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ON THE V-STATES FOR THE GENERALIZED QUASI-GEOSTROPHIC EQUATIONS

ZINEB HASSAINIA AND TAOUFIK HMIDI

ABSTRACT. We prove the existence of the V-states for the generalized inviscid SQG equations with $\alpha \in]0,1[$. These structures are special rotating simply connected patches with $m-$fold symmetry bifurcating from the trivial solution at some explicit values of the angular velocity. This produces, inter alia, an infinite family of non stationary global solutions with uniqueness.

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1. INTRODUCTION

In this paper we shall investigate some special structures of the vortical motions for the generalized inviscid surface quasi-geostrophic equation arising in fluid dynamics. This model

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describes the evolution of the potential temperature θ by the transport equation,

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ u = -\nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\theta, \\ \theta_{|t=0} = \theta_0. \end{cases} \tag{1}$$

Here u refers to the velocity field, $\nabla^\perp = (-\partial_2, \partial_1)$ and α is a real parameter taken in the interval $[0,1[$. The operator $(-\Delta)^{-1+\frac{\alpha}{2}}$ is of convolution type and is defined by

$$(-\Delta)^{-1+\frac{\alpha}{2}}\theta(x) = \frac{C_\alpha}{2\pi}\int_{\mathbb{R}^2}\frac{\theta(y)}{|x-y|^\alpha}dy \tag{2}$$

with $C_\alpha = \frac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$. This model was proposed by Córdoba et al. in [11] as an interpolation between Euler equations and the surface quasi-geostrophic model, hereafter denoted by SQG, corresponding to $\alpha = 0$ and $\alpha = 1$, respectively. The SQG equation was used by Juckes [21] and Held et al. [17] as a concise model of the atmosphere circulation near the tropopause. It was also developed by Lapeyre and Klein [27] to describe the ocean dynamics in the upper layers. We note that there is a strong mathematical and physical analogy with the three-dimensional incompressible Euler equations, and it can be viewed as a simplified model for that system; see [10] for details.

The local well-posedness of classical solutions can be performed in various function spaces. For instance, this was implemented in the framework of Sobolev space [9] by using the commutator theory. However, it is so delicate to extend the Yudovich theory of weak solutions known for the two-dimensional Euler equations [38] to the case $\alpha > 0$ because the velocity is in general below the Lipschitz class. Nonetheless, one can say more about this issue for some special class of concentrated vortices. More precisely, when the initial datum has a vortex patch structure, that is, $\theta_0(x) = \chi_D$ is the characteristic function of a bounded simply connected smooth domain D, then there is a unique local solution in the patch form $\theta(t) = \chi_{D_t}$. In this case, the boundary motion of the domain D_t is described by the contour dynamics formulation; see the papers [15, 32]. The global persistence of the boundary regularity is only known for $\alpha = 0$ according to the result of Chemin [7]. For $\alpha > 0$ there are some numerical simulations showing the singularity formation in finite time, see for instance [11]. The technique of contour dynamics was originally devised by Zabusky et al. [39] and has found many applications in the study of two-dimensional flows. We shall use this technique to track the boundary motion of the patch for the generalized SQG equation. According to Green formula one can recover the velocity from the boundary through the formula,

$$u(t,x) = \frac{C_\alpha}{2\pi}\int_{\partial D_t}\frac{1}{|x-\xi|^\alpha}d\xi \tag{3}$$

where $d\xi$ denotes the complex integration over the positively oriented curve ∂D_t. To write down the equation of the boundary, one can use for instance the Lagrangian parametrization $\gamma_t : [0,2\pi] \to \mathbb{C}$, given by the nonlinear ode,

$$\begin{cases} \partial_t\gamma(t,\sigma) = u(t,\gamma(t,\sigma)), \\ \gamma(0,\sigma) = \gamma_0(\sigma) \end{cases}$$

where γ_0 is a periodic smooth parametrization of the initial boundary and consequently the contour dynamics equation becomes

$$\partial_t\gamma(t,\sigma) = \frac{C_\alpha}{2\pi}\int_0^{2\pi} \frac{\partial_s\gamma(t,s)}{|\gamma(t,\sigma)-\gamma(t,s)|^\alpha} ds. \tag{4}$$

The main objective of this paper is to focus on some special vortices, called V-states or rotating patches, whose dynamics is described by a rigid body transformation. The problem consists in finding some domains D subject to a uniform rotation around their centers of mass. In which case the support of the patch D_t does not change its shape and is given by $D_t = \mathbf{R}_{x_0,\Omega t}D$, where $\mathbf{R}_{x_0,\Omega t}$ stands for the planar rotation with center x_0 and angle Ωt. The parameter Ω is called the angular velocity of the rotating domain.

This problem was investigated first for the two-dimensional Euler equations ($\alpha = 0$) a long time ago and still a subject of intensive research combining analytical and numerical studies. It is worthy noting that explicit non trivial rotating patches are known in the literature and goes back to Kirchhoff [24] who discovered that an ellipse of semi-axes a and b is subject to a perpetual rotation with uniform angular velocity $\Omega = ab/(a+b)^2$; see for instance [3, p. 304] and [26, p. 232]. In the seventies of the last century, Deem and Zabusky [12] wrote an equation for the V-states and gave partial numerical solutions. They put in evidence the existence of the V-states with m-fold symmetry for each integer $m \geq 2$ and in this countable cascade the case $m = 2$ corresponds to the known Kirchhoff's ellipses. Recall that a domain is said m-fold symmetric if it has the same group invariance of a regular polygon with m sides. This means that the domain is invariant by the action of the dihedral group D_m. At each frequency m these V-states can be seen as a continuous deformation of the disc with respect to a hidden bifurcation parameter corresponding to the angular velocity. An analytical proof was given by Burbea in [4] and his approach consists in writing the problem with the conformal mapping of the domain and to look at the non trivial solutions by using the technique of the bifurcation theory. Actually, Burbea's proof is not completely rigorous and one can find a complete one in [18]. In this latter paper Burbea's approach was revisited with more details and explanations. We also studied the boundary regularity of the V-states and showed that they are of class C^∞ and convex close to the disc.

The formulation of the rotating patches can be done in several ways requiring different levels of regularity for the solution. We shall give here a short glimpse with an emphasis on two different approaches. The first one uses the elliptic equation governing the stream function ψ associated to the domain D of the initial patch. As to the second approach, it uses the conformal parametrization of the boundary combined with the contour dynamics formulation. To be more precise, recall that the function ψ is defined by the Newtonian potential through the formula,

$$\psi(x) = \frac{1}{2\pi}\int_D \log|x-y|\,dy, \quad \Delta\psi = \chi_D.$$

Note that a patch with a smooth boundary rotates uniformly around its center, which can be taken equal to zero, means that in its own frame the boundary is stationary. In other words, the relative stream function $x \mapsto \psi(x) - \frac{1}{2}\Omega|x|^2$ should be constant on the boundary and therefore we get the equation

$$\frac{1}{2\pi}\int_D \log|x-y|\,dy - \frac{1}{2}\Omega|x|^2 = \mu, \quad \forall x \in \partial D, \tag{5}$$

with μ a constant. By virtue of this equation, the domains D are in fact defined through a strong interaction between the Newtonian and the quadratic potentials. The issue depends heavily on the sign of Ω. To fix the terminology, we say that the potential is repulsive when $\Omega \leq 0$ and attractive when it has an opposite sign. It seems that the situation in the repulsive case $\Omega \leq 0$ is trivial in the sense that only the discs are solutions of the rotating patch problem. This means that all the V-states must rotate counterclockwise. This result is the subject of a work in progress by the second author [20]. The proof relies on the moving plane method which allows to show that any solution of (5) must be radial with respect to some specific point, which is the center of mass of the domain D, and is strictly monotone. In the attractive case $\Omega > 0$, the interaction between the potentials is more fruitful and leads to infinite nontrivial solutions called the V-states as we have already mentioned. We point out that Burbea shows that for each frequency $m \geq 2$ the V-states V_m can be assimilated to a bifurcating curve from the disc at the angular velocity $\Omega_m = \frac{m-1}{2m}$. His idea is to use the conformal mapping parametrization $\phi : \mathbb{D}^c \to D^c$ which satisfies the nonlinear integral equation

$$
\begin{aligned}
F\big(\Omega, \phi(w)\big) &\triangleq \operatorname{Im}\left\{ \Big((1-2\Omega)\overline{\phi(w)} - \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{\overline{\phi(\tau)-\phi(w)}}{\phi(\tau)-\phi(w)} \phi'(\tau) d\tau \Big) w\, \phi'(w) \right\} \\
&= 0, \quad \forall w \in \mathbb{T}, \qquad (6)
\end{aligned}
$$

where $\mathbb{D}$ denotes the open unit disc and $\mathbb{T}$ its boundary. Now we observe that $F(\Omega, \mathrm{Id}) = 0$ and thus we may try to find non trivial solutions by using the bifurcation theory. For this end Burbea computes the linearized operator of F around this solution and shows that it has a nontrivial kernel if and only if $\Omega \in \{\Omega_m, m \geq 2\}$. In this case $\partial_f F(\Omega, \mathrm{Id})$ is a Fredholm operator with one-dimensional kernel. Consequently, one may apply the bifurcation theory through for instance Crandall-Rabinowitz theorem. This allows to prove the existence of non trivial branch of solutions emerging from the trivial one at each frequency level Ω_m.
One cannot escape mentioning that other explicit vortex solutions are discovered in the literature for the incompressible Euler equations in the presence of an external shear flow; see for instance [8, 22, 28]. A general review about vortex dynamics can be found in the papers [2, 30]. Another closely related subject is to conduct a similar study for the patches with multiple interfaces which is inherently complicated due to the strong interaction between the interfaces. In this context, Flierl and Polvani [14] proved that confocal ellipses with some compatibility relations rotate as a rigid body motion. Recently, we developed a complete characterization of rotating patches with two interfaces provided one of them is prescribed in the ellipses class.
In this paper, we shall address the same problem for the generalized SQG equations and look for the existence of the V-states The question was raised by Diego Córdoba and was the initial motivation for this work. As we shall see later in Proposition 4, the equation (6) becomes

$$
F_\alpha\big(\Omega, \phi(w)\big) \triangleq \operatorname{Im}\left\{ \Big(\Omega\phi(w) - \frac{C_\alpha}{2i\pi} \int_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha} d\tau \Big) \overline{w}\, \overline{\phi'}(w) \right\} = 0, \quad \forall w \in \mathbb{T},
$$

with $C_\alpha = \dfrac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$. Note that the structure of the singular nonlinear part is different from (6). Indeed, the singular kernel is not algebraic with respect to the conformal mapping which is holomorphic outside the unit disc. This property is profoundly important for Euler equations because it yields at different levels of the analysis, especially in the spectral study, to simple computations through Residue Theorem. Another disadvantage of the kernel structure concerns the computations of the regularity of the functional F_α which are heavy and more involved.
The main contribution of this paper is to give a positive answer for the existence of the V-states when $\alpha \in]0,1[$. For the sake of clarity we shall now give an elementary statement and a complete one is postponed to Theorem 3.

Theorem 1. *Let $\alpha \in]0,1[$ and $m \in \mathbb{N}^\star \backslash \{1\}$. Then, there exists a family of m-fold symmetric V-states $(V_m)_{m\geq 2}$ for the equation (1). Moreover, for each $m \geq 2$ the curve V_m bifurcates from the trivial solution $\theta_0 = \chi_{\mathbb{D}}$ at the angular velocity*

$$\Omega_m^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\right),$$

where Γ denotes the gamma function.
In addition, the boundary of the V-states belongs to the Hölder class $C^{2-\alpha}$.

The proof of this theorem will be done in the spirit of the incompressible Euler equations by using the bifurcation theory through Crandall-Rabinowitz Theorem. In the framework of this theory one should understand the structure of the linearized operator around the trivial solution Id and identify the range of Ω where this operator is not invertible. More precisely, we should determine where this operator belongs to the Fredholm class with zero index and possesses a simple kernel. By using some tricky integral formulae summarized in Lemma 2 one finds: for $h(w) = \displaystyle\sum_{n\in\mathbb{N}} b_n \overline{w}^n$

$$\partial_\phi F_\alpha(\Omega, \mathrm{Id})h(w) = \frac{1}{2}b_0\Omega\, i\big(w-\overline{w}\big) + \frac{i}{2}\sum_{n\geq 1}(n+1)\big(\Omega - \Omega_{n+1}^\alpha\big)b_n\Big(w^{n+1} - \overline{w}^{n+1}\Big).$$

Consequently, the linearized operator acts as a Fourier multiplier in the phase space. It behaves as a differential operator of order one because $\sup_n \Omega_n^\alpha < \infty$. Afterwards, we prove that this operator sends $C^{2-\alpha}$ to $C^{1-\alpha}$ and fulfills the required assumptions of Crandall-Rabinowitz Theorem: it is of Fredholm type with zero index and satisfies the transversality assumption. This latter one means that when we look for the linearized operator with Ω close to Ω_m^α, then the eigenvalue which is close to zero (it depends on Ω) must cross the real axis with non zero velocity at the value $\Omega = \Omega_m^\alpha$.
Next, we shall make few comments about the statement of the main theorem.

Remarks. 1) *For the incompressible Euler equations the dilation has no effects on the angular vorticity of the V-states. However, this property fails for the generalized SQG model because we change the homogeneity of the equation. As we shall see later in Proposition 3 the angular velocity depends on the inverse of the dilation parameter raised to the power α. This means that we can find small patches rotating quickly and also big ones rotating very slowly. In addition, the bifurcation set $\{\Omega_m^\alpha, m \geq 2\}$ introduced in Theorem 1 concerns only*

the bifurcation from the unit disc. However, to get a bifurcation from a disc of radius r we have to scale this set as follows $\{r^{-\alpha}\Omega_m^\alpha, m \geq 2\}$.

2) *For the SQG equation corresponding to $\alpha = 1$ the situation is more delicate as we shall discuss later in the end of the paper. Indeed, one can modify the function F_α in order to get a less singular kernel but we note a regularity loss for the linearized operator. This appears more clearly when we compute the linearized operator which is given by*

$$\partial_\phi F_1(\Omega, \mathrm{Id})h(w) = \frac{1}{2}b_0\Omega\, i\big(w - \overline{w}\big) + \frac{i}{2}\sum_{n\geq 1}(n+1)\big(\Omega - \Omega_{n+1}^1\big)b_n\Big(w^{n+1} - \overline{w}^{n+1}\Big),$$

with

$$\Omega_n^1 = \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1}. \tag{7}$$

We see that this operator acts as a Fourier multiplier with an additional logarithmic growth compared to the case $\alpha \in [0, 1[$. As a consequence, this operator does not send $C^{1+\varepsilon}$ to C^ε and it seems complicated to find suitable function spaces X and Y such that Crandall-Rabinowitz Theorem can be applied. More discussion will be brought forward the end of this paper; see Section 10. We also mention that the preceding dispersion relation was computed formally in [1] by using Bessel functions. In Section 10 we shall give another proof of this relation.

3) *The boundary of the rotating patches belongs to Hölder space $C^{2-\alpha}$. For $\alpha = 0$, we get better result as it was shown in [18]; the boundary is C^∞ and convex when the V-states are close to the circle. The proof in this particular case uses in a deep way the algebraic structure of the kernel according to some recurrence formulae. It is not clear whether this approach can be implemented for the generalized SQG equation but we do believe that the boundary is also C^∞.*

4) *The global existence of non stationary solutions for (1) is not known for $\alpha > 0$. The V-states offer a suitable class of initial data with global existence because they generate periodic solutions in time.*

5) *As we shall see later in Lemma 3, there is continuity of the spectrum Ω_m^α with respect to α. This means that,*

$$\lim_{\alpha\to 0}\Omega_m^\alpha = \frac{m-1}{2m} \quad \textit{and} \quad \lim_{\alpha\to 1}\Omega_m^\alpha = \Omega_m^1,$$

where Ω_m^1 is defined in (7).

The paper is organized as follows. In the next section we shall fix some notation. In Section 3, we discuss some general properties of the V-states. In Section 4, we shall introduce and review some background material on the bifurcation theory and singular integrals. In Section 5, we will study the elliptic patches and show that they never rotate. This was recently proved in [6] and we intend to give another proof by using complex analysis formulation. Section 6 is devoted to a general statement of Theorem 1. The proof of Theorem 3 will be discussed in Sections 7, 8 and 9. Last, in Section 10 we will pay a special attention to the SQG model corresponding to the limit case $\alpha = 1$. We shall reformulate the boundary equation in order to kill the violent singularity of the kernel. In this case we give a complete description of the

linearized operator and the dispersion relation. However we are not able to give a complete proof of the bifurcation of the V-states which should require a slightly different mathematical machinery than does the sub-critical case $\alpha \in [0, 1[$.

2. Notation

In this section we shall fix some notation that will be frequently used along this paper.

- We denote by C any positive constant that may change from line to line.
- For any positive real numbers A and B, the notation $A \lesssim B$ means that there exists a positive constant C independent of A and B such that $A \leq CB$.
- We denote by $\mathbb{D}$ the unit disc. Its boundary, the unit circle, is denoted by $\mathbb{T}$.
- Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. We define its mean value by,

$$\fint_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2i\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$

  where $d\tau$ stands for the complex integration.
- Let X and Y be two normed spaces. We denote by $\mathcal{L}(X, Y)$ the space of all continuous linear maps $T : X \to Y$ endowed with its usual strong topology.
- For a linear operator $T : X \to Y$, we denote by $N(T)$ and $R(T)$ the kernel and the range of T, respectively.
- If Y is a vector space and R is a subspace, then Y/R denotes the quotient space.

3. Preliminaries on the V-states

In this introductory section we will focus on some general results on the rotating patches called also V-states according to Deem and Zabusky terminology. These results were proved in [19] for Euler equations and will be likewise extended to the SQG model (1).

3.1. **General facts.** Now we intend to fix some vocabulary and prove in particular that the center of rotation of any V-state should coincide with its center of mass. We shall also deal with the effects of the dilation of the geometry on the angular velocity of the rotating patches.

Definition 1. *Let D_0 be a simply connected domain in the plane with smooth boundary. We say that $\theta_0 = \mathbf{1}_{D_0}$ is a rotating patch if the associated solution of (1) is given by*

$$\theta(t, x) = \mathbf{1}_{D_t} \quad \textit{with} \quad D_t = \mathbf{R}_{x_0,\varphi(t)}D_0.$$

Here we denote by $\mathbf{R}_{x_0,\varphi(t)}$ the planar rotation of center x_0 and angle $\varphi(t)$. In addition, we assume that the function $t \mapsto \varphi(t)$ is smooth and non-constant.

The velocity dynamics in the framework of rotating patches is described as follows.

Proposition 1. *Let θ_0 be a rotating patch as in Definition 1. Then the velocity $u(t)$ can be recovered from its initial value u_0 according to the formula*

$$u(t, x) = \mathbf{R}_{x_0,\varphi(t)}u_0(\mathbf{R}_{x_0,-\varphi(t)}x).$$

Proof. We shall use the formula

$$-(-\Delta)^{1-\frac{\alpha}{2}}u(t, x) = \nabla^{\perp}\theta(t, x).$$

Performing some algebraic computations we get

$$\begin{aligned}\nabla^{\perp}\theta(t,x) &= \mathbf{R}_{x_0,\varphi(t)}\nabla^{\perp}\theta_0(\mathbf{R}_{x_0,-\varphi(t)}x)\\ &= -\mathbf{R}_{x_0,\varphi(t)}(-\Delta)^{1-\frac{\alpha}{2}}v_0(\mathbf{R}_{x_0,-\varphi(t)}x)\\ &= -(-\Delta)^{1-\frac{\alpha}{2}}\big(\mathbf{R}_{x_0,\varphi(t)}v_0(\mathbf{R}_{x_0,-\varphi(t)}x)\big).\end{aligned}$$

Here we have used the commutation between the operator $(-\Delta)^{1-\frac{\alpha}{2}}$ and the rotation transformations which can be checked easily from the integral representation of the fractional Laplacian. Therefore, the result follows from a uniqueness argument. □

Now, we will discuss a special result concerning the evolution of the center of mass of the patch $\theta(t) = \mathbf{1}_{D_t}$, defined by

$$X(t) = \frac{1}{|D_t|}\int_{D_t} x\,dx = \frac{1}{|D_0|}\int_{D_t} x\,dx.$$

We have used the fact that the volume of a patch is an invariant of the motion since the velocity is divergence free. Next, we prove that the center of mass is stationary for any patch solution of (1). This is known for Euler equation and we shall give here a similar proof.

Proposition 2. *Let $\theta(t) = \mathbf{1}_{D_t}$ be a solution of (1) then the center of mass is fixed, that is*

$$X(t) = X(0).$$

Proof. The invariance of the center of mass follows from the constancy of the functions

$$f_j(t) \triangleq \int_{\mathbb{R}^2} x_j\,\theta(t,x)\,dx, \quad j = 1,2.$$

Differentiating this function with respect to the time variable combined with the equation (1) and integration by parts yields

$$\begin{aligned}f_j'(t) &= \int_{\mathbb{R}^2} x_j\,\partial_t\theta(t,x)\,dx\\ &= -\int_{\mathbb{R}^2} x_j\,(u\cdot\nabla\theta)(t,x)\,dx\\ &= \int_{\mathbb{R}^2} u^j(t,x)\,\theta(t,x)\,dx.\end{aligned}$$

Using the relation between u and θ and integrating once again by parts we get

$$\begin{aligned}f_1'(t) &= -\int_{\mathbb{R}^2}\theta\,\nabla^{\perp}(-\Delta)^{-1+\frac{\alpha}{2}}\theta\,dx\\ &= -\int_{\mathbb{R}^2}\big\{(-\Delta)^{-\frac{1}{2}+\frac{\alpha}{4}}\theta\big\}\,\nabla^{\perp}\big\{(-\Delta)^{-\frac{1}{2}+\frac{\alpha}{4}}\theta\big\}\,dx\\ &= 0.\end{aligned}$$

This completes the proof of the desired result. □

In consequence, we obtain the following result.

Corollary 1. *Let $\theta_0 = \mathbf{1}_{D_0}$ be a rotating patch center around some point x_0. Then necescentersarily x_0 is the center of mass of the domain D_0.*

Proof. By a change of variables

$$\begin{aligned} X(t) &= \frac{1}{|D_0|}\int_{\mathbb{R}^2} x\,\theta_0(\mathbf{R}_{x_0,-\varphi(t)}x)dx \\ &= \frac{1}{|D_0|}\int_{\mathbb{R}^2} (\mathbf{R}_{x_0,\varphi(t)}x)\theta_0(x)dx \\ &= \frac{1}{|D_0|}\mathbf{R}_{x_0,\varphi(t)}\left(\int_{\mathbb{R}^2} x\theta_0(x)dx\right) \\ &= \mathbf{R}_{x_0,\varphi(t)}X(0). \end{aligned}$$

Since $X(t) = X(0)$ by Proposition (2), $X(0)$ is fixed by the rotation and thus $X(0) = x_0$, as claimed. □

Next we shall discuss how the dilation affects the angular velocity. We point out that the following notation that we shall use $D_\lambda = \lambda D$ means a dilation of the domain D with respect to its center of mass.

Proposition 3. *Let $\theta_0 = \mathbf{1}_D$ be a rotating patch with constant angular velocity Ω. Let $\lambda > 0$ and denote by $D_\lambda = \lambda D$. Then D_λ is also a rotating patch with angular velocity $\Omega_\lambda = \frac{\Omega}{\lambda^\alpha}$.*

Proof. Without loss of generality we can assume that the center of rotation is the origin. Then according to the equation (13) we have

$$\Omega \mathrm{Re}\{z\overline{z'}\} = \mathrm{Im}\Big\{\frac{C_\alpha}{2\pi}\int_{\partial D}\frac{d\zeta}{|z-\zeta|^\alpha}\overline{z'}\Big\}, \quad \forall\, z \in \partial D.$$

Let $z \in D$ and $\tau = \lambda z$, then multiplying the preceding equation by λ^2 and using the change of variables $w = \lambda\zeta$ we get

$$\big(\lambda^{-\alpha}\Omega\big)\mathrm{Re}\{\tau\overline{\tau'}\} = \mathrm{Im}\Big\{\frac{C_\alpha}{2\pi}\int_{\partial D_\lambda}\frac{dw}{|\tau-w|^\alpha}\overline{\tau'}\Big\}, \quad \forall\, \tau \in \partial D_\lambda.$$

This shows that D_λ rotates with the angular velocity $\lambda^{-\alpha}\Omega$ as it is claimed. □

3.2. **Boundary equation.** Before proceeding further with the consideration of the V-states, we shall recall Riemann mapping theorem which is one of the most important results in complex analysis. To restate this result we shall recall the definition of *simply connected* domains. Let $\widehat{\mathbb{C}} \triangleq \mathbb{C}\cup\{\infty\}$ denote the Riemann sphere. We say that a domain $\Omega \subset \widehat{\mathbb{C}}$ is *simply connected* if the set $\widehat{\mathbb{C}}\backslash\Omega$ is connected.

Riemann Mapping Theorem. Let $\mathbb{D}$ denote the unit open ball and $\Omega \subset \mathbb{C}$ be a simply connected bounded domain. Then there is a unique bi-holomorphic map called also conformal map, $\Phi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{\Omega}$ taking the form

$$\Phi(z) = az + \sum_{n\in\mathbb{N}}\frac{a_n}{z^n} \quad \text{with} \quad a > 0.$$

In this theorem the regularity of the boundary has no effect regarding the existence of the conformal mapping but it contributes in the boundary behavior of the conformal mapping, see for instance [31, 34]. Here, we shall recall the following result.

Kellogg-Warschawski's theorem. It can be found in [34] or in [31, Theorem 3.6]. It asserts that if the boundary $\Phi(\mathbb{T})$ is a Jordan curve of class $C^{n+1+\beta}$, with $n \in \mathbb{N}$ and $0 < \beta < 1$,

then the conformal map $\Phi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{\Omega}$ has a continuous extension to $\mathbb{C}\backslash\mathbb{D}$ which is of class $C^{n+1+\beta}$.
Next, we shall write down the equation governing the boundary of the V-states; it is highly nonlinear and non local as the next proposition shows.

Proposition 4. *Let $\alpha \in]0,1[$, D_0 be a smooth simply connected domain and $D_t = R_{x_0,\varphi(t)}D_0$ be a V-state of the model (1). Then, the following claims hold true.*

(1) *The point x_0 is the center of mass of D_0 and $\dot{\varphi}(t) = \Omega$ is constant.*
(2) *Assume that $x_0 = 0$ and let $\phi : \mathbb{D}^c \to D_0^c$ be the conformal mapping, then*

$$\mathrm{Im}\left\{\left(\Omega\,\phi(w) - C_\alpha \fint_{\mathbb{T}} \frac{\phi'(\tau)d\tau}{|\phi(w)-\phi(\tau)|^\alpha}\right)\overline{w}\,\overline{\phi'(w)}\right\} = 0, \quad \forall w \in \mathbb{T}, \tag{8}$$

$$\text{with } C_\alpha = \frac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}.$$

Proof. (**1**) The first claim was proved in Proposition 1 and so it remains to check that the angular velocity is constant. For this aim we shall start with writing the boundary equation of a V-state. Loosely speaking, the boundary ∂D_t is a material surface and there is no flux matter across it. In other words, it is transpocenterrted by the flow $\psi(t)$ defined in the next few lines. For a smooth initial boundary, say of class C^1, there exists a function $\varphi_0 : \mathbb{R}^2 \to \mathbb{R}$ of class C^1 such that

$$\partial D_0 = \Big\{x \in \mathbb{R}^2; \varphi_0(x) = 0\Big\},$$

with the additional constraints: $\forall x \in \partial D_0,\ \ \nabla\varphi_0(x) \neq 0,$

$$\varphi_0 < 0 \ \text{ on } \ D_0 \quad \text{and} \qquad \varphi_0 > 0 \ \text{ on } \ \mathbb{R}^2 \setminus \overline{D_0}.$$

One says in this case that φ_0 is a defining function for ∂D_0. Set

$$F(t,x) = \varphi_0(\psi^{-1}(t,x)),$$

where ψ is the flow associated to the velocity u and given by the integral equation

$$\psi(t,x) = x + \int_0^t u\big(\tau, \psi(\tau,x)\big)d\tau.$$

It follows that the maps $x \mapsto F(t,x)$ is a defining function for $\partial D_t = \psi(t, \partial D_0)$ and satisfies the transport equation

$$\partial_t F + u \cdot \nabla F = 0.$$

Now, let $\sigma \in [0, 2\pi] \mapsto \gamma_t(\sigma)$ be a parametrization of ∂D_t, continuously differentiable in t, and let $\vec{n}_t$ be the unit outward normal vector to ∂D_t. Differentiating the equation $F(t, \gamma_t(\sigma)) = 0$ with respect to t yields

$$\partial_t F + \partial_t\gamma_t \cdot \nabla F = 0.$$

Since for $x \in \partial D_t$ the vector $\nabla F(t,x)$ is colinear to the normal vector $\vec{n}_t$ then

$$(\partial_t\gamma_t - u(t,\gamma_t)) \cdot \vec{n}_t = 0. \tag{9}$$

The meaning of (9) is that the velocity of the boundary and the the velocity of the fluid particle occupying the same position have the same normal components. We observe that

the equation (9) can be written in a complex form which seems to be more convenient in our case,

$$\mathrm{Im}\Big\{(\partial_t\gamma_t - v(t,\gamma_t))\overline{\gamma_t'}\Big\} = 0, \tag{10}$$

where the "prime" denotes the derivative with respect to the σ variable.
We now take a closer look at the case of a rotating connected patch. Assume that the boundary rotates with the angular velocity $\dot\theta(t)$ around its center of mass which can be assumed to be the origin. According to the Proposition 1 the velocity $u(t)$ can be recovered from the initial velocity u_0 through to the formula

$$u(t,x) = e^{i\theta(t)}u_0(e^{-i\theta(t)}x). \tag{11}$$

Hence

$$\mathrm{Im}\big\{u(t,\gamma_t)\overline{\gamma_t'}\big\} = \mathrm{Im}\big\{u_0(\gamma_0)\overline{\gamma_0'}\big\}.$$

The rotating patch has a standard parametrization given by $\gamma_t = e^{i\theta(t)}\gamma_0$ which yields

$$\mathrm{Im}\big\{\partial_t\gamma_t\overline{\gamma_t'}\big\} = \dot\theta(t)\mathrm{Re}\big\{\gamma_0\overline{\gamma_0'}\big\}.$$

Consequently the equation (10) becomes

$$\dot\theta(t)\mathrm{Re}\big\{\gamma_0\overline{\gamma_0'}\big\} = \mathrm{Im}\big\{u_0(\gamma_0)\overline{\gamma_0'}\big\}$$

which is equivalent to

$$\frac{\dot\theta(t)}{2}\frac{d}{ds}|\gamma_0(s)|^2 = \mathrm{Im}\Big\{u_0(\gamma_0)\,\overline{\gamma_0'}\Big\}.$$

If there exists some s with $\frac{d}{ds}|\gamma_0(s)|^2 \neq 0$ then, since the right-hand side does not depend on the time variable, we conclude that $\dot\theta(t) = \Omega$ is constant. Otherwise, $\frac{d}{ds}|\gamma_0(s)|^2$ vanishes everywhere, which tells us that the initial domain is a disc and therefore it rotates with any angular velocity. Finally we get the boundary equation

$$\Omega\,\mathrm{Re}\big\{z\,\overline{z'}\big\} = \mathrm{Im}\big\{u_0(z)\,\overline{z'}\big\}, \quad \forall z \in D_0. \tag{12}$$

Recall that z' is a tangent vector to the boundary ∂D_0 at the point z.

(**2**) Combining (12) with the velocity formula (3) we get

$$\Omega\,\mathrm{Re}\big\{z\overline{z'}\big\} = C_\alpha\mathrm{Im}\left\{\frac{1}{2\pi}\int_{\partial D_0}\frac{d\zeta}{|z-\zeta|^\alpha}\overline{z'}\right\}, \quad \forall\, z \in \partial D_0. \tag{13}$$

We shall now parametrize the domain with the outside conformal mapping $\phi : \mathbb{D}^c \to D_0^c$.

$$\phi(w) = w + \sum_{n\geq 0}\frac{b_n}{w^n} \tag{14}$$

Setting $z = \phi(w)$ and $\zeta = \phi(\tau)$, then for $w \in \mathbb{T}$ a tangent vector is given by

$$\overline{z'} = -i\overline{w}\,\overline{\phi'(w)}.$$

Inserting this in the equation (13) gives

$$G(\Omega,\phi)(w) \triangleq \mathrm{Im}\left\{\left(\Omega\phi(w) - \frac{C_\alpha}{2i\pi}\int_{\mathbb{T}}\frac{\phi'(\tau)d\tau}{|\phi(w)-\phi(\tau)|^\alpha}\right)\overline{w}\,\overline{\phi'(w)}\right\} = 0, \quad \forall\, w \in \mathbb{T}. \tag{15}$$

This achieves the proof of the proposition. □

4. Tools

The purpose of this introductory section is to review and collect some technical tools that will be used quite often in the remainder of this paper. We will firstly recall some basic elements of the bifurcation theory. We will focus on the Crandall-Rabinowitz's theorem, hereafter referred by C-R Theorem, which is very crucial for the proof of our main result. Secondly, some simple facts about Hölder spaces $C^{n+\gamma}(\mathbb{T})$ will be recalled and we shall also explore some results on the action of singular integral operators on these spaces. Last, we end this section with some integral computations that will be frequently used in the study of the linearized operator.

4.1. **Elements of the bifurcation theory.** We intend now to give some formal explanations and general principles of the bifurcation theory. This discussion will be closed by stating C-R theorem. Roughly speaking, the main objective of this theory is to look for the solutions of the equation

$$F(\lambda, x) = 0$$

where $F : \mathbb{R} \times X \to Y$ is continuous function and satisfies some additional regularity assumptions. The vector spaces X and Y are Banach spaces. We assume in addition that $x = 0$ is a trivial solution for any λ, that is, $F(\lambda, 0) = 0$. Whether close to the solution $(\lambda_0, 0)$ one can find a branch of non trivial ones is the main problem discussed in this theory. If this is the case we say that there is a bifurcation at the point $(\lambda_0, 0)$. As the Implicit Function Theorem tells us, the first test that should be carried out is to analyze the linear operator $\mathcal{L}_\lambda \triangleq \partial_x F(\lambda, 0) : X \to Y$. If this operator is an isomorphism then such non trivial solutions cannot exist. Thus a necessary condition for the bifurcation is to get a nontrivial kernel of $\mathcal{L}_\lambda$. In many instances, the involved Banach spaces are infinite-dimensional and thus the bifurcation analysis is in general very complex. However, if the linearized operator is of Fredholm type one can reduce the problem to finite-dimensional spaces by using the so-called Lyapunov-Schmidt reduction. Recall that a Fredholm operator means that it is continuous and whose kernel $N(\mathcal{L}_\lambda)$ and cokerel $Y/R(\mathcal{L}_\lambda)$ are finite-dimensional, where $R(\mathcal{L}_\lambda)$ denotes the range of $\mathcal{L}_\lambda$. If moreover the index of this operator is zero then the bifurcation may occur despite that some suitable conditions are satisfied. Here we shall only discuss the bifurcation with one dimensional kernel which is the most common one and appears in many dynamical systems as for our generalized SQG model. With the preceding assumptions on the linear operator a one-parameter curve bifurcates from the trivial solution provided a transversality assumption is satisfied. Roughly speaking, this latter assumption means that the linear operator $\mathcal{L}_\lambda$ possesses a one-parameter eigenvalues $\lambda \mapsto \mu(\lambda)$ that should cross the real axis at λ_0 with non zero velocity. This is the classical theorem proved by Crandall and Rabinowitz [5] which is a basic tool in the bifurcation theory and that will be used in this paper. More general results are summarized in the book of Kielhöfer [23]. Now we recall Crandall-Rabinowitz Theorem.

Theorem 2. *Let X, Y be two Banach spaces, V a neighborhood of 0 in X and let $F : \mathbb{R} \times V \to Y$ with the following properties:*

1. *$F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$.*
2. *The partial derivatives F_λ, F_x and $F_{\lambda x}$ exist and are continuous.*

(3) $N(\mathcal{L}_0)$ *and* $Y/R(\mathcal{L}_0)$ *are one-dimensional.*
(4) *Transversality assumption:* $F_{tx}(0,0)x_0 \notin R(\mathcal{L}_0)$, *where*

$$N(\mathcal{L}_0) = span\{x_0\}, \quad \mathcal{L}_0 \triangleq \partial_x F(0,0).$$

If Z *is any complement of* $N(\mathcal{L}_0)$ *in* X, *then there is a neighborhood* U *of* $(0,0)$ *in* $\mathbb{R}\times X$, *an interval* $(-a,a)$, *and continuous functions* $\varphi : (-a,a) \to \mathbb{R}$, $\psi : (-a,a) \to Z$ *such that* $\varphi(0) = 0$, $\psi(0) = 0$ *and*

$$F^{-1}(0) \cap U = \Big\{ \big(\varphi(\xi), \xi x_0 + \xi\psi(\xi)\big)\,;\, |\xi| < a \Big\} \cup \Big\{ (\lambda, 0)\,;\, (\lambda,0) \in U \Big\}.$$

4.2. **Singular integrals.** In this paragraph we shall briefly recall the classical Hölder spaces on the periodic case and state some classical facts on the continuity of singular integrals over these spaces. It is convenient to think of 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ as a function of the complex variable $w = e^{i\eta}$ rather than a function of the real variable η. To be more precise, let $f : \mathbb{T} \to \mathbb{R}^2$, be a continuous function, then it can be assimilated to a $2\pi-$ periodic function $g : \mathbb{R} \to \mathbb{R}$ via the relation

$$f(w) = g(\eta), \quad w = e^{i\eta}.$$

Hence when f is smooth enough we get

$$f'(w) \triangleq \frac{df}{dw} = -ie^{-i\eta} g'(\eta).$$

Because d/dw and $d/d\eta$ differ only by a smooth factor with modulus one we shall in the sequel work with d/dw instead of $d/d\eta$ which appears to be more convenient in the computations. Moreover, if f has real Fourier coefficients and is of class C^1 then we have the identity

$$\{\overline{f}\}'(w) = -\frac{1}{w^2}\overline{f'(w)}. \tag{16}$$

Now we shall introduce Hölder spaces on the unit circle $\mathbb{T}$.

Definition 2. *Let* $0 < \gamma < 1$. *We denote by* $C^\gamma(\mathbb{T})$ *the space of continuous functions* f *such that*

$$\|f\|_{C^\gamma(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \sup_{x\neq y\in\mathbb{T}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

For any integer n *the space* $C^{n+\gamma}(\mathbb{T})$ *stands for the set of functions* f *of class* C^n *whose* $n-th$ *order derivatives are Hölder continuous with exponent* γ. *It is equipped with the usual norm,*

$$\|f\|_{C^{n+\gamma}(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \Big\| \frac{d^n f}{dw^n} \Big\|_{C^\gamma(\mathbb{T})}.$$

Recall that the Lipschitz (semi)-norm is defined as follows.

$$\|f\|_{\mathrm{Lip}(\mathbb{T})} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

Now we list some classical properties that will be used later especially in Section 7.

(1) For $n \in \mathbb{N}, \gamma \in]0,1[$ the space $C^{n+\gamma}(\mathbb{T})$ is an algebra.
(2) For $K \in L^1(\mathbb{T})$ and $f \in C^{n+\gamma}(\mathbb{T})$ we have the convolution law,

$$\|K * f\|_{C^{n+\gamma}(\mathbb{T})} \leq \|K\|_{L^1(\mathbb{T})} \|f\|_{C^{n+\gamma}(\mathbb{T})}.$$

The next result is used often. It deals with singular integrals of the following type,

$$\mathcal{T}(f)(w) = \int_{\mathbb{T}} K(w,\tau)\, f(\tau) d\tau, \tag{17}$$

with $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ a singular kernel satisfying some properties. This problem will appear naturally when we shall deal with the regularity of the nonlinear operator in the rotating patches formalism, see Section 7. The result that we shall discuss with respect to this subject is classical and for the self-containing of the paper we shall provide a complete proof which is similar to [25].

Lemma 1. *Let $0 \leq \alpha < 1$ and consider a function $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ with the following properties. There exits $C_0 > 0$ such that,*

(1) *K is measurable on $\mathbb{T} \times \mathbb{T} \backslash \{(w,w),\, w \in \mathbb{T}\}$ and*

$$\big|K(w,\tau)\big| \leq \frac{C_0}{|w-\tau|^\alpha}, \quad \forall\, w \neq \tau \in \mathbb{T}.$$

(2) *For each $\tau \in \mathbb{T}$, $w \mapsto K(w,\tau)$ is differentiable in $\mathbb{T}\backslash\{\tau\}$ and*

$$\big|\partial_w K(w,\tau)\big| \leq \frac{C_0}{|w-\tau|^{1+\alpha}}, \quad \forall\, w \neq \tau \in \mathbb{T}.$$

Then the operator $\mathcal{T}$ defined by (17) is continuous from $L^\infty(\mathbb{T})$ to $C^{1-\alpha}(\mathbb{T})$. More precisely, there exists a constant C_α depending only on α such that

$$\|\mathcal{T}(f)\|_{1-\alpha} \leq C_\alpha C_0 \|f\|_{L^\infty}.$$

Proof. We first prove that $\mathcal{T}(f)$ is bounded on $\mathbb{T}$. Let $w \in \mathbb{T}$, then by the condition (1),

$$\begin{aligned} |\mathcal{T}(f)(w)| &\leq C_0 \|f\|_{L^\infty} \Big| \int_{\mathbb{T}} \frac{d\tau}{|w-\tau|^\alpha} \Big| \\ &\leq C_\alpha C_0 \|f\|_{L^\infty}. \end{aligned}$$

Next, take $w_1, w_2 \in \mathbb{T}$, set $r = |w_1 - w_2|$ and define $B_r(w_1) = \big\{\tau \in \mathbb{T};\, |\tau - w_1| \leq r\big\}$. Then,

$$\begin{aligned} \Big|\mathcal{T}(f)(w_1) - \mathcal{T}(f)(w_2)\Big| &\leq \Big| \int_{B_{2r}(w_1)} |f(\tau)||K(w_1,\tau)| d\tau \Big| + \Big| \int_{B_{2r}(w_1)} |f(\tau)||K(w_2,\tau)| d\tau \Big| \\ &+ \Big| \int_{B^c_{2r}(w_1)} |f(\tau)||K(w_1,\tau) - K(w_2,\tau)| d\tau \Big| \\ &\triangleq J_1 + J_2 + J_3. \end{aligned}$$

By using again the condition (1), J_1 and J_2 can be estimated by

$$\begin{aligned} J_1 + J_2 &\leq C_0 \|f\|_{L^\infty} \Big(\Big| \int_{B_{2r}(w_1)} \frac{d\tau}{|w_1-\tau|^\alpha} \Big| + \Big| \int_{B_{3r}(w_2)} \frac{d\tau}{|w_2-\tau|^\alpha} \Big| \Big) \\ &\leq C_\alpha C_0 \|f\|_{L^\infty} |w_1 - w_2|^{1-\alpha}. \end{aligned}$$

To estimate the third term J_3 we shall use the condition (2) combined with the Mean Value Theorem,

$$|K(w_1,\tau) - K(w_2,\tau)| \leq C C_0 \frac{|w_1 - w_2|}{|w_1 - \tau|^{1+\alpha}}, \quad \forall\, \tau \in B^c_{2r}(w_1).$$

Consequently we get

$$\begin{aligned} J_3 &\leq CC_0\|f\|_{L^\infty}\Big|\int_{B^c_{2r}(w_1)}\frac{|w_1-w_2|}{|w_1-\tau|^{1+\alpha}}d\tau \\ &\leq C_\alpha C_0\|f\|_{L^\infty}|w_1-w_2|^{1-\alpha}. \end{aligned}$$

This concludes the result. □

As a by-product we obtain the result.

Corollary 2. *Let $0 \leq \alpha < 1$, $\phi : \mathbb{T} \to \phi(\mathbb{T})$ be a bi-Lipschitz function with real Fourier coefficients and define the operator*

$$\mathcal{T}_\phi : f \mapsto \fint_{\mathbb{T}} \frac{f(\tau)}{|\phi(w)-\phi(\tau)|^\alpha}d\tau, \quad w \in \mathbb{T}.$$

Then $\mathcal{T}_\phi : L^\infty(\mathbb{T}) \to C^{1-\alpha}(\mathbb{T})$ is continuous with the estimation,

$$\|\mathcal{T}_\phi(f)\|_{C^{1-\alpha}(\mathbb{T})} \leq C\Big(\|\phi^{-1}\|^\alpha_{\mathrm{Lip}(\mathbb{T})} + \|\phi\|^2_{\mathrm{Lip}(\mathbb{T})}\|\phi^{-1}\|^{1+\alpha}_{\mathrm{Lip}(\mathbb{T})}\Big)\|f\|_{L^\infty(\mathbb{T})},$$

where C is a positive constant depending only on α.

Proof. We set

$$K(w,\tau) = \frac{1}{|\phi(w)-\phi(\tau)|^\alpha}, \quad \forall\, w \neq \tau \in \mathbb{T}.$$

Since ϕ is bi-Lipschitz then we deduce that

$$|K(w,\tau)| \leq \|\phi^{-1}\|^\alpha_{\mathrm{Lip}(\mathbb{T})}\frac{1}{|w-\tau|^\alpha} \quad \forall w \neq \tau \in \mathbb{T}. \tag{18}$$

To get the second assumption (2) of Lemma 1 we shall compute $\partial_w K(w,\tau)$.

$$\begin{aligned} \partial_w K(w,\tau) &= \frac{-\alpha}{2}\left(\phi'(w)\frac{\overline{\phi(w)}-\overline{\phi(\tau)}}{|\phi(w)-\phi(\tau)|^{\alpha+2}} + (\overline{\phi})'(w)\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}\right) \\ &= \frac{-\alpha}{2}\left(\phi'(w)\frac{\overline{\phi(w)}-\overline{\phi(\tau)}}{|\phi(w)-\phi(\tau)|^2} - \frac{\overline{\phi'(w)}}{w^2}\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^2}\right)K(w,\tau) \quad w \neq \tau \in \mathbb{T}. \end{aligned} \tag{19}$$

We have used the fact that the Fourier coefficients of ϕ are real and therefore we can apply the identity (16). It follows that,

$$\begin{aligned} |\partial_w K(w,\tau)| &\leq C\|\phi\|^2_{\mathrm{Lip}(\mathbb{T})}\frac{1}{|\phi(w)-\phi(\tau)|^{\alpha+1}} \\ &\leq C\|\phi\|^2_{\mathrm{Lip}(\mathbb{T})}\|\phi^{-1}\|^{1+\alpha}_{\mathrm{Lip}(\mathbb{T})}\frac{1}{|w-\tau|^{\alpha+1}}. \end{aligned}$$

We can conclude by Lemma 1 and get the desired result. □

4.3. **Basic integrals.** This section presents some basic computations of few integrals that will appear later in the study of the linearized operator. But before going further into the details we shall recall some facts on the gamma function which emerges in a natural way in our computations. The function $\Gamma : \mathbb{C}\backslash(-\mathbb{N}) \to \mathbb{C}$ refers to the gamma function which is the analytic continuation to the negative half plane of the usual gamma function defined on the positive half-plane $\{\mathrm{Re} z > 0\}$ by the integral representation

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t} dt.$$

It satisfies the relation

$$\Gamma(z+1) = z\,\Gamma(z), \quad \forall z \in \mathbb{C}\backslash(-\mathbb{N}). \tag{20}$$

Note that this function does not vanish and its poles $\{-n, n \in \mathbb{N}\}$ are simple and so the reciprocal gamma function $\frac{1}{\Gamma}$ is an entire function. There are some particular values of the gamma function that will be used later,

$$\Gamma(n+1) = n!, \quad \Gamma(1/2) = \sqrt{\pi}. \tag{21}$$

Now we shall introduce another related function called the digamma function which is nothing but the logarithmic derivative of the function gamma and often denoted by $\digamma$. It is given by

$$\digamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For a future use we need the following identity,

$$\forall n \in \mathbb{N}, \quad \digamma(n+\frac{1}{2}) = -\gamma - 2\ln 2 + 2\sum_{k=0}^{n-1} \frac{1}{2k+1}. \tag{22}$$

Now for $x \in \mathbb{R}$ we denote by $(x)_n$ the Pokhhammer's symbol defined by

$$(x)_n = \begin{cases} x(x+1)...(x+n-1), & \text{if} \quad n \geq 1, \\ 1, & \text{if} \quad n = 0. \end{cases} \tag{23}$$

Note that in the literature the above notation is replaced by $(x)^n$ which can introduce in our context a lot of confusion with the power x^n and for this reason we prefer not to use it.
It is obvious that

$$(x)_n = x\,(1+x)_{n-1}, \quad (x)_{n+1} = (x+n)\,(x)_n. \tag{24}$$

From the identity (20) we deduce the relations

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad (x)_n = (-1)^n \frac{\Gamma(1-x)}{\Gamma(1-x-n)}, \tag{25}$$

provided all the quantities in the right terms are well-defined.
In the sequel we shall prove the following lemma which is the main result of this section.

Lemma 2. *Let $n \in \mathbb{N}$ and $\alpha \in (0,1)$. Then for any $w \in \mathbb{T}$ we have the following formulae.*

$$⨍_{\mathbb{T}} \frac{\tau^n}{|\tau - w|^\alpha} d\tau = \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)} \frac{\left(\frac{\alpha}{2}\right)_{n+1}}{(1-\frac{\alpha}{2})_{n+1}}\, w^{n+1}. \tag{26}$$

$$\fint_{\mathbb{T}} \frac{(w-\tau)(w^n-\tau^n)}{|w-\tau|^{\alpha+2}}d\tau = \frac{\left(1+\frac{\alpha}{2}\right)\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\left(1-\frac{\left(2+\frac{\alpha}{2}\right)_n}{\left(2-\frac{\alpha}{2}\right)_n}\right)w^{n+2}. \tag{27}$$

$$\fint_{\mathbb{T}} \frac{(\overline{w}-\overline{\tau})(\overline{w}^n-\overline{\tau}^n)}{|1-\tau|^{\alpha+2}}d\tau = -\frac{\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\left(1-\frac{\left(\frac{\alpha}{2}\right)_n}{(-\frac{\alpha}{2})_n}\right)\overline{w}^n. \tag{28}$$

Proof. We start with the change of variables $\tau = w\zeta$,

$$\begin{aligned}\fint_{\mathbb{T}} \frac{\tau^n}{|\tau-w|^\alpha}d\tau &= w^{n+1}\fint_{\mathbb{T}} \frac{\zeta^n}{|\zeta-1|^\alpha}d\tau \\ &= w^{n+1}\frac{1}{2^{\alpha+1}\pi}\int_0^{2\pi}\frac{e^{i(n+1)\eta}}{|\sin(\eta/2)|^\alpha}d\eta.\end{aligned}$$

Again by the change of variables $\eta/2 \mapsto \eta$ one gets

$$\fint_{\mathbb{T}} \frac{\tau^n}{|\tau-w|^\alpha}d\tau = w^{n+1}\frac{1}{2^\alpha\pi}\int_0^{\pi}\frac{e^{2i(n+1)\eta}}{\sin^\alpha\eta}d\eta.$$

We shall now recall the following identity, see for instance [29, p.8] and [36, p.449].

$$\int_0^\pi \sin^x(\eta)e^{iy\eta}d\eta = \frac{\pi e^{i\frac{\pi y}{2}}\Gamma(x+1)}{2^x\Gamma(1+\frac{x+y}{2})\Gamma(1+\frac{x-y}{2})}, \quad \forall\, x>-1, \quad \forall\, y\in\mathbb{R}. \tag{29}$$

As it was pointed before the gamma function has no real zeros but simple poles located at $-\mathbb{N}$ and therefore the function $\frac{1}{\Gamma}$ admits an analytic continuation on $\mathbb{C}$. Apply this formula with $x=-\alpha$ and $y=2(n+1)$ yields,

$$\frac{1}{2^\alpha\pi}\int_0^\pi\frac{e^{2i(n+1)\eta}}{\sin^\alpha\eta}d\eta = \frac{(-1)^{n+1}\Gamma(1-\alpha)}{\Gamma(n+2-\frac{\alpha}{2})\Gamma(-n-\frac{\alpha}{2})}. \tag{30}$$

It is easy to see that from the relations (25) we may write for any $n\in\mathbb{N}$,

$$\begin{aligned}\Gamma(1+n-\alpha/2) &= \Gamma(1-\alpha/2)\left(1-\frac{\alpha}{2}\right)_n \\ \Gamma(1-n-\alpha/2) &= (-1)^n\frac{\Gamma(1-\alpha/2)}{\left(\frac{\alpha}{2}\right)_n}.\end{aligned}$$

It follows that

$$\Gamma(1-n-\alpha/2)\Gamma(1+n-\alpha/2) = (-1)^n\Gamma^2\big(1-\frac{\alpha}{2}\big)\frac{\left(1-\frac{\alpha}{2}\right)_n}{\left(\frac{\alpha}{2}\right)_n}.$$

By replacing n with $n+1$ we get

$$\Gamma(-n-\alpha/2)\Gamma(2+n-\alpha/2) = (-1)^{n+1}\Gamma^2\big(1-\frac{\alpha}{2}\big)\frac{\left(1-\frac{\alpha}{2}\right)_{n+1}}{\left(\frac{\alpha}{2}\right)_{n+1}}.$$

Inserting this identity into (30) gives

$$\frac{1}{2^\alpha \pi}\int_0^\pi \frac{e^{2i(n+1)\theta}}{\sin^\alpha \theta}d\theta = \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\frac{\left(\frac{\alpha}{2}\right)_{n+1}}{\left(1-\frac{\alpha}{2}\right)_{n+1}}. \tag{31}$$

Consequently

$$⨍_{\mathbb{T}} \frac{\tau^n}{|\tau-w|^\alpha}d\tau = \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\frac{\left(\frac{\alpha}{2}\right)_{n+1}}{\left(1-\frac{\alpha}{2}\right)_{n+1}} w^{n+1}.$$

This completes the proof of (26).
We intend now to compute the second integral. To this end we use a change of variable as before,

$$J_n \triangleq ⨍_{\mathbb{T}} \frac{(w-\zeta)(w^n-\zeta^n)}{|w-\zeta|^{\alpha+2}}d\zeta = w^{n+2}⨍_{\mathbb{T}} \frac{(1-\zeta)(1-\zeta^n)}{|1-\zeta|^{\alpha+2}}d\zeta.$$

Using once again the change of variables $\zeta \mapsto e^{i\eta}$ and $\eta \mapsto 2\eta$ one gets

$$\begin{aligned} J_n &= \frac{w^{n+2}}{2\pi}\int_0^{2\pi} \frac{(1-e^{i\eta})(1-e^{in\eta})e^{i\eta}}{2^{\alpha+2}|\sin(\eta/2)|^{\alpha+2}}d\eta \\ &= \frac{w^{n+2}}{2^{\alpha+2}\pi}\int_0^\pi \frac{(1-e^{i2\theta})(1-e^{i2n\theta})e^{i2\theta}}{\big(\sin\theta\big)^{\alpha+2}}d\theta. \end{aligned}$$

Observe that

$$J_n = \frac{w^{n+2}}{2^{\alpha+1}i\pi}\int_0^\pi \frac{(e^{2i\theta}-e^{i2(n+1)\theta})e^{i\theta}}{\sin^{\alpha+1}\theta}d\theta$$

and therefore

$$J_n = \frac{w^{n+2}}{2^{\alpha+1}i\pi}\left(\int_0^\pi \left(e^{i2\theta}-e^{i2(n+1)\theta}\right)\frac{\cos\theta}{\sin^{\alpha+1}\theta}d\theta + i\int_0^\pi \frac{e^{i2\theta}-e^{i2(n+1)\theta}}{\sin^\alpha\theta}d\theta\right).$$

Integrating by parts implies

$$\int_0^\pi \left(e^{i2\theta}-e^{i2(n+1)\theta}\right)\frac{\cos\theta}{\sin^{\alpha+1}\theta}d\theta = \frac{2i}{\alpha}\int_0^\pi \frac{\left(e^{i2\theta}-(n+1)e^{i2(n+1)\theta}\right)}{\sin^\alpha\theta}d\theta.$$

Note that in this formula the contribution coming from the boundary terms is zero for $\alpha \in [0,1[$. Hence we get

$$J_n = \frac{w^{n+2}}{2^{\alpha+1}\pi}\left(\frac{2+\alpha}{\alpha}\int_0^\pi \frac{e^{i2\theta}}{\sin^\alpha\theta}d\theta - \frac{2(n+1)+\alpha}{\alpha}\int_0^\pi \frac{e^{i2(n+1)\theta}}{\sin^\alpha\theta}d\theta\right).$$

Combining this formula with the identity (31) gives

$$\begin{aligned} J_n &= w^{n+2}\frac{(2+\alpha)\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})} - \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\frac{2n+2+\alpha}{2\alpha}\frac{\left(\frac{\alpha}{2}\right)_{n+1}}{\left(1-\frac{\alpha}{2}\right)_{n+1}} \\ &= w^{n+2}\frac{(1+\frac{\alpha}{2})\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\left(1-\frac{1-\frac{\alpha}{2}}{1+\frac{\alpha}{2}}\frac{n+1+\frac{\alpha}{2}}{\frac{\alpha}{2}}\frac{\left(\frac{\alpha}{2}\right)_{n+1}}{\left(1-\frac{\alpha}{2}\right)_{n+1}}\right). \end{aligned}$$

By (24) we may transform this formula into,

$$J_n = w^{n+2}\frac{(1+\frac{\alpha}{2})\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\left(1-\frac{\left(2+\frac{\alpha}{2}\right)_n}{\left(2-\frac{\alpha}{2}\right)_n}\right).$$

Next we shall now move to the computation of the last integral (28),

$$Z_n \triangleq \fint_{\mathbb{T}} \frac{(\overline{w}-\overline{\tau})(\overline{w}^n-\overline{\tau}^n)}{|1-\tau|^{\alpha+2}}d\tau = \overline{w}^n \fint_{\mathbb{T}} \frac{(1-\overline{\zeta})(1-\overline{\zeta}^n)}{|1-\zeta|^{\alpha+2}}d\zeta.$$

Making a standard change of variables as for the preceding integral we obtain

$$\begin{aligned}
Z_n &= \frac{\overline{w}^n}{2\pi}\int_0^{2\pi}\frac{(1-e^{-i\eta})(1-e^{-in\eta})e^{i\eta}}{2^{\alpha+2}|\sin(\eta/2)|^{\alpha+2}}d\eta \\
&= \frac{\overline{w}^n}{2^{\alpha+2}\pi}\int_0^{\pi}\frac{(1-e^{-i2\eta})(1-e^{-i2n\eta})e^{i2\eta}}{\sin^{\alpha+2}\eta}d\eta \\
&= \frac{i\overline{w}^n}{2^{\alpha+1}\pi}\int_0^{\pi}\frac{(1-e^{-i2n\theta})e^{i\theta}}{\sin^{\alpha+1}\theta}d\theta \\
&= \frac{i\overline{w}^n}{2^{\alpha+1}\pi}\left(\int_0^{\pi}\frac{\cos\theta(1-e^{-i2n\theta})}{\sin^{\alpha+1}\theta}d\theta + i\int_0^{\pi}\frac{(1-e^{-i2n\theta})}{\sin^{\alpha}\theta}d\theta\right).
\end{aligned}$$

Integrating by parts gives

$$\int_0^{\pi}\frac{\cos\theta(1-e^{-i2n\theta})}{\sin^{\alpha+1}\theta}d\theta = \frac{2in}{\alpha}\int_0^{\pi}\frac{e^{-i2n\theta}}{\sin^{\alpha}\theta}d\theta.$$

This implies that

$$Z_n = -\frac{\overline{w}^n}{2^{\alpha+1}\pi}\left(\int_0^{\pi}\frac{1}{\sin^{\alpha}\theta}d\theta + \frac{2n-\alpha}{\alpha}\int_0^{\pi}\frac{e^{-i2n\theta}}{\sin^{\alpha}\theta}d\theta\right).$$

Using once again (31) and (24) we obtain

$$\begin{aligned}
Z_n &= -\overline{w}^n\frac{\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\left(1+\frac{n-\frac{\alpha}{2}}{\frac{\alpha}{2}}\frac{(\frac{\alpha}{2})_n)}{(1-\frac{\alpha}{2})_n}\right) \\
&= -\overline{w}^n\frac{\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\left(1-\frac{(\frac{\alpha}{2})_n}{(-\frac{\alpha}{2})_n}\right).
\end{aligned}$$

Note that we have used the following fact which can be deduced easily from (31) by conjugation,

$$\int_0^{\pi}\frac{e^{-i2n\theta}}{\sin^{\alpha}\theta}d\theta = \int_0^{\pi}\frac{e^{i2n\theta}}{\sin^{\alpha}\theta}d\theta$$

and therefore the proof of the lemma is now completed. □

5. Elliptic patches

Given a simply connected domain, to check whether or not it is a rotating patch can be done through the equation of Proposition 4 provided that a parametric representation of the boundary is known (for example the one given by the conformal mapping) and the computations of the the integral term are feasible. In what follows we shall concretize this program for some elementary domains. We shall prove that the ellipses never rotate except

for the degenerate case where they coincide with discs. We point out this result was recently shown in [6] and we will give here a flexible proof with less computations.

Proposition 5. *The following holds true*

(1) *The discs are rotating patches for any* $\Omega \in \mathbb{R}$.
(2) *The ellipses are not rotating patches.*

Proof. (**1**) Recall from (15) that the conformal mapping of a rotating domain must satisfy the equation

$$G(\Omega, \phi(w)) = 0, \quad \forall w \in \mathbb{T}$$

To check whether or not the unit disc is a solution, it suffices to prove that

$$G(\Omega, \mathrm{Id}) = 0.$$

It is easy to see that,

$$\begin{aligned} G(\Omega, \mathrm{Id})(w) &= \mathrm{Im}\Big\{ \Big(\Omega w - C_\alpha \fint_{\mathbb{T}} \frac{d\tau}{|w-\tau|^\alpha}\Big)\frac{1}{w} \Big\} \\ &= -C_\alpha \mathrm{Im}\Big\{ \fint_{\mathbb{T}} \frac{d\tau}{w|w-\tau|^\alpha} \Big\}. \end{aligned}$$

Using the the formula (26) with $n = 0$ we may conclude tha for any $\Omega \in \mathbb{R}$,

$$G(\Omega, \mathrm{Id}) = 0.$$

We observe that this result is known and expected because the disc corresponds to a stationary solution for (1) and is invariant by rotation.

(**2**) By translation, dilation and rotation we can assume that the ellipse $\mathcal{E}$ is parametrized by the conformal mapping

$$\phi_Q : w \in \mathbb{T} \mapsto w + Q\overline{w}, \quad \text{with} \quad Q = \frac{a-b}{a+b} \in (0,1)$$

where a and b denote the major and minor axes, respectively. This map sends conformally the exterior of the unit disc to the exterior of the ellipse. Performing straightforward computations leads in view of (15) to

$$G\big(\Omega, \phi_Q\big)(w) = -\mathrm{Im}\Big\{ 2Q\Omega w^2 + C_\alpha\big(\overline{w} - Qw\big) \fint_{\mathbb{T}} \frac{\big(1-Q\overline{\tau}^2\big)d\tau}{|w-\tau+Q(\overline{w}-\overline{\tau})|^\alpha} \Big\}.$$

By using the identity

$$\big|z + Q\overline{z}\big|^2 = (1+Q^2)|z|^2 + 2Q\,\mathrm{Re}(z^2), \quad \forall z \in \mathbb{C},$$

one gets

$$G\big(\Omega, \phi_Q\big)(w) = -\mathrm{Im}\Big\{ 2Q\Omega w^2 + C_\alpha\big(\overline{w} - Qw\big) \fint_{\mathbb{T}} \frac{\big(1-Q\overline{\tau}^2\big)d\tau}{\Big[(1+Q^2)|w-\tau|^2 + 2Q\mathrm{Re}\big\{(w-\tau)^2\big\}\Big]^{\alpha/2}} \Big\}.$$

Making the change of variables $\tau = w\zeta$ and using the identity

$$\big(1-z\big)^2 = -z|1-z|^2, \quad \forall\, z \in \mathbb{T}$$

we find

$$G\big(\Omega,\phi_Q\big)(w)=-\mathrm{Im}\Big\{2Q\Omega w^2+\frac{C_\alpha}{(1+Q^2)^{\frac{\alpha}{2}}}\big(1-Qw^2\big)\fint_{\mathbb{T}}\frac{\big(1-Q\overline{w}^2\overline{\zeta}^2\big)d\zeta}{|1-\zeta|^\alpha\Big[1-\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\Big]^{\alpha/2}}\Big\}.$$

We shall transform the last integral term as follows,

$$\fint_{\mathbb{T}}\frac{1-Q\overline{w}^2\overline{\zeta}^2}{|1-\zeta|^\alpha\Big[1-\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\Big]^{\alpha/2}}d\zeta=J(w)-Q\overline{w}^2\overline{J(w)},$$

with

$$J(w)\triangleq\fint_{\mathbb{T}}\frac{d\zeta}{|1-\zeta|^\alpha\big(1-\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\big)^{\alpha/2}}.$$

Therefore we get,

$$\begin{aligned}G\big(\Omega,\phi_Q\big)(w)&=-\mathrm{Im}\Big\{2Q\Omega w^2+\frac{C_\alpha}{(1+Q^2)^{\frac{\alpha}{2}}}\Big(J(w)+Q^2\overline{J(w)}-Q\Big[w^2J(w)+\overline{w}^2\overline{J(w)}\Big]\Big)\Big\}\\(32)\qquad&=-\mathrm{Im}\Big\{2Q\Omega w^2+C_\alpha\frac{1-Q^2}{(1+Q^2)^{\frac{\alpha}{2}}}J(w)\Big\}.\end{aligned}$$

Since $\big|\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\big|<1$ then we can use the Taylor series

$$\Big(1-\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\Big)^{-\alpha/2}=\sum_{n=0}^{\infty}2^nA_n\big(\mathrm{Re}\{w^2\zeta\}\big)^n,$$

with

$$A_n=\frac{(\alpha/2)_n}{n!}\Big(\frac{Q}{1+Q^2}\Big)^n,\quad\forall\,n\in\mathbb{N}.$$

Consequently we get

$$\begin{aligned}J(w)&=\sum_{n=0}^{\infty}2^nA_n\fint_{\mathbb{T}}\frac{\big(\mathrm{Re}\{w^2\zeta\}\big)^n}{|1-\zeta|^\alpha}d\zeta\\&=a_\alpha+\sum_{n=1}^{\infty}A_n\sum_{k=0}^{n}\binom{n}{k}w^{2(n-2k)}\fint_{\mathbb{T}}\frac{\zeta^{n-2k}}{|1-\zeta|^\alpha}d\zeta.\end{aligned}$$

By the Lemma 2 the coefficient a_α is real and therefore it does not contribute in $\mathrm{Im}J(w)$. Our goal now is to compute the coefficients of w^4 and $\overline{w}^4$ of the function between the bracket in (32), denoted by B_4 and B_{-4}, respectively. First we observe that the coefficient B_4 can be obtained by summing over the set

$$\Big\{n\geq1,0\leq k\leq n\setminus n-2k=2\Big\}=\Big\{n\geq1,k\geq0\setminus n=2k+2\Big\}.$$

This is equivalent to write

$$\begin{aligned} B_4 &= \sum_{k=0}^{\infty} A_{2k+2} \begin{pmatrix} 2k+2 \\ k \end{pmatrix} \fint_{\mathbb{T}} \frac{\zeta^2 d\zeta}{|1-\zeta|^\alpha} \\ &= \sum_{k=1}^{\infty} A_{2k} \begin{pmatrix} 2k \\ k-1 \end{pmatrix} a_2, \qquad\qquad a_2 \triangleq \fint_{\mathbb{T}} \frac{\zeta^2 d\zeta}{|1-\zeta|^\alpha}. \end{aligned}$$

Next we shall compute the coefficient of $\overline{w}^4$ denoted by B_{-4}. This may be done by summing over the set

$$\Big\{n \geq 1, 0 \leq k \leq n \setminus n - 2k = -2\Big\} = \Big\{n \geq 1, k \geq 2 \setminus n = 2k - 2\Big\}.$$

Hence by change of variables,

$$\begin{aligned} B_{-4} &= \sum_{k=2}^{\infty} A_{2k-2} \begin{pmatrix} 2k-2 \\ k \end{pmatrix} \fint_{\mathbb{T}} \frac{\xi^{-2} d\xi}{|1-\xi|^\alpha} \\ &= \sum_{k=1}^{\infty} A_{2k} \begin{pmatrix} 2k \\ k+1 \end{pmatrix} \fint_{\mathbb{T}} \frac{d\zeta}{|1-\zeta|^\alpha} \\ &\triangleq \sum_{k=1}^{\infty} A_{2k} \begin{pmatrix} 2k \\ k-1 \end{pmatrix} a_0. \end{aligned}$$

But, in view of Lemma 2, one has

$$\frac{a_2}{a_0} = \frac{(2+\alpha)(4+\alpha)}{(4-\alpha)(6-\alpha)} \neq 1 \quad \text{for } \alpha \neq 1.$$

Thus $B_4 \neq B_{-4}$ and therefore the coefficient of $w^4 - \overline{w}^4$ of $G(\Omega, \phi_Q)(w)$ does not vanish. It follows that the equation $G(\Omega, \phi_Q)(w) = 0, \forall w \in \mathbb{T}$ is not true for any Ω. This concludes the proof of the desired result. □

6. General statement

In this section we shall give a more precise statement of Theorem 1. In particular we shall give a description of the conformal mapping which parametrizes the rotating patches close to the unit disc.

Theorem 3. *Let $\alpha \in]0,1[$ and $m \in \mathbb{N}^* \setminus \{1\}$. Then there exists $a > 0$ and two continuous functions $\Omega : (-a, a) \to \mathbb{R}$, $\phi : (-a, a) \to C^{2-\alpha}\big(\mathbb{T}\big)$ satisfying $\Omega(0) = \Omega_m^\alpha$, $\phi(0) = \mathrm{Id}$, such that $(\phi_s)_{-a<s<a}$ is a one-parameter non trivial solution of the equation (8), where*

$$\Omega_m^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})} \bigg(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})} \bigg),$$

Moreover, ϕ_s admits the expansion

$$\phi_s(w) = w\Big(1 + s\frac{1}{w^m} + s\sum_{n\geq 2} a_{nm-1}(s)\frac{1}{w^{nm}}\Big), \quad \forall\, w \in \mathbb{T},$$

and it is conformal on $\mathbb{C}\setminus\mathbb{D}$ and the complement D_s of $\phi_s\big(\mathbb{C}\setminus\mathbb{D}\big)$ is an $m-$fold rotating patch with the angular velocity $\Omega(s)$. In addition, the boundary of this patch belongs to the class $C^{2-\alpha}$.

• **Outline of the proof**. The proof of this theorem will be divided into several steps. The main key is Crandall Rabinowitz Theorem, sometimes denoted by C-R, which requires to check many properties for the linear and the nonlinear functionals of the equation (8) defining the V-states. Firstly, we shall check the regularity assumptions that will be separated into weak and strong ones. Secondly, we will conduct a spectral study of the linearized operator around the trivial solution. In this context, we are able to describe the complete bifurcation set made of the values Ω such that the linearized operator is Fredholm with one-dimensional kernel. We shall also check in this section the transversality assumption of C-R Theorem. In the last step, we give the complete proof for the existence of the V-states and check their m-fold structure.

7. Regularity of the functional F

This section is devoted to the study of the regularity assumptions stated in C-R Theorem. The object that we shall study is the nonlinear functional G introduced in (15) and given by

$$G(\Omega,\phi)(w) \triangleq \mathrm{Im}\left\{\left(\Omega\phi(w) - C_\alpha \fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha} d\tau\right)\overline{w}\,\overline{\phi'(w)}\right\}, \quad \forall\, w \in \mathbb{T}.$$

Because we are interested in the bifurcation from the disc (corresponding to $\phi = \mathrm{Id}$), it is more convenient to make a translation and study the bifurcation from zero. To this end, we introduce the function F defined by

$$F(\Omega, f)(w) = G(\Omega, w + f(w)), \quad \forall w \in \mathbb{T}.$$

In order to apply C-R Theorem we need first to fix the function spaces and check the regularity of the functional F with respect to these spaces. We should look for Banach spaces X and Y such that $F : \mathbb{R} \times X \mapsto Y$ is well-defined and satisfies the assumptions of Theorem 2. These spaces will be defined in the spirit of the work done for the incompressible Euler equations [18]. They are given by,

$$X = \left\{ f \in C^{2-\alpha}(\mathbb{T}),\ f(w) = \sum_{n\geq 0} b_n \overline{w}^n, b_n \in \mathbb{R},\ w \in \mathbb{T} \right\}$$

and

$$Y = \left\{ g \in C^{1-\alpha}(\mathbb{T}),\ g(w) = i\sum_{n\geq 1} g_n\big(w^n - \overline{w}^n\big), g_n \in \mathbb{R},\ w \in \mathbb{T} \right\}.$$

For $r \in (0,1)$ we denote by B_r the open ball of X with center 0 and radius r,

$$B_r = \Big\{ f \in X, \quad \|f\|_{C^{2-\alpha}} < r \Big\}.$$

It is straightforward that for any $f \in B_r$ the function $w \mapsto \phi(w) = w + f(w)$ is conformal on $\mathbb{C}\backslash\overline{\mathbb{D}}$. Moreover according to Kellog-Warshawski result [34], the boundary of $\phi(\mathbb{C}\backslash\overline{\mathbb{D}})$ is a Jordan curve of class $C^{2-\alpha}$. This gives the proof of the last result of Theorem 3 provided that the regularity of ϕ is shown. Note that we can prove the regularity of the boundary without making appeal to the result [34]. We just look for the conformal parametrization $\theta \mapsto \phi(e^{i\theta})$ which is regular and prove that it belongs to $C^{2-\alpha}$. This last fact is equivalent to $\phi \in C^{2-\alpha}(\mathbb{T})$.

7.1. **Weak regularity.** Our objective is to prove that the functional F is well-defined and admits Gâteaux derivatives for any given direction. More precisely, we shall prove the following result.

Proposition 6. *For any $r \in (0,1)$ the following holds true.*

(1) *$F : \mathbb{R} \times B_r \to Y$ is well-defined.*
(2) *For each point $(\Omega, f) \in \mathbb{R} \times B_r$, the Gâteaux derivative of F, $\partial_f F(\Omega, f) : X \to Y$ exists and belongs to $\mathcal{L}(X, Y)$*

Proof. (**1**) First, because the space $C^{1-\alpha}(\mathbb{T})$ is an algebra, it is clear that the first part of the functional G given by, $w \mapsto \Omega\, \phi(w)\, \overline{w}\, \overline{\phi'(w)}$ belongs to $C^{1-\alpha}(\mathbb{T})$. To prove that the second term of G belongs to $C^{1-\alpha}(\mathbb{T})$ it suffices to check that

$$S(\phi) : w \mapsto \fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w) - \phi(\tau)|^\alpha} d\tau \in C^{1-\alpha}(\mathbb{T}).$$

This follows immediately from Corollary 2. Therefore it remains to check that the Fourier coefficients of $G(\Omega, \phi)$ belong to $i\mathbb{R}$. By the assumption, the Fourier coefficients of $\phi = \mathrm{Id} + f$ are real and thus the coefficients of $\overline{\phi'}$ are real too. Now using the stability of this property under the multiplication and the conjugation we deduce that the Fourier coefficients of $w \mapsto \Omega\phi(w)\overline{\phi'(w)}\overline{w}$ are real. To complete the proof we shall check that the Fourier coefficients of $S(\phi)$ are also real for every $f \in B_r$. From the regularity of $\phi \in C^{1-\alpha}(\mathbb{T})$ we can pointwise expand this function into its Fourier series, that is,

$$S(\phi)(w) = \sum_{n \in \mathbb{Z}} a_n w^n, \quad a_n = \fint_{\mathbb{T}} \frac{S(\phi)(w)}{w^{n+1}} dw = \fint_{\mathbb{T}} \fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(\tau) - \phi(w)|^\alpha} d\tau \frac{dw}{w^{n+1}}.$$

This coefficient can also be written in the form

$$a_n = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\phi'(e^{i\theta}) e^{i\theta} e^{-in\eta}}{|\phi(e^{i\theta}) - \phi(e^{i\eta})|^\alpha} d\theta\, d\eta.$$

By taking the conjugate of a_n and using the properties

$$\overline{\phi(e^{i\theta})} = \phi(e^{-i\theta}), \quad \overline{\phi'(e^{i\theta})} = \phi'(e^{-i\theta}) \quad \text{and} \quad |z| = |\overline{z}|$$

one may obtain by change of variables

$$\begin{aligned} \overline{a_n} &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\phi'(e^{-i\theta}) e^{-i\theta} e^{in\eta}}{|\phi(e^{-i\theta}) - \phi(e^{-i\eta})|^\alpha} d\theta\, d\eta \\ &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\phi'(e^{i\theta}) e^{i\theta} e^{-in\eta}}{|\phi(e^{i\theta}) - \phi(e^{i\eta})|^\alpha} d\theta\, d\eta \\ &= a_n. \end{aligned}$$

Consequently the Fourier coefficients of $S(\phi)$ are real and therefore $F(\Omega, f)$ belongs to Y.

(**2**) We shall compute the Gâteaux derivative of F at the point $f \in B_r$ in the direction $h \in X$. A refined analysis concerning its connection with Fréchet derivative will be developed in the

next section. The Gâteaux derivative of $\partial_f F(\Omega, f)h$ is defined through the formula,

$$\begin{aligned}\partial_f F(\Omega, f)h(w) &= \lim_{t\to 0} \frac{F(\Omega, f(w)+th(w)) - F(\Omega, f(w))}{t} \\ &= \frac{d}{dt}\Big|_{t=0} F(\Omega, f+th)(w).\end{aligned}$$

This limit is taken in the strong topology of $C^{1-\alpha}(\mathbb{T})$. Thus we shall first prove the existence of this limit for every point $w \in \mathbb{T}$ and after check that this limit exists in $C^{1-\alpha}(\mathbb{T})$. With the notation $\phi = \mathrm{Id} + f$,

$$\begin{aligned}\partial_f F(\Omega, f)h(w) &= \frac{d}{dt}\Big|_{t=0} F(\Omega, f+th)(w) \\ &= \Omega \,\mathrm{Im}\Big\{ \phi(w)\, \overline{w}\, \overline{h'(w)} + h(w)\, \overline{w}\, \overline{\phi'(w)} \Big\} \\ &- C_\alpha \,\mathrm{Im}\Big\{ S(\phi(w))\overline{w}\overline{h'(w)} + \overline{w}\, \overline{\phi'(w)} \frac{d}{dt}\Big|_{t=0} S(\phi+th)(w) \Big\} \\ &\triangleq \mathcal{L}(f)(h(w)). \qquad (33)\end{aligned}$$

We shall make use of the following identity: let $A \in \mathbb{C}^\star$, $B \in \mathbb{C}$, $\alpha \in \mathbb{R}$ and introduce the function $K : t \mapsto |A+Bt|^\alpha$ which is smooth close to zero, then we have

$$K'(0) = \alpha |A|^{\alpha-2} \mathrm{Re}(\overline{A}B). \qquad (34)$$

Combining this formula with few easy computations one gets

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} S(\phi+th)(w) &= \fint_{\mathbb{T}} \frac{h'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha} d\tau - \frac{\alpha}{2} \fint_{\mathbb{T}} \frac{\big(\phi(w)-\phi(\tau)\big)\big(\overline{h(w)}-\overline{h(\tau)}\big)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} \phi'(\tau) d\tau \\ &- \frac{\alpha}{2} \fint_{\mathbb{T}} \frac{\big(\overline{\phi(w)}-\overline{\phi(\tau)}\big)\big(h(w)-h(\tau)\big)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} \phi'(\tau) d\tau \\ &\triangleq A(\phi,h)(w) - \frac{\alpha}{2}\Big(B(\phi,h)(w) + C(\phi,h)(w) \Big). \qquad (35)\end{aligned}$$

Therefore we obtain from (33) the identity

$$\begin{aligned}\mathcal{L}(f)(h)(w) &= \mathrm{Im}\Big\{ \Omega \Big[\phi(w)\, \overline{w}\, \overline{h'(w)} + h(w)\, \overline{w}\, \overline{\phi'(w)} \Big] - C_\alpha\, S(\phi(w))\, \overline{w}\, \overline{h'(w)} \Big\} \\ &- C_\alpha \,\mathrm{Im}\Big\{ \overline{w}\, \overline{\phi'(w)} \Big[A(\phi,h)(w) - \frac{\alpha}{2}\Big(B(\phi,h)(w) + C(\phi,h)(w) \Big) \Big] \Big\}. \qquad (36)\end{aligned}$$

Set

$$\mathcal{L}_1(f)h(w) \triangleq \mathrm{Im}\Big\{ \Omega \Big[\phi(w)\, \overline{w}\, \overline{h'(w)} + h(w)\, \overline{w}\, \overline{\phi'(w)} \Big] - C_\alpha\, S(\phi(w))\, \overline{w}\, \overline{h'(w)} \Big\}$$

Since $C^{1-\alpha}(\mathbb{T})$ is an algebra and using some classical Hölder embeddings, we get

$$\|\mathcal{L}_1(f)h\|_{C^{1-\alpha}((T)} \lesssim \|\phi\|_{C^{2-\alpha}} \|h\|_{C^{2-\alpha}} + \|S(\phi)\|_{C^{1-\alpha}} \|h\|_{C^{2-\alpha}}.$$

To estimate $S(\phi)$ we use Corollary 2 combined with the estimate $\|\phi\|_{\mathrm{Lip}} + \|\phi^{-1}\|_{\mathrm{Lip}} \leq C(r)$. Therefore

$$\|S(\phi)\|_{C^{1-\alpha}(\mathbb{T})} \leq C.$$

It follows that

$$\|\mathcal{L}_1(f)h\|_{C^{1-\alpha}((T)} \leq C\|h\|_{C^{2-\alpha}}. \tag{37}$$

Now using once again Corollary 2 we get that $A(\phi, h) \in C^{1-\alpha}$ and

$$\begin{aligned} \|A(\phi,h)\|_{C^{1-\alpha}(\mathbb{T})} &\leq C\|h'\|_{L^\infty} \\ &\leq C\|h\|_{C^{2-\alpha}(\mathbb{T})}. \end{aligned} \tag{38}$$

So, it remains to show that $B(f,h)$ and $C(f,h)$ are of class $C^{1-\alpha}\big(\mathbb{T}\big)$. For this end we set

$$K_1(w,\tau) \triangleq \frac{\big(\phi(w)-\phi(\tau)\big)\big(\overline{h(w)}-\overline{h(\tau)}\big)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}.$$

Clearly we have for $\tau \neq w \in \mathbb{T}$,

$$\begin{aligned} |K_1(w,\tau)| &\leq \frac{\|\phi\|_{\mathrm{Lip}}\|h\|_{\mathrm{Lip}}}{|w-\tau|^{\alpha}} \\ &\leq C\frac{\|h\|_{C^{2-\alpha}(\mathbb{T})}}{|w-\tau|^{\alpha}}. \end{aligned} \tag{39}$$

Moreover, in view of the formula (16) we readily obtain

$$\begin{aligned} \partial_w K_1(w,\tau) &= \phi'(w)\frac{\overline{h(w)}-\overline{h(\tau)}}{|\phi(w)-\phi(\tau)|^{\alpha+2}} - \frac{\overline{h'(w)}}{w^2}\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} \\ &- \frac{\alpha+2}{2}\left(\frac{\phi'(w)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} - \frac{\overline{\phi'(w)}}{w^2}\frac{\big(\phi(w)-\phi(\tau)\big)^2}{|\phi(w)-\phi(\tau)|^{\alpha+4}}\right)\Big(\overline{h(w)}-\overline{h(\tau)}\Big). \end{aligned}$$

Therefore one has

$$\begin{aligned} \big|\partial_w K_1(w,\tau)\big| &\leq C\frac{\|\phi'\|_{L^\infty}\|h'\|_{L^\infty}}{|w-\tau|^{\alpha+1}} \\ &\leq C\frac{\|h\|_{C^{2-\alpha}(\mathbb{T})}}{|w-\tau|^{1+\alpha}}. \end{aligned} \tag{40}$$

Hence, combining the inequalities (39) and (40) with Lemma 1 we get

$$\begin{aligned} \|B(\phi,h)\|_{C^{1-\alpha}(\mathbb{T})} &\leq C\|h\|_{C^{2-\alpha}(\mathbb{T})}\|\phi'\|_{L^\infty(\mathbb{T})} \\ &\leq C\|h\|_{C^{2-\alpha}(\mathbb{T})}. \end{aligned} \tag{41}$$

To estimate the last term $C(\phi,h)$ we observe that

$$C(\phi,h)(w) = \fint_{\mathbb{T}} \overline{K_1(w,\tau)}\phi'(\tau)d\tau$$

and consequently similar proof of the estimate (41) allows to get,

$$\|C(\phi,h)\|_{C^{1-\alpha}(\mathbb{T})} \leq C\|h\|_{C^{2-\alpha}(\mathbb{T})}.$$

By putting together this estimate with (36), (37), (38) and (41) one concludes

$$\|\mathcal{L}(f)h\|_{C^{1-\alpha}(\mathbb{T})} \leq C\|h\|_{C^{2-\alpha}(\mathbb{T})}.$$

This means that $\mathcal{L}(f) \in \mathcal{L}(X,Y)$. To achieve the proof it remains to check that the convergence in (33) towards $\mathcal{L}(f)(h)$ occurs in the strong topology of $C^{1-\alpha}(\mathbb{T})$. The convergence of the quadratic terms containing the parameter Ω can be easily obtained from the algebra structure of $C^{1-\alpha}(\mathbb{T})$. Therefore the problem reduces to verify only the convergence in the formula (35). We shall check only the convergence for the term involving $A(\phi,h)$ and the analysis for the other terms leading to $B(\phi,h)$ and $C(\phi,h)$ is quite similar and we omit here the details. We start with showing

$$\lim_{t\to 0} \fint_{\mathbb{T}} \frac{h'(\tau)}{|\{\phi(w)-\phi(\tau)\}+t\{h(w)-h(\tau)\}|^{\alpha}} d\tau = \fint_{\mathbb{T}} \frac{h'(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha}} d\tau \quad \text{in} \quad C^{1-\alpha}.$$

Set

$$\begin{aligned} K(t,w,\tau) &= \frac{1}{|\{\phi(w)-\phi(\tau)\}+t\{h(w)-h(\tau)\}|^{\alpha}} - \frac{1}{|\phi(w)-\phi(\tau)|^{\alpha}} \\ &\triangleq g(t,w,\tau) - g(0,w,\tau). \end{aligned}$$

Then according to Lemma 1, the convergence happens provided that

$$\big|K(t,w,\tau)\big| \le C|t|\frac{1}{|w-\tau|^{\alpha}}, \quad \big|\partial_w K(t,w,\tau)\big| \le C|t|\frac{1}{|w-\tau|^{1+\alpha}}.$$

Let $t>0$ such that $t\|h\|_{\mathrm{Lip}(\mathbb{T})} \le \frac{1}{2}$ then

$$\begin{aligned} \big|K(t,w,\tau)\big| &\le C\frac{\Big||\{\phi(w)-\phi(\tau)\}+t\{h(w)-h(\tau)\}|^{\alpha} - |\{\phi(w)-\phi(\tau)\}|^{\alpha}\Big|}{|\tau-w|^{2\alpha}} \\ &\le C|t|\|h\|_{\mathrm{Lip}(\mathbb{T})}|\tau-w|\frac{|\tau-w|^{\alpha-1}}{|\tau-w|^{2\alpha}} \\ &\le C|t|\|h\|_{\mathrm{Lip}(\mathbb{T})}\frac{1}{|\tau-w|^{\alpha}}, \end{aligned}$$

where we have used the inequality: for $\alpha\in(0,1)$, there exists $C_\alpha>0$, such that

$$|a^{\alpha}-b^{\alpha}| \le C_{\alpha}\frac{|a-b|}{a^{1-\alpha}+b^{1-\alpha}}, \quad \forall a,b\in\mathbb{R}_+^*. \tag{42}$$

To estimate $\partial_w K(t,w,\tau)$ we shall use the Mean value Theorem,

$$K(t,w,\tau) = \int_0^t \partial_s g(s,w,\tau) ds$$

and therefore

$$\big|\partial_w K(t,w,\tau)\big| \le \int_0^t \big|\partial_w\partial_s g(s,w,\tau)\big| ds,$$

with

$$g(t,w,\tau) = \frac{1}{|\{\phi(w)-\phi(\tau)\}+t\{h(w)-h(\tau)\}|^{\alpha}}.$$

Using (19) leads to

$$\begin{aligned}\partial_w g(t,w,\tau) &= \frac{-\alpha}{2}\frac{g(t,w,\tau)}{|\phi(w)-\phi(\tau)+t\big(h(w)-h(\tau)\big)|^2}\times \\ &\Big\{\big(\phi'(w)+th'(w)\big)\Big(\overline{\phi(w)-\phi(\tau)}+t\overline{h(w)-h(\tau)}\Big) \\ &- \frac{\overline{\phi'(w)+th'(w)}}{w^2}\Big(\phi(w)-\phi(\tau)+t\big(h(w)-h(\tau)\big)\Big)\Big\}.\end{aligned}$$

Using straightforward computations yield for any $s\in[0,t]$,

$$\Big|\partial_s\partial_w g(s,w,\tau)\Big| \le C\frac{1}{|w-\tau|^{1+\alpha}}.$$

Hence we get

$$\big|\partial_w K(t,w,\tau)\big| \le C|t|\frac{1}{|w-\tau|^{1+\alpha}}.$$

This completes the proof of the estimate of the kernel and the required statement follows immediately. □

7.2. **Strong regularity.** In this subsection we shall discuss the existence of Fréchet derivative of F and prove that F is continuously differentiable on the domain $\mathbb{R}\times B_r$. More precisely, we shell establish the following result.

Proposition 7. *For any $r\in(0,1)$ the following holds true.*

(1) *$F:\mathbb{R}\times B_r\to Y$ is of class C^1.*
(2) *The partial derivative $\partial_\Omega\partial_f F:\mathbb{R}\times B_r\to\mathcal{L}(X,Y)$ exists and is continuous.*

Proof. (**1**) This amounts to showing that the partial derivatives $\partial_\Omega F$ and $\partial_f F$ in the Gâteaux sense exist and are continuous. For the first derivative, we observe the the linear dependence of F on Ω allows to get,

$$\partial_\Omega F(\Omega,f)(w) = \mathrm{Im}\Big\{\overline{w}\,\phi(w)\,\overline{\phi'(w)}\Big\}.$$

Obviously this is polynomial on ϕ and ϕ' and therefore it is continuous in the strong topology of X. The next step is to prove that for given Ω, $\partial_f F\big(\Omega,f\big)$ is continuous as a function of f taking values in the space of bounded linear operators from X to Y. In other words, we will show that, for a fixed $f,g\in B_r$,

$$\|\partial_f F(\Omega,f)(h)-\partial_g F(\Omega,g)(h)\|_{C^{1-\alpha}(\mathbb{T})}\le C\|f-g\|_{C^{2-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}. \tag{43}$$

Now because $C^{1-\alpha}$ is an algebra and from (35) and (36) the problem reduces to show the required inequality for the quantities $\mathcal{L}_1(f)h$, $A(\phi,h)$, $B(\phi,h)$ and $C(\phi,h)$. The crucial tool for this task is Lemma 1 which will be frequently used here. We shall start with proving the estimate

$$\|\mathcal{L}_1(f)h-\mathcal{L}_1(g)h\|_{C^{1-\alpha}}\le C\|f-g\|_{C^{2-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}.$$

For this end , it sufficient to establish that

$$\|S(\phi)-S(\psi)\|_{C^{1-\alpha}}\le C\|f-g\|_{C^{2-\alpha}(\mathbb{T})},$$

with $\phi=\mathrm{Id}+f$ and $\psi=\mathrm{Id}+g$. Write

$$
\begin{aligned}
S(\phi)(w) - S(\psi)(w) &= \fint_{\mathbb{T}} \Big(\frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha} - \frac{\psi'(\tau)}{|\psi(w)-\psi(\tau)|^\alpha} \Big) d\tau \\
&= \fint_{\mathbb{T}} \Big(\frac{1}{|\phi(w)-\phi(\tau)|^\alpha} - \frac{1}{|\psi(w)-\psi(\tau)|^\alpha} \Big) \psi'(\tau) d\tau \\
&+ \fint_{\mathbb{T}} \frac{\phi'(\tau)-\psi'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha} d\tau.
\end{aligned}
$$

The estimate of the last term follows immediately from Corollary 2, that is,

$$
\begin{aligned}
\Big\| \fint_{\mathbb{T}} \frac{\phi'(\tau)-\psi'(\tau)}{|\phi(\cdot)-\phi(\tau)|^\alpha} d\tau \Big\|_{C^{1-\alpha}(\mathbb{T})} &\leq C \|f'-g'\|_{L^\infty} \\
&\leq C \|f-g\|_{C^{2-\alpha}(\mathbb{T})}.
\end{aligned}
$$

As to the estimate of first term it can be deduced easily from the next general one: let T be the operator defined by

$$
T\chi(w) = \fint_{\mathbb{T}} \Big(\frac{1}{|\phi(w)-\phi(\tau)|^\alpha} - \frac{1}{|\psi(w)-\psi(\tau)|^\alpha} \Big) \chi(\tau) d\tau
$$

then

$$
\big\| T\chi \big\|_{C^{1-\alpha}(\mathbb{T})} \leq C \|\psi-\phi\|_{\mathrm{Lip}(\mathbb{T})} \|\chi\|_{L^\infty(\mathbb{T})}. \tag{44}
$$

To prove this control we shall introduce the kernel

$$
K_2(w,\tau) \triangleq \frac{1}{|\phi(w)-\phi(\tau)|^\alpha} - \frac{1}{|\psi(w)-\psi(\tau)|^\alpha}
$$

and prove that it satisfies the estimates,

$$
|K_2(w,\tau)| \lesssim \frac{\|\psi'-\phi'\|_{L^\infty}}{|w-\tau|^\alpha} \quad \text{and} \quad |\partial_w K_2(w,\tau)| \lesssim \frac{\|\psi'-\phi'\|_{L^\infty}}{|w-\tau|^{\alpha+1}}.
$$

Whence these estimates are proved we can then apply Lemma 1 and get the desired result. The first estimate is easy to obtain by using (42). On other hand, in view of (16) the derivative of $K_2(w,\tau)$ with respect to w is given by

$$
\partial_w K_2(w,\tau) = -\frac{\alpha}{2} \Big(\overline{\mathcal{I}(w,\tau)} - \frac{\mathcal{I}(w,\tau)}{w^2} \Big)
$$

where

$$
\mathcal{I}(w,\tau) \triangleq \overline{\phi'(w)} \frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} - \overline{\psi'(w)} \frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}}.
$$

We shall transform this quantity into,

$$
\mathcal{I}(w,\tau) = \mathcal{I}_1(w,\tau) + \mathcal{I}_2(w,\tau) + \mathcal{I}_3(w,\tau),
$$

with

$$\mathcal{I}_1(w,\tau) \triangleq \overline{\phi'(w)}\,\frac{(\phi-\psi)(w)-(\phi-\psi)(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}},$$

$$\mathcal{I}_2(w,\tau) \triangleq \big(\overline{\phi'(w)}-\overline{\psi'(w)}\big)\,\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}},$$

and

$$\mathcal{I}_3(w,\tau) \triangleq \overline{\phi'(w)}\big(\psi(\tau)-\psi(w)\big)\frac{|\phi(w)-\phi(\tau)|^{\alpha+2}-|\psi(w)-\psi(\tau)|^{\alpha+2}}{|\phi(w)-\phi(\tau)|^{\alpha+2}|\psi(w)-\psi(\tau)|^{\alpha+2}}.$$

For the first and the second term one readily gets

$$|\mathcal{I}_1(w,\tau)|+|\mathcal{I}_2(w,\tau)| \;\lesssim\; \frac{\|\phi-\psi\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}. \tag{45}$$

Concerning the last term we shall use the following inequality whose proof is classical.

$$|a^{k+1+\alpha}-b^{k+1+\alpha}| \le C(k,\alpha)|a-b|\big(a^{k+\alpha}+b^{k+\alpha}\big), \quad a,b\in\mathbb{R}_+, k\in\mathbb{N}^*,\ 0<\alpha<1. \tag{46}$$

Thus we find

$$\Big||\phi(w)-\phi(\tau)|^{\alpha+2}-|\psi(w)-\psi(\tau)|^{\alpha+2}\Big| \le C\|\phi-\psi\|_{\mathrm{Lip}(\mathbb{T})}\,|\tau-w|^{\alpha+2}$$

and consequently,

$$|\mathcal{I}_3(w,\tau)| \le C\frac{\|\phi-\psi\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}. \tag{47}$$

Putting together (45) and (47) we find,

$$|\mathcal{I}(w,\tau)| \le C\frac{\|\phi-\psi\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}.$$

Therefore

$$|\partial_w K_2(w,\tau)| \le C\frac{\|\phi-\psi\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}.$$

This achieves the suitable estimates for the kernel K_2. Let us now move to the continuity estimate of $A(\phi,h)$. We write from the definition,

$$A(\phi,h)(w)-A(\psi,h)(w) = \fint_{\mathbb{T}}\Big(\frac{1}{|\phi(w)-\phi(\tau)|^{\alpha}}-\frac{1}{|\psi(w)-\psi(\tau)|^{\alpha}}\Big)h'(\tau)d\tau \tag{48}$$

Using (44) we immediately obtain,

$$\begin{aligned}\big\|A(\phi,h)-A(\psi,h)\big\|_{C^{1-\alpha}} &\le C\|\phi-\psi\|_{\mathrm{Lip}(\mathbb{T})}\|h\|_{\mathrm{Lip}(\mathbb{T})}\\ &\le C\|f-g\|_{\mathrm{Lip}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned} \tag{49}$$

This completes the proof of the estimate of the term $A(\phi,h)$ which fits with (43).
Now we shall investigate the continuity estimate of $B(\phi,h)$ defined in (35). According to this definition, one has

$$\big(B(\phi,h)-B(\psi,h)\big)(w) = \fint_{\mathbb{T}} K_3(w,\tau)d\tau,$$

with

$$K_3(w,\tau) \triangleq \left(\phi'(\tau)\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} - \psi'(\tau)\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}}\right)\Big(\overline{h(w)}-\overline{h(\tau)}\Big).$$

We can rewrite this kernel in the form,

$$K_3(w,\tau) = \Big(K_3^1(w,\tau)+K_3^2(w,\tau)+K_3^3(w,\tau)\Big)\Big(\overline{h(w)}-\overline{h(\tau)}\Big),$$

with

$$K_3^1(w,\tau) \triangleq \phi'(\tau)\,\frac{(\phi-\psi)(w)-(\phi-\psi)(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}},$$
$$K_3^2(w,\tau) \triangleq \big(\phi'-\psi'\big)(\tau)\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}},$$

and

$$K_3^3(w,\tau) \triangleq \phi'(\tau)\big(\psi(\tau)-\psi(w)\big)\frac{|\phi(w)-\phi(\tau)|^{\alpha+2}-|\psi(w)-\psi(\tau)|^{\alpha+2}}{|\psi(w)-\psi(\tau)|^{\alpha+2}|\phi(w)-\phi(\tau)|^{\alpha+2}}.$$

In view of the inequality (46) we may conclude that

$$\begin{aligned}|K_3^1(w,\tau)|+|K_3^2(w,\tau)|+|K_3^3(w,\tau)| &\leq C\frac{\|\phi-\psi\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}\\ &\leq C\frac{\|f-g\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}.\end{aligned}$$

Consequently we find for $w\neq\tau\in\mathbb{T}$

$$|K_3(w,\tau)| \leq C\|h\|_{\mathrm{Lip}(\mathbb{T})}\frac{\|f-g\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha}}. \tag{50}$$

Now we intend to estimate $\partial_w K_3(w,\tau)$. Easy computations yield

$$\begin{aligned}\partial_w K_3(w,\tau) &= \Big(-\frac{\alpha}{2}\mathcal{N}_1(w,\tau)-\big(1+\frac{\alpha}{2}\big)\mathcal{N}_2(w,\tau)\Big)\big(\overline{h(w)}-\overline{h(\tau)}\big)\\ &- \mathcal{N}_3(w,\tau)\frac{\overline{h'(w)}}{w^2},\end{aligned} \tag{51}$$

with

$$\mathcal{N}_1(w,\tau) \triangleq \frac{\phi'(\tau)\phi'(w)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} - \frac{\psi'(\tau)\psi'(w)}{|\psi(w)-\psi(\tau)|^{\alpha+2}},$$
$$\mathcal{N}_2(w,\tau) \triangleq \frac{\overline{\phi'(w)}}{w^2}\frac{\phi'(\tau)\big(\phi(w)-\phi(\tau)\big)^2}{|\phi(w)-\phi(\tau)|^{\alpha+4}} - \frac{\overline{\psi'(w)}}{w^2}\frac{\psi'(\tau)\big(\psi(w)-\psi(\tau)\big)^2}{|\psi(w)-\psi(\tau)|^{\alpha+4}},$$

and

$$\mathcal{N}_3(w,\tau) \triangleq \phi'(\tau)\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} - \psi'(\tau)\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}}.$$

The estimate of the last term $\mathcal{N}_3$ can be done exactly as for $\mathcal{I}$. Concerning $\mathcal{N}_1(w,\tau)$ we may write

$$\begin{aligned}\mathcal{N}_1(w,\tau) &= (\phi'(w)-\psi'(w))\frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} + \big(\phi'(\tau)-\psi'(\tau)\big)\frac{\psi'(w)}{|\psi(w)-\psi(\tau)|^{\alpha+2}}\\ &- \phi'(\tau)\psi'(w)\frac{|\phi(w)-\phi(\tau)|^{\alpha+2}-|\psi(w)-\psi(\tau)|^{\alpha+2}}{|\phi(w)-\phi(\tau)|^{\alpha+2}|\psi(w)-\psi(\tau)|^{\alpha+2}}.\end{aligned}$$

Hence, using inequality (46) we immediately deduce that

$$|\mathcal{N}_1(w,\tau)| \;\leq\; \frac{C\|\phi'-\psi'\|_{L^\infty}}{|w-\tau|^{\alpha+2}}.$$

Now we shall split the term $\mathcal{N}_2(w,\tau)$ as follows,

$$\mathcal{N}_2(w,\tau) \;=\; \sum_{k=1}^{4}\mathcal{N}_{2,k}(w,\tau)$$

with

$$\mathcal{N}_{2,1}(w,\tau)=\phi'(\tau)\big(\overline{\phi'(w)}-\overline{\psi'(w)}\big)\frac{\big(\phi(w)-\phi(\tau)\big)^2}{w^2|\phi(w)-\phi(\tau)|^{\alpha+4}},$$

$$\mathcal{N}_{2,2}(w,\tau)=\phi'(\tau)\overline{\psi'(w)}\,\frac{\big(\phi(w)-\phi(\tau)\big)^2-\big(\psi(w)-\psi(\tau)\big)^2}{w|\phi(w)-\phi(\tau)|^{\alpha+4}},$$

$$\mathcal{N}_{2,3}(w,\tau)=\big(\phi'(\tau)-\psi'(\tau)\big)\overline{\psi'(w)}\frac{\big(\psi(w)-\psi(\tau)\big)^2}{w^2|\phi(w)-\phi(\tau)|^{\alpha+4}},$$

and

$$\mathcal{N}_{2,4}(w,\tau) \;=\; \psi'(\tau)\overline{\psi'(w)}\big(\psi(w)-\psi(\tau)\big)^2\frac{|\psi(w)-\psi(\tau)|^{\alpha+4}-|\phi(w)-\phi(\tau)|^{\alpha+4}}{|\psi(w)-\psi(\tau)|^{\alpha+4}|\phi(w)-\phi(\tau)|^{\alpha+4}}.$$

Similar computations as before lead to,

$$|\mathcal{N}_2(w,\tau)| \;\lesssim\; \frac{\|\phi'-\psi'\|_{L^\infty}}{|w-\tau|^{\alpha+2}}.$$

Hence, in view of the identity (51) we obtain

$$|\partial_w K_3(w,\tau)| \;\lesssim\; \frac{\|h'\|_{L^\infty}\|\phi'-\psi'\|_{L^\infty}}{|w-\tau|^{\alpha+1}}. \tag{52}$$

At this stage we can use (50), (52) and Lemma 1,

$$\begin{aligned}\big\|B(\phi,h)-B(\psi,h)\big\|_{C^{1-\alpha}} &\lesssim \|f-g\|_{\mathrm{Lip}(\mathbb{T})}\|h\|_{\mathrm{Lip}(\mathbb{T})}\\ &\lesssim \|f-g\|_{C^{2-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned} \tag{53}$$

Finally, we observe from (35) that

$$C(\phi,h)(w)-C(\psi,h)(w)=\fint_{\mathbb{T}}\overline{K_3(w,\tau)}\phi'(\tau)d\tau$$

and therefore we find similar estimate to (53). This ends the proof of (43).

(**2**) Now we shall compute $\partial_\Omega\partial_f F(\Omega,f)$ and prove the continuity of this function. Let $f\in B_r$ and $h\in C^{2-\alpha}(\mathbb{T})$ be a fixed direction, then in view of (33) one has

$$\partial_\Omega\partial_f F(\Omega,f)h(w)=\mathrm{Im}\Big\{\phi(w)\,\overline{w}\,\overline{h'(w)}+h(w)\,\overline{w}\,\overline{\phi'(w)}\Big\}.$$

It follows that for $f,g\in B_r$,

$$\begin{aligned}\|\partial_\Omega\partial_f F(\Omega,f)h-\partial_\Omega\partial_f F(\Omega,f)h\|_{C^{1-\alpha}(\mathbb{T})} &\leq C\|f-g\|_{C^{1-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}\\ &\leq C\|f-g\|_{C^{2-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned}$$

This proves the continuity of $\partial_\Omega \partial_f F(\Omega, f) : \mathbb{R} \times B_r \to \mathcal{L}(X, Y)$ and therefore the proof of the second point is now achieved. □

8. Spectral study

In this section we concentrate on the spectral study of the linearized operator of F around zero and denoted by $\partial_f F(\Omega, 0)$. We shall peculiarly look for the values of Ω where the kernel is non trivial. We will be seeing that the kernel is necessarily simple and all the required assumptions of the C-R Theorem are satisfied. According to the Proposition 7, the functional $F : \mathbb{R} \times B_r \to Y$ is C^1 and therefore Gâteaux and Fréchet derivatives with respect to f and in the direction $h \in X$ coincide. Now putting together the formulas (35) and (36) with $\phi = \mathrm{Id}$, we find

$$\begin{aligned}
\partial_f F(\Omega, 0)h(w) &= \mathrm{Im}\Big\{\Omega\Big(\overline{h'(w)} + \frac{h(w)}{w}\Big) - C_\alpha \frac{\overline{h'(w)}}{w} \fint_{\mathbb{T}} \frac{d\tau}{|w-\tau|^\alpha} - \frac{C_\alpha}{w} \fint_{\mathbb{T}} \frac{h'(\tau)}{|w-\tau|^\alpha} d\tau \\
&+ \frac{\alpha C_\alpha}{2w} \fint_{\mathbb{T}} \frac{(w-\tau)\big(\overline{h(w)} - \overline{h(\tau)}\big)}{|w-\tau|^{\alpha+2}} d\tau + \frac{\alpha C_\alpha}{2w} \fint_{\mathbb{T}} \frac{(\overline{w}-\overline{\tau})\big(h(w)-h(\tau)\big)}{|w-\tau|^{\alpha+2}} d\tau \Big\} \\
&\triangleq \mathrm{Im}\Big\{\mathrm{I}_1(h(w)) + \mathrm{I}_2(h(w)) + \mathrm{I}_3(h(w)) + \mathrm{I}_4(h(w)) + \mathrm{I}_5(h(w))\Big\}. \qquad (54)
\end{aligned}$$

Recall that the spaces X and Y are successively given by,

$$X = \Big\{ f \in C^{2-\alpha}(\mathbb{T}),\ f(w) = \sum_{n\geq 0} b_n \overline{w}^n, b_n \in \mathbb{R},\ w \in \mathbb{T} \Big\}$$

and

$$Y = \Big\{ g \in C^{1-\alpha}(\mathbb{T}),\ g(w) = i\sum_{n\geq 1} g_n\big(w^n - \overline{w}^n\big), g_n \in \mathbb{R},\ w \in \mathbb{T} \Big\}.$$

To state our main result we shall introduce a special set $\mathcal{S}$ describing the dispersion relation which plays a central role in the bifurcation of non trivial solutions.

$$\mathcal{S} \triangleq \left\{ \Omega \in \mathbb{R},\ \exists\, m \geq 2, \quad \Omega = \Omega_m^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})} \left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})} \right) \right\}. \qquad (55)$$

We shall discuss soon some elementary properties of this set. Now we state our result.

Proposition 8. *The following assertions hold true.*

(1) *The kernel of $\partial_f F(\Omega, 0)$ is non trivial if and only if $\Omega = \Omega_m^\alpha \in \mathcal{S}$ and, in this case, it is one-dimensional vector space generated by*

$$v_m(w) = \overline{w}^{m-1}.$$

(2) *The range of $\partial_f F(\Omega_m^\alpha, 0)$ is closed in Y and is of co-dimension one. It is given by*

$$R\big(\partial_f F(\Omega_m^\alpha, 0)\big) = \Big\{ g \in C^{1-\alpha}(\mathbb{T}); \quad g(w) = i\sum_{\substack{n\geq 1 \\ n\neq m}}^{\infty} g_n(w^n - \overline{w}^n), g_n \in \mathbb{R} \Big\}.$$

(3) *Transversality assumption:*

$$\partial_\Omega \partial_f F(\Omega_m^\alpha, 0)(v_m) \notin R\big(\partial_f F(\Omega_m^\alpha, 0)\big).$$

Before proving this result we collect some properties on the asymptotic behavior of the sequence $\{\Omega_n^\alpha\}$ with respect to α and n. This is summarized in the next lemma.

Lemma 3. *We have the following results.*

(1) *Let $n \geq 2$, then*

$$\lim_{\alpha\to 0} \Omega_n^\alpha = \frac{n-1}{2n}, \qquad \lim_{\alpha\to 1} \Omega_n^\alpha = \frac{2}{\pi}\sum_{k=1}^{n-1} \frac{1}{2k+1}\cdot$$

(2) *For any $\alpha \in (0,1)$, we get $\Omega_n^\alpha > 0$ and $n \mapsto \Omega_n^\alpha$ is strictly increasing. Moreover,*

$$\mathcal{S} \subset \Theta_\alpha \Big[\frac{1-\alpha}{2-\frac{\alpha}{2}}, 1\Big[,$$

with

$$\Theta_\alpha \triangleq \frac{2^\alpha \Gamma(1+\frac{\alpha}{2})\Gamma(1-\alpha)}{(2-\alpha)\Gamma^3(1-\frac{\alpha}{2})}\cdot$$

(3) *For $\alpha \in (0,1)$ fixed and n sufficiently large,*

$$\Omega_n^\alpha = \Theta_\alpha - \big(1-\alpha/2\big)\Theta_\alpha \frac{e^{\alpha\gamma+c_\alpha}}{n^{1-\alpha}} + O\Big(\frac{1}{n^{2-\alpha}}\Big), \tag{56}$$

where γ denotes Euler constant, c_α is the sum of the series

$$c_\alpha \triangleq \sum_{m=1}^{\infty} \frac{\alpha^{2m+1}}{2^{2m-1}(2m+1)}\zeta(2m+1).$$

and $s \mapsto \zeta(s)$ is the Riemann zeta function.

Proof. (**1**) Recall first that for $n \geq 2$,

$$\Omega_n^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}\right).$$

Passing to the limit in this formula when α goes to zero yields

$$\begin{aligned}\lim_{\alpha\to 0} \Omega_n^\alpha &= \frac{1}{2}\Big(\frac{\Gamma(1)}{\Gamma(2)} - \frac{\Gamma(n)}{\Gamma(n+1)}\Big)\\ &= \frac{1}{2}\Big(1 - \frac{(n-1)!}{n!}\Big)\\ &= \frac{n-1}{2n}\cdot\end{aligned}$$

As to the second limit, we shall introduce for a fixed n the function

$$\phi_n(\alpha) = \frac{\Gamma(n+\alpha/2)}{\Gamma(n+1-\alpha/2)}\cdot$$

Therefore we obtain according to (20) and (21) and the relation $\phi_n(1) = 1$,

$$\begin{aligned}\lim_{\alpha\to 1} \Omega_n^\alpha &= \frac{-1}{\Gamma^2(1/2)} \lim_{\alpha\to 1}\big\{(1-\alpha)\Gamma(1-\alpha)\big\} \lim_{\alpha\to 1}\Big\{\frac{\phi_1(\alpha)-\phi_1(1)}{\alpha-1} - \frac{\phi_n(\alpha)-\phi_n(1)}{\alpha-1}\Big\}\\ &= \frac{-1}{\pi}\Big\{\phi_1'(1) - \phi_n'(1)\Big\}.\end{aligned}$$

By applying the logarithm function to ϕ_n and differentiating with respect to α one obtains the relation

$$2\frac{\phi_n'(\alpha)}{\phi_n(\alpha)} = \digamma(n+\alpha/2) + \digamma(n+1-\alpha/2).$$

Now using the fact that $\phi_n(1) = 1$ combined with the preceding identity and (22), we find

$$\begin{aligned}\lim_{\alpha\to 1}\Omega_n^\alpha &= \frac{-1}{\pi}\Big\{\digamma(3/2) - \digamma(n+1/2)\Big\} \\ &= \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1},\end{aligned}$$

which is the desired result.

(**2**) Using the identities (25) we find the alternative formula

$$\Omega_n^\alpha \triangleq \Theta_\alpha\left(1 - \frac{\left(1+\frac{\alpha}{2}\right)_{n-1}}{\left(2-\frac{\alpha}{2}\right)_{n-1}}\right), \tag{57}$$

with

$$\Theta_\alpha \triangleq \frac{2^\alpha\Gamma(1+\frac{\alpha}{2})\Gamma(1-\alpha)}{(2-\alpha)\Gamma^3(1-\frac{\alpha}{2})}.$$

and $(x)_n$ denotes Pokhhammer's symbol introduced in (23). Now because $x \mapsto (x)_{n-1}$ is increasing in the set $\mathbb{R}_+$ provided that $\alpha < 1$ we conclude easily that $\Omega_n^\alpha > 0$.
To prove that $n \mapsto \Omega_n^\alpha$ is strictly increasing, it suffices according to (57) to check that the sequence $n \mapsto u_n = \frac{\left(1+\frac{\alpha}{2}\right)_{n-1}}{\left(2-\frac{\alpha}{2}\right)_{n-1}}$ is strictly decreasing. This follows from the obvious fact that for $\alpha \in (0,1)$, one has

$$\frac{u_{n+1}}{u_n} = \frac{n+\frac{\alpha}{2}}{n+1-\frac{\alpha}{2}} < 1.$$

From this it is apparent that

$$\mathcal{S} \subset \big[\Omega_2^\alpha, \lim_{n\to\infty}\Omega_n^\alpha\big[\subset \Theta_\alpha\Big[\frac{1-\alpha}{2-\frac{\alpha}{2}}, 1\Big[.$$

Note that we have used in the last limit that $\lim_{n\to\infty} u_n = 0$ which can be deduced for instance from the proof of the point (**3**) of this lemma.

(**3**) First recall that Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, s > 1.$$

To get the required asymptotic behavior we shall first study the sequence,

$$U_n \triangleq \log\left(\frac{\left(1+\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n}\right).$$

Making use of the definition of $(x)_n$, we can rewrite this sequence in the manner

$$U_n = \sum_{k=1}^{n} \Big\{ \log\Big(1+\frac{\alpha}{2k}\Big) - \log\Big(1-\frac{\alpha}{2k}\Big)\Big\}.$$

Using the Taylor expansion of $\log(1+x)$ around zero one gets

$$\begin{aligned} U_n &= \sum_{k=1}^{n} \Big\{ 2 \sum_{m=0}^{\infty} \frac{1}{2m+1} \Big(\frac{\alpha}{2k}\Big)^{2m+1} \Big\} \\ &= \sum_{k=1}^{n} \frac{\alpha}{k} + 2 \sum_{m=1}^{\infty} \frac{\alpha^{2m+1}}{2^{2m}(2m+1)} \sum_{k=1}^{n} \frac{1}{k^{2m+1}} \\ &= \sum_{k=1}^{n} \frac{\alpha}{k} + 2 \sum_{m=1}^{\infty} \Big(\frac{\alpha^{2m+1}}{2^{2m}(2m+1)} \zeta(2m+1) + O\Big(\frac{1}{n^{2m}}\Big)\Big) \\ &= \sum_{k=1}^{n} \frac{\alpha}{k} + c_\alpha + O\Big(\frac{1}{n^2}\Big). \end{aligned}$$

Note that we have used the following estimate for the remainder term of the zeta function

$$\sum_{k=n+1}^{\infty} \frac{1}{k^{2m+1}} \leq \frac{1}{n^{2m}}.$$

Now we use the classical expansion of the harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O\Big(\frac{1}{n}\Big)$$

with γ the Euler constant. Therefore we get

$$U_n = \alpha \log n + \alpha\gamma + c_\alpha + O\Big(\frac{1}{n}\Big)$$

and consequently by raising to the exponential we find

$$\begin{aligned} e^{U_n} &= \frac{\big(1+\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n} \\ &= e^{\alpha\gamma + c_\alpha} n^\alpha e^{O(1/n)} \\ &= e^{\alpha\gamma + c_\alpha} n^\alpha + O\Big(\frac{1}{n^{1-\alpha}}\Big). \end{aligned}$$

It is apparent that

$$\begin{aligned} \frac{\big(1+\frac{\alpha}{2}\big)_{n-1}}{\big(2-\frac{\alpha}{2}\big)_{n-1}} &= \frac{1-\frac{\alpha}{2}}{n-\frac{\alpha}{2}} \frac{\big(1+\frac{\alpha}{2}\big)_{n-1}}{\big(1-\frac{\alpha}{2}\big)_{n-1}} \\ &= \frac{1-\frac{\alpha}{2}}{n-\frac{\alpha}{2}} e^{U_{n-1}} \end{aligned}$$

and consequently by making appeal to the formula (57) we obtain

$$\begin{aligned}\Omega_n^{\alpha} &= \Theta_{\alpha}\Big(1-\frac{1-\frac{\alpha}{2}}{n-\frac{\alpha}{2}}e^{U_{n-1}}\Big)\\ &= \Theta_{\alpha}\Big(1-\big(1-\frac{\alpha}{2}\big)\frac{e^{\alpha\gamma+c_{\alpha}}}{n^{1-\alpha}}\Big)+O\Big(\frac{1}{n^{2-\alpha}}\Big).\end{aligned}$$

This concludes the proof of Lemma 3. □

In what follows we shall give the proof of the Proposition 8.

Proof. (**1**) We begin by calculating $\mathrm{I}_1(h)$ in (54) which is easy compared to the other terms. Let $h\in X$ taking the form $h(w)=\sum_{n\geq 0}\frac{b_n}{w^n}$, then straightforward computations give

$$\mathrm{I}_1(h(w))=\Omega\sum_{n\geq 0}\Big(b_n\overline{w}^{n+1}-nb_nw^{n+1}\Big). \tag{58}$$

To compute the second term $I_2(h(w))$ we write

$$\begin{aligned}\mathrm{I}_2(h(w)) &\triangleq -C_{\alpha}\overline{w}\,\overline{h'(w)}\oint_{\mathbb{T}}\frac{d\tau}{|w-\tau|^{\alpha}}\\ &= C_{\alpha}\sum_{n\geq 1}nb_nw^n\oint_{\mathbb{T}}\frac{d\tau}{|w-\tau|^{\alpha}}.\end{aligned}$$

Applying the formula (26) with $n=0$ we get

$$\mathrm{I}_2(h(w))=\frac{\alpha C_{\alpha}\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\alpha/2)}\sum_{n\geq 1}nb_nw^{n+1}. \tag{59}$$

Regarding the third term $\mathrm{I}_3(h(w))$ it may be rewritten in the manner

$$\begin{aligned}\mathrm{I}_3(h(w)) &\triangleq -C_{\alpha}\overline{w}\oint_{\mathbb{T}}\frac{h'(\tau)}{|w-\tau|^{\alpha}}d\tau\\ &= C_{\alpha}\sum_{n\geq 1}nb_n\overline{w}\oint_{\mathbb{T}}\frac{\overline{\tau}^{n+1}}{|w-\tau|^{\alpha}}d\tau.\end{aligned}$$

Using change of variables allows to get

$$\oint_{\mathbb{T}}\frac{\overline{\tau}^{n+1}}{|w-\tau|^{\alpha}}d\tau=\overline{w}^n\oint_{\mathbb{T}}\frac{\tau^{n-1}}{|1-\tau|^{\alpha}}d\tau$$

which yields in view of the formula (26) to the expression

$$\mathrm{I}_3(h(w))=\frac{C_{\alpha}\Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)}\sum_{n\geq 1}nb_n\frac{\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n}\overline{w}^{n+1}. \tag{60}$$

Concerning the term $I_4(h(w))$ we start with the expansion,

$$\begin{aligned} \mathrm{I}_4(h(w)) &\triangleq \frac{\alpha C_\alpha}{2} \fint_{\mathbb{T}} \frac{(w-\tau)\big(\overline{h(w)} - \overline{h(\tau)}\big)}{w|w-\tau|^{\alpha+2}} d\tau \\ &= \frac{\alpha C_\alpha}{2} \sum_{n\geq 1} b_n \fint_{\mathbb{T}} \frac{(w-\tau)(w^n - \tau^n)}{w|w-\tau|^{\alpha+2}} d\tau. \end{aligned} \tag{61}$$

Hence, using the identity (27) one gets

$$\mathrm{I}_4(h(w)) = \frac{\alpha(1+\frac{\alpha}{2})}{2(2-\alpha)} \frac{C_\alpha \Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)} \sum_{n\geq 1} b_n \bigg(1 - \frac{\big(2+\frac{\alpha}{2}\big)_n}{\big(2-\frac{\alpha}{2}\big)_n}\bigg) w^{n+1}. \tag{62}$$

It remains to compute the last term I_5 of (34) which can be written in the form

$$\begin{aligned} \mathrm{I}_5(h(w)) &= \frac{\alpha C_\alpha}{2} \fint_{\mathbb{T}} \frac{(\overline{w}-\overline{\tau})\big(h(w) - h(\tau)\big)}{w|w-\tau|^{\alpha+2}} d\tau \\ &= \frac{\alpha C_\alpha}{2} \sum_{n\geq 1} b_n \fint_{\mathbb{T}} \frac{(\overline{w}-\overline{\tau})(\overline{w}^n - \overline{\tau}^n)}{w|w-\tau|^{\alpha+2}} d\tau. \end{aligned}$$

Using the identity (28) gives

$$\mathrm{I}_5(h(w)) = -\frac{\alpha C_\alpha \Gamma(1-\alpha)}{4\Gamma^2(1-\alpha/2)} \sum_{n\geq 1} b_n \bigg(1 - \frac{\big(\frac{\alpha}{2}\big)_n}{\big(-\frac{\alpha}{2}\big)_n}\bigg) \overline{w}^{n+1}. \tag{63}$$

Collecting the identities (60), (63) and using (24) we find

$$\begin{aligned} \mathrm{I}_3(h(w)) + \mathrm{I}_5(h(w)) &= \frac{C_\alpha \Gamma(1-\alpha)}{2\Gamma^2(1-\alpha/2)} \sum_{n\geq 1} b_n \bigg(\frac{2n\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n} - \frac{\alpha}{2} - \frac{\big(-\frac{\alpha}{2}\big)\big(\frac{\alpha}{2}\big)_n}{\big(-\frac{\alpha}{2}\big)_n} \bigg) \overline{w}^{n+1} \\ &= \frac{C_\alpha \Gamma(1-\alpha)}{2\Gamma^2(1-\alpha/2)} \sum_{n\geq 1} b_n \bigg(\frac{2n\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n} - \frac{\alpha}{2} - \big(n-\frac{\alpha}{2}\big) \frac{\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n} \bigg) \overline{w}^{n+1} \\ &= \frac{C_\alpha \Gamma(1-\alpha)}{2\Gamma^2(1-\alpha/2)} \sum_{n\geq 1} b_n \bigg(-\frac{\alpha}{2} + \big(n+\frac{\alpha}{2}\big) \frac{\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n} \bigg) \overline{w}^{n+1} \\ &= -\frac{C_\alpha \Gamma(1-\alpha)}{2\Gamma^2(1-\alpha/2)} \sum_{n\geq 1} b_n \bigg(\frac{\alpha}{2} - \frac{\big(\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_n} \bigg) \overline{w}^{n+1} \\ &\triangleq -\sum_{n\geq 1} b_n \beta_n \overline{w}^{n+1}, \end{aligned}$$

with

$$\beta_n = \frac{C_\alpha \Gamma(1-\alpha)}{2\Gamma^2(1-\alpha/2)} \bigg(\frac{\alpha}{2} - \frac{\big(\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_n} \bigg).$$

Now by summing up (59) and (62) we deduce that,

$$\begin{aligned}\mathrm{I}_2(h(w))+\mathrm{I}_4(h(w)) &= \frac{\alpha C_\alpha\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\alpha/2)}\sum_{n\geq 1}b_n\bigg(2n+1+\frac{\alpha}{2}-\frac{(1+\frac{\alpha}{2})(2+\frac{\alpha}{2})_n}{(2-\frac{\alpha}{2})_n}\bigg)w^{n+1}\\ &= \frac{\alpha C_\alpha\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\alpha/2)}\sum_{n\geq 1}b_n\bigg(2n+1+\frac{\alpha}{2}-\frac{(1+\frac{\alpha}{2})_{n+1}}{(2-\frac{\alpha}{2})_n}\bigg)w^{n+1}\\ &\triangleq \sum_{n\geq 1}b_n\alpha_n w^{n+1},\end{aligned}$$

with

$$\alpha_n \triangleq \frac{\alpha C_\alpha\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\alpha/2)}\bigg(2n+1+\frac{\alpha}{2}-\frac{(1+\frac{\alpha}{2})_{n+1}}{(2-\frac{\alpha}{2})_n}\bigg).$$

Then inserting (58) and the two preceding identities into (54) one can readily verify that

$$\begin{aligned}\partial_f F(\Omega,0)(h)(w) &= \mathrm{Im}\bigg\{\Omega b_0\overline{w}-\sum_{n\geq 1}b_n\Big(n\Omega-\alpha_n\Big)w^{n+1}+\sum_{n\geq 1}b_n\Big(\Omega-\beta_n\Big)\overline{w}^{n+1}\bigg\}\\ &= \frac{\Omega b_0}{2}i\big(w-\overline{w}\big)+i\sum_{n\geq 1}\frac{b_n}{2}\Big(\big(n+1\big)\Omega-\big(\alpha_n+\beta_n\big)\Big)\Big(w^{n+1}-\overline{w}^{n+1}\Big). \qquad (64)\end{aligned}$$

By using (24) combined with the foregoing expressions for α_n and β_n one may write,

$$\begin{aligned}\alpha_n+\beta_n &= \frac{\alpha C_\alpha\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\alpha/2)}\bigg(2n+2-\frac{(1+\frac{\alpha}{2})_{n+1}}{(2-\frac{\alpha}{2})_n}-\frac{(1-\frac{\alpha}{2})(\frac{\alpha}{2})_{n+1}}{\frac{\alpha}{2}(1-\frac{\alpha}{2})_n}\bigg)\\ &= \frac{\alpha C_\alpha\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\alpha/2)}\bigg(2n+2-\frac{(1+\frac{\alpha}{2})_{n+1}}{(2-\frac{\alpha}{2})_n}-\frac{(1+\frac{\alpha}{2})_n}{(2-\frac{\alpha}{2})_{n-1}}\bigg)\\ &= \frac{\alpha C_\alpha\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\alpha/2)}(n+1)\bigg(1-\frac{(1+\frac{\alpha}{2})_n}{(2-\frac{\alpha}{2})_n}\bigg). \qquad (65)\end{aligned}$$

Coming back to the definition of C_α, see for instance Proposition 4, and setting

$$\Theta_\alpha \triangleq \frac{\alpha C_\alpha\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}=\frac{\alpha\Gamma(\frac{\alpha}{2})\Gamma(1-\alpha)}{2^{1-\alpha}(2-\alpha)\Gamma^3(1-\frac{\alpha}{2})},$$

one finds that

$$\alpha_n+\beta_n=\Theta_\alpha(n+1)\bigg(1-\frac{(1+\frac{\alpha}{2})_n}{(2-\frac{\alpha}{2})_n}\bigg).$$

Making appeal to the definition (57), the linearized operator (64) takes the form,

$$\partial_f F(\Omega,0)(h)(w)=\frac{\Omega b_0}{2}i\big(w-\overline{w}\big)+\frac{1}{2}i\sum_{n\geq 1}\big(n+1\big)b_n\Big(\Omega-\Omega^\alpha_{n+1}\Big)\Big(w^{n+1}-\overline{w}^{n+1}\Big). \qquad (66)$$

We should mention in passing that the linearized operator has a special structure: it acts as a Fourier multiplier and as we shall see this will be very useful in the explicit computations for the kernel and the range of this operator. Now let us look for the values of Ω corresponding to non trivial kernel. It is easy to see that this will be the case if and only if Ω belongs to

the dispersion set $\mathcal{S}$ introduced in (55). This corresponds to the values of Ω such that there exists $m \geq 1$ with

$$\begin{aligned} \Omega &= \Omega^{\alpha}_{m+1} \\ &= \Theta_{\alpha}\Big(1 - \frac{\big(1+\frac{\alpha}{2}\big)_m}{\big(2-\frac{\alpha}{2}\big)_m}\Big). \end{aligned}$$

From Lemma 3-(2) the sequence $n \mapsto \Omega^{\alpha}_n$ is strictly increasing and therefore for any $n \neq m$

$$(1+n)\big(\Omega^{\alpha}_{m+1} - \Omega^{\alpha}_{n+1}\big) \neq 0.$$

From these last facts it is apparent that the kernel of $\partial_f F(\Omega^{\alpha}_{m+1}, 0)$ is one-dimensional vector space generated by the function $v_m(w) = \overline{w}^m$.The claim of Proposition 8 follows by shifting the index m.

(**2**) Now we are going to show that for any $m \geq 2$ the range $R(\partial_f F(\Omega^{\alpha}_m, 0))$ coincides with the subspace

$$Z_m \triangleq \Big\{ g \in C^{1-\alpha}(\mathbb{T}); \quad g(w) = \sum_{\substack{n\geq 1 \\ n\neq m}}^{\infty} i g_n (w^n - \overline{w}^n), g_n \in \mathbb{R} \Big\}.$$

Note that this sub-space is closed and of co-dimension one in the ambient space Y. In addition, one may easily deduce from (64) the trivial inclusion $R(\partial_f F(\Omega^{\alpha}_m, 0)) \subset Z_m$ and therefore it remains to check just the converse. For this end, let $g \in Z_m$ we shall look for a pre-image $h(w) = \sum_{n\geq 0} b_n \overline{w}^n \in X$ satisfying $\partial_f F(\Omega^{\alpha}_m, 0)(h) = g$. From the relation (66) this is equivalent to

$$\frac{\Omega^{\alpha}_m}{2} b_0 = g_1 \quad \text{and} \quad \frac{n b_{n-1}}{2}\Big(\Omega^{\alpha}_m - \Omega^{\alpha}_n\Big) = g_n, \ \forall n \geq 2, \ n \neq m.$$

This determines uniquely the sequence $(b_n)_{n\neq m-1}$ and one has

$$b_0 = \frac{2g_1}{\Omega^{\alpha}_m} \quad \text{and} \quad b_n = \frac{2g_{n+1}}{(n+1)\big(\Omega^{\alpha}_m - \Omega^{\alpha}_{n+1}\big)}, \quad \forall n \neq m-1, n \geq 1.$$

However the value b_{m-1} is free and it can be taken zero. Then the proof of $h \in X$ reduces to show that $h \in C^{2-\alpha}\big(\mathbb{T}\big)$. For this end, it suffices to show that the function $H(w) = \sum_{n\geq m} b_n \overline{w}^n$ belongs to this latter Hölder space. First we shall transform H in the form

$$\begin{aligned} H(w) &= 2\sum_{n\geq m} \frac{g_{n+1}}{(n+1)\big(\Omega^{\alpha}_m - \Omega^{\alpha}_{n+1}\big)}\,\overline{w}^n \\ &= 2w \sum_{n\geq m+1} \frac{g_n}{n\big(\Omega^{\alpha}_m - \Omega^{\alpha}_n\big)}\overline{w}^n. \end{aligned}$$

Using (56) one may write down

$$H(w) = 2w \sum_{n\geq m+1} \frac{g_n}{n\Big(\Omega^{\alpha}_m - \Theta_{\alpha} - d_n\Big)}\overline{w}^n,$$

where

$$d_n = -\Theta_\alpha\big(1-\alpha/2\big)\frac{e^{\alpha\gamma+c_\alpha}}{n^{1-\alpha}} + O\big(\frac{1}{n^{2-\alpha}}\big)\cdot \tag{67}$$

Denote $A = \Omega_m^\alpha - \Theta_\alpha$ then one may use the general decomposition: for $k \in \mathbb{N}$,

$$\frac{1}{A-d_n} = \frac{A^{-k-1}d_n^{k+1}}{A-d_n} + \sum_{j=0}^{k} A^{-j-1}d_n^j.$$

This allows to rewrite $H(w)$ in the manner

$$\begin{aligned} H(w) &= 2A^{-k-1}w\sum_{n\geq m+1}\frac{g_n d_n^{k+1}}{n(A-d_n)}\overline{w}^n + 2w\sum_{j=0}^{k}A^{-j-1}\sum_{n\geq m+1}\frac{g_n d_n^j}{n}\overline{w}^n \\ &\triangleq 2A^{-k-1}wH_{k+1}(\overline{w}) + 2w\sum_{j=0}^{k}A^{-j-1}L_j(\overline{w}). \end{aligned}$$

Fix k such that $(1-\alpha)(k+1) > 2$ then $H_{k+1} \in C^2(\mathbb{T})$. Indeed as the sequence $(g_n)_n$ is bounded then we get by (67)

$$\Big|\frac{g_n d_n^{k+1}}{n(A-d_n)}\Big| \lesssim \frac{|d_n|^{k+1}}{n} \lesssim \frac{1}{n^{1+(1-\alpha)(k+1)}}.$$

Therefore the regularity follows from the polynomial decay of the Fourier coefficients. Concerning the estimate of L_j we shall restrict the analysis to $j=0$ and $j=1$ and the higher terms can be treated in a similar way. We write

$$L_0(w) = \sum_{n\geq m+1}\frac{g_n}{n}w^n.$$

Using Cauchy-Schwarz we deduce that

$$\begin{aligned} \|L_0\|_{L^\infty} &\lesssim \sum_{n\geq m+1}\frac{|g_n|}{n} \\ &\lesssim \Big(\sum_{n\geq 1}\frac{1}{n^2}\Big)^{1/2}\Big(\sum_{n\geq m+1}|g_n|^2\Big)^{1/2} \\ &\lesssim \|g\|_{L^2}. \end{aligned}$$

Hence, by the embedding $C^{1-\alpha}\big(\mathbb{T}\big) \hookrightarrow L^\infty\big(\mathbb{T}\big) \hookrightarrow L^2\big(\mathbb{T}\big)$ we conclude that

$$\|L_0\|_{L^\infty} \lesssim \|g\|_{1-\alpha}.$$

It remains to prove that $L_0' \in C^{1-\alpha}(\mathbb{T})$. For this end one need first to check that one can differentiate the series term by term. Fix $N \geq m+1$ and define

$$L_0^N(w) \triangleq \sum_{n=m+1}^{N}\frac{g_n}{n}w^n.$$

Then it is obvious from Cauchy-Schwarz inequality that

$$\lim_{N\to\infty}\|L_0^N - L_0\|_{L^\infty(\mathbb{T})} = 0. \tag{68}$$

Now differentiating L_0^N term by term one should get

$$\begin{aligned}(L_0^N)'(w) &= \overline{w}\sum_{n=m+1}^{N} g_n w^n \\ &\triangleq \overline{w}\, G_N(w).\end{aligned}$$

Assume for a while that $w \mapsto G(w) = \sum_{n\geq m+1} g_n w^n$ belongs to $C^{1-\alpha}(\mathbb{T})$, then by virtue of a classical result on Fourier series one gets

$$\lim_{N\to\infty} \|G_N - G\|_{L^\infty(\mathbb{T})} = 0$$

and consequently

$$\lim_{N\to\infty} \|(L_0^N)' - \overline{w}\, G\|_{L^\infty(\mathbb{T})} = 0. \tag{69}$$

Putting together (68) and (69) we obtain that L_0 is differentiable and

$$L_0'(w) = \overline{w}\, G(w), \quad w \in \mathbb{T}.$$

This concludes that $L_0 \in C^{2-\alpha}$. Now to complete rigorously the reasoning it remains to prove the preceding claim asserting that $G \in C^{1-\alpha}(\mathbb{T})$. Actually, this is based on the continuity of Szegö projection

$$\Pi : \sum_{n\in\mathbb{Z}} a_n w^n \mapsto \sum_{n\in\mathbb{N}} a_n w^n$$

on Hölder spaces $C^\varepsilon, \varepsilon \in (0,1)$. To see this we write

$$G(w) = \Pi\big(-i\, g(w) - \sum_{n=0}^{m} g_n w^n\big).$$

From which we deduce that

$$\begin{aligned}\|G\|_{C^{1-\alpha}(\mathbb{T})} &\leq C\Big(\|g\|_{C^{1-\alpha}} + \sum_{n=0}^{m} |g_n| \|w^n\|_{C^{1-\alpha}}\Big) \\ &\leq C_m\big(\|g\|_{C^{1-\alpha}} + \|g\|_{L^2}\big) \\ &\leq C_m \|g\|_{C^{1-\alpha}} \qquad (70)\end{aligned}$$

and this concludes the proof of the claim.
As to the term L_1 we write down by the definition

$$L_1(w) = \sum_{n\geq m+1} \frac{g_n d_n}{n} w^n.$$

As before we can easily get $L_1 \in L^\infty$ and we shall check that $L_1' \in C^{1-\alpha}\big(\mathbb{T}\big)$. Arguing in a similar way to L_0 we can differentiate term by term the series defining L_1 leading to

$$w\, L_1'(w) = \sum_{n\geq m+1} g_n d_n w^n.$$

We shall write down this series in the convolution form. With the notation $w = e^{i\theta}$, we may write

$$\begin{aligned} w\, L_1'(w) &= (K * G)(w), \\ &= \frac{1}{2\pi}\int_0^{2\pi} K(e^{i\eta})G(e^{i(\theta-\eta)})d\eta \end{aligned}$$

where

$$K(w) \triangleq \sum_{n\geq m+1} d_n w^n.$$

Making use of the definition (67) we find the expansion

$$\begin{aligned} K(w) &= -\Theta_\alpha\big(1-\alpha/2\big)e^{\alpha\gamma+c_\alpha}\sum_{n\geq m+1}\frac{w^n}{n^{1-\alpha}} + \sum_{n\geq m+1} O\big(\frac{1}{n^{2-\alpha}}\big)w^n. \\ &\triangleq -\Theta_\alpha\big(1-\alpha/2\big)e^{\alpha\gamma+c_\alpha}K_1(w) + K_2(w). \end{aligned}$$

The second term is easy to analyze because we have an absolute series as follows,

$$\|K_2\|_{L^\infty} \lesssim \sum_{n\geq m+1}\frac{1}{n^{2-\alpha}} \leq C$$

and therefore $K_2 \in L^1(\mathbb{T})$. It suffices now to combine this fact with the classical convolution law $L^1(\mathbb{T}) * C^{1-\alpha}(\mathbb{T}) \to C^{1-\alpha}(\mathbb{T})$ with (70). Next, we shall concentrate on the first term $K_1 * G$ and prove that it belongs to $C^{1-\alpha}(\mathbb{T})$. For this end it is enough to show that $K_1 \in L^1(\mathbb{T})$ for $\alpha \in [0,1[$ which is more tricky. This claim is an immediate consequence of a more precise estimate: for any $\beta \in (\alpha, 1)$

$$|K_1(e^{i\theta})| \lesssim \frac{1}{\sin^\beta(\frac{\theta}{2})}, \quad \forall \theta \in (0, 2\pi)\cdot \tag{71}$$

This estimate sounds classical and for the convenience of the reader we shall give here a complete proof. The basic tool is Abel transform. We set

$$K_1^n(w) \triangleq \sum_{k=m+1}^{n}\frac{w^k}{k^{1-\alpha}} \quad \text{and} \quad U_n(w) \triangleq \sum_{k=0}^{n} w^k.$$

Then it is apparent that

$$\begin{aligned} K_1^n(w) &\triangleq \sum_{k=m+1}^{n}\frac{U_k(w)-U_{k-1}(w)}{k^{1-\alpha}} \\ &= \sum_{k=m+1}^{n}\frac{U_k(w)}{k^{1-\alpha}} - \sum_{k=m}^{n-1}\frac{U_k(w)}{(1+k)^{1-\alpha}} \\ &= \sum_{k=m+1}^{n-1} U_k(w)\Big(\frac{1}{k^{1-\alpha}} - \frac{1}{(1+k)^{1-\alpha}}\Big) + \frac{U_n(w)}{n^{1-\alpha}} - \frac{U_m(w)}{(m+1)^{1-\alpha}} \\ &\triangleq K_{1,1}^n(w) + K_{1,2}^n(w) + K_{1,3}(w). \end{aligned}$$

The last term is bounded independently of n and w. For the second term, it converges to zero as n goes to infinity for any $w \in \mathbb{T}\backslash\{1\}$. This follows easily from the estimate,

$$\begin{aligned} |K_{1,2}^n(w)| &\leq \frac{|1-w^{n+1}|}{|1-w|}\frac{1}{n^{1-\alpha}} \\ &\leq \frac{2}{|1-w|}\frac{1}{n^{1-\alpha}}. \end{aligned}$$

As regards the first term $K_{1,1}^n$, we shall use the mean value theorem through the simple fact

$$0 \leq \frac{1}{k^{1-\alpha}} - \frac{1}{(1+k)^{1-\alpha}} \lesssim \frac{1}{k^{2-\alpha}}.$$

Hence we get

$$|K_{1,1}^n(w)| \lesssim \sum_{k=m+1}^{n-1} \frac{|U_k(w)|}{k^{2-\alpha}}.$$

Now we use the classical estimates

$$|U_k(w)| \leq k+1, \quad |U_k(w)| \leq \frac{1}{|\sin\frac{\theta}{2}|}, \ w = e^{i\theta}.$$

By an obvious convexity inequality we get for any $\beta \in [0,1]$

$$|U_k(w)| \leq \frac{(k+1)^{1-\beta}}{|\sin\frac{\theta}{2}|^{\beta}}$$

and therefore

$$|K_{1,1}^n(w)| \lesssim \frac{1}{|\sin\frac{\theta}{2}|^{\beta}} \sum_{k=m+1}^{n-1} \frac{1}{k^{1-\alpha+\beta}}.$$

The partial sum of the series converges provided that we choose $\beta \in (\alpha, 1)$. Collecting the preceding estimates and passing to the limit when n goes to infinity we may write,

$$|K_1(w)| \lesssim \frac{1}{|\sin\frac{\theta}{2}|^{\beta}}$$

and this completes the proof of the inequality (71).

(**3**) Now, we intend to check the transversality assumption. According to the continuity property of the second derivative $\partial_\Omega \partial_f F$ seen in Proposition 7, this assumption reduces to

$$\big\{\partial_\Omega \partial_f F(\Omega,0)(h)\big\}\Big|_{\Omega=\Omega_m^\alpha, h=v_m} \notin R\Big(\partial_f F(\Omega_m^\alpha, 0)\Big).$$

Differentiating (54) with respect to Ω one gets

$$\partial_\Omega \partial_f F(\Omega,0)(h)(w) = \mathrm{Im}\Big\{\overline{h'}(w) + \overline{w}h(w)\Big\}.$$

Then obviously

$$\partial_\Omega \partial_f F(\Omega_m^\alpha, 0)(\overline{w}^{m-1}) = i\frac{m}{2}\big(w^m - \overline{w}^m\big).$$

which is not in the range of $\partial_f F(\Omega_m^\alpha, 0)$ as it was described in the part (2) of Proposition 8. □

9. $m-$FOLD SYMMETRY

Now we are ready to complete the proof of Theorem 3 started and developed throughout the preceding sections. We have gathered all the required elements to apply Theorem 2 of Crandal-Rabinowitz. Combining Proposition 7 and Proposition 8 we deduce the existence of non trivial curves $\{\mathcal{C}_m, m \geq 2\}$ bifurcating at the points Ω_m^α of the dispersion set $\mathcal{S}$ introduced in (55). Each point of the branch $\mathcal{C}_m$ represents a V-state and we shall now see that it is an m-fold symmetric in a similar way to the case of the incompressible Euler equations. This will be done by showing the bifurcation in spaces including the m-fold symmetry. To be more precise, let $m \geq 2$ and define the spaces X_m and Y_m as follows: the space X_m is the set of those functions $f \in X$ with a Fourier expansion of the type

$$f(w) = \sum_{n=1}^{\infty} a_{nm-1} \overline{w}^{nm-1}, \quad w \in \mathbb{T}.$$

equipped with the usual strong topology of $C^{2-\alpha}(\mathbb{T})$. We define the ball of radius $r \in (0,1)$ by

$$B_r^m = \Big\{ f \in X_m,\ \|f\|_{C^{2-\alpha}(\mathbb{T})} \leq r \Big\}.$$

If $f \in B_r^m$ the expansion of the associated conformal mapping ϕ in $\{z : |z| \geq 1\}$ is given by

$$\phi(z) = z + f(z) = z\Big(1 + \sum_{n=1}^{\infty} \frac{a_{nm-1}}{z^{nm}}\Big).$$

This will provide the $m-$fold symmetry of the associated patch $\phi(\mathbb{T})$, via the relation

$$\phi\big(e^{i2\pi/m} z\big) = e^{i2\pi/m} \phi(z), \quad |z| \geq 1. \tag{72}$$

The space Y_m is the subspace of Y consisting of those $g \in Y$ whose Fourier expansion is of the type

$$g(w) = i \sum_{n=1}^{\infty} g_{nm} \big(w^{nm} - \overline{w}^{nm}\big), \quad w \in \mathbb{T}.$$

To apply Crandall-Rabinowitz's Theorem and get the symmetry property stated in Theorem 3 it suffices to show the following result.

Proposition 9. *The following assertions hold true. Let $m \geq 2$ and $r \in (0,1)$, then*

1. *$F : \mathbb{R} \times B_r^m \to Y_m$ is well-defined.*
2. *The kernel of $\partial_f F(\Omega_m^\alpha, 0)$ is one-dimensional and generated by $w \mapsto \overline{w}^{m-1}$.*
3. *The range of $\partial_f F(\Omega_m^\alpha, 0)$ is closed in Y_m and is of co-dimension one.*

Proof. (**1**) Let $f \in B_r^m$, we shall show that $F(\Omega, f) = G(\Omega, \phi) \in Y_m$. Recall that the functional G is defined by

$$G(\Omega, \phi)(w) = \mathrm{Im}\Big\{ \Big(\Omega\, \overline{w}\, \phi(w) - \overline{w}\, S(\phi)(w) \Big)\, \overline{\phi'(w)} \Big\},$$

with

$$S(\phi)(w) = C_\alpha \fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w) - \phi(\tau)|^\alpha} d\tau.$$

It is easy to verify from (72) that the functions ϕ' and $w \mapsto \frac{\phi(w)}{w}$ belong to the space $C^{1-\alpha}(\mathbb{T})$ and their Fourier coefficients vanish at frequencies which are not integer multiples of m. Since this latter space is an algebra and stable by conjugation then the map $w \mapsto \mathrm{Im}\big\{\overline{\phi'(w)}\frac{\phi(w)}{w}\big\}$ belongs to the space Y_m. Therefore, it remains to show that $w \mapsto \mathrm{Im}\big\{\overline{\phi'(w)}\big(\overline{w}S(\phi)(w)\big)\big\} \in Y_m$. This follows easily once we have proved that $w \mapsto \overline{w}S(\phi)(w)$ satisfies (72). For this end, set

$$\begin{aligned}\Phi(w) &\triangleq \overline{w}S(\phi)(w)\\ &= \overline{w}\fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha}}d\tau.\end{aligned}$$

Then

$$\Phi(e^{i2\pi/m}w) = e^{-i2\pi/m}\overline{w}\fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(e^{i2\pi/m}w)-\phi(\tau)|^{\alpha}}d\tau.$$

By the change of variables $\tau = e^{i2\pi/m}\zeta$ and according to (72) we get for any $w \in \mathbb{T}$

$$\begin{aligned}\Phi(e^{i2\pi/m}w) &= \frac{1}{w}\fint_{\mathbb{T}} \frac{\phi'(e^{i2\pi/m}\zeta)}{|\phi(e^{i2\pi/m}w)-\phi(e^{i2\pi/m}\zeta)|^{\alpha}}d\zeta\\ &= \frac{1}{w}\fint_{\mathbb{T}} \frac{\phi'(\zeta)}{|\phi(w)-\phi(\zeta)|^{\alpha}}d\zeta\\ &= \Phi(w).\end{aligned}$$

Hence, The Fourier coefficients of Φ vanish at frequencies which are not integer multiples of m and this concludes the proof of the result,

$$f \in B^m_r \Longrightarrow F(\Omega, f) = G(\Omega, \phi) \in Y_m.$$

(**2**) Since the generator $w \mapsto \overline{w}^{m-1}$ of the kernel of $\partial_f F(\Omega^{\alpha}_m, 0)$ belongs to X_m, we still have that the dimension of the kernel is 1.

(**3**) Using Proposition 8, we deduce that

$$\begin{aligned}R\Big(\partial_f F(\Omega^{\alpha}_m, 0)\Big) &= \Big\{g \in C^{1-\alpha}(\mathbb{T}); \quad g(w) = i\sum_{n\geq 1, n\neq m}^{\infty} g_n(w^n - \overline{w}^n), g_n \in \mathbb{R}\Big\} \cap Y_m\\ &= \Big\{g \in C^{1-\alpha}(\mathbb{T}); \quad g(w) = i\sum_{n\geq 2}^{\infty} g_{nm}(w^{nm} - \overline{w}^{nm}), g_{nm} \in \mathbb{R}\Big\}.\end{aligned}$$

Obviously, the range of $\partial_f F(\Omega_m, 0)$ is of co-dimension 1 in the space Y_m.
Therefore we can apply Crandall-Rabinowitz's Theorem to X_m and Y_m and obtain the existence of the $m-$fold symmetric patches for each integer $m \geq 2$. This achieves the proof of Theorem 3.

□

10. Limiting case $\alpha = 1$

In this section we shall discuss the limiting case $\alpha = 1$ corresponding to the SQG model. This case was excluded from Theorem 1 at least because the rotating patch model seen in (15) does not work due to the higher singularity of the kernel. Thus we shall modify a little

bit this model as in [11] and give an equation of the boundary of the V-states. Although the model seems to be coherent and satisfactory, it is completely different from the sub-critical one $\alpha \in [0,1[$ and generates more technical difficulties in studying the rotating patches. As we shall see later when we compute formally the linearized operator $\mathcal{L}_\Omega$ around the trivial solution we find that it behaves as a Fourier multiplier with an extra loss compared to the case $\alpha \in [0,1[$ which is of logarithmic type. Thus the property $\mathcal{L}_\Omega : C^{1+\varepsilon} \to C^{\varepsilon}$ fails and one should find other suitable spaces X and Y satisfying the assumptions of C-R Theorem. We do believe that such spaces must exist but certainly this would require more sophisticated analysis than what we shall do here. Among our objective is to describe in details the dispersion relation which tells us where the bifurcating curves emerge from the trivial one and also shed light on the main difficulties encountered in this case.

10.1. **Rotating patch model.** First recall from in the equation (12) one can change the velocity at the boundary by subtracting a tangential vector to the boundary without changing the full equation. Thus we shall work with the following modified velocity: let γ_0 be a 2π periodic parametrization of the boundary of D, and define

$$u_0(\gamma_0(\sigma)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial_s\gamma_0(s) - \partial_\sigma\gamma_0(\sigma)}{|\gamma_0(s)-\gamma_0(\sigma)|}ds. \tag{73}$$

Then, by substituting the expression of the velocity in the equation of the boundary (12) one gets

$$\Omega \mathrm{Re}\big\{z\ \overline{z'}\big\} = \mathrm{Im}\Big\{\frac{1}{2\pi}\int_{\partial D} \frac{(\zeta'-z')}{|z-\zeta|^\alpha}\frac{d\zeta}{\zeta'}\ \overline{z'}\Big\}, \quad \forall z \in \partial D. \tag{74}$$

Next, we parametrize the domain with the outside conformal mapping $\phi : \mathbb{D}^c \to D^c$,

$$\phi(z) = z + \sum_{n\geq 0}\frac{b_n}{z^n} \tag{75}$$

by setting $z = \phi(w)$ and $\zeta = \phi(\tau)$. Then, we obtain the equation

$$G(\Omega,\phi)(w) \triangleq \mathrm{Im}\left\{\left(\Omega\phi(w) - \fint_{\mathbb{T}} \frac{\tau\phi'(\tau) - w\phi'(w)}{|\phi(w)-\phi'(\tau)|}\frac{d\tau}{\tau}\right)\frac{\overline{\phi'(w)}}{w}\right\} = 0, \quad \forall w \in \mathbb{T}, \tag{76}$$

which is nothing but the boundary equation of the rotating patches. As for the sub-critical case we define,

$$F(\Omega,f)(w) \triangleq G(\Omega, \mathrm{Id}+f)(w), \quad f(w) = \sum_{n\geq 0}\frac{b_n}{w^n},\ w \in \mathbb{T},\ b_n \in \mathbb{R}.$$

We point out that the Rankine vortices correspond to the trivial solutions $F(\Omega,0) = 0$. This can be checked as follows.

$$\begin{aligned} F(\Omega,0)(w) &= \mathrm{Im}\left\{\left(\Omega w - \fint_{\mathbb{T}}\frac{\tau - w}{|w-\tau|}\frac{d\tau}{\tau}\right)\frac{1}{w}\right\} \\ &= -\mathrm{Im}\left\{\frac{1}{w}\fint_{\mathbb{T}}\frac{\tau-w}{|w-\tau|}\frac{d\tau}{\tau}\right\}. \end{aligned}$$

In view of the next identity (77) applied with $n = 1$ we conclude that

$$F(\Omega,0)(w) = 0, \quad \forall \Omega \in \mathbb{R},\ \forall w \in \mathbb{T}.$$

We should mention in passing that we can get a similar result to the Proposition 5 and prove that the ellipses never rotate.

10.2. **Integral computations.** We shall discuss some elementary integrals that will appear later in the computations of the linearized operator.

Lemma 4. *Let $n \geq 1$ and $w \in \mathbb{T}$, then we have*

$$\fint_{\mathbb{T}} \frac{\tau^n - w^n}{|w-\tau|}\frac{d\tau}{\tau} = -\frac{2w^n}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+1}. \tag{77}$$

$$\fint_{\mathbb{T}} \frac{(\tau-w)^2(\tau^n - w^n)}{|w-\tau|^3}\frac{d\tau}{\tau} = \frac{2w^{n+2}}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1}. \tag{78}$$

Proof. To prove (77) we use successively the change of variables $\tau = w\zeta$ and $\zeta = e^{i\eta}$

$$\begin{aligned}
\fint_{\mathbb{T}} \frac{\tau^n - w^n}{|w-\tau|}\frac{d\tau}{\tau} &= w^n \fint_{\mathbb{T}} \frac{\zeta^n - 1}{|1-\zeta|}\frac{d\zeta}{\zeta} \\
&= -w^n \frac{1}{2\pi}\int_0^{2\pi} \frac{1-e^{in\eta}}{|1-e^{i\eta}|}d\eta \\
&= -w^n \frac{1}{\pi}\int_0^{\pi} \frac{1-e^{i2n\eta}}{|1-e^{i2\eta}|}d\eta.
\end{aligned}$$

Using the identity

$$|1-e^{i2\eta}| = i(1-e^{i2\eta})e^{-i\eta}, \quad \eta \in [0,\pi].$$

we find easily

$$\begin{aligned}
\fint_{\mathbb{T}} \frac{\tau^n - w^n}{|w-\tau|}\frac{d\tau}{\tau} &= -w^n \frac{1}{i\pi}\int_0^{\pi} \frac{1-e^{i2n\eta}}{1-e^{i2\eta}}e^{i\eta}d\eta \\
&= -w^n \frac{1}{i\pi}\sum_{k=0}^{n-1}\int_0^{\pi} e^{i(2k+1)\eta}d\eta \\
&= -w^n \frac{2}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+1}.
\end{aligned}$$

To compute the second integral we argue as before by using suitable change of variables,

$$\begin{aligned}
\fint_{\mathbb{T}} \frac{(\tau-w)^2(\tau^n - w^n)}{|w-\tau|^3}\frac{d\tau}{\tau} &= -w^{n+2} \fint_{\mathbb{T}} \frac{(\zeta-1)^2(1-\zeta^n)}{|1-\zeta|^3}\frac{d\zeta}{\zeta} \\
&= -w^{n+2} \fint_{\mathbb{T}} \frac{(1-\zeta)(1-\zeta^n)}{(1-\bar{\zeta})|1-\zeta|}\frac{d\zeta}{\zeta} \\
&= -w^{n+2} \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-e^{i\eta})(1-e^{in\eta})}{(1-e^{-i\eta})|1-e^{i\eta}|}d\eta.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\fint_{\mathbb{T}} \frac{(\tau-w)^2(\tau^n-w^n)}{|w-\tau|^3}\frac{d\tau}{\tau} &= -w^{n+2}\frac{i}{\pi}\int_0^\pi \frac{e^{i3\eta}(1-e^{i2n\eta})}{1-e^{i2\eta}}d\eta \\
&= -w^{n+2}\frac{i}{\pi}\sum_{k=0}^{n-1}\int_0^\pi e^{i(2k+3)\eta}d\eta \\
&= w^{n+2}\frac{2}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+3}.
\end{aligned}$$

This concludes the proof of the lemma. □

10.3. **Dispersion relation.** We shall now compute the Gâteaux derivative of F with respect to f in the direction h, denoted as before by $\partial_f F(\Omega, f)h$. Afterwards, we shall exhibit the dispersion set corresponding to the values of Ω where the kernel of the linearized operator around $f = 0$ is non trivial. Using (76)

$$\begin{aligned}
\partial_f F(\Omega,f)h(w) &= \frac{d}{dt}F(\Omega,f+th)(w)_{|t=0} \\
&= \mathrm{Im}\Big\{\Omega\Big(\phi(w)\overline{h'(w)}+\frac{\overline{\phi'(w)}}{w}h(w)\Big) \\
&- \frac{\overline{h'(w)}}{w}\fint_{\mathbb{T}}\frac{\tau\phi'(\tau)-w\phi'(w)}{|\phi(w)-\phi(\tau)|}\frac{d\tau}{\tau} - \frac{\overline{\phi'(w)}}{w}\fint_{\mathbb{T}}\frac{\tau h'(\tau)-wh'(w)}{|\phi(w)-\phi(\tau)|}\frac{d\tau}{\tau} \\
&+ \frac{\overline{\phi'(w)}}{w}\fint_{\mathbb{T}}\frac{\big(\tau\phi'(\tau)-w\phi'(w)\big)\mathrm{Re}\Big\{\big(\overline{h(\tau)}-\overline{h(w)}\big)\big(\phi(\tau)-\phi(w)\big)\Big\}}{|\phi(w)-\phi(\tau)|^3}\frac{d\tau}{\tau}\Big\} \qquad (79)
\end{aligned}$$

with the notation $\phi = \mathrm{Id} + f$.

In the particular case $f = 0$ one has

$$\begin{aligned}
\partial_f F(\Omega,0)h(w) &= \frac{d}{dt}F(\Omega,th)(w)_{|t=0} \\
&= \mathrm{Im}\Big\{\Omega\big(\overline{h'(w)}+\overline{w}h(w)\big) - \fint_{\mathbb{T}}\frac{\tau h'(\tau)-wh'(w)}{w|w-\tau|}\frac{d\tau}{\tau} - \frac{\overline{h'(w)}}{w}\fint_{\mathbb{T}}\frac{\tau-w}{|w-\tau|}\frac{d\tau}{\tau} \\
&+ \frac{1}{2w}\fint_{\mathbb{T}}\frac{h(\tau)-h(w)}{|w-\tau|}\frac{d\tau}{\tau} + \frac{1}{2w}\fint_{\mathbb{T}}\frac{(\tau-w)^2\big(\overline{h(\tau)}-\overline{h(w)}\big)}{|w-\tau|^3}\frac{d\tau}{\tau}\Big\}. \qquad (80)
\end{aligned}$$

with $h(w) = \sum_{n\geq 0}\frac{b_n}{w^n}$ and $b_n \in \mathbb{R}$ for all $n \geq 1$.

Our next goal is to look for the values Ω where the linearized operator fails to be injective. We will be seeing that the function spaces that we shall use differs from the ones of the case $\alpha \in (0,1)$. We will abandon the use of Hölder spaces which generate more technical difficulties. We introduce the spaces,

$$B^s(\mathbb{T}) = \Big\{ f(w) = \sum_{n\geq 0} b_n \overline{w}^n, b_n \in \mathbb{R},\ \|f\|_{B^s} < \infty \Big\},\ \|f\|_{B^s} = |b_0| + \sum_{n\geq 1} n^s |b_n|$$

and

$$B^s_{\mathrm{Log}}(\mathbb{T}) = \Big\{ g(w) = i \sum_{n\geq 1} g_n \big(w^n - \overline{w}^n\big), g_n \in \mathbb{R},\ \|g\|_{B^s_{\mathrm{Log}}} < \infty \Big\},\ \|g\|_{B^s_{\mathrm{Log}}} = \sum_{n\geq 1} \frac{n^s}{1+\ln n} |g_n|.$$

First we define the dispersion set

$$\mathcal{S}_1 = \Big\{ \Omega = \frac{2}{\pi} \sum_{k=1}^{m-1} \frac{1}{2k+1} \quad m \geq 2 \Big\}.$$

The main result of this section reads as follows.

Proposition 10. *Let $s \geq 1$, $m \geq 2$.*

(1) *For any $\Omega \in \mathbb{R}$, $\partial_f F(\Omega, 0) : B^s(\mathbb{T}) \to B^{s-1}_{\mathrm{Log}}(\mathbb{T})$ is continuous.*

(2) *The kernel of $\partial_f F(\Omega, 0)$ is non trivial if and only if $\Omega \in \mathcal{S}$ and, in this case, it is a one-dimensional vector space generated by*

$$v_m(w) = \overline{w}^{m-1} \quad \textit{with} \quad \Omega = \Omega^1_m \triangleq \frac{2}{\pi} \sum_{k=1}^{m-1} \frac{1}{2k+1}.$$

(3) *The range of $\partial_f F(\Omega^1_m, 0)$ is closed in $B^{s-1}_{\mathrm{Log}}(\mathbb{T})$ and is of co-dimension one. It is given by*

$$R\big(\partial_f F(\Omega^1_m, 0)\big) = \Big\{ g \in B^s_{\mathrm{Log}}(\mathbb{T}), \quad g(w) = i \sum_{\substack{n\geq 1 \\ n\neq m}}^{\infty} g_n (w^n - \overline{w}^n),\ g_n \in \mathbb{R} \Big\}.$$

(4) *Transversality assumption:*

$$\partial_\Omega \partial_f F(\Omega^1_m, 0)(v_m) \notin R\big(\partial_f F(\Omega^1_m, 0)\big).$$

Before giving the proof of this result, we should make few comments.

Remark 1. (1) *The dispersion relation was discovered in [1] by using another analytical approach based on Bessel functions. The proof that we shall present is different and is somehow elementary.*

(2) *The spaces $B^s(\mathbb{T})$ and $B^{s-1}_{\mathrm{Log}}(\mathbb{T})$ introduced above are well-adapted to the study of the linear operator but it is not at all clear whether the nonlinear function F sends $B^s(\mathbb{T})$ into $B^{s-1}_{\mathrm{Log}}(\mathbb{T})$ and satisfies the regularity properties required by C-R Theorem. If this is the case then one can show the existence of the V-states for the SQG equation.*

Proof. **(1) – (2).** We shall prove in the same time the two points. We start with replacing h and $\overline{h'}$ in the identity (80) by their Fourier expansions,

$$h(w) = \sum_{n\geq 0} b_n \overline{w}^n \quad \text{and} \quad \overline{h'(w)} = -\sum_{n\geq 0} n b_n w^{n+1}.$$

Therefore we get

$$\begin{aligned}\partial_f F(\Omega,0)h(w) &= \mathrm{Im}\Big\{\Omega\sum_{n\geq 0} b_n\big(\overline{w}^{n+1}-nw^{n+1}\big)+\sum_{n\geq 1} nb_n\overline{w}\fint_{\mathbb{T}}\frac{\overline{\tau}^n-\overline{w}^n}{|w-\tau|}\frac{d\tau}{\tau}\\ &+ \sum_{n\geq 1} nb_n w^n\fint_{\mathbb{T}}\frac{\tau-w}{|w-\tau|}\frac{d\tau}{\tau}+\frac{1}{2}\sum_{n\geq 1} b_n\overline{w}\fint_{\mathbb{T}}\frac{\overline{\tau}^n-\overline{w}^n}{|w-\tau|}\frac{d\tau}{\tau}\\ &+ \frac{1}{2}\sum_{n\geq 1} b_n\overline{w}\fint_{\mathbb{T}}\frac{(w-\tau)^2(\tau^n-w^n)}{|w-\tau|}\frac{d\tau}{\tau}\Big\}.\end{aligned}$$

Note that

$$\fint_{\mathbb{T}}\frac{\overline{\tau}^n-\overline{w}^n}{|w-\tau|}\frac{d\tau}{\tau}=\overline{\fint_{\mathbb{T}}\frac{\tau^n-w^n}{|w-\tau|}\frac{d\tau}{\tau}}, \tag{81}$$

and consequently we can rewrite in view of Lemma 4 the linear operator as follows,

$$\begin{aligned}\partial_f F(\Omega,0)h(w) &= \mathrm{Im}\Big\{\Omega\sum_{n\geq 0} b_n\big(\overline{w}^{n+1}-nw^{n+1}\big)-\frac{2}{\pi}\sum_{n\geq 1} nb_n\overline{w}^{n+1}\sum_{k=0}^{n-1}\frac{1}{2k+1}\\ &- \frac{2}{\pi}\sum_{n\geq 1} nb_n w^{n+1}-\frac{1}{\pi}\sum_{n\geq 0} b_n\overline{w}^{n+1}\sum_{k=0}^{n-1}\frac{1}{2k+1}+\frac{1}{\pi}\sum_{n\geq 0} b_n w^{n+1}\sum_{k=1}^{n}\frac{1}{2k+1}\Big\}\\ &= \frac{b_0\Omega}{2}i(w-\overline{w})+\mathrm{Im}\Big\{\sum_{n\geq 1} b_n\Big(\Omega-\alpha_n\Big)\overline{w}^{n+1}-\sum_{n\geq 1} b_n\Big(n\Omega-\beta_n\Big)w^{n+1}\Big\}\\ &= \frac{b_0\Omega}{2}i(w-\overline{w})+\frac{1}{2i}\sum_{n\geq 1}(n+1)\Big(\Omega-\frac{\alpha_n+\beta_n}{n+1}\Big)b_n\big(\overline{w}^{n+1}-w^{n+1}\big), \qquad (82)\end{aligned}$$

where

$$\alpha_n\triangleq\frac{2n+1}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+1}$$

and

$$\beta_n\triangleq-\frac{2n}{\pi}+\frac{1}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1}.$$

It is plain to see that

$$\begin{aligned}\alpha_n+\beta_n &= \frac{2n+1}{\pi}\Big(1-\frac{1}{2n+1}+\sum_{k=1}^{n}\frac{1}{2k+1}\Big)-\frac{2n}{\pi}+\frac{1}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1}\\ &= \frac{2(n+1)}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1}\\ &\triangleq (n+1)\Omega_{n+1}^1.\end{aligned}$$

Inserting this formula into (82) we obtain

$$\partial_f F(\Omega,0)h(w) = \frac{b_0\Omega}{2}i(w-\overline{w})+\frac{1}{2}i\sum_{n\geq 1}(n+1)\Big(\Omega-\Omega_{n+1}^1\Big)b_n\big(w^{n+1}-\overline{w}^{n+1}\big). \tag{83}$$

To check that $\partial_f F(\Omega,0) : B^s(\mathbb{T}) \to B^{s-1}_{\rm Log}(\mathbb{T})$ is continuous we write

$$\begin{aligned}\big\|\partial_f F(\Omega,0)h\big\|_{B^{s-1}_{\rm Log}} &= \frac{1}{2}|b_0\Omega| + \frac{1}{2}\sum_{n\geq 1}\frac{(1+n)^s}{1+\ln(1+n)}\big|\Omega-\Omega^1_{n+1}\big|\,|b_n| \\ &\leq \frac{1}{2}|b_0\Omega| + C\sum_{n\geq 1}\frac{n^s}{1+\ln n}\big|\Omega-\Omega^1_{n+1}\big|\,|b_n| \\ &\leq C\Omega\|h\|_{B^s} + C\sum_{n\geq 1}\frac{n^s}{1+\ln n}\sum_{k=1}^{n}\frac{1}{2k+1}\,|b_n|.\end{aligned}$$

To estimate the last term we shall use the asymptotic behavior of the harmonic series

$$\sum_{k=1}^{n}\frac{1}{2k+1} = \frac{1}{2}\ln n + \frac{1}{2}\gamma + \ln 2 - 1 + O(\frac{1}{n}) \tag{84}$$

which yields,

$$\big\|\partial_f F(\Omega,0)h\big\|_{B^{s-1}_{\rm Log}} \leq C\|h\|_{B^s}.$$

This concludes the continuity of the linear operator $\partial_f F(\Omega,0) : B^s(\mathbb{T}) \to B^{s-1}_{\rm Log}$.
Now we shall study the kernel of this operator. From the formulae (83) we immediately deduce that the kernel of $\partial_f F(\Omega,0)$ is non trivial if and only if there exists $m \geq 2$ such that

$$\Omega = \Omega^1_m = \frac{2}{\pi}\sum_{k=1}^{m-1}\frac{1}{2k+1}.$$

In which case the kernel contains the eigenfunction $w \mapsto \overline{w}^{m-1}$. Moreover, it is one-dimensional vector space because the sequence $n \mapsto \Omega_n$ is strictly increasing and therefore the Fourier coefficients in (83) satisfy

$$(1+n)\Big(\Omega^1_m - \Omega^1_{n+1}\Big) \neq 0, \quad \forall\, n \neq m-1.$$

This achieves the proof of a simple kernel.
(**3**) Denote by

$$Z_m \triangleq \Big\{g \in B^s_{\rm Log}(\mathbb{T}), \quad g(w) = i\sum_{\substack{n\geq 1\\ n\neq m}}^{\infty} g_n(w^n - \overline{w}^n)\Big\}.$$

Clearly Z_m is a closed subspace of $B^s_{\rm Log}(\mathbb{T})$ and from (83) we deduce the obvious embedding $R(\partial_f F(\Omega_m,0)) \subset Z_m$. Thus it remains to check the converse, that is, for any $g \in Z_m$ there exists $w \mapsto h(w) = \sum_{n\geq 0} b_n\overline{w}^n \in B^s(\mathbb{T})$ such that $\partial_f F(\Omega_m,0)h = g$. In terms of Fourier coefficients this is equivalent to

$$b_0\Omega^1_m = 2g_0, \quad n\Big(\Omega^1_m - \Omega^1_n\Big)b_{n-1} = 2g_n, \quad \forall n \geq 2,$$

This defines only one sequence $(b_n)_{n\neq m-1}$ and the coefficient b_{m-1} is free. To check the regularity of h it suffices to prove that

$$w \mapsto H(w) = \sum_{n\geq m} b_n\overline{w}^n \in B^s(\mathbb{T}).$$

According to the definition of the norm of B^s and (84) one gets

$$\begin{aligned}\|H\|_{B^s} &= \sum_{n\geq m} n^s |b_n| \\ &= 2\sum_{n\geq m} n^s \frac{|g_{n+1}|}{(1+n)(\Omega^1_{n+1}-\Omega^1_m)} \\ &\leq C\sum_{n\geq m}\frac{n^{s-1}}{\ln n}|g_{n+1}| \\ &\leq \|g\|_{B^s_{\mathrm{Log}}}.\end{aligned}$$

This completes the proof of $Z_m = R(\partial_f F(\Omega^1_m, 0))$.
(**4**) We shall now check the transversality assumption

$$\partial_\Omega \partial_f F(\Omega^1_m, 0)(v_m) \notin R\Big(\partial_f F(\Omega^1_m, 0)\Big).$$

Differentiating (83) one gets

$$\partial_\Omega \partial_f F(\Omega, 0)(h)(w) = \mathrm{Im}\Big\{\overline{h'(w)} + \overline{w}h(w)\Big\}.$$

Then

$$\partial_\Omega \partial_f F(\Omega^1_m, 0)(v_m) = i\frac{m}{2}\big(w^m - \overline{w}^m\big),$$

which is not clearly in the subspace $Z_m = \partial_f F(\Omega_m, 0)$ as it was claimed. The proof of Proposition 10 is now completed.

□

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References

[1] M. H. P. Ambaum and B. J. Harvey, *Perturbed Rankine vortices in surface quasi-geostrophic dynamics*. Geophysical and Astrophysical Fluid Dynamics, 105 (4-5) (2011), 377–391.
[2] H. Aref, *Integrable, chaotic, and turbulent vortex motion in two-dimensional flows*, Ann. Rev. Fluid Mech., **15** (1983), 345–389.
[3] A.L. Bertozzi and A.J. Majda, *Vorticity and Incompressible Flow*, Cambridge texts in applied Mathematics, Cambridge University Press, Cambridge, (2002).
[4] J. Burbea, *Motions of vortex patches*, Lett. Math. Phys. **6** (1982), 1–16.
[5] M. G. Crandall and P.H. Rabinowitz, *Bifurcation from simple eigenvalues*, J. of Func. Analysis **8** (1971), 321–340.
[6] A. Castro, D. Córdoba, J. Gómez-Serrano, A. Martín Zamora, *Remarks on geometric properties of SQG sharp fronts and α-patches*, arXiv:1401.5376 .
[7] J.Y. Chemin, *Fluides parfaits incompressibles*, Astérisque **230**, Société Mathématique de France (1995).
[8] S. A. Chaplygin, *On a pulsating cylindrical vortex.* Translated from the 1899 Russian original by G. Krichevets, edited by D. Blackmore and with comments by V. V. Meleshko. Regul. Chaotic Dyn. 12 (2007), no. 1, 101–116.
[9] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, and J. Wu, *Generalized surface quasi-geostrophic equations with singular velocities.* Comm. Pure Appl. Math., 65(8) (2012),1037–1066.
[10] P. Constantin, A. J. Majda, and E. Tabak, *Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar.* Nonlinearity, 7(6) (1994),1495–1533.

[11] D. Córdoba, M. A. Fontelos, A. M. Mancho and J. L. Rodrigo, *Evidence of singularities for a family of contour dynamics equations*. Proc. Natl. Acad. Sci. USA 102 (2005) 5949–5952.
[12] G.S. Deem and N. J. Zabusky, *Vortex waves : Stationary "V-states", Interactions, Recurrence, and Breaking*, Phys. Rev. Lett. **40** 13 (1978), 859–862.
[13] P. L. Duren, *Univalent functions*, Grundlehren der mathematischen Wissenschaften 259. Springer-Verlag, New York, 1983.
[14] G. R. Flierl, L. M. Polvani, *Generalized Kirchhoff vortices*, Phys. Fluids 29 (1986) 2376–2379.
[15] F. Gancedo, *Existence for the α-patch model and the QG sharp front in Sobolev spaces.* Adv. Math., 217,**6** (2008) 2569–2598.
[16] B. J. Harvey, M. H. P. Ambaum,*Perturbed Rankine vortices in surface quasi-geostrophic dynamics.* Geophys. Astrophys. Fluid Dyn. 105 (2011), no. 4-5, 377–391.
[17] I. Held, R. Pierrehumbert, S. Garner,K. Swanson, *Surface quasi-geostrophic dynamics.* J. Fluid Mech. 282 (1995), 1–20.
[18] T. Hmidi, J. Mateu, J. Verdera, *Boundary Regularity of Rotating Vortex Patches*. Arch. Ration. Mech. Anal. 209 (2013), no. 1, 171–208.
[19] T. Hmidi, J. Mateu, J. Verdera, *On rotating doubly connected vortices*. arxiv:1310.0335.
[20] T. Hmidi, Notes on the clockwise rotating patches, in progress.
[21] M. Juckes, *Quasigeostrophic dynamics of the tropopause/* J. Armos. Sci. (1994) 2756–2768.
[22] S. Kida, *Motion of an elliptical vortex in a uniform shear flow,* J. Phys. Soc. Japan. 50 (1981) 3517–3520.
[23] H. Kielhöfer, *Bifurcation Theory: An Introduction With Applications to Partial Differential Equations,* Springer (2011)
[24] G. Kirchhoff, Vorlesungen uber mathematische Physik (Leipzig, 1874).
[25] J. Mateu, J. Orobitg and J. Verdera, *Extra cancellation of even Calderón-Zygmund operators and quasiconformal mappings*, J. Math. Pures Appl. **91** (4)(2009), 402–431.
[26] H. Lamb, *Hydrodynamics*, Dover Publications, New York, (1945).
[27] G. Lapeyre, P. Klein, *Dynamics of the upper oceanic layers in terms of surface quasigeostrophic theory.* J. Phys. Oceanogr. **36** (2006), 165–176.
[28] J. Neu, *The dynamics of columnar vortex in an imposed strain,* Phys. Fluids 27 (1984) 12397–2402.
[29] W. Magnus, F. Oberhettinger *Formeln und satze fur die speziellen funktionen der mathematischen physik*, Berlin.Gottingen.Heidelberg : Springer , 1948 .
[30] P. K. Newton, *The N-Vortex Problem. Analytical Techniques,* (Springer, New York, 2001).
[31] Ch. Pommerenke, *Boundary behaviour of conformal maps,* Springer-Verlag, Berlin, 1992.
[32] J. L. Rodrigo, *On the evolution of sharp fronts for the quasi-geostrophic equation,* Comm. Pure Appl. Math., 58 (6) (2005) 821–866.
[33] J. Verdera, *L^2 boundedness of the Cauchy Integral and Menger curvature,* Contemporary Mathematics **277** (2001), 139–158.
[34] S. E. Warschawski, *On the higher derivatives at the boundary in conformal mapping,* Trans. Amer. Math. Soc. **38** (1935), no. 2, 310–340.
[35] Watson, *A Treatise on the Theory of Bessel Functions,* Cambridge University Press , 1944.
[36] R. Wittmann, *Application of a Theorem of M.G. Krein to singular integrals,* Trans. Amer. Math. Soc. **299**(2) (1987), 581–599.
[37] H.M. Wu, E.A. Overman II and N.J. Zabusky, *Steady-state solutions of the Euler equations in two dimensions : rotating and translating V-states with limiting cases I. Algorithms ans results*, J. Comput. Phys. **53** (1984), 42–71.
[38] Y. Yudovich, *Nonstationary flow of an ideal incompressible liquid*. Zh. Vych. Mat., **3** (1963), 1032–1066.
[39] N. Zabusky, M. H. Hughes, K.V. Roberts, *Contour dynamics for the Euler equations in two dimensions.* J. Comput. Phys. 30 (1979), no. 1, 96–106.

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