Estimation of multivariate critical layers: Applications to rainfall data

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Abstract

Calculating return periods and critical layers (i.e. multivariate quantile curves) in a multivariate environment is a difficult problem. A possible consistent theoretical framework for the calculation of the return period, in a multi-dimensional environment, is essentially based on the notion of copula and level sets of the multivariate probability distribution. In this paper we propose a fast and parametric methodology to estimate the multivariate critical layers of a distribution and its associated return periods. The model is based on transformations of the marginal distributions and transformations of the dependence structure within the class of Archimedean copulas. The model has a tunable number of parameters, and we show that it is possible to get a competitive estimation without any global optimum research. We also get parametric expressions for the critical layers and return periods. The methodology is illustrated on rainfall 5-dimensional real data. On this real data-set we obtain a good quality of estimation and we compare the obtained results with some classical parametric competitors.

Keyword: Multivariate probability transformations , level sets estimation , copulas , hyperbolic conversion functions , risk assessment , multivariate return periods.

1 Introduction

1.1 Return Periods

The notion of *Return Period* (RP) is frequently used in environmental sciences for the identification of dangerous events, and provides a means for rational decision making and risk assessment. Roughly speaking, the RP can be considered as an analogue of the "Value-at-Risk" in Economics and Finance, since it is used to quantify and assess the risk (see, e.g., Nappo and Spizzichino (2009)). In engineering practice, finance, insurance and environmental science the choice of the RP depends on the impact/magnitude of the considered event and the consequences of its realisation.

Equally important is the related concept of *design quantile*, usually defined as "the value of the variable characterizing the event associated with a given RP". In the univariate case the design quantile is usually identified without ambiguity. Conversely in the multivariate setting different definitions are possible (see Serfling (2002)). For this reason, the identification problem of design events in a multivariate context has recently attracted the attention of many researches. The interested reader is referred for example to Embrechts and Puccetti (2006), Belzunce et al. (2007), Nappo and Spizzichino (2009), in the economics and finance context; to Chebana and Ouarda (2009), Chebana and Ouarda (2011) (and references therein) in the hydrological context.

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During the last years, researchers in environmental fields joined efforts to properly answer the following crucial question: "How is it possible to calculate the critical design event(s) in the multivariate case?" (see for instance Salvadori et al. (2007)). In this sense, a possible consistent theoretical framework for the calculation of the design event(s) and the associated return period(s) in a multi-dimensional environment, is proposed, e.g., by Salvadori et al. (2011), Salvadori et al. (2012), Gräler et al. (2013). In particular the authors define the multivariate return period using the notion of upper and lower level sets of multivariate probability distribution F and of the associated Kendall's measure.

In the following, we will consider a sequence $\mathbf{X} = \{\mathbf{X}_1, \mathbf{X}_2, \ldots\}$ of independent and identically distributed $d-$dimensional random vectors, with $d > 1$. Thus each $\mathbf{X}_k$, $k \in \mathbb{N}$, has the same multivariate distribution $F_{\mathbf{X}} : \mathbb{R}_+^d \to [0,1]$ as the nonnegative real-valued random vector $\mathbf{X} \sim F_{\mathbf{X}} = C(F_{X_1}, \ldots, F_{X_d})$ describing the hydrological phenomenon under investigation. The function C is the d-dimensional *copula* associated to F (see Nelsen (1999)). We write $I = \{1, \ldots, d\}$ the set of indexes of the considered random variables and of their associated cumulative distribution functions, i.e., $F_{X_i}(x_i) = P(X_i \leq x_i)$, for $i \in I$.

In the following we will consider multivariate distribution functions $F_{\mathbf{X}}$ satisfying these *regularity conditions*

- for all $u \in [0,1]$, the diagonal of the copula C, i.e. $C(u, \ldots, u)$, is a strictly increasing function of u;
- for any $i \in I$, the marginal F_{X_i} is continuous and strictly monotonic distribution function.

In this setting, we introduce the notion of *critical layer* (see, e.g., Salvadori et al. (2011), Salvadori et al. (2012), Gräler et al. (2013)).

Definition 1.1 (Critical layer)**.** *The critical layer $\partial L(\alpha)$ associated to the multivariate distribution function $F_{\boldsymbol{X}}$ of level $\alpha \in (0,1)$ is defined as*

$$\partial L(\alpha) = \{\boldsymbol{x} \in \mathbb{R}^d : F_{\boldsymbol{X}}(\boldsymbol{x}) = \alpha\}.$$

Then $\partial L(\alpha)$ is the iso-hyper-surface (with dimension $d-1$) where F equals the constant value α. Thus, $\partial L(\alpha)$ is a (iso)line for bivariate distributions, a (iso)surface for trivariate ones, and so on. The critical layer $\partial L(\alpha)$ partitions $\mathbb{R}^d$ into three non-overlapping and exhaustive regions:

$$\begin{cases} L^{<}(\alpha) & = & \{\mathbf{x} \in \mathbb{R}^d : F_{\mathbf{X}}(\mathbf{x}) < \alpha\}, \\ \partial L(\alpha) & = & \text{the critical layer itself}, \\ L^{>}(\alpha) & = & \{\mathbf{x} \in \mathbb{R}^d : F_{\mathbf{X}}(\mathbf{x}) > \alpha\}. \end{cases}$$

Practically, at any occurrence of the hydrological considered phenomenon, only three mutually exclusive events may happen: either a realization of the considered hydrological event lies in one of these 3 Borel sets $L^{<}(\alpha)$, $\partial L(\alpha)$, or $L^{>}(\alpha)$.

In the applications, usually, the event of interest is of the type $\{X \in \mathcal{A}\}$, where $\mathcal{A}$ is a non-empty Borel set in $\mathbb{R}^d$ collecting all the values judged to be "dangerous" according to some suitable criterion. A natural choice for $\mathcal{A}$ is the set $L^{>}(\alpha)$ (see Salvadori et al. (2011), Gräler et al. (2013)). The first random index N where $\mathbf{X}_k$ reaches the set $L^{>}(\alpha)$ is $N = \min_{k \in \mathbb{N}} \{\mathbf{X}_k \in L^{>}(\alpha)\}$. Assuming $\mathbb{P}[\mathbf{X} \in L^{>}(\alpha)] \in (0,1)$, one easily shows that N is a geometric random variable. The Return Period is defined as the average time required for reaching the set $L^{>}(\alpha)$, that is:

$$\mathrm{RP}^{>}(\alpha) = \Delta_t \cdot \mathbb{E}[N] = \frac{\Delta_t}{\mathbb{P}[\mathbf{X} \in L^{>}(\alpha)]}, \tag{1}$$

where $\Delta_t > 0$ is the (deterministic) average time elapsing between $\mathbf{X}_k$ and $\mathbf{X}_{k+1}$, $k \in \mathbb{N}$. The probability that a realization of this vector belongs to $L^{<}(\alpha)$ is given by the Kendall's function, which only depends on the copula C of this random vector, i.e.,

$$K_C(\alpha) = \mathbb{P}\left[X \in L^{<}(\alpha)\right] = \mathbb{P}\left[C(U_1, \ldots, U_d) \leq \alpha\right], \quad \text{for } \alpha \in (0,1). \tag{2}$$

Then, the considered Return Period can be expressed using Kendall's function in (2), $\mathrm{RP}^{>}(\alpha) = \Delta_t \cdot \frac{1}{1-K_C(\alpha)}$. Obviously, Return Periods can naturally be associated to other sets than $L^{>}(\alpha)$, the interested reader is referred for example to Salvadori et al. (2011).

This paper aims at:

- giving a parametric representation of the multivariate distribution F of a random vector $\mathbf{X}$, here representing rain measurements (for applications see Section 5),
- giving direct estimation procedure for this representation,
- giving closed parametric expressions, both for critical layers in Definition 1.1 and Return Periods in (1),
- adapting this methodology to some asymmetric dependencies (as, for instance, non-exchangeable random vectors; for a possible investigation in this sense see Section 6).

In the next section, we introduce the model used to answer the issues introduced above.

1.2 The model

We consider the following model, which is detailed in Di Bernardino and Rullière (2013a),

$$\widetilde{F}(x_1, \ldots, x_d) = T \circ C_0(T_1^{-1} \circ F_1(x_1), \ldots, T_d^{-1} \circ F_d(x_d)), \tag{3}$$

or equivalently

$$\left\{ \begin{array}{lcl} \widetilde{F}(x_1, \ldots, x_d) & = & \widetilde{C}(\widetilde{F}_1(x_1), \ldots, \widetilde{F}_d(x_d)), \text{ with} \\ \widetilde{C}(u_1, \ldots, u_d) & = & T \circ C_0(T^{-1}(u_1), \ldots, T^{-1}(u_1)) \\ \widetilde{F}_i(x) & = & T \circ T_i^{-1} \circ F_i(x), \quad \text{for} \quad i \in I, \end{array} \right. \tag{4}$$

where $F_1, \ldots, F_d$ are given parametric *initial marginal cumulative distribution functions*, and where C_0 is a given *initial copula*. Hence the distribution $\widetilde{F}(x_1, \ldots, x_d)$ is built from transformed marginals $\widetilde{F}_i, i \in I$ and from a transformed copula $\widetilde{C}$, under regularity conditions. Transformation T permits to transform the initial dependence structure C_0. For a given T, transformations T_i permit to transform marginals, $i \in I$. All these transformations are described hereafter.

Furthermore, as in Di Bernardino and Rullière (2013b), we will assume in the following that C_0 is an Archimedean copula. This means that in this paper, we mainly consider copulas that can be written as

$$C_\phi(u_1, \ldots, u_d) = \phi(\phi^{-1}(u_1) + \ldots + \phi^{-1}(u_d)),$$

where the function ϕ is called the generator of the Archimedean copula C_ϕ. Some conditions like $d-$monotony are given in McNeil and Nešlehová (2009). Here we choose *strict* generator, i.e., $\phi(t) > 0$, $\forall\, t \geq 0$ and $\lim\limits_{t \to +\infty} \phi(t) = 0$, with proper inverse ϕ^{-1} such that $\phi \circ \phi^{-1}(t) = t$.

The function $T : [0, 1] \to [0, 1]$ is a continuous and increasing function on the interval $[0, 1]$, with $T(0) = 0$, $T(1) = 1$, with supplementary assumptions that will be chosen to guarantee that $\widetilde{C}$ is also a copula (detailed hereafter). Internal transformations $T_i : [0, 1] \to [0, 1]$ are continuous non-decreasing functions, such that $T_i(0) = 0$, $T_i(1) = 1$, for $i \in I$. Conditions on transformations such that $\widetilde{C}$ is a copula are discussed for example in Durante et al. (2010), Di Bernardino and Rullière (2013a), Di Bernardino and Rullière (2013b).

In the following we detail the proposed semi-parametric estimation procedure in order to easily fit the model in (3). We show in particular how to estimate the transformations T and T_i, $i \in I$.

Organization of the paper
The paper is organized as follows. In Section 2 we developed the estimation procedure for the chosen model in (3). In Section 2.1 we focus on a central tool in our estimation procedure: the estimation of the diagonal section of a

copula. This estimated diagonal section is used in the non-parametric estimation of the external transformation T and the internal transformations T_i (see Section 2.2). The parametric estimation using composite hyperbolic transformations is detailed in Section 3. Then in Section 4 we propose the parametric form for the desired quantities: the multivariate distribution function F, its critical layers $\partial L(\alpha)$ and the associated Kendall's function $K_C(\alpha)$, for $\alpha \in (0,1)$. Finally, Section 5 is devoted to a detailed study of a $5-$dimensional rainfall data-set. Furthermore, a nested model is proposed for this $5-$dimensional data-set, using the whole estimation procedure presented in the paper (see Section 6). Directions for future research and the conclusion are in Section 7.

2 Nonparametric estimation

2.1 Initial diagonal section estimation

In order to give a non-parametric estimation of the external transformation T, we will propose a non-parametric estimator of the copula within the Archimedean class of copulas. Several non-parametric estimators of the generator of an Archimedean copula are available. One can cite for example those of Dimitrova et al. (2008) or Genest et al. (2011) both based on empirical Kendall's function. Here, proposed estimations will rely on an initial estimation of the diagonal section of the empirical copula. Firstly, we propose some (classical) estimators of the diagonal section δ_1 of a copula,

$$\delta_1(u) = C(u, \ldots, u), \quad u \in [0,1].$$

Remark that if $(U_1, U_2, \ldots, U_d)$ has as distribution function the copula C then

$$\mathbb{P}[U_1 \le u, \ldots, U_d \le u] = C(u, \ldots, u) = \mathbb{P}[\max\{U_1, U_2, \ldots, U_d\} \le u] = \delta_1(u).$$

As a consequence, estimators based on the diagonal section of a copula are relying on the distribution of the maximum of $U_1, \ldots, U_d$, as explained in Sungur and Yang (1996). As pointed out in Di Bernardino and Rullière (2013b), the diagonal of an Archimedean copula, under some suitable conditions, is essential to describe the copula. So, in the following we recall important assumptions (which are fulfilled for many Archimedean copulas, including the independent copula) for the unique determination of an Archimedean copulas starting from its diagonal section (see, for instance, Erdely et al. (2013) and references therein). Some constructions of copulas starting from the *diagonal section* are given for example in Nelsen et al. (2008) and Wysocki (2012).

Proposition 2.1 (Identity of Archimedean copulas, Theorem 3.5 by Erdely et al. (2013))**.** *Let C a $d-$dimensional Archimedean copula whose diagonal section δ_C satisfies $\delta_C^{'}(1^-) = d$. Then C is uniquely determined by its diagonal.*

Condition in Proposition 2.1 is referred to as *Frank's condition* in Erdely et al. (2013) (see their Theorems 1.2 and 3.5). Note that if $|\phi^{'}(0)| < +\infty$ then the condition on the diagonal in Proposition 2.1 is automatically satisfied. In this case, the function ϕ can be reconstructed from the diagonal δ (see also Segers (2011)). As pointed out by Embrechts and Hofert (2011) a possible limitation is that if ϕ has finite right-hand derivative at zero, the Archimedean copula generated by ϕ has upper tail independent structure. To show that the situation of many Archimedean copulas having the same diagonal is far from exceptional, a recipe to construct further examples is given in Segers (2011). For further details the interested reader is referred to Section 3 in Di Bernardino and Rullière (2013b).
It has been shown however that if one aims at fitting both lower and upper tail dependence, then some specific dependence structures can be suited to this purpose, see Di Bernardino and Rullière (2014). Here we will not focus on very extreme tail behavior, but on the main part of the distribution. Using the necessary few data in the tails would require some specific estimators of tail dependence coefficients, as detailed in above mentioned article.

In the following we present different estimation of the diagonal section. Some empirical copula estimators for δ_1 are given in Deheuvels (1979). A comparison between several more recent estimators is presented in Omelka et al. (2009).

Empirical diagonal
We present here an estimator that is detailed in Omelka et al. (2009), and directly inspired by the one of Deheuvels (1979). Consider observations $\{\mathbf{X}^{(k)} = (X_1^{(k)}, \ldots, X_d^{(k)})\}_{k \in \{1,\ldots,n\}}$ of the random vector $\mathbf{X}$. Define pseudo-

observations $\mathbf{U}^{(k)} = (U_1^{(k)}, \dots, U_d^{(k)})$ by setting every component i for observation number k to

$$U_i^{(k)} = \frac{1}{n+1} \sum_{j \in I} 1_{\left\{X_i^{(j)} \leq X_i^{(k)}\right\}},$$

with $i \in I$, $k \in \{1, \dots, n\}$. One can check that for any $i \in I$, $k \in \{1, \dots, n\}$, $U_i^{(k)} \in (0, 1)$.
The empirical estimator $\widehat{C}$ of the copula of vector $\mathbf{X}$, is

$$\widehat{C}(u_1, \dots, u_d) = \frac{1}{n} \sum_{k=1}^{n} 1_{\left\{U_1^{(k)} \leq u_1, \dots, U_d^{(k)} \leq u_d\right\}}.$$

We thus obtain for any $u \in [0, 1]$, the empirical estimation for the diagonal of C and its inverse, i.e.,

$$\begin{cases} \widehat{\delta}_1^{\text{emp}}(u) & = & \widehat{C}(u, \dots, u), \\ \widehat{\delta}_{-1}^{\text{emp}}(u) & = & \text{arginf}\left\{x \in [0, 1];\ \widehat{\delta}_1^{\text{emp}}(x) \geq u\right\}. \end{cases}$$

Smooth empirical diagonal
We do not present here all possible smooth estimators of a copula and the associated smoothed estimates of the diagonal section. We will restrict ourselves in estimators based on transformations, in the same spirit as the other parts of this paper. These estimators perform reasonably well (see Omelka et al. (2009)), especially considering Cramér–von Mises distance.

Using a smooth estimation of the empirical cumulative distribution of $\mathbf{U}$ could create some bias since the distribution support must be $[0, 1]$, this is the reason why we consider a transformation of pseudo-observations. For given smoothing coefficients $h_1, \dots, h_d$, we can define $\mathbf{L}^{(k)} = (L_1^{(k)}, \dots, L_d^{(k)})$ where

$$L_i^{(k)} = G^{-1}(U_i^{(k)}),$$

with $i \in I$, $k \in \{1, \dots, n\}$, and where G^{-1} is the inverse of a cumulative distribution function, for example $G^{-1}(x) = \text{logit}(x)$, and one can check that $((G)'(x))^2/G(x)$ is bounded, which is a required condition detailed in Omelka et al. (2009).
A smooth version of $\widehat{C}$ can be defined by:

$$\overline{C}(u_1, \dots, u_d) = \frac{1}{n} \sum_{k=1}^{n} \prod_{i \in I} K\left(\frac{G^{-1}(u_i) - L_i^{(k)}}{h_i}\right),$$

where K is a suited kernel function (we took here a multiplicative multivariate kernel).
We thus obtain for any $u \in [0, 1]$, a smooth estimator $\widehat{\delta}$ of the diagonal section of C, and a smooth estimator $\widehat{\delta}_{-1}$ of the inverse function of this diagonal section:

$$\begin{cases} \widehat{\delta}_1(u) & = & \overline{C}(u, \dots, u) \\ \widehat{\delta}_{-1}(u) & = & \text{arginf}\left\{x \in [0, 1];\ \widehat{\delta}_1(x) \geq u\right\}. \end{cases} \tag{5}$$

Note that initial pseudo-data can also be smoothed, by defining

$$\overline{U}_i^{(k)} = \frac{1}{n+1} \sum_{j \in \{1, \dots, n\}} \Phi\left(\frac{X_i^{(k)} - X_i^{(j)}}{h_i}\right).$$

Discussions on this last choices can be found in Chen and Huang (2007) and Omelka et al. (2009).
A discussion on the possible choices for $h_1, \dots, h_d$ is given in Chiu (1996) (for $d = 1$), Wand and Jones (1993), Wand and Jones (1994), Zhang et al. (2006) (for $d \geq 2$) and references therein. However, according to Omelka et al. (2009), "*a good bandwidth selection rule is missing, for the moment, and is subject of further research.*"

A summary of the needed input parameter and the estimation of δ_1 and δ_1^{-1} is given in Algorithm 1.

Algorithm 1 Smooth initial diagonal estimation

Input parameters
Choose G^{-1} inverse of a cdf, *e.g.,* $G^{-1}(x) = \text{logit}(x)$
Choose h_i, $i \in I$ bandwidth sizes, *e.g., the ones proposed by the Silverman's Rule of Thumb*

Estimation
For any $x \in [0, 1]$, get $\widehat{\delta}_1(x)$ and $\widehat{\delta}_{-1}(x)$ by Equation (5)

2.2 Nonparametric estimation of transformations T and T_i, $i \in I$

A non parametric estimator of the external transformation T in (3) is given in Di Bernardino and Rullière (2013b), for $x \in (0, 1)$, by

$$\widehat{T}(x) = \widehat{\delta}_{r(x)}(y_0), \qquad \text{with } r(x) \text{ such that } \delta^0_{r(x)}(x_0) = x, \tag{6}$$

where $\widehat{\delta}_r$ refers to the estimator of the self-nested diagonal δ_r of the target copula C, and $\delta^0_{r(x)}(x_0)$ refers to the self-nested diagonal of the initial copula C_0. These estimators $\widehat{\delta}_r$ and self-nested diagonals are defined hereafter.

In the case where the initial copula C_0 is an Archimedean copula of generator ϕ_0, then

$$\delta^0_{r(x)}(x_0) = \phi_0\left(d^{r(x)}\phi_0^{-1}(x_0)\right) \quad \text{and} \quad r(x) = \frac{1}{\ln d}\ln\left(\frac{\phi_0^{-1}(x)}{\phi_0^{-1}(x_0)}\right).$$

In particular, if C_0 is the independence copula, with generator $\phi_0(x) = \exp(-x)$, and setting for example $x_0 = y_0 = \exp(-1)$, then

$$\widehat{T}(x) = \widehat{\delta}_{\ln(-\ln(x))/\ln d}(\mathrm{e}^{-1}).$$

Let $\widehat{\delta}_1$ be an estimator of δ_1, and $\widehat{\delta}_{-1}$ be an estimator of the inverse function δ_{-1} as in Equation (5). At a relative integer order $k \in \mathbb{Z}$, the self-nested diagonals estimators are defined as

$$\begin{cases} \widehat{\delta}_k(u) &= \widehat{\delta}_1 \circ \ldots \circ \widehat{\delta}_1(u), & (k \text{ times}), & k \in \mathbb{N} \\ \widehat{\delta}_{-k}(u) &= \widehat{\delta}_{-1} \circ \ldots \circ \widehat{\delta}_{-1}(u), & (k \text{ times}), & k \in \mathbb{N} \\ \widehat{\delta}_0(u) &= u. \end{cases} \tag{7}$$

At any real order $r \in \mathbb{R}$, an estimator $\widehat{\delta}_r$ of the self-nested diagonal is

$$\widehat{\delta}_r(x) = z\left(\left(z^{-1} \circ \widehat{\delta}_k(x)\right)^{1-\alpha}\left(z^{-1} \circ \widehat{\delta}_{k+1}(x)\right)^{\alpha}\right), \quad x \in [0, 1]$$

with $\alpha = r - \lfloor r \rfloor$ and $k = \lfloor r \rfloor$, where $\lfloor r \rfloor$ denotes the integer part of r, and where z is a function driving the interpolation, ideally (if known) the generator of the considered copula C. Some consistency results about this estimator $\widehat{T}$ are detailed in Di Bernardino and Rullière (2013b).

For invertible marginal transformations T_i, one easily shows

$$T_i = F_i \circ \widetilde{F}_i^{-1} \circ T, \quad i \in I. \tag{8}$$

Then, by replacing the transformed marginal distribution $\widetilde{F}_i$ in (8) by an estimator of the target i-th marginal distribution, denoted by $\widehat{F}_i$ (e.g., empirical cdf), and given the external transformation or its consistent estimation $\widehat{T}$, a non-parametric estimation of T_i, for $i \in I$, is provided by

$$\widehat{T}_i(u) = F_i \circ \widehat{F}_i^{-1} \circ \widehat{T}(u), \quad u \in (0, 1), \tag{9}$$

where F_i is the chosen $i-$th initial marginal.
Following Definition 2.1 summaries the expression of non parametric estimators for both T and T_i, $i \in I$.

Definition 2.1 (Non-parametric estimators of T and T_i)**.** *For a given arbitrary couple $(x_0, y_0) \in (0,1)^2$, a non-parametric estimator of T is given by*

$$\widehat{T}(x) = \widehat{\delta}_{r(x)}(y_0), \text{ for all } x \in (0,1)$$
$$\text{with } r(x) \text{ such that } \delta^0_{r(x)}(x_0) = x,$$

where $\delta^0_{r(x)}$ refers to the self nested diagonal of the initial copula C_0. In particular, if the initial copula C_0 is the independence copula, $r(x) = \frac{1}{\ln d} \ln\left(\frac{-\ln x}{-\ln x_0}\right)$. For any $i \in I$, non-parametric estimators T_i are

$$\widehat{T}_i(x) = F_i \circ \widehat{F}_i^{-1} \circ \widehat{T}(x).$$

A summary of the steps for the non-parametric estimation of T and T_i is given in Algorithm 2.

Algorithm 2 Non-parametric estimation of $\widehat{T}$ and $\widehat{T}_i$

Input parameters
Choose (x_0, y_0) in $(0,1)^2$, initial copula C_0 and initial marginals F_i, for $i \in I$
Choose $k_{\max}$ pre-computation range

Eventual pre-calculations
Get $\widehat{\delta}_1$ and $\widehat{\delta}_{-1}$ by Algorithm 1
Calculate and store $\widehat{\delta}_k(y_0)$, $k \in \{-k_{\max}, \ldots, 0, \ldots, k_{\max}\}$ by Equation (7)

Non-Parametric estimation
Get $\widehat{T}$ by Equation (6), using previous precomputations
Get $\widehat{T}_i$ by Equation (9)

2.3 Subset of points of transformations

In practice, we will propose in Section 3 some parametric estimators for the transformations T and T_i, by requiring that these transformations are passing through a finite set of points. The interested reader is also referred to Di Bernardino and Rullière (2013a). We aim here at determining some good set of points to be chosen.

Firstly, we define some sets of points to be chosen, starting from given sets of quantile levels. The chosen expressions for these sets is motivated by several interesting properties (see Proposition 2.2 below).

Definition 2.2 (Set Ω)**.** *Let $J \subset \mathbb{N}$ be a finite set of indexes and $\mathcal{Q} = \left\{q_j^{(T)}\right\}_{j \in J}$ be a given set of targeted percentiles, $q_j^{(T)} \in (0,1)$, $j \in J$. One defines $\Omega(\mathcal{Q}) = \{(\alpha_j, \beta_j\}_{j \in J}$, with*

$$\begin{cases} \alpha_j &=& q_j^{(T)} \\ \beta_j &=& \widehat{T}(\alpha_j), \end{cases}$$

where $\widehat{T}$ is the estimator in Equation (6).

Definition 2.3 (Sets Ω_i, $i \in I$)**.** *Let $J_i \subset \mathbb{N}$ be finite sets of indexes and $\mathcal{Q}_i = \left\{q_j^{(i)}\right\}_{j \in J_i}$ be finite given sets of targeted percentiles, $q_j^{(i)} \in (0,1)$, $j \in J_i$, $i \in I$. For a given transformation τ, one defines the sets $\Omega_i(\mathcal{Q}_i, \tau) = \left\{(\alpha^i_j, \beta^i_j)\right\}_{j \in J_i}$ where*

$$\begin{cases} \alpha^i_j &=& \tau^{-1}(q_j^{(i)}), \\ \beta^i_j &=& F_i \circ \widehat{F}_i^{-1}(q_j^{(i)}), \end{cases} \tag{10}$$

where $\widehat{F}_i^{-1}$ is an estimator of the target i-th marginal distribution, denoted by $\widehat{F}_i$.

Given Ω and Ω_i, for $i \in I$, some set of points as in Definitions 2.2 and 2.3, we firstly derive some properties of some parametric transformations T_Ω and T_{Ω_i}, $i \in I$ that are chosen to pass through these sets Ω and Ω_i, for $i \in I$ (see Proposition 2.2 below). Theses properties justify the choice of sets Ω and Ω_i in our estimation procedure.

Proposition 2.2 (Set of points for T_i)**.** *Assume that estimators $\widehat{T}$ and $\widehat{F}_i$ are given continuous and invertible functions. Let $J_i \subset \mathbb{N}$ be finite sets of indexes and $\mathcal{Q}_i = \left\{q_j^{(i)}\right\}_{j \in J_i}$ be a finite given set of targeted percentiles, $q_j^{(i)} \in (0,1)$, $j \in J_i$, $i \in I$. Then for transformations $\widehat{T}_i$ defined in Equation (9), $i \in I$*

$$\widehat{T}_i \text{ is passing through all points of } \Omega_i(\mathcal{Q}_i, \widehat{T}),$$

where $\Omega_i(\mathcal{Q}_i, \widehat{T})$ is defined as in Definition 2.3. Furthermore, for any invertible transformation T_{Ω_i} passing through all points of $\Omega_i(\mathcal{Q}_i, \widehat{T})$, $i \in I$

$$\widetilde{F}_{\Omega_i}^{-1}(q) = \widehat{F}_i^{-1}(q) \text{ for all } q \in \mathcal{Q}_i, \tag{11}$$

where $\widetilde{F}_{\Omega_i} = \widehat{T} \circ T_{\Omega_i}^{-1} \circ F_i$ is the transformed marginal distribution which uses T_{Ω_i}.

Proof: Let $i \in I$, for any $q \in \mathcal{Q}_i$, one can define $(\alpha, \beta) \in \Omega_i(\mathcal{Q}_i, \widehat{T})$ by Equation (10), with $\alpha = \widehat{T}^{-1}(q)$ and $\beta = F_i \circ \widehat{F}_i^{-1}(q)$. We assume here that all estimators are continuous and invertible functions, so that for example $\widehat{T} \circ \widehat{T}^{-1} = Id$.
Firstly, one can check that $\widehat{T}_i(\alpha) = \widehat{T}_i \circ \widehat{T}^{-1}(q)$ and from Equation (9), $\widehat{T}_i(\alpha) = F_i \circ \widehat{F}_i^{-1} \circ \widehat{T} \circ \widehat{T}^{-1}(q) = F_i \circ \widehat{F}_i^{-1}(q) = \beta$, so that $\widehat{T}_i(\alpha) = \beta$ and the first result holds.
Secondly, assuming all estimators are invertible, one can write $\widetilde{F}_{\Omega_i}^{-1} = F_i^{-1} \circ T_{\Omega_i} \circ \widehat{T}^{-1}$, so that $\widetilde{F}_{\Omega_i}^{-1}(q) = F_i^{-1} \circ T_{\Omega_i}(\alpha)$. By assumption $T_{\Omega_i}(\alpha) = \beta$ since T_{Ω_i} is passing through all points of $\Omega_i(\mathcal{Q}_i, \widehat{T})$, so that finally $\widetilde{F}_{\Omega_i}^{-1}(q) = F_i^{-1}(\beta) = F_i^{-1} \circ F_i \circ \widehat{F}_i^{-1}(q)$ and $\widetilde{F}_{\Omega_i}^{-1}(q) = \widehat{F}_i^{-1}(q)$ which gives the second result. □

Once chosen the thresholds $q_j^{(i)}$, $j \in J_i$, for which we want to identify the transformed margins with targeted margins, one thus get a finite set of passage points for T_i. Indeed, Proposition 2.2 shows that, using Ω_i from Definition 2.3, one can select a further parametric estimator T_{Ω_i} passing trough points of Ω_i and such that quantiles of the target $\widehat{F}_i$ are identified with quantiles of the transformed margin using T_{Ω_i} (see Equation (11)), for each quantile level $q \in \mathcal{Q}_i$ and for any $i \in I$. Figure 1 illustrates this identification of quantile levels.
Such a result would be difficult to establish for the external transformation T. A weaker form is given by following Proposition 2.3.

Proposition 2.3 (Set of points for T)**.** *Let $J \subset \mathbb{N}$ be a finite set of indexes and $\mathcal{Q} = \left\{q_j^{(T)}\right\}_{j \in J}$ be a given set of targeted percentiles, $q_j^{(T)} \in (0,1)$, $j \in J$. Define $\Omega(\mathcal{Q})$ as in Definition 2.2. Then obviously*

$$\widehat{T} \text{ is passing through all points of } \Omega(\mathcal{Q}).$$

Furthermore, if T_Ω is another transformation passing through all points of $\Omega(\mathcal{Q})$, then

$$\max_{(\alpha_j, \beta_j) \in \Omega} \left\| \widehat{C}(\beta_j, \ldots, \beta_j) - \widetilde{C}_\Omega(\beta_j, \ldots, \beta_j) \right\| \leq \sup_{u \in [0,1]} \left\| \widehat{T}(u) - T_\Omega(u) \right\|$$

where $\widetilde{C}_\Omega(u_1, \ldots, u_d) = T_\Omega \circ C_0(T_\Omega^{-1}(u_1), \ldots, T_\Omega^{-1}(u_d))$ is the transformed copula using transformation T_Ω, $\widehat{C}_\Omega(u_1, \ldots, u_d) = \widehat{T} \circ C_0(\widehat{T}^{-1}(u_1), \ldots, \widehat{T}^{-1}(u_d))$ is the transformed copula using the full non-parametric transformation $\widehat{T}$ and C_0 is the initial copula.

Proof: From Definition 2.2, it is obvious that $\widehat{T}$ is passing through all points of $\Omega(\mathcal{Q})$. For any $(\alpha_j, \beta_j) \in \Omega(\mathcal{Q})$, $\widetilde{C}_\Omega(\beta_j, \ldots, \beta_j) = T_\Omega \circ C_0(T_\Omega^{-1}(\beta_j), \ldots, T_\Omega^{-1}(\beta_j))$ and $\widehat{C}_\Omega(\beta_j, \ldots, \beta_j) = \widehat{T} \circ C_0(\widehat{T}^{-1}(\beta_j), \ldots, \widehat{T}^{-1}(\beta_j))$. Since both $\widehat{T}$ and T_Ω are passing trough all points of $\Omega(\mathcal{Q})$, it follows that $T_\Omega^{-1}(\beta_j) = \widehat{T}^{-1}(\beta_j) = \alpha_j$. As a consequence,

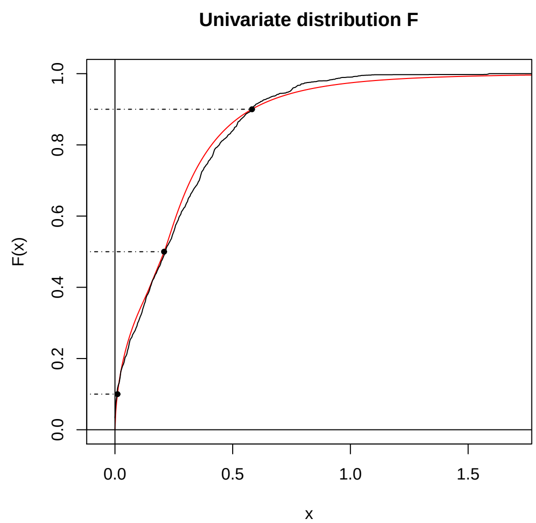

Figure 1: Graphical illustration of procedure described in Proposition 2.2, for $d = 1$. Here an univariate data-set and the set $\mathcal{Q} = \{0.1, 0.5, 0.9\}$ are chosen. The black line represents the empirical cdf of data, $\widehat{F}$, and the red one the cdf $\widetilde{F}$ using the procedure in Proposition 2.2. As proved before, the quantiles of $\widehat{F}$ are identified with quantiles of the obtained transformed $\widetilde{F}$ for each quantile level $q \in \mathcal{Q}$ (black points and associated dotted lines).

$\left\|\widehat{C}(\beta_j, \dots, \beta_j) - \widetilde{C}_\Omega(\beta_j, \dots, \beta_j)\right\| = \left\|\widehat{T}(v_j) - T_\Omega(v_j)\right\|$, with $v_j = C_0(\alpha_j, \dots, \alpha_j)$, $v_j \in [0, 1]$. Hence the result. $\square$

Proposition 2.3 gives an interpretation of the set $\mathcal{Q}$ defined in Definition 2.2. Indeed, $\mathcal{Q}$ can be interpreted as a set of percentiles for which we want to minimize the difference between diagonals of the transformed non-parametric copula and a transformed parametric copula using a transformation T_Ω instead of $\widehat{T}$.

Once obtained these finite sets of reaching points for $\widehat{T}_i$ and $\widehat{T}$, one can find parametric estimators without any optimization procedure. A summary of the estimation procedure for obtaining piecewise linear estimators of T and T_i, $i \in I$, is given in Algorithm 3.

Algorithm 3 Subsets from non-parametric estimation

Input parameters
Choose $\mathcal{Q}$ and $\mathcal{Q}_i$, $i \in I$, initial set of quantile levels, *e.g.* $\mathcal{Q} = \mathcal{Q}_1 = \ldots = \mathcal{Q}_d = \{0.25, 0.5, 0.75\}$.

Subsets of points
Get $\hat{T}$ and $\hat{T}_i$, $i \in I$, by Algorithm 2
Get $\Omega(\mathcal{Q})$ by Definition 2.2
Get $\Omega_i(\mathcal{Q}_i, \widehat{T})$ by Definition 2.3

We have seen that, starting from given thresholds $q_j^{(i)} \in (0, 1)$, it was possible to propose some set of points Ω_i, $i \in I$ and Ω, as in Proposition 2.2 and 2.3, such that each $\widehat{T}_i$ is passing trough points of Ω_i, and each $\widehat{T}$ is passing trough points of Ω. Starting from these considerations, one can also introduce in the following definition for some piecewise linear estimators of T and T_i, $i \in I$.

Definition 2.4 (Piecewise linear estimators of T and T_i). *One can define two piecewise linear estimators of the external and internal transformations T and T_i, $i \in I$.*

- $\widehat{T}^{\mathrm{PL}}$ *is defined as a piecewise linear function passing trough the points* $(0, 0)$*, all points of* $\Omega(\mathcal{Q})$*, and* $(1, 1)$*, where* $\Omega(\mathcal{Q})$ *is given in Definition 2.2.*

- *for each* $i \in I$, $\widehat{T}_i^{\mathrm{PL}}$ *is defined as a piecewise linear function passing through the points* $(0,0)$*, all points of* $\Omega_i(\mathcal{Q}_i, \widehat{T}^{\mathrm{PL}})$*, and* $(1,1)$*, where* $\Omega_i(\mathcal{Q}_i, \widehat{T}^{\mathrm{PL}})$*,* $i \in I$ *is given in Definition 2.3.*

Piecewise linear estimators $\widehat{T}^{\mathrm{PL}}$ and $\widehat{T}_i^{\mathrm{PL}}$, $i \in I$, are estimators relying on a chosen finite number of parameters. However, these estimators are not differentiable everywhere on $(0,1)$. In the following, we will try to find differentiable parametric estimators passing through the points of considered subsets Ω and Ω_i, $i \in I$.

3 Parametric estimation

3.1 Composite hyperbolic transformations

As initially defined in Bienvenüe and Rullière (2011) and Bienvenüe and Rullière (2012) we recall here the definition of hyperbolic composite transformations.

Definition 3.1 (Conversion and transformation functions)**.** *Let* f *any bijective increasing function,* $f : \mathbb{R} \to \mathbb{R}$*. It is said to be a conversion function. Furthermore the associated transformation function* T *to* f *is defined by:* $T_f : [0,1] \to [0,1]$ *such that*

$$T_f(u) = \begin{cases} 0 & \text{if } u = 0, \\ \operatorname{logit}^{-1}(f(\operatorname{logit}(u))) & \text{if } 0 < u < 1, \\ 1 & \text{if } u = 1. \end{cases}$$

Remark that T_f is a continuous non-decreasing function, such that $T_f(0) = 0$, $T_f(1) = 1$. Furthermore we remark that the transformation functions in Definition 3.1 are chosen in a way to be easily invertible. In particular in a way such that $T_f \circ T_g = T_{f \circ g}$, $T_f^{-1} = T_{f^{-1}}$. When f is easily invertible, these readily invertible transformations help sampling transformed distributions (see Bienvenüe and Rullière (2011)).

In this section we consider the following particular class of hyperbolic conversion function (for further details see Bienvenüe and Rullière (2012)).

Definition 3.2 (A class of hyperbole)**.** *The considered hyperbole* H *is*

$$H_{m,h,\rho_1,\rho_2,\eta}(x) = m - h + (e^{\rho_1} + e^{\rho_2})\frac{x - m - h}{2} - (e^{\rho_1} - e^{\rho_2})\sqrt{\left(\frac{x - m - h}{2}\right)^2 + e^{\eta - \frac{\rho_1 + \rho_2}{2}}},$$

with $m, h, \rho_1, \rho_2 \in \mathbb{R}$*, and one smoothing parameter* $\eta \in \mathbb{R}$*.*

After some calculations, one can check that

$$H^{-1}_{m,h,\rho_1,\rho_2,\eta}(x) \quad = \quad H_{m,-h,-\rho_1,-\rho_2,\eta}(x).$$

Functions H in Definition 3.2 are thus readily invertible: a simple change of the sign of some parameters leads to the inverse function. As a consequence, transformations T_H based on conversion functions H will also be readily invertible. In the following, we consider the generic hyperbolic conversion function defined in Definition 3.2. First remark that when the smoothing parameter η tends to $-\infty$, the hyperbole H tends to the angle function:

$$A_{m,h,\rho_1,\rho_2}(x) = m - h + (x - m - h)\left(e^{\rho_1} 1_{\{x<m+h\}} + e^{\rho_2} 1_{\{x>m+h\}}\right). \tag{12}$$

As remarked in Bienvenüe and Rullière (2011), it thus appears that hyperbolic transformations have the advantage of being smooth versions of angle functions. They show in their paper that initial parameters for the estimation are easy to obtain with angle compositions.
For sake of clarity, we recall below some definitions in Di Bernardino and Rullière (2013a). These definitions of composite transformations and suited parameters will be useful in the following.

Definition 3.3 (Composite transformations)**.** *Let $k \in \mathbb{N}$. Consider $\eta \in \mathbb{R}$ and a given parameter vector $\theta = (m, h, \rho_1, \rho_2, a_1, r_1, \ldots, a_k, r_k)$ if $k \geq 1$, or $\theta = (m, h, \rho_1, \rho_2)$ if $k = 0$. We define the angle composite transformation $\mathcal{A}_\theta$ as:*

$$\mathcal{A}_\theta = T_{f_\theta}, \text{ with } f_\theta = \begin{cases} A_{a_k,0,0,r_k} \circ \cdots \circ A_{a_1,0,0,r_1} \circ A_{m,h,\rho_1,\rho_2} & \text{if } k \geq 1, \\ A_{m,h,\rho_1,\rho_2} & \text{if } k = 0, \end{cases}$$

and the hyperbolic composite transformation $\mathcal{H}_{\theta,\eta}$ as:

$$\mathcal{H}_{\theta,\eta} = T_{f_{\theta,\eta}}, \text{ with } f_{\theta,\eta} = \begin{cases} H_{a_k,0,0,r_k,\eta} \circ \cdots \circ H_{a_1,0,0,r_1,\eta} \circ H_{m,h,\rho_1,\rho_2,\eta} & \text{if } k \geq 1, \\ H_{m,h,\rho_1,\rho_2,\eta} & \text{if } k = 0, \end{cases}$$

with A_{m,h,ρ_1,ρ_2} as in Equation (12) and $H_{m,h,\rho_1,\rho_2,\eta}$ as in Definition 3.2.

Remark 1 (Inverse composite transformations)**.** Let $k \in \mathbb{N}$. Consider $\eta \in \mathbb{R}$ and a given parameter vector $\theta = (m, h, \rho_1, \rho_2, a_1, r_1, \ldots, a_k, r_k)$ if $k \geq 1$, or $\theta = (m, h, \rho_1, \rho_2)$ if $k = 0$. Since $T_f^{-1} = T_{f^{-1}}$, the angle composite transformation $\mathcal{A}_\theta^{-1}$ is such that:

$$\mathcal{A}_\theta^{-1} = T_{f_\theta}, \text{ with } f_\theta = \begin{cases} A_{m,-h,-\rho_1,-\rho_2} \circ A_{a_1,0,0,-r_1} \circ \cdots \circ A_{a_k,0,0,-r_k} & \text{if } k \geq 1, \\ A_{m,-h,-\rho_1,-\rho_2} & \text{if } k = 0. \end{cases}$$

The hyperbolic inverse composite transformation $\mathcal{H}_{\theta,\eta}$ is such that:

$$\mathcal{H}_{\theta,\eta}^{-1} = T_{f_{\theta,\eta}}, \text{ with } f_{\theta,\eta} = \begin{cases} H_{m,-h,-\rho_1,-\rho_2,\eta} \circ H_{a_1,0,0,-r_1,\eta} \circ \cdots \circ H_{a_k,0,0,-r_k,\eta} & \text{if } k \geq 1, \\ H_{m,-h,-\rho_1,-\rho_2,\eta} & \text{if } k = 0. \end{cases}$$

Definition 3.4 (Suited parameters from Ω)**.** *Let $k \in \mathbb{N}$. Consider one given set $\Omega = \{\omega_1, \ldots, \omega_{3+k}\}$, $\omega_j \in (0,1)^2$. Denote by u_j and v_j the two respective components of each ω_j in the logit scale, such that $\omega_j = (\text{logit}^{-1} u_j, \text{logit}^{-1} v_j)$, $j \in \{1, \ldots, 3+k\}$. Assume that u_j and v_j are increasing sequences of j. We define:*

$$\Theta(\Omega) = \begin{cases} (m, h, \rho_1, \rho_2, a_1, r_1, \ldots, a_k, r_k) & \text{if } k \geq 1, \\ (m, h, \rho_1, \rho_2) & \text{if } k = 0, \end{cases}$$

where $m = \frac{u_2+v_2}{2}$, $h = \frac{u_2-v_2}{2}$, $\rho_1 = \ln\left(\frac{v_2-v_1}{u_2-u_1}\right)$, $\rho_2 = \ln\left(\frac{v_3-v_2}{u_3-u_2}\right)$, $r_k = \ln\left(\frac{v_{3+k}-v_{2+k}}{u_{3+k}-u_{2+k}} \frac{u_{2+k}-u_{1+k}}{v_{2+k}-v_{1+k}}\right)$, $a_k = v_{2+k}$, $k \geq 1$.

Proposition 3.1 (Suited composite transformations)**.** *Let $k \in \mathbb{N}$. Consider one given set $\Omega = \{\omega_1, \ldots, \omega_{3+k}\}$, $\omega_j \in (0,1)^2$ and a smoothing parameter $\eta \in \mathbb{R}$. Set $\theta = \Theta(\Omega)$, then*

- *the transformation $\mathcal{A}_\theta(x)$ is piecewise linear in the logit scale and will be called logit-piecewise linear. It links point $(0,0)$, points of Ω, and point $(1,1)$.*
- *the transformation $\mathcal{H}_{\theta,\eta}$ converges pointwise to $\mathcal{A}_\theta$ as η tends to $-\infty$. It results that the continuous and differentiable transformation $\mathcal{H}_{\theta,\eta}$ can fit as precisely as desired the set of points Ω when η tends to $-\infty$.*

Proof: The first result is proved in Bienvenüe and Rullière (2011). It simply comes from the fact that $\mathcal{A}_{\Theta(\Omega)}(u_j) = v_j$ for all $j \in \{1, \ldots, 3+k\}$, where $\omega_j = (\text{logit}^{-1} u_j, \text{logit}^{-1} v_j)$. The convergence of the hyperbole composite transformation toward the angle composite transformation is straightforward and also evoked in Bienvenüe and Rullière (2011). □

3.2 Smoothed parametric estimators

For the estimation of transformations T and T_i, we assume that are given:

- Initial estimators of δ and δ^{-1} (see Section 2.1).

- The sets of quantile levels $\mathcal{Q} = \left\{q_j^{(T)}\right\}_{j\in J}$, $\mathcal{Q}_i = \left\{q_j^{(i)}\right\}_{j\in J_i}$, $i \in I$ (see Section 2.3).
- Some smoothing parameters $\eta \in \mathbb{R}$ and $\eta_i \in \mathbb{R}$, $i \in I$ (see Section 3.1).

In the following we introduce the smooth parametric estimators $\overline{T}$ and $\overline{T}_i$, $i \in I$, for external and internal transformations respectively.

Estimation of T

Using estimators of δ and δ^{-1} (see Section 2.1), one easily gets the resulting set $\Omega(\mathcal{Q})$ by Proposition 2.3 and suited parameters by Definition 3.4:

$$\widehat{\theta} = \Theta(\widehat{\Omega}(\mathcal{Q})). \tag{13}$$

Then one defines: $\overline{T} = \mathcal{H}_{\widehat{\theta},\eta}$.

Estimation of T_i, $i \in I$

Once T estimated by $\overline{T}$, one gets the passage set Ω_i, for $i \in I$ by Proposition 2.2 and suited associated parameters θ_i by Definition 3.4, i.e.,

$$\widehat{\theta}_i = \Theta(\widehat{\Omega}_i(\mathcal{Q}_i, \overline{T}))). \tag{14}$$

Note that once given thresholds sets $\mathcal{Q}$ and $\mathcal{Q}_i$ and smooth parameters η and η_i, all estimated parameters are directly and analytically defined, so that we do not need here any inversion or optimization procedure.
Then one defines: $\overline{T}_i = \mathcal{H}_{\widehat{\theta}_i,\eta_i}$, for $i \in I$.
A summary of the expressions of smooth estimators $\overline{T}$ and $\overline{T}_i$, $i \in I$, is given in following definition.

Definition 3.5 (Smooth estimation of T and T_i, $i \in I$)**.** *Let $\mathcal{Q}$ and $\mathcal{Q}_i$, $i \in I$ be given sets of quantile levels. Let $\eta \in \mathbb{R}$ and $\eta_i \in \mathbb{R}$, $i \in I$ be given smoothing parameters. One defines*

$$\begin{cases} \overline{T} &= \mathcal{H}_{\widehat{\theta},\eta}, \\ \overline{T}_i &= \mathcal{H}_{\widehat{\theta}_i,\eta_i}, \ i \in I, \end{cases} \tag{15}$$

where parameters $\widehat{\theta}$ and $\widehat{\theta}_i$ are given by

$$\begin{cases} \widehat{\theta} &= \Theta(\widehat{\Omega}(\mathcal{Q}))), \\ \widehat{\theta}_i &= \Theta(\widehat{\Omega}_i(\mathcal{Q}_i, \overline{T})), \ i \in I. \end{cases}$$

The sets Ω and Ω_i are defined in Definitions 2.2 and 2.3. The function Θ is defined in Definition 3.4.

Estimation procedure for obtaining smoothed parametric estimators $\overline{T}$ and $\overline{T}_i$, $i \in I$ is gathered in Algorithm 4.

Algorithm 4 Parametric estimation

Input parameters
Choose $\mathcal{Q}$ and $\mathcal{Q}_i$, $i \in I$ the sets of quantile levels, *e.g.* $\mathcal{Q} = \mathcal{Q}_1 = \ldots = \mathcal{Q}_d = \{0.25, 0.5, 0.75\}$,
Choose smoothing parameters $\eta \in \mathbb{R}$ and $\eta_i \in \mathbb{R}$, $i \in I$,

Parametric estimation
Get $\widehat{\Omega}(\mathcal{Q}))$ and $\widehat{\Omega}_i(\mathcal{Q}_i, \overline{T})$, $i \in I$ by Algorithm 3
Get $\widehat{\theta}$ by Equation (13)
Get $\widehat{\theta}_i$ by Equation (14)
Get smooth estimators $\overline{T}$ and $\overline{T}_i$, $i \in I$ by Equation (15).

One can define the complete vector parameter presented in Algorithms 3 and 4:

$$\boldsymbol{\Theta} = (\widehat{\theta}_1, \ldots, \widehat{\theta}_d, \widehat{\theta}, \eta_1, \ldots, \eta_d, \eta).$$

What is noticeable here is that given thresholds $\mathcal{Q}$, $\mathcal{Q}_i$ and smoothing parameters η, η_i, one have direct expressions for the parametric estimators $\widehat{\theta}$ and $\widehat{\theta}_i$, $i \in I$. The estimation can also change, depending on some other estimation choices (generator among its equivalence class (x_0, y_0), bandwidths h_i, $i \in I$ or kernel function K), but one aims here at finding good estimators whatever the choice of any reasonable value of these parameters. In numerical Section 5 we will illustrate this point. Finding a global optimum in high dimension would lead to a curse a dimensionality and numerical problems. As a consequence, it is very important to get correct estimators without jointly optimizing a lot of parameters in high dimension.

4 Final parametric results

Once all parameters estimated as in Section 3, the previous parametric model allows to get various analytical results for both the multivariate distribution function, its associated critical layers, Kendall's function and multivariate return periods.

Firstly we obtain the parametric expression for the transformed copula $\widetilde{C}$:

$$\widetilde{C}(u_1, \ldots, u_d) = \widetilde{\phi}(\widetilde{\phi}^{-1}(u_1) + \ldots + \widetilde{\phi}^{-1}(u_d)), \tag{16}$$

where $\widetilde{\phi}$ is the final parametric transformed Archimedean generator (see (4)). This generator can be easily written in terms of the external transformation T, i.e.,

$$\widetilde{\phi}(t) = \overline{T}(\phi_0(t)), \tag{17}$$

where ϕ_0 is the generator associated to the initial copula C_0 in (4) and $\overline{T}$ is the smooth estimator of the external transformation presented in Section 3.2.

Using model in (4), the corresponding estimated transformed multivariate distribution is given by:

$$\widetilde{F}_{\mathbf{\Theta}}(x_1, \ldots, x_d) = \mathcal{H}_{\widehat{\theta},\eta} \circ C_0(\mathcal{H}^{-1}_{\widehat{\theta}_1,\eta_1} \circ F_1(x_1), \ldots, \mathcal{H}^{-1}_{\widehat{\theta}_d,\eta_d} \circ F_d(x_d)), \tag{18}$$

where $\mathbf{\Theta}$ is the complete estimated vector parameter presented in Algorithms 3 and 4.

Furthermore, using expression in (18) for the estimated transformed multivariate distribution $\widetilde{F}$, the associated parametric α critical-layers are given by

$$\partial\widetilde{L}_{\mathbf{\Theta}}(\alpha) = \{(F_1^{-1} \circ \mathcal{H}_{\widehat{\theta}_1,\eta_1}(u_1), \ldots, F_d^{-1} \circ \mathcal{H}_{\widehat{\theta}_d,\eta_d}(u_d)), (u_1, \ldots, u_d) \in (0,1)^d, C_0(u_1, \ldots, u_d) = \mathcal{H}^{-1}_{\widehat{\theta},\eta}(\alpha)\},$$

where a direct analytic expression $\mathcal{H}^{-1}_{\widehat{\theta},\eta}$ is given by Remark 1 (see also Proposition 2.4 in Di Bernardino and Rullière (2013a)).

As presented in the introduction, we aim at providing a parametric estimation of the multivariate Return Period

$$\mathrm{RP}^{>}(\alpha) = \Delta_t \cdot \tfrac{1}{1-K_C(\alpha)}.$$

Genest and Rivest (2001) obtained the following explicit expression for the Kendall distribution in the case of multivariate Archimedean copulas with a given generator φ, i.e.,

$$K_C(\alpha) = \alpha + \sum_{i=1}^{d-1} \frac{1}{i!} \left(-\varphi^{-1}(\alpha)\right)^i \varphi^{(i)}\left(\varphi^{-1}(\alpha)\right), \quad \text{for } \alpha \in (0,1),$$

where the notation $f^{(i)}$ corresponds to the $i-$th derivatives of a function f.

Consider the case of transformed generator $\widetilde{\phi}(t) = \overline{T}(\phi_0(t))$, in (16)-(17). Then, the analytical formula for the estimated transformed Kendall distribution $\widetilde{K}_C$ can be easily written as:

$$\widetilde{K}_C(\alpha) \quad = \quad \alpha + \sum_{i=1}^{d-1} \frac{1}{i!} \left(-\phi_0^{-1}(\overline{T}^{-1}(\alpha))\right)^i \left(\overline{T} \circ \phi_0\right)^{(i)} \left(\phi_0^{-1}(\overline{T}^{-1}(\alpha)\right), \quad \text{for } \alpha \in (0,1). \tag{19}$$

Remark that the hyperbolic transformation $\overline{T} = \mathcal{H}_{\widehat{\theta},\eta}$, in this paper are chosen in a way to be easily invertible. Then estimated smooth inverse transformation $\overline{T}^{-1}$ in (19) is straightforwardly obtained by changing the signs of some parameters of the hyperbolic composition (see Remark 1).

5 Numerical results on the rainfall real data

In the following we illustrate our methodology presented in the previous sections using a 5−dimensional rainfall data-set.

5.1 Presentation of the data-set

Data comes from the website *CISL Research Data Archive (RDA)*, http://rda.ucar.edu, and is available for registered users. *The user is granted the right to use the Site for non-commercial, non-profit research, or educational purposes only, without any fee or cost*, as specified in the terms of use of the website (January 2014). The name of the data in this website is: *ds570.0*, and the direct url to this data is http://rda.ucar.edu/datasets/ds570.0/. The whole citation information for this data is:

> *National Climatic Data Center/NESDIS/NOAA/U.S. Department of Commerce, Harvard College Observatory/Harvard University, Meteorology Department/Florida State University, and Climate Analysis Section/Climate and Global Dynamics Division/National Center for Atmospheric Research/University Corporation for Atmospheric Research (1981): World Monthly Surface Station Climatology. Research Data Archive at the National Center for Atmospheric Research, Computational and Information Systems Laboratory. Dataset. http://rda.ucar.edu/datasets/ds570.0. Accessed 08 nov 2013.*

We detail here data selection in order to help researchers to retrieve the same dataset. We have taken all monthly rainfall data available from CSV files for five chosen stations of India and Sri-Lanka. After a first importation step, only numerical or missing values of the field Precip(mm) were kept, and the field Date was considered as a numerical primary key in order to avoid repeated data. Considered stations and corresponding lines number after this step are:

- X1: Colombo (station Id: 434660, 1722 lines from 1870-01 to 2013-06, 0 lines excluded)
- X2: Pamban (station Id: 433630, 1470 lines from 1891-01 to 2013-06, 3 lines excluded)
- X3: Puttalam (station Id: 434240, 1734 lines from 1869-01 to 2013-06, 3 lines excluded)
- X4: Thiruvananthapuram (station Id: 433710, 1926 lines from 1853-01 to 2013-06, 8 lines excluded)
- X5: Trincomalee (station Id: 434180, 1734 lines from 1869-01 to 2013-06, 4 lines excluded)

In a second step, all five stations were grouped by date, and we have selected dates for which all fields Precip (mm) of the five station were non-missing and non-zero (the suppression of zero precipitation data aims at easing the parametric representation of margins, as detailed in Koning and Philip (2005)). At last precipitations have been expressed in decimetres: all fields Precip (mm) have been divided by 100. As a result, the data has 797 lines giving monthly precipitation in decimetres, for some dates in the period from 1893-01 to 2013-06. Excluded dates in this period are those for which at least one field Precip (mm) was missing, zero or non-numerical.

As one can see part of the data is very old and should certainly be interpreted with caution. We have drawn on Figure 2 the autocorrelation functions to show that a small autocorrelation is still remaining in considered data. Indeed as remarked in Reiss and Thomas (2007), on the one hand, considering annual data is a way to avoid the problems of serial correlation and seasonal variation. On the other hand, it may represent a lost of information contained in the data. A compromise can be to base the inference on seasonal or monthly maxima (see Reiss and Thomas (2007), Section 14.1). This is the approach followed in this paper. Geographical position of 5 stations and the scatter plot of ranks of data are provided in Figure 3.

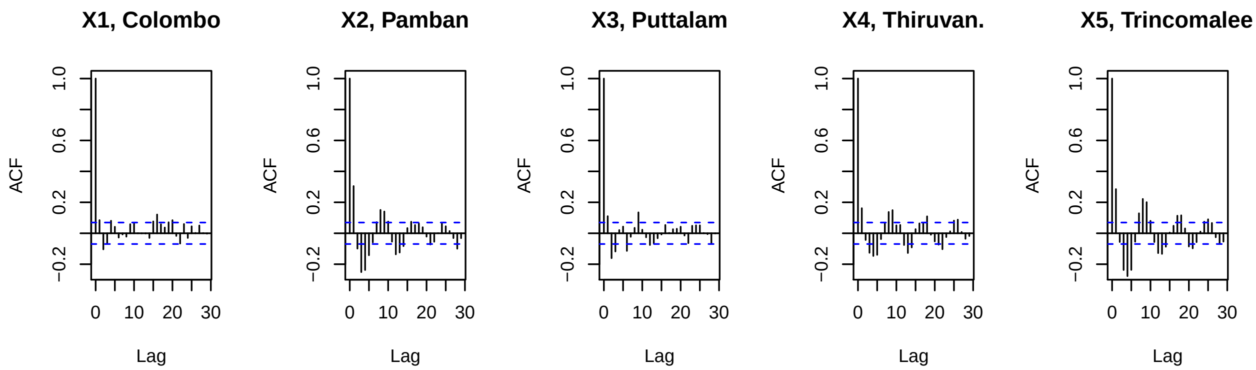

Figure 2: Estimates of the autocorrelation functions for the considered 5−dimensional rainfall data. Dotted horizontal lines give indicative 10% autocorrelation thresholds.

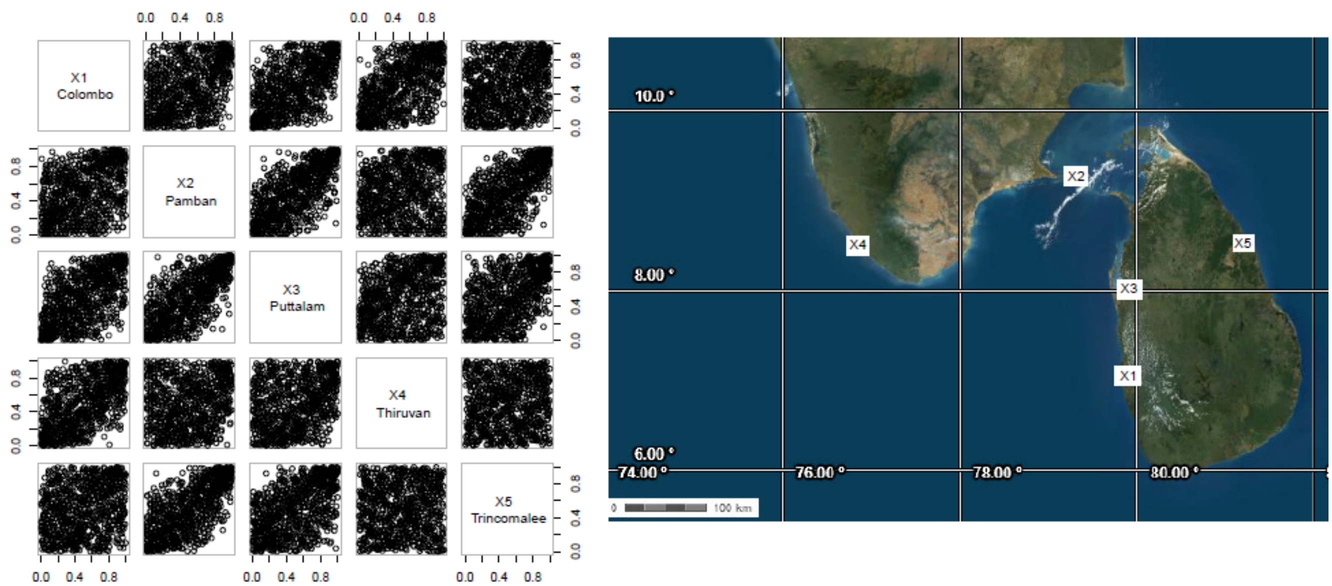

Figure 3: Left: Scatter plot of ranks for the considered 797 monthly rainfall measurements (in decimeter) in 5 stations of Sri-Lanka and India between January 1893 and June 2013. Right: Geographical positions of 5 considered stations.

Remark that the numerical illustrations presented here aims at showing the feasibility of the estimation but do not aim at furnishing a complete hydrological study, which would require more data treatments for handling seasonality, analysis of peak and durations of rainfall, etc. The interested reader is referred to Salvadori et al. (2011), Salvadori et al. (2012), Gräler et al. (2013).

5.2 Estimation results

We consider the model as in Equation (3). For the sake of clarity, each transformation is here involving only one hyperbola, thus requiring the choice of three quantiles thresholds per hyperbola for the estimation. We take as initial copula C_0 the independent one, and the initial margins $F_i(x) = 1 - e^{-x}$, $i \in I$. Concerning the impact of the choice of the initial copula C_0 on transformed generator $\widetilde{\phi}$, the interested reader is referred to Proposition 3.12 in Di Bernardino and Rullière (2013b).

For the estimation, we have chosen very small smoothing bandwidths h_i, $i \in I$ (see Equation (5)). Diagonal section of the copula and smooth empirical margins are thus very close to empirical ones, but small smoothing allows to get proper invertible functions. As stated in Algorithm 2, the estimation relies on an arbitrary chosen initial point (x_0, y_0), which corresponds to the arbitrary choice of one point of the Archimedean generator among its equivalence class (see Remarks 7 and 8 in Di Bernardino and Rullière (2013b)). The choice of this point has an impact on the estimation, and given an arbitrary abscissa $x_0 = 0.5$, we have kept the value of y_0 giving the best results, here $y_0 = 0.24$. The non-parametric estimation relies on the choice of quantile thresholds (see Algorithm 3). Here we have chosen:

$$\mathcal{Q} = \{5\%, 50\%, 95\%\} \quad \text{and} \quad \mathcal{Q}_i = \{20\%, 50\%, 80\%\}, \quad i \in I = \{1, \ldots, 5\}.$$

At last, for the parametric estimation (see Algorithm 4), one have to choose smoothing parameters. Selected smoothing parameters are:

$$\eta = -1 \quad \text{and} \quad \eta_i = -3, \quad i \in I = \{1, \ldots, 5\},$$

(see Table 1 below). It would be naturally possible to optimize all these parameters, $\mathcal{Q}$, $\mathcal{Q}_i$, η, η_i, for $i \in I$, but we have chosen here fixed threshold and smoothing parameters in order to show the feasibility of the estimation. This also shows that it is possible to get good fits with non optimized parameters, which is important when the dimension is high, since a global optimization procedure would have to face a curse of dimensionality.

As presented in Sections 2 and 3, we obtain the complete estimated vector of parameters $\boldsymbol{\Theta}$ given in Table 1. Then we get the transformed multivariate copula $\widetilde{C}$ and distribution function $\widetilde{F}_{\boldsymbol{\Theta}}$, obtained by Equations (16) and (18) respectively.

Parameters $\boldsymbol{\Theta}$	m	h	ρ_1	ρ_2	η
θ for external T	-0.576	0.576	-0.0566	-0.185	-1
θ for T_1	1.509	-0.089	-0.211	0.0624	-3
θ for T_2	0.532	0.888	0.216	0.244	-3
θ for T_3	0.921	0.499	-0.0057	-0.083	-3
θ for T_4	1.097	0.323	0.067	-0.001	-3
θ for T_5	1.147	0.274	-0.102	0.116	-3

Table 1: Complete estimated vector of parameters $\boldsymbol{\Theta}$.

We perform a goodness-of-fit test based on the empirical process in order to test the quality of the adjustment of copula $\widetilde{C}$ on these multivariate data. In the large scale Monte Carlo experiments carried out by Genest et al. (2009), the statistic S_n gave the best results overall (see Section 4 in Ivan Kojadinovic and Jun Yan (2010)). An approximate p-value for S_n can be obtained by means of a parametric bootstrap-based procedure (see Section 4.1 in Ivan Kojadinovic and Jun Yan (2010)), and whose validity was recently shown by Genest and Rémillard (2008).

In order to apply the bootstrap-based procedure, we need to generate random samples from the transformed copula $\widetilde{C}$ with generator as in (17). To this aim we use the Marshall and Olkin's algorithm with a numerical inversion of the Laplace Transform of generater $\widetilde{\phi}$ using the Talbot method. We obtain a $p-$value$= 0.37129$. Furthermore, we test different competitor copula families, with maximum likelihood estimated parameters. Obtained $p-$values, with different goodness-of-fit tests, are gathered in Table 2. Therefore, among all the copula families that we have tested, the transformed copula $\widetilde{C}$ is the only one that is not rejected at the 5% significance level (see Ivan Kojadinovic and Jun Yan (2010)).
To appreciate the quality of multivariate parametric adjustments we evaluate the Supremum Absolute Error on a lattice G (see (20)) and the Mean Absolute Error on the data (see (21)), respectively defined as:

Copula under H_0	S_n	S_n^B	S_n^C	A_n
Gumbel–Hougaard	0.00331	0.00495	0.00454	0.03465
Clayton	0.00381	0.00980	0.00704	0.00981
Frank	0.00617	0.00941	0.00819	0.08416
t-Student	0.00495	0.00592	0.00498	0.00963
Normal	0.00980	0.00719	0.00454	0.00205
Joe	0.00819	0.00495	0.00454	0.00916

Table 2: The bootstrapped $p-$values for different goodness-of-fit tests (see Genest et al. (2009)) for competitor copula families on the considered 5-dimensional rainfall data, with $n = 797$. In all cases, the number of Monte Carlo experiments is fixed at $N = 1000$.

$$\text{SAE} = \sup_{(x_1,x_2,x_3,x_4,x_5)\in G} |F(x_1, x_2, x_3, x_4, x_5) - F_n(x_1, x_2, x_3, x_4, x_5)|, \tag{20}$$

$$\text{MAE} = \frac{1}{n}\sum_{k=1}^{n} \left|F(X_1^{(k)}, X_2^{(k)}, X_3^{(k)}, X_4^{(k)}, X_5^{(k)}) - F_n(X_1^{(k)}, X_2^{(k)}, X_3^{(k)}, X_4^{(k)}, X_5^{(k)})\right|, \tag{21}$$

with $n = 797$, F_n the 5−dimensional empirical distribution function and F a parametric model on the multivariate rainfall data. The first SAE criterion is a classical Kolmogorov-Smirnov statistic evaluated on a particular lattice G, and the same choice of an absolute value (L_1-norm) has been done for the second MAE criterion in order to use comparable norms, and to get results in accordance with further Figures 4-6 showing absolute differences. This second criterion can be linked to L_1-variant Cramér-von-Mises distances like $\int |F(\mathbf{x}) - F_n(\mathbf{x})|\, dF(\mathbf{x})$, where the continuous measure $dF(\mathbf{x})$ has been replaced by the empirical one $dF_n(\mathbf{x})$ thus leading to a sum. This way, the MAE criterion can be efficiently computed without integration on a multivariate domain, while the difference between the sum and the integral can be bounded using the difference $|F(\mathbf{x}) - F_n(\mathbf{x})|$.

We consider our transformed model (see (18)) and three different parametric models using Frank, Gumbel and Clayton copulas and parametric marginals. The dependence parameters of copulas, fitted by Maximum likelihood, are: $\theta = 2.45$ (Frank copula), Gumbel $\theta = 1.33$ (Gumbel copula) and $\theta = 0.47$ (Clayton copula). For marginals we have tested 15 different classes of distributions and we have fitted the best model using the Akaike Information Criterion. Following this criterion, the best parametric marginals were Gamma distributed. We recall here that zero valued precipitations have been excluded from this data. Results are gathered in Table 3.

Gamma distributions	X_1	X_2	X_3	X_4	X_5
shape	1.7553	0.6344	0.9600	1.0373	1.0116
rate	0.7994	0.5968	0.7844	0.6724	0.5799

Table 3: Shape and rate MLE parameters of fitted marginal Gamma distributions for X_i for $i = 1, \ldots, 5$.

Then the obtained Supremum Absolute Error in (20) and Mean Absolute Errors in (21) are given in Table 4. As one can see, our model performs better both on the Supremum Absolute Error (SAE) and the Mean Absolute Error (MAE) criteria. However, quantifying the goodness of fit on a multivariate data is a difficult problem. Some models with small Mean Absolute Error criteria may behave poorly when considering a specific projection of the 5-dimensional space. A control of the performance of the model for the distribution fit of each pair of random variable is highly recommended.

Using parametric multivariate models introduced above, we now consider the fit of the bivariate distributions $F_{(X_i,X_j)}$ and of the marginals F_{X_i}, for $i, j = 1, \ldots, 5$. They are particular projections of the 5−dimensional

Models	$\widetilde{F}$	Frank	Gumbel	Clayton
SAE	**0.0791**	0.0941	0.0995	0.1367
MAE	**0.0095**	0.0138	0.0169	0.0236

Table 4: Supremum Absolute Errors (SAE) and Mean Absolute Errors (MAE) as in (20)-(21) for the considered parametric 5−dimensional models. Best results are indicated in bold font.

distribution. We consider the errors in (20)-(21) for (X_i, X_j) data, for $i, j = 1, \ldots, 5$, i.e.,

$$\mathrm{SAE}_{i,j} = \sup_{(x,y)\in G} \left| F_{(X_i,X_j)}(x,y) - F_n(x,y) \right|, \tag{22}$$

$$\mathrm{MAE}_{i,j} = \frac{1}{n}\sum_{k=1}^{n} \left| F_{(X_i,X_j)}(X_i^{(k)}, X_j^{(k)}) - F_n(X_i^{(k)}, X_j^{(k)}) \right|, \tag{23}$$

with F_n the bivariate empirical distribution function and $F_{(X_i,X_j)}$ the projection of the multivariate parametric model on (X_i, X_j). Errors in (22)-(23), evaluated using our transformed model $\widetilde{F}$ and classical parametric models on the considered rainfall data, are gathered in Tables 5 and 6.

Errors model $\widetilde{F}$					
	X_1	X_2	X_3	X_4	X_5
X_1	**0.0110** **0.0052**				
X_2	**0.0306** **0.0088**	**0.0157** **0.0078**			
X_3	**0.0316** **0.0138**	**0.0677** **0.0327**	**0.0247** **0.0076**		
X_4	**0.0328** **0.0117**	0.0643 0.0247	0.0536 0.0160	**0.0193** **0.0060**	
X_5	**0.0437** 0.0155	**0.0643** **0.0296**	0.0356 0.0123	0.0867 0.0318	0.0299 0.0115

Errors Frank copula and Gamma marginals					
	X_1	X_2	X_3	X_4	X_5
X_1	0.0119 0.0053				
X_2	0.0410 0.0195	0.0389 0.0225			
X_3	0.0363 0.0147	0.0742 0.0362	0.0351 0.0175		
X_4	0.0396 0.0137	0.0567 0.0180	0.0672 0.0205	0.0282 0.0129	
X_5	0.0493 0.0174	0.0838 0.0356	**0.0344** **0.0101**	0.0882 0.0323	**0.0246** **0.0055**

Table 5: Left: Supremum Absolute Errors $\mathrm{SAE}_{i,j}$ in (22) (first lines) and Mean Absolute Errors $\mathrm{MAE}_{i,j}$ in (23) (second lines) using the transformed model $\widetilde{F}_5$ with parameters as in Table 1. Right: Errors using parametric model with Frank copula $\theta = 2.450$ and Gamma marginals with parameters as in Table 3. Best results are indicated in bold font.

Errors Gumbel copula and Gamma marginals					
	X_1	X_2	X_3	X_4	X_5
X_1	0.0119 0.0053				
X_2	0.0406 0.0121	0.0389 0.0225			
X_3	0.0401 0.0168	0.0820 0.0373	0.0351 0.0175		
X_4	0.0462 0.0170	0.0521 0.0172	0.0621 0.0195	0.0282 0.0129	
X_5	0.0444 0.0161	0.0880 0.0379	0.0352 0.0102	0.0798 0.0309	**0.0246** **0.0055**

Errors Clayton copula and Gamma marginals					
	X_1	X_2	X_3	X_4	X_5
X_1	0.0119 0.0053				
X_2	0.0482 0.0158	0.0389 0.0225			
X_3	0.0621 0.0229	0.0959 0.0448	0.0351 0.0175		
X_4	0.0583 0.0221	**0.0403** **0.0119**	**0.0429** **0.0144**	0.0282 0.0129	
X_5	0.0456 **0.0114**	0.1104 0.0437	0.0534 0.0187	**0.0628** **0.0237**	**0.0246** **0.0055**

Table 6: Supremum Absolute Errors $\mathrm{SAE}_{i,j}$ in (22) (first lines) and Mean Absolute Errors $\mathrm{MAE}_{i,j}$ in (23) (second lines) using parametric model using Gumbel copula $\theta = 1.33$ (left), and Clayton copula $\theta = 0.47$ (right) and Gamma marginals with parameters as in Table 3. Best results are indicated in bold font.

Model	$\widetilde{F}$	Frank	Gumbel	Clayton
SSAE	**0.086**	0.088	0.088	0.110
MSAE	**0.011**	0.012	0.012	0.012
SMAE	**0.032**	0.036	0.039	0.045
MMAE	**0.004**	0.005	0.005	0.005

Table 7: Syntectic statistics associated to Tables 5 and 6, where SSAE= $\sup_{ij}$ SAE$_{ij}$, SMAE= $\sup_{ij}$ MAE$_{ij}$, MSAE= mean$_{ij}$SAE$_{ij}$ and MMAE= mean$_{ij}$MAE$_{ij}$.

Remark that, without any optimization procedure, the transformed model performs better in terms of errors in (22)-(23) for almost all the couples (X_i, X_j). In order to make the reading of the Tables 5 and 6 easier, we give in Table 7 the associated synthetic statistics. As we can seen the best values (displayed in bold font) are provided by the transformed model $\widetilde{F}$.
As remarked above, some graphical illustrations of Tables 5 and 6 are provided in Figures 4 for (X_1, X_4), Figure 5 for (X_2, X_5) and Figure 6 for (X_3, X_4). The maximal range for these figures corresponds to the 95th percentile of each random variable, in order to focus on main part of the data and to preserve the readability of each figure.

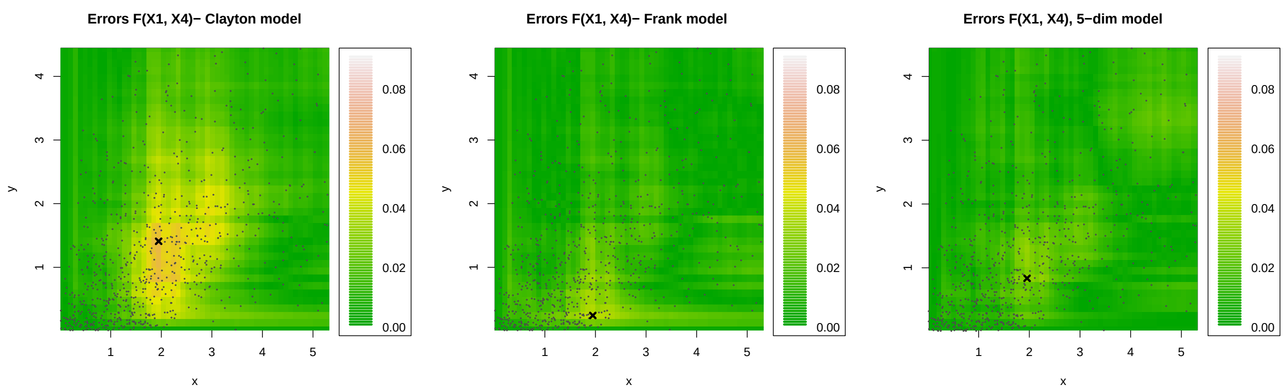

Figure 4: Errors $|F_{(X_1,X_4)}(x,y) - F_n(x,y)|$, for (x,y) in a lattice of 100×100 points, where F_n is the empirical distribution function and $F_{(X_1,X_4)}$ is parametric model with Gamma marginals and Clayton copula (left), Frank copula (center). Errors of our transformed model $\widetilde{F}$ are displayed in the right plot. Black cross represents the maximum error $SAE_{1,4}$ in the considered lattice (see (22)). Black dots represent the associated rainfall data (X_1, X_4).

Furthermore, from (4), we get $\tilde{F}_i = T \circ T_i^{-1} \circ F_i$, for $i \in I$. Then, using the smooth estimation of external and internal transformations $\overline{T}$ and $\overline{T}_i^{-1}$, for $i \in I$, one can obtain the transformed parametric marginal distributions. Results for some margins are presented in Figure 7 below (results for other margins are completely analogous).
Finally we consider another projections of $\widetilde{F}$, i.e. the $5-$dimensional diagonal. In Figure 8 (left) we present the parametric estimation of the $5-$dimensional diagonal using the transformed model and the classical parametric competitors introduced above.

In Figure 8 (right) we present the Kendall distribution for the considered 5-dimensional rainfall data-set using our transformed generator $\widetilde{\phi}(t) = T \circ \exp(-t)$. We also display $K_C(\alpha)$, for Gumbel, Frank and Clayton copulas.

6 A nested model

6.1 Choice of clusters

In this section we intend to show the flexibility of the proposed model and associated estimation procedure. In particular, we adapt our methodology in the case of some asymmetric dependencies (as, for instance, non-exchangeable random vectors). The correlation matrix of the considered rainfall data is displayed in Figure 9 (left).

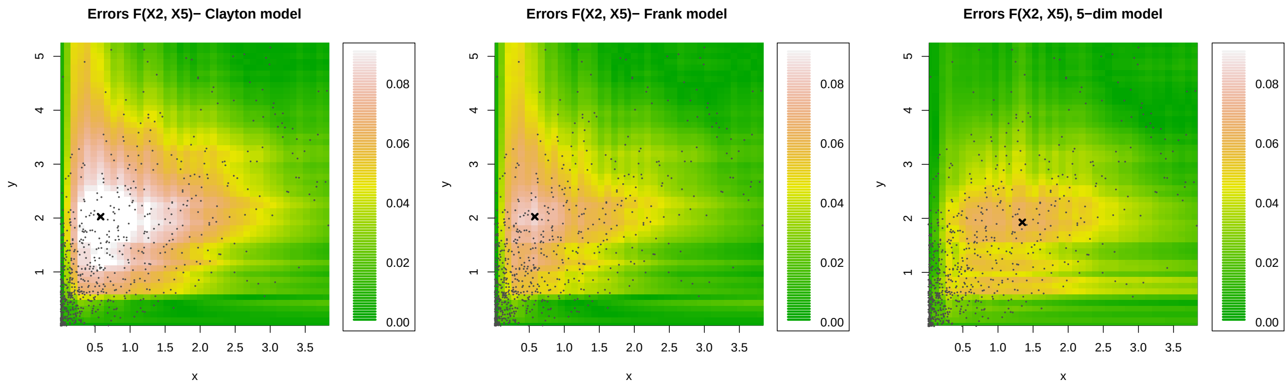

Figure 5: Errors $|F_{(X_2,X_5)}(x, y) - F_n(x, y)|$, for (x, y) in a lattice of 100×100 points, where F_n is the empirical distribution function and $F_{(X_2,X_5)}$ is parametric model with Gamma marginals and Clayton copula (left), Frank copula (center). Errors of our transformed model $\widetilde{F}$ are displayed in the right plot. Black cross represents the maximum error $SAE_{2,5}$ in the considered lattice (see (22)). Black dots represent the associated rainfall data (X_2, X_5).

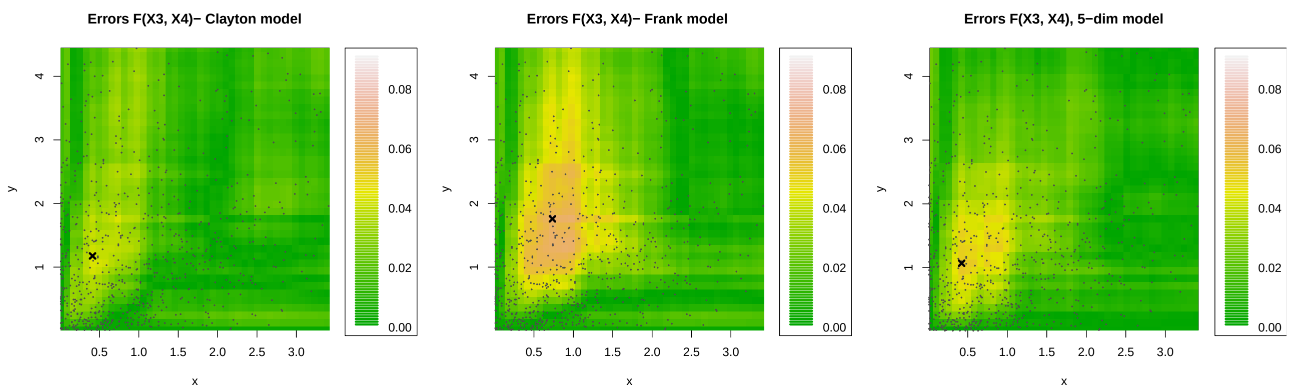

Figure 6: Errors $|F_{(X_3,X_4)}(x, y) - F_n(x, y)|$, for (x, y) in a lattice of 100×100 points, where F_n is the empirical distribution function and $F_{(X_3,X_4)}$ is parametric model with Gamma marginals and Clayton copula (left), Frank copula (center). Errors of our transformed model $\widetilde{F}$ are displayed in the right plot. Black cross represents the maximum error $SAE_{3,4}$ in the considered lattice (see (22)). Black dots represent the associated rainfall data (X_3, X_4).

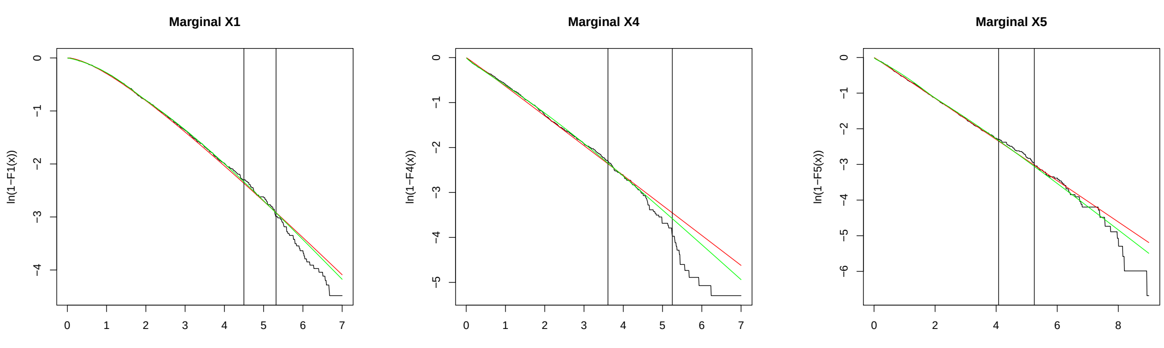

Figure 7: Fit of marginals. Black full lines are the empirical marginal tail distribution functions (in log scale). The vertical lines show estimates of 90% and 95% univariate marginal quantiles. Red lines the Gamma distribution with parameters as in Table 3. Green lines are $\ln(1 - \widetilde{F}_i)$, for $i = 1, 4, 5$ as in the 5−dimensional transformed model with parameters as in Table 1.

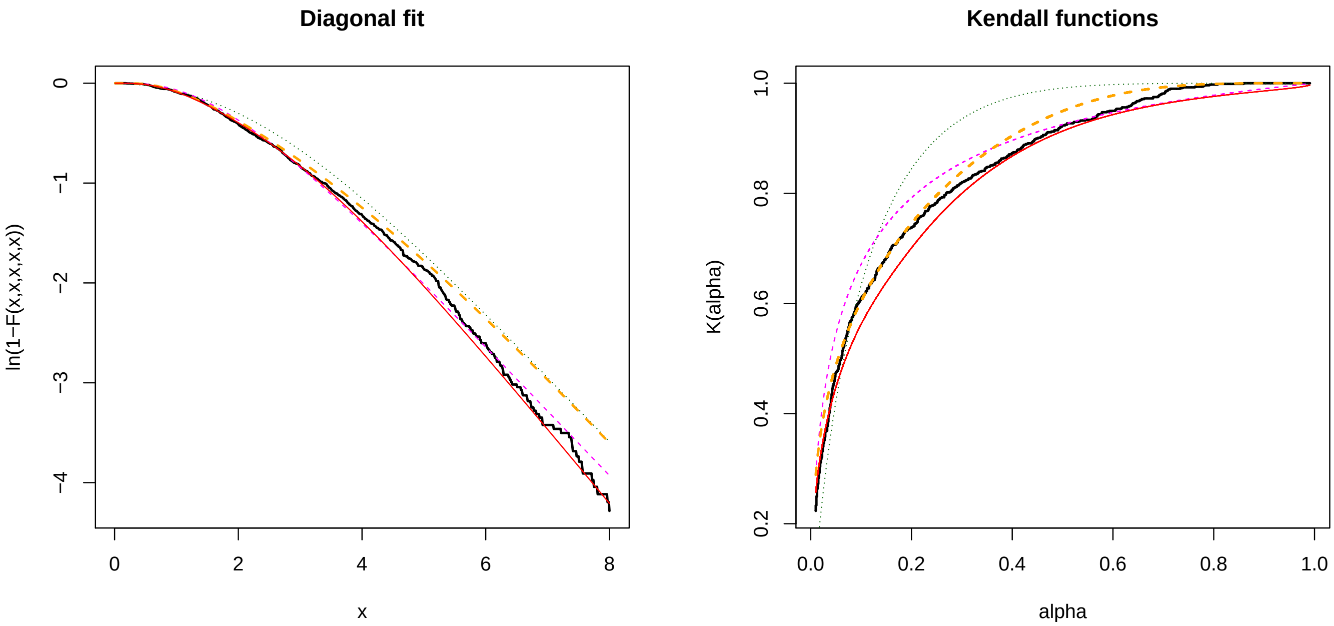

Figure 8: Left: Estimation of the 5-dimensional survival diagonal in logarithmic scale. Right: Kendall distribution function for the 5-dimensional rainfall data. Empirical diagonal and empirical Kendall (as in Barbe et al. (1996)) are presented in black thick line; diagonal of the transformed model $\widetilde{F}$ and associated $K_{\widetilde{\phi}}$ with $\widetilde{\phi} = \overline{T} \circ \exp(-t)$ in full red line; diagonal of parametric Gumbel model and $K_{\phi_{Gumbel}}$ with $\phi_{Gumbel}(t) = \exp(-t^{1/1.33})$ in magenta dashed line; diagonal of parametric Frank model and $K_{\phi_{Frank}}$ with $\phi_{Frank}(t) = -(\log(\exp(-x)(\exp(-2.45) - 1) + 1))/2.45$ in orange thick dashed line; diagonal of parametric Clayton model and $K_{\phi_{Clayton}}$ where $\phi_{Clayton}(t) = (1 + t)^{-1/0.47}$ in green dotted line.

	X_1	X_2	X_3	X_4	X_5
X_1	**1**	0.377	0.552	0.504	0.207
X_2	0.377	**1**	**0.698**	0.178	**0.660**
X_3	0.552	**0.698**	**1**	0.276	0.523
X_4	0.504	0.178	0.276	**1**	0.018
X_5	0.207	**0.660**	0.523	0.018	**1**

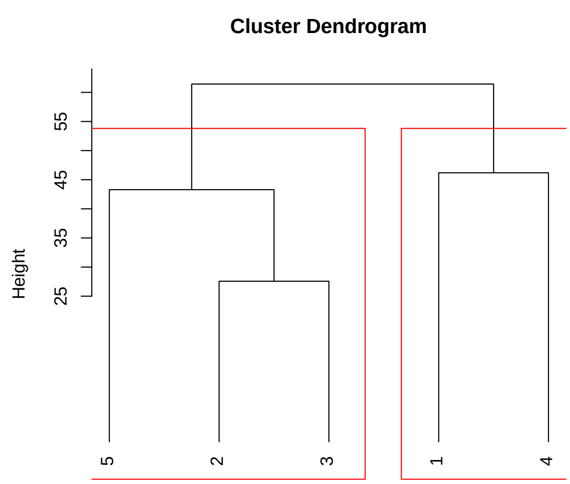

Figure 9: Left: Correlation matrix of the considered rainfall data. Correlations greater than 60% are indicated in bold font. Right: Dendrogram resulting to the hierarchical cluster analysis on the set of dissimilarities produced by the Euclidian distance on the rainfall data. Red boxes show the two considered clusters.

As we can see, some pairs of stations present a bigger correlation. To illustrate how our model can be adapted to this situation we have decided to create 2 different clusters. We have grouped together pairs of variables presenting correlation greater than 66%, this leads to a first (tri-variate) cluster composed by stations (X_2, X_3, X_5). The remaining second (bivariate) cluster is (X_1, X_4). One can check that all correlations inside each cluster are greater than 50%. Figure 3 gives the geographical position of each station and helps visualizing each cluster. Furthermore an hierarchical cluster analysis on the set of dissimilarities produce by the distance of the X_i is developed. We use different types of distance to create the dissimilarities (Euclidian, maximum, Manhattan, Canberra, Binary, Minkowski). In all cases we obtain the result in Figure 9. As one can see, whatever the distance chosen for dissimilarities, the dendrogram gives a justification to chosen clusters of station indexes $\{2, 3, 5\}$ and $\{1, 4\}$. However, chosen nested structure has not been optimized; the choice of an optimal nested structure is analyzed for instance in Segers and Uyttendaele (2014).

Then, following these considerations, we firstly fit a 3−dimensional model for the first group and a 2−dimensional one for the second one. We generate the pseudo-data coming from these two models and finally we construct the joint copula for these bivariate data-set. As in Section 5, in the following we take as initial copula C_0 the independent one, and the initial margins $F_i(x) = 1 - \mathrm{e}^{-x}$, $i \in A, B$.

The multivariate distribution for the cluster $A = \{2, 3, 5\}$ is assumed to be written:

$$\begin{aligned} F_A(x_2, x_3, x_5) &= T_A \circ C_0(T_A^{-1} \circ \widetilde{F}_2(x_2),\, T_A^{-1} \circ \widetilde{F}_3(x_3),\, T_A^{-1} \circ \widetilde{F}_5(x_5)), \\ \text{with } \tilde{F}_i &= T_A \circ T_{A_i}^{-1} \circ F_i\,, \quad \text{for } i \in A. \end{aligned} \tag{24}$$

The multivariate distribution for the cluster $B = \{1, 4\}$ is assumed to be written:

$$\begin{aligned} F_B(x_1, x_4) &= T_B \circ C_0(T_B^{-1} \circ \widetilde{F}_1(x_1),\, T_B^{-1} \circ \widetilde{F}_4(x_4)), \\ \text{with } \tilde{F}_i &= T_B \circ T_{B_i}^{-1} \circ F_i\,, \quad \text{for } i \in B. \end{aligned} \tag{25}$$

The whole 5−dimensional distribution is assumed to be written:

$$\widetilde{F}(x_1, x_2, x_3, x_4, x_5) = T \circ C_0 \left(T^{-1} \circ F_A(x_2, x_3, x_5),\, T^{-1} \circ F_B(x_1, x_4)\right), \tag{26}$$

where $\tilde{C}(u, v) = T \circ C_0 \left(T^{-1}(u),\, T^{-1}(v)\right)$ is referred as the *root copula* at point (u, v). It is effectively a proper copula if the transformation T satisfies admissibility conditions that are given in Proposition 2.1 of Di Bernardino and Rullière (2013b), in order to satisfy 2−monotony as detailed in McNeil and Nešlehová (2009). Model in (26) corresponds to a *Nested Archimedean Copula model*, with two nested levels, as described in Hofert and Pham (2013). In this article authors give also conditions such that the resulting nested copula is a proper distribution function. Despite it may not be the case in general, we will assume that $\widetilde{F}$ with expression as in (26) is a proper multivariate distribution. In the following we will check the admissibility for considered transformation (see Figure 12).

As one will see in Figure 13, conditions like the admissibility of used copulas and of the final nested distribution should be carefully checked. Here the copula $\tilde{C}$ is used without uniform margins to create a multivariate distribution when the final nested density remains positive on the whole domain. Other constructions ensuring this admissibility could be investigated, like the use of Hierarchical Kendall Copula, as in Brechmann (2014). However such supplementary investigations are beyond the scope of this paper.

Remark that in the model presented here, for the sake of simplicity, the root copula $\widetilde{C}$ in (26) has only two arguments (two child copulas). The methodology detailed here is however applicable with more arguments. The hierarchical copula detailed here is also a two-step hierarchical copula, with only one imbrication level, but the methodology can be extended to more levels. The interest reader is referred to Hofert and Pham (2013) or Brechmann (2014).

6.2 Estimation results

The parameters of each cluster copula transformation T_A and T_B in (24) and (25) will be respectively denoted by θ_A and θ_B. At last, parameter of marginal transformation T_{A_i} will be denoted θ_{A_i}, for $i \in A$, and parameter of marginal transformation T_{B_i} will be denoted θ_{B_i}, for $i \in B$. The obtained results are gathered in tables below.

Parameters F_A in (24)	m	h	ρ_1	ρ_2	η
θ_A	-2.418	-0.168	-0.831	-0.517	-2
θ_{A_2}	-0.262	0.747	-0.136	-0.135	-4
θ_{A_3}	1.711	0.576	-0.389	-0.315	-4
θ_{A_5}	2.061	0.300	-0.401	-0.104	-4

Parameters F_B in (25)	m	h	ρ_1	ρ_2	η
θ_B	0.168	0.776	-0.168	-0.271	-2
θ_{B_1}	1.823	0.094	-0.277	-0.003	-4
θ_{B_4}	-0.449	0.491	0.032	-0.029	-4

To estimate the external transformation T of model (26) we firstly construct a bivariate pseudo data-set:

$$\begin{aligned} Z_1 &= F_A(X_1, X_4), \\ Z_2 &= F_B(X_2, X_3, X_5). \end{aligned}$$

Then we fit on this bivariate data-set a model

$$\begin{aligned} \widetilde{F}_{(Z_1,Z_2)}(z_1, z_2) &= T \circ C_0(T^{-1} \circ \tilde{F}_1(z_1), T^{-1} \circ \tilde{F}_2(z_2)), \\ \text{with } \tilde{F}_i &= T \circ T_1^{-1} \circ F_i\,, \quad \text{for } i = 1, 2. \end{aligned} \tag{27}$$

The parameter of transformation T in (27) will be denoted θ, parameters of transformation T_1 and T_2 will be respectively denoted by θ_1 and θ_2. The obtained values are gathered in table below.

Parameters $\widetilde{F}_{(Z_1,Z_2)}$ in (27)	m	h	ρ_1	ρ_2	η
θ	-1.225	0.126	-0.243	-0.150	-2
θ_1	0.401	0.682	-0.055	-0.384	-4
θ_2	0.478	0.864	0.081	-0.250	-4

Both data-set (Z_1, Z_2) and a graphical illustration of the fit of model (27) on this data using the estimated parameters above, are given in Figure 10.

We evaluate the Mean Absolute Error in (21) for final model in (26) and we obtain MAE = **0.0088** and SAE = **0.0636**. This value is smaller than all values in Table 4. Furthermore, as in Tables 5 and 6, we consider the fit of the bivariate distributions $F_{(X_i,X_j)}$ and of the marginals F_{X_i}, for $i, j = 1, \ldots, 5$. To this aim we evaluate the $\text{SAE}_{i,j}$ and $\text{MAE}_{i,j}$ errors in (22)-(23) using the nested model in (26). Results are gathered in Table 8. Remark that errors in Table 8 are smaller than all values previously obtained in Tables 5-6. Analogously to Figures 4-6, and for the same couples of random variables, graphical illustrations of Table 8 are provided in Figure 11.
As one can see in Table 8, results with this nested model are very good. In particular, one can see in Figure 11 and in Table 8 that absolute errors in every bi-dimensional projection remain very small. This model is however a simple illustration to feasible improvements of the initial model, when the dimension is greater than 2, without using heavy optimization algorithms.

As remarked above, one have to ensure that the Equations (24), (25) and (26) are proper distribution functions. We verify here the admissibility for external transformations T, T_A and T_B from Proposition 2.1 in Di Bernardino and Rullière (2013b).

Consider an external transformation τ, as T_A, T_B or T in respective Equations (24), (25) and (26). Denote by d_τ the dimension of this external transformation, and $\tau^{(i)}$ the $i-$th derivative of the function τ, $i = 1, \ldots, d_\tau$. For

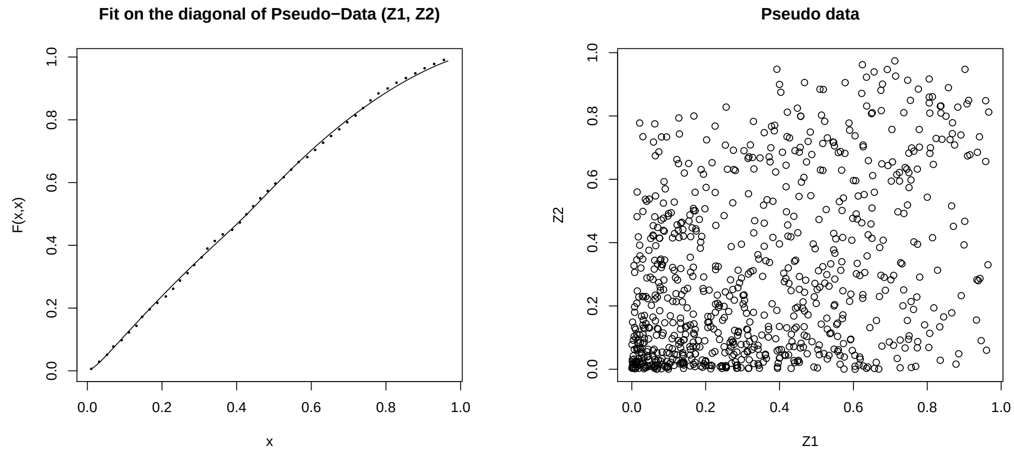

Figure 10: Left: Adjustment on the diagonal section for the pseudo-data $(Z_1,\, Z_2)$; empirical estimation of the diagonal $\widehat{F}_{n,(Z_1,\,Z_2)}(x,x)$ (dotted line) versus our parametrical model (full line). Right: Plot of the pseudo-data $(Z_1,\, Z_2)$.

Nested model $\widetilde{F}$ in (26) with optimized thresholds					
	X_1	X_2	X_3	X_4	X_5
X_1	**0.0088** **0.0039**				
X_2	**0.0308** **0.0080**	**0.0146** **0.0051**			
X_3	**0.0305** **0.0114**	**0.0320** **0.0066**	**0.0209** **0.0058**		
X_4	**0.0245** **0.0045**	**0.0393** **0.0100**	**0.0351** **0.0100**	**0.0139** **0.0053**	
X_5	**0.0342** **0.0071**	**0.0376** **0.0079**	**0.0342** **0.0103**	**0.0619** **0.0204**	**0.0237** **0.0041**

Table 8: Supremum Absolute Errors $\mathrm{SAE}_{i,j}$ (first lines) and Mean Absolute Errors $\mathrm{MAE}_{i,j}$ (second lines) using the nested model $\widetilde{F}$ in (26) with parameters as in table above. Better results than those in Tables 5 and 6 are indicated in bold font.

any considered external transformation τ, the transformation is admissible if and only if each quantity $f_{\tau,i}(x)$ is nonnegative for $x \in (0,1)$ and for each $i = 1,\ldots,d_\tau$, where :

$$\left\{\begin{array}{lcl} f_{\tau,1}(x) & = & \tau^{(1)}(x) \\ f_{\tau,2}(x) & = & \tau^{(1)}(x) + x\,\tau^{(2)}(x), \\ f_{\tau,3}(x) & = & \tau^{(1)}(x) + 3\,x\,\tau^{(2)}(x) + x^2\tau^{(3)}(x). \end{array}\right.$$

In Figure 12 we have drawn, for $x \in (0,1)$,

$$m_\tau = \min\left\{\ln f_{\tau,i}(x),\ i = 1,\ldots,d_\tau\right\}, \tag{28}$$

for $\tau \equiv T$ in (26) and $d_\tau = 2$ (left), for $\tau \equiv T_B$ in (25) and $d_\tau = 2$ (center), for $\tau \equiv T_A$ in (24) and $d_\tau = 3$ (right).

When functions $f_{\tau,i}$ are continuous, $i \le d_\tau$, their logarithm tends to $-\infty$ before $f_{\tau,i}$ becomes negative, and a lower bound of the logarithm ensure that m_τ is well-defined for $x \in (0,1)$, so that the quantity m_τ helps checking the admissibility of τ, especially for very small values of $f_{\tau,i}$. As one can see in Figure 12, for each transformation T, T_B, T_A, the quantity in (28) is well-defined for each $x \in (0,1)$ and inferiorly bounded. Then, we can deduce the admissibility for considered external transformations T, T_B and T_A.

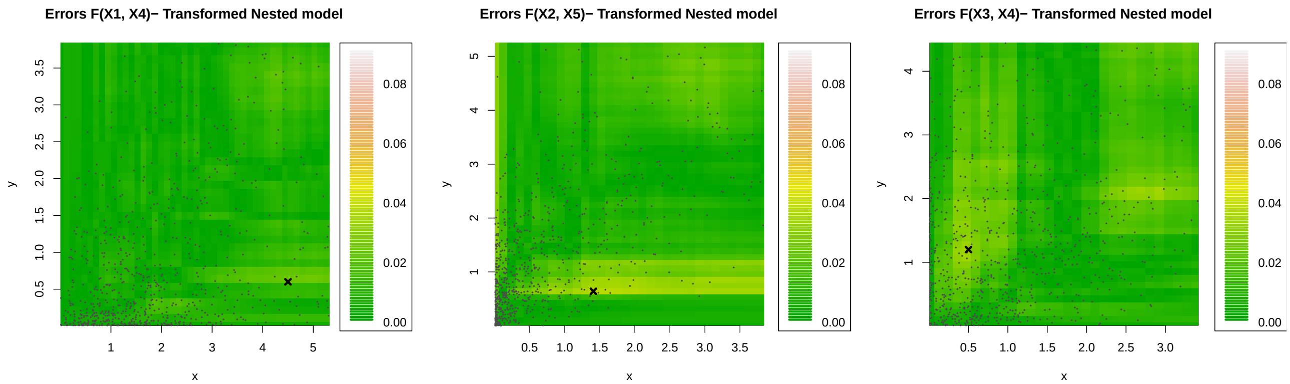

Figure 11: Errors $|F_{(X_i,X_j)}(x,y) - F_n(x,y)|$, for (x,y) in a lattice of 100×100 points, where F_n is the empirical distribution function and $F_{(X_i,X_j)}$ is the parametric nested model in (26) for (i,j)= (1, 4) (left), (i,j)= (2, 5) (center), (i,j)= (3, 4) (right). Black cross represents the maximum error $SAE_{i,j}$ in the considered lattice (see (22)). Black dots represent the associated rainfall data (X_i, X_j).

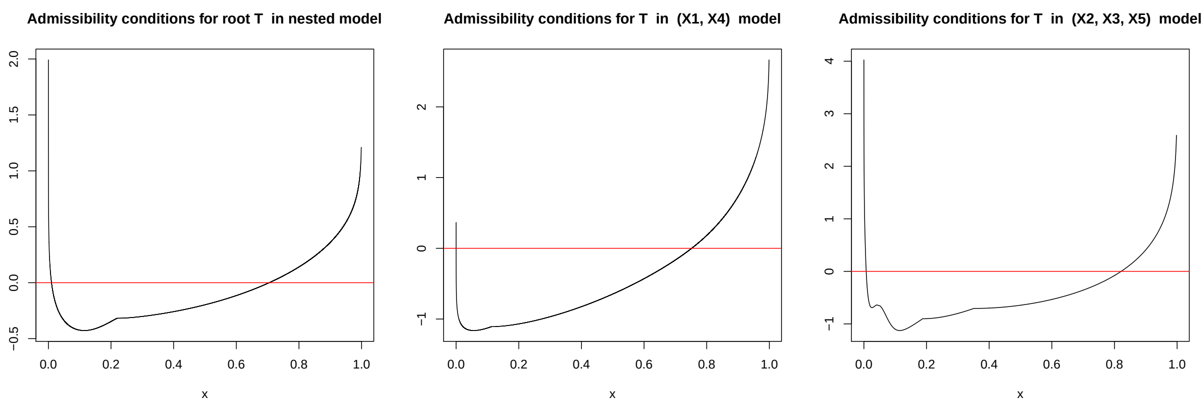

Figure 12: Admissibility conditions in (28) for transformations T in (26) (left) and T_B in (25) (center) and T_A in (24) (right).

To illustrate the danger to use a single criterion, and the need to check admissibility conditions, we propose in Figure 13 an illustration of situations that can happen with deliberately non admissible transformations. In the left panel of the Figure 13, we have drawn the function m_τ in (28) for a typical non-admissible external transformation τ (in this example associated to the cluster A, see (24)). What is noticeable is that this non-admissible transformation is however leading to a good value of global MAE and SAE criteria for the final nested model in (26), MAE $= 0.0057$ and SAE $= 0.0654$. These error values have to be compared with a MAE $= 0.0088$ and SAE $= 0.0636$ previously obtained using admissible transformations.

In the right panel of Figure 13, we have drawn errors $|F_{(X_3,X_4)}(x,y) - F_n(x,y)|$, for (x,y) in a lattice of 100×100 points, where F_n is the empirical distribution function and $F_{(X_3,X_4)}$ is the parametric nested model in (26) using this non-admissible external transformation τ. The maximum error is $\mathrm{SAE}_{3,4} = 0.0906$ in the considered lattice (at the black cross point). Furthermore $\mathrm{MAE}_{3,4} = 0.0201$. These values are larger then associated values in Table 8, i.e., $\mathrm{SAE}_{3,4} = 0.0351$ and $\mathrm{MAE}_{3,4} = 0.0100$. Despite good global criteria MAE and SAE, admissibility conditions are not fulfilled, and projected criteria $\mathrm{MAE}_{3,4}$ and $\mathrm{SAE}_{3,4}$ are disappointing. This shows that projected criteria may behave differently than global criteria, and that the admissibility conditions have to be checked carefully to avoid undesirable behavior of the nested adjustment.

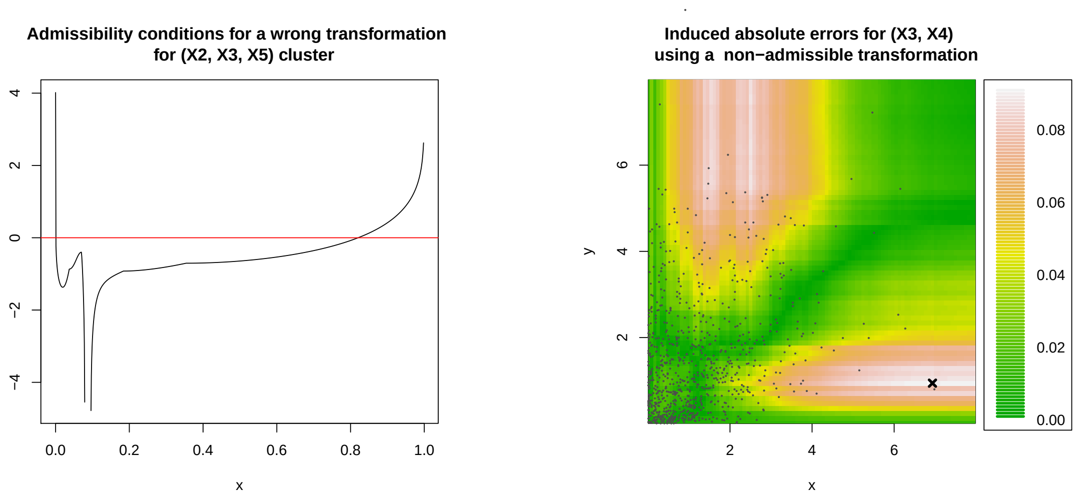

Figure 13: Left: Function m_τ in (28) for a typical non-admissible external transformation τ. This non-admissible transformation τ corresponds here to a non-admissible external transformation T_A in (24) however leading to a good value of global MAE and SAE criteria for the final nested model in (26), MAE $= 0.0057$ and SAE $= 0.0654$. Right: Errors $|F_{(X_3,X_4)}(x,y) - F_n(x,y)|$, for (x,y) in a lattice of 100×100 points, where F_n is the empirical distribution function and $F_{(X_3,X_4)}$ is the parametric nested model in (26) using the non-admissible external transformation τ for the transformation T_A. Black cross represents the maximum error $\text{SAE}_{3,4} = 0.0906$ in the considered lattice. Black dots represent the associated rainfall data (X_3, X_4).

6.3 Critical Layers for nested model

Let $\alpha \in (0,1)$ be a targeted level for a critical layer. Let C_0 be the initial copula to be transformed, and assume that C_0 is the independent copula.

The analytical critical layers of the distributions F_B and F_A are easy to obtain. For F_B in (25), we have

$$\begin{aligned}
\partial L_B(\alpha) &= \{(x_1,x_4) : F_B(x_1,x_4) = \alpha\} \\
&= \left\{(x_1,x_4) : T_B \circ C_0(T_B^{-1} \circ \widetilde{F}_1(x_1), T_B^{-1} \circ \widetilde{F}_4(x_4)) = \alpha\right\} \\
&= \left\{(x_1,x_4) : T_B^{-1} \circ \widetilde{F}_1(x_1) \cdot T_B^{-1} \circ \widetilde{F}_4(x_4) = T_B^{-1}(\alpha)\right\}.
\end{aligned}$$

Choosing p such that $T_B^{-1} \circ \widetilde{F}_1(x_1) = (T_B^{-1}(\alpha))^p$, one gets $T_B^{-1} \circ \widetilde{F}_4(x_4) = (T_B^{-1}(\alpha))^{1-p}$. Finally,

$$\partial L_B(\alpha) = \left\{(x_1,x_4) : x_1 = \widetilde{F}_1^{-1} \circ T_B\left((T_B^{-1}(\alpha))^p\right), x_4 = \widetilde{F}_4^{-1} \circ T_B\left((T_B^{-1}(\alpha))^{1-p}\right), p \in (0,1)\right\}.$$

Analytical expressions of the inverse of any transformed margins $\widetilde{F}_i = T \circ T_i^{-1} \circ F_i$, for $i \in B$, are available since inverse transformations are given and since the initial distribution F_i is chosen to be readily invertible.

Analogously, we get, for F_A in (24)

$$\begin{aligned}
\partial L_A(\alpha) = \{(x_2,x_3,x_5) : x_2 = \widetilde{F}_1^{-1} \circ T_B\left((T_B^{-1}(\alpha))^{p_1}\right), x_3 = \widetilde{F}_3^{-1} \circ T_B\left((T_B^{-1}(\alpha))^{p_2}\right), \\
x_5 = \widetilde{F}_5^{-1} \circ T_B\left((T_B^{-1}(\alpha))^{1-p_1-p_2}\right),\ p_1, p_2 \in (0,1),\ p_1 + p_2 < 1\}.
\end{aligned}$$

For the nested distribution $\widetilde{F}$ in (26), one can write,

$$\begin{aligned}
\partial L(\alpha) &= \left\{(x_1,\ldots,x_5) : \widetilde{F}(x_1,x_2,x_3,x_4,x_5) = \alpha\right\} \\
&= \left\{(x_1,\ldots,x_5) : T \circ C_0\left(T^{-1} \circ F_A(x_2,x_3,x_5), T^{-1} \circ F_B(x_1,x_4)\right) = \alpha\right\} \\
&= \left\{(x_1,\ldots,x_5) : T^{-1} \circ F_A(x_2,x_3,x_5) \cdot T^{-1} \circ F_B(x_1,x_4) = T^{-1}(\alpha)\right\}
\end{aligned}$$

Now choosing $s_1 \in (0,1)$ such that $T^{-1} \circ F_B(x_1, x_4) = (T^{-1}(\alpha))^{s_1}$, one must have $(T^{-1} F_A(x_2, x_3, x_5))^{1-s_1} = (T^{-1}(\alpha))^{1-s_1}$, so that

$$\begin{aligned}\partial L(\alpha) &= \left\{(x_1, \ldots, x_5) : F_B(x_1, x_4) = T\left((T^{-1}(\alpha))^{s_1}\right), F_A(x_2, x_3, x_5) = T\left((T^{-1}(\alpha))^{1-s_1}\right),\ s_1 \in (0,1)\right\} \\ &= \left\{(x_1, x_2, x_3, x_4, x_5) : (x_1, x_4) \in \partial L_B\left(T\left((T^{-1}(\alpha))^{s_1}\right)\right), (x_2, x_3, x_5) \in \partial L_A\left(T\left((T^{-1}(\alpha))^{1-s_1}\right)\right),\ s_1 \in (0,1)\right\}.\end{aligned}$$

An illustration of critical-layers $\partial L_A(\alpha)$ and $\partial L_B(\alpha)$ derived above is provided in Figure 14.

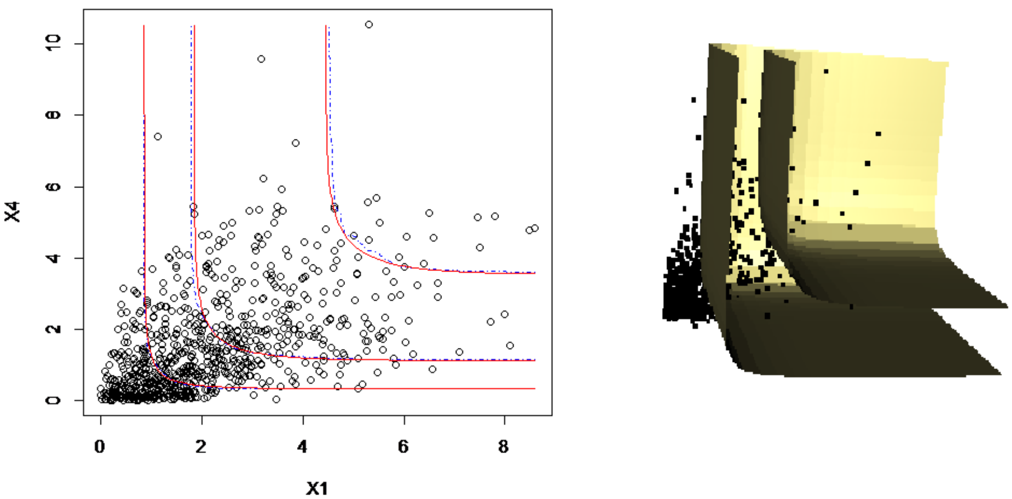

Figure 14: Left: 2-dimensional critical-layers $\partial L_B(\alpha)$ with $\alpha = 0.2, 0.5, 0.9$. Associated non-parametric empirical critical-layers are drawn in blue dashed lines. Right: 3-dimensional critical layers $\partial L_A(\alpha)$ with $\alpha = 0.3, 0.9$. Black dots represent rainfall data (X_1, X_4) (left) and (X_2, X_3, X_5) (right).

7 Conclusion

We described an estimation procedure for multivariate distribution functions. This methodology provides also parametric expressions of associated quantities as critical layers, Kendall's function, return periods. The considered model is based on transformations of the marginals and of the dependence structure within the class of Archimedean copulas. The proposed estimation is straightforward, it has a tunable number of parameters and it does not rely on any optimization procedure. Furthermore the proposed adjustment is flexible, it can be adapted to different types of data (multimodal distribution or non-exchangeable vectors, see for instance illustration in Section 6). Numerical illustrations are provided using a real rainfall data-set.

Some perspectives for future work are the following ones. Firstly, as we remarked above our procedure does not require any optimization procedure. However an optimization can improve the quality of the estimation. Firstly the choice of thresholds $\mathcal{Q}_i$ and the smoothing parameters η_i can be optimized (in this paper these sets are arbitrarily chosen). Also the parameters linked to the estimation of the Archimedean copula can be optimized, as the choice of a generator among its equivalence class via the point (x_0, y_0), or the kernel for smoothing empirical diagonal of the copula.

Furthermore, the impact on the tail of transformed copulas has to be investigated (see for instance Durante et al. (2010)). In particular the relationship between the asymptote of the parametric transformation T and the regular variation of the transformed tails represents an open interesting point. A good understanding of the tail behavior is indeed required to estimate the shape of the transformation near 0 and 1, in extreme quantiles where there is a lack of data. More precisely, some parameters of hyperbola are linked to the upper tail dependence coefficient. This implies the possibility to modify the tail dependency of the transformed distribution without changing the global adjustment.

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