Boundary Behavior of the Ginzburg-Landau Order Parameter in the Surface Superconductivity Regime

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Abstract

We study the 2D Ginzburg-Landau theory for a type-II superconductor in an applied magnetic field varying between the second and third critical value. In this regime the order parameter minimizing the GL energy is concentrated along the boundary of the sample and is well approximated to leading order (in L^2 norm) by a simplified 1D profile in the direction perpendicular to the boundary. Motivated by a conjecture of Xing-Bin Pan, we address the question of whether this approximation can hold uniformly in the boundary region. We prove that this is indeed the case as a corollary of a refined, second order energy expansion including contributions due to the curvature of the sample. Local variations of the GL order parameter are controlled by the second order term of this energy expansion, which allows us to prove the desired uniformity of the surface superconductivity layer.

Contents

1 Introduction

The Ginzburg-Landau (GL) theory of superconductivity, originating in [GL], provides a phenomenological, macroscopic, description of the response of a superconductor to an applied magnetic field. Several years after it was introduced, it turned out that it could be derived from the microscopic BCS theory [BCS, Gor] and should thus be seen as a mean-field/semiclassical approximation of many-body quantum mechanics. A mathematically rigorous derivation starting from BCS theory has been provided recently [FHSS].

Within GL theory, the state of a superconductor is described by an order parameter $\Psi : \mathbb{R}^2 \to \mathbb{C}$ and an induced magnetic vector potential $\kappa\sigma\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ generating an induced magnetic field

$$h = \kappa\sigma \,\mathrm{curl}\, \mathbf{A}.$$

The ground state of the theory is found by minimizing the energy functional[1]

$$\mathcal{G}^{\mathrm{GL}}_{\kappa,\sigma}[\Psi,\mathbf{A}] = \int_{\Omega} \mathrm{d}\mathbf{r} \, \left\{ \left| \left(\nabla + i\kappa\sigma\mathbf{A} \right) \Psi \right|^2 - \kappa^2 |\Psi|^2 + \tfrac{1}{2}\kappa^2 |\Psi|^4 + (\kappa\sigma)^2 \left| \mathrm{curl}\mathbf{A} - 1 \right|^2 \right\}, \tag{1.1}$$

where $\kappa > 0$ is a physical parameter (penetration depth) characteristic of the material, and $\kappa\sigma$ measures the intensity of the external magnetic field, that we assume to be constant throughout the sample. We consider a model for an infinitely long cylinder of cross-section $\Omega \subset \mathbb{R}^2$, a compact simply connected set with regular boundary.

Note the invariance of the functional under the gauge transformation

$$\Psi \to \Psi e^{-i\kappa\sigma\varphi}, \qquad \mathbf{A} \to \mathbf{A} + \nabla\varphi, \tag{1.2}$$

which implies that the only physically relevant quantities are the gauge invariant ones such as the induced magnetic field h and the density $|\Psi|^2$. The latter gives the local relative density of electrons bound in Cooper pairs. It is well-known that a minimizing Ψ must satisfy $|\Psi|^2 \leq 1$. A value $|\Psi| = 1$ (respectively, $|\Psi| = 0$) corresponds to the superconducting (respectively, normal) phase where all (respectively, none) of the electrons form Cooper pairs. The perfectly superconducting state with $|\Psi| = 1$ everywhere is an approximate ground state of the functional for small applied field and the normal state where Ψ vanishes identically is the ground state for large magnetic field. In between these two extremes, different mixed phases can occur, with normal and superconducting regions varying in proportion and organization.

A vast mathematical literature has been devoted to the study of these mixed phases in type-II superconductors (characterized by $\kappa > 1/\sqrt{2}$), in particular in the limit $\kappa \to \infty$ (extreme type-II). Reviews and extensive lists of references may be found in [FH3, SS2, Sig]. Two main phenomena attracted much attention:

- The formation of hexagonal vortex lattices when the applied magnetic field varies between the first and second critical field, first predicted by Abrikosov [Abr], and later experimentally observed (see, e.g., [H *et al*]). In this phase, vortices (zeros of the order parameter with quantized phase circulation) sit in small normal regions included in the superconducting phase and form regular patterns.

[1] Here we use the units of [FH3], other choices are possible, see, e.g., [SS2].

- The occurrence of a surface superconductivity regime when the applied magnetic fields varies between the second and third critical fields. In this case, superconductivity is completely destroyed in the bulk of the sample and survives only at the boundary, as predicted in [SJdG]. We refer to [N *et al*] for experimental observations.

We refer to [CR] for a more thorough discussion of the context. We shall be concerned with the surface superconductivity regime, which in the above units translates into the assumption

$$\sigma = b\kappa \tag{1.3}$$

for some fixed parameter b satisfying the conditions

$$1 < b < \Theta_0^{-1} \tag{1.4}$$

where Θ_0 is a spectral parameter (minimal ground state energy of the shifted harmonic oscillator on the half-line, see [FH3, Chapter 3]):

$$\Theta_0 := \inf_{\alpha \in \mathbb{R}} \inf \left\{ \int_{\mathbb{R}^+} \mathrm{d}t \left(|\partial_t u|^2 + (t+\alpha)^2 |u|^2 \right), \; \|u\|_{L^2(\mathbb{R}^+)} = 1 \right\}. \tag{1.5}$$

From now on we introduce more convenient units to deal with the surface superconductivity phenomenon: we define the small parameter

$$\varepsilon = \frac{1}{\sqrt{\sigma\kappa}} = \frac{1}{b^{1/2}\kappa} \ll 1 \tag{1.6}$$

and study the asymptotics $\varepsilon \to 0$ of the minimization of the functional (1.1), which in the new units reads

$$\mathcal{G}_\varepsilon^{\mathrm{GL}}[\Psi, \mathbf{A}] = \int_\Omega \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \Psi \right|^2 - \frac{1}{2b\varepsilon^2} \left(2|\Psi|^2 - |\Psi|^4 \right) + \frac{b}{\varepsilon^4} \left| \mathrm{curl} \mathbf{A} - 1 \right|^2 \right\}. \tag{1.7}$$

We shall denote

$$E_\varepsilon^{\mathrm{GL}} := \min_{(\Psi, \mathbf{A}) \in \mathscr{D}^{\mathrm{GL}}} \mathcal{G}_\varepsilon^{\mathrm{GL}}[\Psi, \mathbf{A}], \tag{1.8}$$

with

$$\mathscr{D}^{\mathrm{GL}} := \left\{ (\Psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \right\}, \tag{1.9}$$

and denote by $(\Psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})$ a minimizing pair (known to exist by standard methods [FH3, SS2]).

The salient features of the surface superconductivity phase are as follows:

- The GL order parameter is concentrated in a thin boundary layer of thickness $\sim \varepsilon = (\kappa\sigma)^{-1/2}$. It decays exponentially to zero as a function of the distance from the boundary.
- The applied magnetic field is very close to the induced magnetic field, $\mathrm{curl}\, \mathbf{A} \approx 1$.
- Up to an appropriate choice of gauge and a mapping to boundary coordinates, the ground state of the theory is essentially governed by the minimization of a 1D energy functional in the direction perpendicular to the boundary.

A review of rigorous statements corresponding to these physical facts may be found in [FH3]. One of their consequences is the energy asymptotics

$$E_\varepsilon^{\mathrm{GL}} = \frac{|\partial\Omega| E_0^{\mathrm{1D}}}{\varepsilon} + \mathcal{O}(1), \tag{1.10}$$

where $|\partial\Omega|$ is the length of the boundary of Ω, and E_0^{1D} is obtained by minimizing the functional

$$\mathcal{E}_{0,\alpha}^{\mathrm{1D}}[f] := \int_0^{+\infty} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \frac{1}{2b}\left(2f^2 - f^4\right) \right\}, \tag{1.11}$$

both with respect to the function f and the real number α. We proved recently [CR] that (1.10) holds in the full surface superconductivity regime, i.e. for $1 < b < \Theta_0^{-1}$. This followed a series of partial results due to several authors [Alm1, AH, FH1, FH2, FHP, LP, Pan], summarized in [FH3, Theorem 14.1.1]. Some of these also concern the limiting regime $b \nearrow \Theta_0^{-1}$. The other limiting case $b \searrow 1$ where the transition from boundary to bulk behavior occurs is studied in [FK, Kac], whereas results in the regime $b \nearrow 1$ may be found in [AS, Alm2, SS1].

The rationale behind (1.10) is that, up to a suitable choice of gauge, any minimizing order parameter Ψ^{GL} for (1.1) has the structure

$$\Psi^{\mathrm{GL}}(\mathbf{r}) \approx f_0\left(\tfrac{\tau}{\varepsilon}\right) \exp\left(-i\alpha_0 \tfrac{s}{\varepsilon}\right) \tag{1.12}$$

where (f_0, α_0) is a minimizing pair for (1.11) and (s,τ) = (tangent coordinate, normal coordinate) are boundary coordinates defined in a tubular neighborhood of $\partial\Omega$ with $\tau = \mathrm{dist}(\mathbf{r}, \partial\Omega)$ for any point $\mathbf{r}$ there. Results in the direction of (1.12) may be found in the above references, in particular [AH, FHP, FH2]. In [CR, Theorem 2.1] we proved that

$$\left\| |\Psi^{\mathrm{GL}}|^2 - f_0^2\left(\tfrac{\tau}{\varepsilon}\right) \right\|_{L^2(\Omega)} \leq C\varepsilon \ll \left\| f_0^2\left(\tfrac{\tau}{\varepsilon}\right) \right\|_{L^2(\Omega)} \tag{1.13}$$

for any $1 < b < \Theta_0^{-1}$ in the limit $\varepsilon \to 0$. A very natural question is whether the above estimate may be improved to a uniform control (in L^∞ norm) of the local discrepancy between the modulus of the true GL minimizer and the simplified normal profile $f_0\left(\frac{\tau}{\varepsilon}\right)$. Indeed, (1.13) is still compatible with the vanishing of Ψ^{GL} in small regions, e.g., vortices, inside of the boundary layer. Proving that such local deviations from the normal profile do not occur would explain the observed uniformity of the surface superconducting layer (see again [N *et al*] for experimental pictures). Interest in this problem (stated as Open Problem number 4 in the list in [FH3, Page 267]) originates from a conjecture of X.B. Pan [Pan, Conjecture 1] and an affirmative solution has been provided in [CR] for the particular case of a disc sample. The purpose of this paper is to extend the result to general samples with regular boundary (the case with corners is known to require a different analysis [FH3, Chapter 15]).

Local variations (on a scale $\mathcal{O}(\varepsilon)$) in the tangential variable are compatible with the energy estimate (1.10), and thus the uniform estimate obtained for disc samples in [CR] is based on an expansion of the energy to the next order:

$$E_\varepsilon^{\mathrm{GL}} = \frac{2\pi E_\star^{\mathrm{1D}}(k)}{\varepsilon} + \mathcal{O}(\varepsilon|\log\varepsilon|), \tag{1.14}$$

where $E_\star^{\mathrm{1D}}(k)$ is the minimum (with respect to both the real number α and the function f) of the ε-dependent functional

$$\mathcal{E}_{k,\alpha}^{\mathrm{1D}}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1-\varepsilon k t) \left\{ |\partial_t f|^2 + \frac{(t+\alpha-\frac{1}{2}\varepsilon k t^2)^2}{(1-\varepsilon k t)^2} f^2 - \frac{1}{2b}\left(2f^2 - f^4\right) \right\}, \tag{1.15}$$

where the constant c_0 has to be chosen large enough and $k = R^{-1}$ is the curvature of the disc under consideration, whose radius we denote by R. Of course, (1.11) is simply the above functional where one sets $k = 0$, $\varepsilon = 0$, which amounts to neglect the curvature of the boundary. When the curvature is constant, (1.14) in fact follows from a next order expansion of the GL order parameter beyond (1.12):

$$\Psi^{\mathrm{GL}}(\mathbf{r}) \approx f_k\left(\tfrac{\tau}{\varepsilon}\right) \exp\left(-i\alpha(k)\tfrac{s}{\varepsilon}\right) \tag{1.16}$$

where $(\alpha(k), f_k)$ is a minimizing pair for (1.15). Note that for any fixed k

$$f_k = f_0(1 + \mathcal{O}(\varepsilon)), \qquad \alpha(k) = \alpha_0(1 + \mathcal{O}(\varepsilon)), \tag{1.17}$$

so that (1.16) is a slight refinement of (1.12) but the $\mathcal{O}(\varepsilon)$ correction corresponds to a contribution of order 1 beyond (1.10) in (1.14), which turns out to be the order that controls local density variations.

As suggested by the previous results in the disc case, the corrections to the energy asymptotics (1.10) must be curvature-dependent. The case of a general sample where the curvature of the boundary is not constant is then obviously harder to treat than the case of a disc, where one obtains (1.14) by a simple variant of the proof of (1.10), as explained in our previous paper [CR].

In fact, we shall obtain below the desired uniformity result for the order parameter in general domains as a corollary of the energy expansion (γ is a fixed constant)

$$\boxed{E_{\varepsilon}^{\mathrm{GL}} = \frac{1}{\varepsilon} \int_0^{|\partial\Omega|} \mathrm{d}s \, E_{\star}^{\mathrm{1D}} \left(k(s)\right) + \mathcal{O}(\varepsilon |\log \varepsilon|^{\gamma})} \tag{1.18}$$

where the integral runs over the boundary of the sample, $k(s)$ being the curvature of the boundary as a function of the tangential coordinate s. Just as the particular case (1.14), (1.18) contains the leading order (1.10), but $\mathcal{O}(1)$ corrections are also evaluated precisely. As suggested by the energy formula, the GL order parameter has in fact small but fast variations in the tangential variable which contribute to the subleading order of the energy. More precisely, one should think of the order parameter as having the approximate form (again, up to a suitable choice of gauge)

$$\boxed{\Psi^{\mathrm{GL}}(\mathbf{r}) = \Psi^{\mathrm{GL}}(s,\tau) \approx f_{k(s)}\left(\tfrac{\tau}{\varepsilon}\right) \exp\left(-i\alpha(k(s))\tfrac{s}{\varepsilon}\right)} \tag{1.19}$$

with $f_{k(s)}, \alpha(k(s))$ a minimizing pair for the energy functional (1.15) at curvature $k = k(s)$. The main difficulty we encounter in the present paper is to precisely capture the subtle curvature dependent variations encoded in (1.19). What our new result (we give a rigorous statement below) (1.19) shows is that curvature-dependent deviations to (1.12) do exist but are of limited amplitude and can be completely understood via the minimization of the family of 1D functionals (1.15). A crucial input of our analysis is therefore a detailed inspection of the k-dependence of the ground state of (1.15).

We can deduce from (1.18) a uniform density estimate settling the general case of [Pan, Conjecture 1] and [FH3, Open Problem 4, page 267]. We believe that the energy estimate (1.18) is of independent interest since it helps in clarifying the role of domain curvature in surface superconductivity physics. It was previously known (see [FH3, Chapters 8 and 13] and references therein) that corrections to the value of the third critical field depend on the domain's curvature, but applications of these results are limited to the regime where $b \to \Theta_0^{-1}$ when $\varepsilon \to 0$. The present paper seems to contain the first results indicating the role of the curvature in the regime $1 < b < \Theta_0^{-1}$. This role may seem rather limited since it only concerns the second order in the energy asymptotics but it is in fact crucial in controlling local variations of the order parameter and allowing to prove a strong form of uniformity for the surface superconductivity layer.

Our main results are rigorously stated and further discussed in Section 2, their proofs occupy the rest of the paper. Some material from [CR] is recalled in Appendix A for convenience.

Notation. In the whole paper, C denotes a generic fixed positive constant independent of ε whose value changes from formula to formula. A $\mathcal{O}(\delta)$ is always meant to be a quantity whose absolute value is bounded by $\delta = \delta(\varepsilon)$ in the limit $\varepsilon \to 0$. We use $\mathcal{O}(\varepsilon^{\infty})$ to denote a quantity (like $\exp(-\varepsilon^{-1})$) going to 0 faster than any power of ε and $|\log \varepsilon|^{\infty}$ to denote $|\log \varepsilon|^{a}$ where $a > 0$ is some unspecified, fixed but possibly large constant. Such quantities will always appear multiplied by a power of ε, e.g., $\varepsilon|\log\varepsilon|^\infty$ which is a $\mathcal{O}(\varepsilon^{1-c})$ for any $0 < c < 1$, and hence we usually do not specify the precise power a.

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2 Main Results

2.1 Statements

We first state the refined energy and density estimates that reveal the contributions of the domain's boundary. As suggested by (1.19), we now introduce a reference profile that includes these variations. A piecewise constant function in the tangential direction is sufficient for our purpose and we thus first introduce a decomposition of the superconducting boundary layer that will be used in all the paper. The thickness of this layer in the normal direction should roughly be of order ε, but to fully capture the phenomenon at hand we need to consider a layer of size $c_0\varepsilon|\log\varepsilon|$ where c_0 is a fixed, large enough constant. By a passage to boundary coordinates and dilation of the normal variable on scale ε (see [FH3, Appendix F] or Section 4 below), the surface superconducting layer

$$\tilde{\mathcal{A}}_\varepsilon := \{\mathbf{r} \in \Omega \mid \tau \leq c_0\varepsilon|\log\varepsilon|\}, \tag{2.1}$$

where

$$\tau := \mathrm{dist}(\mathbf{r}, \partial\Omega), \tag{2.2}$$

can be mapped to

$$\mathcal{A}_\varepsilon := \{(s,t) \in [0, |\partial\Omega|] \times [0, c_0|\log\varepsilon|]\}. \tag{2.3}$$

We split this domain into $N_\varepsilon = \mathcal{O}(\varepsilon^{-1})$ rectangular cells $\{\mathcal{C}_n\}_{n=1,\ldots,N_\varepsilon}$ of constant side length $\ell_\varepsilon \propto \varepsilon$ in the s direction. We denote s_n, $s_{n+1} = s_n + \ell_\varepsilon$ the s coordinates of the boundaries of the cell $\mathcal{C}_n$:

$$\mathcal{C}_n = [s_n, s_{n+1}] \times [0, c_0|\log\varepsilon|]$$

and we may clearly choose

$$\ell_\varepsilon = \varepsilon|\partial\Omega|\,(1 + \mathcal{O}(\varepsilon))$$

for definiteness. We will approximate the curvature $k(s)$ by its mean value k_n in each cell:

$$k_n := \ell_\varepsilon^{-1} \int_{s_n}^{s_{n+1}} \mathrm{d}s\, k(s).$$

We also denote

$$f_n := f_{k_n}, \qquad \alpha_n := \alpha(k_n)$$

respectively the optimal profile and phase associated to k_n, obtained by minimizing (1.15) first with respect to[2] f and then to α.

The reference profile is then the piecewise continuous function

$$g_{\mathrm{ref}}(s,t) := f_n(t), \qquad \text{for } s \in [s_n, s_{n+1}], \tag{2.4}$$

and we compare the density of the full GL order parameter to g_{ref} in the next theorem. Note that because of the gauge invariance of the energy functional, the phase of the order parameter is not an observable quantity, so the next statement is only about the density $|\Psi^{\mathrm{GL}}|^2$.

[2] We are free to impose $f_n \geq 0$, which we always do in the sequel.

Theorem 2.1 (Refined energy and density asymptotics).
Let $\Omega \subset \mathbb{R}^2$ be any smooth, bounded and simply connected domain. For any fixed $1 < b < \Theta_0^{-1}$, in the limit $\varepsilon \to 0$, it holds

$$E_\varepsilon^{\mathrm{GL}} = \frac{1}{\varepsilon} \int_0^{|\partial\Omega|} \mathrm{d}s \, E_\star^{\mathrm{1D}}(k(s)) + \mathcal{O}(\varepsilon |\log \varepsilon|^\infty). \tag{2.5}$$

and

$$\left\| |\Psi^{\mathrm{GL}}|^2 - g_{\mathrm{ref}}^2 \left(s, \varepsilon^{-1}\tau\right) \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon^{3/2} |\log \varepsilon|^\infty) \ll \left\| g_{\mathrm{ref}}^2 \left(s, \varepsilon^{-1} t\right) \right\|_{L^2(\Omega)}. \tag{2.6}$$

Remark 2.1 [The energy to subleading order]
The most precise result prior to the above is [CR, Theorem 2.1] where the leading order is computed and the remainder is shown to be at most of order 1. Such a result had been obtained before in [FHP] for a smaller range of parameters, namely for $1.25 \leq b < \Theta_0^{-1}$, see also [FH3, Chapter 14] and references therein. The above theorem evaluates precisely the $\mathcal{O}(1)$ term, which is better appreciated in light of the following comments:

1. In the effective 1D functional (1.15), the parameter k that corresponds to the local curvature of the sample appears with an ε prefactor. As a consequence, one may show (see Section 3.1 below) that for all $s \in [0, |\partial\Omega|]$

   $$E_\star^{\mathrm{1D}}(k(s)) = E_\star^{\mathrm{1D}}(0) + \mathcal{O}(\varepsilon)$$

   so that (2.5) contains the previously known results. More generally we prove below that

   $$\left| E_\star^{\mathrm{1D}}(k(s)) - E_\star^{\mathrm{1D}}(k(s')) \right| \leq C\varepsilon |s - s'|$$

   so that $E_\star^{\mathrm{1D}}(k(s))$ has variations of order ε on the scale of the boundary layer. These contribute to a term of order 1 that is included in (2.5).

2. Undoing the mapping to boundary coordinates, one should note that $g_{\mathrm{ref}}(s, \varepsilon^{-1} t)$ has fast variations (at scale ε) in both the t direction and s directions. The latter are of limited amplitude however, which explains that they enter the energy only at subleading order, and why a piecewise constant profile is sufficient to capture the physics.

3. We had previously proved the density estimate (1.13), which is less precise than (2.6). Note in particular that (2.6) does not hold at this level of precision if one replaces $g_{\mathrm{ref}}^2 \left(s, \varepsilon^{-1} t\right)$ by the simpler profile $f_0^2(\varepsilon^{-1} t)$.

4. Strictly speaking the function g_{ref} is defined only in the boundary layer $\tilde{\mathcal{A}}_\varepsilon$, so that (2.6) should be interpreted as if g_{ref} would vanish outside $\tilde{\mathcal{A}}_\varepsilon$. However the estimate there is obviously true thanks to the exponential decay of Ψ^{GL}.

□

We now turn to the uniform density estimates that follow from the above theorem. Here we can be less precise than before. Indeed, as suggested by the previous discussion, a density deviation of order ε on a length scale of order ε only produces a $\mathcal{O}(\varepsilon^2)$ error in the energy. Thus, using (2.5) we may only rule out local variations of a smaller order than the tangential variations included in (2.4), and for this reason we will compare $|\Psi^{\mathrm{GL}}|$ in L^∞ norm only to the simplified profile $f_0(\varepsilon^{-1}\tau)$, since by (1.17) $f_0(t) - f_k(t) = \mathcal{O}(\varepsilon)$. Also, the result may be proved only in a region where the density is relatively large[3], namely in

$$\mathcal{A}_{\mathrm{bl}} := \left\{ \mathbf{r} \in \Omega \, : \, f_0\left(\varepsilon^{-1}\tau\right) \geq \gamma_\varepsilon \right\} \subset \left\{ \mathrm{dist}(\mathbf{r}, \partial\Omega) \leq \tfrac{1}{2}\varepsilon \sqrt{|\log \gamma_\varepsilon|} \right\}, \tag{2.7}$$

[3] Recall that it decays exponentially far from the boundary.

where bl stands for "boundary layer" and $0 < \gamma_\varepsilon \ll 1$ is any quantity such that

$$\gamma_\varepsilon \gg \varepsilon^{1/6} |\log \varepsilon|^a, \tag{2.8}$$

where $a > 0$ is a suitably large constant related[4] to the power of $|\log\varepsilon|$ appearing in (2.5). The inclusion in (2.7) follows from (A.6) below and ensures we are really considering a significant boundary layer: recall that the physically relevant region has a thickness roughly of order $\varepsilon|\log\varepsilon|$.

Theorem 2.2 (Uniform density estimates and Pan's conjecture).
Under the assumptions of the previous theorem, it holds

$$\left\| \left| \Psi^{\mathrm{GL}}(\mathbf{r}) \right| - f_0\left(\varepsilon^{-1}\tau\right) \right\|_{L^\infty(\mathcal{A}_{\mathrm{bl}})} \leq C \gamma_\varepsilon^{-3/2} \varepsilon^{1/4} |\log\varepsilon|^\infty \ll 1. \tag{2.9}$$

In particular for any $\mathbf{r} \in \partial\Omega$ we have

$$\left| \left| \Psi^{\mathrm{GL}}(\mathbf{r}) \right| - f_0(0) \right| \leq C \varepsilon^{1/4} || \log\varepsilon|^\infty| \ll 1, \tag{2.10}$$

where C does not depend on $\mathbf{r}$.

Estimate (2.10) solves the original form of Pan's conjecture [Pan, Conjecture 1]. In addition, since f_0 is strictly positive, the stronger estimate (2.9) ensures that Ψ^{GL} does not vanish in the boundary layer (2.7). A physical consequence of the theorem is thus that normal inclusions such as vortices in the surface superconductivity phase may not occur. This is very natural in view of the existing knowledge on type-II superconductors but had not been proved previously.

We now return to the question of the phase of the order parameter. Of course, the full phase cannot be estimated because of gauge invariance but gauge invariant quantities linked to the phase can. One such quantity is the winding number (a.k.a. phase circulation or topological degree) of Ψ^{GL} around the boundary $\partial\Omega$ defined as

$$2\pi \deg\left(\Psi, \partial\Omega\right) := -i \int_{\partial\Omega} \mathrm{d}s \, \frac{|\Psi|}{\Psi} \partial_s \left(\frac{\Psi}{|\Psi|} \right), \tag{2.11}$$

∂_s standing for the tangential derivative. Theorem 2.2 ensures that $\deg\left(\Psi, \partial\mathcal{B}_R\right) \in \mathbb{Z}$ is well-defined. Roughly, this quantity measures the number of quantized phase singularities (vortices) that Ψ^{GL} has inside Ω. Our estimate is as follows:

Theorem 2.3 (Winding number of Ψ^{GL} on the boundary).
Under the previous assumptions, any GL minimizer Ψ^{GL} satisfies

$$\deg\left(\Psi^{\mathrm{GL}}, \partial\Omega\right) = \frac{|\alpha_0|}{\varepsilon} + \mathcal{O}(\varepsilon^{-3/4} |\log\varepsilon|^\infty) \tag{2.12}$$

in the limit $\varepsilon \to 0$.

Note that the remainder term in (2.12) is much larger than $\varepsilon^{-1}|\alpha(k) - \alpha_0| = \mathcal{O}(1)$ so that the above result does not allow to estimate corrections due to curvature. We believe that, just as we had to expand the energy to second order to obtain the refined first order results Theorems 2.2 and 2.3, obtaining uniform density estimates and degree estimates at the second order would require to expand the energy to the third order, which goes beyond the scope of the present paper.

We had proved Theorems 2.2 and 2.3 before in the particular, significantly easier, case where Ω is a disc. The next subsection contains a sketch of the proof of the general case, where new ingredients enter, due to the necessity to take into account the non-trivial curvature of the boundary.

[4]Assuming that (2.5) holds true with an error of order $\varepsilon|\log\varepsilon|^\gamma$, for some given $\gamma > 0$, the constant a can be any number satisfying $a > \frac{1}{6}(\gamma + 3)$.

2.2 Sketch of proof

In the regime of interest to this paper, the GL order parameter is concentrated along the boundary of the sample and the induced magnetic field is extremely close to the applied one. The tools allowing to prove these facts are well-known and described at length in the monograph [FH3]. We shall thus not elaborate on this and the formal considerations presented in this subsection take as starting point the following effective functional

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] := \int_0^{|\partial\Omega|} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\,(1-\varepsilon k(s)t) \left\{ |\partial_t\psi|^2 + \frac{1}{(1-\varepsilon k(s)t)^2} \left|(\varepsilon\partial_s + i a_\varepsilon(s,t))\,\psi\right|^2 \right.$$
$$\left. - \frac{1}{2b}\left[2|\psi|^2 - |\psi|^4\right] \right\}, \quad (2.13)$$

where (s,t) represent boundary coordinates in the original domain Ω, the normal coordinate t having been dilated on scale ε, and ψ can be thought of as $\Psi^{\mathrm{GL}}(\mathbf{r}(s,\varepsilon t))$, i.e., the order parameter restricted to the boundary layer. We denote $k(s)$ the curvature of the original domain and have set

$$a_\varepsilon(s,t) := -t + \tfrac{1}{2}\varepsilon k(s)t^2 + \varepsilon\delta_\varepsilon, \quad (2.14)$$

with

$$\delta_\varepsilon := \frac{\gamma_0}{\varepsilon^2} - \left\lfloor \frac{\gamma_0}{\varepsilon^2} \right\rfloor, \qquad \gamma_0 := \frac{1}{|\partial\Omega|}\int_\Omega \mathrm{d}\mathbf{r}\,\mathrm{curl}\,\mathbf{A}^{\mathrm{GL}}, \quad (2.15)$$

$\lfloor\,\cdot\,\rfloor$ standing for the integer part. Note that a specific choice of gauge has been made to obtain (2.13).

Thanks to the methods exposed in [FH3], one can show that the minimization of the above functional gives the full GL energy in units ε^{-1}, up to extremely small remainder terms, provided c_0 is chosen lare enough. To keep track of the fact that the domain $\mathcal{A}_\varepsilon = [0,|\partial\Omega|]\times[0,c_0|\log\varepsilon|]$ corresponds to the unfolded boundary layer of the original domain and ψ to the GL order parameter in boundary coordinates, one should impose periodicity of ψ in the s direction.

Here we shall informally explain the main steps of the proof that

$$G_{\mathcal{A}_\varepsilon} = \int_0^{|\partial\Omega|} \mathrm{d}s\, E^{\mathrm{1D}}_\star\left(k(s)\right) + \mathcal{O}(\varepsilon^2|\log\varepsilon|^\infty). \quad (2.16)$$

where $G_{\mathcal{A}_\varepsilon}$ is the ground state energy associated to (2.13). When $k(s)\equiv k$ is constant (the disc case), one may use the ansatz

$$\psi(s,t) = f(t)e^{-i\left(\varepsilon^{-1}\alpha s - \varepsilon\delta_\varepsilon s\right)}. \quad (2.17)$$

and recover the functional (1.15). It is then shown in [CR] that the above ansatz is essentially optimal if one chooses $\alpha = \alpha(k)$ and $f = f_k$. An informal sketch of the proof in the case $k=0$ is given in Section 3.2 therein. The main insight in the general case is to realize that the above ansatz stays valid *locally in s*. Indeed, since the terms involving $k(s)$ in (2.13) come multiplied by an ε factor, it is natural to expect variations in s to be weak and the state of the system to be roughly of the form (1.19), directly inspired by (2.17).

As usual the upper and lower bound inequalities in (2.16) are proved separately.

Upper bound. To recover the integral in the energy estimate (2.16), we use a Riemann sum over the cell decomposition $\mathcal{A}_\varepsilon = \bigcup_{n=1}^{N_\varepsilon}\mathcal{C}_n$ introduced at the beginning of Section 2.1. Indeed, as already suggested in (2.4), a piecewise constant approximation in the s-direction will be sufficient. Our trial state roughly has the form

$$\psi(s,t) = f_n(t)e^{-i\left(\varepsilon^{-1}\alpha_n s - \varepsilon\delta_\varepsilon s\right)}, \qquad \text{for } s_n \le s \le s_{n+1}. \quad (2.18)$$

Of course, we need to make this function continuous to obtain an admissible trial state, and we do so by small local corrections, described in more details in Section 4.1. We may then approximate the curvature by its mean value in each cell, making a relative error of order ε^2 per cell. Evaluating the energy of the trial state in this way we obtain an upper bound of the form

$$G_{\mathcal{A}_\varepsilon} \leq \sum_{n=1}^{N_\varepsilon} |s_{n+1} - s_n| E^{1\mathrm{D}}_\star(k_n)(1 + o(1)) + \mathcal{O}(\varepsilon^2) \tag{2.19}$$

where the $o(1)$ error is due to the necessary modifications to (2.18) to make it continuous. The crucial point is to be able to control this error by showing that the modification needs not be a large one. This requires a detailed analysis of the k dependence of the relevant quantities $E^{1\mathrm{D}}_\star(k)$, $\alpha(k)$ and f_k obtained by minimizing (1.15). Indeed, we prove in Section 3.1 below that

$$\left|E^{1\mathrm{D}}_\star(k) - E^{1\mathrm{D}}_\star(k')\right| \leq C\varepsilon|\log\varepsilon|^\infty|k-k'|, \qquad |\alpha(k)-\alpha(k')| \leq C\varepsilon^{1/2}|\log\varepsilon|^\infty|k-k'|^{1/2}$$

and, in a suitable norm,

$$f_{k'} = f_k + \mathcal{O}\left(\varepsilon^{1/2}|\log\varepsilon|^\infty|k-k'|^{1/2}\right),$$

which will allow to obtain the desired control of the $o(1)$ in (2.19) and conclude the proof by a Riemann sum argument.

Lower bound. In view of the argument we use for the upper bound, the natural idea to obtain the corresponding lower bound is to use the strategy for the disc case we developed in [CR] locally in each cell. In the disc case, a classical method of energy decoupling and Stokes' formula lead to the lower bound[5]

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] \gtrapprox E^{1\mathrm{D}}_\star(k) + \int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t\ (1-\varepsilon k t)\, K_k(t)\left(|\partial_t v|^2 + \tfrac{\varepsilon^2}{(1-\varepsilon k t)^2}|\partial_s v|^2\right) \tag{2.20}$$

where we have used the strict positivity of f_k to write

$$\psi(s,t) = f_k(t)e^{-i\left(\varepsilon^{-1}\alpha(k)s - \varepsilon\delta_\varepsilon s\right)}v(s,t) \tag{2.21}$$

and the "cost function" is

$$\begin{aligned} K_k(t) &= f_k^2(t) + F_k(t), \\ F_k(t) &= 2\int_0^t d\eta\, \frac{\eta + \alpha(k) - \frac{1}{2}\varepsilon k\eta^2}{1-\varepsilon k\eta} f_k^2(\eta). \end{aligned}$$

This method is inspired from our previous works on the related Gross-Pitaevskii theory of rotating Bose-Einstein condensates [CRY, CPRY1, CPRY2] (informal summaries may be found in [CPRY3, CPRY4]). Some of the steps leading to (2.20) have also been used before in this context [AH]. The desired lower bound in the disc case follows from (2.20) and the fact that K_k is essentially *positive*[6] for any k. This is proved by carefully exploiting special properties of f_k and $\alpha(k)$.

To deal with the general case where the curvature is not constant, we again split the domain $\mathcal{A}_\varepsilon$ into small cells, approximate the curvature by a constant in each cell and use the above strategy locally. A serious new difficulty however comes from the use of Stokes' formula in the derivation of (2.20). We need to reduce the terms produced by Stokes' formula to expressions involving only first order derivatives of the order parameter, using further integration by parts. In the disc case, boundary terms associated with this operation vanish due to the periodicity of ψ in the s

[5] We simplify the argument for pedagogical purposes.

[6] More precisely it is positive except possibly for large t, a region that can be handled using the exponential decay of GL minimizers (Agmon estimates).

variable. When doing the integrations by parts in each cell, using different f_k and $\alpha(k)$ in (2.21), the boundary terms do not vanish since we artificially introduce some (small) discontinuity by choosing a cell-dependent profile f_{k_n} as reference.

To estimate these boundary terms we proceed as follows: the term at $s = s_{n+1}$, made of one part coming from the cell $\mathcal{C}_n$ and one from the cell $\mathcal{C}_{n+1}$ is integrated by parts back to become a bulk term in the cell $\mathcal{C}_n$. In this sketch we ignore a rather large amount of technical complications and state what is essentially the conclusion of this procedure:

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] \gtrapprox \sum_{n=1}^{N_\varepsilon} \left[|s_{n+1} - s_n| E_\star^{1\mathrm{D}}(k_n) + \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\, (1 - \varepsilon k_n t)\, \tilde{K}_n \left(|\partial_t u_n|^2 + \tfrac{\varepsilon^2}{(1-\varepsilon k_n t)^2} |\partial_s u_n|^2 \right) \right] \tag{2.22}$$

where

$$u_n(s,t) = f_{k_n}^{-1}(t) e^{i\left(\varepsilon^{-1}\alpha(k)s + \varepsilon\delta_\varepsilon s\right)} \psi(s,t) \tag{2.23}$$

and the "modified cost function" is

$$\begin{aligned} \tilde{K}_n(s,t) &= K_{k_n}(t) - |\partial_s \chi_n(s)| |I_{n,n+1}(t)| - |\chi_n(s)|\, |\partial_t I_{n,n+1}(t)|\,, \\ I_{n,n+1}(t) &= F_{k_n}(t) - F_{k_{n+1}}(t) \frac{f_{k_n}^2(t)}{f_{k_{n+1}}^2(t)}, \end{aligned}$$

and χ_n is a suitable localization function supported in $\mathcal{C}_n$ with $\chi_n(s_{n+1}) = 1$ that we use to perform the integration by parts in $\mathcal{C}_n$. Note that the dependence of the new cost function on both k_n and k_{n+1} is due to the fact that the original boundary terms at s_{n+1} that we transform into bulk terms in $\mathcal{C}_n$ involved both u_n and u_{n+1}.

The last step is to prove a bound of the form

$$|I_{n,n+1}(t)| + |\partial_t I_{n,n+1}(t)| \leq C\varepsilon |\log \varepsilon|^\infty f_{k_n}^2(t) \tag{2.24}$$

on the "correction function" $I_{n,n+1}$, so that

$$\tilde{K}_n(t) \geq (1 - C\varepsilon|\log\varepsilon|^\infty)\, f_{k_n}^2(t) + F_{k_n}(t).$$

This allows us to conclude that (essentially) $\tilde{K}_n \geq 0$ by a perturbation of the argument applied to K_{k_n} in [CR] and thus concludes the lower bound proof modulo the same Riemann sum argument as in the upper bound part. Note the important fact that the quantity in the l.h.s. of (2.24) is proved to be small *relatively to* $f_{k_n}^2(t)$, including in a region where the latter function is exponentially decaying. This bound requires a thorough analysis of auxiliary functions linked to (1.15) and is in fact a rather strong manifestation of the continuity of this minimization problem as a function of k.

The rest of the paper is organized as follows: Section 3 contains the detailed analysis of the effective, curvature-dependent, 1D problem. The necessary continuity properties as function of the curvature are given in Subsection 3.1 and the analysis of the associated auxiliary functions in Subsection 3.2. The details of the energy upper bound are then presented in Section 4 and the energy lower bound is proved in Section 5. We deduce our other main results in Section 6. Appendix A recalls for the convenience of the reader some material from [CR] that we use throughout the paper.

3 Effective Problems and Auxiliary Functions

This section is devoted to the analysis of the 1D curvature-dependent reduced functionals whose minimization allows us to reconstruct the leading and sub-leading order of the full GL energy. We shall prove results in two directions:

- We carefully analyse the dependence of the 1D variational problems as a function of curvature in Subsection 3.1. Our analysis, in particular the estimate of the subleading order of the GL energy, requires some quantitative control on the variations of the optimal 1D energy, phase and density when the curvature parameter is varied, that is when we move along the boundary layer of the original sample along the transverse direction.
- In our previous paper [CR] we have proved the positivity property of the cost function which is the main ingredient in the proof of the energy lower bound in the case of a disc (constant curvature). As mentioned above, the study of general domains with smooth curvature that we perform here will require to estimate more auxiliary functions, which is the subject of Subsection 3.2.

We shall use as input some key properties of the 1D problem at fixed k that we proved in [CR]. These are recalled in Appendix A below for the convenience of the reader.

3.1 Effective 1D functionals

We take for granted the three crucial but standard steps of reduction to the boundary layer, replacement of the vector potential and mapping to boundary coordinates. Our considerations thus start from the following reduced GL functional giving the original energy in units of ε^{-1}, up to negligible remainders:

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] := \int_0^{|\partial\Omega|} \mathrm{d}s \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\,(1-\varepsilon k(s)t)\left\{|\partial_t\psi|^2 + \frac{1}{(1-\varepsilon k(s)t)^2}\left|(\varepsilon\partial_s + ia_\varepsilon(s,t))\,\psi\right|^2 - \frac{1}{2b}\left[2|\psi|^2 - |\psi|^4\right]\right\}, \tag{3.1}$$

where $k(s)$ is the curvature of the original domain. We have set

$$a_\varepsilon(s,t) := -t + \tfrac{1}{2}\varepsilon k(s)t^2 + \varepsilon\delta_\varepsilon, \tag{3.2}$$

and

$$\delta_\varepsilon := \frac{\gamma_0}{\varepsilon^2} - \left\lfloor \frac{\gamma_0}{\varepsilon^2} \right\rfloor, \qquad \gamma_0 := \frac{1}{|\partial\Omega|}\int_\Omega \mathrm{d}\mathbf{r}\,\mathrm{curl}\,\mathbf{A}^{\mathrm{GL}}, \tag{3.3}$$

$\lfloor\,\cdot\,\rfloor$ standing for the integer part. The boundary layer in rescaled coordinates is denoted by

$$\mathcal{A}_\varepsilon := \left\{\mathbf{r}\in\Omega \mid \mathrm{dist}(\mathbf{r},\partial\Omega) \le c_0\varepsilon|\log\varepsilon|\right\}. \tag{3.4}$$

The effective functionals that we shall be concerned with in this section are obtained by computing the energy (3.1) of certain special states. In particular we have to go beyond the simple ansätze considered so far in the literature, e.g., in [FH3, CR], and obtain the following effective energies:

- *2D functional with definite phase.* Inserting the ansatz

$$\psi(s,t) = g(s,t)e^{-i\left(\varepsilon^{-1}S(s) - \varepsilon\delta_\varepsilon s\right)} \tag{3.5}$$

in (3.1), with g and S respectively real valued density and phase, we obtain

$$\mathcal{E}^{\mathrm{2D}}_S[g] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \int_0^{|\partial\Omega|} \mathrm{d}s\,(1-\varepsilon k(s)t)\left\{|\partial_t g|^2 + \frac{\varepsilon^2}{(1-\varepsilon k(s)t)^2}|\partial_s g|^2 + \frac{\left(t + \partial_s S - \frac{1}{2}\varepsilon t^2 k(s)\right)^2}{(1-\varepsilon t k(s))^2}g^2 - \frac{1}{2b}\left(2g^2 - g^4\right)\right\}. \tag{3.6}$$

In the particular case where $\partial_s S = \alpha \in 2\pi\mathbb{Z}$ we may obtain a simpler functional of the density alone

$$\mathcal{E}^{\mathrm{2D}}_{\alpha}[g] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \int_0^{|\partial\Omega|} \mathrm{d}s \, (1-\varepsilon k(s)t) \left\{ |\partial_t g|^2 + \frac{\varepsilon^2}{(1-\varepsilon k(s)t)^2} |\partial_s g|^2 \right. \\ \left. + W_\alpha(s,t) g^2 - \frac{1}{2b}\left(2g^2 - g^4\right) \right\}, \quad (3.7)$$

where

$$W_\alpha(s,t) = \frac{\left(t + \alpha - \frac{1}{2}k(s)\varepsilon t^2\right)^2}{(1-k(s)\varepsilon t)^2}. \quad (3.8)$$

However to capture the next to leading order of (3.1) we do consider a non-constant $\partial_s S$ to accommodate curvature variations, which is in some sense the main novelty of the present paper. In particular, (3.7) does *not* provide the $\mathcal{O}(\varepsilon)$ correction to the full GL energy. On the opposite (3.6) does, once minimized over the phase factor S as well as the density g. We will not prove this directly although it follows rather easily from our analysis.

- *1D functional with given curvature and phase.* If the curvature $k(s) \equiv k$ is constant (the disc case), the minimization of (3.7) reduces to the 1D problem

$$\mathcal{E}^{\mathrm{1D}}_{k,\alpha}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t (1-\varepsilon k t) \left\{ |\partial_t f|^2 + V_{k,\alpha}(t) f^2 - \tfrac{1}{2b}\left(2f^2 - f^4\right) \right\}, \quad (3.9)$$

with

$$V_{k,\alpha}(t) := \frac{(t + \alpha - \frac{1}{2}\varepsilon k t^2)^2}{(1-\varepsilon k t)^2}. \quad (3.10)$$

In the sequel we shall denote

$$I_\varepsilon = [0, c_0|\log\varepsilon|] =: [0, t_\varepsilon]. \quad (3.11)$$

Note that (3.9) includes $\mathcal{O}(\varepsilon)$ corrections due to curvature. As explained above our approach is to approximate the curvature of the domain as a piecewise constant function and hence an important ingredient is to study the above 1D problem for different values of k, and prove some continuity properties when k is varied. For $k = 0$ (the half-plane case, sometimes referred to as the half-cylinder case) we recover the familiar

$$\mathcal{E}^{\mathrm{1D}}_{0,\alpha}[f] := \int_0^{c_0|\log\varepsilon|} \mathrm{d}t \left\{ |\partial_t f|^2 + (t+\alpha)^2 f^2 - \tfrac{1}{2b}\left(2f^2 - f^4\right) \right\}, \quad (3.12)$$

which has been known to play a crucial role in surface superconductivity physics for a long time (see [FH3, Chapter 14] and references therein).

In this section we provide details about the minimization of (3.9) that go beyond our previous study [CR, Section 3.1]. We will use the following notation:

- Minimizing (3.9) with respect to f at fixed α we get a minimizer $f_{k,\alpha}$ and an energy $E^{\mathrm{1D}}(k,\alpha)$.
- Minimizing the latter with respect to α we get some (a priori non-unique) $\alpha(k)$ and some energy $E^{\mathrm{1D}}_{\star}(k)$.
- Corresponding to $E^{\mathrm{1D}}_{\star}(k) := E^{\mathrm{1D}}(k,\alpha(k))$ we have an optimal density f_k, which minimizes $E^{\mathrm{1D}}(k,\alpha(k))$, and a potential

$$V_k(t) := V_{k,\alpha(k)}(t).$$

The following Proposition contains the crucial continuity properties (as a function of k) of these objects:

Proposition 3.1 (Dependence on curvature of the 1D minimization problem).
Let $k, k' \in \mathbb{R}$ be bounded independently of ε and $1 < b < \Theta_0^{-1}$, then the following holds:

$$\left| E_\star^{1\mathrm{D}}(k) - E_\star^{1\mathrm{D}}(k') \right| \leq C\varepsilon |k - k'| |\log \varepsilon|^\infty \tag{3.13}$$

and

$$|\alpha(k) - \alpha(k')| \leq C \left(\varepsilon |k - k'|\right)^{1/2} |\log \varepsilon|^\infty. \tag{3.14}$$

Finally, for all $n \in \mathbb{N}$,

$$\left\| f_k^{(n)} - f_{k'}^{(n)} \right\|_{L^\infty(I_\varepsilon)} \leq C \left(\varepsilon |k - k'|\right)^{1/2} |\log \varepsilon|^\infty. \tag{3.15}$$

We first prove (3.13) and (3.14) and explain that these estimates imply the following lemma:

Lemma 3.1 (Preliminary estimate on density variations).
Under the assumptions of Proposition 3.1 it holds

$$\left\| f_k^2 - f_{k'}^2 \right\|_{L^2(I_\varepsilon)} \leq C \left(\varepsilon |k - k'|\right)^{1/2} |\log \varepsilon|^\infty. \tag{3.16}$$

Proof of Lemma 3.1. We proceed in three steps:

Step 1. Energy decoupling. We use the strict positivity of f_k recalled in the appendix to write any function f on I_ε as

$$f = f_k v.$$

We can then use the variational equation (A.1) satisfied by f_k to decouple the α', k' functional in the usual way, originating in [LM]. Namely, we integrate by parts and use the fact that f_k satisfies Neumann boundary conditions to write

$$\begin{aligned}\int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1 - \varepsilon k' t)(\partial_t f)^2 &= \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1 - \varepsilon k' t) \left[v^2 (\partial_t f_k)^2 + f_k^2 (\partial_t v)^2 + 2 f_k \partial_t f_k v \partial_t v \right] \\ &= \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1 - \varepsilon k' t) \left[f_k^2 (\partial_t v)^2 + \left(\tfrac{\varepsilon k'}{1 - \varepsilon k' t} - \tfrac{\varepsilon k}{1 - \varepsilon k t} \right) v^2 f_k \partial_t f_k - f_k^2 v^2 \left(V_k + \tfrac{1}{b}(f_k^2 - 1) \right) \right].\end{aligned}$$

Inserting this into the definition of $\mathcal{E}_{k',\alpha'}^{1\mathrm{D}}$ and using (A.3), we obtain for any f

$$\begin{aligned}\mathcal{E}_{k',\alpha'}^{1\mathrm{D}}[f] = E_\star^{1\mathrm{D}}(k) &+ \mathcal{F}_{\mathrm{red}}[v] \\ &+ \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1 - \varepsilon k' t) \left(V_{k',\alpha'}(t) - V_k(t) \right) f_k^2 v^2 \\ &+ \frac{1}{b} \varepsilon (k' - k) \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, t f_k^4 + \varepsilon \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, \left(k' - k \tfrac{1 - \varepsilon k' t}{1 - \varepsilon k t} \right) |v|^2 f_k \partial_t f_k\end{aligned} \tag{3.17}$$

with

$$\mathcal{F}_{\mathrm{red}}[v] = \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1 - \varepsilon k' t) \left\{ f_k^2 (\partial_t v)^2 + \tfrac{1}{2b} f_k^4 \left(1 - v^2\right)^2 \right\}. \tag{3.18}$$

In the case $\alpha' = \alpha(k)$ we can insert the trial state $v \equiv 1$ in the above, which gives

$$E_\star^{1\mathrm{D}}(k') \leq E_{k',\alpha(k)}^{1\mathrm{D}} \leq E_\star^{1\mathrm{D}}(k) + C\varepsilon |k - k'| |\log \varepsilon|^\infty \tag{3.19}$$

in view of the bounds on f_k recalled in Appendix A and the easy estimate

$$\left|V_{k',\alpha(k)}(t) - V_k(t)\right| \leq C\varepsilon|k-k'||\log\varepsilon|^\infty$$

for any $t \in I_\varepsilon$. Changing the role of k and k' in (3.19) we obtain the reverse inequality

$$E_\star^{\mathrm{1D}}(k) \leq E_\star^{\mathrm{1D}}(k') + C\varepsilon|k-k'||\log\varepsilon|^\infty,$$

and hence (3.13) is proved.

Step 2. Use of the cost function. We now consider the case $\alpha' = \alpha(k'), f = f_{k'}$ and bound from below the term on the second line of (3.17). A simple computation gives

$$\begin{aligned}
&\int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1-\varepsilon k' t)\left(V_{k',\alpha(k')} - V_{k,\alpha(k)}\right) f_k^2 v^2 \\
&\quad = \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1-\varepsilon k t)^{-1} \left(\alpha(k') - \alpha(k)\right)\left(2t + \alpha(k) + \alpha(k') - \varepsilon k t^2\right) f_k^2 v^2 + \mathcal{O}(\varepsilon|k-k'|) \\
&\qquad = \left(\alpha(k') - \alpha(k)\right)^2 \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, (1-\varepsilon k' t)^{-1} f_k^2 v^2 \\
&\qquad + 2(\alpha(k') - \alpha(k)) \int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, \frac{t + \alpha(k) - \frac{1}{2}\varepsilon k t^2}{1-\varepsilon k t} f_k^2 v^2 + \mathcal{O}(\varepsilon|k-k'|).
\end{aligned} \tag{3.20}$$

We may now follow closely the procedure of [CR, Section 5.2]: with the potential function F_k defined in (A.8) below we have

$$2\frac{t + \alpha(k) - \frac{1}{2}\varepsilon k t^2}{1-\varepsilon k t} f_k^2 = \partial_t F_k(t)$$

and hence an integration by parts yields (boundary terms vanish thanks to Lemma A.3)

$$2\int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, \frac{t + \alpha(k) - \frac{1}{2}\varepsilon k t^2}{1-\varepsilon k t} f_k^2 v^2 = -2\int_0^{c_0|\log\varepsilon|} \mathrm{d}t\, F_k v \partial_t v. \tag{3.21}$$

We now split the integral into one part running from 0 to $\bar{t}_{k,\varepsilon}$ and a boundary part running from $\bar{t}_{k,\varepsilon}$ to $c_0|\log\varepsilon|$, where $\bar{t}_{k,\varepsilon}$ is defined in (A.12) and (A.13) below. For the second part, it follows from the decay estimates of Lemma A.2 that

$$\int_{\bar{t}_{k,\varepsilon}}^{c_0|\log\varepsilon|} \mathrm{d}t\, F_k v \partial_t v = \mathcal{O}(\varepsilon^\infty). \tag{3.22}$$

To see this, one can simply adapt the procedure in [CR, Eqs. (5.21) – (5.28)]. The bound (3.22) is in fact easier to derive than the corresponding estimate in [CR] because the decay estimates in Lemma A.2 are stronger than the Agmon estimates we had to use in that case. Details are thus omitted.

We turn to the main part of the integral (3.21), which lives in $[0, \bar{t}_{k,\varepsilon}]$. Since F_k is negative we have, using Lemma A.4 and Cauchy-Schwarz,

$$\begin{aligned}
&\left| 2(\alpha(k') - \alpha(k)) \int_0^{\bar{t}_{k,\varepsilon}} \mathrm{d}t\, F_k v \partial_t v \right| \\
&\quad \leq (\alpha(k') - \alpha(k))^2 \int_0^{\bar{t}_{k,\varepsilon}} \mathrm{d}t\, (1-\varepsilon k' t)^{-1} |F_k|\, v^2 + \int_0^{\bar{t}_{k,\varepsilon}} \mathrm{d}t\, (1-\varepsilon k' t)\, |F_k|\, (\partial_t v)^2 \\
&\quad \leq (1-d_\varepsilon)(\alpha(k') - \alpha(k))^2 \int_0^{\bar{t}_{k,\varepsilon}} \mathrm{d}t\, (1-\varepsilon k' t)^{-1} f_k^2 v^2 + (1-d_\varepsilon) \int_0^{\bar{t}_{k,\varepsilon}} \mathrm{d}t (1-\varepsilon k' t) f_k^2 (\partial_t v)^2
\end{aligned}$$

for any $0 < d_\varepsilon \leq C|\log\varepsilon|^{-4}$. Inserting this bound and (3.22) in (3.17), using (3.20) and (3.21), yields the lower bound

$$E^{\rm 1D}_\star(k') \geq E^{\rm 1D}_\star(k) + \int_0^{c_0|\log\varepsilon|} {\rm d}t\,(1-\varepsilon t k')\left\{d_\varepsilon f_k^2(\partial_t v)^2 + d_\varepsilon \frac{(\alpha'-\alpha(k))^2}{(1-\varepsilon t k')^2} f_k^2 v^2 + \frac{f_k^4}{2b}\left(1-v^2\right)^2\right\}$$
$$+\,\varepsilon\int_0^{c_0|\log\varepsilon|} {\rm d}t\, v^2 f_k \partial_t f_k\left(k' - k\frac{1-\varepsilon t k'}{1-\varepsilon t k}\right) - C\varepsilon|k-k'||\log\varepsilon|^\infty \tag{3.23}$$

where $v = f_{k'}/f_k$ and we also used the uniform bound (A.2) to estimate the fourth term of the r.h.s. of (3.17).

Step 3. Conclusion. We still have to bound the first term in the second line of (3.23):

$$\varepsilon\int_0^{c_0|\log\varepsilon|} {\rm d}t\, v^2 f_k\partial_t f_k\left(k' - k\frac{1-\varepsilon k' t}{1-\varepsilon k t}\right) = \frac{1}{2}\left[v^2 f_k^2\left(\varepsilon k' - \varepsilon k\frac{1-\varepsilon k' t}{1-\varepsilon k t}\right)\right]_0^{c_0|\log\varepsilon|}$$
$$+\int_0^{c_0|\log\varepsilon|} {\rm d}t\, v^2 f_k^2\frac{\varepsilon k(k'-k)}{(1-\varepsilon k t)^2} - \int_0^{c_0|\log\varepsilon|} {\rm d}t\, v\partial_t v f_k^2\left(\varepsilon k' - \varepsilon k\frac{1-\varepsilon k' t}{1-\varepsilon k t}\right).$$

The first two terms are both $\mathcal{O}(\varepsilon|k-k'||\log\varepsilon|^\infty)$ thanks to (A.2) applied to $f_{k'}^2 = f_k^2 v^2$. For the third one we write

$$\left|\int_0^{c_0|\log\varepsilon|} {\rm d}t\, v\partial_t v f_k^2\left(\varepsilon k' - \varepsilon k\frac{1-\varepsilon k' t}{1-\varepsilon k t}\right)\right| \leq C\varepsilon|k-k'||\log\varepsilon|^\infty\int_0^{c_0|\log\varepsilon|} {\rm d}t\, v|\partial_t v| f_k^2$$
$$\leq C\varepsilon|k-k'||\log\varepsilon|^\infty\left[\int_0^{c_0|\log\varepsilon|} {\rm d}t\, f_k^2 v^2 + \int_0^{c_0|\log\varepsilon|} {\rm d}t\, f_k^2(\partial_t v)^2\right].$$

Inserting this in (3.23), using again (A.2) and dropping a positive term, we finally get

$$E^{\rm 1D}_\star(k') \geq E^{\rm 1D}_\star(k) + |\log\varepsilon|^{-5}(\alpha(k')-\alpha(k))^2\int_0^{c_0|\log\varepsilon|} {\rm d}t\,(1-\varepsilon k' t) f_{k'}^2$$
$$+\frac{1}{2b}\int_0^{c_0|\log\varepsilon|} {\rm d}t\,(1-\varepsilon k' t)\left(f_k^2 - f_{k'}^2\right)^2 - C\varepsilon|k-k'||\log\varepsilon|^\infty \tag{3.24}$$

where we have chosen $d_\varepsilon = |\log\varepsilon|^{-5}$, which is compatible with the requirement $0 < d_\varepsilon \leq C|\log\varepsilon|^{-4}$. Combining with the estimate (3.13) that we proved in Step 1 concludes the proof of (3.14). To get (3.16) one has to use in addition (A.6), which guarantees that under the assumptions $1 < b < \Theta_0^{-1}$

$$\|f_{k'}\|_{L^2(I_\varepsilon)} \geq C > 0$$

for some constant C independent of ε. □

To conclude the proof of Proposition 3.1 there only remains to discuss (3.15). We shall upgrade the estimate (3.16) to better norms, taking advantage of the 1D nature of the problem and using a standard bootstrap argument.

Proof of Proposition 3.1. We write $f_k = f_{k'} + (f_k - f_{k'})$ and expand the energy $E^{\rm 1D}_\star(k) = \mathcal{E}^{\rm 1D}_k[f_k]$, using the variational equation (A.1) for $f_{k'}$:

$$
\begin{aligned}
E^{\rm 1D}_\star(k) \geq E^{\rm 1D}_\star(k') &+ \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) |\partial_t (f_k - f_{k'})|^2 + \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) V_k (f_k - f_{k'})^2 \\
&+ \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t)(V_k - V_{k'}) f_{k'}^2 + 2 \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) f_{k'} (f_k - f_{k'})(V_k - V_{k'}) \\
&+ \frac{1}{2b} \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) \left[6 f_{k'}^2 (f_k - f_{k'})^2 + 4 f_{k'} (f_k - f_{k'})^3 + (f_k - f_{k'})^4 - 2 (f_k - f_{k'})^2 \right] \\
&- C \varepsilon |k - k'| |\log \varepsilon|^\infty
\end{aligned}
$$

where the $\mathcal{O}(\varepsilon|k-k'||\log\varepsilon|^\infty)$ is as before due to the replacement of the curvature $k \leftrightarrow k'$. Using the same procedure to expand $E^{\rm 1D}_\star(k') = \mathcal{E}^{\rm 1D}_{k'}[f_{k'}]$ and combining the result with the above we obtain

$$
\begin{aligned}
E^{\rm 1D}_\star(k) \geq E^{\rm 1D}_\star(k) &+ 2 \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) |\partial_t (f_k - f_{k'})|^2 + \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) (V_k + V_{k'}) (f_k - f_{k'})^2 \\
&+ \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t)(V_k - V_{k'}) (f_{k'}^2 - f_k^2) \\
&+ 2 \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) (f_{k'} (f_k - f_{k'}) - f_k (f_{k'} - f_k))(V_k - V_{k'}) \\
&+ \frac{1}{2b} \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) (f_k - f_{k'})^2 \left[4 f_{k'}^2 + 4 f_k^2 + 4 f_{k'} f_k - 4 \right] \\
&- C \varepsilon |k - k'| |\log \varepsilon|^\infty.
\end{aligned}
$$

Hence it holds

$$
\begin{aligned}
C \varepsilon |k - k'| |\log \varepsilon|^\infty \geq\ & 2 \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) |\partial_t (f_k - f_{k'})|^2 \\
&+ \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t)(V_k - V_{k'}) (f_k^2 - f_{k'}^2) \\
&+ \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t) (f_k - f_{k'})^2 \left[V_k + V_{k'} + \frac{2}{b} \left(f_{k'}^2 + f_k^2 + f_{k'} f_k - 2 \right) \right].
\end{aligned}
\tag{3.25}
$$

Next we note that thanks to (3.14)

$$
\sup_{I_\varepsilon} |V_k - V_{k'}| \leq C \left(|\alpha(k) - \alpha(k')| + \varepsilon |k - k'| \right) |\log \varepsilon|^\infty \leq C \left(\varepsilon |k - k'| \right)^{1/2} |\log \varepsilon|^\infty
$$

as revealed by an easy computation starting from the expression (3.10). Thus, using (3.16) and the Cauchy-Schwartz inequality,

$$
\begin{aligned}
&\left| \int_{I_\varepsilon} \mathrm{d}t (1-\varepsilon k t)(V_k - V_{k'}) (f_k^2 - f_{k'}^2) \right| \leq \\
&\qquad C |\log \varepsilon|^{1/2} \sup_{I_\varepsilon} |V_k - V_{k'}| \left\| f_k^2 - f_{k'}^2 \right\|_{L^2(I_\varepsilon)} \leq C \varepsilon |k - k'| |\log \varepsilon|^\infty.
\end{aligned}
\tag{3.26}
$$

For the term on the third line of (3.25) we notice that, using the growth of the potentials V_k and $V_{k'}$ for large t, the integrand is positive in

$$
\tilde{I}_\varepsilon := \left[c_1 (\log |\log \varepsilon|)^{1/2}, c_0 |\log \varepsilon| \right]
$$

for any constant c_1 and ε small enough. On the other hand, combining (3.16) and the pointwise lower bound in (A.6) we have

$$\|f_k - f_{k'}\|_{L^2(\tilde{I}_\varepsilon)} \leq C\left(\varepsilon|k-k'|\right)^{1/2} |\log\varepsilon|^{\infty}.$$

Splitting the integral into two pieces we thus have

$$\int_{I_\varepsilon} \mathrm{d}t(1-\varepsilon kt)(f_k - f_{k'})^2 \left[V_k + V_{k'} + \tfrac{2}{b}\left(f_{k'}^2 + f_k^2 + f_{k'}f_k - 2\right)\right] \geq -C\varepsilon|k-k'||\log\varepsilon|^{\infty}.$$

Using this and (3.26) we deduce from (3.25) that

$$\int_{I_\varepsilon} \mathrm{d}t(1-\varepsilon kt)|\partial_t(f_k - f_{k'})|^2 \leq C\varepsilon|k-k'||\log\varepsilon|^{\infty} \tag{3.27}$$

and combining with the previous L^2 bound this gives

$$\|f_k - f_{k'}\|_{H^1(\tilde{I}_\varepsilon)} \leq C\left(\varepsilon|k-k'|\right)^{1/2} |\log\varepsilon|^{\infty}.$$

Since we work on a 1D interval, the Sobolev inequality implies

$$\|f_k - f_{k'}\|_{L^\infty(\tilde{I}_\varepsilon)} \leq C\left(\varepsilon|k-k'|\right)^{1/2} |\log\varepsilon|^{\infty}. \tag{3.28}$$

In particular

$$\left|f_k(c_1(\log|\log\varepsilon|)^{1/2}) - f_{k'}(c_1(\log|\log\varepsilon|)^{1/2})\right| \leq C\left(\varepsilon|k-k'|\right)^{1/2} |\log\varepsilon|^{\infty}.$$

Then, integrating the bound (3.27) from $c_1(\log|\log\varepsilon|)^{1/2}$ to $c_0|\log\varepsilon|$ we can extend (3.28) to the whole interval I_ε:

$$\|f_k - f_{k'}\|_{L^\infty(I_\varepsilon)} \leq C\left(\varepsilon|k-k'|\right)^{1/2} |\log\varepsilon|^{\infty},$$

which is (3.15) for $n = 0$. The bounds on the derivatives follow by a standard bootstrap argument, inserting the L^∞ bound in the variational equations. □

3.2 Estimates on auxiliary functions

In this Section we collect some useful estimates of other quantities involving the 1D densities as well as the optimal phases. It turns out that we need an estimate of the k-dependence of $\partial_t \log(f_k)$, provided in the following

Proposition 3.2 (**Estimate of logarithmic derivatives**).
Let $k, k' \in \mathbb{R}$ be bounded independently of ε and $1 < b < \Theta_0^{-1}$, then the following holds:

$$\left\|\frac{f_k'}{f_k} - \frac{f_{k'}'}{f_{k'}}\right\|_{L^\infty(I_\varepsilon)} \leq C\left(\varepsilon|k-k'|\right)^{1/2} |\log\varepsilon|^{\infty}. \tag{3.29}$$

Proof. Let us denote for short

$$g(t) := \frac{f_k'(t)}{f_k(t)} - \frac{f_{k'}'(t)}{f_{k'}(t)}. \tag{3.30}$$

We first notice that the estimate is obviously true in the region where $f_k \geq |\log\varepsilon|^{-M}$ for any $M > 0$ finite, thanks to (3.15) and (A.7):

$$|g(t)| \leq \frac{|f_k' - f_{k'}'|}{f_k} + \frac{|f_{k'}'|\,|f_k - f_{k'}|}{f_k f_{k'}} \leq |\log\varepsilon|^M\,|f_k' - f_{k'}'| + |\log\varepsilon|^{M+3}\,|f_k - f_{k'}|$$
$$\leq C\left(\varepsilon|k-k'|\right)^{1/2} |\log\varepsilon|^{\infty}.$$

Let t_* be the unique solution to $f_k(t_*) = |\log\varepsilon|^{-M}$ (uniqueness follows from the properties of f_k discussed in Proposition A.1). To complete the proof it thus suffices to prove the estimate in the region $[t_*, c_0|\log\varepsilon|]$. Notice also that thanks to (A.6), it must be that $t_* \to \infty$ when $\varepsilon \to 0$.

At the boundary of the interval $[t_*, t_\varepsilon]$ (recall (3.11)), one has

$$g(t_*) = \mathcal{O}\left((\varepsilon|k-k'|)^{1/2}|\log\varepsilon|^M\right), \qquad g(t_\varepsilon) = 0, \tag{3.31}$$

because of Neumann boundary conditions. Hence if the supremum of $|g|$ is reached at the boundary there is nothing to prove. Let us then assume that $\sup_{t\in[t_*,t_\varepsilon]}|g| = |g(t_0)|$, for some $t_* < t_0 < t_\varepsilon$, such that $g'(t_0) = 0$, i.e.,

$$\frac{f_k''(t_0)}{f_k(t_0)} - \frac{f_{k'}''(t_0)}{f_{k'}(t_0)} + \frac{(f_k'(t_0))^2}{f_k^2(t_0)} - \frac{(f_{k'}'(t_0))^2}{f_{k'}^2(t_0)} = 0. \tag{3.32}$$

Since f_k and $f_{k'}$ are both decreasing in $[t_*, t_\varepsilon]$ (see again Proposition A.1) we also have

$$\frac{(f_k'(t_0))^2}{f_k^2(t_0)} - \frac{(f_{k'}'(t_0))^2}{f_{k'}^2(t_0)} = \left[\frac{|f_k'(t_0)|}{f_k(t_0)} + \frac{|f_{k'}'(t_0)|}{f_{k'}(t_0)}\right] g(t_0). \tag{3.33}$$

The variational equations satisfied by f_k and $f_{k'}$ on the other hand imply

$$\left|\frac{f_k''(t_0)}{f_k(t_0)} - \frac{f_{k'}''(t_0)}{f_{k'}(t_0)}\right| = \left|\frac{\varepsilon k f_k'(t_0)}{(1-\varepsilon kt)f_k(t_0)} - \frac{\varepsilon k' f_{k'}'(t_0)}{(1-\varepsilon k't)f_{k'}(t_0)} + V_k(t_0) - V_{k'}(t_0)\right.$$
$$\left. - \frac{1}{b}\left(f_k^2(t_0) - f_{k'}^2(t_0)\right)\right| \leq C\left[(\varepsilon|k-k'|)^{1/2}|\log\varepsilon|^\infty + \varepsilon|g(t_0)|\right], \tag{3.34}$$

thanks to (3.14) and (3.15). For the first two terms the estimate (A.7) has also been used for the derivatives f_k' and $f_{k'}'$:

$$\frac{\varepsilon k f_k'(t_0)}{(1-\varepsilon kt)f_k(t_0)} - \frac{\varepsilon k' f_{k'}'(t_0)}{(1-\varepsilon k't)f_{k'}(t_0)} = \mathcal{O}(\varepsilon)g(t_0) + \frac{f_{k'}'(t_0)}{f_{k'}(t_0)}\left(\frac{\varepsilon k}{1-\varepsilon kt} - \frac{\varepsilon k'}{1-\varepsilon k't}\right)$$
$$= \mathcal{O}(\varepsilon)g(t_0) + \mathcal{O}(\varepsilon|k-k'|).$$

Plugging (3.33) and (3.34) into (3.32), we get the estimate

$$\left[\frac{|f_k'(t_0)|}{f_k(t_0)} + \frac{|f_{k'}'(t_0)|}{f_{k'}(t_0)} + \mathcal{O}(\varepsilon)\right] g(t_0) = \mathcal{O}\left((\varepsilon|k-k'|)^{1/2}|\log\varepsilon|^\infty\right). \tag{3.35}$$

Now if

$$\frac{|f_k'(t_0)|}{f_k(t_0)} + \frac{|f_{k'}'(t_0)|}{f_{k'}(t_0)} \geq |\log\varepsilon|^{-2},$$

the result follows immediately. Therefore we can assume that

$$\frac{|f_k'(t_0)|}{f_k(t_0)} + \frac{|f_{k'}'(t_0)|}{f_{k'}(t_0)} \leq |\log\varepsilon|^{-2}, \tag{3.36}$$

but we claim that this also implies

$$\frac{|f_k'(t)|}{f_k(t)} + \frac{|f_{k'}'(t)|}{f_{k'}(t)} \leq |\log\varepsilon|^{-2} \text{ for any } t \in [t_0, t_\varepsilon]. \tag{3.37}$$

Indeed, setting

$$h_k(t) := -f_k'(t)/f_k(t),$$

a simple computation involving the variational equation (A.1) yields

$$h_k'(t) = -\frac{\varepsilon k f_k'(t)}{(1-\varepsilon k t) f_k(t)} - V_k(t) + \frac{1}{b}\left(1 - f_k^2(t)\right) + h_k^2(t) = -V_k(t) + h_k^2(t) + \mathcal{O}(1),$$

using (A.7) again. Hence $h_k'(t_0) < 0$, since $V_k(t_0) \gg 1$, which follows from $t_0 > t_* \gg 1$, and therefore (3.37) holds. An identical argument applies to $h_{k'}$ and thus to the sum

$$h_k + h_{k'} =: h.$$

Finally, the explicit expression of $g'(t)$ in combination with (3.37) gives for $t \geq t_0$

$$\begin{aligned} |g(t)| = \left| \int_t^{t_\varepsilon} \mathrm{d}\eta \, g'(\eta) \right| &\leq \int_t^{t_\varepsilon} \mathrm{d}\eta \, \left[(h(\eta) + \mathcal{O}(\varepsilon)) \, |g(\eta)| + \mathcal{O}\left((\varepsilon|k-k'|)^{1/2} \, |\log\varepsilon|^{\infty} \right) \right] \\ &\leq C|\log\varepsilon|^{-1} \sup_{t\in[t_0,t_\varepsilon]} |g(t)| + \mathcal{O}\left((\varepsilon|k-k'|)^{1/2} \, |\log\varepsilon|^{\infty} \right), \end{aligned} \tag{3.38}$$

which implies the result. □

The above estimate is mainly useful in providing bounds on quantities of the form

$$I_{k,k'}(t) := F_k(t) - F_{k'}(t) \frac{f_k^2(t)}{f_{k'}^2(t)}, \tag{3.39}$$

alluded to in Subsection 2.2. As announced there, the main difficulty is that we need to show that $I_{k,k'}$ is small *relatively to* f_k^2, which is the content of the following corollary. We need the following notation

$$[0, \bar{t}_{k,\varepsilon}] := \left\{ t : f_k(t) \geq |\log\varepsilon|^3 f_k(t_\varepsilon) \right\}. \tag{3.40}$$

Note that the monotonicity for large t of f_k guarantees that the above set is indeed an interval and that

$$\bar{t}_{k,\varepsilon} = t_\varepsilon + \mathcal{O}(\log|\log\varepsilon|). \tag{3.41}$$

Corollary 3.1 (Estimates on the correction function).
Under the assumptions of Proposition 3.2, it holds

$$\sup_{t\in[0,t_\varepsilon]} \left| \frac{I_{k,k'}}{f_k^2} \right| \leq C\,(\varepsilon|k-k'|)^{1/2}\, |\log\varepsilon|^{\infty} \tag{3.42}$$

and, setting $\bar{t}_\varepsilon := \min\{\bar{t}_{k,\varepsilon}, \bar{t}_{k',\varepsilon}\}$,

$$\sup_{t\in[0,\bar{t}_\varepsilon]} \left| \frac{\partial_t I_{k,k'}}{f_k^2} \right| \leq C\,(\varepsilon|k-k'|)^{1/2}\, |\log\varepsilon|^{\infty}. \tag{3.43}$$

Proof. We write

$$\frac{I_{k,k'}(t)}{f_k^2(t)} = \frac{F_k(t)}{f_k^2(t)} - \frac{F_{k'}(t)}{f_{k'}^2(t)}$$

Using the definition of the potential function (A.8) and its properties (A.9), we can rewrite

$$\begin{aligned} \frac{F_k(t)}{f_k^2(t)} - \frac{F_{k'}(t)}{f_{k'}^2(t)} &= -\int_t^{t_\varepsilon} \mathrm{d}\eta \left[b_k(\eta) \frac{f_k^2(\eta)}{f_k^2(t)} - b_{k'}(\eta) \frac{f_{k'}^2(\eta)}{f_{k'}^2(t)} \right] \\ &= \int_t^{t_\varepsilon} \mathrm{d}\eta \left[b_k(\eta) \left(\frac{f_{k'}^2(\eta)}{f_{k'}^2(t)} - \frac{f_k^2(\eta)}{f_k^2(t)} \right) + (b_{k'}(\eta) - b_k(\eta)) \, \frac{f_{k'}^2(\eta)}{f_{k'}^2(t)} \right]. \end{aligned} \tag{3.44}$$

We first observe that for any $\eta \geq t$

$$\frac{f_{k'}(\eta)}{f_{k'}(t)} \leq C, \tag{3.45}$$

as it easily follows by combining the monotonicity of f_k for t large with its strict positivity close to the origin (see Proposition A.1 and Lemma A.2 for the details). Hence we can bound the last term on the r.h.s. of (3.44) as

$$\left| \int_t^{t_\varepsilon} \mathrm{d}\eta \, \left(b_{k'}(\eta) - b_k(\eta) \right) \frac{f_{k'}^2(\eta)}{f_{k'}^2(t)} \right| \leq C |\log \varepsilon| \left\| b_{k'} - b_k \right\|_{L^\infty(I_\varepsilon)} = \mathcal{O}\left(\left(\varepsilon |k - k'| \right)^{1/2} |\log \varepsilon|^\infty \right), \tag{3.46}$$

since by (3.14)

$$b_{k'}(t) - b_k(t) = (1 + \mathcal{O}(\varepsilon)) \left(\mathcal{O}(\varepsilon |k - k'| t^2) + \alpha(k) - \alpha(k') \right) = \mathcal{O}\left(\left(\varepsilon |k - k'| \right)^{1/2} |\log \varepsilon|^\infty \right).$$

For the first term on the r.h.s. of (3.44) we exploit the estimate

$$\frac{f_{k'}(\eta)}{f_{k'}(t)} - \frac{f_k(\eta)}{f_k(t)} = \mathcal{O}\left(\left(\varepsilon |k - k'| \right)^{1/2} |\log \varepsilon|^\infty \right),$$

which can be proven by writing

$$\frac{f_k(\eta)}{f_k(t)} = \exp\left\{ \int_t^\eta \mathrm{d}\tau \, \frac{f_k'(\tau)}{f_k(\tau)} \right\},$$

which implies

$$\left| \frac{f_{k'}(\eta)}{f_{k'}(t)} - \frac{f_k(\eta)}{f_k(t)} \right| = \frac{f_{k'}(\eta)}{f_{k'}(t)} \left| 1 - \exp\left\{ \int_t^\eta \mathrm{d}\tau \left[\frac{f_k'(\tau)}{f_k(\tau)} - \frac{f_{k'}'(\tau)}{f_{k'}(\tau)} \right] \right\} \right|$$
$$\leq C \int_t^\eta \mathrm{d}\tau \left| \frac{f_k'(\tau)}{f_k(\tau)} - \frac{f_{k'}'(\tau)}{f_{k'}(\tau)} \right| \exp\left\{ \int_t^\eta \mathrm{d}\tau \left| \frac{f_k'(\tau)}{f_k(\tau)} - \frac{f_{k'}'(\tau)}{f_{k'}(\tau)} \right| \right\} \leq C \left(\varepsilon |k - k'| \right)^{1/2} |\log \varepsilon|^\infty, \tag{3.47}$$

where we have used (3.45), the estimate $|1 - e^\delta| \leq |\delta| e^{|\delta|}$, $\delta \in \mathbb{R}$, and (3.29). Putting together (3.44) with (3.46) and (3.47), we conclude the proof of (3.42).

To obtain (3.29) we first note that since $F_k'(t) \leq 0$, the positivity of K_k in $[0, \bar{t}_{k,\varepsilon}]$ recalled in Lemma A.4 ensures that

$$\left| \frac{F_k(t)}{f_k^2(t)} \right| \leq 1$$

in $[0, \bar{t}_{k,\varepsilon}]$. Then we may use (3.29) again to estimate

$$\sup_{t \in [0, \bar{t}_\varepsilon]} \left| \frac{\partial_t I_{k,k'}}{f_k^2} \right| = \sup_{t \in [0, \bar{t}_\varepsilon]} \left[\left| (1 - \varepsilon k t) \, b_k - (1 - \varepsilon k' t) \, b_{k'} \right| + 2 \left| \frac{F_{k'}}{f_{k'}^2} \right| \left| \frac{f_k'}{f_k} - \frac{f_{k'}'}{f_{k'}} \right| \right]$$
$$\leq C \left(\varepsilon |k - k'| \right)^{1/2} |\log \varepsilon|^\infty,$$

and the proof is complete. □

4 Energy Upper Bound

We now turn to the proof of the energy upper bound corresponding to (2.5), namely we prove the following:

Proposition 4.1 (**Upper bound to the full GL energy**).
Let $1 < b < \Theta_0^{-1}$ and ε be small enough. Then it holds

$$E_\varepsilon^{\mathrm{GL}} \leq \frac{1}{\varepsilon}\int_0^{|\partial\Omega|} \mathrm{d}s\, E_\star^{\mathrm{1D}}(k(s)) + C\varepsilon|\log\varepsilon|^\infty \tag{4.1}$$

where $s \mapsto k(s)$ is the curvature function of the boundary $\partial\Omega$ as a function of the tangential coordinate.

This result is proven as usual by evaluating the GL energy of a trial state having the expected physical features. As is well-known [FH3], such a trial state should be concentrated along the boundary of the sample, and the induced magnetic field should be chosen close to the applied one. Before entering the heart of the proof, we briefly explain how these considerations allow us to reduce to the proof of an upper bound to the reduced functional (3.1). We define

$$G_{\mathcal{A}_\varepsilon} := \inf\left\{\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi], \psi(0,t) = \psi(|\partial\Omega|,t)\right\}, \tag{4.2}$$

the infimum of the reduced functional under periodic boundary conditions in the tangential direction and prove

Lemma 4.1 (**Reduction to the boundary functional**).
Under the assumptions of Proposition 4.1, it holds

$$E_\varepsilon^{\mathrm{GL}} \leq \frac{1}{\varepsilon} G_{\mathcal{A}_\varepsilon} + C\varepsilon^\infty. \tag{4.3}$$

Proof. This is a standard reduction for which more details may be found in [FH3, Section 14.4.2] and references therein. See also [CR, Sections 4.1 and 5.1]. We provide a sketch of the proof for completeness.

We first pick the trial vector potential as

$$\mathbf{A}_{\mathrm{trial}} = \mathbf{F}$$

where $\mathbf{F}$ is the induced vector potential written in a gauge where $\mathrm{div}\,\mathbf{F} = 0$, namely the unique solution of

$$\begin{cases} \mathrm{div}\,\mathbf{F} = 0, & \text{in } \Omega, \\ \mathrm{curl}\,\mathbf{F} = 1, & \text{in } \Omega, \\ \mathbf{F}\cdot\boldsymbol{\nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

Next we introduce boundary coordinates as described in [FH3, Appendix F]: let

$$\boldsymbol{\gamma}(\xi) : \mathbb{R}\setminus(|\partial\Omega|\mathbb{Z}) \to \partial\Omega$$

be a counterclockwise parametrization of the boundary $\partial\Omega$ such that $|\boldsymbol{\gamma}'(\xi)| = 1$. The unit vector directed along the inward normal to the boundary at a point $\boldsymbol{\gamma}(\xi)$ will be denoted by $\boldsymbol{\nu}(\xi)$. The curvature $k(\xi)$ is then defined through the identity

$$\boldsymbol{\gamma}''(\xi) = k(\xi)\boldsymbol{\nu}(\xi).$$

Our trial state will essentially live in the region

$$\tilde{\mathcal{A}}_\varepsilon := \left\{\mathbf{r}\in\Omega \mid \mathrm{dist}(\mathbf{r},\partial\Omega) \leq c_0\varepsilon|\log\varepsilon|\right\}, \tag{4.4}$$

and in such a region we can introduce tubular coordinates $(s,\varepsilon t)$ (note the rescaling of the normal variable) such that, for any given $\mathbf{r}\in\tilde{\mathcal{A}}_\varepsilon$, $\varepsilon t = \mathrm{dist}(\mathbf{r},\partial\Omega)$, i.e.,

$$\mathbf{r}(s,\varepsilon t) = \boldsymbol{\gamma}'(s) + \varepsilon t\boldsymbol{\nu}(s), \tag{4.5}$$

which can obviously be realized as a diffeomorphism for ε small enough. Hence the boundary layer becomes in the new coordinates (s,t)

$$\mathcal{A}_\varepsilon := \{(s,t) \in [0, |\partial\Omega|] \times [0, c_0|\log\varepsilon|]\}\,. \tag{4.6}$$

We now pick a function $\psi(s,t)$ defined on $\mathcal{A}_\varepsilon$, satisfying periodic boundary conditions in the s variable. Using a smooth cut-off function $\chi(t)$ with $\chi(t) \equiv 1$ for $t \in [0, c_0|\log\varepsilon|]$ and $\chi(t)$ exponentially decreasing for $t > c_0|\log\varepsilon|$, we associate to ψ the GL trial state

$$\Psi_{\mathrm{trial}}(\mathbf{r}) := \psi(s,t)\chi(t).$$

Then, with the definition of $\mathcal{G}_{\mathcal{A}_\varepsilon}$ as in (3.1), a relatively straightforward computation gives

$$E^{\mathrm{GL}}\left[\Psi_{\mathrm{trial}}, \mathbf{A}_{\mathrm{trial}}\right] \leq \frac{1}{\varepsilon}\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] + C\varepsilon^\infty,$$

and the desired result follows immediately. Note that this computation uses the gauge invariance of the GL functional, e.g., through [FH3, Lemma F.1.1]. □

The problem is now reduced to the construction of a proper trial state for $\mathcal{G}_{\mathcal{A}_\varepsilon}$. To capture the $\mathcal{O}(\varepsilon)$ correction (which depends on curvature) to the leading order of the GL energy (which does not depend explicitly on curvature), we need a more elaborate function than has been considered so far. The construction is detailed in Subsection 4.1 and the computation completing the proof of Proposition 4.1 is given in Subsection 4.2.

4.1 The trial state in boundary coordinates

We start by recalling the splitting of the domain $\mathcal{A}_\varepsilon$ defined in (3.4) into $N_\varepsilon \propto \varepsilon^{-1}$ rectangular cells $\{\mathcal{C}_n\}_{n=1\ldots N_\varepsilon}$ with boundaries s_n, s_{n+1} in the s-coordinate such that

$$s_{n+1} - s_n = \ell_\varepsilon \propto \varepsilon,$$

so that

$$N_\varepsilon = \frac{|\partial\Omega|}{\ell_\varepsilon}.$$

We denote

$$\mathcal{C}_n = [s_n, s_{n+1}] \times [0, c_0|\log\varepsilon|], \tag{4.7}$$

with the convention that $s_1 = 0$, for simplicity. We will approximate the curvature $k(s)$ inside each cell by its mean value and set

$$k_n := \ell_\varepsilon^{-1}\int_{s_n}^{s_{n+1}} \mathrm{d}s\, k(s). \tag{4.8}$$

We also denote by

$$\alpha_n = \alpha(k_n) \tag{4.9}$$

the optimal phase associated to k_n, obtained by minimizing $E^{1\mathrm{D}}(\alpha, k_n)$ with respect to α as in Section 3.1.

The assumption about the smoothness of the boundary guarantees that

$$k_n - k_{n+1} = \mathcal{O}(\varepsilon). \tag{4.10}$$

Indeed if we assume that $\sup_{s\in[0,2\pi]}|\partial_s k(s)| \leq C < \infty$ (independent of ε), one gets

$$\ell_\varepsilon^{-1}\left|\int_{k_n}^{k_{n+1}} \mathrm{d}s\, k(s) - \int_{k_{n+1}}^{k_{n+2}} \mathrm{d}s\, k(s)\right| = \ell_\varepsilon^{-1}\left|\int_{k_n}^{k_{n+1}} \mathrm{d}s \int_s^{k_{n+1}} \mathrm{d}\eta\, \partial_\eta k(\eta) + \int_{k_{n+1}}^{k_{n+2}} \mathrm{d}s \int_{k_{n+1}}^{s} \mathrm{d}\eta\, \partial_\eta k(\eta)\right|$$
$$\leq C\ell_\varepsilon = \mathcal{O}(\varepsilon).$$

We can then apply Proposition 3.1 to obtain

$$\alpha_n - \alpha_{n+1} = \mathcal{O}(\varepsilon|\log\varepsilon|^\infty), \tag{4.11}$$

$$\left\| f_n^{(m)} - f_{n+1}^{(m)} \right\|_{L^\infty(I_\varepsilon)} = \mathcal{O}(\varepsilon|\log\varepsilon|^\infty), \tag{4.12}$$

for any finite $m \in \mathbb{N}$.

Our trial state has the form

$$\psi_{\mathrm{trial}}(s,t) = g(s,t)\exp\left\{-i\left(\varepsilon^{-1}S(s) - \varepsilon\delta_\varepsilon s\right)\right\} \tag{4.13}$$

where δ_ε is the number (3.3). The density g and phase factor S are defined as follows:

- The density. The modulus of our wave function is constructed to be essentially piecewise constant in the s-direction, with the form $f_{k_n}(t)$ in the cell $\mathcal{C}_n$. The admissibility of the trial state requires that g be continuous and we thus set:

  $$g(s,t) := f_{k_n} + \chi_n, \tag{4.14}$$

  where the function χ_n satisfies

  $$\chi_n(s,t) = \begin{cases} 0, & \text{at } s = s_n, \\ f_{k_{n+1}}(t) - f_{k_n}(t), & \text{at } s = s_{n+1}, \end{cases} \tag{4.15}$$

  the continuity at the s_n boundary being ensured by χ_{n-1}. A simple choice is given by

  $$\chi_n(s,t) = \left(f_{k_{n+1}}(t) - f_{k_n}(t)\right)\left(1 - \frac{s - s_{n+1}}{s_n - s_{n+1}}\right). \tag{4.16}$$

  Note that $|k_n - k_{n+1}| \leq C|s_n - s_{n+1}| \leq C\varepsilon$ since the curvature is assumed to be a smooth function of s. Clearly, in view of Proposition 3.1 we can impose the following bounds on χ_n:

  $$|\chi_n| \leq C\varepsilon|\log\varepsilon|^\infty, \qquad |\partial_t\chi_n| \leq C\varepsilon|\log\varepsilon|^\infty, \qquad |\partial_s\chi_n| \leq C|\log\varepsilon|^\infty, \tag{4.17}$$

  so that χ_n is indeed only a small correction to the desired density f_{k_n} in $\mathcal{C}_n$.

- The phase. We let

  $$S = S(s) = S_{\mathrm{loc}}(s) + S_{\mathrm{glo}}(s)$$

  where S_{loc} varies locally (on the scale of a cell) and S_{glo} varies globally (on the scale of the full interval $[0, |\partial\Omega|]$) and is chosen to enforce the periodicity on the boundary of the trial state. The term S_{loc} is the main one, and its s derivative should be equal to α_n in each cell $\mathcal{C}_n$ in order that the evaluation of the energy be naturally connected to the 1D functional we studied before. We define S_{loc} recursively by setting:

  $$S_{\mathrm{loc}}(s) = \begin{cases} \alpha_1 s, & \text{in } \mathcal{C}_1, \\ \alpha_n(s - s_n) + S_{\mathrm{loc}}(s_n), & \text{in } \mathcal{C}_n, n \geq 2, \end{cases} \tag{4.18}$$

  which in particular guarantees the continuity of S_{loc} on $[s_1, s_{N_\varepsilon+1}[$. Moreover we easily compute (recall that $s_1 = 0$)

  $$S_{\mathrm{loc}}(s_n) = \sum_{m=2}^{n-1}\alpha_m\left(s_{m+1} - s_m\right) + \alpha_1 s_2 = \int_0^{s_n} \mathrm{d}s\,\alpha(s) + \mathcal{O}(\varepsilon|\log\varepsilon|^\infty). \tag{4.19}$$

The factor S_{glo} ensures that

$$S(s_{N_\varepsilon+1}) - S(s_1) = S(s_{N_\varepsilon+1}) \in 2\pi\varepsilon\mathbb{Z},$$

which is required for (4.13) to be periodic in the s-direction and hence to correspond to a single-valued wave function in the original variables. The conditions we impose on S_{glo} are thus

$$\begin{aligned} S_{\mathrm{glo}}(s_1) &= 0 \\ S_{\mathrm{glo}}(s_{N_\varepsilon+1}) &= 2\pi\varepsilon \left(\alpha_{N_\varepsilon}\left(s_{N_\varepsilon+1} - s_{N_\varepsilon}\right) + S_{\mathrm{loc}}(s_{N_\varepsilon}) - \lfloor \alpha_{N_\varepsilon}\left(s_{N_\varepsilon+1} - s_{N_\varepsilon}\right) + S_{\mathrm{loc}}(s_{N_\varepsilon})\rfloor\right) \end{aligned} \tag{4.20}$$

with $\lfloor\, . \,\rfloor$ standing for the integer value. Thanks to (4.19), we have

$$\alpha_{N_\varepsilon}\left(s_{N_\varepsilon+1} - s_{N_\varepsilon}\right) + S_{\mathrm{loc}}(s_{N_\varepsilon}) = \mathcal{O}(1)$$

and we can thus clearly impose that S_{glo} be regular and

$$|S_{\mathrm{glo}}| \leq C\varepsilon, \qquad |\partial_s S_{\mathrm{glo}}| \leq C\varepsilon. \tag{4.21}$$

Remark 4.1 (s-dependence of the trial state)
The main novelty here is the fact that the density and phase of the trial state have (small) variations on the scale of the cells which are of size $\mathcal{O}(\varepsilon)$ in the s-variable. A noteworthy point is that the phase needs not have a t-dependence to evaluate the energy at the level of precision we require. Basically this is associated with the fact that the t^2 term in (3.2) comes multiplied with an ε factor. The main point that renders the computation of the energy doable is (4.17) and this is where the analysis of Subsection 3.1 enters heavily. □

4.2 The energy of the trial state

We may now complete the proof of Proposition 4.1 by proving

Lemma 4.2 (Upper bound for the boundary functional).
With ψ_{trial} given by the preceding construction, it holds

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi_{\mathrm{trial}}] \leq \int_0^{|\partial\Omega|} \mathrm{d}s E^{1\mathrm{D}}_\star(k(s)) + \mathcal{O}(\varepsilon^2|\log\varepsilon|^\infty). \tag{4.22}$$

The upper bound (4.1) follows from Lemmas 4.1 and 4.2 since ψ_{trial} is periodic in the s-variable and hence an admissible trial state for $G_{\mathcal{A}_\varepsilon}$.

Proof. As explained in Subsection 3.1, inserting (4.13) into (3.1) yields

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi_{\mathrm{trial}}] = \mathcal{E}^{2\mathrm{D}}_S[g] \tag{4.23}$$

where $\mathcal{E}^{2\mathrm{D}}_S[g]$ is defined in (3.6). For clarity we split the estimate of the r.h.s. of the above equation into several steps. We use the shorter notation f_n for f_{k_n} when this generates no confusion.

Step 1. Approximating the curvature. In view of the continuity of the trial function, the energy is the sum of the energies restricted to each cell. We approximate $k(s)$ by k_n in $\mathcal{C}_n$ as announced, and note that since k is regular we have $|k(s) - k(s_n)| \leq C\varepsilon$ in each cell, with a constant C independent of j. We thus have

$$\begin{aligned} \mathcal{E}^{2\mathrm{D}}_S[g] \leq \sum_{n=1}^{N_\varepsilon} \int_{\mathcal{C}_n} \mathrm{d}t\, \mathrm{d}s\, \left(1 - \varepsilon k_n t\right) \Bigg\{ |\partial_t g|^2 + \frac{\varepsilon^2}{(1-\varepsilon k_n t)^2} |\partial_s g|^2 \\ + \frac{\left(t + \partial_s S - \frac{1}{2}\varepsilon t^2 k_n\right)^2}{(1-\varepsilon k_n t)^2} g^2 - \frac{1}{2b}\left(2g^2 - g^4\right) \Bigg\} \left(1 + \mathcal{O}(\varepsilon^2)\right) \end{aligned} \tag{4.24}$$

since each k-dependent term comes multiplied with an ε factor.

Step 2. Approximating the phase. In $\mathcal{C}_n$ we have

$$\partial_s S = \alpha_n + \partial_s S_{\mathrm{glo}} = \alpha_n + \mathcal{O}(\varepsilon).$$

We can thus expand the potential term:

$$\int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \frac{\left(t+\partial_s S - \frac{1}{2}\varepsilon t^2 k_n\right)^2}{1-\varepsilon k_n t} g^2 = \int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \frac{\left(t+\alpha_n - \frac{1}{2}\varepsilon t^2 k_n\right)^2}{1-\varepsilon k_n t} g^2$$
$$+ 2\int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \partial_s S_{\mathrm{glo}} \frac{t+\alpha_n - \frac{1}{2}\varepsilon t^2 k_n}{1-\varepsilon k_n t} g^2 + \int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \frac{\left(\partial_s S_{\mathrm{glo}}\right)^2}{1-\varepsilon k_n t} g^2 \quad (4.25)$$

and obviously

$$\int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \frac{\left(\partial_s S_{\mathrm{glo}}\right)^2}{1-\varepsilon k_n t} g^2 \leq C\varepsilon^3 |\log\varepsilon|^\infty,$$

because of (4.21) and the size of $\mathcal{C}_n$ in the s direction. Next we note that in $\mathcal{C}_n$

$$g^2 = f_n^2 + 2 f_n \chi_n + \chi_n^2$$

so that, using (A.5) and the fact that $\partial_s S_{\mathrm{glo}}$ only depends on s we have

$$\int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \partial_s S_{\mathrm{glo}} \frac{t+\alpha_n - \frac{1}{2}\varepsilon t^2 k_n}{1-\varepsilon k_n t} g^2 = \int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \partial_s S_{\mathrm{glo}} \frac{t+\alpha_n - \frac{1}{2}\varepsilon t^2 k_n}{1-\varepsilon k_n t} \left(2 f_n \chi_n + \chi_n^2\right),$$

which is easily bounded by $C\varepsilon^3|\log\varepsilon|^\infty$ using (4.17), (4.21) and the fact that $|s_{n+1} - s_n| \leq C\varepsilon$. All in all:

$$\int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \frac{\left(t+\partial_s S - \frac{1}{2}\varepsilon t^2 k_n\right)^2}{1-\varepsilon k_n t} g^2 = \int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\, \frac{\left(t+\alpha_n - \frac{1}{2}\varepsilon t^2 k_n\right)^2}{1-\varepsilon k_n t} g^2 + \mathcal{O}(\varepsilon^3|\log\varepsilon|^\infty). \quad (4.26)$$

Step 3. The 1D functional inside each cell. We now have to estimate an essentially 1D functional in each cell, closely related to (3.9):

$$\int_{\mathcal{C}_n} \mathrm{d}t\,\mathrm{d}s\,(1-\varepsilon k_n t) \left\{ |\partial_t g|^2 + \frac{\varepsilon^2}{(1-\varepsilon k_n t)^2} |\partial_s g|^2 + \frac{\left(t+\alpha_n - \frac{1}{2}\varepsilon t^2 k_n\right)^2}{(1-\varepsilon k_n t)^2} g^2 - \frac{1}{2b}\left(2g^2 - g^4\right) \right\}. \quad (4.27)$$

We may now expand g according to (4.14) in the above expression and use the variational equation (A.1) to cancel the first order terms in χ_n. This yields

$$\int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\ (1-\varepsilon k_n t) \left\{ |\partial_t g|^2 + \tfrac{\varepsilon^2}{(1-\varepsilon k_n t)^2} |\partial_s g|^2 + V_{k_n}(t) g^2 - \tfrac{1}{2b}\left(2g^2 - g^4\right) \right\} = \ell_\varepsilon E^{\mathrm{1D}}_\star(k_n)$$
$$+ \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\ (1-\varepsilon k_n t) \left\{ |\partial_t \chi_n|^2 + \tfrac{\varepsilon^2}{(1-\varepsilon t k_n)^2} |\partial_s \chi_n|^2 + V_{k_n} \chi_n^2 + \tfrac{1}{2b}\left(6\chi_n^2 f_n^2 + 4\chi_n^3 f_n + \chi_n^4 - 2\chi_n^2\right) \right\}$$
$$= \ell_\varepsilon E^{\mathrm{1D}}_\star(k_n) + O(\varepsilon^3|\log\varepsilon|^\infty), \quad (4.28)$$

where we only have to use (4.17) to obtain the final estimate.

Step 4, Riemann sum approximation. Gathering all the above estimates we obtain

$$\mathcal{E}^{\mathrm{2D}}_S[g] \leq \ell_\varepsilon \sum_{n=1}^{N_\varepsilon} E^{\mathrm{1D}}_\star(k_n)\left(1+\mathcal{O}(\varepsilon^2)\right) + \mathcal{O}(\varepsilon^2|\log\varepsilon|^\infty) = \int_0^{|\partial\Omega|} \mathrm{d}s\, E^{\mathrm{1D}}_\star(k(s)) + \mathcal{O}(\varepsilon^2|\log\varepsilon|^\infty). \quad (4.29)$$

Indeed, (3.13) implies that inside $\mathcal{C}_n$

$$\left|E_\star^{1\mathrm{D}}(k_n) - E_\star^{1\mathrm{D}}(k(s))\right| \leq C\varepsilon\ell_\varepsilon |\log\varepsilon|^\infty \leq C\varepsilon^2 |\log\varepsilon|^\infty. \tag{4.30}$$

Recognizing a Riemann sum of $N_\varepsilon \propto \varepsilon^{-1}$ terms in (4.29) and recalling that $E_\star^{1\mathrm{D}}(k_n)$ is of order 1, irrespective of n, thus leads to (4.29). Combining (4.23) and (4.29) we obtain (4.22) which concludes the proof of Lemma 4.2 and hence that of Proposition 4.1, via Lemma 4.1. □

5 Energy Lower Bound

The main result proven in this section is the following

Proposition 5.1 (**Energy lower bound**).
Let $\Omega \subset \mathbb{R}^2$ be any smooth simply connected domain. For any fixed $1 < b < \Theta_0^{-1}$, in the limit $\varepsilon \to 0$, it holds

$$E_\varepsilon^{\mathrm{GL}} \geq \frac{1}{\varepsilon}\int_0^{|\partial\Omega|} \mathrm{d}s\, E_\star^{1\mathrm{D}}\left(k(s)\right) - C\varepsilon|\log\varepsilon|^\infty. \tag{5.1}$$

We first reduce the problem to the study of decoupled functionals in the boundary layer in Subsection 5.1 and then provide lower bounds to these in Subsection 5.2, which contains the main new ideas of our proof.

5.1 Preliminary reductions

As in Section 4, the starting point is a restriction to the boundary layer together with a replacement of the vector potential. We refer to the proof of Lemma 4.1 and in particular (4.5) for the definition of the boundary coordinates.

Lemma 5.1 (**Reduction to the boundary functional**).
Under the assumptions of Proposition 5.1, it holds

$$E_\varepsilon^{\mathrm{GL}} \geq \frac{1}{\varepsilon}\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] - C\varepsilon^2|\log\varepsilon|^2, \tag{5.2}$$

with $\psi(s,t) = \Psi^{\mathrm{GL}}(\mathbf{r}(s,\varepsilon t))$ in $\mathcal{A}_\varepsilon$ and $\mathcal{G}_{\mathcal{A}_\varepsilon}$ is the boundary functional defined in (3.1)

Proof. A simplified version of the result for disc samples is proven in [CR, Proposition 4.1], where a rougher lower bound is also derived for general domains. This latter result is obtained by dropping the curvature dependent terms from the energy, which was sufficient for the analysis contained there. Here we need more precision in order to obtain a remainder term of order $o(\varepsilon)$. We highlight here the main steps and skip most of the technical details.

A suitable partition of unity together with the standard Agmon estimates (see [FH1, Section 14.4]) allow to restrict the integration to the boundary layer:

$$E_\varepsilon^{\mathrm{GL}} \geq \int_{\tilde{\mathcal{A}}_\varepsilon} \mathrm{d}\mathbf{r} \left\{ \left| \left(\nabla + i\tfrac{\mathbf{A}^{\mathrm{GL}}}{\varepsilon^2} \right) \Psi_1 \right|^2 - \tfrac{1}{2b\varepsilon^2}\left[2|\Psi_1|^2 - |\Psi_1|^4\right] \right\} + \mathcal{O}(\varepsilon^\infty). \tag{5.3}$$

where Ψ_1 is given in terms of Ψ^{GL} in the form $\Psi_1 = f_1\Psi^{\mathrm{GL}}$ for some radial $0 \leq f_1 \leq 1$ with support containing the set $\tilde{\mathcal{A}}_\varepsilon$ defined by (4.4) and contained in

$$\{\mathbf{r} \in \Omega \mid \mathrm{dist}(\mathbf{r}, \partial\Omega) \leq C\varepsilon|\log\varepsilon|\}$$

for a possibly large constant C. The constant c_0 in the definition (4.4) of the boundary layer has to be chosen large enough, but the choice of the support of f_1 remains to any other extent arbitrary and one can clearly pick f_1 in such a way that $f_1 = 1$ in $\tilde{\mathcal{A}}_\varepsilon$ and going smoothly to 0 outside of it.

The second ingredient of the proof is the replacement of the magnetic potential $\mathbf{A}^{\mathrm{GL}}$ but this can be done following the same strategy applied to disc samples in [CR, Eqs. (4.18) – (4.26)], whose estimates are not affected by the dependence of the curvature on s. The crucial properties used there are indeed provided by the Agmon estimates, see below.

The overall prefactor ε^{-1} is then inherited from the rescaling of the normal coordinate $\tau = \varepsilon t$ in the tubular neighborhood of the boundary. Note here the use of a different convention with respect to both [CR, FH1], where the tangential coordinate s was rescaled too. □

We need to rephrase some well-known decay estimates in a form suited to our needs. The Agmon estimates proven in [FH2, Eq. (12.9)] can be translated is analogous bounds applying to $\psi(s,t) = \Psi^{\mathrm{GL}}(\mathbf{r}(s,\varepsilon t))$ in $\mathcal{A}_\varepsilon$: for some constant $A > 0$ it holds

$$\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t\, (1 - \varepsilon k(s) t)\, e^{At} \left\{ |\psi(s,t)|^2 + \left| \left((\varepsilon \partial_s, \partial_t) + i \tfrac{\mathbf{A}^{\mathrm{GL}}(\mathbf{r}(s,\varepsilon t))}{\varepsilon} \right) \psi(s,t) \right|^2 \right\} = \mathcal{O}(1). \tag{5.4}$$

In addition we are going to use two additional bounds proven in [FH2, Eq. (10.21) and (11.50)]:

$$\|\psi\|_{L^\infty(\mathcal{A}_\varepsilon)} \leq 1, \qquad \|(\varepsilon\partial_s, \partial_t)\psi\|_{L^\infty(\mathcal{A}_\varepsilon)} \leq C. \tag{5.5}$$

These bounds imply the following

Lemma 5.2 (**Useful consequences of Agmon estimates**).
Let $\bar{t} = c_0|\log\varepsilon|(1 + o(1))$ for some c_0 large enough, then for any $a, b, s_0 \in [0, 2\pi)$,

$$\int_a^b \mathrm{d}s\, |\psi(s,\bar{t})| = \mathcal{O}(\varepsilon^\infty), \qquad \int_{\bar{t}}^{c_0|\log\varepsilon|} \mathrm{d}t\, |\psi(s_0,t)| = \mathcal{O}(\varepsilon^\infty). \tag{5.6}$$

Proof. We start by considering the first estimate: let $\chi(t)$ be a suitable smooth function with support in $[t_1, \bar{t}]$, with $t_1 = \bar{t} - c$, for some $c > 0$, and such that $0 \leq \chi \leq 1$, $\chi(\bar{t}) = 1$ and $|\partial_t \chi| \leq C$. Then one has

$$\int_a^b \mathrm{d}s\, |\psi(s,\bar{t})| = \int_a^b \mathrm{d}s\, \chi(\bar{t})|\psi(s,\bar{t})| = \int_a^b \mathrm{d}s \int_{t_1}^{\bar{t}} \mathrm{d}t\, \left[\chi(t)\partial_t|\psi(s,t)| + |\psi(s,t)|\partial_t\chi(t)\right]$$
$$\leq C e^{-\frac{1}{2}At_1} \left\{ \left[\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t\, e^{At} \left|\partial_t|\psi(s,t)|\right|^2 \right]^{1/2} + \left[\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t\, e^{At} |\psi(s,t)|^2 \right]^{1/2} \right\} = \mathcal{O}(\varepsilon^\infty), \tag{5.7}$$

by (5.4) and the assumption on t_1 and $\bar{t}$. Indeed the factor $e^{-\frac{1}{2}At_1} = \varepsilon^{\frac{1}{2}Ac_0(1+o(1))}$ can be made smaller than any power of ε by taking c_0 large enough.

For the second estimate we use a tangential cut-off function, i.e., a smooth function $\chi(s)$ with support[7] in $[s_0, 2\pi]$, such that $0 \leq \chi \leq 1$, $\chi(s_0) = 1$ and $|\partial_s\chi| \leq C$. Then as in the estimate above (recall that $t_\varepsilon := c_0|\log\varepsilon|$)

$$\int_{\bar{t}}^{t_\varepsilon} \mathrm{d}t\, |\psi(s_0,t)| = \int_{\bar{t}}^{t_\varepsilon} \mathrm{d}t\, \chi(s_0)|\psi(s_0,t)| \leq \int_{s_0}^{2\pi} \mathrm{d}s \int_{\bar{t}}^{t_\varepsilon} \mathrm{d}t\, \left|\chi(s)\partial_s|\psi(s,t)| + |\psi(s,t)|\partial_s\chi(s)\right|$$
$$\leq C e^{-\frac{1}{2}A\bar{t}} \left\{ \varepsilon^{-1} \left[\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t\, e^{At} \left|\varepsilon\partial_s|\psi(s,t)|\right|^2 \right]^{1/2} + \left[\int_{\mathcal{A}_\varepsilon} \mathrm{d}s\mathrm{d}t\, e^{At} |\psi(s,t)|^2 \right]^{1/2} \right\} = \mathcal{O}(\varepsilon^\infty), \tag{5.8}$$

where the main ingredient is again (5.4) and the assumption on $\bar{t}$. □

[7]Let us assume that $s_0 - 2\pi > C > 0$, otherwise one can take as a support for χ the complement set, i.e., $[0, s_0]$.

We now introduce some reduced energy functionals defined over the cells we have introduced before, see Subsection 4.1 for the notation. In each cell we write

$$\psi(s,t) =: u_n(s,t) f_n(t) \exp\left\{-i\left(\tfrac{\alpha_n}{\varepsilon} + \delta_\varepsilon\right) s\right\}, \tag{5.9}$$

and introduce the reduced functionals

$$\mathcal{E}_n[u] := \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\, (1-\varepsilon k_n t)\, f_n^2 \left\{ |\partial_t u|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2} |\varepsilon \partial_s u|^2 - 2\varepsilon b_n(t) J_s[u] + \tfrac{1}{2b} f_n^2 \left(1-|u|^2\right)^2 \right\}, \tag{5.10}$$

with

$$b_n(t) := \frac{t + \alpha_n - \frac{1}{2}\varepsilon k_n t^2}{(1-\varepsilon k_n t)^2}, \tag{5.11}$$

and

$$J_s[u] := (iu, \partial_s u) = \tfrac{i}{2}\left(u^* \partial_s u - u \partial_s u^*\right). \tag{5.12}$$

Note that in (5.10) the curvature is approximated by its mean value in the cell $\mathcal{C}_n$. These objects play a crucial role in the sequel, as per

Lemma 5.3 (**Lower bound in terms of the reduced functionals**).
With the previous notation

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] \geq \int_0^{|\partial\Omega|} \mathrm{d}s\, E^{\mathrm{1D}}_\star(k(s)) + \sum_{n=1}^{N_\varepsilon} \mathcal{E}_n[u_n] - C\varepsilon^2 |\log\varepsilon|^\infty \tag{5.13}$$

Proof. With the above cell decomposition, we can estimate

$$\mathcal{G}_{\mathcal{A}_\varepsilon}[\psi] \geq \sum_{n=1}^{N_\varepsilon} \mathcal{E}_n^{\mathrm{GL}}[\psi] - C\varepsilon^2 |\log\varepsilon|^\infty, \tag{5.14}$$

where

$$\mathcal{E}_n^{\mathrm{GL}}[\psi] := \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\, (1-\varepsilon k_n t) \left\{ |\partial_t \psi|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2} \left|(\varepsilon \partial_s + i a_n(t))\, \psi\right|^2 - \tfrac{1}{2b}\left[2|\psi|^2 - |\psi|^4\right] \right\}, \tag{5.15}$$

and

$$a_n(t) := -t + \tfrac{1}{2}\varepsilon k_n t^2 + \varepsilon \delta_\varepsilon. \tag{5.16}$$

The remainder term has been estimated as follows: the replacement of $k(s)$ by k_n produces two different rests which can be estimated separately, i.e.,

$$\int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\; (k(s) - k_n)\, t \left\{ |\partial_t \psi|^2 - \tfrac{1}{2b}\left[2|\psi|^2 - |\psi|^4\right] \right\} = \mathcal{O}(\varepsilon^2 |\log\varepsilon|^\infty), \tag{5.17}$$

$$\frac{1}{\varepsilon}\int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\; \left\{ \tfrac{1}{1-\varepsilon k(s) t} \left|(\varepsilon\partial_s + i a_\varepsilon(s,t))\, \psi\right|^2 - \tfrac{1}{1-\varepsilon k_n t}\left|(\varepsilon \partial_s + i a_n(t))\,\psi\right|^2 \right\} = \mathcal{O}(\varepsilon^2|\log\varepsilon|^\infty). \tag{5.18}$$

In estimating the first error term (5.17), we use the fact that

$$k(s) - k_n = \mathcal{O}(\varepsilon)$$

and the bounds (5.5) together with the cell size. For the second estimate the same ingredients are sufficient as well, in addition to the simple bound

$$\sup_{(s,t)\in\mathcal{C}_n} |a_\varepsilon(s,t) - a_n(t)| \leq C\varepsilon \sup_{(s,t)\in\mathcal{C}_n} |k(s) - k_n|\, |\log\varepsilon|^2 = \mathcal{O}(\varepsilon^2|\log\varepsilon|^2).$$

Inside any given cell $\mathcal{C}_n$ we can then decouple the functional in the usual way (see [CR, Lemma 5.2] for a statement in this context) to obtain

$$\mathcal{E}_n^{\mathrm{GL}}[\psi] = E_\star^{\mathrm{1D}}(k_n)\ell_\varepsilon + \mathcal{E}_n[u_n]. \tag{5.19}$$

The first term in (5.19) is a Riemann sum approximation of the leading order term in (5.1): using (4.30), we immediately get

$$\sum_{n=1}^{N_\varepsilon} E_\star^{\mathrm{1D}}(k_n)\ell_\varepsilon = \sum_{n=1}^{N_\varepsilon} E_\star^{\mathrm{1D}}(k_n)(s_{n+1}-s_n)$$
$$= \sum_{n=1}^{N_\varepsilon} \int_{s_n}^{s_{n+1}} \mathrm{d}s \left[E_\star^{\mathrm{1D}}(k(s)) + \mathcal{O}(\varepsilon^2|\log\varepsilon|^\infty)\right] = \int_0^{|\partial\Omega|} \mathrm{d}s\, E_\star^{\mathrm{1D}}(k(s)) + \mathcal{O}(\varepsilon^2|\log\varepsilon|^\infty), \tag{5.20}$$

which concludes the proof. □

5.2 Lower bounds to reduced functionals

In view of our previous reductions, the final lower bound (5.1) is a consequence of the following lemma

Lemma 5.4 (Lower bound on the reduced functionals).
With the previous notation, we have

$$\sum_{n=1}^{N_\varepsilon} \mathcal{E}_n[u_n] \geq |\log\varepsilon|^{-4} \sum_{n=1}^{N_\varepsilon} \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\ (1-\varepsilon k_n t)\, f_n^2 \left[|\partial_t u_n|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2}\, |\varepsilon\partial_s u_n|^2\right]$$
$$+ \frac{1}{2b\varepsilon} \sum_{n=1}^{N_\varepsilon} \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\ (1-\varepsilon k_n t)\, f_n^4 \left(1-|u_n|^2\right)^2 - C\varepsilon^2|\log\varepsilon|^\infty \tag{5.21}$$

Proposition 5.1 now follows by a combination of Lemmas 5.1, 5.3 and 5.4 because the two sums in the right-hand side of (5.21) are positive. These terms will prove useful to obtain our density and degree estimates in Section 6.

We can now focus on the proof of Lemma 5.4, which is the core argument of the proof of Proposition 5.1.

Proof of Lemma 5.4. The proof is split into two rather different steps. In the first one we essentially follow the strategy of [CR, Section 5.2] to control the main part of the only potentially negative term in (5.10). This is done locally inside each cell and uses mainly the positivity of the cost function, Lemma A.4. This strategy however involves an application of Stokes' formula and subsequent further integrations by parts to put the so obtained terms in such a form (involving only first order derivatives, see (5.26)) that they can be compared with the kinetic one. This produces unphysical surface terms located on the boundaries of the (rather artificial) cells we have introduced. The second step of the proof consists in controlling those, which requires to sum them all and reorganize the sum in a convenient manner. It is in this step only that we cease working locally inside each cell.

Step 1. Lower bound inside each cell. First, we split the integration over two regions, one where a suitable lower bound to the density f_n holds true and another one yielding only a very small contribution. More precisely we set

$$\mathcal{R}_n := \left\{(s,t) \in \mathcal{C}_n : f_n(t) \geq |\log\varepsilon|^3 f_n(t_\varepsilon)\right\}. \tag{5.22}$$

Note that the monotonicity for large t of f_n (see Proposition A.1) guarantees that

$$\mathcal{R}_n := [s_n, s_{n+1}] \times [\bar{t}_{n,\varepsilon}, t_\varepsilon], \qquad \bar{t}_{n,\varepsilon} = t_\varepsilon + \mathcal{O}(\log|\log\varepsilon|). \tag{5.23}$$

Now we use the potential function $F_n(t)$ defined as

$$F_n(t) := 2\int_0^t \mathrm{d}\eta\,(1-\varepsilon k_n\eta) f_n^2(\eta) b_n(\eta) = 2\int_0^t \mathrm{d}\eta\, f_n^2(\eta)\frac{\eta+\alpha_n-\frac{1}{2}\varepsilon k_n\eta^2}{1-\varepsilon k_n\eta}, \tag{5.24}$$

and compute

$$-2\varepsilon\int_{\mathcal{C}_n}\mathrm{d}s\mathrm{d}t\;(1-\varepsilon k_n t)\, f_n^2(t) b_k(t) J_s[u_n] = \varepsilon\int_{\mathcal{C}_n}\mathrm{d}s\mathrm{d}t\, F_n(t)\partial_t J_s[u_n], \tag{5.25}$$

where we have exploited the vanishing of F_n at $t=0$ and $t=t_\varepsilon$. Now we split the r.h.s. of the above expression into an integral over $\mathcal{D}_n := \mathcal{C}_n \setminus \mathcal{R}_n$ and a rest. In order to compare the first part with the kinetic energy and show that the sum is positive, we have to perform another integration by parts:

$$\varepsilon\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\, F_n(t)\partial_t J_s[u_n] = 2\varepsilon\int_0^{\bar{t}_{n,\varepsilon}}\mathrm{d}t\int_{s_n}^{s_{n+1}}\mathrm{d}s\, F_n(t)\,(i\partial_t u_n, \partial_s u_n) \\ + \varepsilon\int_0^{\bar{t}_{n,\varepsilon}}\mathrm{d}t F_n(t)\left[J_t[u_n](s_{n+1},t) - J_t[u_n](s_n,t)\right]. \tag{5.26}$$

The first term in (5.26) can be bounded by using some kinetic energy:

$$2\varepsilon\int_{\mathcal{D}_n}\mathrm{d}t\mathrm{d}s\, F_n(t)\,(i\partial_t u_n, \partial_s u_n) \geq -2\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\;|F_n(t)|\,|\partial_t u_n|\,|\varepsilon\partial_s u_n| \\ \geq -\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\,(1-\varepsilon k_n t) F_n(t)\left[|\partial_t u_n|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2}\,|\varepsilon\partial_s u_n|^2\right], \tag{5.27}$$

where we have used the inequality $ab \leq \frac{1}{2}(\delta a^2 + \delta^{-1}b^2)$ and the negativity of $F_n(t)$ (see Lemma A.3). Combining the above lower bound with (5.10) and (5.14) and dropping the part of the kinetic energy located in $\mathcal{R}_n$, we get

$$\mathcal{E}_n[u_n] \geq \int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\;(1-\varepsilon k_n t)\, K_n(t)\left[|\partial_t u_n|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2}\,|\varepsilon\partial_s u_n|^2\right] \\ + \varepsilon\int_0^{\bar{t}_{n,\varepsilon}}\mathrm{d}t F_n(t)\left[J_t[u_n](s_{n+1},t) - J_t[u_n](s_n,t)\right] + \varepsilon\int_{\mathcal{R}_n}\mathrm{d}s\mathrm{d}t\, F_n(t)\partial_t J_s[u_n] \\ + d_\varepsilon\int_{\mathcal{C}_n}\mathrm{d}s\mathrm{d}t\;(1-\varepsilon k_n t)\, f_n^2\left[|\partial_t u_n|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2}\,|\varepsilon\partial_s u_n|^2\right] \\ + \frac{1}{2b}\int_{\mathcal{C}_n}\mathrm{d}s\mathrm{d}t\;(1-\varepsilon k_n t)\, f_n^4\left(1-|u_n|^2\right)^2, \tag{5.28}$$

where

$$K_n(t) := K_{k_n}(t), \tag{5.29}$$

is the cost function defined in (A.10), for some given d_ε, satisfying (A.11). The third term in (5.28) is bounded from below by a quantity smaller than any power of ε, provided c_0 is chosen large enough. This is shown using the same strategy as in [CR, Eq. (5.21) and following discussion] and we skip the details for the sake of brevity. For the first term we use the positivity of K_n provided by Lemma A.4. We then conclude

$$\mathcal{E}_n[u_n] \geq \varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t F_n(t) \left[J_t[u_n](s_{n+1},t) - J_t[u_n](s_n,t) \right] \\ + d_\varepsilon \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t \; (1-\varepsilon k_n t) \, f_n^2 \left[|\partial_t u_n|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2} |\varepsilon \partial_s u_n|^2 \right] \\ + \frac{1}{2b} \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t \; (1-\varepsilon k_n t) \, f_n^4 \left(1 - |u_n|^2\right)^2 + \mathcal{O}(\varepsilon^\infty), \quad (5.30)$$

and there only remains to bound the first term on the r.h.s. from below. We are not actually able to bound the term coming from cell n separately, so in the next step we put back the sum over cells.

Step 2. Summing and controlling boundary terms. We now conclude the proof of (5.21) by proving the following inequality:

$$\varepsilon \sum_{n=1}^{N_\varepsilon} \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t F_n(t) \left[J_t[u_n](s_{n+1},t) - J_t[u_n](s_n,t) \right] \geq \\ - C|\log\varepsilon|^{-5} \sum_{n=1}^{N_\varepsilon} \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t \; (1-\varepsilon k_n t) \, f_n^2 \left[|\partial_t u_n|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2} |\varepsilon \partial_s u_n|^2 \right] - C\varepsilon^2 |\log\varepsilon|^\infty. \quad (5.31)$$

Grouping (5.30) and (5.31), choosing $d_\varepsilon = 2|\log\varepsilon|^{-4}$ (which we are free to do) concludes the proof.

We turn to our claim (5.31). Once we have put back the sum over all cells the idea is to associate the two terms evaluated on the same boundary, which come from two adjacent cells and therefore contain two different densities:

$$\varepsilon \sum_{n=1}^{N_\varepsilon} \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t F_n(t) \left[J_t[u_n](s_{n+1},t) - J_t[u_n](s_n,t) \right] \\ = \varepsilon \sum_{n=1}^{N_\varepsilon} \left[\int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \left[F_n(t) J_t[u_n](s_n,t) - F_{n+1}(t) J_t[u_{n+1}](s_n,t) \right] + R_n \right], \quad (5.32)$$

where, assuming without loss of generality that $\bar{t}_{n,\varepsilon} < \bar{t}_{n+1,\varepsilon}$,

$$R_n := - \int_{\bar{t}_{n,\varepsilon}}^{\bar{t}_{n+1,\varepsilon}} \mathrm{d}t \; F_{n+1}(t) J_t[u_{n+1}](s_{n+1},t), \quad (5.33)$$

If on the other hand $\bar{t}_{n,\varepsilon} > \bar{t}_{n+1,\varepsilon}$, in (5.32) $\bar{t}_{n,\varepsilon}$ should be replaced with $\bar{t}_{n+1,\varepsilon}$ and in place of R_n one would find

$$\int_{\bar{t}_{n+1,\varepsilon}}^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \; F_j(t) J_t[u_j](s_{n+1},t).$$

In other words the remainder R_n is inherited from the fact that the decomposition $\mathcal{C}_n = \mathcal{D}_n \cup \mathcal{R}_n$ clearly depends on n and the boundary terms in (5.32) do not compensate exactly. However it is clear from what follows that the estimate of such a boundary term is the same in both cases and essentially relies on the second inequality in (5.6): recalling that

$$f_{n+1}^2(t) + F_{n+1}(t) \geq 0$$

for any $t \leq \bar{t}_{n+1,\varepsilon}$, we have

$$|R_n| = \int_{\bar{t}_{n,\varepsilon}}^{\bar{t}_{n+1,\varepsilon}} \mathrm{d}t\, |F_{n+1}(t)|\, |J_t[u_{n+1}](s_{n+1},t)| \leq \int_{\bar{t}_{n,\varepsilon}}^{\bar{t}_{n+1,\varepsilon}} \mathrm{d}t f_{n+1}^2(t)\, |u_{n+1}(s_{n+1},t)|\, |\partial_t u_{n+1}(s_{n+1},t)|$$
$$\leq \int_{\bar{t}_{n,\varepsilon}}^{\bar{t}_{n+1,\varepsilon}} \mathrm{d}t\ |\psi(s_{n+1},t)| \left[|\partial_t \psi(s_{n+1},t)| + |u_{n+1}(s_{n+1},t)|\, |\partial_t f_{n+1}(t)| \right]$$
$$\leq C |\log\varepsilon|^3 \int_{\bar{t}_{n,\varepsilon}}^{\bar{t}_{n+1,\varepsilon}} \mathrm{d}t\ |\psi(s_{n+1},t)| = \mathcal{O}(\varepsilon^\infty), \quad (5.34)$$

where we have used the bounds (5.5) and (A.7), i.e., $|f_{n+1}'| \leq |\log\varepsilon|^3 f_{n+1}(t)$. The identity (5.32) hence yields

$$\varepsilon \sum_{n=1}^{N_\varepsilon} \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t F_n(t) \left[J_t[u_n](s_{n+1},t) - J_t[u_n](s_n,t) \right]$$
$$= \varepsilon \sum_{n=1}^{N_\varepsilon} \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \left[F_n(t) J_t[u_n](s_n,t) - F_{n+1}(t) J_t[u_{n+1}](s_n,t) \right] + \mathcal{O}(\varepsilon^\infty). \quad (5.35)$$

Using now the definitions (5.9) of u_n and u_{n+1}, we get

$$u_{n+1}(s,t) = \frac{f_n(t)}{f_{n+1}(t)} e^{i(\alpha_{n+1}-\alpha_n)s} u_n(s,t), \quad (5.36)$$

so that

$$J_t[u_{n+1}](s_n,t) = i G_{n,n+1}(t) G_{n,n+1}'(t)\, |u_n(s_n,t)|^2 + G_{n,n+1}^2(t) J_t[u_n](s_n,t), \quad (5.37)$$

where we have set

$$G_{n,n+1}(t) := \frac{f_n(t)}{f_{n+1}(t)}. \quad (5.38)$$

Then we can compute

$$\varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \left[F_n J_t[u_n](s_n,t) - F_{n+1} J_t[u_{n+1}](s_n,t) \right] = \varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \left[F_n(t) - F_{n+1}(t) G_{n,n+1}^2(t) \right] J_t[u_n](s_n,t)$$
$$- \frac{i\varepsilon}{2} \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t\ F_{n+1}(t) \partial_t \left(G_{n,n+1}^2(t) \right) |u_n(s_n,t)|^2,$$

but we know that the l.h.s. of the above expression is real, so that we can take the real part of the identity above obtaining

$$\varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \left[F_n J_t[u_n](s_n,t) - F_{n+1} J_t[u_{n+1}](s_n,t) \right] = \varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t\ \left[F_n - F_{n+1} G_{n,n+1}^2 \right] J_t[u_n](s_n,t). \quad (5.39)$$

To estimate the r.h.s. we integrate by parts back by introducing a suitable cut-off function. Let, for any given $n = 1, \ldots, N_\varepsilon$, $\chi_n(s)$ be a suitable smooth function, such that

$$\chi_n(s_n) = 1, \qquad \chi\left(\tfrac{1}{2}(s_n + s_{n+1})\right) = 0$$

and

$$\left[s_n, \tfrac{1}{2}\left(s_n + s_{n+1}\right) \right) \subset \mathrm{supp}(\chi_n), \qquad |\partial_s \chi_n| \leq C. \quad (5.40)$$

We can rewrite

$$\varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \, \left[F_n - F_{n+1} G_{n,n+1}^2 \right] J_t[u_n](s_n,t) = \varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \, \chi_n(s_n) \left[F_n - F_{n+1} G_{n,n+1}^2 \right] J_t[u_n](s_n,t)$$
$$= \varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \Big\{ \chi_n(s) I_{n,n+1}(t) \partial_s \left(J_t[u_n] \right) + \partial_s \left(\chi_n(s) \right) I_{n,n+1}(t) J_t[u_n] \Big\}, \quad (5.41)$$

where we have set for short (compare with (3.39))

$$I_{n,n+1}(t) := F_n(t) - F_{n+1}(t) G_{n,n+1}^2(t) = F_n(t) - F_{n+1}(t) \frac{f_n^2(t)}{f_{n+1}^2(t)}. \quad (5.42)$$

The first contribution to (5.41) can be cast in a form analogous to (5.27):

$$\varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, \chi_n(s) I_{n,n+1}(t) \partial_s \left(J_t[u_n] \right)$$
$$= \varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, \chi_n(s) \left\{ 2 I_{n,n+1}(t) \left(i \partial_s u_n, \partial_t u_n \right) - \partial_t \left(I_{n,n+1}(t) \right) J_s[u_n] \right\}$$
$$+ \varepsilon \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, \chi_n(s) I_{n,n+1}(\bar{t}_{n,\varepsilon}) J_s[u_n](s, \bar{t}_{n,\varepsilon}). \quad (5.43)$$

The first term on the r.h.s. can be handled as we did for (5.27):

$$2\varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, \chi_n(s) I_{n,n+1}(t) \left(i \partial_s u_n, \partial_t u_n \right) \geq -2 \int_{\mathcal{D}_n} \mathrm{d}s \mathrm{d}t \, \left| I_{n,n+1}(t) \right| \left| \varepsilon \partial_s u_n \right| \left| \partial_t u_n \right|$$
$$\geq -C \varepsilon |\log \varepsilon|^{\infty} \int_{\mathcal{D}_n} \mathrm{d}s \mathrm{d}t \, (1 - \varepsilon k_n t) f_n^2 \left[\left| \partial_t u_n \right|^2 + \tfrac{1}{(1 - \varepsilon k_n t)^2} \left| \varepsilon \partial_s u_n \right|^2 \right], \quad (5.44)$$

where we have used (3.42) with $k = k_n$, $k' = k_{n+1}$ and recalled that $|k_n - k_{n+1}| \leq C\varepsilon$ to bound $I_{n,n+1}$. The last term in (5.43) can be easily shown to provide a small correction: using (3.42) again yields

$$\left| I_{n,n+1}(\bar{t}_{n,\varepsilon}) \right| \leq C \varepsilon |\log \varepsilon|^{\infty} f_n^2(\bar{t}_{n,\varepsilon}),$$

so that by (5.6) and (5.23)

$$\left| \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, \chi_n(s) I_{n,n+1}(\bar{t}_{n,\varepsilon}) J_s[u_n](s, \bar{t}_{n,\varepsilon}) \right| \leq \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, \left| I_{n,n+1}(\bar{t}_{n,\varepsilon}) \right| \left| J_s[u_n] \right|$$
$$\leq C \varepsilon |\log \varepsilon|^{\infty} \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, f_n^2(\bar{t}_{n,\varepsilon}) \left| u_n(s, \bar{t}_{n,\varepsilon}) \right| \left| \partial_s u_n(s, \bar{t}_{n,\varepsilon}) \right|$$
$$\leq C |\log \varepsilon|^{\infty} \left\| \varepsilon \partial_s \psi \right\|_{\infty} \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, \left| \psi(s, \bar{t}_{n,\varepsilon}) \right| = \mathcal{O}(\varepsilon^{\infty}), \quad (5.45)$$

where we have estimated the s-derivative of ψ by means of (5.5). Hence, combining (5.43) with (5.44) and (5.46), we can bound from below (5.41) as

$$\varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \, \left[F_n - F_{n+1} G_{n,n+1}^2 \right] J_t[u_n](s_n,t)$$
$$\geq \varepsilon \int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s \, \left\{ -\partial_t I_{n,n+1} J_s[u_n] + \partial_s \chi_n I_{n,n+1} J_t[u_n] \right\}$$
$$- C \varepsilon |\log \varepsilon|^{\infty} \int_{\mathcal{D}_n} \mathrm{d}s \mathrm{d}t \, (1 - \varepsilon k_n t) f_n^2 \left[\left| \partial_t u_n \right|^2 + \tfrac{1}{(1 - \varepsilon k_n t)^2} \left| \varepsilon \partial_s u_n \right|^2 \right] + \mathcal{O}(\varepsilon^{\infty}). \quad (5.46)$$

To complete the proof it only remains to estimate the first two terms on the r.h.s. of the expression above, which again requires to borrow a bit of the kinetic energy. Using (3.29) we have

$$\sup_{t\in[0,\bar{t}_{n,\varepsilon}]}\left|\frac{\partial_t I_{n,n+1}}{f_n^2}\right| \leq C\varepsilon|\log\varepsilon|^{\infty},$$

so that

$$\begin{aligned}
\varepsilon\left|\int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s\, \partial_t I_{n,n+1} J_s[u_n]\right| &\leq C\varepsilon|\log\varepsilon|^{\infty}\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\, f_n^2\,|u_n|\,|\varepsilon\partial_s u_n| \\
&\leq C\varepsilon|\log\varepsilon|^{\infty}\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\,\left[\tfrac{1}{\delta}\tfrac{1}{1-\varepsilon k_n t} f_n^2\,|\varepsilon\partial_s u_n|^2+\delta|\psi|^2\right] \\
&\leq C|\log\varepsilon|^{-5}\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\,\tfrac{1}{1-\varepsilon k_n t} f_n^2\,|\varepsilon\partial_s u_n|^2+\mathcal{O}(\varepsilon^3|\log\varepsilon|^{\infty}), \qquad (5.47)
\end{aligned}$$

where we have chosen $\delta = \varepsilon|\log\varepsilon|^a$, for some suitably large $a > 0$ to compensate the $|\log\varepsilon|$ prefactor (this generates the coefficient $|\log\varepsilon|^{-5}$), and used (5.5) to estimate the remaining term. For the second term on the r.h.s. of (5.46) we proceed in the same way, using first (3.42) and the assumption $|\partial_s\chi| \leq C$, to get

$$\begin{aligned}
\varepsilon\left|\int_0^{\bar{t}_{n,\varepsilon}} \mathrm{d}t \int_{s_n}^{\frac{1}{2}(s_n+s_{n+1})} \mathrm{d}s\, \partial_s\chi_n I_{n,n+1} J_t[u_n]\right| &\leq C\varepsilon^2|\log\varepsilon|^{\infty}\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\, f_n^2\,|u_n|\,|\partial_t u_n| \\
&\leq C\varepsilon^2|\log\varepsilon|^{\infty}\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\,\left[(1-\varepsilon k_n t)\, f_n^2\,|\partial_t u_n|^2+|\psi|^2\right] \\
&\leq C\varepsilon^2|\log\varepsilon|^{\infty}\int_{\mathcal{D}_n}\mathrm{d}s\mathrm{d}t\,\,(1-\varepsilon k_n t)\, f_n^2\,|\partial_t u_n|^2+\mathcal{O}(\varepsilon^3|\log\varepsilon|^{\infty}). \qquad (5.48)
\end{aligned}$$

Collecting all the previous estimates yields our claim (5.31) (recall that there are $N_\varepsilon \propto \varepsilon^{-1}$ terms to be summed, whence the final error of order $\varepsilon^2|\log\varepsilon|^{\infty}$). □

6 Density and Degree Estimates

In this section we prove the main results about the behavior of $|\Psi^{\mathrm{GL}}|$ close to the boundary of the sample $\partial\Omega$ and an estimate of its degree at $\partial\Omega$.

We first notice that the L^2 estimate stated in (2.6) is in fact a trivial consequence of the energy asymptotics (2.5): putting together the lower bounds (5.2), (5.13) and (5.21) with the upper bound (4.1), we obtain

$$\frac{1}{2\varepsilon b}\sum_{n=1}^{N_\varepsilon}\int_{\mathcal{C}_n}\mathrm{d}s\mathrm{d}t\,\,(1-\varepsilon k_n t)\, f_n^4\left(1-|u_n|^2\right)^2 \leq C\varepsilon|\log\varepsilon|^{\gamma}, \qquad (6.1)$$

for some power γ large enough (recall the meaning of the notation $|\log\varepsilon|^{\infty}$). Now, using the fact that $k_n = k(s)\,(1+\mathcal{O}(\varepsilon))$ inside $\mathcal{C}_n$, we can easily reconstruct (2.6), once everything has been expressed in the original unscaled variables and the definitions (2.4) and (5.9) has been exploited (recall also that $\psi(s,t) = \Psi^{\mathrm{GL}}(\mathbf{r}(s,\varepsilon t))$. See also [CR, Section 4.2] for further details.

We now focus on the refined density estimate discussed in Theorem 2.2 and the proof of Pan's conjecture. The result is obtained via an adaptation of the arguments used in [CR, Section 5.3], originating in [BBH1]. The general idea is now rather standard so we will mainly comment on the changes needed to make those argument work in the present setting.

Proof of Theorem 2.2. The two main ingredients of the proof are the above estimate (6.1) and a pointwise bound on the gradient of u_n. Once combined, the two estimates imply that the function $|u_n|$ cannot be too far from 1 anywhere in the boundary layer $\mathcal{A}_{\mathrm{bl}}$ (see (2.7) for its precise definition).

In the present setting one has first to show that $|u_n|$ is almost continuous from one cell to the adjacent one. This is a straightforward consequence of the pointwise estimate (3.15) and the continuity of $\psi(s,t)$: from the definition of u_n in (5.9), one has, for any $n = 1, \ldots, N_\varepsilon$,

$$|u_n(s_{n+1},t)| = \frac{f_n(t)}{f_{n+1}(t)}\,|u_{n+1}(s_{n+1})| = \left(1 + \mathcal{O}\left(\varepsilon\gamma_\varepsilon^{-1}|\log\varepsilon|^\infty\right)\right)|u_{n+1}(s_{n+1})| \\ = (1+o(1))\,|u_{n+1}(s_{n+1})|\,, \quad (6.2)$$

since

$$\varepsilon\gamma_\varepsilon^{-1}|\log\varepsilon|^\infty \ll \varepsilon^{5/6}|\log\varepsilon|^\infty \ll 1,$$

thanks to the assumption (2.8).

Another minor difference with the setting in [CR, Section 5.3] is due to the convention we used to avoid a scaling of the tangential coordinate s. This is just a matter of notation and by following exactly [CR, Proof of Lemma 5.3], it is easy to show that, for any $n = 1, \ldots, N_\varepsilon$,

$$|\partial_t|u_n|| \le C f_n^{-1}(t)|\log\varepsilon|^3, \qquad |\partial_s|u_n|| \le C f_n^{-1}(t)\varepsilon^{-1}. \quad (6.3)$$

Notice the second estimate above, which is a consequence of not scaling the coordinate s.

Now we can apply the standard argument and show that $|u_n|$ can not differ too much from 1. The proof is done by contradiction: suppose that there exists a point (s_0,t_0) in $\mathcal{C}_n \cap \mathcal{A}_{\mathrm{bl}}$ such that

$$|1 - |u_n(s_0,t_0)|| \ge \sigma_\varepsilon,$$

for

$$\sigma_\varepsilon := \varepsilon^{1/4}\gamma_\varepsilon^{-3/2}|\log\varepsilon|^c \ll |\log\varepsilon|^{c-3a/2} \ll 1, \quad (6.4)$$

if c is chosen in such a way that $c < \frac{3}{2}a$. Then by (6.3) we can construct a rectangle-like region $R_\varepsilon \subset \mathcal{A}_{\mathrm{bl}}$ of tangential length $\varepsilon\gamma_\varepsilon\sigma_\varepsilon$ and longitudinal length ϱ_ε with

$$\varrho_\varepsilon := \gamma_\varepsilon\sigma_\varepsilon|\log\varepsilon|^{-3}, \quad (6.5)$$

which intersects several cells $\mathcal{C}_n$ but inside each one of them

$$|1 - |u_n(s,t)|| \ge \tfrac{1}{2}\sigma_\varepsilon.$$

Note that under the assumptions on γ_ε and σ_ε, we have

$$\varrho_\varepsilon = \varepsilon^{1/4}\gamma_\varepsilon^{-1/2}|\log\varepsilon|^{c-a-3} \ll \varepsilon^{1/6}|\log\varepsilon|^\infty \ll |\log\varepsilon|^{1/2},$$

which is the order of the width of $\mathcal{A}_{\mathrm{bl}}$ (see (2.7)).

To complete the proof it suffices to estimate from below

$$\frac{1}{\varepsilon}\sum_{n=1}^{N_\varepsilon}\int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\;(1-\varepsilon k_n t)\,f_n^4\left(1-|u_n|^2\right)^2 \ge \frac{1}{\varepsilon}\sum_{n=1}^{N_\varepsilon}\int_{R_\varepsilon\cap\mathcal{C}_n} \mathrm{d}s\mathrm{d}t\;(1-\varepsilon k_n t)\,f_n^4\left(1-|u_n|^2\right)^2 \\ \ge \gamma_\varepsilon^5\sigma_\varepsilon^3\varrho_\varepsilon = \gamma_\varepsilon^6\sigma_\varepsilon^4|\log\varepsilon|^{-3} = \varepsilon|\log\varepsilon|^{4c-3} \gg \varepsilon|\log\varepsilon|^\gamma, \quad (6.6)$$

where γ is the power of $|\log\varepsilon|$ appearing in the r.h.s. of (6.1) and we have chosen c so that $c \ge \frac{1}{4}(\gamma+3)$. Recalling the condition $a > \frac{2}{3}c$ this also implies $a > \frac{1}{6}(\gamma+3)$. Under such conditions the estimate above contradicts the upper bound (6.1) and the result is proven.

Once one knows that $|u_n| \ge 1 - \sigma_\varepsilon = 1 + o(1)$, then it is easy to translate this estimate in an analogous one for $|\psi(s,t)|$ and therefore $|\Psi^{\mathrm{GL}}|$. Such an estimate would then involve the reference profile g_{ref} but the bound $\|f_n - f_0\|_\infty = \mathcal{O}(\varepsilon)$ allows the replacement of g_{ref} with f_0. □

We can now turn to the proof of the estimate of the winding number of Ψ^{GL} along $\partial\Omega$.

Proof of Theorem 2.3. Thanks to the positivity of g_{ref} at $t=0$ (see Lemma A.2) and the result discussed above, Ψ^{GL} never vanishes on $\partial\Omega$ and therefore its winding number is well defined. The rest of the proof follows the lines of [CR, Proof of Theorem 2.4]. A minor modification is due to the cell decomposition and the use of a different decoupling in each cell: the analogue of [CR, Lemma 5.4] is the following

$$\sum_{n=1}^{N_\varepsilon} \int_{s_n}^{s_{n+1}} \mathrm{d}s \; J_s[u_n](s,0) = \mathcal{O}(|\log\varepsilon|^\infty). \tag{6.7}$$

To see that, we introduce a tangential cut-off function $\chi(t)$ with support contained in $[0,|\log\varepsilon|^{-1}]$ and such that $0\le\chi\le 1$, $\chi(0)=1$ and $|\partial_t\chi| \; = \mathcal{O}(|\log\varepsilon|)$. Then we compute

$$\int_{s_n}^{s_{n+1}} \mathrm{d}s \; J_s[u_n](s,0) = \int_{s_n}^{s_{n+1}} \mathrm{d}s \int_0^{\frac{1}{|\log\varepsilon|}} \mathrm{d}t \; \left[\partial_t\chi J_s[u_n](s,t) + \chi\partial_t J_s[u_n](s,t)\right] =$$
$$\int_0^{\frac{1}{|\log\varepsilon|}} \mathrm{d}t \left\{ \int_{s_n}^{s_{n+1}} \mathrm{d}s \; \left[\partial_t\chi J_s[u_n](s,t) + 2\chi\left(i\partial_t u_n, \partial_s u_n\right)\right] + J_t[u_n](s_{n+1},t) - J_t[u_n](s_n,t) \right\} \tag{6.8}$$

and after a rearrangement of the boundary terms

$$\sum_{n=1}^{N_\varepsilon} \int_{s_n}^{s_{n+1}} \mathrm{d}s \; J_s[u_n](s,0) = \sum_{n=1}^{N_\varepsilon} \int_0^{|\log\varepsilon|^{-1}} \mathrm{d}t \int_{s_n}^{s_{n+1}} \mathrm{d}s \; \left[\left(\partial_t\chi\right) J_s[u_n](s,t) + 2\chi\left(i\partial_t u_n, \partial_s u_n\right)\right]$$
$$- \sum_{n=1}^{N_\varepsilon} \int_0^{|\log\varepsilon|^{-1}} \mathrm{d}t \; \left[J_t[u_{n+1}](s_{n+1},t) - J_t[u_n](s_{n+1},t)\right]. \tag{6.9}$$

The three terms on the r.h.s. of the above expression are going to be bounded independently. We first observe that, exactly like we derived (6.1), one can also extract from the comparison between the energy upper and lower bounds (see (5.21)) the following estimate:

$$\sum_{n=1}^{N_\varepsilon} \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t \; \left(1-\varepsilon k_n t\right) f_n^2 \left\{ \left|\partial_t u_n\right|^2 + \tfrac{1}{(1-\varepsilon k_n t)^2} \left|\varepsilon\partial_s u_n\right|^2 \right\} \le C\varepsilon^2 |\log\varepsilon|^\infty. \tag{6.10}$$

Then we can estimate the absolute value of the first two terms on the r.h.s. of (6.9) by using the Cauchy-Schwarz inequality

$$\sum_{n=1}^{N_\varepsilon} \int_0^{|\log\varepsilon|^{-1}} \mathrm{d}t \int_{s_n}^{s_{n+1}} \mathrm{d}s \; \left[C|\log\varepsilon||u_n|\left|\partial_s u_n\right| + 2\left|\partial_t u_n\right|\left|\partial_s u_n\right|\right]$$
$$\le C \sum_{n=1}^{N_\varepsilon} \int_{\mathcal{C}_n} \mathrm{d}s\mathrm{d}t \, (1-\varepsilon k_n t) f_n^2 \left[|\log\varepsilon| \, |u_n|^2 + \tfrac{2}{(1-\varepsilon k_n t)^2} \left|\partial_s u_n\right|^2 + \left|\partial_t u_n\right|^2 \right], \tag{6.11}$$

where we have exploited the pointwise lower bound (A.6), which implies $f_n(t) \ge C > 0$ for any $t \in [0, |\log\varepsilon|^{-1}]$ and $n = 1, \ldots, N_\varepsilon$, to put back the density f_n^2 in the expression. Now the bound

$$f_n|u_n| = |\psi| \le 1$$

together with (6.10) yield

$$\sum_{n=1}^{N_\varepsilon} \left| \int_0^{|\log\varepsilon|^{-1}} \mathrm{d}t \int_{s_n}^{s_{n+1}} \mathrm{d}s \; \left[\left(\partial_t\chi\right) J_s[u_n](s,t) + 2\chi\left(i\partial_t u_n, \partial_s u_n\right)\right] \right| \le C|\log\varepsilon|^\infty. \tag{6.12}$$

On the other hand the definition (5.9) of u_n implies that

$$\left| \int_0^{|\log\varepsilon|^{-1}} \mathrm{d}t\; J_t[u_{n+1}](s_{n+1},t) - J_t[u_n](s_{n+1},t) \right| = \left| \int_0^{|\log\varepsilon|^{-1}} \mathrm{d}t \left(\frac{1}{f_{n+1}^2} - \frac{1}{f_n^2} \right) J_t[|\psi|](s_{n+1},t) \right|$$
$$\leq C\varepsilon |\log\varepsilon|^{\infty} \int_0^{|\log\varepsilon|^{-1}} \mathrm{d}t\; |\psi||\partial_s|\psi|| \leq C|\log\varepsilon|^{\infty}, \quad (6.13)$$

thanks to (3.15), the already mentioned lower bound on f_n in $[0, |\log\varepsilon|^{-1}]$ and the standard bound $\|\nabla\psi\|_\infty \leq C\varepsilon^{-1}$ (see, e.g., [FH3, Eq. (11.50)]).

Hence (6.7) is proven and the rest of the proof is just a repetition of the estimates in [CR, Proof of Theorem 2.4]. Note that, as already anticipated in the comments after Theorem 2.3 $\alpha_n = \alpha_0(1+\mathcal{O}(\varepsilon))$, so that the optimal phases α_n can all be replaced with α_0. □

A Useful Estimates on 1D Functionals

Here we recall some preliminary results obtained in [CR]. We start in Subsection A.1 with elementary properties of the minimizing 1D profiles and carry on in Subsection A.2 by recalling the crucial positivity property of the cost function we mentioned in Subsection 2.2.

A.1 Properties of optimal phases and densities

This subsection contains a summary of results on the 1D minimization problem that follow from relatively standard methods. We start with the well-posedness of the minimization problem at fixed α. The following is [CR, Proposition 3.1].

Proposition A.1 (**Optimal density** $f_{k,\alpha}$)**.**
For any given $\alpha \in \mathbb{R}$, $k \geq 0$ and ε small enough, there exists a minimizer $f_{k,\alpha}$ to $\mathcal{E}^{\mathrm{1D}}_{k,\alpha}$, unique up to sign, which we choose to be non-negative. It solves the variational equation

$$-f''_{k,\alpha} + \tfrac{\varepsilon k}{1-\varepsilon k t} f'_{k,\alpha} + V_{k,\alpha}(t) f_{k,\alpha} = \tfrac{1}{b}\left(1 - f_{k,\alpha}^2\right) f_{k,\alpha} \quad (A.1)$$

with boundary conditions $f'_{k,\alpha}(0) = f'_{k,\alpha}(c_0|\log\varepsilon|) = 0$. Moreover $f_{k,\alpha}$ satisfies the estimate

$$\|f_{k,\alpha}\|_{L^\infty(I_\varepsilon)} \leq 1 \quad (A.2)$$

and it is monotonically decreasing for $t \geq \max\left[0, -\alpha + \frac{1}{\sqrt{b}} - C\varepsilon\right]$. In addition $E^{\mathrm{1D}}_{k,\alpha}$ is a smooth function of $\alpha \in \mathbb{R}$ and

$$E^{\mathrm{1D}}_{k,\alpha} = -\frac{1}{2b}\int_{I_\varepsilon} \mathrm{d}t\,(1-\varepsilon k t) f_{k,\alpha}^4(t). \quad (A.3)$$

Next we consider the minimization problem as a function of the phase α, dealt with in [CR, Lemma 3.1]. Here Θ_0^{-1} is defined as in (1.5).

Lemma A.1 (**Optimal phase** $\alpha(k)$)**.**
For any $1 < b < \Theta_0^{-1}$, $k \geq 0$ and ε small enough, there exists at least one $\alpha(k)$ minimizing $E^{\mathrm{1D}}_{k,\alpha}$:

$$\inf_{\alpha\in\mathbb{R}} E^{\mathrm{1D}}_{k,\alpha} = E^{\mathrm{1D}}_{k,\alpha(k)} =: E^{\mathrm{1D}}_{\star}(k). \quad (A.4)$$

Setting $f_k := f_{k,\alpha(k)}$ we have that $f_k > 0$ everywhere and

$$\int_{I_\varepsilon} \mathrm{d}t\, \frac{t + \alpha(k) - \frac{1}{2}\varepsilon k t^2}{1 - \varepsilon k t} f_k^2(t) = 0. \quad (A.5)$$

We also use some decay and gradient estimates for the minimizing density. The following is a combination of [CR, Proposition 3.3 and Lemma A.1]

Lemma A.2 (**Useful bounds on** $f_{k,\alpha}$).
For any $1 < b < \Theta_0^{-1}$, $k \in \mathbb{R}$ *and* ε *sufficiently small, there exist two positive constants* $c, C > 0$ *independent of* ε *such that*

$$c \exp\left\{ -\tfrac{1}{2}\big(t+\sqrt{2}\big)^2 \right\} \leq f_k(t) \leq C \exp\left\{ -\tfrac{1}{2}\left(t+\alpha\right)^2 \right\}, \tag{A.6}$$

for any $t \in I_\varepsilon$.

Moreover there exists a finite constant C *such that*

$$|f_k'(t)| \leq C \begin{cases} 1, & \text{for } t \in \left[0, |\alpha| + \frac{2}{\sqrt{b}}\right], \\ |\log\varepsilon|^3 f_k(t), & \text{for } t \in \left[|\alpha| + \frac{2}{\sqrt{b}}, c_0|\log\varepsilon|\right]. \end{cases} \tag{A.7}$$

A.2 Positivity of the cost function

A less standard part of our analysis in [CR] is the introduction of a cost function K_k whose positivity is one of the crucial ingredients of the energy lower bounds in the present paper.

Let us first recall the definition of the potential function associated with f_k:

$$F_k(t) := 2\int_0^t \mathrm{d}\eta\, (1-\varepsilon k\eta) f_k^2(\eta) \frac{\eta + \alpha(k) - \frac{1}{2}\varepsilon k \eta^2}{(1-\varepsilon k \eta)^2}, \tag{A.8}$$

which has the following properties [CR, Lemma 3.2]:

Lemma A.3 (**Properties of the potential function** F_k).
For any $1 < b < \Theta_0^{-1}$, $k \in \mathbb{R}$ *and* ε *sufficiently small, we have*

$$F_k(t) \leq 0, \quad \text{in } I_\varepsilon, \qquad F_k(0) = F_k(t_\varepsilon) = 0. \tag{A.9}$$

The cost function that naturally enters the analysis is then

$$K_k(t) = (1-d_\varepsilon) f_k^2(t) + F_k(t) \tag{A.10}$$

where d_ε is any parameter satisfying

$$0 < d_\varepsilon \leq C|\log\varepsilon|^{-4}, \quad \text{as } \varepsilon \to 0. \tag{A.11}$$

The positivity property we exploit is proved in [CR, Proposition 3.5]. Let

$$\bar{I}_{k,\varepsilon} := \left\{ t \in I_\varepsilon \ : f_k\left(t\right) \geq |\log\varepsilon|^3 f_k(t_\varepsilon) \right\}, \tag{A.12}$$

which is an interval in the t variable, i.e.,

$$\bar{I}_{k,\varepsilon} = [0, \bar{t}_{k,\varepsilon}], \tag{A.13}$$

with

$$\bar{t}_{k,\varepsilon} \geq t_\varepsilon - C\log|\log\varepsilon| = c_0|\log\varepsilon|\left(1 - \mathcal{O}\left(\tfrac{\log|\log\varepsilon|}{|\log\varepsilon|}\right)\right). \tag{A.14}$$

We then have

Lemma A.4 (**Positivity of the cost function**).
For any $d_\varepsilon \in \mathbb{R}^+$ *satisfying* (A.11), $1 < b < \Theta_0^{-1}$, $k > 0$ *and* ε *sufficiently small, we have*

$$K_k(t) \geq 0, \qquad \text{for any } t \in \bar{I}_{k,\varepsilon}. \tag{A.15}$$

References

[Abr] A. Abrikosov, On the magnetic properties of superconductors of the second type, *Soviet Phys. JETP* **5**, 1174–1182 (1957).

[AS] A. Aftalion, S. Serfaty, Lowest Landau level approach in superconductivity for the Abrikosov lattice close to H_c2, *Selecta Math.* **2**, 13, (2007).

[Alm1] Y. Almog, Nonlinear Surface Superconductivity in the Large κ Limit, *Rev. Math. Phys.* **16**, 961–976 (2004).

[Alm2] Y. Almog, Abrikosov lattices in finite domains, *Commun. Math. Phys.* **262**, 677-702 (2006).

[AH] Y. Almog, B. Helffer, The Distribution of Surface Superconductivity along the Boundary: on a Conjecture of X.B. Pan, *SIAM J. Math. Anal.* **38**, 1715–1732 (2007).

[BCS] J. Bardeen, L. Cooper, J. Schrieffer, Theory of Superconductivity, *Phys. Rev.* **108**, 1175–1204 (1957).

[BBH1] F. Béthuel, H. Brézis, F. Hélein, Asymptotics for the Minimization of a Ginzburg-Landau Functional, *Calc. Var. Partial Differential Equations* **1**, 123–148 (1993).

[BBH2] F. Béthuel, H. Brézis, F. Hélein, *Ginzburg-Landau Vortices*, Progress in Nonlinear Differential Equations and their Applications **13**, Birkhäuser, Basel, 1994.

[CPRY1] M. Correggi, F. Pinsker, N. Rougerie, J. Yngvason, Rotating Superfluids in Anharmonic Traps: From Vortex Lattices to Giant Vortices, *Phys. Rev. A* **84**, 053614 (2011).

[CPRY2] M. Correggi, F. Pinsker, N. Rougerie, J. Yngvason, Critical Rotational Speeds for Superfluids in Homogeneous Traps, *J. Math. Phys.* **53**, 095203 (2012).

[CPRY3] M. Correggi, F. Pinsker, N. Rougerie, J. Yngvason, Giant vortex phase transition in rapidly rotating trapped Bose-Einstein condensates, *Eur. J. Phys. Special Topics* **217**, 183–188 (2013).

[CPRY4] M. Correggi, F. Pinsker, N. Rougerie, J. Yngvason, Vortex Phases of Rotating Superfluids, Proceedings of the 21st International Laser Physics Workshop, Calgary, July 23-27, (2012).

[CR] M. Correggi, N. Rougerie, On the Ginzburg-Landau Functional in the Surface Superconductivity Regime, *Commun. Math. Phys.* to appear.

[CRY] M. Correggi, N. Rougerie, J. Yngvason, The Transition to a Giant Vortex Phase in a Fast Rotating Bose-Einstein Condensate, *Commun. Math. Phys.* **303**, 451–508 (2011).

[FH1] S. Fournais, B. Helffer, Energy asymptotics for type II superconductors, *Calc. Var. Partial Differential Equations* **24**, 341–376 (2005).

[FH2] S. Fournais, B. Helffer, On the third critical field in Ginzburg-Landau theory, *Commun. Math. Phys.* **266**, 153–196 (2006).

[FH3] S. Fournais, B. Helffer, *Spectral Methods in Surface Superconductivity*, Progress in Nonlinear Differential Equations and their Applications **77**, Birkhäuser, Basel, 2010.

[FHP] S. Fournais, B. Helffer, M. Persson, Superconductivity between H_{c_2} and H_{c_3}, *J. Spectr. Theory* **1**, 273–298 (2011).

[FK] S. Fournais, A. Kachmar, Nucleation of bulk superconductivity close to critical magnetic field, *Adv. Math.* **226**, 1213–1258 (2011).

[FHSS] R.L. Frank, C. Hainzl, R. Seiringer, J.P. Solovej, Microscopic Derivation of Ginzburg-Landau Theory, *J. Amer. Math. Soc.* **25**, 667–713 (2012).

[GL] V.L. Ginzburg, L.D. Landau, On the theory of superconductivity, *Zh. Eksp. Teor. Fiz.* **20**, 1064–1082 (1950).

[Gor] L.P. Gor'kov, Microscopic derivation of the Ginzburg-Landau equations in the theory of superconductivity, *Zh. Eksp. Teor. Fiz.* **36**, 1918–1923 (1959); english translation *Soviet Phys. JETP* **9**, 1364–1367 (1959).

[H *et al*] H. F. Hess, R. B. Robinson, R. C. Dynes, J. M. Valles Jr., J. V. Waszczak, Scanning-Tunneling-Microscope Observation of the Abrikosov Flux Lattice and the Density of States near and inside a Fluxoid, *Phys. Rev. Lett.* **62**, 214 (1989).

[LM] L. Lassoued, P. Mironescu, Ginzburg-Landau Type Energy with Discontinuous Constraint, *J. Anal. Math.* **77** (1999), 1–26.

[Kac] A. Kachmar, The Ginzburg-Landau order parameter near the second critical field, preprint *arXiv:1308.4236* (2013).

[LP] K. Lu, X.B. Pan, Estimates of the upper critical field for the Ginzburg-Landau equations of superconductivity, *Physica D* **127**, 73–104 (1999).

[N *et al*] Y.X. Ning, C.L. Song, Z.L. Guan, X.C. Ma, X. Chen, J.F. Jia, Q.K. Xue, Observation of surface superconductivity and direct vortex imaging of a Pb thin island with a scanning tunneling microscope, *Europhys. Lett.* **85**, 27004 (2009).

[Pan] X. B. Pan, Surface Superconductivity in Applied Magnetic Fields above H_{c2}, *Commun. Math. Phys.* **228**, 327–370 (2002).

[SS1] E. Sandier, S. Serfaty, The decrease of bulk-superconductivity near the second critical field in the Ginzburg-Landau model, *SIAM J. Math. Anal.* **34**, 939-956 (2003).

[SS2] E. Sandier, S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Progress in Nonlinear Differential Equations and their Applications **70**, Birkhäuser, Basel, 2007.

[Sig] I.M. Sigal, Magnetic Vortices, Abrikosov Lattices and Automorphic Functions, preprint *arXiv:1308.5446* (2013).

[SJdG] D. Saint-James, P.G. de Gennes, Onset of superconductivity in decreasing fields, *Phys. Lett.* **7**, 306–308 (1963).