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ON ESTIMATES FOR WEIGHTED BERGMAN PROJECTIONS

P. CHARPENTIER, Y. DUPAIN & M. MOUNKAILA

ABSTRACT. In this note we show that the weighted L^2 -Sobolev estimates obtained by P. Charpentier, Y. Dupain & M. Mounkaila for the weighted Bergman projection of the Hilbert space $L^2(\Omega, d\mu_0)$ where Ω is a smoothly bounded pseudoconvex domain of finite type in \mathbb{C}^n and $\mu_0 = (-\rho_0)^r d\lambda$, λ being the Lebesgue measure, $r \in \mathbb{Q}_+$ and ρ_0 a special defining function of Ω , are still valid for the Bergman projection of $L^2(\Omega, d\mu)$ where $\mu = (-\rho)^r d\lambda$, ρ being any defining function of Ω . In fact a stronger directional Sobolev estimate is established. Moreover similar generalizations are obtained for weighted L^p -Sobolev and Lipschitz estimates in the case of pseudoconvex domain of finite type in \mathbb{C}^2 and for some convex domains of finite type.

1. INTRODUCTION

Let Ω be a smoothly bounded domain in \mathbb{C}^n . A non negative measurable function ν on Ω is said to be an admissible weight if the space of holomorphic functions square integrable for the measure $\nu d\lambda$ ($d\lambda$ being the Lebesgue measure) is a closed subspace of the Hilbert space $L^2(\nu d\lambda)$ of square integrable functions on Ω (see, for example, [PW90]). In complex analysis, ν being admissible, the regularity of the Bergman projection associated to $\nu d\lambda$ (i.e. the orthogonal projection of $L^2(\nu d\lambda)$ onto the subspace of holomorphic functions) is a fundamental question. It has been intensively studied when $\nu \equiv 1$ and specially when Ω is pseudoconvex.

If η is a smooth strictly positive function on $\overline{\Omega}$ it is well known that the regularity properties of the Bergman projections of the Hilbert spaces $L^2(\eta \nu d\lambda)$ and $L^2(\nu d\lambda)$ can be very different. For example in [Koh72] J. J. Kohn proved that if Ω is pseudoconvex, for any integer k there exists $t > 0$ such that the Bergman projection of $L^2(e^{-t|z|^2} d\lambda)$ maps the Sobolev space $L^2_k(\Omega)$ into itself, and, in [Chr96] M. Christ showed that there exists a smoothly bounded pseudoconvex domain such that the Bergman projection of $L^2(\Omega) = L^2(e^{t|z|^2} e^{-t|z|^2} d\lambda)$ is not L^2 -Sobolev regular.

In this paper we show that some of the (weighted) estimates obtained in [CDMb] for pseudoconvex domains of finite type remain true when the weight is multiplied by a function which is smooth and strictly positive in $\overline{\Omega}$. This shows that the corresponding estimates obtained in [CDMb] for the Bergman projection of $L^2((-\rho_0)^r d\lambda)$, where ρ_0 is a special defining function of Ω and r a non negative rational number, are valid for the Bergman projection of $L^2((-\rho)^r d\lambda)$ where ρ is any defining function of the domain. Moreover, we show that these Bergman projections satisfy a stronger directional Sobolev estimate.

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2. NOTATIONS AND MAIN RESULTS

Throughout all the paper $d\lambda$ denotes the Lebesgue measure. Let D be a smoothly bounded open set in \mathbb{C}^l . Recall that d is said to be a defining function of D if it is a real function in $\mathcal{C}^\infty(\mathbb{C}^l)$ such that $D = \{\zeta \in \mathbb{C}^l \text{ s. t. } d(\zeta) < 0\}$ and ∇d does not vanish on ∂D .

Let ν be an admissible weight on D .

For $1 \leq p < +\infty$ we denote by $L^p(D, \nu d\lambda)$ the L^p space for the measure $\nu d\lambda$. When $\nu \equiv 1$ we write, as usual, $L^p(D)$.

We denote by P_ν^D the orthogonal projection of the Hilbert space $L^2(D, \nu d\lambda)$ (i. e. for the scalar product $\langle f, g \rangle = \int_D f \bar{g} \nu d\lambda$) onto the closed subspace of holomorphic functions. If $\nu \equiv 1$ we simply write P^D . In this paper, P_ν^D is called the (weighted) *Bergman projection of $L^2(D, \nu d\lambda)$* .

For $k \in \mathbb{N}$ and $1 < p < +\infty$, we define the weighted Sobolev space $L_k^p(D, \nu d\lambda)$ by

$$L_k^p(D, \nu d\lambda) = \left\{ u \in L^p(D, \nu d\lambda) \text{ such that } \left. \begin{aligned} \|u\|_{L_k^p(D, \nu d\lambda)}^p &= \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(D, \nu d\lambda)}^p < +\infty \end{aligned} \right\}.$$

If $\nu \equiv 1$ this space is the classical Sobolev space $L_k^p(D)$.

Let d be a smooth defining function of D . We denote by T_d the vector field

$$T_d = \sum_i \frac{\partial d}{\partial \bar{z}_i} \frac{\partial}{\partial z_i} - \frac{\partial d}{\partial z_i} \frac{\partial}{\partial \bar{z}_i}.$$

Thus T_d is a vector field tangent to d (i.e. $T_d d \equiv 0$) which is transverse to the complex tangent space to d near the boundary of D .

Following a terminology introduced in [HMS14] a vector field T with coefficients in $\mathcal{C}^\infty(\bar{D})$ is said to be *tangential and complex transversal* to $\partial\Omega$ if it can be written $T = aT_d + L$ where $a \in \mathcal{C}^\infty(\bar{D})$ is nowhere vanishing on ∂D and $L = L_1 + \bar{L}_2$ where L_1 and L_2 are $(1, 0)$ -type vector fields tangential to ∂D . Note that this definition is independent of the choice of the defining function d (see the beginning of Section 4). Then, for all non negative integer k , and $1 < p < +\infty$, we denote by $L_{k,T}^p(D, \nu d\lambda)$ the weighted directional Sobolev space

$$L_{k,T}^p(D, \nu d\lambda) = \left\{ u \in L^p(D, \nu d\lambda) \text{ such that } \left. \|u\|_{L_{k,T}^p(D, \nu d\lambda)}^p = \sum_{l \leq k} \|T^l u\|_{L^p(D, \nu d\lambda)}^p < +\infty \right\}.$$

Our first result extends Theorem 2.2 of [CDMb] and, for finite type domains, Theorem 1.1 obtained by A.-K. Herbig, J. D. McNeal and E. J. Straube in [HMS14] for the standard Bergman projection:

Theorem 2.1. *Let Ω be a smooth bounded pseudoconvex domain of finite type in \mathbb{C}^n . Let ρ be a smooth defining function of Ω . Let $r \in \mathbb{Q}_+$, be a non negative rational number and $\eta \in \mathcal{C}^\infty(\bar{\Omega})$, strictly positive. Let T be a $\mathcal{C}^\infty(\bar{\Omega})$ vector field tangential*

and complex transversal to $\partial\Omega$. Define $\omega = \eta(-\rho)^r$. Then, for any integer k , P_ω^Ω maps continuously the weighted directional Sobolev space $L_{k,T}^2(\Omega, \omega d\lambda)$ into $L_k^2(\Omega, \omega d\lambda)$.

Note that r is allowed to be 0.

Corollary. *In the conditions stated in the theorem, P_ω^Ω maps continuously*

$$\cap_{k \in \mathbb{N}} L_{k,T}^2(\Omega, \omega_0 d\lambda)$$

into $C^\infty(\overline{\Omega})$.

Our second result is inspired by Theorem 1.10 of [HMS14]:

Theorem 2.2. *Let Ω , η and ω as in Theorem 2.1. Let $f \in L^2(\Omega, \omega d\lambda)$ such that \bar{f} is holomorphic and let $h \in C^\infty(\overline{\Omega})$. Then $P_\omega^\Omega(fh) \in C^\infty(\overline{\Omega})$.*

The proofs are done in Section 4.

Our other results are partial generalizations of Theorem 2.1 of [CDMb] for domains in \mathbb{C}^2 and for convex domains. As the results for convex domains are not general we state them separately and we will only indicate the articulations of the proof at the end of Section 5.

The first result extends the results obtained by A. Bonami and S. Grellier in [BG95]:

Theorem 2.3. *Let Ω , η and ω as in Theorem 2.1. Assume moreover that, at every point of $\partial\Omega$, the rank of the Levi form is $\geq n - 2$. Then:*

- (1) *For $1 < p < +\infty$ and $k \in \mathbb{N}$, P_ω^Ω maps continuously the Sobolev space $L_k^p(\Omega, \omega d\lambda)$ into itself;*
- (2) *For $\alpha < 1$, P_ω^Ω maps continuously the Lipschitz space $\Lambda_\alpha(\Omega)$ into itself.*

The results for convex domains are identical but under an additional condition on the existence of a special defining function. To state it we recall a terminology introduced in Section 2.4 of [CDMb]:

If g is a real or complex valued smooth function defined in a neighborhood of the origin in \mathbb{R}^d , we call the *order* of g at the origin the integer $\text{ord}_0(g)$ defined by $\text{ord}_0(g) = \infty$ if $g^{(\alpha)}(0) = 0$ for all multi-index $\alpha \in \mathbb{N}^d$ and

$$\text{ord}_0(g) = \min \left\{ k \in \mathbb{N} \text{ such that there exists } \alpha \in \mathbb{N}^d, \right. \\ \left. |\alpha| = \sum \alpha_i = k \text{ such that } g^{(\alpha)}(0) \neq 0 \right\}$$

otherwise. If ψ is a smooth function defined in a neighborhood of the origin in \mathbb{C}^m , then, for all function φ from the unit disc of the complex plane into \mathbb{C}^m such that $\varphi(0) = 0$, $\psi \circ \varphi$ is smooth in a neighborhood of the origin in \mathbb{C} . Then we call the *type* of ψ at the origin the supremum of $\frac{\text{ord}_0(\psi \circ \varphi)}{\text{ord}_0(\varphi)}$, taken over all non zero holomorphic function φ from the unit disc of the complex plane into \mathbb{C}^m such that $\varphi(0) = 0$. If this supremum is finite, we say that ψ is of *finite type* at the origin and we denote this supremum by $\text{typ}_0(\psi)$. Moreover, if ϑ is a smooth function defined in a neighborhood of a point $z_0 \in \mathbb{C}^m$, the type $\text{typ}_{z_0}(\vartheta)$ of ϑ at z_0 is $\text{typ}_0(\vartheta_k)$ where $\vartheta_k(z) = \vartheta(z_0 + z)$ and we say that ϑ is of finite type at z_0 if $\text{typ}_{z_0}(\vartheta) < +\infty$. If ϑ is defined on a neighborhood of a set S we say that ϑ is of finite type on S if $\sup_{z \in S} \text{typ}_z(\vartheta) < +\infty$.

Proposition 2.1. *Let Ω , η and ω as in Theorem 2.1. Assume that Ω is convex and admits a defining function which is smooth, convex and of finite type in $\overline{\Omega}$. Then:*

- (1) *For $1 < p < +\infty$ and $k \in \mathbb{N}$, P_ω^Ω maps continuously the Sobolev space $L_k^p(\Omega, \omega d\lambda)$ into itself;*
- (2) *For $\alpha < 1$, P_ω^Ω maps continuously the Lipschitz space $\Lambda_\alpha(\Omega)$ into itself.*

Remark. The defining function choosen in [CDMb] for a general convex domain of finite type is smooth convex of finite type everywhere except at one point.

Theorem 2.3 is proved in Section 5 as a special case of a stronger directional L_k^p estimate (Theorem 5.1).

The general scheme of the proofs of these results is as follows. Recall that in [CDMb] we obtain the estimates in the above theorems for the projections $P_{\omega_0}^\Omega$ where $\omega_0 = (-\rho_0)^r$, ρ_0 being the following special defining function of Ω :

- For Theorem 2.1 and for Theorem 2.3, using a celebrated theorem of K. Diederich & J. E. Fornæss ([DF77, Theorem 1]), ρ_0 is chosen so that there exists $t \in]0, 1[$ such that $-(-\rho_0)^t$ is strictly plurisubharmonic in Ω .
- If Ω is convex ρ_0 is assumed to be convex and of finite type in $\overline{\Omega}$ (hypothesis of Proposition 2.1).

Then we obtain the results for P_ω^Ω comparing P_ω^Ω and $P_{\omega_0}^\Omega$ as explained in the next section. The restriction imposed to Ω in Proposition 2.1 comes from the fact that this comparison uses estimates with gain for solutions of the $\bar{\partial}$ -equation in a domain $\tilde{\Omega}$ in \mathbb{C}^{n+m} , $n + m \geq 3$, which are known only under strong hypothesis on $\tilde{\Omega}$. As ρ and ρ_0 are two smooth defining functions of Ω there exists a function φ smooth and strictly positive on $\overline{\Omega}$ such that $\rho = \varphi\rho_0$. Then, there exists a function $\eta_1 \in C^\infty(\overline{\Omega})$, $\eta_1 > 0$, such that $\omega = \eta_1(-\rho_0)^r$.

Thus, from now on, ρ_0 and ω_0 are fixed as above and, to simplify the notations, we write $\omega = \eta(-\rho_0)^r$ where η is a strictly positive function in $C^1(\overline{\Omega})$.

3. COMPARING P_ω^Ω AND $P_{\omega_0}^\Omega$

This comparison is based on the following simple formula:

Proposition 3.1. *With the previous notations for D and P_ν^D , let η be a strictly positive function in $C^\infty(\overline{D})$ (so that $\eta\nu$ is an admissible weight). Let $L_{(0,1)}^2(D, \nu d\lambda)$ be the space of $(0, 1)$ -forms with coefficients in $L^2(D, \nu d\lambda)$. If there exists a continuous linear operator A_ν from $L_{(0,1)}^2(D, \nu d\lambda) \cap \ker \bar{\partial}$ into $L^2(D, \nu d\lambda)$ such that, for $f \in L_{(0,1)}^2(D, \nu d\lambda) \cap \ker \bar{\partial}$, $A_\nu(f)$ is orthogonal to holomorphic functions in $L^2(D, \nu d\lambda)$ and $\bar{\partial}A_\nu(f) = f$ then, for all $u \in L^2(D, \nu d\lambda)$ we have*

$$\eta P_{\eta\nu}^D(u) = P_\nu^D(\eta u) + A_\nu(P_{\eta\nu}^D(u)\bar{\partial}\eta).$$

Proof. This is almost immediate: from the second hypothesis on A_ν both sides of the formula have same $\bar{\partial}$, and, from the first hypothesis, both sides have same scalar product, in $L^2(D, \nu d\lambda)$, against holomorphic functions. \square

We use this formula in the context developed in [CDMb].

For $h(w) = |w|^{2q}$, $w \in \mathbb{C}^m$, $r = q/m$ or $h(w) = \sum |w_i|^{2q_i}$, $w_i \in \mathbb{C}$, $r = \sum 1/q_i$ (c.f. [CDMb]), ρ_0 and ω_0 as introduced in the preceding section, we consider the

domain in \mathbb{C}^{n+m} defined by

$$\tilde{\Omega} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m, \text{ s. t. } r(z, w) = \rho_0(z) + h(w) < 0\}.$$

Then (c.f. [CDMb]) $\tilde{\Omega}$ is smooth, bounded and pseudoconvex. Therefore the $\bar{\partial}$ -Neumann operator $\mathcal{N}_{\tilde{\Omega}}$ is well defined. Let us introduce two notations:

- If $u \in L^p(\Omega, \omega_0 d\lambda)$, $1 \leq p < +\infty$, we denote by $I(u)$ the function, belonging to $L^p(\tilde{\Omega})$, defined by $I(u)(z, w) = u(z)$ (the fact that $I(u) \in L^p(\tilde{\Omega})$ follows Fubini's theorem). We extend this notations to forms $f = \sum f_i d\bar{z}_i$ in $L^p_{(0,1)}(\Omega, \omega_0 d\lambda)$ by $I(f) = \sum I(f_i) d\bar{z}_i$ (so that $I(f) \in L^p_{(0,1)}(\tilde{\Omega})$ and, $I(f)$ is $\bar{\partial}$ -closed if f is so).
- If $v \in L^p(\tilde{\Omega})$, $1 \leq p < +\infty$, is holomorphic in w we denote by $R(v)$ the function, belonging to $L^p(\Omega, \omega_0 d\lambda)$ (by the mean value property applied to the subharmonic function $w \mapsto |v(z, w)|^p$), defined by $R(v)(z) = v(z, 0)$.

Then:

Proposition 3.2. *For any function $u \in L^2(\Omega, \omega d\lambda)$, we have*

$$(3.1) \quad \eta P_\omega^\Omega(u) = P_{\omega_0}^\Omega(\eta u) + R \circ \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I \left(P_\omega^\Omega(u) \bar{\partial}(\eta) \right).$$

Proof. By the preceding proposition, it suffices to note that the operator $R \circ \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I$ is continuous from $L^2_{(0,1)}(\Omega, \omega_0 d\lambda) \cap \ker \bar{\partial}$ into $L^2(\Omega, \omega_0 d\lambda)$, solves the $\bar{\partial}$ -equation and gives the solution which is orthogonal to holomorphic functions in that space. But if $f \in L^2_{(0,1)}(\Omega, \omega_0 d\lambda) \cap \ker \bar{\partial}$ then, by Fubini's theorem, $I(f) \in L^2_{(0,1)}(\tilde{\Omega}) \cap \ker \bar{\partial}$, and $\left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f)$ is the solution of $\bar{\partial}u = I(f)$ which is orthogonal to holomorphic functions in $L^2(\tilde{\Omega})$ and satisfies

$$\left\| \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f) \right\|_{L^2(\tilde{\Omega})} \lesssim \|I(f)\|_{L^2_{(0,1)}(\tilde{\Omega})} = C \|f\|_{L^2_{(0,1)}(\Omega, \omega_0 d\lambda)}$$

(recall that $\tilde{\Omega}$ is pseudoconvex and that the volume of $\{h(w) < -\rho_0(z)\}$ is equal to $C\omega_0(z)$). As $I(f)$ is independent of the variable w , $\left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f)$ is holomorphic in w and

$$\bar{\partial}_z \left(\left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f) \right) (z, 0) = f(z)$$

so $\bar{\partial} \left(R \circ \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f) \right) = f$, and, by the mean value property (applied to the subharmonic function $w \mapsto \left| \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f)(z, w) \right|^2$),

$$\left\| R \circ \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f) \right\|_{L^2(\Omega, \omega_0 d\lambda)} \leq C \left\| \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f) \right\|_{L^2(\tilde{\Omega})} \lesssim \|f\|_{L^2_{(0,1)}(\Omega, \omega_0 d\lambda)}.$$

Moreover, if g is a holomorphic function in $L^2(\Omega, \omega_0 d\lambda)$, by the mean value property,

$$\begin{aligned} \int_{\Omega} R \circ \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f) \bar{g} \omega_0 d\lambda &= \int_{\Omega} \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f)(z, 0) \bar{g}(z) \omega_0(z) d\lambda(z) \\ &= C \int_{\Omega} \left(\int_{\{h(w) < -\rho_0(z)\}} \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \right. \\ &\quad \left. \circ I(f)(z, w) d\lambda(w) \right) \bar{g}(z) d\lambda(z) \\ &= C \int_{\tilde{\Omega}} \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \right) \circ I(f)(z, w) \bar{g}(z) d\lambda(z, w) = 0. \end{aligned}$$

□

An immediate density argument shows that:

Corollary. *Let $p \in]1, +\infty[$. Assume that the following properties are satisfied:*

- P_{ω}^{Ω} and $P_{\omega_0}^{\Omega}$ map continuously $L^p(\Omega, \omega d\lambda)$ into itself;
- $\bar{\partial} \mathcal{N}_{\tilde{\Omega}}$ maps continuously $L^p(\tilde{\Omega})$ into itself.

Then equation (3.1) is valid for any function $u \in L^p(\Omega, \omega d\lambda)$.

In the proofs of the theorems we need to use weighted Sobolev spaces $L_s^p(D, \nu d\lambda)$ defined for all $s \geq 0$ and some directional Sobolev spaces.

It is well known that for $s \in [k, k+1]$, $k \in \mathbb{N}$, the fractional Sobolev space $L_s^p(D)$ is obtained using the complex interpolation method between $L_k^p(D)$ and $L_{k+1}^p(D)$ (see, for example, [Tri78]). By analogy, we extend the definition of the weighted Sobolev spaces $L_s^p(D, \nu d\lambda)$ to any index $s \geq 0$ using the complex interpolation method:

$$L_s^p(D, \nu d\lambda) = [L_k^p(D, \nu d\lambda), L_{k+1}^p(D, \nu d\lambda)]_{s-k}, \text{ if } s \in [k, k+1].$$

Note that if ν_1 and ν_2 are two admissible weights such that $\nu_2 = \eta \nu_1$ with η a strictly positive function in $\mathcal{C}^{[s]+1}(\tilde{\Omega})$ then the Banach spaces $L_s^p(D, \nu_1 d\lambda)$ and $L_s^p(D, \nu_2 d\lambda)$ are identical.

For all $s \geq 0$, we extend the definition of $L_{k,T}^p(D, \nu d\lambda)$ to $L_{s,T}^p(D, \nu d\lambda) = [L_{k,T}^p(D, \nu d\lambda), L_{k+1,T}^p(D, \nu d\lambda)]_{s-k}$, $k \leq s \leq k+1$ by complex interpolation between two consecutive integers. Clearly, the spaces $L_{s,T}^2(D, \nu d\lambda)$ are Hilbert spaces and $L_{s,T}^p(D, \nu d\lambda)$ are Banach spaces. When $\nu \equiv 1$ we denote this space $L_{s,T}^p(D)$.

Note that, $r = \rho_0 + h$ being the defining function of $\tilde{\Omega}$, we have $T_r = T_{\rho_0} + T_h$, with $T_h = |w|^{2q-2} \sum \left(w_i \frac{\partial}{\partial w_i} - \bar{w}_i \frac{\partial}{\partial \bar{w}_i} \right)$ when $h(w) = |w|^{2q}$, $w \in \mathbb{C}^m$ and $T_h = \sum_{i=1}^m |w_i|^{2q_i-2} \left(w_i \frac{\partial}{\partial w_i} - \bar{w}_i \frac{\partial}{\partial \bar{w}_i} \right)$ when $h(w) = \sum_{i=1}^m |w_i|^{2q_i}$, $w_i \in \mathbb{C}$.

Remark 3.1. The spaces $L_{s,T}^p(D, \nu d\lambda)$ depend on the choice of the vector field T (see Section 5 of [HM12]).

We now state some elementary properties of the operators I and R introduced before and related to these Sobolev spaces. It is convenient to introduce other

spaces: for $1 < p < +\infty$ and $s \geq 0$, let

$$L_s^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w = \left\{ u(z, w) \in L_s^p(\tilde{\Omega}) \text{ such that } \frac{\partial u}{\partial \bar{w}_i} \equiv 0, 1 \leq i \leq m \right\}.$$

Lemma 3.1. *With the previous notations and for $1 < p < +\infty$, we have:*

- (1) *For all $s \geq 0$, I maps continuously $L_s^p(\Omega, \omega_0 d\lambda)$ into $L_s^p(\tilde{\Omega})$.*
- (2) *For all non negative integer k , R maps continuously $L_k^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w$ into $L_k^p(\Omega, \omega_0 d\lambda)$.*

Proof. As $D_z^\alpha I(h) = I(D^\alpha h)$ for any derivative D^α , Fubini's Theorem implies $\|D_z^\alpha I(h)\|_{L^p(\tilde{\Omega})}^p = C \|D^\alpha h\|_{L^p(\Omega, \omega_0 d\lambda)}^p$ and (1) follows for $s \in \mathbb{N}$ and for all s by the interpolation theorem.

The second point of the lemma is also very simple. If $u \in L_k^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w$, then, for all derivative D^α , $D^\alpha (Ru)(z) = D_z^\alpha u(z, 0)$, and $w \mapsto |D_z^\alpha u(z, w)|^p$ is subharmonic. Therefore the mean value property gives

$$C |D^\alpha (Ru)(z)|^p \omega_0(z) \leq \int_{\{h(w) < -\rho_0(z)\}} |D_z^\alpha u(z, w)|^p d\lambda(w).$$

Integrating this inequality over Ω finishes the proof. \square

In Sections 4 and 5 we will need estimates for R on the spaces $L_s^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w$ for all $s \geq 0$. Unfortunately the two spaces $L_s^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w = \left[L_k^p(\tilde{\Omega}), L_{k+1}^p(\tilde{\Omega}) \right]_{s-k} \cap \ker \bar{\partial}_w$ and $\left[L_k^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w, L_{k+1}^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w \right]_{s-k}$ may be different. This difficulty is circumvented by the following lemma:

Lemma 3.2. *With the previous notations and for $1 < p < +\infty$, we have:*

- (1) *For all $s \geq 0$, I maps continuously $L_{s, T_{\rho_0}}^p(\Omega, \omega_0 d\lambda)$ into $L_{s, T_r}^p(\tilde{\Omega})$.*
- (2) *For all $s \geq 0$, R maps continuously $L_s^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w$ into $L_{s, T_{\rho_0}}^p(\Omega, \omega_0 d\lambda)$.*

Proof. As $T_r^l(I(h)) = I(T_{\rho_0}^l(h))$ Fubini's Theorem gives (1) when s is an integer. Therefore (1) follows by interpolation.

To see the second point of the lemma, let us denote by $M_0 u$ the mean with respect to the variable w of a function u in $L^p(\tilde{\Omega})$

$$M_0 u(z) = \frac{1}{C\omega_0(z)} \int_{\{h(w) < -\rho_0(z)\}} u(z, w) d\lambda(w).$$

As T_{ρ_0} is tangent to ρ_0 , we have $T_{\rho_0}(\omega_0) \equiv T_{\rho_0}(\rho_0) \equiv 0$, and, for all integer l we get

$$T_{\rho_0}^l M_0 u(z) = \frac{1}{C\omega_0(z)} \int_{\{h(w) < -\rho_0(z)\}} T_{\rho_0}^l u(z, w) d\lambda(w).$$

Then, by Hölder inequality we have

$$C |T_{\rho_0}^l M_0 u(z)|^p \omega_0(z) \leq \int_{\{h(w) < -\rho_0(z)\}} |T_{\rho_0}^l u(z, w)|^p d\lambda$$

and, integrating this inequality over Ω , we get that M_0 maps continuously $L_k^p(\tilde{\Omega})$ into $L_{k, T_{\rho_0}}^p(\Omega, \omega_0 d\lambda)$. Therefore, by the interpolation theorem, M_0 maps continuously $L_s^p(\tilde{\Omega})$ into $L_{s, T_{\rho_0}}^p(\Omega, \omega_0 d\lambda)$ for all $s \geq 0$.

This proves (2) of the lemma because, by the mean value property for holomorphic functions, $M_0u = Ru$ when $u \in L_s^p(\tilde{\Omega}) \cap \ker \bar{\partial}_w$. \square

4. PROOF OF THEOREMS 2.1 AND 2.2

For convenience, we extend the notation of the vector field T_d given at the beginning of Section 2 denoting by T_ψ the vector field

$$T_\psi = \sum \frac{\partial \psi}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} - \frac{\partial \psi}{\partial z_j} \frac{\partial}{\partial \bar{z}_j}$$

where ψ is any function in $\mathcal{C}^1(\bar{D})$.

If $T = aT_\rho + L$ is the vector field given in Theorem 2.1 then (writing $\rho = \varphi\rho_0$) we have $T = a\varphi T_{\rho_0} + (\rho_0 T_\varphi + L) = a\varphi T_{\rho_0} + L'$ with $\varphi > 0$ on $\bar{\Omega}$ and $L' = L'_1 + L'_2$ where L'_1 and L'_2 are (1,0)-type vector fields tangential to $\partial\Omega$. Moreover, writing $a = a' + b$ where a' is nowhere vanishing on $\bar{\Omega}$ and b identically 0 in a neighborhood of $\partial\Omega$, we get $T = a'\varphi T_{\rho_0} + L''$ with $L'' = L''_1 + L''_2$ where L''_1 and L''_2 are (1,0)-type vector fields tangential to $\partial\Omega$ and $a'\varphi$ is nowhere vanishing on $\bar{\Omega}$.

We now prove the following reformulation of Theorem 2.1:

Theorem 4.1. *Let Ω be as in Theorem 2.1. Let k be a non negative integer. Let ρ_0 , ω_0 and ω be as at the end of Section 2 with $\eta \in \mathcal{C}^{k+1}(\bar{\Omega})$. Let $\varphi \in \mathcal{C}^\infty(\bar{\Omega})$ a function which is nowhere vanishing on $\bar{\Omega}$ and let $T = \varphi T_{\rho_0} + L$ with $L = L_1 + L_2$ where L_1 and L_2 are $\mathcal{C}^\infty(\bar{\Omega})$ vector fields of type (1,0) tangential to $\partial\Omega$. Then for $s \in [0, k]$ the weighted Bergman projection P_{ω}^Ω maps continuously the directional weighted Sobolev space $L_{s,T}^2(\Omega, \omega_0 d\lambda)$ into $L_s^2(\Omega, \omega_0 d\lambda)$.*

Proof. With the notations of the end of the preceding section, we choose $h(w) = |w|^{2q}$, $w \in \mathbb{C}^m$ with $r = m/q$. Then, by results of [CDMb], $\tilde{\Omega}$ is a smoothly bounded pseudoconvex domain in \mathbb{C}^{n+m} of finite type. First we note that the estimate of the theorem for $P_{\omega_0}^\Omega$ is a consequence of a theorem of A.-K. Herbig, J. D. McNeal and E. Straube:

Lemma 4.1. *The Bergman projection $P_{\omega_0}^\Omega$ maps continuously the directional space $L_{s,T}^2(\Omega, \omega_0 d\lambda)$ into $L_s^2(\Omega, \omega_0 d\lambda)$.*

Proof of the lemma. According to [CDMb], Section 3, we have

$$P_{\omega_0}^\Omega = R \circ P^{\tilde{\Omega}} \circ I,$$

where $P^{\tilde{\Omega}}$ is the standard Bergman projection of $\tilde{\Omega}$.

Lemma 4.2. *There exists a vector field $W = \sum a_i \frac{\partial}{\partial w_i} + b_i \frac{\partial}{\partial \bar{w}_i}$ with coefficients a_i and b_i in $\mathcal{C}^\infty(\tilde{\Omega})$ such that $T+W$ is smooth in $\tilde{\Omega}$, tangential and complex transversal to $\partial\tilde{\Omega}$.*

Proof of Lemma 4.2. This is a very simple calculus. $T + W = \varphi T_{\rho_0} + L + W = \varphi T_r - \varphi T_{|w|^{2q}} + L + W$, where r denotes the defining function of $\tilde{\Omega}$. As φT_r is tangential and complex transversal to $\partial\tilde{\Omega}$ it is enough to see that the coefficients of

W can be chosen so that the $(1, 0)$ and $(0, 1)$ parts of $-\varphi T_{|w|^{2q}} + L + W$ are both tangential to $\partial\tilde{\Omega}$. For example, the $(1, 0)$ part of this vector field is

$$-q\varphi |w|^{2q-2} \sum w_i \frac{\partial}{\partial w_i} + L_1 + \sum a_i \frac{\partial}{\partial w_i},$$

and it is tangential to $\partial\tilde{\Omega}$ if $q^2\varphi |w|^{4q-2} - L_1\rho_0 \equiv \sum qa_i |w|^{2q-2} \bar{w}_i$ on $\partial\tilde{\Omega}$. As L_1 is tangential to $\partial\Omega$, $L_1\rho_0$ vanishes at $\partial\Omega$ and there exists a function $\psi \in \mathcal{C}^\infty(\bar{\Omega})$ such that $-L_1\rho_0 = \psi_1(-\rho_0)$. If $(z, w) \in \partial\tilde{\Omega}$ then $-\rho_0(z) = |w|^{2q}$, and, it suffices to choose

$$a_i = \frac{1}{q} w_i \left[q^2\varphi |w|^{2q-2} + \psi_1 \right].$$

Similarly, the $(0, 1)$ part of $-\varphi T_{|w|^{2q}} + L + W$ is tangential to $\partial\tilde{\Omega}$ choosing $b_i = \frac{1}{q} \bar{w}_i \left[q^2\varphi |w|^{2q-2} + \psi_2 \right]$. \square

Let us now finish the proof of Lemma 4.1. By Theorem 1.1 of [HMS14], for any non negative integer k , P^Ω maps continuously $L^2_{k, T+W}(\tilde{\Omega})$ into $L^2_k(\tilde{\Omega})$. As $(T+W)^l(I(u)) = T^l(u)$, for each $u \in L^2_{k, T}(\Omega, \omega_0 d\lambda)$, we have $I(u) \in L^2_{k, T+W}(\tilde{\Omega})$, and, for $s = k$, the Lemma follows (2) of Lemma 3.1. The general case $s \geq 0$ is therefore obtained by interpolation. \square

Now we use the formula of Proposition 3.2 to prove Theorem 4.1, by induction, for $s \in \{0, \dots, k\}$, the general case $s \in [0, k]$ being then a consequence of the interpolation theorem. Let us assume the Theorem true for $s-1$, $0 < s \leq k$ and let us prove it for s . Let N be an integer whose inverse is smaller than the index of subellipticity of the $\bar{\partial}$ -Neumann problem of $\tilde{\Omega}$ (recall that we show in [CDMb] that $\tilde{\Omega}$ is of finite type). Let $u \in L^2_{s, T}(\Omega, \omega_0 d\lambda)$. To prove that $P^\Omega_\omega(u) \in L^2_s(\Omega, \omega_0 d\lambda)$, let us prove, by induction over $l \in \{0, 1, \dots, N\}$ that $P^\Omega_\omega(u) \in L^2_{s-1+l/N}(\Omega, \omega_0 d\lambda)$. Assume $P^\Omega_\omega(u) \in L^2_{s-1+l/N}(\Omega, \omega_0 d\lambda)$, $l \leq N-1$. As $\eta \in \mathcal{C}^{k+1}(\bar{\Omega})$, by (1) of Lemma 3.2, $I(P^\Omega_\omega(u)\bar{\partial}(\eta)) \in L^2_{s-1+l/N}(\tilde{\Omega})$. By subelliptic estimates for the $\bar{\partial}$ -Neumann problem on $\tilde{\Omega}$,

$$\left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}}\right) \circ I(P^\Omega_\omega(u)\bar{\partial}(\eta)) \in L^2_{s-1+(l+1)/N}(\tilde{\Omega}).$$

By (2) of Lemma 3.2,

$$R \circ \left(\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}}\right) \circ I(P^\Omega_\omega(u)\bar{\partial}(\eta)) \in L^2_{s-1+(l+1)/N, T\rho_0}(\Omega, \omega_0 d\lambda).$$

By Lemma 4.1 $P^\Omega_{\omega_0}(\eta u) \in L^2_s(\Omega, \omega_0 d\lambda)$, thus, as $\eta^{-1} \in \mathcal{C}^{k+1}(\bar{\Omega})$, Proposition 3.2 gives

$$P^\Omega_\omega(u) \in L^2_{s-1+(l+1)/N, T\rho_0}(\Omega, \omega_0 d\lambda),$$

and, as $P^\Omega_{\omega_0} \circ P^\Omega_\omega = P^\Omega_\omega$, Lemma 4.1 implies $P^\Omega_\omega(u) \in L^2_{s-1+(l+1)/N}(\Omega, \omega_0 d\lambda)$ finishing the proof. \square

Proof of the corollary of Theorem 2.1. It is enough to see that, if l_r is a positive integer such that $2l_r \geq r$ then, for any integer $k \geq l_r + 1$, for $u \in L^2_{k, T}(\Omega, \omega_0 d\lambda)$ we have $P^\Omega_\omega(u) \in L^2_{k-l_r}(\Omega)$. But this is a consequence of the theorem and of Theorem

1.1 of [CK03]: $P_\omega^\Omega(u) \in L_k^2(\Omega, \omega_0 d\lambda) \subset L_k^2(\Omega, \delta_{\partial\Omega}^{2l_r} d\lambda)$, $\delta_{\partial\Omega}$ being the distance to the boundary of Ω , and a harmonic function in $L_k^2(\Omega, \delta_{\partial\Omega}^{2l_r} d\lambda)$ is in $L_{k-l_r}^2(\Omega)$. \square

Remark.

- (1) As noted in [HM12], $\cap_{k \in \mathbb{N}} L_{k, T_{\varphi\rho_0}}^2(\Omega, \omega_0 d\lambda)$ is, in general, strictly larger than $C^\infty(\overline{\Omega})$.
- (2) In Remark 4.1 (2) of [CDMb] we notice that, if Ω is a smoothly bounded pseudoconvex domain in \mathbb{C}^n (not assumed of finite type) admitting a defining function ρ_1 pluri-subharmonic in Ω then, by a result of H. Boas and E. Straube ([BS91]) the weighted Bergman projection $P_{\omega_1}^\Omega$, $\omega_1 = (-\rho_1)^r$, maps continuously the Sobolev space $L_s^2(\Omega, \omega_1 d\lambda)$ into themselves. Then, using Theorem 1.1 of [HMS14], the proof of Lemma 4.1 shows that $P_{\omega_1}^\Omega$ maps continuously $L_{s, T_{\rho_1}}^2(\Omega, \omega_1 d\lambda)$ into $L_s^2(\Omega, \omega_1 d\lambda)$, $s \geq 0$. Moreover, the arguments of the proof of the above corollary show that $P_{\omega_1}^\Omega$ maps continuously $\cap_{k \in \mathbb{N}} L_{k, T_{\rho_1}}^2(\Omega, \omega_1 d\lambda)$ into $C^\infty(\overline{\Omega})$.
If ρ is another defining function of such a domain, we do not know if P_ν^Ω , $\nu = (-\rho)^r$, is $L_s^2(\Omega, \nu d\lambda)$ regular.

Proof of Theorem 2.2. It is very similar to the proof of Theorem 4.1. First, by Theorem 1.10 of [HMS14], $P_{\tilde{\omega}_0}^\Omega(I(\eta fh)) \in C^\infty(\overline{\Omega})$ so that $P_{\omega_0}^\Omega(\eta fh) \in C^\infty(\overline{\Omega})$. Then, by induction, the arguments used in the proof of Theorem 4.1 show that, for all non negative integer k , $P_\omega^\Omega(fh) \in L_k^2(\Omega, \omega d\lambda)$. Then, arguing as in the proof of the corollary of Theorem 4.1 we conclude that $P_\omega^\Omega(fh) \in L_{k-l_r}^2(\Omega)$ which completes the proof. \square

5. PROOF OF THEOREM 2.3 AND PROPOSITION 2.1

The proof, based on the formula of Proposition 3.2 and on estimates for solutions of the $\bar{\partial}$ -equation, is very similar to the one given in the previous section. As only the case of domains with rank of the Levi form $\geq n - 2$ is general (due to the restriction on the defining function for convex domains) we will give the proof with some details in this case and only indicate the steps for the convex case.

As in the preceding section we obtain the Sobolev estimates of Theorem 2.3 proving a stronger directional estimate:

Theorem 5.1. *Let Ω be as Theorem 2.3. Let ρ_0 , ω_0 and ω be as at the end of Section 2. Then:*

- (1) *Let k be a non negative integer. Assume $\eta \in C^{k+1}(\overline{\Omega})$. Then, for $1 < p < +\infty$ and $s \in [0, k]$ the weighted Bergman projection P_ω^Ω maps continuously the directional weighted Sobolev space $L_{s, T_{\rho_0}}^p(\Omega, \omega_0 d\lambda)$ into $L_s^p(\Omega, \omega_0 d\lambda)$.*
- (2) *Let $\alpha \leq 1$. Assume $\eta \in C^{[\alpha]+1}(\overline{\Omega})$. Then the weighted Bergman projection P_ω^Ω maps continuously the lipschitz space $\Lambda_\alpha(\Omega)$ into itself.*

We know ([CDMb]) that the Levi form of the domain $\tilde{\Omega}$ is locally diagonalizable at every point of $\partial\tilde{\Omega}$. Thus we use the estimates for the $\bar{\partial}$ -Neumann problem obtained by C. L. Fefferman, J. J. Kohn and M. Machedon in 1990 and by K. Koenig in 2004 for these domains:

Theorem 5.2 ([FKM90, Koe04]). *Let D be a smoothly bounded pseudoconvex domain in \mathbb{C}^n of finite type whose Levi form is locally diagonalizable at every boundary point. Then there exists a positive integer N such that:*

- (1) *For every $\alpha \geq 0$, $\bar{\partial}^* \mathcal{N}_D$ maps continuously the lipschitz space $\Lambda_\alpha(D)$ into $\Lambda_{\alpha+1/N}(D)$;*
- (2) *For $1 < p < +\infty$ and $s \geq 0$, $\bar{\partial}^* \mathcal{N}_D$ maps continuously the Sobolev space $L_s^p(D)$ into $L_{s+1/N}^p(D)$;*
- (3) *For $1 < p < +\infty$, $\bar{\partial}^* \mathcal{N}_D$ maps continuously $L^p(D)$ into $L^{p+1/N}(D)$;*
- (4) *For p sufficiently large $\bar{\partial}^* \mathcal{N}_D$ maps continuously $L^p(D)$ into $\Lambda_0(D)$.*

The first statement is explicitly stated in [FKM90], for N strictly larger than the type of D , for the $\bar{\partial}_b$ -Neumann problem at the boundary, and exactly stated in [Koe04] (Corollary 6.3, p. 286). In [Koe04] it is also proved that $\bar{\partial}^* \mathcal{N}_D$ maps continuously the Sobolev space $L_s^p(D)$ into $L_{s+1/m-\varepsilon}^p(D)$, where m is the type of D and $\varepsilon > 0$. Therefore the third and fourth statements of the theorem follow the Sobolev embedding theorems (see, for example, [AF03]).

We need also directional Sobolev estimates for the standard Bergman projection $P_{\tilde{\Omega}}$. Such estimates have been obtained for finite type domains in \mathbb{C}^2 by A. Bonami, D.-C. Chang and S. Grellier ([BCG96]) and by D.-C. Chang and B. Q. Li ([CL97]) in the case of decoupled domains of finite type in \mathbb{C}^n .

Following the proof of Lemma 3.4 of [CD06] but using the integral curve of the real normal to the boundary of D as in the proof of Theorem 4.2.1 of [BCG96], instead of a coordinate in a special coordinate system (also used in [MS94]), we easily write $\nabla^k P^D = \sum P_i^D T_d^i$ with “good” operators P_i^D and obtain the following estimate for P^D :

Theorem 5.3. *Let D be a smoothly bounded pseudoconvex domain of finite type in \mathbb{C}^n whose Levi form is locally diagonalizable at every point of ∂D .*

If d is a defining function of D , let $T_d = \sum_i \frac{\partial d}{\partial \bar{z}_i} \frac{\partial}{\partial z_i} - \frac{\partial d}{\partial z_i} \frac{\partial}{\partial \bar{z}_i}$. Then, for $1 < p < +\infty$ and $s \geq 0$, the Bergman projection P^D of D maps continuously the space $L_{s, T_d}^p(D)$ into $L_s^p(D)$.

Proof of Theorem 5.1. Let us first prove the weighted L^p regularity of P_ω^Ω . Let $u \in L^p(\Omega, \omega d\lambda)$. Assume for the moment $p > 2$. Let N_p be an integer such that $p-2/N_p < 1/N$ where N is the integer of Theorem 5.2 and let us prove, by induction over $l \in \{0, \dots, N_p\}$ that $P_\omega^\Omega(u) \in L^{2+l(p-2)/N_p}(\Omega, \omega d\lambda)$. Assume that $P_\omega^\Omega(u) \in L^{2+l(p-2)/N_p}(\Omega, \omega d\lambda)$ for $l < N_p$. Then Lemma 3.1 and Theorem 5.2 give $\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \circ I(P_\omega^\Omega(u) \bar{\partial} \eta) \in L^{p+(l+1)(p-2)/N_p}(\tilde{\Omega})$, and the second part of Lemma 3.1 gives the result. The L^p regularity of P_ω^Ω for $1 < p < 2$ is then obtained using the fact that P_ω^Ω is self-adjoint.

The Λ_α regularity is proved similarly. Suppose $u \in \Lambda_\alpha(\Omega)$. Then u belongs to all $L^p(\Omega, \omega d\lambda)$ spaces, $p < +\infty$, and, the $L^p(\Omega, \omega d\lambda)$ regularity of P_ω^Ω , Lemma 3.1 and the last assertion of Theorem 5.2 show that $\bar{\partial}^* \mathcal{N}_{\tilde{\Omega}} \circ I(P_\omega^\Omega(u) \bar{\partial} \eta) \in \Lambda_0(\tilde{\Omega})$ therefore $P_\omega^\Omega(u) \in \Lambda_0(\Omega)$. Then, using the first assertion of Theorem 5.2 it is easy to prove, by induction, that $P_\omega^\Omega(u) \in \Lambda_{l\alpha/N_\alpha}(\Omega)$, $l \in \{1, \dots, N_\alpha\}$, where N_α is a sufficiently large integer.

For the $L_s^p(\Omega, \omega d\lambda)$ regularity, we deduce from Theorem 5.3 the following extension of Lemma 4.1:

Lemma 5.1. $P_{\omega_0}^\Omega$ maps continuously the space $L_{s, T_{\rho_0}}^p(\Omega, \omega_0 d\lambda)$ into $L_s^p(\Omega, \omega_0 d\lambda)$.

As we already know that $P_{\omega_0}^\Omega$ maps $L^p(\Omega, \omega_0 d\lambda)$ into itself, the proof of the $L_{s, T_{\rho_0}}^p(\Omega, \omega_0 d\lambda)$ - $L_s^p(\Omega, \omega_0 d\lambda)$ regularity of $P_{\omega_0}^\Omega$ is identical to the end of the proof of Theorem 4.1. \square

For the convex domains considered in Proposition 2.1 the scheme of the proof is strictly identical.

For the L^p estimate we use the estimates for solutions of the $\bar{\partial}$ -equation given by A. Cumenge:

Theorem ([Cum01a, Cum01b]). *Let D be a smoothly bounded convex domain in \mathbb{C}^l of finite type τ_D . Then:*

- (1) *For $1 \leq p < \tau_D l + 2$ the restriction of $\bar{\partial}^* \mathcal{N}_D$ to $\bar{\partial}$ -closed $(0, 1)$ -forms maps continuously $L_{(0,1)}^p(D) \cap \ker \bar{\partial}$ into $L^s(D)$ with $1/s = 1/p - 1/\tau_D l + 2$;*
- (2) *For $\tau_D l + 2 < p \leq +\infty$, the restriction of $\bar{\partial}^* \mathcal{N}_D$ to $\bar{\partial}$ -closed $(0, 1)$ -forms maps continuously $L_{(0,1)}^p(D) \cap \ker \bar{\partial}$ into the lipschitz space $\Lambda_\alpha(D)$ with $\alpha = 1/\tau_D - (l+2/\tau_D)/p$.*

For the lipschitz estimate, as the type of $\tilde{\Omega}$ is larger than the type of Ω , we need a general lipschitz spaces estimate for the solutions of the $\bar{\partial}$ -equation. Using techniques developed in [MS94, MS97, CD06] and formulas introduced in [CDMa] the following result can be proved for lineally convex domains of finite type (as the detailed proof is long, technical and not new, we will not write it here):

Theorem. *Under the conditions of the preceding theorem, for $\alpha \geq 0$, the restriction of $\bar{\partial}^* \mathcal{N}_D$ to $\bar{\partial}$ -closed $(0, 1)$ -forms maps continuously the lipschitz spaces $\Lambda_\alpha(D)$ into $\Lambda_{\alpha+1/\tau_D}$.*

Finally for the Sobolev $L_s^p(\Omega, \omega d\lambda)$ estimate, using techniques similar to those used in the previous estimate it can be shown that, for $1 < p < +\infty$ and $s \geq 0$, the restriction of $\bar{\partial}^* \mathcal{N}_D$ to $\bar{\partial}$ -closed $(0, 1)$ -forms maps continuously $L_s^p(D)$ into $L_{s+1/\tau_D}^p(D)$.

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