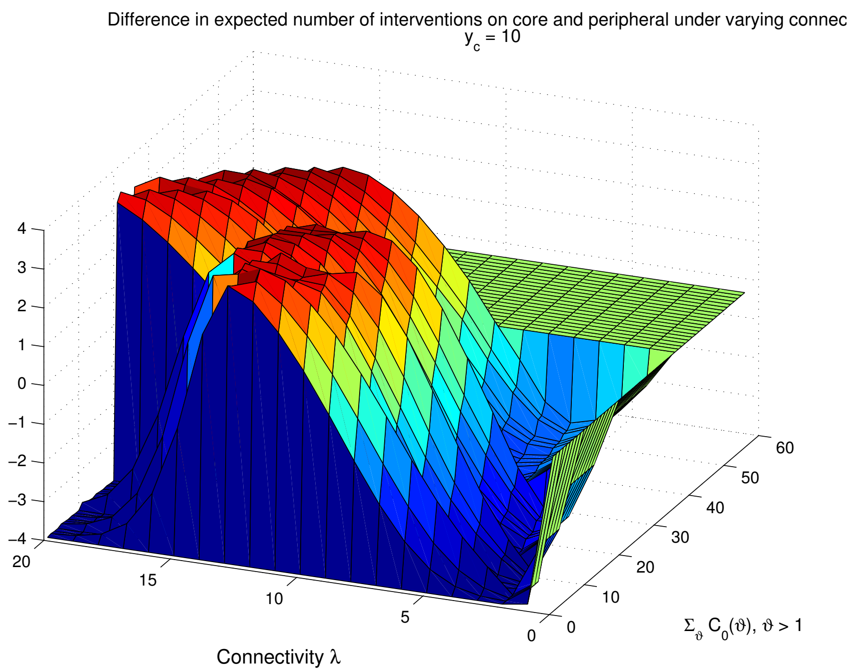


Figure 5: Difference between the expected number of interventions on core and peripheral banks under varying connectivity. The horizontal axis represents the triplet giving the initial state of the core banks $(C_0(1), C_0(2), C_0(3))$ in increasing order of $\sum_{\vartheta=1}^{3} C_0(\vartheta)$ (the number of non-failed core banks). $P_0 = p$. The different slices correspond to different values of $\sum_{\vartheta=1}^{3} C_0(\vartheta) \in [0, c]$. For connectivity 0, the network is absent.

We define the value of adapted intervention as follows.

Definition 9. The **value of adapted intervention** is defined as the difference

$$\Phi_0 - \tilde{\Phi}_0,$$

with

$$\tilde{\Phi}_0 = \max_{u \in [1,M]^n, \sum_{i \in [n]} u(i) \leq M} \mathbb{E}\big(J(T^u, A^u_{T^u})\big). \tag{18}$$

It is easy to see that the value $\tilde{\Phi}_0$ results as an optimization problem in dimension ϑ_{max}, over the increases in the initial distances to failure, under budget constraints. Let us denote by ΔC the increase in the initial number of core banks at different distances to failure. After intervention, the increase in the initial number of banks at different distances to failure verify $\sum_{\vartheta \geq 1} \Delta C_0(\vartheta) = 0$, since the number of initially failed banks does not change. Moreover, since banks can only increase their distances to failure after intervention: $\forall\, 2 \leq \vartheta \leq \vartheta_{max}, \sum_{k > \vartheta} \Delta C_0(\vartheta) \geq 0$.

Thus, we have the following proposition.

Proposition 10. *The solution of the optimization Problem* (18) *is given by*

$$\tilde{\Phi}_0 = \max_{\Delta C_0} \phi_0(A_0 = (C_0 + \Delta C_0, P_0, 0, 0))$$

$$\textit{subject to: } \sum_{\vartheta=1}^{\vartheta_{max}} \Delta C_0(\vartheta) \cdot \vartheta \leq M,$$

$$\sum_{\vartheta \geq 1} \Delta C_0(\vartheta) = 0, \ \forall\, 2 \leq \vartheta \leq \vartheta_{max}, \sum_{k > \vartheta} \Delta C_0(k) \geq 0.$$

Figure 6 shows the difference between the value of the system with an open-loop control and the value of the system without intervention.

Figure 7 shows that the adapted intervention strategy performs *significantly* better than the open-loop control, constrained to take place at time 0. This shows that the gain from intervention, plotted in Figure 4 is mostly due to the existence of an *adapted* control.

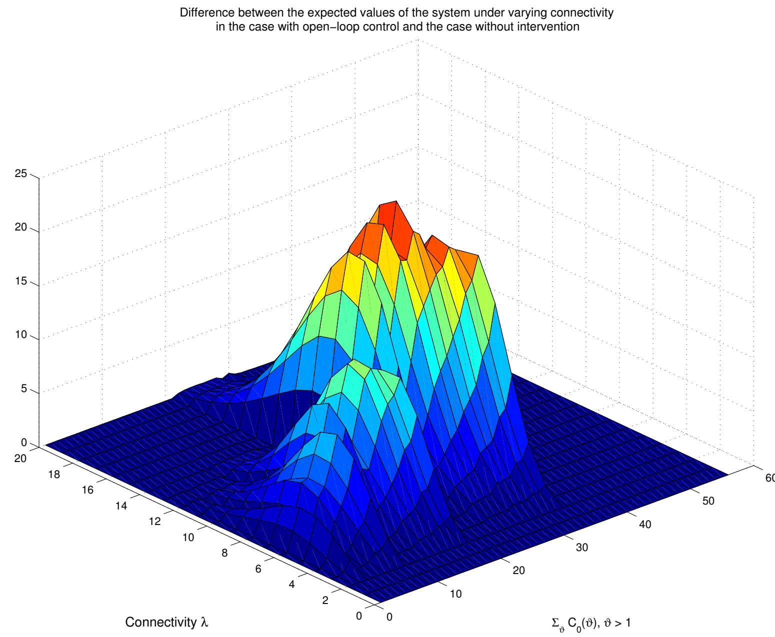


Figure 6: Difference between the value function with open-loop control and the case with no intervention. $\sum_\vartheta C_0(\vartheta)$ is increasing in the number of failed core banks at time 0.

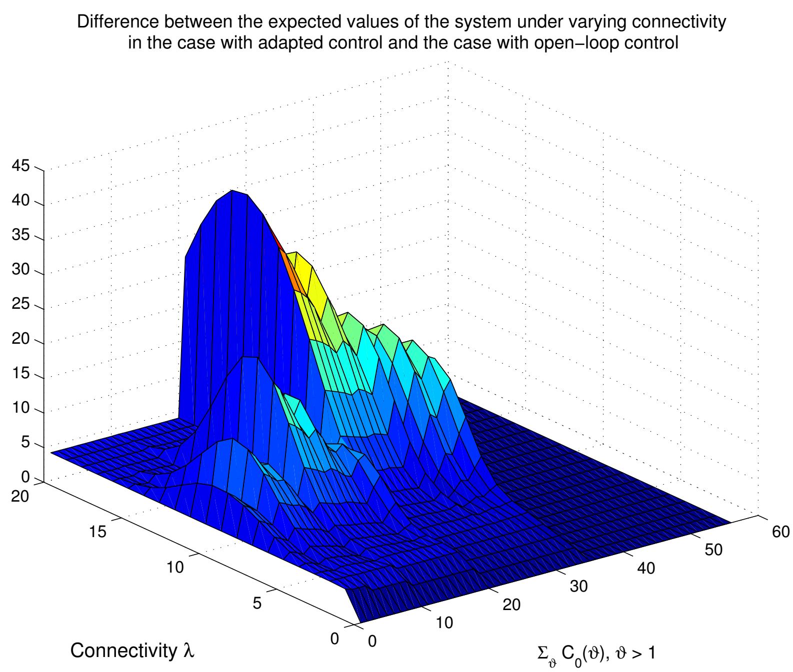


Figure 7: Difference between value function open-loop control and with $\mathcal{G}_t$-adapted intervention. $\sum_\vartheta C_0(\vartheta)$ is increasing in the number of failed core banks at time 0.

5 Conclusions and further research directions

The most important insight of the paper relates the value of the financial system, connectivity and optimal intervention. Up to a certain connectivity, the value of the financial system increases with connectivity. This implies that a connected system prone to contagion and thus depending on intervention is preferable than a disconnected system. However, this is no longer the case if connectivity becomes too large and even in the case there is intervention, the value of the system may fall below the value of the disconnected system. We identify the optimal policy of a controller that injects liquidity into the system so as to maximize its expected value. We find that in the low connectivity regime the controller prefers to inject directly into the periphery. In the medium connectivity regime, the policy switches and it is preferable to inject liquidity into the core banks.

Somewhat surprisingly, this is no longer true if the connectivity is high: in the highest connectivity regime, it is again optimal to intervene into the periphery rather than the core. This insight shows that it is far from obvious that connectivity of a core bank should always be brought forward as an argument for priority intervention and it may be sometimes preferable to invest in non-core banks that lend directly to the economy.

We have further analyzed the reduction in the magnitude of contagion. We find that contagion is significantly mitigated by intervention, provided the controller uses an adapted strategy.

We have introduced a stylized model, which despite its simplicity suggests that there is an optimal level of connectivity for the financial system. However, the system is dependent on intervention at that level of connectivity, in case there are initial failures. The natural question remains how to create incentives for the banks to attain an optimal level of connectivity and how to design a guarantee fund of the banks that would represent the intervention fund.

There are also may ways in which more realism can be added to the model. Of course, the mechanic behavior of banks that withdraw all credit lines and fully liquidate their portfolios as soon as they reach a distance to failure zero is highly stylized as banks are strategic about which lines to withdraw and how much to liquidate. For now, this is required for the tractability of the optimal control problem under partial network observation, which to our knowledge has not been tackled before. Relaxing the assumption of full liquidation and full withdrawal of credit lines in case of failures is left for future research.

Other extensions include the design of optimal sharing rules of the benefit from connectivity and control among the controller and the financial system, for example in the form of interest. In this model, we do not consider the possibility that the distance to failure of a bank can exogenously rise. Therefore the model is more adapted to a short term crisis. It would be an interesting extension to allow for an exogenous rise in the distance to failure.

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