Left properness and Strictification of co-Segal dg-categories

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Abstract

We study co-Segal dg-categories over a general commutative ring that is not necessarily a field. We show that for any set X there is a model structure on co-Segal dg-categories over X that is always left proper. We show that the corresponding homotopy category is equivalent to the homotopy category of usual dg-categories over X. This result will be shown to extend naturally when we vary the set of objects. We also extend the co-Segal formalism to algebras and categories over an operad $\mathcal{P}$. The results of the paper are established for co-Segal $\mathscr{M}$-categories enriched over any symmetric monoidal model category $\mathscr{M} = (\underline{M}, \otimes, I)$ whose underlying model category is combinatorial and left proper. We conjecture that the co-Segal formalism for algebras over an operad should bypass the limitation of having a field of characteristic zero to get a homotopy theory for commutative (co-Segal) dg-algebras.

Contents

1 Introduction

Given a symmetric monoidal model category $\mathscr{M} = (\underline{M}, \otimes, I)$, we've started in [4, 7] a theory of homotopy enriched category with the notion of *co-Segal $\mathscr{M}$-category*. We describe briefly this theory in the next few lines of this introduction, to give an overview of the subject.

The basic idea is that if a, b, c are objects of a co-Segal $\mathscr{M}$-category $\mathcal{D}$, then instead of having a direct composition map $\mathcal{D}(a,b) \otimes \mathcal{D}(b,c) \longrightarrow \mathcal{D}(a,c)$, we have the following data.

- A *precomposition* $\mathcal{D}(a,b) \otimes \mathcal{D}(b,c) \longrightarrow \mathcal{D}(a,b,c)$, that lands into a third party object $\mathcal{D}(a,b,c)$;
- A weak equivalence $\mathcal{D}(a,c) \xrightarrow{\sim} \mathcal{D}(a,b,c)$;
- Some other data and axioms that ensure that this precomposition is somehow associative and compatible with the various weak equivalences.

If we put these data together, we get a zigzag in $\mathscr{M}$

$$\mathcal{D}(a,b) \otimes \mathcal{D}(b,c) \longrightarrow \mathcal{D}(a,b,c) \xleftarrow{\sim} \mathcal{D}(a,c),$$

that defines a map in the homotopy category of $\mathscr{M}$ from $\mathcal{D}(a,b) \otimes \mathcal{D}(b,c)$ to $\mathcal{D}(a,c)$. In the end we see that this map defines an *up-to-homotopy* composition in $\mathcal{D}$.

The precise definition of such $\mathcal{D}$ is an $\mathscr{M}$-valued lax functor that satisfies some conditions that we refer to as *co-Segal conditions*. These conditions are the ones that demand the map $\mathcal{D}(a,c) \to \mathcal{D}(a,b,c)$ to be a weak equivalence. This map is in fact part of a diagram that has the shape of a simplicial object in $\mathscr{M}$:

$$\mathcal{D}(a,c) \to \mathcal{D}(a,b,c) \rightrightarrows \cdots \mathcal{D}(a, ..., c) \cdots .$$

The co-Segal conditions demand the entire diagram to be a diagram of weak equivalences, therefore one might think of it as a *resolution* of $\mathcal{D}(a,c)$. If we drop the co-Segal conditions then the structure we get will be called *co-Segal $\mathscr{M}$-precategory*.

As expected, any usual $\mathscr{M}$-category is a co-Segal $\mathscr{M}$-category and they are exactly the co-Segal categories such that every such diagram is constant (of value $\mathcal{D}(a,c)$). So philosophically if we start with a usual strict $\mathscr{M}$-category $\mathcal{D}$ and take a Reedy (simplicial) resolution of each hom-object $\mathcal{D}(a,c)$, then one gets a co-Segal structure as long as we can multiply a term in the resolution of $\mathcal{D}(a,b)$ and a term in the resolution of $\mathcal{D}(b,c)$ to get 'something' in the resolution of $\mathcal{D}(a,c)$. And it turns out that this philosophy fits in *Dugger's presentation theorem* applied to $\mathscr{M}$-**Cat**.

The major characteristics in a co-Segal structure $\mathcal{D}$ are the following.

- The objects $\mathcal{D}(a,b)$, $\mathcal{D}(b,c)$, $\mathcal{D}(a,c)$, that we call *initial entries*, <u>do not</u> receive any algebraic data in the sense that there is no map involving $\otimes$ like

$$\mathcal{D}(-) \otimes \mathcal{D}(-) \longrightarrow \mathcal{D}(a,c).$$

– There is no map either $\mathcal{D}(-) \longrightarrow \mathcal{D}(a,c)$ that is part of the diagram except the identity. This is due to the fact that the simplicial diagram $\{\mathcal{D}(a,c) \longrightarrow \cdots\}$ is controlled by the direct category $\Delta_{\text{epi}}^{\text{op}}$, therefore there is no *going back* to $\mathcal{D}(a,c)$ using codegeneracies. Using Δ_{epi} rather than the entire Δ is the cornerstone of the homotopy theory as we explain later.

– For every a, the initial entry $\mathcal{D}(a,a)$ contains an identity map $I_a : I \longrightarrow \mathcal{D}(a,a)$. But the constraint of invariance under composition with I_a is not on the other initial entries $\mathcal{D}(a,c)$ themselves; but on their *homotopy clones* $\mathcal{D}(a,a,c), \mathcal{D}(a,c,c)$, etc. Indeed a left unital constraint is expressed by a commutative diagram of the form:

$$\begin{tikzcd}
I \otimes \mathcal{D}(a,c) \arrow[r, "\cong"] \arrow[d, "I_a \otimes \mathrm{Id}"'] & \mathcal{D}(a,c) \arrow[d, "\sim"] \\
\mathcal{D}(a,a) \otimes \mathcal{D}(a,c) \arrow[r, "\varphi"] & \mathcal{D}(a,a,c)
\end{tikzcd}$$

We see that the constraint of commutativity is on the object $\mathcal{D}(a,a,c)$ i.e, the two maps have to be equal when they arrive there. And even if the diagram doesn't commute at $\mathcal{D}(a,a,c)$ we can take the coequalizer of the two maps to replace $\mathcal{D}(a,a,c)$ by the *coequalizing object*, say $\widehat{\mathcal{D}}(a,a,c)$. This operation doesn't change the initial entries $\mathcal{D}(a,a), \mathcal{D}(a,c)$. And this is true even when $a = c$.

– A direct consequence of the preceding characteristic is the existence of a *unitalization functor* from the category of preunital $\mathscr{M}$-precategories to unital $\mathscr{M}$-precategories, and this functor doesn't change the initial entries. This functor is obtained by first taking the coequalizing object and then expanding the data (to entries of higher degree).

With these facts in mind, the homotopy theory of co-Segal $\mathscr{M}$-categories can be drastically simplified if we concentrate it at the initial entries. And this is what we do, starting with the following definition.

Definition. Say that a map $(\sigma, f) : \mathcal{C} \longrightarrow \mathcal{D}$ is a *local easy weak equivalence* (resp. a local easy fibration) if for all (a,b) the initial component $\mathcal{C}(a,b) \longrightarrow \mathcal{D}(fa, fb)$ is a weak equivalence in $\mathscr{M}$ (resp. a fibration).

With a little thinking, we see that this definition coincides with the notion of local weak equivalences and fibrations for usual strict $\mathscr{M}$-categories (see for example [9]). More generally, if we use the 3-for-2 property of weak equivalences in $\mathscr{M}$, it's not hard to see that if $(\sigma, f) : \mathcal{C} \longrightarrow \mathcal{D}$ is a morphism between precategories that satisfy the co-Segal conditions, then σ is level-wise weak equivalence if and only if it's an easy weak equivalence.

For the remaining part of this introduction we're going to present briefly the results of the paper. To avoid a very long paper, we only study first in this paper, the homotopy theory of co-Segal $\mathscr{M}$-categories when we fix the set of objects. This include in particular the homotopy theory of co-Segal algebras which are by definition co-Segal categories with a single object. The next part of the project that can partially be found in [5] treats the homotopy theory when we vary the set of objects. A surprising fact to the author is the existence of the Dwyer-Kan model structure under few hypotheses on $\mathscr{M}$.

The main result of the present paper is the content of Theorem 9.2, Theorem 9.6 and Theorem 10.2. We announce them at the end of the next part of this introduction because some preliminary notations are needed.

1.1 Outline of the content

Given a set X, there is a category $\mathscr{M}_{\mathcal{S}}(X)_u$ whose objects are called unital co-Segal $\mathscr{M}$-precategories over X. Similarly we will denote by $\mathscr{M}\text{-}\mathbf{Cat}(X)$ (resp $\mathscr{M}\text{-}\mathbf{Graph}(X)$) the category of $\mathscr{M}$-categories (resp. $\mathscr{M}$-graphs) over X. We have a fully faithful inclusion:

$$\iota : \mathscr{M}\text{-}\mathbf{Cat}(X) \hookrightarrow \mathscr{M}_{\mathcal{S}}(X)_u .$$

We also have functor that projects the initial entries:

$$\mathrm{Ev}_{\leq 1} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow (I_X \downarrow \mathscr{M}\text{-}\mathbf{Graph}(X)).$$

Here $(I_X \downarrow \mathscr{M}\text{-}\mathbf{Graph}(X))$ represents the category of $\mathscr{M}$-graphs $m = \{m(a,b)\}_{(a,b)\in X^2}$ equipped with a map $I \longrightarrow m(a,a)$ for every a. The following are important categorical facts (see Proposition 3.8 and Lemma 3.20).

Lemma. *Let $\mathscr{M}$ be a symmetric monoidal closed category that is complete and cocomplete. Then we have:*

1. *$\mathscr{M}\text{-}\mathbf{Cat}(X)$ is a full reflective subcategory of $\mathscr{M}_{\mathcal{S}}(X)_u$. In particular there is a left adjoint $|-| : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X)$.*

2. *The functor $Ev_{\leq 1} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow (I_X \downarrow \mathscr{M}\text{-}\boldsymbol{Graph}(X))$ is a left adjoint. In particular it preserves colimits.*

We can already notice that since the inclusion $\mathscr{M}\text{-}\mathbf{Cat}(X) \hookrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$ is a right adjoint, it won't always preserve colimits. This means that colimits of usual $\mathscr{M}$-categories are not computed the same way in $\mathscr{M}_{\mathcal{S}}(X)_u$ as we compute them in $\mathscr{M}\text{-}\mathbf{Cat}(X)$. And therefore it's not hard to believe that $\mathscr{M}_{\mathcal{S}}(X)_u$ can be left proper even when $\mathscr{M}\text{-}\mathbf{Cat}(X)$ is not. And in fact thanks to the second assertion, pushouts are taken level-wise at the initial entries. Now since the homotopy theory is concentrated at the initial entries, left properness is straightforward.

We remind the reader that the model structure on $\mathscr{M}\text{-}\mathbf{Cat}(X)$ is the *transferred* model structure from $\mathscr{M}\text{-}\mathbf{Graph}(X)$ through the usual monadic adjunction. This model structure exists under some hypotheses on $\mathscr{M}$. Our preliminary result on the homotopy theory of $\mathscr{M}_{\mathcal{S}}(X)_u$ is the following theorem (see Theorem 5.1 and Corollary 5.2).

Theorem. *Let $\mathscr{M}$ be a symmetric monoidal model category that is combinatorial. Then the following hold.*

1. *There is is a combinatorial model structure on the category $\mathscr{M}_{\mathcal{S}}(X)_u$ characterized as follows.*

– *The weak equivalences are the easy weak equivalences;*

– *The fibrations are the easy fibrations.*

– *The cofibrations are the maps that have the left lifting property (LLP) against any map that is simultaneously a fibration and a weak equivalence.*

We will denote by $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ this model category.

2. *This model structure if furthermore left proper if $\mathscr{M}$ is.*

3. *If the transferred model structure on $\mathscr{M}\text{-}\mathbf{Cat}(X)$ exists then we have a Quillen adjunction in which the inclusion $\iota : \mathscr{M}\text{-}\mathbf{Cat}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$ is right Quillen.*

$$|-| : \mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X) : \iota$$

This theorem is an intermediate step toward the correct homotopy theory on $\mathscr{M}_{\mathcal{S}}(X)_u$. By 'correct' homotopy theory, we mean a model structure on $\mathscr{M}_{\mathcal{S}}(X)_u$ such that the fibrant objects satisfy the co-Segal conditions. To this end, we want to follow the classical method and localize the above model structure with respect to some set of maps. The set of maps we want is characterized in the following proposition (see Section 6).

Proposition. *Let X be set and let $\mathscr{M}$ be a combinatorial model category with a generating set $\mathbf{I}$ for the cofibrations. Then there exists a set $\mathbf{K}_X(\mathbf{I})$ of morphisms in $\mathscr{M}_{\mathcal{S}}(X)_u$ such that the following hold.*

1. *Given a unital precategory $\mathcal{F}$, if the unique map $\mathcal{F} \longrightarrow *$ going to the terminal object has the right lifting property (RLP) with respect to any map in $\mathbf{K}_X(\mathbf{I})$, then $\mathcal{F}$ satisfies the co-Segal conditions.*

2. *For any $\sigma \in \mathbf{K}_X(\mathbf{I})$, the component of σ at any initial entry (a, b), is a cofibration in $\mathscr{M}$.*

3. *If the transferred model structure on $\mathscr{M}\text{-}\mathbf{Cat}(X)$ exists then the left adjoint $|-| : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X)$ sends elements of $\mathbf{K}_X(\mathbf{I})$ to weak equivalences in $\mathscr{M}\text{-}\mathbf{Cat}(X)$.*

The set $\mathbf{K}_X(\mathbf{I})$ was already outlined in [4, 7], but the third assertion is the surprising fact which turns out to play an important role. Indeed since we have a left Quillen functor $|-| : \mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X)$ that maps $\mathbf{K}_X(\mathbf{I})$ to weak equivalences, the universal property of the left Bousfield localization says that this functor descends to the left Bousfield localization of $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ with respect to $\mathbf{K}_X(\mathbf{I})$. The Bousfield localization exists by Smith's theorem for combinatorial and left proper model categories. A direct consequence of this is the following result (Lemma 10.1).

Lemma. *Let $\mathscr{M}$ be a symmetric monoidal model category that is combinatorial and left proper. Assume that the transferred model category on $\mathscr{M}\text{-}\mathbf{Cat}(X)$ exists.*

Let $\sigma : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor of $\mathscr{M}$-categories that is also regarded as a morphism in $\mathscr{M}_{\mathcal{S}}(X)_u$. Then σ is a $\mathbf{K}_X(\mathbf{I})$-local equivalence in $\mathscr{M}_{\mathcal{S}}(X)_u$ if and only if σ is a weak equivalence in $\mathscr{M}\text{-}\mathbf{Cat}(X)$.

Now the model structure on $\mathscr{M}_{\mathcal{S}}(X)_u$ we want is not exactly the left Bousfield localization of $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ with respect to $\mathbf{K}_X(\mathbf{I})$, because $\mathbf{K}_X(\mathbf{I})$ is, a priori, not in the class of cofibrations in the model structure $\mathscr{M}_{\mathcal{S}}(X)_{ue}$. The idea is that we want $\mathbf{K}_X(\mathbf{I})$ to be among the trivial cofibrations in the Bousfield localization, so that every fibrant object will be in particular $\mathbf{K}_X(\mathbf{I})$-injective, whence a co-Segal $\mathscr{M}$-category.

However if $\mathscr{M}$ is *tractable* then there exists a set $\widetilde{\mathbf{K}}_X(\mathbf{I})$ of cofibrations in $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ such that every $\widetilde{\mathbf{K}}_X(\mathbf{I})$-injective is a co-Segal category. But our motivation in this paper is not to demand any further hypotheses on $\mathscr{M}$ other than combinatorial and left proper.

Our next step is to augment the generating set of cofibrations by including $\mathbf{K}_X(\mathbf{I})$. We keep the same weak equivalences and we get a model structure '+' as follows (see Section 8).

Theorem. *Let $\mathscr{M}$ be a symmetric monoidal model category that is combinatorial. There is is a combinatorial model structure on the category $\mathscr{M}_{\mathcal{S}}(X)_u$ such that the following hold.*

1. *The weak equivalences are the easy weak equivalences.*
2. *Every element of $\mathbf{K}_X(\mathbf{I})$ is generating cofibration.*
3. *We will denote by $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$ this model category.*
4. *This model structure if furthermore left proper if $\mathscr{M}$ is.*
5. *The left Quillen functor $\mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_{ue+}$ defined by the identity is a Quillen equivalence.*

With this model structure at hand, we are able to prove the following result which is the most important ingredient to compare the homotopy theories of $\mathscr{M}_{\mathcal{S}}(X)_u$ and $\mathscr{M}\text{-}\mathbf{Cat}(X)$ (see Proposition 9.5).

Proposition. *For every $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_u$, the unit $\mathcal{F} \longrightarrow \iota(|\mathcal{F}|)$ of the adjunction $|-| \dashv \iota$ is a $\mathbf{K}_X(\mathbf{I})$-local equivalence.*

This proposition relies on the material of Section 7. Finally the main results of this paper are Theorem 9.2 and Theorem 10.2. We summarize their content as follows.

Theorem. *Let $\mathscr{M}$ be symmetric monoidal model category whose underlying category is combinatorial and left proper. Let X be a set and let $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue}$ and $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ be respective left Bousfield localization with respect to the same set $\mathbf{K}_X(\mathbf{I})$. Then the following hold*

1. *In $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ every fibrant object $\mathcal{F}$ is a co-Segal $\mathscr{M}$-category.*
2. *The left Quillen functor $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ defined by the identity, is a Quillen equivalence.*
3. *If the transferred model structure on $\mathscr{M}\text{-}\mathbf{Cat}(X)$ exists, then the Quillen adjunction*

$$|-|^{\mathbf{c}} : \mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue} \leftrightarrows \mathscr{M}\text{-}\mathbf{Cat}(X) : \iota,$$

   is a Quillen equivalence.

4. *The diagram* $\mathscr{M}_{\mathbb{S}}(X)^{\mathbf{c}}_{ue+} \leftarrow \mathscr{M}_{\mathbb{S}}(X)^{\mathbf{c}}_{ue} \xrightarrow{|-|^{\mathbf{c}}} \mathscr{M}\text{-}\mathbf{Cat}(X)$ *is a zigzag of Quillen equivalences. In particular we have the following equivalences between the homotopy categories.*
$$\mathbf{ho}[\mathscr{M}_{\mathbb{S}}(X)^{\mathbf{c}}_{ue+}] \xleftarrow{\simeq} \mathbf{ho}[\mathscr{M}_{\mathbb{S}}(X)^{\mathbf{c}}_{ue}] \xrightarrow{\simeq} \mathbf{ho}[\mathscr{M}\text{-}\mathbf{Cat}(X)].$$

1.2 Why the theory works

As mentioned previously, the cornerstone of this theory is the fact that we use the category Δ^+_{epi} rather than Δ^+. To explain this briefly let's consider the case of a co-Segal $\mathscr{M}$-category with one object: we call such object a co-Segal associative algebra (or monoid).

If A is an object of $\mathscr{M}$, we usually define an algebra structure on A by specifying a multiplication $\mu : A \otimes A \longrightarrow A$ and a unit $e : I \longrightarrow A$. And we demand the usual axiom of associativity for μ and the unitality axiom for e and μ. If we think in the co-Segal world, we will regard this as a co-Segal algebra structure on a object A that is *static* i.e, it doesn't "change *with time*".

More precisely, having a co-Segal algebra structure on A is the possibility of having a multiplication $A \otimes A \longrightarrow A(2)$ that does <u>not</u> necessarily land to A itself but to another object $A(2)$ that possesses the same homotopy information as A.

The object A is the initial entry ("initial state") of a diagram of weak equivalences, defined over $(\Delta^+_{\text{epi}})^{op}$:
$$A = A(1) \xrightarrow{\sim} A(2) \rightrightarrows \cdots A(n) \cdots .$$
The category $(\Delta^+_{\text{epi}})^{op}$ is a direct category, and being direct is a concept that reflects a one-way *evolution*. Therefore we can think of this diagram as an evolution of A without the possibility to go back in time (can we ?). And since this is a diagram of weak equivalences, these changes preserve the entire homotopy information of the initial entry A.

It turns out that having a pseudo-multiplication $A \otimes A \longrightarrow A(2)$ plus a direct diagram of weak equivalences (co-Segal conditions) give more flexibility for homotopy theory purposes. Indeed all we have to do is to concentrate first the homotopy theory at the initial entries and then use Bousfield localizations.

To close this introduction, we describe briefly in the next section how this idea can be carried out to algebras over an operad $\mathcal{P}$ and more generally to categories over an operad, e.g A_∞-categories. Such considerations are not just "general nonsense", but there is a good reason in doing that. One reason is the existence of a homotopy theory for these weak algebras under minimal hypotheses on the monoidal category $\mathscr{M}$. Moreover the homotopy theory for the weak algebras always inherits some of the interesting properties of $\mathscr{M}$ such as left properness.

Another reason is that this approach fixes many issues that occur with usual $\mathcal{P}$-algebras and $\mathcal{P}$-categories. For example the Hinich model structure for commutative dg-algebras only works over a field of characteristic 0 (see [12]). Another example is the failure of the category of (linear) A_∞-categories to be a genuine model category as developed by Lefèvre-Hasegawa

[15]. The reason being that A_∞-categories are not closed under limits. In fact the subcategory of precategories that satisfy the co-Segal conditions is not closed under (co)limits either, that's why we work with precategories and then enforce the co-Segal conditions in the left Bousfield localization.

We list below the hypotheses on $\mathscr{M}$ that are needed to have a homotopy theory of co-Segal $\mathscr{M}$-categories. We can conjecture already that the same hypotheses are enough for the general case discussed in the next section, when a general operad $\mathcal{P}$ is involved.

- We need a symmetric closed monoidal category $\mathscr{M} = (\underline{M}, \otimes, I)$ that is combinatorial and left proper. Being closed is not necessary, we just need $\otimes$ to distributes over (filtered) colimits.
- We don't use the axioms of a monoidal model category i.e, we don't use the interaction of the tensor product $\otimes$ and the model structure on the underlying category $\underline{M}$. This is due to the fact that the initial entries don't receive data involving $\otimes$.
- In particular we don't require all objects to be cofibrant, not even the unit I.

However if we want to iterate the enrichment process as in [21], we might need the axioms of a monoidal model category.

1.3 Generalization to $\mathcal{P}$-co-Segal $\mathscr{M}$-category for an operad $\mathcal{P}$

Let $\mathcal{P}$ be a nonsymmetric operad enriched over $\mathscr{M}$. The idea of a *co-Segal $\mathcal{P}$-algebra* is actually simple. Basically instead of having an action map $\mathcal{P}(l) \otimes A^{\otimes l} \longrightarrow A$, that lands in A itself, we replace by a pseudo-action map $\mathcal{P}(l) \otimes A^{\otimes l} \longrightarrow A(l)$, together with a weak equivalence $A \xrightarrow{\sim} A(l)$. These two maps give a zig-zag in $\mathscr{M}$ that defines a weak $\mathcal{P}$-algebra structure on A:

$$\mathcal{P}(l) \otimes A^{\otimes l} \longrightarrow A(l) \xleftarrow{\sim} A.$$

Here is a sketchy definition for such structure in the nonunital case.

Definition. Let $\mathcal{P}$ be a nonsymmetric operad enriched over the symmetric monoidal category $\mathscr{M} = (\underline{M}, \otimes, I)$. A nonsymmetric *co-Segal $\mathcal{P}$-algebra* $\mathcal{A}$ is given by the following data and axioms.

<u>Data:</u>

1. A diagram $\mathcal{A} : (\Delta_{\text{epi}}^{+})^{op} \longrightarrow \mathscr{M}$ of weak equivalences that takes $n \in \Delta_{\text{epi}}^{+}$ to $\mathcal{A}(n) \in \mathscr{M}$.
2. For $l \in \mathbb{N}$ such that $l \geq 2$ and for every l-tuple $(n_1, ..., n_l)$ of objects of Δ_{epi}^{+}, we have a map in $\mathscr{M}$:
$$\varphi : \mathcal{P}(l) \otimes \mathcal{A}(n_1) \otimes \cdots \otimes \mathcal{A}(n_l) \longrightarrow \mathcal{A}(n_1 + \cdots + n_l).$$
Here $n_1 + \cdots + n_l$ represents the ordinal addition in Δ_{epi}^{+}.
3. For $l = 1$, and for every $n \in \Delta_{\text{epi}}^{+}$ such that $n \geq 2$, there is a map
$$\mathcal{P}(1) \otimes \mathcal{A}(n) \longrightarrow \mathcal{A}(n).$$

4. For $l = 0$ there is a map $\mathcal{P}(0) \longrightarrow \mathcal{A}(0)$. Since $0 \in \Delta^+_{\text{epi}}$ is isolated, we might always take $\mathcal{A}(0) = I$ or $\mathcal{A}(0) = *$, where $*$ is the terminal object of $\mathscr{M}$.

5. For every l-tuple of morphisms $(u_1, ..., u_l)$ in $(\Delta^+_{\text{epi}})^{op}$, with $u_i : n_i \longrightarrow m_i$, the following commutes.

$$\begin{tikzcd}
\mathcal{P}(l) \otimes \mathcal{A}(n_1) \otimes \cdots \otimes \mathcal{A}(n_l) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Id} \otimes (\mathcal{A}(u_i))"] & \mathcal{A}(n_1 + \cdots + n_l) \arrow[d, "\mathcal{A}(u_1 + \ldots + u_l)"] \\
\mathcal{P}(l) \otimes \mathcal{A}(m_1) \otimes \cdots \otimes \mathcal{A}(m_l) \arrow[r, "\varphi"] & \mathcal{A}(m_1 + \cdots + m_l)
\end{tikzcd}$$

6. The previous diagram also commutes if $l = 1$ and $n_1 \geq 2$ (whence $m_1 \geq 2$).

<u>Axioms:</u>

1. Assume we are given the following data.

   – $\varphi_{l_i} : \mathcal{P}(l_i) \otimes \mathcal{A}(n_{i,1}) \otimes \cdots \otimes \mathcal{A}(n_{i,l_i}) \longrightarrow \mathcal{A}(n_{i,1} + \cdots + n_{i,l_i})$, with $1 \leq i \leq r$.

   – $\varphi_{(l_1,...,l_r)} : \mathcal{P}(l_1 + \cdots + l_r) \otimes \mathcal{A}(n_{1,1}) \otimes \cdots \otimes \mathcal{A}(n_{r,l_r}) \longrightarrow \mathcal{A}(n_{1,1} + \cdots + n_{r,l_r})$

   – $\varphi_r : \mathcal{P}(r) \otimes \mathcal{A}(n_{1,1} + \cdots + n_{1,l_1}) \otimes \cdots \otimes \mathcal{A}(n_{r,1} + \cdots + n_{r,l_r}) \longrightarrow \mathcal{A}(n_{1,1} + \cdots + n_{r,l_r})$

   – $\gamma : \mathcal{P}(r) \otimes \mathcal{P}(l_1) \otimes \cdots \otimes \mathcal{P}(l_r) \longrightarrow \mathcal{P}(l_1 + \cdots + l_r)$.

   We require an equality:

$$\varphi_{(l_1,...,l_r)} \circ \gamma = \varphi_r \circ \{[\mathrm{Id}_{\mathcal{P}(r)} \otimes (\varphi_{l_1} \otimes \cdots \otimes \varphi_{l_r})] \circ \text{shuffle}\}.$$

2. If $u_n : 1 \longrightarrow n$ is the unique morphism in $(\Delta^+_{\text{epi}})^{op}$ with $n > 1$, we require a commutative diagram (at least for $n = 2$).

$$\begin{tikzcd}
I \otimes \mathcal{A}(1) \arrow[r, "\cong"] \arrow[d, "\eta \otimes \mathcal{A}(u_n)"'] & \mathcal{A}(1) \arrow[d, "\mathcal{A}(u_n))"] \\
\mathcal{P}(1) \otimes \mathcal{A}(n) \arrow[r, "\varphi"] & \mathcal{A}(n)
\end{tikzcd}$$

If $\mathcal{B}$ is another co-Segal $\mathcal{P}$-algebra, a map of $\mathcal{P}$-algebras $\sigma : \mathcal{A} \longrightarrow \mathcal{B}$ is simply a natural transformation σ such that the following commutes.

$$\begin{tikzcd}
\mathcal{P}(l) \otimes \mathcal{A}(n_1) \otimes \cdots \otimes \mathcal{A}(n_l) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Id} \otimes (\sigma_{n_i})"] & \mathcal{A}(n_1 + \cdots + n_l) \arrow[d, "\sigma_{(n_1 + \ldots + n_l)}"] \\
\mathcal{P}(l) \otimes \mathcal{B}(n_1) \otimes \cdots \otimes \mathcal{B}(n_l) \arrow[r, "\varphi"] & \mathcal{B}(n_1 + \cdots + n_l)
\end{tikzcd}$$

Remark 1.1. 1. The reader can see that we are careful not to have a map $\mathcal{P}(1) \otimes \mathcal{A}(1) \longrightarrow \mathcal{A}(1)$. And this is the whole point of having a co-Segal structure.

2. In the second axiom we can limit everything to $n = 2$. Having more commutative diagrams won't change the outcome of the homotopy theory !

3. If we want symmetric algebra when $\mathcal{P}$ is symmetric, then it suffices to replace Δ^+_{epi} by the category $(\Phi_{\text{epi}})^{op}$ where Φ is the cousin of Segal's category Γ (see [20]). In fact there is a symmetric monoidal category $(\Phi, +, 0)$ of finite sets with all functions, where '+' represents the disjoint union (see [16]).

4. To define $\mathcal{P}$-co-Segal $\mathscr{M}$-categories, one needs to replace Δ^+_{epi} by the 2-category $\mathcal{S}_{\overline{\mathrm{X}}}$ that depends on a set X that will be the set of objects. This 2-category is described briefly in the upcoming sections. The objects n of Δ^+_{epi} are replaced by finite ordered sequences $s = (a_0, ..., a_n)$ of elements of X, and the addition $+$ is replaced by a concatenation of sequences. We didn't not develop the theory here because it wasn't not our main motivation. However we will treat separately the homotopy theory of co-Segal commutative dg-algebras in a future work.

5. As the reader will check, the theory of this paper is when $\mathcal{P} = I$ is the trivial operad.

We close this introduction with the following conjecture.

Conjecture 1.2. *1. The results of Theorem 9.2 and Theorem 10.2 hold for $\mathcal{P}$-co-Segal $\mathscr{M}$-categories.*

2. Let $\mathscr{M}$ be a symmetric monoidal category whose underlying category is a combinatorial and left proper model category. If $\mathcal{P}$ is a symmetric operad, there exists a model structure on co-Segal $\mathcal{P}$-pre-algebras such that every fibrant object is a symmetric co-Segal $\mathcal{P}$-algebra. This model structure extends the Hinich model structure in all characteristics.

2 Unital co-Segal categories

Let $\mathscr{M} = (\underline{M}, \otimes, I)$ be a symmetric monoidal closed category, regarded as a 2-category with a single object. Denote by $\emptyset$ the initial object of $\mathscr{M}$ (actually of $\underline{M}$). Recall that $\emptyset$ is the colimit of the empty diagram, and since $\mathscr{M}$ is closed monoidal, then any functor $m \otimes -$ preserves colimits. In particular it must preserve the initial object.

Notation 2.1. Let X be a set.

1. Denote by I_X the *discrete enriched category* associated to X. The set of objects is X and given $(a, b) \in X^2$ the hom-object $I_X(a, b)$ is given by the formulas:

$$I_X(a, b) = I \quad \text{if } a = b;$$

$$I_X(a, b) = \emptyset \quad \text{if } a \neq b.$$

The composition map is the usual one. It's given essentially by one of the two maps:

$$I \otimes I \cong I, \quad \emptyset \longrightarrow I.$$

2. Denote by $\overline{X}$ the *groupoid of pairs* associated to X. In this category, there is exactly one morphism between any two objects (=elements of X) . There are other names for this category: *indiscrete category*, *chaotic category*, *coarse category*, etc.

3. Let $(\Delta^+, +, 0)$ be the category of finite ordinals and order preserving maps. This is the classifying category for monoids.

4. Let $(\Delta^+_{\text{epi}}, +, 0)$ be the subcategory of epimorphisms. $(\Delta^+_{\text{epi}}, +, 0)$ is the classifying category for nonunital monoids.

Warning. In this paper we assume that the hypothesis hereafter holds.

Hypothesis 2.1. In the model structure on $\mathscr{M}$, the unique map $\emptyset \longrightarrow I$ is <u>not</u> a weak equivalence.

2.1 Nonunital precategories

The following material can be found in a complete form in [7]. We only include a short paragraph.

For a set X, there is a 2-category $\mathscr{P}_{\overline{X}}$ that is a decorated version of $(\Delta^+, +, 0)$, in the sense that if X has a single element then we have an isomorphism $\mathscr{P}_{\overline{X}} \cong (\Delta^+, +, 0)$. The 2-category $\mathscr{P}_{\overline{X}}$ is characterized by the fact that for any 2-category $\mathscr{B}$, we have a functorial isomorphism of sets

$$\text{Lax}(\overline{X}, \mathscr{B}) \cong \text{2-Func}(\mathscr{P}_{\overline{X}}, \mathscr{B}).$$

This is a small version of an adjunction due to Bénabou (see [10] for a detailed account). There is a 2-functor $\textbf{deg} : \mathscr{P}_{\overline{X}} \longrightarrow (\Delta^+, +, 0)$ which is locally a cofibred category. As mentioned in the introduction we don't want Δ^+ but rather Δ^+_{epi}.

Definition 2.2. *Let X be a set. Define the 2-category $\mathcal{S}_{\overline{X}}$ to be the 2-category obtained by forming the genuine pullback of* $\textbf{deg} : \mathscr{P}_{\overline{X}} \longrightarrow (\Delta^+, +, 0)$ *along the inclusion*

$$(\Delta^+_{epi}, +, 0) \hookrightarrow (\Delta^+, +, 0).$$

In particular if X has a single element we have an isomorphism:

$$\mathcal{S}_{\overline{X}} \cong (\Delta^+_{epi}, +, 0).$$

Given $(a, b) \in X^2$, we have a category $\mathcal{S}_{\overline{X}}(a, b)$ of 1-morphisms in $\mathcal{S}_{\overline{X}}$ from a to b. The objects of this category can be identified with finite ordered sequences $s = (a, ...a_i, ..b)$, starting at a and ending at b. There is a degree functor $\deg : \mathcal{S}_{\overline{X}}(a, b) \longrightarrow \Delta^+_{\text{epi}}$ that takes a sequence s to its length. With this functor $\mathcal{S}_{\overline{X}}(a, b)$ become an inverse (Reedy) category, since any nonidentity epimorphism will decrease the degree. The composition in $\mathcal{S}_{\overline{X}}$ is given by the concatenation of sequences.

It follows that if we pass to the respective opposite categories we have a degree functor $\mathcal{S}_{\overline{X}}(a, b)^{op} \longrightarrow (\Delta^+_{\text{epi}})^{op}$ that makes $\mathcal{S}_{\overline{X}}(a, b)^{op}$ a direct (Reedy) category. The sequence (a, b) was the terminal object in $\mathcal{S}_{\overline{X}}(a, b)$ and therefore it becomes the initial object in $\mathcal{S}_{\overline{X}}(a, b)^{op}$

Warning. The object $0 \in \Delta_{\text{epi}}$ is totally isolated, since there is no epimorphism out of it except the identity. We can ignore it since it won't play any role. We implicitly keep it because we want to avoid a language of semi-monoidal categories, semi-2-categories and so on. And saying that the object 1 is terminal in Δ_{epi} is not entirely accurate because there is no morphism from 0 to 1 in that category.

Notation 2.3. For a set X, denote by $(\mathcal{S}_{\overline{X}})^{2\text{-}op}$ the 2-category whose category of morphisms between a and b is $\mathcal{S}_{\overline{X}}(a,b)^{op}$. In particular if X has one element then we have an isomorphism

$$(\mathcal{S}_{\overline{X}})^{2\text{-}op} \cong ((\Delta^{+}_{\text{epi}})^{op}, +, 0).$$

Definition 2.4. *Let $\mathscr{M}$ be a symmetric monoidal (model) category. A nonunital co-Segal $\mathscr{M}$-precategory $\mathcal{F}$ is a normal lax functor:*

$$\mathcal{F} : (\mathcal{S}_{\overline{X}})^{2\text{-}op} \longrightarrow \mathscr{M}.$$

For a set X, let $\mathscr{M}_{\mathcal{S}}(X)$ be the category of all these lax functors and transformations between them. There is a forgetful functor

$$\mathcal{U} : \mathscr{M}_{\mathcal{S}}(X) \longrightarrow \prod_{(a,b)} \text{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}].$$

An object on the right hand side will be called co-Segal $\mathscr{M}$-graph.

Let $\mathscr{M}\text{-}\mathbf{Cat}(X)$ and $\frac{1}{2}\mathscr{M}\text{-}\mathbf{Cat}(X)$ be respectively, the category of $\mathscr{M}$-categories and semi-$\mathscr{M}$-categories. There is a fully faithful functor

$$\frac{1}{2}\mathscr{M}\text{-}\mathbf{Cat}(X) \hookrightarrow \mathscr{M}_{\mathcal{S}}(X).$$

2.2 Preunital and unital precategories

For any set X the discrete category I_X can be regarded as an object of $\frac{1}{2}\mathscr{M}\text{-}\mathbf{Cat}(X)$, and therefore as an object of $\mathscr{M}_{\mathcal{S}}(X)$.

Definition 2.5. *A* preunital co-Segal $\mathscr{M}$-precategory *is an object of the under category*

$$I_X \downarrow \mathscr{M}_{\mathcal{S}}(X).$$

We will write for simplicity $\mathscr{M}_{\mathcal{S}}(X)_\star$ the category $I_X \downarrow \mathscr{M}_{\mathcal{S}}(X)$ and will denote by

$$\mathcal{U} : \mathscr{M}_{\mathcal{S}}(X)_\star \longrightarrow \mathscr{M}_{\mathcal{S}}(X)$$

the forgetful functor.

Being preunital simply boils down to have a morphism $I_a : I \longrightarrow \mathcal{F}(a,a)$ for every a, such that I_a together with the induced map $I \longrightarrow F(a,a) \longrightarrow F(A,A,A)$ fit in a commutative diagram:

$$\begin{tikzcd}
I \otimes I \arrow[r, "\cong"] \arrow[d, "I_a \otimes I_a"'] & I \arrow[d] \\
\mathcal{F}(a,a) \otimes \mathcal{F}(a,a) \arrow[r, "\varphi"] & \mathcal{F}[(a,a,a)]
\end{tikzcd}$$

Definition 2.6. *A preunital precategory $\mathcal{F}$ is said to be* unital *if for every $(a,b) \in X^2$ the following diagrams commute.*

$$\begin{tikzcd}
I \otimes \mathcal{F}(a,b) \arrow[r, "\cong"] \arrow[d, "I_a \otimes \mathrm{Id}"] & \mathcal{F}(a,b) \arrow[d] & \mathcal{F}(a,b) \otimes I \arrow[r, "\cong"] \arrow[d, "\mathrm{Id} \otimes I_b"] & \mathcal{F}(a,b) \arrow[d] \\
\mathcal{F}(a,a) \otimes \mathcal{F}(a,b) \arrow[r, "\varphi"] & \mathcal{F}[(a,a,b)] & \mathcal{F}(a,b) \otimes \mathcal{F}(b,b) \arrow[r, "\varphi"] & \mathcal{F}[(a,b,b)]
\end{tikzcd}$$

We will denote by $\mathscr{M}_{\mathcal{S}}(X)_u$ the full subcategory of $\mathscr{M}_{\mathcal{S}}(X)_\star$, consisting of unital precategories. There is a fully faithful forgetful functor:

$$\mathcal{U} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_\star .$$

Note. At this point we have a chain of forgetful functors

$$\mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_\star \longrightarrow \mathscr{M}_{\mathcal{S}}(X) \longrightarrow \prod_{(a,b)} \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}].$$

We take a moment to show in the next section that each functor as a left adjoint. And more importantly, we will establish that the left adjoint to the whole composite

$$\mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \prod_{(a,b)} \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}],$$

doesn't change much the initial entries.

3 Categorical properties of precategories

Scholium 3.1.

1. Let $\mathcal{J}$ be a direct category with an initial object e and let $F : \mathcal{J} \longrightarrow \mathscr{B}$ be a functor. Then given any morphism $h : b \longrightarrow F(e)$ in $\mathscr{B}$, we can define a diagram $h^\star F : \mathcal{J} \longrightarrow \mathscr{B}$ by simply changing the value $F(e)$ to b. More precisely $h^\star F$ is the composite of the chain of functors:

   $$\mathcal{J} \xrightarrow{\simeq} (e \downarrow \mathcal{J}) \xrightarrow{F} (F(e) \downarrow \mathscr{B}) \xrightarrow{h^\star} (b \downarrow \mathscr{B}) \longrightarrow \mathscr{B}.$$

   We have a natural transformation $h^\star F \longrightarrow F$ whose component at e is h. Every other component $[h^\star F](j) \longrightarrow F(j)$ is the identity map.

2. Let $\mathcal{C}$ be 2-category such that each hom-category $\mathcal{C}(a,b)$ is a direct category with an initial object e_{ab}. Assume furthermore that the composition in $\mathcal{C}$ adds the degrees. Then each e_{ab} is not *decomposable* in the sense that there is no nonidentity map $s \otimes t \longrightarrow e_{ab}$ such that both s and t have nonzero degrees.

   Let $F : \mathcal{C} \longrightarrow \mathscr{M}$ be a normal lax functor of 2-categories and for each (a,b) let $h_{ab} : m_{ab} \longrightarrow F(e_{ab})$ be any map in $\mathscr{M}$. Then the functors $h^\star_{ab} F_{ab}$ form a normal lax functor $h^\star F : \mathcal{C} \longrightarrow \mathscr{M}$ and there is a transformation of lax functors $h^\star F \longrightarrow F$ induced by the natural transformations $h^\star_{ab} F_{ab} \longrightarrow F_{ab}$.

One gets the laxity maps involving the new object m_{ab}, m_{bc} as:

$$m_{ab} \otimes F(s) \xrightarrow{h_{ab} \otimes \mathrm{Id}} F(e_{ab}) \otimes F(s) \longrightarrow F[e_{ab} \otimes s];$$

$$m_{ab} \otimes m_{bc} \xrightarrow{h_{ab} \otimes h_{bc}} F(e_{ab}) \otimes F(e_{bc}) \longrightarrow F[e_{ab} \otimes e_{bc}];$$

We do the same thing when m_{ab} is on the right. It takes a little effort to see that these laxity maps remain associative.

The previous constructions can be applied to any object $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)$ but we can also apply them to any object of $\mathscr{M}_{\mathcal{S}}(X)_\star$ or $\mathscr{M}_{\mathcal{S}}(X)_u$ as we outline hereafter.

Scholium 3.2.

1. Let $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_\star$ be a preunital precategory and let $\{h_{ab} : m_{ab} \longrightarrow \mathcal{F}(a,b)\}_{(a,b) \in X^2}$ be a family of morphisms in $\mathscr{M}$. Assume that for every a the map $I_a : I \longrightarrow \mathcal{F}(a,a)$ factors through a map $\widetilde{I_a} : I \longrightarrow m_{aa}$ as:

$$I \xrightarrow{I_a} \mathcal{F}(a,a) = I \xrightarrow{\widetilde{I_a}} m_{aa} \xrightarrow{h_{aa}} \mathcal{F}(a,a).$$

   Then the object $h^\star \mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)$ lifts to a preunital precategory i.e, an object of $\mathscr{M}_{\mathcal{S}}(X)_\star$ and we have a natural map $h^\star \mathcal{F} \longrightarrow \mathcal{F}$ in $\mathscr{M}_{\mathcal{S}}(X)_\star$.

2. Let $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_u$ be a unital precategory. Then under the same assumption as in the previous situation, $h^\star \mathcal{F}$ lifts to a unital precategory. For example, the diagram of (left) invariance is given by the fact that everything commutes in the diagram below. And we have the same thing for the right invariance.

$$\begin{tikzcd}
 & I \otimes m_{ab} \arrow[r, "\cong"] \arrow[dl, "\widetilde{I_a} \otimes \mathrm{Id}"'] \arrow[d, "\mathrm{Id} \otimes h_{ab}"] & m_{ab} \arrow[d, "h_{ab}"] \\
m_{aa} \otimes m_{ab} \arrow[dr, "h_{aa} \otimes h_{ab}"'] & I \otimes \mathcal{F}(a,b) \arrow[r, "\cong"] \arrow[d, "I_a \otimes \mathrm{Id}"] & \mathcal{F}(a,b) \arrow[d] \\
 & \mathcal{F}(a,a) \otimes \mathcal{F}(a,b) \arrow[r] & \mathcal{F}(a,a,b)
\end{tikzcd}$$

3.1 From co-Segal $\mathscr{M}$-graphs to nonunital precategories

The following result can be found in [7]. For simplicity we will denote by $\mathscr{M}_{\mathcal{S}}$-**Graph**$(X)$ the category $\prod_{(a,b)} \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}]$ of co-Segal $\mathscr{M}$-graphs.

Theorem 3.3. *Let $\mathscr{M}$ be a (co)complete symmetric monoidal closed category and let X be a set. Then the following hold.*

1. *There is a left adjoint* $\Gamma : \mathscr{M}_{\mathcal{S}}$-**Graph**$(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)$ *to the forgetful functor* $\mathcal{U} : \mathscr{M}_{\mathcal{S}}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}$-**Graph**$(X)$.

2. *The adjunction is monadic and* $\mathscr{M}_{\mathcal{S}}(X)$ *is also (co)complete.*

3. *The category* $\mathscr{M}_{\mathcal{S}}(X)$ *is furthermore locally presentable if* $\mathscr{M}$ *is.*

Given a co-Segal $\mathscr{M}$*-graph* $\mathcal{H}$*, the object* $\Gamma\mathcal{H}$ *will be called the free nonunital precategory generated by* $\mathcal{H}$*.*

Remark 3.4. 1. If $\mathcal{H}$ is a co-Segal $\mathscr{M}$-graph and $z = (a_0, \cdots, a_n)$ is a 1-morphism of $(\mathcal{S}_{\overline{X}})^{2\text{-}op}$, then $\Gamma\mathcal{H}$ is given by

$$[\Gamma\mathcal{H}]z = \mathcal{H}(z) \bigsqcup \Big(\coprod_{(s_1,\ldots,s_l)\in\otimes^{-1}(z)} \mathcal{H}(s_1) \otimes \cdots \otimes \mathcal{H}(s_l) \Big).$$

Here $\otimes^{-1}(z)$ is the set of all subdivisions of z i.e, all l-tuples of composable s_i whose composite (concatenation) is z, and such that $\mathbf{deg}(s_i) > 0$ for all i.

2. In particular for $z = (a, b)$, $[\Gamma\mathcal{H}](a, b) = \mathcal{H}(a, b)$

3. If $\eta : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a morphism of co-Segal graphs then for every (a, b) the component

$$\Gamma\eta : [\Gamma\mathcal{H}_1](a, b) \longrightarrow [\Gamma\mathcal{H}_2](a, b)$$

between the initial entries is just (essentially) the same map $\eta : \mathcal{H}_1(a, b) \longrightarrow \mathcal{H}_2(a, b)$.

A direct consequence of this is the obvious proposition:

Proposition 3.5. *Let* $\mathscr{M}$ *be a symmetric monoidal model category and let* $\eta : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ *be a map of co-Segal* $\mathscr{M}$*-graphs. Assume that every component* $\eta : \mathcal{H}_1(a, b) \longrightarrow \mathcal{H}_2(a, b)$ *is a (trivial) cofibration in* $\mathscr{M}$*. Then every component* $\Gamma\eta : [\Gamma\mathcal{H}_1](a, b) \longrightarrow [\Gamma\mathcal{H}_2](a, b)$ *is also a (trivial) cofibration in* $\mathscr{M}$*.*

3.2 From $\mathscr{M}$-graphs to unital precategories

Let $\mathscr{M}$ be as before and let $*$ be its terminal object. Recall that $\mathscr{M}$- **Graph**(X) represents the category of usual $\mathscr{M}$-graphs.

Notation 3.6. Let X be a set.

1. Let $[*]_X$ be the terminal object of $\mathscr{M}_{\mathcal{S}}(X)$. The underlying co-Segal $\mathscr{M}$-graph is the family of constant diagrams $[*] : \mathcal{S}_{\overline{X}}(a, b)^{op} \longrightarrow \mathscr{M}$ of value $*$. The laxity map $* \otimes * \longrightarrow *$, is the unique map going to the terminal object.

2. It's not hard to see that if we consider the unique map $I \longrightarrow *$, then $[*]_X$ lifts to an object of $\mathscr{M}_{\mathcal{S}}(X)_{\star}$ and then to an object of $\mathscr{M}_{\mathcal{S}}(X)_u$.

3. We will identify I_X with its underlying $\mathscr{M}$-graph. Then $(I_X \downarrow \mathscr{M}\text{-}\mathbf{Graph}(X))$ is equivalent to the category of pointed $\mathscr{M}$-graphs.

Definition 3.7. *Let $\mathscr{M}$ be a symmetric monoidal category and let $m = (m_{ab})_{(a,b)\in X^2} \in \mathscr{M}\text{-}\boldsymbol{Graph}(X)$ be a usual $\mathscr{M}$-graph. For each (a,b), let $h_{ab} : m_{ab} \longrightarrow *$ be the unique map going to $*$.*

1. *Define the tautological nonunital precategory associated to m to be the precategory $taut(m)$ obtained from the construction described in Scholium 3.1 applied to $[*]_X \in \mathscr{M}_{\mathcal{S}}(X)$ with respect to the maps $h_{ab} : m_{ab} \longrightarrow *$. We have a functor:*

$$taut : \mathscr{M}\text{-}\boldsymbol{Graph}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)\,.$$

2. *If m is pointed with a map $I \longrightarrow m_{aa}$ for every a, define the tautological (pre)unital precategory $taut_u(m)$ as the precategory obtained from the construction described in Scholium 3.2 applied to $[*]_X \in \mathscr{M}_{\mathcal{S}}(X)_u$ (resp. $\mathscr{M}_{\mathcal{S}}(X)_\star$) with respect to the maps $h_{ab} : m_{ab} \longrightarrow *$. This gives a functor:*

$$taut_u : (I_X \downarrow \mathscr{M}\text{-}\boldsymbol{Graph}(X)) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u\,.$$

If we unwind the definition of a morphism between (non)unital precategories, it's not hard to see that:

Proposition 3.8. *Let $\mathscr{M}$ be a symmetric monoidal category.*

1. *The functor $taut : \mathscr{M}\text{-}\boldsymbol{Graph}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)$ is right adjoint to the functor that projects the initial entries $Ev_{\leq 1} : \mathscr{M}_{\mathcal{S}}(X) \longrightarrow \mathscr{M}\text{-}\boldsymbol{Graph}(X)$.*

2. *The functor $taut_u : (I_X \downarrow \mathscr{M}\text{-}\boldsymbol{Graph}(X)) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$ is right adjoint to the functor that projects the initial entries $Ev_{\leq 1} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow (I_X \downarrow \mathscr{M}\text{-}\boldsymbol{Graph}(X))$.*

In particular in $\mathscr{M}_{\mathcal{S}}(X)$ colimits are computed level-wise at the initial entries. In $\mathscr{M}_{\mathcal{S}}(X)_u$ pushouts are computed level-wise at these entries.

Proof. Left adjoints preserve all kind of colimits. In $\mathscr{M}\text{-}\mathbf{Graph}(X)$ all colimits are just point-wise. However due to the presence of the maps $I \longrightarrow \mathcal{F}(a,a)$ for objects of $\mathscr{M}_{\mathcal{S}}(X)_u$, general colimits are not computed level-wise at the entry $\mathcal{F}(a,a)$ on the diagonal. But pushouts in the under $(I_X \downarrow \mathscr{M}\text{-}\mathbf{Graph}(X))$ are computed in $\mathscr{M}\text{-}\mathbf{Graph}(X)$. ∎

3.3 From nonunital to preunital precategories

Proposition 3.9. *The functor $\mathcal{U} : \mathscr{M}_{\mathcal{S}}(X)_\star \longrightarrow \mathscr{M}_{\mathcal{S}}(X)$ has a left adjoint*

$$\mathscr{R}_* : \mathscr{M}_{\mathcal{S}}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_\star\,.$$

Proof. Define $\mathscr{R}_*(\mathcal{F}) = I_X \coprod \mathcal{F}$ equipped with the canonical map $I_X \longrightarrow I_X \coprod \mathcal{F}$ going to the coproduct. The coproduct exists since $\mathscr{M}_{\mathcal{S}}(X)$ is cocomplete. Following Proposition 3.8 the initial entries of $\mathscr{R}_*(\mathcal{F})$ are given as follows.

$$\mathscr{R}_*(\mathcal{F})(a,b) = \mathcal{F}(a,b) \quad \text{if } a \neq b$$

and

$$\mathscr{R}_*(\mathcal{F})(a,a) = \mathcal{F}(a,a) \coprod I \quad \text{if } a = b.$$

∎

Corollary 3.10. *Let $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism in $\mathscr{M}_{\mathcal{S}}(X)$ such that for every $(a,b) \in X^2$, the component $\sigma_{(a,b)} : \mathcal{F}(a,b) \longrightarrow \mathcal{G}(a,b)$ is a (trivial) cofibration in $\mathscr{M}$. Then for every $(a,b) \in X^2$, the map $\mathscr{R}_*(\sigma)_{(a,b)} : \mathscr{R}_*(\mathcal{F})(a,b) \longrightarrow \mathscr{R}_*(\mathcal{G})(a,b)$ is also a (trivial) cofibration.*

Proof. The component $\mathscr{R}_*(\mathcal{F})(a,b) \longrightarrow \mathscr{R}_*(\mathcal{G})(a,b)$ is either $\sigma_{(a,b)}$ or the coproduct $\mathrm{Id}_I \coprod \sigma_{(a,a)}$. The identity Id_I is simultaneously a cofibration and a trivial cofibration. The corollary follows from the fact that (trivial) cofibrations are closed under coproduct. ■

Since $\mathscr{M}_{\mathcal{S}}(X)_\star$ is a comma category, it inherits most of the properties that $\mathscr{M}_{\mathcal{S}}(X)$ has. And clearly one has the following proposition.

Proposition 3.11. *1. If $\mathscr{M}_{\mathcal{S}}(X)$ is complete (resp. cocomplete, locally presentable) then so is $\mathscr{M}_{\mathcal{S}}(X)_\star$ respectively.*

2. If $\mathscr{M}_{\mathcal{S}}(X)$ carries a model structure then $\mathscr{M}_{\mathcal{S}}(X)_\star$ carries the under model structure.

3.4 Unitalization of precategories

We list first some facts about lax functors. These facts are tedious to check but simply follows from universal properties and some diagrams chasing. The reader should think of lax functors as generalizations of categories and hence as generalizations of algebras. The important thing is that limits for both unital and non unital associative algebras are computed level-wise. The same thing happens with lax functors in general.

Scholium 3.12. Let $\mathcal{C}$ be a 2-category and let $\mathscr{M}$ be a monoidal category regarded as a 2-category with a single object.

1. In the category $\mathrm{Lax}(\mathcal{C}, \mathscr{M})_n$ of normal lax functors, limits are computed level-wise.
2. The unitality conditions in $\mathscr{M}_{\mathcal{S}}(X)_\star$ is closed under level-wise limits and filtered colimits.
3. Coequalizers of reflexive pairs in $\mathscr{M}_{\mathcal{S}}(X)$ are computed level-wise in the category of co-Segal $\mathscr{M}$-graphs. And more generally coequalizers of reflexive pairs in any category $\mathrm{Lax}(\mathcal{C}, \mathscr{M})$ are computed level-wise.

Having these facts in minds we deduce that:

Proposition 3.13. *For a complete and cocomplete monoidal category $\mathscr{M}$ the following hold.*

1. *The category $\mathscr{M}_{\mathcal{S}}(X)_u$ is complete and limits are computed level-wise.*
2. *The category $\mathscr{M}_{\mathcal{S}}(X)_u$ is closed under filtered colimits and they are computed level-wise.*
3. *The category $\mathscr{M}_{\mathcal{S}}(X)_u$ is closed under coequalizer of reflexive pairs and they are computed level-wise.*

For the rest of the discussion we assume that $\mathscr{M} = (\underline{M}, \otimes, I)$ is locally presentable. This is only to make the proof easier and shorter. Following Theorem 3.3, the category $\mathscr{M}_{\mathcal{S}}(X)$ is also locally presentable.

Proposition 3.14. *Let $\mathscr{M}$ be a symmetric monoidal closed category which is locally presentable and let $\mathcal{U} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_\star$ be the forgetful functor. Then the following hold.*

1. *The functor $\mathcal{U}$ has a left adjoint $\Phi : \mathscr{M}_{\mathcal{S}}(X)_\star \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$.*
2. *The induced adjunction is a monadic adjunction.*
3. *The category $\mathscr{M}_{\mathcal{S}}(X)_u$ is locally presentable.*

Proof. Assertion (1) is an application of the main theorem in [2] which asserts that for a locally presentable category, any full subcategory that is complete and closed under directed colimits is reflective.

For Assertion (2) we use Beck monadicity theorem (see for example [1]). The only thing that we need to check is that the $\mathscr{M}_{\mathcal{S}}(X)_u$ has coequalizers of reflexive pairs and that $\mathcal{U} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_\star$ preserves them. But this is contained in the previous proposition.

Finally Assertion (3) follows from the fact that the induced monad is finitary i.e, preserves directed colimits. Indeed both $\mathcal{U}$ and Φ preserves directed colimits. Now as a finitary monad defined on the locally presentable category $\mathscr{M}_{\mathcal{S}}(X)_\star$, we conclude by [3, Remark 2.78] that $\mathscr{M}_{\mathcal{S}}(X)_u$, which is equivalent to the category of algebras of the monad, is also locally presentable. ∎

Unfortunately the above proof doesn't tell us much about Φ. One of the important properties of the adjoint that will be needed is the following:

Theorem 3.15. *Let $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_\star$ be a preunital precategory and let (a,b) be a pair of objects of $\mathcal{S}_{\overline{X}}$. Then the left adjoint Φ doesn't change, up-to an isomorphism, the value of $\mathcal{F}$ at the 1-morphism (a,b) i.e,*

$$\Phi(\mathcal{F})(a,b) \cong \mathcal{F}(a,b).$$

And if $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism in $\mathscr{M}_{\mathcal{S}}(X)_\star$ then the component

$$\Phi(\sigma)_{(a,b)} : \Phi(\mathcal{F})(a,b) \longrightarrow \Phi(\mathcal{G})(a,b),$$

is isomorphic to the component $\sigma_{(a,b)}$, in the category $\mathscr{M}^{[1]}$ (of morphisms of $\mathscr{M}$).

Philosophically the theorem is obvious because the unital conditions don't affect the first entries. But a concrete proof of the theorem is long and involve some technicalities that are not conceptually useful. Instead we give a light version of the theorem that contains all we need for this paper.

But first let's recall a basic category theory result.

Lemma 3.16. *Let $F : \mathcal{A} \longrightarrow \mathscr{B}$ a left adjoint to $U : \mathscr{B} \longrightarrow \mathcal{A}$. For every $a \in \mathcal{A}$, let $\eta_a : a \longrightarrow UF(a)$ be the unit of the adjunction. Then the following hold.*

1. *Let $f : b \longrightarrow F(a)$ be a morphism in $\mathscr{B}$ and let $U(f) : U(b) \longrightarrow UF(a)$ be its projection in $\mathcal{A}$. Assume that the unit η_a factors through $U(f)$ as:*

$$a \xrightarrow{\eta_a} UF(a) = a \longrightarrow U(b) \xrightarrow{U(f)} UF(a).$$

Then $F(a)$ is a retract of b i.e, there is a unique map $s_a : F(a) \longrightarrow b$, adjunct to $a \longrightarrow U(b)$, such that the composite $f \circ s_a$ is the identity map of $F(a)$.

2. *Let $\alpha : a \longrightarrow a'$ be a morphism in $\mathcal{A}$ regarded as an object of $\mathcal{A}^{[1]}$ and let $h : b \longrightarrow b'$ be a morphism in $\mathscr{B}$ viewed as an object of $\mathscr{B}^{[1]}$. Let $\eta_\alpha : \alpha \longrightarrow UF(\alpha)$ be the morphism in $\mathcal{A}^{[1]}$ induced by the unit. Assume that there is a morphism $\theta : h \longrightarrow UF(\alpha)$ in $\mathscr{B}^{[1]}$ such that we have a factorization in $\mathcal{A}^{[1]}$:*

$$\alpha \xrightarrow{\eta_\alpha} UF(\alpha) = \alpha \longrightarrow U(h) \xrightarrow{U(\theta)} UF(\alpha).$$

Then the morphism $F(\alpha) \in \mathscr{B}^{[1]}$ is a retract of $h : b \longrightarrow b'$.

Proof. First observe that Assertion (2) is a consequence of Assertion (1) applied to the induced adjunction $F^{[1]} : \mathcal{A}^{[1]} \longrightarrow \mathscr{B}^{[1]} : U^{[1]}$ between the arrow-categories. So it suffices to prove the first assertion. And this simply follows from the fact that the adjunction is natural in both variable. Indeed we have a commutative square as follows.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{B}}(F(a), b) \arrow[r, "\cong"] \arrow[d, "f\circ"'] & \mathrm{Hom}_{\mathcal{A}}(a, U(b)) \arrow[d, "U(f)\circ"] \\
\mathrm{Hom}_{\mathscr{B}}(F(a), F(a)) \arrow[r, "\cong"'] & \mathrm{Hom}_{\mathcal{A}}(a, UF(a))
\end{tikzcd}$$

Using this commutative square and the hypothesis of the lemma we see that the two elements $f \circ s_a$ and $\mathrm{Id}_{F(a)}$ of $\mathrm{Hom}_{\mathscr{B}}(F(a), F(a))$ have the same adjunct map $\eta_a : a \longrightarrow UF(a)$. The injectivity of the isomorphism

$$\mathrm{Hom}_{\mathscr{B}}(F(a), F(a)) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{B}}(a, UF(a))$$

gives the equality $\mathrm{Id}_{F(a)} = f \circ s_a$. ■

Since we have not provided a proof of Theorem 3.15, we want to convince the reader though that the left adjoint Φ preserves the homotopy theory at the initial entries.

Theorem 3.17. *Let $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_\star$ be a preunital precategory and let (a, b) be a pair of objects of $\mathcal{S}_{\overline{\mathrm{X}}}$. Then the following hold.*

1. *$\Phi(\mathcal{F})(a, b)$ is a retract of $\mathcal{F}(a, b)$.*

2. *If $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism in $\mathscr{M}_{\mathcal{S}}(X)_\star$ then the component*

$$\Phi(\sigma)_{(a,b)} : \Phi(\mathcal{F})(a, b) \longrightarrow \Phi(\mathcal{G})(a, b),$$

is a retract of the component $\sigma_{(a,b)}$.

3. *In particular if the component $\sigma_{(a,b)}$ is a (trivial) cofibration in the model category $\mathscr{M}$, then so is the component $\Phi(\sigma)_{(a,b)}$.*

Proof. Let $\eta_F : \mathcal{F} \longrightarrow \mathcal{U}\Phi(\mathcal{F})$ be the unit of the adjunction. Let $h_{ab} : \mathcal{F}(a,b) \longrightarrow \Phi(\mathcal{F})(a,b)$ the component of this map at the entry (a,b).

Let $h^\star[\Phi(\mathcal{F})]$ be the unital precategory obtained with the construction described in Scholium 3.2 applied to $\Phi(\mathcal{F})$ with respect to the family $h = \{h_{ab}\}_{(a,b)\in X^2}$.

We have a canonical map $h^\star[\Phi(\mathcal{F})] \xrightarrow{\delta} \Phi(\mathcal{F})$ which is the identity outside the initial entries. Then it's not hard to see that the unit $\eta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{U}\Phi(\mathcal{F})$ factors through the map δ as:

$$\mathcal{F} \xrightarrow{\eta_{\mathcal{F}}} \mathcal{U}\Phi(\mathcal{F}) = \mathcal{F} \longrightarrow \mathcal{U}h^\star[\Phi(\mathcal{F})] \xrightarrow{\mathcal{U}(\delta)} \mathcal{U}\Phi(\mathcal{F}). \tag{3.4.1}$$

The projection of this factorization at the entry (a,b) is

$$\mathcal{F}(a,b) \longrightarrow \Phi(\mathcal{F})(a,b) = \mathcal{F}(a,b) \xrightarrow{\mathrm{Id}} \mathcal{F}(a,b) \xrightarrow{h_{ab}} \Phi(\mathcal{F})(a,b).$$

And at every other entry s, this factorization corresponds to the other (tautological) factorization

$$\mathcal{F}(s) \longrightarrow \Phi(\mathcal{F})(s) = \mathcal{F}(s) \xrightarrow{\eta_s} \Phi(\mathcal{F})(s) \xrightarrow{\mathrm{Id}} \Phi(\mathcal{F})(s).$$

With the equality (3.4.1), we get the first assertion from Lemma 3.16. The second assertion follows also from Lemma 3.16, since the construction in Scholium 3.2 is functorial. ■

3.5 From co-Segal $\mathscr{M}$-graphs to unital precategories

The material we present in this section is one of the most important part of the paper. We remind the reader that we have a chain of forgetful functors:

$$\mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_\star \longrightarrow \mathscr{M}_{\mathcal{S}}(X) \longrightarrow \prod_{(a,b)} \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}] = \mathscr{M}_{\mathcal{S}}\text{-}\mathbf{Graph}(X)\,.$$

By the results of the preceding sections, we know that each functor has a left adjoint. More over each left adjoint preserves (most) of the homotopy theory at the initial entries.

Theorem 3.18. *Let $\mathscr{M}$ be a symmetric closed monoidal model category and let X be a set. Then the following hold.*

1. *The forgetful functor $\mathcal{U} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}_{\mathcal{S}}$-**Graph**$(X)$ has a left adjoint*

   $$\mathcal{Q} : \mathscr{M}_{\mathcal{S}}\text{-}\boldsymbol{Graph}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u\,.$$

   This adjunction is monadic.

2. *Let $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is a map in $\mathscr{M}_{\mathcal{S}}$-**Graph**$(X)$ such that every component $\alpha : \mathcal{F}(A,B) \longrightarrow \mathcal{G}(a,b)$ is a (trivial) cofibration in $\mathscr{M}$. Then the component $\mathcal{Q}\,\alpha : \mathcal{Q}(\mathcal{F})(a,b) \longrightarrow \mathcal{Q}(\mathcal{G})(a,b)$ is also a (trivial) cofibration in $\mathscr{M}$.*

Proof. We have $\mathcal{Q} = \Phi \mathscr{R}_* \Gamma$. Everything follows from Proposition 3.5, Corollary 3.10 and Theorem 3.17.

Finally the functor $\mathcal{U}$ creates coequalizers of $\mathcal{U}$-split pairs and it clearly reflects isomorphisms. It follows from Beck monadicity that the adjunction is monadic. ∎

3.6 Ingredients for the homotopy theory: some pushouts

The following result is also important to establish the model structure.

Proposition 3.19. *Let $\mathscr{M}$ be a symmetric closed monoidal model category and let $\mathcal{Q}$: $\mathscr{M}_{\mathcal{S}}$-**Graph**$(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$ be the left adjoint to the forgetful functor. Let $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ be a map of co-Segal $\mathscr{M}$-graphs and let $\mathcal{Q}(\alpha)$ be its image in $\mathscr{M}_{\mathcal{S}}(X)_u$. Consider a pushout square in $\mathscr{M}_{\mathcal{S}}(X)_u$:*

$$\begin{tikzcd}
\mathcal{Q}\,\mathcal{F} \arrow[r, "j"] \arrow[d, "\mathcal{Q}\,\alpha"'] & \mathcal{B} \arrow[d, "\overline{\mathcal{Q}\,\alpha}"] \\
\mathcal{Q}\,\mathcal{G} \arrow[r, "\bar{j}"] & \mathcal{Q}\,\mathcal{G} \cup^{\mathcal{Q}\,\mathcal{F}} \mathcal{B}
\end{tikzcd}$$

Then the following hold.

1. *If the component $\alpha_{(a,b)} : \mathcal{F}(a,b) \longrightarrow \mathcal{G}(a,b)$ is a (trivial) cofibration in $\mathscr{M}$ then so is the component*
$$\overline{\mathcal{Q}\,\alpha}_{(a,b)} : \mathcal{B}(a,b) \longrightarrow [\mathcal{Q}\,\mathcal{G} \cup^{\mathcal{Q}\,\mathcal{F}} \mathcal{B}](a,b).$$

2. *Assume that $\mathscr{M}$ is left proper and that the component $\alpha_{(a,b)} : \mathcal{F}(a,b) \longrightarrow \mathcal{G}(a,b)$ is a cofibration. If $j_{(a,b)} : (\mathcal{Q}\,\mathcal{F})(a,b) \longrightarrow \mathcal{B}(a,b)$ is a weak equivalence then so is the map:*
$$\bar{j}_{(a,b)} : (\mathcal{Q}\,\mathcal{G})(a,b) \longrightarrow [\mathcal{Q}\,\mathcal{G} \cup^{\mathcal{Q}\,\mathcal{F}} \mathcal{B}](a,b)$$

Proof. Thanks to Theorem 3.18 we know that the component $\mathcal{Q}(\alpha)_{(a,b)} : \mathcal{Q}(\mathcal{F})(a,b) \longrightarrow \mathcal{Q}(\mathcal{G})(a,b)$ is a (trivial) cofibration if the corresponding component of α is. Then both assertions follows from Proposition 3.8, which says that pushouts in $\mathscr{M}_{\mathcal{S}}(X)_u$ are computed level-wise at the 1-morphism (a,b). ∎

3.7 Strict $\mathscr{M}$-categories and unital precategories

The following lemma is of great importance; it can be found in a general version in [6]. As said in the introduction, unital co-Segal precategories that are locally constant are precisely $\mathscr{M}$-categories. This provides an inclusion of $\iota : \mathscr{M}\text{-}\mathbf{Cat}(X) \hookrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$.

Lemma 3.20. *Let $\mathscr{M}$ be symmetric monoidal closed category that is also locally presentable. Then the inclusion:*

$$\mathscr{M}\text{-}\mathbf{Cat}(X) \hookrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$$

has a left adjoint

$$|-|: \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X)\,.$$

Proof. From our previous results, we know that limits and direct colimits are computed level-wise in $\mathscr{M}\text{-}\mathbf{Cat}(X)$ and in $\mathscr{M}_{\mathcal{S}}(X)_u$. It follows that $\mathscr{M}\text{-}\mathbf{Cat}(X)$ is a full subcategory of $\mathscr{M}_{\mathcal{S}}(X)_u$ that is closed under limits and directed colimits. Therefore by the main theorem in [2], $\mathscr{M}\text{-}\mathbf{Cat}(X)$ is a reflective subcategory of $\mathscr{M}_{\mathcal{S}}(X)_u$.

The argument is basically the adjoint functor theorem between locally presentable categories. Indeed we know already that $\mathscr{M}\text{-}\mathbf{Cat}(X)$ and $\mathscr{M}_{\mathcal{S}}(X)_u$ are locally presentable ∎

Remark 3.21. There is a direct proof that doesn't use the general theory of adjoint functor and the idea goes as follows. If we have only a nonunital precategory $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)$, then we can define a nonunital category $|\mathcal{F}|$ whose hom-object at (a,b) is the colimit of the component:

$$\mathcal{F}_{ab}: \mathcal{S}_{\overline{X}}{}^{op}(a,b) \longrightarrow \underline{M}.$$

Since $\mathscr{M}$ is monoidal closed, colimits distribute over $\otimes$. Using this fact one can show that the laxity maps for $\mathcal{F}$ induce a composition map between the respective colimits. One can verify that the composition is associative using the uniqueness of maps out of a colimit.

Now if $\mathcal{F}$ is a unital, then there is an operation that has to be done before taking the colimit of $\mathcal{F}_{ab}$. The idea is to *increase the unitality conditions* by imposing more conditions in a specific way.

4 The easy model structure for co-Segal $\mathscr{M}$-graphs

We provided in [7] two model structures on $\mathscr{M}_{\mathcal{S}}(X)$, denoted $\mathscr{M}_{\mathcal{S}}(X)_{\text{inj}}$ and $\mathscr{M}_{\mathcal{S}}(X)_{\text{proj}}$, in which the weak equivalences are the level-wise weak equivalences. These model structures lack of *left properness*, a property that is needed to guarantee the existence of a left Bousfield localization.

Note. We assume that all our categories are κ-small for a sufficiently large regular cardinal κ.

Definition 4.1. *Let $\mathscr{M}$ be a model category and $\mathcal{J}$ be a small category with an initial object e. A natural transformation $\eta: \mathcal{F} \longrightarrow \mathcal{G}$ in $\operatorname{Hom}(\mathcal{J}, \mathscr{M})$ is called an* easy weak equivalence *if the component $\eta_e: \mathcal{F}(e) \longrightarrow \mathcal{G}(e)$ at the initial object is a weak equivalence in $\mathscr{M}$.*

Clearly one has that:

Proposition 4.2. *1. Every level-wise weak equivalence is an easy weak equivalence.*

2. If $\mathcal{F}$ and $\mathcal{G}$ take their value in the subcategory of weak equivalences, then $\eta: \mathcal{F} \longrightarrow \mathcal{G}$ is a level-wise weak equivalence if and only if it is an easy weak equivalence.

Recall that for any object $i \in \mathcal{J}$, there is an evaluation functor Ev_i at i. This functor has a left adjoint $\mathbf{F}^i$ that is given by the following formula. For $U \in \mathscr{M}$, $\mathbf{F}^i_U$ is the diagram whose value at $j \in \mathcal{J}$ is:

$$\mathbf{F}^i_U(j) = U \otimes \mathrm{Hom}(i, j) = \coprod_{i \longrightarrow j} U.$$

We have a similar formula on morphisms (see [13]). The adjunction

$$\mathrm{Ev}_i : \mathrm{Hom}(\mathcal{J}, \mathscr{M}) \rightleftarrows \mathscr{M} : \mathbf{F}^i,$$

will later be automatically a Quillen adjunction (with the projective model structures).

4.1 Easy projective model structure

Recall that for a cofibrantly generated model category $\mathscr{M}$, there is a *projective model structure* on $\mathrm{Hom}(\mathcal{J}, \mathscr{M})$ in which the fibrations and weak equivalences are object wise. This is a cofibrantly generated model category and we can explicitly say what are the generating set cofibrations and trivial cofibrations.

Indeed if $\mathbf{I}$ and $\mathbf{J}$ are respectively the set of generating cofibrations and trivial cofibrations for $\mathscr{M}$, then the following sets are the corresponding generating set for $\mathrm{Hom}(\mathcal{J}, \mathscr{M})_{proj}$.

$$\mathbf{I}_{proj} = \coprod_{i \in \mathcal{J}} \{\mathbf{F}^i_f; f \in \mathbf{I}\}$$

$$\mathbf{J}_{proj} = \coprod_{i \in \mathcal{J}} \{\mathbf{F}^i_f; f \in \mathbf{J}\}$$

Notation 4.3. We will denote by:

1. $\mathbf{W}_{\text{easy}}$ = the class of easy weak equivalences;
2. $\mathbf{I}_{easy} = \{\mathbf{F}^e_f; f \in \mathbf{I}\}$;
3. $\mathbf{J}_{easy} = \{\mathbf{F}^e_f; f \in \mathbf{J}\}$.

If we use the adjunction $\mathbf{F}^e \dashv \mathrm{Ev}_e$ we can prove the following classical results.

Proposition 4.4. *Let $\eta : \mathcal{F} \longrightarrow \mathcal{G}$ be morphism in $\mathrm{Hom}(\mathcal{J}, \mathscr{M})$ and let η_e be the component at e. With the previous notation, the following hold.*

1. *η is in $\mathbf{I}_{easy}$-inj if and only if η_e is a trivial fibration.*
2. *η is in $\mathbf{J}_{easy}$-inj if and only if η_e is a fibration.*

In particular there is an inclusion $\mathbf{J}_{easy} \subset \mathbf{I}_{easy}$-cof$\cap \mathbf{W}_{\text{easy}}$, whence an inclusion $\mathbf{J}_{easy}$-cell $\subset$ $\mathbf{I}_{easy}$-cof$\cap \mathbf{W}_{\text{easy}}$.

For simplicity we assume $\mathscr{M}$ to be locally presentable i.e, a combinatorial model category.

Theorem 4.5. *Let $\mathscr{M}$ be a combinatorial model category and $\mathcal{J}$ be a category with an initial object e such that the comma category $\mathcal{J} \downarrow e$ is reduced to Id_e. Then there is a combinatorial model structure on $\mathrm{Hom}(\mathcal{J}, \mathscr{M})$, denoted $\mathrm{Hom}(\mathcal{J}, \mathscr{M})_{easy}$ where:*

1. *$\mathbf{W}_{\text{easy}}$ is the class of weak equivalences;*
2. *$\mathbf{I}_{easy}$ is the set of generating cofibrations;*
3. *$\mathbf{J}_{easy}$ is the set of generating trivial cofibrations;*

The identity functor $\mathrm{Id} : \mathrm{Hom}(\mathcal{J}, \mathscr{M})_{easy} \longrightarrow \mathrm{Hom}(\mathcal{J}, \mathscr{M})_{proj}$ is a left Quillen functor.

To prove the theorem we will use the recognition theorem that we recall hereafter as stated in [14].

Theorem 4.6. *Suppose $\mathcal{C}$ is a category with all small colimits and limits. Suppose $\mathscr{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $\mathscr{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathscr{W}$ has the two out of three property and is closed under retracts.*
2. *The domains of I are small relative to I-cell.*
3. *The domains of J are small relative to J-cell.*
4. *J-cell $\subseteq \mathscr{W} \cap I$-cof.*
5. *I-inj $\subseteq \mathscr{W} \cap J$-inj.*
6. *Either $\mathscr{W} \cap I$-cof $\subseteq J$-cof or $\mathscr{W} \cap J$-inj $\subseteq I$-inj.*

Proof of Theorem 4.5. The first three conditions are fulfilled as the reader can check.

Condition (4) is outlined in Proposition 4.4. Condition (5) is also part of Proposition 4.4. For Condition (6) it's not hard to see that we have an inclusion $\mathbf{W}_{\text{easy}} \cap \mathbf{J}_{easy}-\text{inj} \subseteq \mathbf{I}_{easy}-\text{inj}$. Indeed consider $\eta \in \mathbf{W}_{\text{easy}} \cap \mathbf{J}_{easy}$-inj. Then on the one hand $\eta \in \mathbf{J}_{easy}$-inj, which means that η_e is a fibration according to Proposition 4.4. And since $\eta \in \mathbf{W}_{\text{easy}}$, then component η_e is a weak equivalence, thus a trivial fibration. Then by the first assertion of Proposition 4.4, we see that η is in $\mathbf{I}_{easy}$-inj. ■

Remark 4.7. Note that by definition a fibrant object in $\mathrm{Hom}(\mathcal{J}, \mathscr{M})_{easy}$ is just a diagram $\mathcal{F}$ such that the object $\mathcal{F}(e)$ is fibrant in $\mathscr{M}$.

A direct consequence of the previous theorem is:

Corollary 4.8. *For any pair (a,b) of objects of $(\mathcal{S}_{\overline{X}})^{2\text{-}op}$, there is a combinatorial model structure* $\mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}]_{easy}$ *on* $\mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}]$*, in which the weak equivalences are the easy weak equivalences and the cofibrations are the easy projective cofibrations.*

The fibrations are the maps σ such that the component $\sigma_{(a,b)}$ is a fibration. The generating set of trivial cofibrations is $\mathbf{J}_{easy}$ and the generating set of cofibration is $\mathbf{I}_{easy}$.

The identity functor $\mathrm{Id} : \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}]_{easy} \longrightarrow \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}]_{proj}$ *is a left Quillen functor.*

4.1.1 The model structure on co-Segal $\mathscr{M}$-graphs

Remark 4.9.

1. Recall that we have an isomorphism:
$$\mathscr{M}_{\mathcal{S}}\text{-}\mathbf{Graph}(X) := \prod_{(a,b)} \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}] \cong \mathrm{Hom}[\coprod_{(a,b)} \mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}].$$
2. The inclusion $\mathcal{S}_{\overline{X}}(a,b)^{op} \hookrightarrow \coprod_{(a,b)} \mathcal{S}_{\overline{X}}(a,b)^{op}$ determines a functor
$$p_{ab} : \mathscr{M}_{\mathcal{S}}\text{-}\mathbf{Graph}(X) \longrightarrow \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}].$$
3. It's not hard to see that the functor p_{ab} has a left adjoint δ_{ab} that we call *Dirac mass.*
4. Let $\mathbf{I}_{ab}$ and $\mathbf{J}_{ab}$ be respectively set of generating cofibrations and trivial cofibrations of $\mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}]_{easy}$.

If we combine the above remark and the previous theorem we get:

Theorem 4.10. *Let $\mathscr{M}$ be a combinatorial model category and let $\mathscr{M}_{\mathcal{S}}$-**Graph**$(X)$ be the category of co-Segal $\mathscr{M}$-graphs (over X). Then there is a combinatorial model structure on $\mathscr{M}_{\mathcal{S}}$-**Graph**$(X)$ which may be described as follows.*

1. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a weak equivalence (resp. fibration) if for every $(a,b) \in X^2$ the component $\sigma : \mathcal{F}(a,b) \longrightarrow \mathcal{G}(a,b)$ is a weak equivalence (resp. fibration) in $\mathscr{M}$.*
2. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a cofibration if it has the LLP with respect to any map that is a fibration and a weak equivalence.*
3. *The following set constitute respectively the set of generating cofibration and generating trivial cofibrations.*
$$\coprod_{(a,b)} \{\delta_{ab}(\alpha); \alpha \in \mathbf{I}_{ab}\}$$
$$\coprod_{(a,b)} \{\delta_{ab}(\alpha); \alpha \in \mathbf{J}_{ab}\}$$

This model structure is the product model structure of the model categories
$$\mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}]_{easy}.$$

4.1.2 From the arrow category $\mathscr{M}^{[1]}$ to co-Segal $\mathscr{M}$-graphs

We have another corollary of Theorem 4.5, when $\mathcal{J} = [1] = \{0 \longrightarrow 1\}$ is the *walking-morphism category.* Note that by definition $\mathscr{M}^{[1]} = \mathrm{Hom}[\mathcal{J}, \mathscr{M}]$ and that the left adjoint $\mathbf{F}^0 : \mathscr{M} \longrightarrow \mathscr{M}^{[1]}$ is the natural embedding that takes an object m to Id_m; it takes a morphism $f : m \longrightarrow m'$ to the morphism $[f] : \mathrm{Id}_m \longrightarrow \mathrm{Id}_{m'}$ whose components are both equal to f.

Corollary 4.11. *There is combinatorial model structure on the category $\mathscr{M}^{[1]}$ of morphisms of $\mathscr{M}$ where a weak equivalence (resp. fibration) is a map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ such that the component $\sigma_0 : \mathcal{F}(0) \longrightarrow \mathcal{G}(0)$ is a weak equivalence (resp. fibration).*

The set of generating cofibrations is

$$\{[\alpha] : \mathrm{Id}_U \longrightarrow \mathrm{Id}_V\}_{\alpha: U \longrightarrow V \in \mathbf{I}}.$$

The set of generating trivial cofibrations is

$$\{[\alpha] : \mathrm{Id}_U \longrightarrow \mathrm{Id}_V\}_{\alpha: U \longrightarrow V \in \mathbf{J}}.$$

We will denote by $\mathscr{M}^{[1]}_{easy}$ this model structure.

Proposition 4.12. *Let $s = (A, ..., B)$ be an object of $\mathcal{S}_{\overline{\mathrm{X}}}(a,b)^{op}$, and let $u_s : (a,b) \longrightarrow s$ be the unique morphism therein. Then the following hold.*

1. *The evaluation $Ev_{u_s} : \mathrm{Hom}[\mathcal{S}_{\overline{\mathrm{X}}}(a,b)^{op}, \mathscr{M}] \longrightarrow \mathscr{M}^{[1]}$ has a left adjoint*

$$\Psi_s : \mathscr{M}^{[1]} \longrightarrow \mathrm{Hom}[\mathcal{S}_{\overline{\mathrm{X}}}(a,b)^{op}, \mathscr{M}].$$

2. *This adjunction is moreover a Quillen adjunction between the respective easy model structures.*

3. *If $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism in $\mathscr{M}^{[1]}$ such that the component $\sigma_0 : \mathcal{F}(0) \longrightarrow \mathcal{G}(0)$ is an isomorphism (resp. cofibration), then the component $\Psi_s(\sigma)_{(a,b)} : \Psi_s(\mathcal{F})(a,b) \longrightarrow \Psi_s(\mathcal{G})(a,b)$ is also an isomorphism (resp cofibration).*

Proof. Ψ_s is the left Kan extension along the functor $[1] \xrightarrow{u_s} \mathcal{S}_{\overline{\mathrm{X}}}(a,b)^{op}$ that picks u_s; this gives Assertion (1).

The evaluation sends (trivial) fibrations to (trivial) fibrations. In fact it's easily seen that Ψ_s sends generating (trivial) cofibrations to generating (trivial) cofibrations; we get Assertion (2).

Assertion (3) follows from the formula of the left Kan extension as the reader can check. In fact $\Psi_s(\mathcal{F})(a,b) \longrightarrow \Psi_s(\mathcal{G})(a,b)$ is isomorphic to $\sigma_0 : \mathcal{F}(0) \longrightarrow \mathcal{G}(0)$ as objects in $\mathscr{M}^{[1]}$. ■

5 The easy model structure on unital precategories

Let $\mathscr{M}_{\mathcal{S}}$-$\mathbf{Graph}(X)_e$ be the model category obtained in Theorem 4.10. Denote by $\mathbf{I}_{\bullet}$ and $\mathbf{J}_{\bullet}$ the respective generating sets of cofibrations and trivial cofibrations therein. Let $\mathcal{Q}$ be the functor $\Phi \mathscr{R}_* \Gamma : \mathscr{M}_{\mathcal{S}}\text{-}\mathbf{Graph}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$.

Theorem 5.1. *Let $\mathscr{M}$ be a symmetric monoidal model category that is combinatorial. For any set X the following hold.*

1. *There is a combinatorial model structure on $\mathscr{M}_{\mathcal{S}}(X)_u$ which may be described as follows.*
   - *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a weak equivalence if it's an easy weak equivalences.*
   - *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a fibration if it's an easy fibration i.e; if the component $\sigma_{(a,b)} : \mathcal{F}(a,b) \longrightarrow \mathcal{G}(a,b)$ is a fibration for every $(a,b) \in X^2$.*
   - *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a cofibration if it has the LLP against any map that is a weak equivalence and a fibration.*
2. *This model structure if furthermore left proper if $\mathscr{M}$ is.*
3. *The sets $\mathcal{Q}\mathbf{I}_{\bullet}$ is a set of generating cofibrations and the set $\mathcal{Q}\mathbf{J}_{\bullet}$ is a set of generating trivial cofibrations.*
4. *We will denote by $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ this model category. The monadic adjunction*

$$\mathcal{Q} : \mathscr{M}_{\mathcal{S}}\text{-}\boldsymbol{Graph}(X)_e \rightleftarrows \mathscr{M}_{\mathcal{S}}(X)_{ue} : \mathcal{U},$$

is a Quillen adjunction where $\mathcal{Q}$ is left Quillen and $\mathcal{U}$ is right Quillen.

Proof. Thanks to Proposition 3.19, we know that the pushout of a generating trivial cofibration is an easy weak equivalence and this the key condition for the transfer lemma of Schwede-Shipley [19] with the respect to the monadic adjunction $\mathcal{Q} \dashv \mathcal{U}$.

The left properness is given by the second assertion of Proposition 3.19. ■

Note. We can avoid the use of the transfer lemma of [19] and do everything directly with Theorem 4.6. With both methods, the proof boils down to checking that the pushout of a generating trivial cofibration is a weak equivalence.

5.1 A Quillen adjunction

A direct consequence of Theorem 5.1 is the following.

Corollary 5.2. *If the transferred model structure on $\mathscr{M}$-$\mathbf{Cat}(X)$ exists then the adjunction*

$$| - | : \mathscr{M}_{\mathcal{S}}(X)_{ue} \leftrightarrows \mathscr{M}\text{-}\mathbf{Cat}(X) : \iota,$$

is a Quillen adjunction.

Proof. Indeed a local (trivial) fibration in $\mathscr{M}$-$\mathbf{Cat}(X)$ is also a (trivial) fibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue}$. ■

We also have the following corollary about the adjunction

$$taut_u : (I_X \downarrow \mathscr{M}\text{-}\mathbf{Graph}(X)) \leftrightarrows \mathscr{M}_{\mathcal{S}}(X)_u : \mathrm{Ev}_{\leq 1}$$

Corollary 5.3. *The adjunction*

$$taut_u : (I_X \downarrow \mathscr{M}\text{-}\boldsymbol{Graph}(X)) \leftrightarrows \mathscr{M}_{\mathcal{S}}(X)_{ue} : Ev_{\leq 1}$$

is a Quillen adjunction where $Ev_{\leq 1}$ is left Quillen.

We also expect the evaluation $\mathrm{Ev}_{\leq 1} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}\text{-}\mathbf{Graph}(X)$ to be a right Quillen functor since it sends (trivial) fibration to (trivial) fibrations. This means that $\mathrm{Ev}_{\leq 1}$ must have a left adjoint which is indeed true and is given by the composite hereafter.

$$\mathscr{M}\text{-}\mathbf{Graph}(X) \hookrightarrow \mathscr{M}_{\mathcal{S}}\text{-}\mathbf{Graph}(X) \xrightarrow{\mathcal{Q}} \mathscr{M}_{\mathcal{S}}(X)_u\,.$$

The first inclusion means that any usual $\mathscr{M}$-graph is a locally constant co-Segal $\mathscr{M}$-graph, just like any $\mathscr{M}$-category is a locally constant co-Segal $\mathscr{M}$-category. This gives the following.

Corollary 5.4. *The functor $Ev_{\leq 1} : \mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}\text{-}\boldsymbol{Graph}(X)$ is a right Quillen functor.*

6 Localizing set

Notation 6.1.

1. If $\alpha : U \longrightarrow V$ is a morphism of $\mathscr{M}$, we will denote by

$$\alpha_{\downarrow \mathrm{Id}_V} : \alpha \longrightarrow \mathrm{Id}_V,$$

the morphism in the arrow category $\mathscr{M}^{[1]}$ which is identified with the following commutative square.

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & V \arrow[d, "\mathrm{Id}"] \\ V \arrow[r, "\mathrm{Id}"] & V \end{tikzcd}$$

2. If u is a 2-morphism in $(\mathcal{S}_{\overline{X}})^{2\text{-}op}$, denote by Ev_u the evaluation at u

$$\mathrm{Ev}_u : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}^{[1]},$$

that takes $\mathcal{F}$ to $\mathcal{F}(u)$.

3. The functor Ev_u has a left adjoint that will be denoted by $\Psi_u : \mathscr{M}^{[1]} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$ or simply Ψ if there is no potential confusion. It follows that if α is a morphism in $\mathscr{M}$ (=object of $\mathscr{M}^{[1]}$) and $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_u$, we have functorial isomorphism of sets

$$\mathrm{Hom}_{\mathscr{M}^{[1]}}(\alpha, \mathcal{F}(u)) \cong \mathrm{Hom}_{\mathscr{M}_{\mathcal{S}}(X)_u}(\Psi(\alpha), \mathcal{F}).$$

For the record if $u \in \mathcal{S}_{\overline{X}}(a,b)$, Ψ is obtained as a composite of left adjoints as follows.

$$\mathscr{M}^1 \longrightarrow \mathrm{Hom}[\mathcal{S}_{\overline{X}}(a,b)^{op}, \mathscr{M}] \xrightarrow{\delta_{ab}} \mathscr{M}_{\mathcal{S}}\text{-}\mathbf{Graph}(X) \xrightarrow{\Gamma} \mathscr{M}_{\mathcal{S}}(X) \xrightarrow{\mathscr{R}_*} \mathscr{M}_{\mathcal{S}}(X)_\star \xrightarrow{\Phi} \mathscr{M}_{\mathcal{S}}(X)_u\,.$$

4. We will denote by $\Psi(\alpha_{\downarrow \mathrm{Id_V}}) : \Psi(\alpha) \longrightarrow \Psi(\mathrm{Id}_V)$ the image of $\alpha_{\downarrow \mathrm{Id_V}}$ by Ψ.
5. For every 1-morphism $s = (A, ..., B)$ in $\mathcal{S}_{\overline{\mathrm{X}}}(a, b)^{op}$, we have a unique 2-morphism

$$u_s : (a, b) \longrightarrow s.$$

For simplicity we will denote again like in Proposition 4.12 by

$$\Psi_s : \mathscr{M}^{[1]} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$$

the previous left adjoint when $u = u_s$.

Definition 6.2. *Let* $\mathbf{I}$ *be set of generating cofibrations of* $\mathscr{M}$.

1. *Define the* localizing set *for* $\mathscr{M}_{\mathcal{S}}(X)_u$ *as*

$$\mathbf{K}_X(\mathbf{I}) := \{ \coprod_{s \in 1-Mor(\mathcal{S}_{\overline{\mathrm{X}}}), \boldsymbol{deg}(s) \geq 2} \{\Psi_s(\alpha_{\downarrow \mathrm{Id_V}}); \quad \alpha \in \mathbf{I}\}\}.$$

2. *Let* $*$ *be the coinitial (or terminal) object of* $\mathscr{M}_{\mathcal{S}}(X)_u$ *and let* σ *be map in* $\mathscr{M}_{\mathcal{S}}(X)_u$. *Say that an object* $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_u$ *is* σ*-injective if the unique map* $\mathcal{F} \longrightarrow *$ *has the RLP with respect to* σ.
3. *Say that* $\mathcal{F}$ *is* $\mathbf{K}_X(\mathbf{I})$*-injective if* $\mathcal{F}$ *is* σ*-injective for all* $\sigma \in \mathbf{K}_X(\mathbf{I})$.

One can easily establish the following proposition.

Proposition 6.3. *Let* $\theta = (f, g) : \alpha \longrightarrow p$ *be a morphism in* $\mathscr{M}^{[1]}$ *which is represented by the following commutative square.*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\alpha"'] & X \arrow[d, "p"] \\ V \arrow[r, "g"] & Y \end{tikzcd}$$

Then the following are equivalent.

- *There is a lifting in the commutative square above i.e there exists* $k : V \longrightarrow X$ *such that:* $k \circ \alpha = f$, $p \circ k = g$.
- *There is a lifting in the following square of* $\mathscr{M}^{[1]}$.

$$\begin{tikzcd} \alpha \arrow[r, "\theta"] \arrow[d, "\alpha_{\downarrow \mathrm{Id_V}}"'] & p \arrow[d] \\ \mathrm{Id}_V \arrow[r] & * \end{tikzcd}$$

That is, there exists $\beta = (k, l) : \mathrm{Id}_V \longrightarrow p$ *such that* $\beta \circ \alpha_{\downarrow \mathrm{Id_V}} = \theta$.

Using that proposition, the adjunction, and the fact that trivial fibrations are the **I**-injective maps; we get the following.

Lemma 6.4. *Let $\mathcal{F}$ be an object of $\mathscr{M}_{\mathcal{S}}(X)_u$. Then the following hold.*

1. *$\mathcal{F}$ is $\mathbf{K}_X(\mathbf{I})$-injective if and only if for every $s = (A_0, ..., A_n)$ the map*

$$\mathcal{F}(u_s) : \mathcal{F}(a, b) \longrightarrow \mathcal{F}(s),$$

is a trivial fibration in $\mathscr{M}$. In particular if $\mathcal{F}$ is $\mathbf{K}_X(\mathbf{I})$-injective, then $\mathcal{F}$ is a co-Segal $\mathscr{M}$-category.

2. *Every a strict $\mathscr{M}$-category $\mathcal{F} \in \mathscr{M}\text{-}\mathbf{Cat}(X)$ is $\mathbf{K}_X(\mathbf{I})$-injective.*

Proof. If $\mathcal{F}$ is $\mathbf{K}_X(\mathbf{I})$-injective, by definition, $\mathcal{F}$ is $\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})$-injective for all generating cofibration α in $\mathscr{M}$. And by adjointness we find that $\mathcal{F}(u_s)$ is $\alpha_{\downarrow \mathrm{Id_V}}$-injective thanks to the previous proposition. This is equivalent to saying that any lifting problem defined by α and $\mathcal{F}(u_s)$ has a solution. Consequently $\mathcal{F}(u_s)$ has the RLP with respect to all maps in **I** and we find that $\mathcal{F}(u_s)$ is a trivial fibration as claimed. This proves Assertion (1).

Assertion (2) is a corollary of Assertion (1) since categories are the constant lax diagrams, therefore $\mathcal{F}(u_s)$ is an identity, in particular a trivial fibration. ■

6.1 Projecting the localizing set to $\mathscr{M}\text{-}\mathbf{Cat}(X)$

The following lemma says that the left Quillen functor

$$| - | : \mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X),$$

sends elements in $\mathbf{K}_X(\mathbf{I})$ to trivial cofibration, so in particular to weak equivalences.

Lemma 6.5. *Let $s = (a, ..., b)$ be a 1-morphism in $\mathcal{S}_{\overline{\mathrm{X}}}$ and let $\alpha : U \longrightarrow V$ be a generating cofibration of $\mathscr{M}$. Then the image in $\mathscr{M}\text{-}\mathbf{Cat}(X)$ of the map $\Psi_s(\alpha_{\downarrow \mathrm{Id_V}}) : \Psi_s(\alpha) \longrightarrow \Psi_s(\mathrm{Id}_V)$ by the functor*

$$| - | : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X),$$

is a trivial cofibration in the transferred model structure on $\mathscr{M}\text{-}\mathbf{Cat}(X)$. In particular it's a weak equivalence therein.

Proof. We're going to show that $|\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})|$ has the left lifting property with respect to any fibration in $\mathscr{M}\text{-}\mathbf{Cat}(X)$. In fact it has the LLP with respect to any functor in $\mathscr{M}\text{-}\mathbf{Cat}(X)$.

Let $p : \mathcal{Z} \longrightarrow \mathcal{T}$ be a local fibration in $\mathscr{M}\text{-}\mathbf{Cat}(X)$. By adjointness, a lifting problem in $\mathscr{M}\text{-}\mathbf{Cat}(X)$:

$$\begin{tikzcd} |\Psi_s(\alpha)| \arrow[r] \arrow[d, "|\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})|"'] & \mathcal{Z} \arrow[d, "p"] \\ |\Psi_s(\mathrm{Id}_V)| \arrow[r] & \mathcal{T} \end{tikzcd} \qquad (6.1.1)$$

is equivalent to a lifting problem in $\mathscr{M}_{\mathcal{S}}(X)_u$:

$$\begin{tikzcd}
\Psi_s(\alpha) \arrow[r] \arrow[d, "\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})"'] & \iota(\mathcal{Z}) \arrow[d, "\iota(p)"] \\
\Psi_s(\mathrm{Id}_V) \arrow[r] & \iota(\mathcal{T})
\end{tikzcd}$$

And one of them has a solution if and only if the other one has a solution. Again, using the adjunction

$$\Psi_s : \mathscr{M}^{[1]} \leftrightarrows \mathscr{M}_{\mathcal{S}}(X)_u : \mathrm{Ev}_{u_s},$$

the previous lifting problem in $\mathscr{M}_{\mathcal{S}}(X)_u$ is equivalent to the following one in $\mathscr{M}^{[1]}$.

$$\begin{tikzcd}
\alpha \arrow[r] \arrow[d, "\alpha_{\downarrow \mathrm{Id_V}}"'] & \mathcal{Z}(u_s) = \mathrm{Id}_{\mathcal{Z}(a,b)} \arrow[d, "p"] \\
\mathrm{Id}_V \arrow[r] & \mathcal{T}(u_s) = \mathrm{Id}_{\mathcal{T}(a,b)}
\end{tikzcd} \tag{6.1.2}$$

As shown above $\mathcal{Z}(u_s) = \mathrm{Id}_{\mathcal{Z}(a,b)}$, and similarly $\mathcal{T}(u_s) = \mathrm{Id}_{\mathcal{T}(a,b)}$ since they are categories (therefore locally constant). The morphism $\alpha \longrightarrow \mathrm{Id}_{\mathcal{Z}(a,b)}$ is simply given by two maps $f : U \longrightarrow \mathcal{Z}(a,b)$ and $g : V \longrightarrow \mathcal{Z}(a,b)$ satisfying:

$$f = g \circ \alpha. \tag{6.1.3}$$

The map $\mathrm{Id}_V \longrightarrow \mathrm{Id}_{\mathcal{T}(a,b)}$ is simply a map $h : V \longrightarrow \mathcal{T}(a,b)$. Since (6.1.2) commutes we have two equalities:

$$p \circ f = h \circ \alpha, \quad \text{and} \quad h = p \circ g.$$

Then it's easily seen that the map $\mathrm{Id}_V \longrightarrow \mathrm{Id}_{\mathcal{Z}(a,b)}$ given by g is a lifting to our problem (6.1.2); and by adjointness we find a lifting to the original problem (6.1.1) which means that $|\Psi_s(\alpha)|$ is a trivial cofibration as desired. ■

Let $\mathscr{S}_{\mathrm{can}} : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$ be the $\mathbf{K}_X(\mathbf{I})$-injective replacement functor obtained by the small object argument. We have a natural transformation $\eta_{can} : \mathrm{Id} \longrightarrow \mathscr{S}_{\mathrm{can}}$ whose component $\mathcal{F} \longrightarrow \mathscr{S}_{\mathrm{can}}(\mathcal{F})$ is a $\mathbf{K}_X(\mathbf{I})$-cell complex.

As a consequence of the previous lemma we get:

Proposition 6.6. *If a transferred model structure on $\mathscr{M}$-$\mathbf{Cat}(X)$ exists then the image*

$$|\eta_{can}| : |\mathcal{F}| \longrightarrow |\mathscr{S}_{\mathrm{can}}(\mathcal{F})|,$$

is a trivial cofibration of strict $\mathscr{M}$-categories.

7 Subcategory of 2-constant precategories

We take a moment to outline an important class of precategories that will be needed later. If (a,b) is a pair of objects of $\mathcal{S}_{\overline{X}}$, we will denote by $\mathcal{S}_{\overline{X}}(a,b)_{\geq 2} \subset \mathcal{S}_{\overline{X}}(a,b)$ the full subcategory of 1-morphisms of degree ≥ 2. This simply means that we remove the 1-morphism (a,b) which is the co-initial object in $\mathcal{S}_{\overline{X}}(a,b)$. Similarly we have the dual category $\mathcal{S}_{\overline{X}}(a,b)^{op}_{\geq 2}$.

As both $\mathcal{S}_{\overline{X}}(a,b)$ and $\mathcal{S}_{\overline{X}}(a,b)^{op}$ are Reedy 1-categories, it's not hard to see that $\mathcal{S}_{\overline{X}}(a,b)$ is isomorphic to the *latching category* of $\mathcal{S}_{\overline{X}}(a,b)$ at (a,b) and dually $\mathcal{S}_{\overline{X}}(a,b)^{op}_{\geq 2}$ is isomorphic to the *matching category* of $\mathcal{S}_{\overline{X}}(a,b)^{op}$ at (a,b).

Definition 7.1. *Say that a precategory $\mathcal{F} : (\mathcal{S}_{\overline{X}})^{2\text{-}op} \longrightarrow \mathscr{M}$ is 2-constant if for every pair (a,b) of objects of $\mathcal{S}_{\overline{X}}$, the restriction to $\mathcal{S}_{\overline{X}}(a,b)^{op}_{\geq 2}$ of the component*

$$\mathcal{F}_{ab} : \mathcal{S}_{\overline{X}}(a,b)^{op} \longrightarrow \underline{M},$$

is a constant functor.

Remark 7.2.
1. It follows from the definition that any 2-constant unital precategory has an underlying nonunital $\mathscr{M}$-category $\mathcal{F}_{\geq 2}$. This precategory $\mathcal{F}_{\geq 2}$ inherits of the pseudo-identity maps I_a but they don't necessarily satisfy the usual unitality conditions for strict $\mathscr{M}$-categories. We may change our definition of unital precategories so that $\mathcal{F}_{\geq 2}$ becomes automatically a usual unital $\mathscr{M}$-categories but this won't change the outcome of the homotopy theory.
2. The fact that the outcome of the homotopy theory won't change if we increase the unitality conditions reflects the validity of *Simpson's conjecture* about weak identity morphisms in a higher category.
3. The unital 2-constant precategories we will consider in a moment, have the property that $\mathcal{F}_{\geq 2}$ is already a unital $\mathscr{M}$-category.

The upcoming material is motivated by our desire to analyze the unit of the adjunction $\mathscr{M}_{\mathcal{S}}(X)_u \leftrightarrows \mathscr{M}\text{-}\mathbf{Cat}(X)$. For some reasons that will be clear in a moment we shall outline a subcategory of the category of 2-constant precategories.

Definition 7.3. *Say that a unital 2-constant precategory $\mathcal{F}$ is* perfectly 2*-constant if the preunital precategory $\mathcal{F}_{\geq 2}$ is a usual unital $\mathscr{M}$-category.*

Warning. From now on, when we say 2-constant we mean perfectly 2-constant. This is for simplicity only.

7.1 Associated 2-constant precategory

Let's consider again the adjunction $\mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X)$.

Definition 7.4. *Let $\mathcal{F}$ be a unital precategory and let $\eta : \mathcal{F} \longrightarrow \iota(|\mathcal{F}|)$ be the unity of the adjunction. Let $h_{ab} : \mathcal{F}(a,b) \longrightarrow \iota(|\mathcal{F}|)(a,b)$ be the component of this map at the initial entry (a,b).*

1. *Define the* associated 2-constant unital precategory *of $\mathcal{F}$ to be unital precategory $h^{\star}\iota(|\mathcal{F}|)$ obtained using the construction described in Scholium 3.2 applied to $\iota(|\mathcal{F}|)$ with respect to the maps h_{ab}. In particular $[h^{\star}\iota(|\mathcal{F}|)]_{\geq 2} = \iota(|\mathcal{F}|)$ is a usual unital $\mathscr{M}$-category.*

2. *Define the fundamental factorization for $\eta : \mathcal{F} \longrightarrow \iota(|\mathcal{F}|)$ as:*

$$\mathcal{F} \xrightarrow{\rho} h^{\star}\iota(|\mathcal{F}|) \xrightarrow{\epsilon} \iota(|\mathcal{F}|).$$

The map $\mathcal{F} \xrightarrow{\rho} h^{\star}\iota(|\mathcal{F}|)$ is the identity at the initial entries and the map $h^{\star}\iota(|\mathcal{F}|) \xrightarrow{\epsilon} \iota(|\mathcal{F}|)$ is the identity everywhere outside the initial entries.

Proposition 7.5. *With the previous notation, the following hold.*

1. *The map $\mathcal{F} \xrightarrow{\rho} h^{\star}\iota(|\mathcal{F}|)$ is an easy weak equivalence.*

2. *Let $L : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathcal{B}$ be a functor that takes easy weak equivalences to isomorphisms in $\mathcal{B}$. Then $L(\mathcal{F} \xrightarrow{\eta} \iota(|\mathcal{F}|))$ is an isomorphism in $\mathcal{B}$ if and only if $L(h^{\star}\iota(|\mathcal{F}|) \xrightarrow{\epsilon} \iota(|\mathcal{F}|))$ is an isomorphism in $\mathcal{B}$*

Proof. The component of ρ at (a, b) is the identity which is a wonderful weak equivalence, this gives Assertion (1). Assertion (2) is a consequence of Assertion (1) together with the fact that isomorphisms in any category $\mathcal{B}$ have the 3-for-2 property. ■

Remark 7.6. Note that the proposition holds also for functors from $\mathscr{M}_{\mathcal{S}}(X)$ (resp. $\mathscr{M}_{\mathcal{S}}(X)_{\star}$) to $\mathcal{B}$ that takes easy weak equivalences to isomorphisms.

7.2 The small object argument for 2-constant precategories

In the following we are interested in factoring the map $h^{\star}\iota(|\mathcal{F}|) \xrightarrow{\epsilon} \iota(|\mathcal{F}|)$ using the small object argument with respect to a subset of the localizing set $\mathbf{K}_X(\mathbf{I})$. We refer the reader to [11], [14] for a detailed account on the small object argument. The idea amounts to take sequentially pushout of coproduct of maps in $\mathbf{K}_X(\mathbf{I})$.

It is then important to have a careful analysis of such pushout. We start below with a proposition that allows us to reduce our analysis to the case of a pushout of a single map in $\mathbf{K}_X(\mathbf{I})$.

Lemma 7.7. *Let $\mathscr{B}$ be a category with all small colimits and let $f : A \longrightarrow E$ be the pushout of the coproduct $\coprod_{k\in K} C_k \xrightarrow{\coprod g_k} \coprod_{k\in K} D_k$:*

$$\begin{tikzcd}
\coprod_{k\in K} C_k \arrow[r, "q"] \arrow[d, "\coprod g_k"'] & A \arrow[d, "f"] \\
\coprod_{k\in K} D_k \arrow[r, "p"] & E
\end{tikzcd}$$

For every $k \in K$, let $f_k : A \longrightarrow E_k$ be the pushout of g_k along the attaching map $C_k \overset{i_k}{\hookrightarrow} \coprod_{k \in K} C_k \xrightarrow{q} A$:

$$\begin{tikzcd} C_k \arrow[r] \arrow[d, "g_k"'] & \coprod_k C_k \arrow[r] & A \arrow[d, "f_k"] \\ D_k \arrow[rr] & & E_k \end{tikzcd}$$

Let O be the colimit of the wide pushout $\{A \xrightarrow{f_k} E_k\}_{k \in K}$ and let $\delta : A \longrightarrow O$ be the canonical map going to the colimit.

Then we have an isomorphism $f \cong \delta$ in the comma category $(A \downarrow \mathscr{B})$. In particular we have an isomorphism $O \cong E$ in $\mathscr{B}$.

Proof. Let $\tau_k : E_k \longrightarrow O$ be the canonical map going to the colimit of the wide pushout. By definition of the colimit we have an equality $\delta = \tau_k \circ f_k$. Using τ_k we can extend the pushout square defining f_k to have the following commutative square.

$$\begin{tikzcd} C_k \arrow[r] \arrow[d, "g_k"'] & \coprod_k C_k \arrow[r] & A \arrow[d, "\delta"] \\ D_k \arrow[rr] & & O \end{tikzcd}$$

Let's regard this commutative square as a map $g_k \longrightarrow \delta$ in the arrow category $\mathscr{B}^{[1]}$. The universal property of the coproduct implies that there is a unique induced map $\coprod_k g_k \longrightarrow \delta$ satisfying the usual factorizations. The later map represents a commutative square in $\mathscr{B}$ as:

$$\begin{tikzcd} \coprod_{k \in K} C_k \arrow[r] \arrow[d, "\coprod g_k"] & A \arrow[d, "\delta"] \\ \coprod_{k \in K} D_k \arrow[r] & O \end{tikzcd}$$

But since coproducts in $\mathscr{B}^{[1]}$ are taking point-wise, it's not hard to see that the attaching map $\coprod_{k \in K} C_k \longrightarrow A$ is exactly the map q in the original diagram.

We leave the reader to check that this commutative square is the universal pushout square. That is, the object O equipped with the map δ and the other one, satisfies the universal property of the pushout of $\coprod_k g_k$ along the attaching map q. ∎

7.3 *co-Segalification* for 2-constant precategories

If we want to define a functor $\mathscr{S}$ that takes a 2-constant precategory $\mathcal{F}$ to a precategory that satisfies the co-Segal conditions the natural thing to do is to factor the map $\mathcal{F}(a, b) \longrightarrow |\mathcal{F}|(a, b)$ as a cofibration followed by a trivial fibration as follows.

$$\mathcal{F}(a, b) \hookrightarrow m \xrightarrow{\sim} |\mathcal{F}|(a, b).$$

After this we want to set $\mathscr{S}(\mathcal{F})(a,b) = m$ and $|\mathscr{S}(\mathcal{F})| = |\mathcal{F}|$. This gives a 2-constant diagram that satisfies the co-Segal conditions. The purpose of the following discussion is to show that this can be done as $\mathbf{K}_2$-injective replacement in $\mathscr{M}_{\mathcal{S}}(X)_u$ where $\mathbf{K}_2$ is a subset of the localizing set $\mathbf{K}_X(\mathbf{I})$.

Definition 7.8. *Define the* minimal localizing set *for 2-constant precategories as:*

$$\mathbf{K}_2 = \bigsqcup_{(a,b)\in X^2} \; \bigsqcup_{s\in \mathcal{S}_{\overline{X}}(a,b)^{op} |\, \boldsymbol{deg}(s)=2} \{\Psi_s(\alpha_{\downarrow \mathrm{Id_V}}); \quad \alpha \in \mathbf{I}\}.$$

7.3.1 Pushout of an element of $\mathbf{K}_2$

Let $\Psi_s(\alpha)$ be an element of $\mathbf{K}_2$. We want to calculate the pushout of such morphism. But before doing this, let's recall some facts about the adjunction $\mathrm{Ev}_{u_s} : \mathscr{M}_{\mathcal{S}}(X)_u \leftrightarrows \mathscr{M}^{[1]} : \Psi_s$.

Remark 7.9. Let $\mathcal{F}$ be a unital 2-constant precategory and let $\mathcal{G}$ be any unital precategory.

1. A commutative square in $\mathscr{M}_{\mathcal{S}}(X)_u$

$$\begin{tikzcd} \Psi_s(\alpha) \arrow[r, "\sigma"] \arrow[d, "\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})"'] & \mathcal{F} \arrow[d, "\theta"] \\ \Psi_s(\mathrm{Id}_V) \arrow[r] & \mathcal{G} \end{tikzcd} \tag{7.3.1}$$

   is equivalent by adjointness to a commutative square in $\mathscr{M}^{[1]}$:

$$\begin{tikzcd} \alpha \arrow[r, "\sigma"] \arrow[d, "\alpha_{\downarrow \mathrm{Id_V}}"'] & \mathcal{F}(u_s) \arrow[d, "\theta"] \\ \mathrm{Id}_V \arrow[r] & \mathcal{G}(u_s) \end{tikzcd} \tag{7.3.2}$$

2. The commutative diagram (7.3.2) in $\mathscr{M}^{[1]}$ is equivalent to a commutative cube in $\mathscr{M}$. And if we write this cube we find that (7.3.2) is equivalent to having the commutative diagram (7.3.3) below, together with a lifting for the square defined by α and $\mathcal{G}(u_s)$.

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\alpha"'] & \mathcal{F}(a,b) \arrow[r] \arrow[d] & \mathcal{G}(a,b) \arrow[d] \\ V \arrow[r] \arrow[urr, dotted] & |\mathcal{F}|(a,b) \arrow[r] & \mathcal{G}(s) \end{tikzcd} \tag{7.3.3}$$

   In this diagram, the square on the left represents σ, and the one on the right represents θ. The whole commutative square represents the composite $\theta \circ \sigma$. The lifting $V \longrightarrow \mathcal{G}(a,b)$ and the commutativity of the lower triangle determine the map $\mathrm{Id}_V \longrightarrow \mathcal{G}(u_s)$ in the diagram (7.3.2).

Lemma 7.10. *Let $s = (a, -, b)$ be a generic 1-morphism of degree 2 and let and $\Psi_s : \mathscr{M}^{[1]} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$ be the left adjoint to the evaluation at $u_s : (a, b) \longrightarrow s$. Let $\alpha : U \longrightarrow V$ be a morphism of $\mathscr{M}$ and let $\mathcal{F}$ be a unital <u>2-constant</u> precategory such that $\mathcal{F}_{\geq 2}$ is a strict unital $\mathscr{M}$-category.*

Let $\sigma : \Psi_s(\alpha_{\downarrow \mathrm{Id_V}}) \longrightarrow \mathcal{F}$ be a morphism in $\mathscr{M}_{\mathcal{S}}(X)_u$ and let $\mathcal{E} = \Psi_s(\mathrm{Id}_V) \cup^{\Psi_s(\alpha)} \mathcal{F}$ be the object obtained by the following pushout diagram in $\mathscr{M}_{\mathcal{S}}(X)_u$.

$$\begin{tikzcd}
\Psi_s(\alpha) \arrow[rr, "\sigma"] \arrow[d, "\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})"'] & & \mathcal{F} \arrow[d, "\varepsilon"] \\
\Psi_s(\mathrm{Id}_V) \arrow[r] & \Psi_s(\mathrm{Id}_V) \cup^{\Psi_s(\alpha)} \mathcal{F} &
\end{tikzcd}$$

Then $\mathcal{E} = \Psi_s(\mathrm{Id}_V) \cup^{\Psi_s(\alpha)} \mathcal{F}$ is also a <u>2-constant</u> precategory and has the following properties.

1. *If $(a', b') \neq (a, b)$ then the natural transformation $\varepsilon : \mathcal{F}_{a'b'} \longrightarrow \mathcal{E}_{a'b'}$ is an isomorphism.*
2. *For the entry (a, b) the natural transformation $\varepsilon_{\geq 2} : \mathcal{F}_{ab,\geq 2} \longrightarrow \mathcal{E}_{ab,\geq 2}$ is an isomorphism in $\mathrm{Hom}[\mathcal{S}_{\overline{\mathrm{X}}}(a, b)^{op}_{\geq 2}, \mathscr{M}]$.*
3. *In particular we have an isomorphism $\mathcal{F}_{\geq 2} \cong \mathcal{E}_{\geq 2}$ of unital $\mathscr{M}$-categories*

Proof. We will construct the 2-constant diagram $\mathcal{E}$ and show that it satisfies the universal property of the pushout. For simplicity we will denote by $|\mathcal{F}|(a, b)$ the hom-object of the $\mathscr{M}$-category defined by $\mathcal{F}_{\geq 2}$.

First observe that since $\mathcal{F}$ is 2-constant, the map $\mathcal{F}(u_s)$ is just a map $\mathcal{F}(a, b) \longrightarrow |\mathcal{F}|(a, b)$. The idea of the proof is that taking the pushout is equivalent to factoring this map through a pushout of α.

Consider the attaching map $\sigma : \Psi_s(\alpha) \longrightarrow \mathcal{F}$. By adjointness, this map corresponds to a unique morphism in $\mathscr{M}^{[1]}$, which in turn corresponds to a commutative square in $\mathscr{M}$:

$$\begin{tikzcd}
U \arrow[r, "q"] \arrow[d, "\alpha"'] & \mathcal{F}(a, b) \arrow[d, "\mathcal{F}(u_s)"] \\
V \arrow[r, "p"] & {|\mathcal{F}|(a, b)}
\end{tikzcd} \tag{7.3.4}$$

Define $\mathcal{E}(a, b)$ as the object we get when we take the pushout of α along q:

$$V \xleftarrow{\alpha} U \xrightarrow{q} \mathcal{F}(a, b). \tag{7.3.5}$$

Let $\varepsilon : \mathcal{F}(a, b) \longrightarrow \mathcal{E}(a, b)$ and $i_V : V \longrightarrow \mathcal{E}(a, b)$ be the canonical maps. The map $\varepsilon : \mathcal{F}(a, b) \longrightarrow \mathcal{E}(a, b)$ is by definition the pushout of the map α along q.

If we use the universal property of the pushout with respect to the commutative square (7.3.4) above, we find a unique map $\gamma : \mathcal{E}(a,b) \longrightarrow |\mathcal{F}|(a,b)$ such that the factorizations hereafter hold.

$$\mathcal{F}(u_s) = \gamma \circ \varepsilon; \tag{7.3.6}$$

$$p = \gamma \circ i_V. \tag{7.3.7}$$

Given $(a',b') \in X^2$, let $h_{a'b'} : \mathcal{E}(a',b') \longrightarrow |\mathcal{F}|(a',b')$ be the identity map if $(a',b') \neq (a,b)$. And if $(a',b') = (a,b)$ we let $h_{ab} : \mathcal{E}(a,b) \longrightarrow |\mathcal{F}|(a,b)$ be the universal map γ above.

Define $\mathcal{E} \in \mathscr{M}_{\mathcal{S}}(X)_u$ to be the unital precategory obtained with the construction described in Scholium 3.2 applied to the $\mathscr{M}$-category $\mathcal{F}_{\geq 2}$ with respect to the maps $\{h_{a'b'}\}_{(a',b') \in X^2}$.

In particular we have the following characteristics.

- If $(a',b') \neq (a,b)$ we have $\mathcal{E}_{a'b'} = \mathcal{F}_{a'b'}$.
- And for the entry (a,b) we have $\mathcal{E}_{ab,\geq 2} = \mathcal{F}_{ab,\geq 2}$,
- The map $\mathcal{F}_{\geq 2} \longrightarrow \mathcal{E}_{\geq 2}$ is the identity.
- By definition the map $\mathcal{E}(u_s)$ is just $\gamma = h_{ab}$

It's important to notice that the identity map $\mathcal{F}_{\geq 2} \longrightarrow \mathcal{E}_{\geq 2}$ extends to a map $\Upsilon : \mathcal{F} \longrightarrow \mathcal{E}$, whose component at the entry (a,b) is the map $\varepsilon : \mathcal{F}(a,b) \longrightarrow \mathcal{E}(a,b)$, which is the pushout of α along q. The component of Υ at every other entry is the identity map.

If we incorporate the pushout object $\mathcal{E}(a,b)$ in the diagram (7.3.4), we find that the map $\Upsilon : \mathcal{F} \longrightarrow \mathcal{E}$ fits in the following commutative diagram.

$$\begin{tikzcd}
U \arrow[rr, "q"] \arrow[dd, "\alpha"'] & & \mathcal{F}(a,b) \arrow[dl, dotted, "\varepsilon"] \arrow[dd, "\mathcal{F}(u_s)"] \\
& \mathcal{E}(a,b) \arrow[dr, dotted, "\gamma"] & \\
V \arrow[ur, dotted, "i_V"] \arrow[rr, "p"'] & & {|\mathcal{F}|(a,b)}
\end{tikzcd}
\iff
\begin{tikzcd}
U \arrow[r, "q"] \arrow[d, "\alpha"'] & \mathcal{F}(a,b) \arrow[r, "\Upsilon=\varepsilon"] \arrow[d] & \mathcal{E}(a,b) \arrow[d, "\gamma=\mathcal{E}(u_s)"] \\
V \arrow[urr, dotted] \arrow[r, "p"'] & {|\mathcal{F}|(a,b)} \arrow[r, "\Upsilon", "\mathrm{Id}"'] & {|\mathcal{F}|(a,b) = \mathcal{E}(s)}
\end{tikzcd} \tag{7.3.8}$$

Now if we look at the right hand side of this diagram, then following Remark 7.9, we get by adjointness, a commutative square in $\mathscr{M}_{\mathcal{S}}(X)_u$:

$$\begin{tikzcd}
\Psi_s(\alpha) \arrow[r, "\sigma"] \arrow[d, "\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})"'] & \mathcal{F} \arrow[d, "\Upsilon"] \\
\Psi_s(\mathrm{Id}_V) \arrow[r] & \mathcal{E}
\end{tikzcd}$$

The rest of the proof is to show that this is the universal commutative square i.e, that $\mathcal{E}$ equipped with the appropriate maps satisfies the universal property of the pushout.

Let $\mathcal{G}$ be an arbitrary unital precategory equipped with a copushout data $\Psi_s(\mathrm{Id}_V) \xrightarrow{\chi} \mathcal{G} \xleftarrow{\theta} \mathcal{F}$ that completes

$$\Psi_s(\mathrm{Id}_V) \xleftarrow{\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})} \Psi_s(\alpha) \xrightarrow{\sigma} \mathcal{F}$$

into a commutative square. Then following Remark 7.9, having such commutative square is uniquely equivalent to having the commutative square below and a lifting.

$$\begin{tikzcd}
U \arrow[r, "q"] \arrow[d, "\alpha"'] & \mathcal{F}(a,b) \arrow[r] \arrow[d] & \mathcal{G}(a,b) \arrow[d] \\
V \arrow[r, "p"'] \arrow[urr, dotted] & {|\mathcal{F}|(a,b)} \arrow[r] & \mathcal{G}(s)
\end{tikzcd}$$

Since the upper half triangle that ends at $\mathcal{G}(a,b)$ is commutative, the universal property of the pushout of α along q, gives a *unique map* $\zeta : \mathcal{E}(a,b) \longrightarrow \mathcal{G}(a,b)$ that satisfies the usual factorizations.

Another application of the universal property of the pushout with respect to the whole commutative square that ends at $\mathcal{G}(s)$, gives a unique map $\rho : \mathcal{E}(a,b) \longrightarrow \mathcal{G}(s)$ that satisfies the usual factorizations. But the maps $\theta \circ \gamma$ and $\mathcal{G}(u_s) \circ \zeta$ both solve for ρ the required factorizations, therefore by uniqueness of ρ we have an equality:

$$\theta \circ \gamma = \mathcal{G}(u_s) \circ \zeta.$$

The above facts can be summarized by saying that everything commutes in the following diagram.

$$\begin{tikzcd}
U \arrow[r, "q"] \arrow[d, "\alpha"'] & \mathcal{F}(a,b) \arrow[r] \arrow[d, "\mathcal{F}(u_s)"'] & \mathcal{E}(a,b) \arrow[r, "\zeta"] \arrow[d, "\mathcal{E}(u_s)"] & \mathcal{G}(a,b) \arrow[d] \\
V \arrow[r, "p"'] \arrow[urr, dotted] \arrow[urrr, dotted] & {|\mathcal{F}|(a,b)} \arrow[r, "\mathrm{Id}"'] & {|\mathcal{F}|(a,b)} \arrow[r] & \mathcal{G}(s)
\end{tikzcd} \tag{7.3.9}$$

Since $\mathcal{E} = \mathcal{F}$ everywhere except at the entry (a,b), we see that the map ζ and the data for the map $\theta : \mathcal{F} \longrightarrow \mathcal{G}$ determine a unique map $\zeta : \mathcal{E} \longrightarrow \mathcal{G}$ such that:

$$\theta = \zeta \circ \Upsilon. \tag{7.3.10}$$

Note also that the lifting $V \longrightarrow \mathcal{G}(a,b)$ is precisely the composite $\zeta \circ i_V$, where $i_V : V \longrightarrow \mathcal{E}(a,b)$ is the lifting for the square associated to $\mathcal{E}$. It follows that if we look at the diagram (7.3.9) in the arrow category $\mathscr{M}^{[1]}$, we get a commutative diagram:

$$\begin{tikzcd}
\alpha \arrow[rr, "\sigma"] \arrow[dd, "\alpha_{\downarrow \mathrm{Id_V}}"] & & \mathcal{F}(u_s) \arrow[dl, dotted, "\Upsilon"] \arrow[dd, "\theta"] \\
& \mathcal{E}(u_s) \arrow[dr, dotted, "\zeta"] & \\
\mathrm{Id}_V \arrow[rr] \arrow[ur, dotted] & & \mathcal{G}(u_s)
\end{tikzcd} \tag{7.3.11}$$

Now the uniqueness of the adjunct map implies that the given map $\Psi_s(\mathrm{Id}_V) \xrightarrow{\chi} \mathcal{G}$ is the composite of the canonical map $\Psi_s(\mathrm{Id}_V) \longrightarrow \mathcal{E}$ and ζ. Putting this together with the previous factorization (7.3.10), we see that $\mathcal{E}$ satisfies the universal property of the pushout and the lemma follows. ■

The next lemma tells us how to calculate the wide pushout of maps $\mathcal{F} \longrightarrow \mathcal{E}$ such as the one we've just constructed.

Lemma 7.11. *Let $\{\Upsilon_i : \mathcal{F} \longrightarrow \mathcal{E}_i\}_{i \in S}$ be a small family of morphisms between 2-constant precategories in $\mathscr{M}_{\mathcal{S}}(X)_u$. Assume that each morphism $\Upsilon_i : \mathcal{F} \longrightarrow \mathcal{E}_i$ is such that the induced $\mathscr{M}$-functor $\Upsilon_{i,\geq 2} : \mathcal{F}_{\geq 2} \longrightarrow \mathcal{E}_{i,\geq 2}$ is an isomorphism of $\mathscr{M}$-categories.*

Let $\mathcal{E}_\infty$ be the wide pushout in $\mathscr{M}_{\mathcal{S}}(X)_u$ of the maps Υ_i. Then the following hold.

1. *$\mathcal{E}_\infty$ is also a unital 2-constant precategory.*
2. *The canonical maps $\mathcal{E}_i \longrightarrow \mathcal{E}_\infty$ and $\mathcal{F} \longrightarrow \mathcal{E}_\infty$ induce isomorphisms between the respective categories.*
$$\mathcal{E}_{i,\geq 2} \xrightarrow{\cong} \mathcal{E}_{\infty,\geq 2}, \quad \mathcal{F}_{\geq 2} \xrightarrow{\cong} \mathcal{E}_{\infty,\geq 2}$$

Proof. Take $\mathcal{E}_{\infty,\geq 2}$ to be the object obtained by taking the wide pushout of the isomorphisms $\Upsilon_{i,\geq 2} : \mathcal{F}_{\geq 2} \longrightarrow \mathcal{E}_{i,\geq 2}$. Clearly the canonical maps $\mathcal{E}_{i,\geq 2} \xrightarrow{\cong} \mathcal{E}_{\infty,\geq 2}$ and $\mathcal{F}_{\geq 2} \xrightarrow{\cong} \mathcal{E}_{\infty,\geq 2}$ are isomorphisms.

One gets the initial entry $\mathcal{E}_\infty(a,b)$ together with the unique map $\mathcal{E}_\infty(a,b) \to \mathcal{E}_{\infty,\geq 2}(a,b)$ by taking the wide pushout of $\mathcal{F}(u_s) \longrightarrow \mathcal{E}_i(u_s)$ in the arrow category $\mathscr{M}^{[1]}$. Just like before $\mathcal{E}_\infty$ is also a unital precategory. The canonical maps $\mathcal{E}_{i,\geq 2} \xrightarrow{\cong} \mathcal{E}_{\infty,\geq 2}$ and $\mathcal{F}_{\geq 2} \xrightarrow{\cong} \mathcal{E}_{\infty,\geq 2}$ extend to morphisms in $\mathscr{M}_{\mathcal{S}}(X)_u$
$$\mathcal{E}_i \longrightarrow \mathcal{E}_\infty; \quad \mathcal{F} \longrightarrow \mathcal{E}_\infty.$$

Moreover for each i, the canonical map $\mathcal{F} \longrightarrow \mathcal{E}_\infty$ is the composite of $\mathcal{F} \longrightarrow \mathcal{E}_i$ and $\mathcal{E}_i \longrightarrow \mathcal{E}_\infty$. This means that we have a natural cocone that ends at $\mathcal{E}_\infty$. The reader can easily check that this cocone is the universal one i.e, $\mathcal{E}_\infty$ equipped with this cocone satisfies the universal property of the wide pushout. ■

Proposition-Definition 7.12. *Let $\mathscr{S}_2 : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_u$ be the $\mathbf{K}_2$-injective replacement functor obtained by applying the gluing construction and the small object argument. Denote by $\tau : Id \longrightarrow \mathscr{S}$ the induced natural transformation. Let $\mathcal{F}$ be a unital 2-constant precategory such that $\mathcal{F}_{\geq 2}$ is a strict unital $\mathscr{M}$-category. Then the following hold.*

1. *$\mathscr{S}(\mathcal{F})$ is also a unital 2-constant precategory such that $\mathscr{S}(\mathcal{F})_{\geq 2}$ is unital $\mathscr{M}$-category. The map $\tau_{\geq 2} : \mathcal{F}_{\geq 2} \longrightarrow \mathscr{S}(\mathcal{F})_{\geq 2}$ is an isomorphism of $\mathscr{M}$-categories.*
2. *$\mathscr{S}(\mathcal{F})$ satisfies the co-Segal conditions. And the canonical map $\mathscr{S}(\mathcal{F}) \longrightarrow |\mathscr{S}(\mathcal{F})|$ is an easy weak equivalence. Indeed we have an isomorphism $|\mathscr{S}(\mathcal{F})| \cong \mathscr{S}(\mathcal{F})_{\geq 2}$.*
3. *Let $L : \mathscr{M}_{\mathcal{S}}(X)_u \longrightarrow \mathcal{B}$ be a functor that sends easy weak equivalences to isomorphisms and takes any $\mathbf{K}_2$-cell complex to an isomorphism. Then for all $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_u$, the image of the unit $\eta : \mathcal{F} \longrightarrow \iota(|\mathcal{F}|)$ by L is an isomorphism in $\mathcal{B}$.*

The functor $\mathscr{S}$ will be called the 2-constant co-Segalification functor*.*

Proof. The full subcategory of 2-constant precategories is closed under directed colimits (and limits) since they are computed level-wise. Thanks to Lemma 7.10, we know that if $\mathcal{F}$ is 2-constant, then the pushout of any $\Psi_s(\alpha_{\downarrow \mathrm{Id_V}})$ along any $\Psi_s(\alpha) \longrightarrow \mathcal{F}$ is a morphism of 2-constant precategories which is moreover an isomorphism on the *underlying categories*.

Given any pushout data defined by a coproduct of maps in $\mathbf{K}_2$

$$(\coprod \Psi_s(\mathrm{Id}_V)) \xleftarrow{\coprod \Psi_s(\alpha_{\downarrow \mathrm{Id_V}})} (\coprod \Psi_s(\alpha)) \longrightarrow \mathcal{F},$$

we know thanks to Lemma 7.11 and Lemma 7.7 that the canonical map $\mathcal{F} \longrightarrow \mathcal{F}_1$ toward the pushout-object is again a map of 2-constant precategories. Moreover the induced map $|\mathcal{F}| \longrightarrow |\mathcal{F}_1|$ is an isomorphism of unital $\mathscr{M}$-categories. But these pushouts are precisely the ones we use to construct $\mathscr{S}(\mathcal{F})$.

Therefore $\mathscr{S}(\mathcal{F})$ is a 2-constant precategory as a directed colimit of 2-constant precategories; and the map $\eta : \mathcal{F} \longrightarrow \mathscr{S}(\mathcal{F})$ is a $\mathbf{K}_2$-cell complex with the property that the induced map $|\eta| : |\mathcal{F}| \longrightarrow |\mathscr{S}(\mathcal{F})|$ is an isomorphism of unital $\mathscr{M}$-categories. This proves Assertion (1).

Assertion (2) follows from the fact that $\mathscr{S}(\mathcal{F})$ in $\mathbf{K}_2$-injective, and by adjointness this means that the unique map $\mathscr{S}(\mathcal{F})(a,b) \longrightarrow |\mathscr{S}(\mathcal{F})|(a,b)$, viewed as an object of $\mathscr{M}^{[1]}$, is $\alpha_{\downarrow \mathrm{Id_V}}$-injective for all generating cofibration α.
But this in turn simply means that we have a lifting to any problem defined by α and $\mathscr{S}(\mathcal{F})(a,b) \longrightarrow |\mathscr{S}(\mathcal{F})|(a,b)$ (see Proposition 6.3). Consequently $\mathscr{S}(a,b) \longrightarrow |\mathscr{S}|(a,b)$ is a trivial fibration, in particular a weak equivalence, therefore $\mathscr{S}(\mathcal{F})$ is a co-Segal category. This also proves at the same time that the canonical map $\mathscr{S}(\mathcal{F}) \longrightarrow |\mathscr{S}(\mathcal{F})|$, whose components at the initial entries are precisely the maps $\mathscr{S}(\mathcal{F})(a,b) \longrightarrow |\mathscr{S}(\mathcal{F})|(a,b)$ is an easy weak equivalence.

Now for Assertion (3) it suffices to use the factorization $\mathcal{F} \longrightarrow \iota(|\mathcal{F}|)$ given in Definition 7.4 and observe that we can factor again $\mathcal{F} \longrightarrow \iota(|\mathcal{F}|)$ as follows.

$$\mathcal{F} \xrightarrow{\rho} h^{\star}\iota(|\mathcal{F}|) \xrightarrow{\tau} \mathscr{S}[h^{\star}\iota(|\mathcal{F}|)] \xrightarrow{\sim} \mathscr{S}[h^{\star}\iota(|\mathcal{F}|)]_{\geq 2} \xrightarrow{\cong} \iota(|\mathcal{F}|).$$

Since L sends every $\mathbf{K}_2$-cell complex to an isomorphism, then L sends the map τ to an isomorphism in $\mathcal{B}$. The map $\mathscr{S}[h^{\star}\iota(|\mathcal{F}|)] \xrightarrow{\sim} \mathscr{S}[h^{\star}\iota(|\mathcal{F}|)]_{\geq 2}$ is an easy weak equivalence by the previous assertion we've just proved. The map ρ is alway an easy weak equivalence thanks to Proposition 7.5. In the end we find that the image of $\mathcal{F} \longrightarrow |\mathcal{F}|$ by L is also an isomorphism. ■

8 New model structure on unital precategories

8.1 Enlarging the cofibrations

In the following we would like to have a left proper model structure $\mathscr{M}_{\mathbb{S}}(X)_u$ such that the set of generating cofibrations contains the localizing set $\mathbf{K}_X(\mathbf{I})$ introduced above.
Denote by $\mathbf{I}^+_{\mathscr{M}_{\mathbb{S}}(X)_u}$ the set

$$\mathbf{I}_{\mathscr{M}_{\mathbb{S}}(X)_u} \coprod \mathbf{K}_X(\mathbf{I}).$$

Lemma 8.1. *Given any pair $(a,b) \in X^2$, for all $\sigma \in \mathbf{I}^+_{\mathscr{M}_{\mathbb{S}}(X)_u}$ the component $\sigma_{(a,b)}$ is a cofibration.*

Proof. The statement is clear if $\sigma \in \mathbf{I}_{\mathscr{M}_{\mathbb{S}}(X)_u}$. If $\sigma \in \mathbf{K}_X(\mathbf{I})$, the component $\sigma_{(a,b)}$ is a retract of either α, Id_I or $\mathrm{Id}_I \coprod \alpha$ with $\alpha \in \mathbf{I}$ (see Proposition 4.12 and Theorem 3.18). ■

8.2 The model structure

We show below that there is a left proper combinatorial model structure on $\mathscr{M}_{\mathbb{S}}(X)_u$ with $\mathbf{I}^+_{\mathscr{M}_{\mathbb{S}}(X)_u}$ as the set of generating cofibrations and $\mathscr{W}_{\mathscr{M}_{\mathbb{S}}(X)_{ue}}$ as the class of weak equivalences.

We use Smith's recognition Theorem for combinatorial model categories (see for example Barwick [8, Proposition 2.2]). This theorem gives the possibility to construct a combinatorial model category out of two data consisting of a class $\mathscr{W}$ of morphisms whose elements are called *weak equivalences*; and a set $\mathbf{I}$ of *generating cofibrations*.

Our method is classical and the argument is present in Pellissier's PhD thesis [18]; it is also used by Lurie [17], Simpson [21] and others. But in doing so, we actually reprove (implicitly) a derived version of Smith's theorem that has been outlined by Lurie [17, Proposition A.2.6.13]. This version asserts that the resulting combinatorial model structure is automatically left proper. So we will just use that proposition that we recall hereafter with the same notation as in Lurie's book.

Proposition 8.2. *Let $\mathbf{A}$ be a presentable category. Suppose we are given a class W of morphisms of A, which we will call weak equivalences, and a (small) set C_0 of morphisms of $\mathbf{A}$, which we will call generating cofibrations. Suppose furthermore that the following assumptions are satisfied:*

(1) *The class W of weak equivalences is perfect ([17, Definition A.2.6.10]).*

(2) *For any diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] \arrow[d, "g"] & Y' \arrow[d, "g'"] \\ X'' \arrow[r] & Y'' \end{tikzcd}$$

in which both squares are coCartesian (=pushout square), f belongs to C_0, and g belongs W, the map g' also below to W.

(3) *If $g : X \longrightarrow Y$ is a morphism in $\mathbf{A}$ which has the right lifting property with respect to every morphism in C_0, then g belongs to W.*

Then there exists a left proper combinatorial model structure on $\mathbf{A}$ which may be described as follows:

(C) *A morphism $f : X \longrightarrow Y$ in $\mathbf{A}$ is a cofibration if it belongs to the weakly saturated class of morphisms generated by C_0.*

(W) *A morphism $f : X \longrightarrow Y$ in $\mathbf{A}$ is a weak equivalence if it belongs to W.*

(F) *A morphism $f : X \longrightarrow Y$ in $\mathbf{A}$ is a fibration if it has the right lifting property with respect to every map which is both a cofibration and a weak equivalence.*

Note. Here *perfectness* is a property of stability under filtered colimits and a generation by a small set W_0 (which is more often the intersection of W and the set of maps between presentable objects). The reader can find the exact definition in [17, Definition A.2.6.10].

Warning. We've used so far the letters f, g as functions so to avoid any confusion we will use σ, σ' instead.

Applying the previous proposition we get the following theorem.

Theorem 8.3. *Let $\mathscr{M}$ be a combinatorial monoidal model category which is left proper. Then for any set X there exists a combinatorial model structure on $\mathscr{M}_{\mathcal{S}}(X)_u$ which is left proper and which may be described as follows.*

1. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a weak equivalence if it's an easy weak equivalence i.e, if it's in $\mathscr{W}_{\mathscr{M}_{\mathcal{S}}(X)_{ue}}$.*
2. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a cofibration if it belongs to the weakly saturated class of morphisms generated by $\mathbf{I}^+_{\mathscr{M}_{\mathcal{S}}(X)_u}$.*
3. *A morphism $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a fibration if it has the right lifting property with respect to every map which is both a cofibration and a weak equivalence*

We will denote this model category by $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$. The identity functor

$$\mathrm{Id} : \mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_{ue+},$$

is a left Quillen equivalence

Proof. Condition (1) is straightforward because $\mathscr{W}_{\mathscr{M}_{\mathcal{S}}(X)_{ue}}$ is the class of weak equivalence in the combinatorial model category $\mathscr{M}_{\mathcal{S}}(X)_{ue}$. We also have Condition (3) since a map σ in $\mathbf{I}^+_{\mathscr{M}_{\mathcal{S}}(X)_u}$-inj is in particular in $\mathbf{I}_{\mathscr{M}_{\mathcal{S}}(X)_u}$-inj, therefore it's a trivial fibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ and thus an easy weak equivalence.

It remains to check that Condition (2) is also satisfied. Consider the following diagram as in the proposition.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\sigma"] \arrow[d] & \mathcal{G} \arrow[d] \\
\mathcal{F}' \arrow[r] \arrow[d, "\theta"] & \mathcal{G}' \arrow[d, "\theta'"] \\
\mathcal{F}'' \arrow[r] & \mathcal{G}''
\end{tikzcd}$$

If $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is in $\mathbf{I}^{+}_{\mathscr{M}_{\mathcal{S}}(X)_u}$, we have from Lemma 8.1 that each top-component

$$\sigma_{(a,b)} : \mathcal{F}(a,b) \longrightarrow \mathcal{G}(a,b),$$

is a cofibration in $\mathscr{M}$. Now as mentioned several times in the paper, pushouts in $\mathscr{M}_{\mathcal{S}}(X)_u$ are computed level-wise at each 1-morphisms (a,b). It follows that the top components in that diagram are obtained by pushout in $\mathscr{M}$; and since $\mathscr{M}$ is left proper we get that every top-component $\theta'_{(a,b)}$ is a weak equivalence, which means that θ' is an easy weak equivalence as desired.

A cofibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ is a cofibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$ and since we have the same weak equivalences then the identity functor is a Quillen equivalence. ■

9 Bousfield localizations

Warning. We would like to warn the reader about our upcoming notation for the left Bousfield localizations. We choose to include a small letter **c** (for "correct") as a superscript in both $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$ and $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ to mean that we are taking the left Bousfield localization with respect to the (same) set $\mathbf{K}_X(\mathbf{I})$. A more suggestive notation would be $\mathrm{L}^{\mathbf{K}_X(\mathbf{I})}\mathscr{M}_{\mathcal{S}}(X)_{ue}$ or $\mathbf{K}_X(\mathbf{I})^{-1}\mathscr{M}_{\mathcal{S}}(X)_{ue}$, but as the reader can see, this is too heavy to work with.

Instead we will use the notation $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue}$ and $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$.

Remark 9.1.

1. Since we have the same class of weak equivalences in $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ and in $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$, we have an equivalence of function complexes on these model structures. It follows that a map σ in $\mathscr{M}_{\mathcal{S}}(X)_u$ is a $\mathbf{K}_X(\mathbf{I})$-local equivalence in the model structure $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ if and only if it's a $\mathbf{K}_X(\mathbf{I})$-local equivalence in the model structure $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$.
2. A direct consequence of this is that the left Quillen equivalence $\mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_{ue+}$ given by the identity, will pass to a left Quillen equivalence $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ between the respective Bousfield localizations.

9.1 The first localized model category

We start with the localization of $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$.

Theorem 9.2. *Let $\mathscr{M}$ be a combinatorial monoidal model category which is left proper. Then for any set X there exists a combinatorial model structure on $\mathscr{M}_{\mathcal{S}}(X)_u$ which is left proper and which may be described as follows.*

1. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a weak equivalence if and only if it's a $\mathbf{K}_X(\mathbf{I})$-local equivalence.*

2. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is cofibration if it's a cofibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$.*

3. *Any fibrant object $\mathcal{F}$ is a co-Segal category.*

4. *We will denote this model category by $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$.*

5. *The identity of $\mathscr{M}_{\mathcal{S}}(X)_u$ determines the universal left Quillen functor*

$$L_+ : \mathscr{M}_{\mathcal{S}}(X)_{ue+} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+} .$$

This model structure is the left Bousfield localization of $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$ with the respect to the set $\mathbf{K}_X(\mathbf{I})$.

Definition 9.3. *Define a* co-Segalification functor *for $\mathscr{M}_{\mathcal{S}}(X)_u$ to be any fibrant replacement functor in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$.*

Proof of Theorem 9.2. The existence of the left Bousfield localization and the left properness is guaranteed by Smith's theorem on left Bousfield localization for combinatorial model categories. We refer the reader to Barwick [8, Theorem 4.7] for a precise statement. This model structure is again combinatorial.

For the rest of the proof we will use the following facts on Bousfield localization and the reader can find them in Hirschhorn's book [13].

1. A weak equivalence in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ is a $\mathbf{K}_X(\mathbf{I})$-*local weak equivalence*; we will refer them as *new weak equivalence*. And any easy weak equivalence (old one) is a new weak equivalence.

2. The new cofibrations are the same as the old ones and therefore the new trivial fibrations are just the old ones too. In particular a trivial fibration in the left Bousfield localization is an easy weak equivalence.

3. The fibrant objects are the $\mathbf{K}_X(\mathbf{I})$-local objects that are fibrant in the original model structure.

4. Every map in $\mathbf{K}_X(\mathbf{I})$ becomes a weak equivalence in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue}$, therefore an isomorphism in the homotopy category.

Let $\mathcal{F}$ be a fibrant object in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$, this means that the unique map $\mathcal{F} \longrightarrow *$ has the RLP with respect to any trivial cofibration. Now observe elements of $\mathbf{K}_X(\mathbf{I})$ are trivial cofibrations in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ because they were old cofibrations and become weak equivalences. So any fibrant $\mathcal{F}$ must be in particular $\mathbf{K}_X(\mathbf{I})$-injective and thanks to Lemma 6.4, we know that $\mathcal{F}$ satisfies the co-Segal conditions. ■

Corollary 9.4. *Let $\mathscr{S}$ be a co-Segalification functor, i.e a fibrant replacement in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$. Then a map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a weak equivalence in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ if and only if the map*

$$\mathscr{S}(\sigma) : \mathscr{S}(\mathcal{F}) \longrightarrow \mathscr{S}(\mathcal{G}),$$

is a level-wise weak equivalence of co-Segal $\mathscr{M}$-categories.

Proof. Since $\mathscr{S}$ is a fibrant replacement functor in the new model structure, then $\mathscr{S}(\mathcal{F})$ is a co-Segal category for all $\mathcal{F}$, by the second assertion of the previous theorem.

By the 3-for-2 property of weak equivalences in any model category, a map σ is a weak equivalence if and only if $\mathscr{S}(\sigma)$ is a weak equivalence. But $\mathscr{S}(\sigma)$ is a weak equivalence of fibrant objects in the Bousfield localization, therefore it's a weak equivalence in the original model structure.

In the end we see that σ is a weak equivalence in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ if and only if $\mathscr{S}(\sigma)$ is an easy weak equivalence in $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$. Now an easy weak equivalence between co-Segal categories is just a level-wise weak equivalence. ∎

Proposition 9.5. *For any set X and any $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_u$, the canonical map $\mathcal{F} \longrightarrow |\mathcal{F}|$ is an equivalence in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$ i.e, it's a $\mathbf{K}_X(\mathbf{I})$-local equivalence in $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$ (whence in $\mathscr{M}_{\mathcal{S}}(X)_{ue}$).*

Proof. Every element $\Psi_s(\alpha_{\downarrow \mathrm{Id}_V}) \in \mathbf{K}_X(\mathbf{I})$ becomes a trivial cofibration in $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue+}$, since they were cofibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$. In particular every $\mathbf{K}_X(I)$-cell complex (whence $\mathbf{K}_2$-cell complex) is a trivial cofibration. The proposition follows from Assertion (3) of Proposition 7.12. ∎

9.2 The second localized model category

We now localize the original model category $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ that doesn't contain a priori the set $\mathbf{K}_X(\mathbf{I})$ among the class of cofibrations. The theorem we give below is also a straightforward application of Smith's theorem for left proper combinatorial model category.

Theorem 9.6. *Let $\mathscr{M}$ be a combinatorial monoidal model category which is left proper. Then for any set X there exists a combinatorial model structure on $\mathscr{M}_{\mathcal{S}}(X)_u$which is left proper and which may be described as follows.*

1. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is a weak equivalence if and only if it's a $\mathbf{K}_X(\mathbf{I})$-local equivalence.*

2. *A map $\sigma : \mathcal{F} \longrightarrow \mathcal{G}$ is cofibration if it's a cofibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue}$. We will denote this model category by $\mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue}$*

3. *If a transferred model structure on $\mathscr{M}\text{-}\mathbf{Cat}(X)$ exists then the adjunction*

$$|-| : \mathscr{M}_{\mathcal{S}}(X)_{ue} \leftrightarrows \mathscr{M}\text{-}\mathbf{Cat}(X) : \iota,$$

descends to a Quillen adjunction:

$$|-|^{\mathbf{c}} : \mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue} \leftrightarrows \mathscr{M}\text{-}\mathbf{Cat}(X) : \iota.$$

In particular the inclusion $\iota : \mathscr{M}\text{-}\mathbf{Cat}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue}$ is again a right Quillen functor.

4. *The identity of $\mathscr{M}_{\mathcal{S}}(X)_u$ determines the universal left Quillen functor*

$$L : \mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)^{\mathbf{c}}_{ue}.$$

This model structure is the left Bousfield localization of $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ with the respect to the set $\mathbf{K}_X(\mathbf{I})$.

Proof. The existence and characterization of the Bousfield localization follows also from Smith theorem. The first two assertions are clear. Assertion (3) is a consequence of the universal property of the left Bousfield localization. Indeed we have a left Quillen functor $|-|: \mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X)$ that takes elements of $\mathbf{K}_X(I)$ to weak equivalences in $\mathscr{M}\text{-}\mathbf{Cat}(X)$ (Lemma 6.5). Therefore there exists a unique left Quillen functor $|-|^{\mathbf{c}}$: $\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}} \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X)$ such that we have an equality

$$|-|: \mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}\text{-}\mathbf{Cat}(X) = \mathscr{M}_{\mathcal{S}}(X)_{ue} \xrightarrow{\mathrm{L}} \mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}} \xrightarrow{|-|^{\mathbf{c}}} \mathscr{M}\text{-}\mathbf{Cat}(X)$$

■

10 The Quillen equivalence

We start first with the following lemma which is useful to establish the Quillen equivalence.

Lemma 10.1. *Let $\mathscr{M}$ be a symmetric monoidal model category. Assume that the transferred model structure on $\mathscr{M}\text{-}\mathbf{Cat}(X)$ exists. Let $\sigma : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between $\mathscr{M}$-categories regarded also as morphism in $\mathscr{M}_{\mathcal{S}}(X)_u$.*

Then σ is a $\mathbf{K}_X(\mathbf{I})$-local equivalence in $\mathscr{M}_{\mathcal{S}}(X)_{ue}$ (whence in $\mathscr{M}_{\mathcal{S}}(X)_{ue+}$) if and only if it's a weak equivalence in $\mathscr{M}\text{-}\mathbf{Cat}(X)$.

Proof. The if part is clear since a weak equivalence in $\mathscr{M}\text{-}\mathbf{Cat}(X)$ is an easy weak equivalence in $\mathscr{M}_{\mathcal{S}}(X)_u$ and therefore it's also a weak equivalence in the Bousfield localization. But the weak equivalences in the Bousfield localization are precisely the $\mathbf{K}_X(\mathbf{I})$-local equivalences.

Let's now assume that $\sigma : \mathcal{C} \longrightarrow \mathcal{D}$ becomes a $\mathbf{K}_X(\mathbf{I})$-local equivalence.

Use the axiom of factorization in the model category $\mathscr{M}\text{-}\mathbf{Cat}(X)$ to factor σ as trivial cofibration followed by a fibration:

$$\sigma = \mathcal{C} \underset{\sim}{\xhookrightarrow{\sigma_1}} \mathcal{E} \xrightarrow{\sigma_2} \mathcal{D}.$$

Since we know that inclusion $\iota : \mathscr{M}\text{-}\mathbf{Cat}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_{ue}$ is a right Quillen functor when we pass to the left Bousfield localization $\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}$, it follows that the map $\sigma_2 : \mathcal{E} \longrightarrow \mathcal{D}$ is a fibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}$. Now as σ_1 is a weak equivalence of $\mathscr{M}$-categories, by the if part, it's also a $\mathbf{K}_X(\mathbf{I})$-local equivalence.

By the 3-for-2 property of $\mathbf{K}_X(\mathbf{I})$-local equivalences applied to the equality $\sigma = \sigma_2 \circ \sigma_1$, we find that σ_2 is also a $\mathbf{K}_X(\mathbf{I})$-local equivalence.

In the end we see that σ_2 is simultaneously a fibration in the Bousfield localization $\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}$ and a $\mathbf{K}_X(\mathbf{I})$-local equivalence, therefore it's a trivial fibration in $\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}$. But a trivial fibration in this left Bousfield localization is the same as a trivial fibration in the original model structure. This means that σ_2 is usual local trivial fibration.

Then $\sigma = \sigma_2 \circ \sigma$ is the composite of weak equivalence of $\mathscr{M}$-categories and the lemma follows. ■

10.1 The main Theorem

Theorem 10.2. *Let $\mathscr{M}$ be a symmetric monoidal model category that is combinatorial and left proper. For any set X the following hold.*

1. *The left Quillen equivalence $\mathscr{M}_{\mathcal{S}}(X)_{ue} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_{ue+}$ induced by the identity of $\mathscr{M}_{\mathcal{S}}(X)_u$ descends to a left Quillen equivalence between the respective left Bousfield localizations with respect to $\mathbf{K}_X(\mathbf{I})$:*
$$\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}} \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_{ue+}^{\mathbf{c}}.$$

2. *If the transferred model structure on $\mathscr{M}\text{-}\mathbf{Cat}(X)$ exists, then the adjunction*
$$|-|^{\mathbf{c}} : \mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}} \leftrightarrows \mathscr{M}\text{-}\mathbf{Cat}(X) : \iota,$$
is a Quillen equivalence.

3. *The diagram $\mathscr{M}_{\mathcal{S}}(X)_{ue+}^{\mathbf{c}} \leftarrow \mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}} \xrightarrow{|-|^{\mathbf{c}}} \mathscr{M}\text{-}\mathbf{Cat}(X)$ is a zigzag of Quillen equivalences. In particular we have the following equivalences between the homotopy categories.*
$$\mathbf{ho}[\mathscr{M}_{\mathcal{S}}(X)_{ue+}^{\mathbf{c}}] \xleftarrow{\simeq} \mathbf{ho}[\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}] \xrightarrow{\simeq} \mathbf{ho}[\mathscr{M}\text{-}\mathbf{Cat}(X)].$$

Proof. We only need to prove the second assertion, namely that we have a Quillen equivalence
$$|-|^{\mathbf{c}} : \mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}} \leftrightarrows \mathscr{M}\text{-}\mathbf{Cat}(X) : \iota.$$

For this it suffices to show that if $\mathcal{F} \in \mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}$ is cofibrant and if $\mathcal{C} \in \mathscr{M}\text{-}\mathbf{Cat}(X)$ is fibrant, then a map $\sigma : \mathcal{F} \longrightarrow \iota(\mathcal{C})$ is a weak equivalence in $\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}$ if and only if the adjunct map $\overline{\sigma} : |\mathcal{F}| \longrightarrow \mathcal{C}$ is a weak equivalence in $\mathscr{M}\text{-}\mathbf{Cat}(X)$.

Since the inclusion $\iota : \mathscr{M}\text{-}\mathbf{Cat}(X) \longrightarrow \mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}$ is fully faithful, we will identity $\mathcal{C}$ and $\iota(\mathcal{C})$ and $|\mathcal{F}|$ with $\iota(|\mathcal{F}|)$. Let $\eta : \mathcal{F} \longrightarrow |\mathcal{F}|$ be the unit of the adjunction. Then all three maps fit in a commutative triangle in $\mathscr{M}_{\mathcal{S}}(X)_{ue}^{\mathbf{c}}$:

$$\begin{tikzcd} \mathcal{F} \arrow[d, "\eta"'] \arrow[dr, "\sigma"] & \\ |\mathcal{F}| \arrow[r, "\overline{\sigma}"'] & \mathcal{C} \end{tikzcd} \tag{10.1.1}$$

Thanks to Proposition 9.5 we know that $\eta : \mathcal{F} \longrightarrow |\mathcal{F}|$ is always a $\mathbf{K}_X(\mathbf{I})$-local equivalence. Then by 3-for-2 we see that σ is a $\mathbf{K}_X(\mathbf{I})$-local equivalence if and only if $\overline{\sigma}$ is. Now thanks to Lemma 10.1 we know that $\overline{\sigma}$ is a $\mathbf{K}_X(\mathbf{I})$-local equivalence if and only if it's an equivalence in $\mathscr{M}\text{-}\mathbf{Cat}(X)$ and the theorem follows. ■

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