Compensated fragmentation processes and limits of dilated fragmentations

Jean Bertoin

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Abstract

A new class of fragmentation-type random processes is introduced, in which, roughly speaking, the accumulation of small dislocations which would instantaneously shatter the mass into dust, is compensated by an adequate dilation of the components. An important feature of these compensated fragmentations is that the dislocation measure ν which governs their evolutions has only to fulfill the integral condition $\int_{\mathcal{P}}(1-p_1)^2\nu(\mathrm{d}\mathbf{p}) < \infty$, where $\mathbf{p} = (p_1, \ldots)$ denotes a generic mass-partition. This is weaker than the necessary and sufficient condition $\int_{\mathcal{P}}(1-p_1)\nu(\mathrm{d}\mathbf{p}) < \infty$ for ν to be the dislocation measure of a homogeneous fragmentation. Our main results show that such compensated fragmentations naturally arise as limits of homogeneous dilated fragmentations, and bear close connexions to spectrally negative Lévy processes.

Key words: Homogeneous fragmentation, dilation, compensation, dislocation measure.

Classification: 60F17; 60G51; 60G80.

1 Introduction

Fragmentation processes form a class of stochastic models taking values in the space of mass-partitions

$$\mathcal{P} = \left\{ \mathbf{p} = (p_1, p_2, \ldots) : p_1 \geq p_2 \geq \cdots \geq 0 \text{ and } \sum_{1}^{\infty} p_i \leq 1 \right\},$$

where $\mathbf{p}$ can be thought of as the ordered sequence of the masses of atoms of some probability measure, and then $1 - \sum_{i=1}^{\infty} p_i$ as the mass of the continuous component, i.e. the dust. They are meant to describe the evolution of some measurable set with unit mass which breaks into pieces, randomly and repeatedly as time passes. More precisely, we shall assume here the branching property, that is that distinct fragments evolve independently, and homogeneity, in the sense that if the size of each

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component in a fragmentation process starting from a single unit mass is rescaled by a factor $m > 0$, then the result has the same distribution as the process started from a single mass m.

It is well-known that homogeneous fragmentations bear close similarities and connexions with subordinators. Recall that the law of a subordinator is determined by its drift coefficient and its Lévy measure Π, which is a measure on $(0,\infty)$ such that

$$\int_{(0,\infty)} (1 \wedge x)\Pi(\mathrm{d}x) < \infty. \tag{1}$$

Further, the Lévy-Itō decomposition shows that the drift coefficient corresponds to the deterministic linear component of the subordinator, and that the Lévy measure Π describes the Poissonian intensity of the jumps. In turn, the law of a homogeneous fragmentation process is determined by its erosion coefficient and its dislocation measure ν, which is a measure on $\mathcal{P}$ that assigns no mass to the neutral mass-partition $\mathbf{1} = (1, 0, 0, \ldots)$ and fulfills the condition

$$\int_{\mathcal{P}} (1 - p_1)\nu(\mathrm{d}\mathbf{p}) < \infty. \tag{2}$$

The erosion coefficient determines the rate at which each fragment fades away continuously, and ν describes the statistics of the sudden dislocations in the process. More precisely, when ν is finite, the initial unit mass remains stable during an exponential time with parameter $\nu(\mathcal{P})$ and then splits, the distribution after the split being given by $\nu(\cdot)/\nu(\mathcal{P})$. Note however that (2) allows ν to be infinite. Furthermore, homogeneous fragmentations have a Poissonian construction, which is, in spirit, close to the Lévy-Itō decomposition of subordinators. See Section 3.1 in [9] and [5].

In the present work, we are mainly interested in the situation when condition (2) is not fulfilled. It is well-known that if one applies the Lévy-Itō construction for subordinators with a measure Π for which (1) fails, then one gets a process which explodes (i.e. jumps to $+\infty$) instantaneously. Nonetheless, if the Lévy measure fulfills the weaker condition

$$\int_{(0,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty, \tag{3}$$

then, informally, the explosion phenomenon can be prevented by a deterministic compensation. Specifically, consider for each integer n, a subordinator S_n with Lévy measure Π_n which fulfills (1), and, for simplicity, suppose S_n has no drift. Assume that as $n \to \infty$, there is the weak convergence

$$(1 \wedge x^2)\Pi_n(\mathrm{d}x) \Longrightarrow \sigma^2\delta_0(\mathrm{d}x) + (1 \wedge x^2)\Pi(\mathrm{d}x), \tag{4}$$

where $\sigma^2 \geq 0$ and Π is a measure on $(0,\infty)$ which fulfills (3). Then, if we set $b_n = \int_{(0,1)} x\Pi_n(\mathrm{d}x)$, there is the weak convergence

$$(b_n t - S_n(t))_{t\geq 0} \Longrightarrow (\xi(t))_{t\geq 0}, \tag{5}$$

say in the sense of finite dimensional distributions. The limit process ξ is a spectrally negative Lévy process, i.e. a process with independent and stationary increments which has only negative jumps.

Similarly, the Poissonian construction of homogeneous fragmentations for a dislocation measure ν such that (2) fails, would produce a process in which the entire mass is instantaneously reduced to dust (i.e. the process is immediately absorbed at the degenerate mass-partition $\mathbf{0} = (0, 0, \ldots)$). However, the similarities between subordinators and homogeneous fragmentations suggest that for the latter as well, an analogue of compensation, namely a deterministic dilation of fragments, might prevent instantaneous shattering to dust.

Specifically, we shall consider here a measure ν on $\mathcal{P}$ such that

$$\int_{\mathcal{P}} (1 - p_1)^2 \nu(\mathrm{d}\mathbf{p}) < \infty, \tag{6}$$

and a sequence ν_n of measures on $\mathcal{P}$ which fulfill (2). By analogy with (4), we assume that there is the weak convergence[1]

$$(1 - p_1)^2 \nu_n(\mathrm{d}\mathbf{p}) \Longrightarrow \sigma^2 \delta_{\mathbf{1}}(\mathrm{d}\mathbf{p}) + (1 - p_1)^2 \nu(\mathrm{d}\mathbf{p}) \tag{7}$$

where $\sigma^2 \geq 0$ and $\mathbf{1} = (1, 0, \ldots)$. Our main goal is to establish an analogue of (5); in this direction, we consider for each n a homogeneous fragmentation $\mathbf{X}_n$ with dislocation measure ν_n and no erosion. We shall show that one can choose adequately coefficients $c_n > 0$ and dilate each fragment of $\mathbf{X}_n$ at constant rate c_n so that

$$(\exp(c_n t)\mathbf{X}_n(t))_{t \geq 0} \Longrightarrow (\mathbf{Z}(t))_{t \geq 0}, \tag{8}$$

where $\mathbf{Z}$ is a Markov process with values in $\ell^2(\mathbb{N})$ which we shall call a compensated fragmentation. Before sketching the idea of our approach, let us briefly present a couple of natural obstructions which probably explain why this problem has not been addressed before, even though it looks very natural.

Firstly, the fundamental Lévy-Khintchine formula expresses the Laplace transform of a subordinator in terms of its drift coefficient and its Lévy measure, and the same holds for spectrally negative Lévy processes provided that one takes also into account the Gaussian coefficient. However, there is no analogue of the Lévy-Khintchine formula for homogeneous fragmentations, and this impedes the use of analytic techniques to establish the convergence of distributions via Fourier-Laplace transforms.

Secondly, even though the definition of homogeneous fragmentations with a finite dislocation measure is rather elementary (the first studies in this area go back to Kolmogorov [19] in 1941), the very construction in the case of an infinite dislocation measure with (2) is much more subtle. The latter relies on powerful techniques of sampling and exchangeability which enable to relate mass-partitions to exchangeable random partitions of $\mathbb{N}$ via Kingman's paintboxes (which also lies at the heart of Pitman's construction of Λ-coalescent processes, see [22]). Now this requires conservation of the total

[1] As usual, $\mathcal{P}$ is endowed with the supremum distance, and is then a compact metric space.

mass, taking into account the possible dust, whereas the total mass grows exponentially when dilation is incorporated to fragmentation. Sampling techniques applied to dilated fragmentations may yield strong laws for the empirical measure of the components (see, e.g. Theorem 1.2 in [9], and also Kolmogorov for the first result in this vein), but are inadequate to establish weak limits such as (8) or to construct general compensated fragmentations.

The starting point of our approach relies on the elementary observation that considering the logarithm of components in a homogeneous fragmentation with a finite dislocation measure yields a continuous-time branching random walk, and then, incorporating dilation simply amounts to letting each atom of the branching random walk drift at constant speed. Roughly speaking, we shall first discard the small masses resulting from small dislocation events, so that in such events, dislocations merely induce a loss of mass of the component without creation of new fragments. By modifying our point of view of the genealogy of atoms, we interpret the latter events as jumps of particles. In other words, this induces random spatial displacements of atoms between branching events, whereas in standard branching random walks, atoms stay still between consecutive branching events. We exploit the known convergence of compensated Poisson processes to spectrally negative Lévy processes and finally re-incorporate small dislocations events by considering projective limits. Technically, the crucial property that needs to be verified, is that the point process in the increasing limit has only finitely many atoms in half-lines $[x, \infty)$.

The present work can thus also be viewed as a contribution to the study of extremes of certain branching random walks, a topic in which important developments have been made recently (see in particular Aidekon [1], Arguin *et al.* [3], Aidekon *et al.* [2] and references therein). There is however a major difference with the works just cited, namely the latter deal with large time asymptotics for a given branching random walk, whereas we rather work at fixed times but consider a sequence of branching random walks with an increasing reproduction intensity.

In short, our main purpose in this article is to construct a new class of fragmentation-type processes, which we name compensated fragmentations, for which the dislocation measure ν fulfills (6) but not necessarily (2). Our main result shows that these compensated fragmentations arise as limits of rescaled homogeneous fragmentations as in (8). This opens the way to a number of interesting problems which have been considered previously only for usual fragmentation processes. To name just a few, we mention the connexion with deterministic fragmentation and growth-fragmentation equations (see, for instance, [4], [14] or [17] and references therein), the study of traveling waves and new versions of the Fisher-Kolmogorov-Petrowski-Piscounov (FKPP) equation (see, e.g., [7]), additive martingales and spine-decompositions for compensated fragmentations, asymptotic behavior for large times, extensions to self-similar fragmentations, etc. In a subsequent work, we shall also point out that compensated fragmentations arise naturally in dynamical percolation on certain families of random trees.

The rest of this article is organized as follows. In Section 2, we briefly explain how homogeneous dilated fragmentation processes can be viewed as branching compound Poisson processes with drift, and discuss a basic construction of the latter in terms of a process on the Ulam tree. In Section 3, we introduce more general branching Lévy processes, relying on a key embedding property which enables us to consider projective limits. Compensated fragmentations are introduced in Section 4; we shall establish there some important qualitative and quantitative properties. Finally, in Section 5, we study continuity properties of compensated fragmentations as a function of their characteristics; the convergence (8) then arises as a simple by-product.

2 Branching compound Poisson processes with drift

Throughout this section, we consider a homogeneous fragmentation $\mathbf{X} = (\mathbf{X}(t))_{t\geq 0}$ with a *finite* dislocation measure ν and no erosion. This means that each component $x > 0$ in $\mathbf{X}$ stays stable during an amount of time which is exponentially distributed with parameter $\nu(\mathcal{P})$ and then splits into $xm_1, xm_2, \ldots$, where $(m_1, m_2, \ldots)$ is a random mass-partition with law $\nu(\cdot)/\nu(\mathcal{P})$ and which is further independent of the waiting time. We implicitly assume that the process starts from a single unit mass, i.e. $\mathbf{X}(0) = \mathbf{1} = (1, 0, \ldots)$, and then for all $t \geq 0$, $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots) \in \mathcal{P}$. At the heart of our approach lies the connexion between homogeneous fragmentations and branching random walks; we refer to Bertoin and Rouault [11] for details.

When we assign a Dirac point mass at $\ln x$ for every component x of $\mathbf{X}(t)$, we obtain a random process of point measures on the negative half-line

$$\mathcal{X}_t = \sum_{i\in\mathbb{N}} \delta_{\ln X_i(t)}, \qquad t \geq 0, \tag{9}$$

which evolves as a branching random walk in continuous time, i.e. of the type considered first by Uchiyama [23]. That is, in $\mathcal{X}$, each atom $a = \ln x$ is replaced at rate $\nu(\mathcal{P})$ by a family of atoms $a + \gamma_i$, where the sequence $(\gamma_1, \gamma_2, \ldots)$ is distributed as the image of the normalized probability law $\nu(\cdot)/\nu(\mathcal{P})$ by the map $\mathbf{p} \mapsto \ln \mathbf{p} = (\ln p_1, \ln p_2, \ldots)$, and to different atoms correspond independent sequences. We stress that γ_i may take the value $-\infty$, and in that case, we implicitly agree that the atom $a + \gamma_i$ does not contribute to $\mathcal{X}$.

It will be convenient to formalize the construction of the branching random walk $\mathcal{X}$, and in this direction, we introduce the universal tree $\mathbb{U} = \bigcup_{i=0}^{\infty} \mathbb{N}^i$ with $\mathbb{N} = \{1, 2, \ldots\}$ and the usual convention $\mathbb{N}^0 = \{\emptyset\}$. A node $u \in \mathbb{U}$ is thus a finite sequence $u = (u_1, \ldots, u_i)$ of positive integers where $i = |u|$ is the generation of u, and the children of u are given by the nodes $uk = (u_1, \ldots, u_i, k)$ for $k \in \mathbb{N}$. We encode $\mathcal{X}$ by a process indexed by $\mathbb{U}$, $(\lambda_u, a_u)_{u\in\mathbb{U}}$, where λ_u corresponds the lifetime of the atom of $\mathcal{X}$ labelled by u and a_u its spatial location. The birth-time b_u and death-time d_u of this atom are thus given by

$$b_u = \sum_{j=0}^{|u|-1} \lambda_{(u_1,\dots,u_j)} \quad \text{and} \quad d_u = b_u + \lambda_u. \tag{10}$$

We further agree that for every $u \in \mathbb{U}$, the sequence $(a_{uj})_{j\in\mathbb{N}}$ is always ranked in the decreasing order, that is the j-th largest atom among the offsprings of the atom labelled by u receives the label uj; see Figure 1. We express the point measure at time t in the form

$$\mathcal{X}_t = \sum_{u\in\mathbb{U}} \mathbb{1}_{\{b_u \le t < d_u\}} \delta_{a_u}. \tag{11}$$

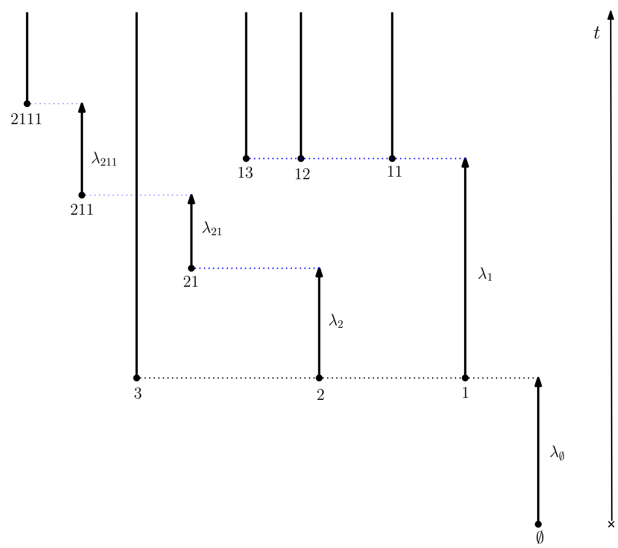

Figure 1 : Representation of a fragmentation as a process on $\mathbb{U}$.
Atoms • are labelled lexicographically and vertical arrows represent their lifetimes.
Horizontal dotted segments correspond to branching events.

It is convenient to introduce the space

$$\mathcal{R} = \{\mathbf{r} = (r_1, r_2, \dots) : r_i \in [-\infty, 0) \text{ and } r_1 \ge r_2 \ge \cdots\}$$

and write μ for the image of the dislocation measure ν by the map $\mathbf{p} \mapsto \ln \mathbf{p} = (\ln p_1, \ln p_2, \dots)$ from $\mathcal{P}$ to $\mathcal{R}$. The distribution of $(\lambda_u, a_u)_{u\in\mathbb{U}}$ induced by (9) and (11) for the homogeneous fragmentation $\mathbf{X}$ is simple to describe in terms of μ. The processes $(\lambda_u)_{u\in\mathbb{U}}$ and $(a_u)_{u\in\mathbb{U}}$ are independent, and the first consists of a family of i.i.d. exponential variables with parameter $\nu(\mathcal{P}) = \mu(\mathcal{R})$. Further, if we write $\Delta a_{uj} = a_{uj} - a_u$ for the displacement of the j-th child of u relative to its parent, then the sequences[2] $\mathbf{\Delta} a_u = (\Delta a_{u1}, \Delta a_{u2}, \ldots)$ for $u \in \mathbb{U}$ are i.i.d., each being distributed according to the normalized probability measure $\mu(\cdot)/\mu(\mathcal{R})$. Again, the displacement $|\Delta a_{uj}|$ may be infinite, in which case the atom corresponding to the node uj as well as all its descendants are not taken into account in the point process $\mathcal{X}$, see (11).

We next introduce a dilation coefficient $c \geq 0$, set

$$\mathbf{Y}(t) = \mathrm{e}^{ct}\mathbf{X}(t), \text{ i.e. } Y_i(t) = \mathrm{e}^{ct} X_i(t) \text{ for all } i \in \mathbb{N},$$

and consider the point measure on $\mathbb{R}$

$$\mathcal{Y}_t = \sum_{i \in \mathbb{N}} \delta_{\ln Y_i(t)}.$$

In other words, $\mathcal{Y}_t$ is simply the image of $\mathcal{X}_t$ by the linear shift $x \mapsto x + ct$, as dilation merely induces a linear motion with constant speed c for every atom of the branching random walk. Note also that a negative coefficient of dilation would precisely correspond to an erosion, cf. [5].

At the heart of our analysis lies the following representation of $\mathcal{Y}$ as a process indexed by $\mathbb{U}$. This representation differs slightly -but crucially- from the one for $\mathcal{X}$; in particular, a same node $u \in \mathbb{U}$ may label different atoms for $\mathcal{X}$ and $\mathcal{Y}$. The key point is that we now distinguish between two types of dislocation events, namely, those for which a component gives rise to a single fragment, and the others. We shall disregard the former as branching events for $\mathcal{Y}$ and rather view them as displacements of atoms.

Specifically, denote the number (possibly infinite) of components of a mass-partition $\mathbf{p} \in \mathcal{P}$ by

$$\#\mathbf{p} = \sup\{i \in \mathbb{N} : p_i > 0\},$$

and write $\mathcal{P}_1 = \{\mathbf{p} \in \mathcal{P} : \#\mathbf{p} = 1\}$, that is $\mathbf{p} \in \mathcal{P}_1$ if and only if $\mathbf{p} = (p, 0, \ldots)$ for some $p \in (0, 1]$. Similarly, we write $\mathcal{R}_1$ for the subspace of $\mathcal{R}$ consisting of sequences $\mathbf{r} = (r, -\infty, -\infty, \ldots)$ with $r \in (-\infty, 0]$, that is $\mathcal{R}_1$ is the image of $\mathcal{P}_1$ by the map $\mathbf{p} \mapsto \ln \mathbf{p}$. Rather than systematically interpreting a dislocation $y \mapsto y\mathbf{p}$ for the fragmentation-dilatation process $\mathbf{Y}$ as a branching event for the atom of $\mathcal{Y}$ located at $\ln y$, in the case when $\mathbf{p} \in \mathcal{P}_1$, we rather see such event as a jump from $\ln y$ to $\ln y + \ln p$ of the trajectory of this same atom. In terms of the genealogical tree of $\mathcal{Y}$, with spatial locations recorded and edge-lengths corresponding to the lifetime of atoms, this amounts to removing all the inner nodes with degree 2. See Figure 2 below.

It should be plain that with this point of view, each atom of $\mathcal{Y}$ now evolves during its lifetime as a *compound Poisson process* with constant linear drift c, where the intensity of the jumps is given by the measure μ restricted to sequences $\mathbf{r}$ in $\mathcal{R}_1$. The lifetime of each atom is exponentially distributed with parameter $\nu(\mathcal{P} \backslash \mathcal{P}_1) = \mu(\mathcal{R} \backslash \mathcal{R}_1)$, and at its death, this atom produces a random sequence of offsprings,

[2] Beware of the notation: for $|u| \geq 1$, $\mathbf{\Delta} a_u$ denotes a sequence of negative random variables, while Δa_u is a single random variable !

which is never a singleton. The relative locations of the offsprings with respect to the location of that atom when it dies, is distributed according to the conditional probability $\mu(\cdot \mid \mathcal{R}\backslash\mathcal{R}_1)$.

To formalize the discussion above, we shall encode $\mathcal{Y}$ as a process indexed by $\mathbb{U}$. For the sake of convenience, we shall use again the same notation λ_u and a_u to represent quantities which may however differ for $\mathcal{X}$ and $\mathcal{Y}$ (since our point of view on the genealogy has changed). Specifically, we now consider the process $(\lambda_u, a_u, \xi_u)_{u\in\mathbb{U}}$, where λ_u represents the lifetime of the atom of $\mathcal{Y}$ indexed by u, a_u its location at birth, and $\xi_u = (\xi_u(s))_{s\geq 0}$ a process which governs its displacements after its birth[3]. The birth-time b_u and the death-time d_u of this atom are still given by (10), and its spatial location at time $b_u + s$ by $a_u + \xi_u(s)$ for $0 \leq s < \lambda_u$, so that

$$\mathcal{Y}_t = \sum_{u\in\mathbb{U}} \mathbb{1}_{\{b_u\leq t<d_u\}} \delta_{a_u+\xi_u(t-b_u)}. \tag{12}$$

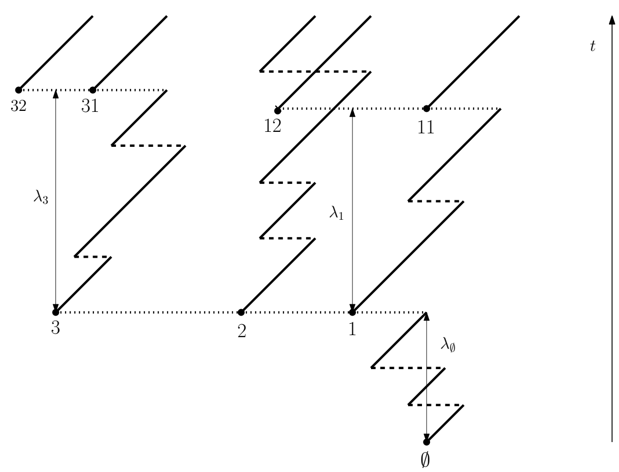

Figure 2 : Representation of a dilated fragmentation as a process on $\mathbb{U}$.
Birth locations of atoms are represented by • and are labelled lexicographically.
Oblique segments correspond to linear motion of atoms and horizontal dashed segments to jumps.
Horizontal dotted segments correspond to branching events.

The location at death of the atom labelled by u is $a_u + \xi_u(\lambda_u)$ (observe that, since ξ_u and λ_u are independent, ξ_u is continuous at time λ_u, a.s.). We further write

$$\Delta a_{ui} = a_{ui} - a_u - \xi_u(\lambda_u), \qquad i \in \mathbb{N}$$

for the initial displacement of the i-th child of u at its birth time, and set $\mathbf{\Delta} a_u = (\Delta a_{ui})_{i\in\mathbb{N}}$. Plainly,

[3]Formally, we only need to know ξ_u on the time interval $[0, \lambda_u)$; however it is convenient to consider ξ_u as a process on the whole time-interval $[0, \infty)$.

the process $(a_u)_{u\in\mathbb{U}}$ can be recovered from $(\lambda_u, \xi_u, \mathbf{\Delta} a_u)_{u\in\mathbb{U}}$, and then we recover $\mathbf{Y}(t)$ from $\mathcal{Y}_t$ by ranking in the decreasing order the exponentials of the atoms of $\mathcal{Y}_t$.

We now summarize this discussion.

Proposition 1 *Suppose $(\lambda_u)_{u\in\mathbb{U}}$, $(\xi_u)_{u\in\mathbb{U}}$ and $(\mathbf{\Delta} a)_{u\in\mathbb{U}}$ are three independent processes such that:*

- *$(\lambda_u)_{u\in\mathbb{U}}$ is a family of i.i.d. exponential variables with parameter $\nu(\mathcal{P}\backslash\mathcal{P}_1)$.*
- *$(\xi_u)_{u\in\mathbb{U}}$ is a family of i.i.d. compound Poisson processes with drift c and Lévy measure given by the restriction $\mu_{|\mathcal{R}_1}$ of μ to $\mathcal{R}_1$.*
- *$(\mathbf{\Delta} a_u)_{u\in\mathbb{U}}$ is a family of i.i.d. sequences, each sequence being distributed according to the conditional probability $\mu(\cdot \mid \mathcal{R}\backslash\mathcal{R}_1)$.*

The atoms of the random point measure $\mathcal{Y}_t$ defined by (12)*, repeated according to their multiplicity, can be ranked in the decreasing order, say, $\ln Y_1(t) \geq \ln Y_2(t) \geq \ldots$, and the process $\mathbf{Y}(t) = (Y_1(t), Y_2(t), \ldots)$ is then a homogeneous dilated fragmentation with dislocation measure ν and dilation coefficient c.*

We mention that a related description of a growth-fragmentation model in which divisions are always binary but where the division rates may depend on the size of the particle, appears in [15].

We conclude this section by introducing the notion of the *selected* fragment, which will be a useful guideline for the intuition. Specifically, imagine that at the first dislocation event, we select the largest component of $\mathbf{Y}$ which results from the dislocation, and so on for the next dislocation events. We denote the size of this selected fragment at time t by $Y_*(t)$. Beware that this quantity may of course differ from $Y_1(t)$, the largest of all the components of $\mathbf{Y}(t)$. The selected fragment can be expressed in terms of the process $(\lambda_u, a_u, \xi_u)_{u\in\mathbb{U}}$ restricted to its oldest branch, namely the branch with nodes $\emptyset, (1), (1,1), (1,1,1), \ldots$. More precisely, we have

$$Y_*(t) = \exp(a_{u(t)} + \xi_{u(t)}(t - b_{u(t)}))$$

where $u(t)$ stands for the unique node on the oldest branch such that $b_u \leq t < d_u$.

The elementary dynamics of fragmentations with a finite dislocation measure readily yield the following.

Lemma 1 *The process $\ln Y_*$ is a compound Poisson process with drift c and its Lévy measure is given by the image of the dislocation measure ν by the map $\mathbf{p} \mapsto \ln p_1$. In particular*

$$\mathbb{E}\left(Y_*(t)^q\right) = \exp\left(t\left(cq - \int_{\mathcal{P}} (1 - p_1^q)\nu(\mathrm{d}\mathbf{p})\right)\right), \qquad q \geq 0.$$

3 Branching Lévy processes

Roughly speaking, we have seen in the preceding section that homogeneous dilated fragmentation processes with a finite dislocation measure can be described in terms of certain branching compound Poisson processes with drift. The purpose of this section is to construct more generally a fairly general class of branching Lévy processes. This will be used in the next section to define general compensated fragmentations. Informally, branching Lévy processes can be thought of as non-interacting particle systems in which particles move in $\mathbb{R}$ according to the dynamics of some Lévy process, and branch. The two difficulties to be overcome are that branching rates could be infinite, and that displacements and branching events are correlated. We stress that in the literature on branching Markov processes, either the branching rate is finite (e.g. for branching random walks) or spatial motions and reproduction events are independent (e.g. for Dawson-Watanabe superprocesses).

As a starting point, consider first the situation of a homogeneous dilated fragmentation $\mathbf{Y}$ with a finite dislocation measure ν and dilation coefficient c. We first rewrite the formula for the q-th moment of the selected fragment $Y_*(t)$ in Lemma 1 as

$$\mathbb{E}\left(Y_*(t)^q\right) = \exp\left(t\left(c'q + \int_{\mathcal{P}} (p_1^q - 1 + q(1-p_1))\nu(\mathrm{d}\mathbf{p})\right)\right),$$

with $c' = c - \int_{\mathcal{P}}(1-p_1)\nu(\mathrm{d}\mathbf{p})$, and point out that the integral above remains finite for $q \geq 1$ even when ν is infinite, provided that (6) holds. In the same direction, note that the Laplace transform of the compound Poisson process with drift $\xi = \xi_u$ occurring in Proposition 1 is given by the Lévy-Khintchine formula

$$\mathbb{E}\left(\exp(q\xi(t))\right) = \exp\left(t\left(c''q + \int_{\mathcal{P}_1} (p_1^q - 1 + q(1-p_1))\nu(\mathrm{d}\mathbf{p})\right)\right) \tag{13}$$

(we stress that the integral is taken over $\mathcal{P}_1$ rather than $\mathcal{P}$ in the formula for the selected fragment), with

$$c'' = c - \int_{\mathcal{P}_1} (1-p_1)\nu(\mathrm{d}\mathbf{p}) = c' + \int_{\mathcal{P}\backslash\mathcal{P}_1} (1-p_1)\nu(\mathrm{d}\mathbf{p}).$$

Roughly speaking, this suggests that we can replace the compound Poisson process with drift ξ in Proposition 1 by a more general Lévy process with no positive jumps. As we shall focus on branching measure-valued processes in this section, rather than considering a measure ν on $\mathcal{P}$, we shall work directly with a measure μ on $\mathcal{R}$ (we may think that μ is the image of ν by the map $\mathbf{p} \mapsto \ln \mathbf{p}$, however this would induce some unnecessary restriction). We shall always assume that

$$\int_{\mathcal{R}} (1 - \mathrm{e}^{r_1})^2 \,\mu(\mathrm{d}\mathbf{r}) < \infty\,, \tag{14}$$

which is the analog of (6), and, in a first step, we further impose

$$\mu(\mathcal{R}\backslash\mathcal{R}_1) < \infty. \tag{15}$$

Definition 1 *Let $\sigma^2 \geq 0$, $c \in \mathbb{R}$ and μ a measure on $\mathcal{R}$ which fulfills (14) and (15). Consider three independent processes $(\lambda_u)_{u\in\mathbb{U}}$, $(\xi_u)_{u\in\mathbb{U}}$ and $(\mathbf{\Delta}a)_{u\in\mathbb{U}}$ such that:*

- *$(\lambda_u)_{u\in\mathbb{U}}$ is a family of i.i.d. exponential variables with parameter $\mu(\mathcal{R}\backslash\mathcal{R}_1)$.*
- *$(\xi_u)_{u\in\mathbb{U}}$ is a family of i.i.d. spectrally negative Lévy processes with Laplace exponent*

$$\Psi(q) = \frac{1}{2}\sigma^2 q^2 + \left(c + \int_{\mathcal{R}\backslash\mathcal{R}_1} (1 - \mathrm{e}^{r_1})\mu(\mathrm{d}\mathbf{r})\right) q + \int_{\mathcal{R}_1} (\mathrm{e}^{qr_1} - 1 + q(1 - \mathrm{e}^{r_1}))\mu(\mathrm{d}\mathbf{r}), \tag{16}$$

that is

$$\mathbb{E}\left(\exp(q\xi_u(t))\right) = \exp(t\Psi(q)), \qquad \textit{for all } t \geq 0, q \geq 0 \textit{ and } u \in \mathbb{U}.$$

- *$(\mathbf{\Delta}a_u)_{u\in\mathbb{U}}$ is a family of i.i.d. sequences, each sequence being distributed according to the conditional probability $\mu(\cdot \mid \mathcal{R}\backslash\mathcal{R}_1)$.*

Define the birth-time b_u and the death-time d_u by (10), $a_\emptyset = 0$, and then iteratively for every $u \in \mathbb{U}$ and $i \in \mathbb{N}$

$$a_{ui} = a_u + \xi_u(\lambda_u) + \Delta a_{ui}.$$

Finally, introduce for every $t \geq 0$ the point measure on $\mathbb{R}$

$$\mathcal{Z}_t = \sum_{u\in\mathbb{U}} \mathbb{1}_{\{b_u \leq t < d_u\}} \delta_{a_u + \xi_u(t-b_u)}.$$

We call the process $\mathcal{Z} = (\mathcal{Z}_t)_{t\geq 0}$ a branching Lévy process with characteristics (σ^2, c, μ).

Remarks. 1. The formula (16) for the Laplace exponent of ξ has its root in the expression (13) [beware that c in (16) plays the role of c' in (13)]. This explains the rather awkward expression for the drift coefficient .

2. In the special case $\sigma^2 = 0$, $\mu(\mathcal{R}) < \infty$ and $c = -\int_{\mathcal{R}}(1 - \mathrm{e}^{r_1})\mu(\mathrm{d}\mathbf{r})$, $\mathcal{Z}$ is merely a branching random walk in continuous time, i.e. atoms stand still between consecutive branching events.

In words, given a measure μ on $\mathcal{R}$ with $\mu(\mathcal{R}\backslash\mathcal{R}_1) < \infty$, the atoms of a branching Lévy process with characteristics (σ^2, c, μ) form a non-interacting particle system in which each atom moves in $\mathbb{R}$ with the dynamics of a spectrally negative Lévy process with Laplace exponent Ψ given by (16), independently of the other atoms. Further, each atom dies at rate $\mu(\mathcal{R}\backslash\mathcal{R}_1)$ and, at the instant of its death, gives birth to a random sequence of children whose spatial position relative to the parent is distributed according to the conditional probability $\mu(\cdot \mid \mathcal{R}\backslash\mathcal{R}_1)$. One deduces from standard arguments (see, for instance Equation (4) in [10]) that the one-dimensional distributions of $\mathcal{Z}$ can be characterized as follows.

For a given measurable function $f : [-\infty, \infty) \to (0, 1]$ with $f(-\infty) = 1$, define first

$$v_t(z) = \mathbb{E}\left(\exp\langle \mathcal{Z}_t, \ln f(z + \cdot)\rangle\right),$$

with the classical notation $\langle \mathcal{Z}_t, g(z+\cdot)\rangle = \int g(z+r)\mathcal{Z}_t(\mathrm{d}r)$. Then there is the identity

$$v_t(z) = \mathcal{E}\left(f(\xi(t)+z) + \int_0^t \mathrm{d}s \int_{\mathcal{R}\backslash\mathcal{R}_1} \mu(\mathrm{d}\mathbf{r}) \left(\prod_{i=1}^{\infty} v_{t-s}(r_i + \xi(s) + z) - v_{t-s}(\xi(s)+z)\right)\right), \tag{17}$$

where $\mathcal{E}$ refers to the mathematical expectation with respect to the spectrally negative Lévy process ξ that appears in Definition 1. It is seen from Gronwall's lemma that this equation has a unique solution and thus this determines the law of $\mathcal{Z}$.

When we compare the construction of the branching Lévy process $\mathcal{Z}$ in Definition 1 with that of a branching random walk in continuous time in Section 2, we realize that $\mathcal{Z}$ can simply be obtained by superposing spatial Lévy motions to the atoms of the latter. More precisely, let μ be a measure on $\mathcal{R}$ which fulfills (14) and (15). Consider $\mathcal{X}$, a branching random walk in continuous time on $\mathbb{R}_-$, such that $\mathcal{X}$ is started from a single atom located at 0 which dies after an exponential time with parameter $\mu(\mathcal{R}\backslash\mathcal{R}_1)$, and then gives birth instantaneously to a sequence of particles distributed according to the conditional law $\mu(\cdot \mid \mathcal{R}\backslash\mathcal{R}_1)$. Let also $\mathcal{Z}$ denote a branching Lévy process with characteristics (σ^2, c, μ). Fix a time $t \geq 0$ and write $(\alpha_i)_{i\in I}$ for the atoms of $\mathcal{X}$ at time t repeated according to their multiplicity, i.e.

$$\mathcal{X}_t = \sum_{i\in I} \delta_{\alpha_i}.$$

The following statement is tailored for our future purpose.

Lemma 2 *In the notation above, there exists a family of real valued random variables $(\beta_i)_{i\in I}$ such that the random point measure*

$$\sum_{i\in I} \delta_{\alpha_i+\beta_i}$$

has the same law as $\mathcal{Z}_t$, and conditionally on $\mathcal{X}$, each β_i has Laplace transform

$$\mathbb{E}(\exp(q\beta_i)) = \exp(t\Psi(q)), \qquad q \geq 0$$

with Ψ given by (16).

Proof: Write $\partial\mathbb{U}$ for the boundary of $\mathbb{U}$, that is a leaf $\bar{u} \in \partial\mathbb{U}$ is an infinite sequence $(u_1, \ldots)$ of positive integers. For every $j \in \mathbb{N}$, write $\bar{u}_j = (u_1, \ldots, u_j)$ for the ancestor of $\bar{u}$ at generation

j. In the framework of Definition 1, write $\xi_{\bar{u}}$ for the process obtained by concatenating the paths $(\xi_{\bar{u}_j}(s) : 0 \le s < \lambda_{\bar{u}_j})$ for $j = 0, 1, \cdots$. It follows from the simple Markov property of Lévy processes that each $\xi_{\bar{u}}$ is a spectrally negative Lévy process with Laplace exponent Ψ. For each $\bar{u} \in \partial\mathbb{U}$, introduce also the compound Poisson process $\eta_{\bar{u}}$ which makes a jump of size $\Delta a_{\bar{u}_j}$ at time $\lambda_\emptyset + \sum_{i=1}^{j} \lambda_{\bar{u}_i}$ for every $j \ge 0$.

We next equip the edges of $\mathbb{U}$ with random lengths, λ_u being the length of the edge connecting the node u to any of its children uj. We then cut this tree at height t, denote generically by $\Lambda \subseteq \partial\mathbb{U}$ a subset of leaves which stem from a cut-point, and write $\mathcal{L}$ for the family of such subsets Λ (so that the $\Lambda \in \mathcal{L}$ forms a partition of $\partial\mathbb{U}$). Clearly, the values $\xi_{\bar{u}}(t)$ (respectively, $\eta_{\bar{u}}(t)$) are the same for all $\bar{u} \in \Lambda$, and we can thus define unambiguously

$$\xi_\Lambda(t) = \xi_{\bar{u}}(t) \text{ and } \eta_\Lambda(t) = \eta_{\bar{u}}(t), \qquad \bar{u} \in \Lambda.$$

With this notations at hand, we have from the very construction of $\mathcal{X}$ and $\mathcal{Z}$ the expressions

$$\mathcal{X}(t) = \sum_{\Lambda \in \mathcal{L}} \delta_{\eta_\Lambda(t)} \text{ and } \mathcal{Z}(t) = \sum_{\Lambda \in \mathcal{L}} \delta_{\eta_\Lambda(t) + \xi_\Lambda(t)}$$

and our claim is justified (with $\alpha_\Lambda = \eta_\Lambda(t)$ and $\beta_\Lambda = \xi_\Lambda(t)$). $\square$

Our next purpose is to get rid of the condition $\mu(\mathcal{R} \backslash \mathcal{R}_1) < \infty$ in Definition 1; this requires a monotonicity argument that we now develop (this is somehow reminiscent of the construction of Lévy trees by Duquesne and Winkel [16] by growing a family of Galton-Watson trees which is consistent under percolation). For every $b \ge 0$ and $r \in [-\infty, 0]$, we set

$$r^{(b)} = \begin{cases} r & \text{if } r > -b \\ -\infty & \text{otherwise.} \end{cases}$$

For $\mathbf{r} = (r_1, r_2, \ldots) \in \mathcal{R}$, we write $\mathbf{r}^{(b)} = (r_1, r_2^{(b)}, r_3^{(b)}, \ldots)$, i.e. $\mathbf{r}^{(b)}$ results from $\mathbf{r}$ by keeping the first element r_1 unchanged, and then replacing the next elements smaller than or equal to $-b$ by $-\infty$.

We still consider a measure μ on $\mathcal{R}$ which fulfills (14) and such that $\mu(\mathcal{R} \backslash \mathcal{R}_1) < \infty$. For every $b \ge 0$, we write $\mu^{(b)}$ for the image of μ by the map $\mathbf{r} \mapsto \mathbf{r}^{(b)}$; plainly $\mu^{(b)}$ also fulfills (14). We shall now show that by suppressing adequately certain atoms of $\mathcal{Z}$ together with their offsprings, we can construct a branching Lévy process $\mathcal{Z}^{(b)}$ with characteristics $(\sigma^2, c, \mu^{(b)})$.

Let $(\lambda_u, \xi_u, \mathbf{\Delta} a_u)_{u \in \mathbb{U}}$ be as in Definition 1. For every node $u = (u_1, \ldots, u_i)$ at generation $i \ge 1$, write $u \in B(b)$ and think of the node u as being b-bad if and only if

$$\Delta a_{(u_1, \ldots, u_j)} \le -b \text{ and } u_j \ge 2 \qquad \text{for some } j \in \{1, \ldots, i\}.$$

Lemma 3 *The process of point measures*

$$\mathcal{Z}_t^{(b)} = \sum_{u \in \mathbb{U}} \mathbb{1}_{\{u \notin B(b)\}} \mathbb{1}_{\{b_u \le t < d_u\}} \delta_{a_u + \xi_u(t - b_u)}, \qquad t \ge 0$$

is a version of a branching Lévy process with characteristics $(\sigma^2, c, \mu^{(b)})$.

Proof: Roughly speaking, one obtains $\mathcal{Z}^{(b)}$ from $\mathcal{Z}$ by keeping systematically at each birth event the child particle (if any) which is the closest to its parent, and by suppressing the other child particles which are born at distance $\ge b$ from their parent, together with their offsprings. By applications of the branching property of $\mathcal{Z}$, it is elementary (though somewhat burdensome) to check that $\mathcal{Z}^{(b)}$ also fulfills the formalism of Definition 1 for some parameters that we shall now specify by analyzing the preceding transformation.

Recall that in $\mathcal{Z}$, a single particle starts from 0 at the initial time and evolves in $\mathbb{R}$ according to a spectrally negative Lévy process $\xi_\emptyset$ with Laplace exponent Ψ given by (16). After a random time $\lambda_\emptyset$ with the exponential distribution with parameter $\mu(\mathcal{R} \backslash \mathcal{R}_1)$, this particle dies at location $\xi_\emptyset(\lambda_\emptyset)$ and is replaced by a family of particles located at $\xi_\emptyset(\lambda_\emptyset) + r_i$, where $(r_1, \ldots) = \mathbf{r} = \boldsymbol{\Delta} a_\emptyset$ is independent of $\lambda_\emptyset$ and distributed according to $\mu(\cdot \mid \mathcal{R} \backslash \mathcal{R}_1)$.

Consider the event $\Lambda_b = \{\mathbf{r}^{(b)} \notin \mathcal{R}_1\}$, and observe that if $\mathbf{r}^{(b)} \notin \mathcal{R}_1$, then *a fortiori* $\mathbf{r} \notin \mathcal{R}_1$. Hence there is the identity

$$\mathbb{P}(\Lambda_b) = \frac{\mu^{(b)}(\mathcal{R} \backslash \mathcal{R}_1)}{\mu(\mathcal{R} \backslash \mathcal{R}_1)}.$$

On the event Λ_b, the lifetime $\lambda_\emptyset^{(b)}$ of the ancestor particle of $\mathcal{Z}^{(b)}$ coincides with $\lambda_\emptyset$. At its death, it is replaced by a family of particles located at $\xi_\emptyset(\lambda_\emptyset) + \mathbf{r}^{(b)}$. Plainly, the law of $\mathbf{r}^{(b)}$ conditionally on Λ_b is $\mu^{(b)}(\cdot \mid \mathcal{R} \backslash \mathcal{R}_1)$.

On the complementary event Λ_b^{c}, $\lambda_\emptyset$ corresponds to a jump time of the ancestor particle of $\mathcal{Z}^{(b)}$; more precisely, on the time-interval $[\lambda_\emptyset, \lambda_\emptyset + \lambda_1)$, the ancestor particle of $\mathcal{Z}^{(b)}$ is given by the particle labelled by the node 1 in $\mathcal{Z}$. In particular, the jump at time $\lambda_\emptyset$ is given by r_1, and has the law

$$\mu(r_1 \in \mathrm{d}r, \mathbf{r}^{(b)} \in \mathcal{R}_1, \mathbf{r} \notin \mathcal{R}_1) / \mu(\mathcal{R} \backslash \mathcal{R}_1), \qquad r \in (-\infty, 0).$$

We then iterate this reasoning at the lifetime of the particles labelled by $1, 11, \ldots$ (i.e. on the oldest branch) in the process $\mathcal{Z}$ until reaching the death-time of the ancestor particle of $\mathcal{Z}^{(b)}$. The number of steps has the geometric distribution with parameter $1 - \mathbb{P}(\Lambda_b)$.

From basic properties of independent exponential variables, we now see that the lifetime $\lambda_\emptyset^{(b)}$ of the ancestor particle of $\mathcal{Z}^{(b)}$ has the exponential distribution with parameter

$$\mu(\mathcal{R} \backslash \mathcal{R}_1) \times \mathbb{P}(\Lambda_b) = \mu^{(b)}(\mathcal{R} \backslash \mathcal{R}_1).$$

In the notation of the proof of Lemma 2, the displacement of this particle during its lifetime is governed by the superposition of the spectrally negative Lévy process $\xi_{\bar{u}}$ for $\bar{u} = (1, 1, 1, \cdots) \in \partial\mathbb{U}$ the oldest leaf, and an independent compound Poisson process on $(-\infty, 0)$ with Lévy measure

$$\mu(r_1 \in \mathrm{d}r, \mathbf{r}^{(b)} \in \mathcal{R}_1, \mathbf{r} \notin \mathcal{R}_1), \qquad r \in (-\infty, 0).$$

This is thus a spectrally negative Lévy process with Laplace exponent

$$\begin{aligned}\Psi^{(b)}(q) &= \Psi(q) + \int (\mathrm{e}^{qr} - 1)\mu(r_1 \in \mathrm{d}r, \mathbf{r}^{(b)} \in \mathcal{R}_1, \mathbf{r} \notin \mathcal{R}_1) \\ &= \frac{1}{2}\sigma^2 q^2 + \left(c + \int_{\mathcal{R}\backslash\mathcal{R}_1} (1 - \mathrm{e}^{r_1})\mu^{(b)}(\mathrm{d}\mathbf{r})\right) q + \int_{\mathcal{R}_1} (\mathrm{e}^{qr_1} - 1 + q(1 - \mathrm{e}^{r_1}))\mu^{(b)}(\mathrm{d}\mathbf{r}),\end{aligned}$$

where for the second line, we used the facts that μ and $\mu^{(b)}$ have the same projection on the first coordinate and that $\mathbf{r} \in \mathcal{R}_1$ implies $\mathbf{r}^{(b)} \in \mathcal{R}_1$.

Comparing with (16) and putting the pieces together, this completes the proof of our statement. □

We now assume that the measure μ fulfills (14) and replace (15) by the weaker

$$\mu^{(b)}(\mathcal{R}\backslash\mathcal{R}_1) < \infty \qquad \text{for all } b > 0. \tag{18}$$

Plainly, for every $b \geq b' \geq 0$ and $\mathbf{r} \in \mathcal{R}$, we have $(\mathbf{r}^{(b)})^{(b')} = \mathbf{r}^{(b')}$, and it follows that $(\mu^{(b)})^{(b')} = \mu^{(b')}$. By Lemma 3 and Kolmogorov's extension theorem, we can construct on the same probability space a sequence of processes of point measures, which for simplicity we still denote by $\left(\mathcal{Z}^{(b)}\right)_{b\geq 0}$, such that each $\mathcal{Z}^{(b)}$ is a branching Lévy process with characteristics $(\sigma^2, c, \mu^{(b)})$ and

$$\left(\mathcal{Z}^{(b)}\right)^{(b')} = \mathcal{Z}^{(b')} \qquad \text{for every } b' \leq b. \tag{19}$$

Definition 2 *Let $\sigma^2 \geq 0$, $c \in \mathbb{R}$ and μ be a measure on $\mathcal{R}$ which fulfills* (14) *and* (18)*. In the notation above, the process*

$$\mathcal{Z}_t := \lim_{b\to\infty} \uparrow \mathcal{Z}_t^{(b)}, \qquad t \geq 0$$

is called a branching Lévy process with characteristics (σ^2, c, μ).

We stress that, by the lack of memory of exponential variables and the Markov property of Lévy processes, a skeleton $(\mathcal{Z}_{tn})_{n\geq 0}$ of a branching Lévy process $\mathcal{Z}$ as constructed in Definition 1 (where $t > 0$ is arbitrary) is a branching random walk in discrete-time. By monotonicity, this feature also holds for the more general branching Lévy processes of Definition 2. However, $\mathcal{Z}$ itself is in general not a branching random walk in continuous time in the sense of Uchiyama [23].

In the special case $b = 0$, we have $\mathbf{r}^{(0)} = (r_1, -\infty, \ldots)$ and, in the notation for Lemma 3, $\mathbb{U}\backslash B(0)$

coincides with the oldest branch. For every branching Lévy process with characteristics (σ^2, c, μ) in the sense of Definition 2 and every $t \geq 0$, the point measure $\mathcal{Z}_t^{(0)}$ has thus at most one atom in $(-\infty, \infty)$, say $\xi_*(t)$ with the convention that $\xi_*(t) = -\infty$ when $\mathcal{Z}_t^{(0)} = 0$. The process ξ_* corresponds to the selected atom of $\mathcal{Z}$, in the sense that at each birth-event, we select the oldest (i.e. right-most) child. Its law is readily described, extending Lemma 1 in the case $\sigma^2 = 0$ and $\mu(\mathcal{R}) < \infty$.

Corollary 1 *The process of the selected atom ξ_* is a spectrally negative Lévy process, possibly killed at some independent exponential time with parameter $\mu(\{\varnothing\})$, where $\varnothing = (-\infty, -\infty, \dots) \in \mathcal{R}$. More precisely its Laplace exponent is given in terms of μ by*

$$\Psi_*(q) = \frac{1}{2}\sigma^2 q^2 + cq + \int_{\mathcal{R}} (\mathrm{e}^{qr_1} - 1 + q(1 - \mathrm{e}^{r_1}))\mu(\mathrm{d}\mathbf{r}), \qquad q \geq 0, \tag{20}$$

i.e.

$$\mathbb{E}\left(\exp(q\xi_*(t))\right) = \exp(t\Psi_*(q)), \qquad q \geq 0,$$

with the convention that $\exp(-q\infty) = 0$.

Proof: From Lemma 3 and Definition 1, we get indeed that ξ_* is distributed as a spectrally negative Lévy process ξ with Laplace exponent given by (16) with $\mu^{(0)}$ replacing μ, and killed at some independent exponential time with parameter $\mu(\{\varnothing\})$. Since $\mu^{(0)}$ is supported by $\mathcal{R}_1 \cup \{\varnothing\}$, we find that the Laplace exponent Ψ_* of ξ_* can be expressed in the form

$$\Psi_*(q) = \mu(\{\varnothing\}) + \frac{1}{2}\sigma^2 q^2 + (c + \mu(\{\varnothing\}))\, q + \int_{\mathcal{R}\setminus\{\varnothing\}} (\mathrm{e}^{qr_1} - 1 + q(1 - \mathrm{e}^{r_1}))\mu(\mathrm{d}\mathbf{r}),$$

which yields our claim. □

Lemma 3 also shows that the selected atom plays the rôle of a *spine* (or backbone) for the branching Lévy process, in the sense that, roughly speaking, one recovers $\mathcal{Z}$ from ξ_* by grafting adequately shifted independent copies of $\mathcal{Z}$ to the trajectory of the selected atom ξ_* (obviously, one needs to re-incorporate the atoms which have been suppressed in the transformation $\mathcal{Z} \mapsto \mathcal{Z}^{(0)}$). Techniques of spine-decompositions have been introduced in the pioneer work of Lyons *et al.* [21] and have played since then a fundamental role in the study of branching processes. Applications of this spine-decomposition to branching Lévy processes will be developed in a subsequent work.

4 Compensated fragmentations

In this section, we construct general compensated fragmentations from branching Lévy processes. Specifically, we consider a measure ν on $\mathcal{P}$ which fulfills (6) and write as usual μ for the image of ν

by the map $\mathbf{p} \mapsto \ln \mathbf{p}$. Then, μ satisfies (14) and also (18) since

$$\mu^{(b)}(\mathcal{R}\backslash\mathcal{R}_1) = \mu(\mathbf{r}^{(b)} \notin \mathcal{R}_1) = \nu(\ln p_1 = -\infty \text{ or } \ln p_2 > -b) \leq \nu(p_1 < 1 - \mathrm{e}^{-b}) < \infty.$$

Hence we may consider a branching Lévy process $\mathcal{Z}$ with characteristics (σ^2, c, μ).

For every $q \geq 2$, we set

$$\kappa(q) = \frac{1}{2}\sigma^2 q^2 + cq + \int_{\mathcal{P}} \left(\sum_{i=1}^{\infty} p_i^q - 1 + q(1 - p_1) \right) \nu(\mathrm{d}\mathbf{p}) \tag{21}$$

and observe that $\kappa(q)$ is well defined since for every $\mathbf{p} \in \mathcal{P}$,

$$|p_1^q - 1 + q(1-p_1)| = O((1-p_1)^2) \quad \text{and} \quad \sum_{i=2}^{\infty} p_i^q \leq (1-p_1)^q.$$

The next statement is a cornerstone of this work; it will be convenient to use the notation

$$\langle \mathcal{Z}_t, \mathrm{e}^{qz} \rangle = \int_{\mathbb{R}} \mathrm{e}^{qz} \mathcal{Z}_t(\mathrm{d}z).$$

Theorem 1 *For every $q \geq 2$ and $t \geq 0$, we have*

$$\mathbb{E}\left(\langle \mathcal{Z}_t, \mathrm{e}^{qz} \rangle\right) = \exp(t\kappa(q)).$$

Proof: We first assume that μ fulfills (15). The restriction of ν to $\mathcal{P}\backslash\mathcal{P}_1$ is then a finite measure, and we consider a homogeneous fragmentation $\mathbf{X}$ with no erosion and dislocation measure $\nu_{|\mathcal{P}\backslash\mathcal{P}_1}$. We write $\mathcal{X}$ for the branching random walk associated to $\mathbf{X}$ by (9). According to Equation (7) in [11], one has for every $t \geq 0$ and $q \geq 1$

$$\mathbb{E}\left(\sum_{j=1}^{\infty} X_j^q(t) \right) = \mathbb{E}(\langle \mathcal{X}_t, \mathrm{e}^{qx} \rangle) = \exp\left(t \int_{\mathcal{P}\backslash\mathcal{P}_1} \left(\sum_{j=1}^{\infty} p_j^q - 1 \right) \nu(\mathrm{d}\mathbf{p}) \right).$$

We next invoke Lemma 2 and use the notation there to write

$$\langle \mathcal{X}_t, \mathrm{e}^{qx} \rangle = \sum_{i \in I} \mathrm{e}^{q\alpha_i} \quad \text{and} \quad \langle \mathcal{Z}_t, \mathrm{e}^{qz} \rangle = \sum_{i \in I} \mathrm{e}^{q(\alpha_i + \beta_i)}.$$

Recall that conditionally on $\mathcal{X}$, each β_i has Laplace transform $\mathbb{E}(\exp(q\beta_i)) = \exp(t\Psi(q))$, so

$$\mathbb{E}\left(\langle \mathcal{Z}_t, \mathrm{e}^{qz} \rangle\right) = \exp(t\Psi(q))\mathbb{E}\left(\langle \mathcal{X}_t, \mathrm{e}^{qx} \rangle\right) = \exp\left[t \left(\int_{\mathcal{P}\backslash\mathcal{P}_1} \left(\sum_{j=1}^{\infty} p_j^q - 1 \right) \nu(\mathrm{d}\mathbf{p}) + \Psi(q) \right) \right].$$

Recall also from (16) that

$$\Psi(q)=\frac{1}{2}\sigma^2q^2+\left(c+\int_{\mathcal{P}\setminus\mathcal{P}_1}(1-p_1)\nu(\mathrm{d}\mathbf{p})\right)q+\int_{\mathcal{P}_1}(p_1^q-1+q(1-p_1))\nu(\mathrm{d}\mathbf{p}),$$

so we conclude that $\mathbb{E}\left(\langle\mathcal{Z}_t,\mathrm{e}^{qz}\rangle\right)=\exp(t\kappa(q))$.

Finally, the case when μ does not fulfill (15) follows by monotone approximations. More precisely, for every $b>0$, denote by $\nu^{(b)}$ the image of ν by the map $\mathbf{p}\mapsto\mathbf{p}^{(b)}$, where

$$\mathbf{p}^{(b)}=(p_1,p_2\mathbb{1}_{\{p_2>\mathrm{e}^{-b}\}},p_3\mathbb{1}_{\{p_3>\mathrm{e}^{-b}\}},\ldots),\tag{22}$$

so that the image of $\nu^{(b)}$ by the map $\mathbf{p}\mapsto\ln\mathbf{p}$ is $\mu^{(b)}$. Then, in the obvious notation

$$\kappa^{(b)}(q)=\frac{1}{2}\sigma^2q^2+cq+\int_{\mathcal{P}}\left(p_1^q-1+q(1-p_1)+\sum_{i=2}^{\infty}\mathbb{1}_{\{p_i>\mathrm{e}^{-b}\}}p_i^q\right)\nu(\mathrm{d}\mathbf{p}),\tag{23}$$

we have according to the first part of the proof that

$$\mathbb{E}\left(\langle\mathcal{Z}_t^{(b)},\mathrm{e}^{qz}\rangle\right)=\exp(t\kappa^{(b)}(q)).$$

It is immediate to check that $\lim_{b\uparrow\infty}\uparrow\kappa^{(b)}(q)=\kappa(q)$, and we conclude by monotone convergence that $\mathbb{E}\left(\langle\mathcal{Z}_t,\mathrm{e}^{qz}\rangle\right)=\exp(t\kappa(q))$. □

It is interesting to observe that there is the identity

$$\kappa(q)=\Psi_*(q)+\int_{\mathcal{P}}\sum_{i=2}^{\infty}p_i^q\ \nu(\mathrm{d}\mathbf{p}),\tag{24}$$

where Ψ_* is given by (20), i.e. is the Laplace exponent of the selected atom. In view of Theorem 1, the identity (24) is of course closely related to the spine decomposition of the branching Lévy process $\mathcal{Z}$ with respect to the trajectory of the selected atom, which was briefly discussed at the end of the preceding section.

Theorem 1 entails that for each $t\geq 0$, the random measure $\mathcal{Z}_t$ is locally finite a.s., and more precisely, its atoms can be ranked and form a decreasing sequence (in which of course, each atom is repeated according to its multiplicity). We first introduce the space

$$\ell^{2\downarrow}=\left\{\mathbf{z}=(z_1,\ldots):z_1\geq z_2\geq\cdots\geq 0\text{ and }\sum_{i=1}^{\infty}z_i^2<\infty\right\},$$

endowed with the usual ℓ^2-distance and arrive at the following definition.

Definition 3 *Let ν be a measure on $\mathcal{P}$ which fulfills (6) and μ its image by the map $\mathbf{p}\mapsto\ln\mathbf{p}$. Let* *also $\sigma^2 \geq 0$ and $c \in \mathbb{R}$, and denote by $\mathcal{Z}$ a branching Lévy process with characteristics (σ^2, c, μ). For every $t \geq 0$, the atoms of $\mathcal{Z}_t$ repeated according to their multiplicities can be ranked into a decreasing sequence $\ln Z_1(t) \geq \ln Z_2(t) \geq \cdots$, so that*

$$\mathcal{Z}_t = \sum_{i=1}^{\infty} \delta_{\ln Z_i(t)}.$$

The process $\mathbf{Z} = (\mathbf{Z}(t))_{t\geq 0}$ with $\mathbf{Z}(t) = (Z_1(t), \ldots)$ takes its values in $\ell^{2\downarrow}$ and is called a compensated fragmentation with characteristics (σ^2, c, ν).

Throughout the rest of this section, $\mathbf{Z}$ designates a compensated fragmentation with characteristics (σ^2, c, ν) as defined above. We first point out that $\mathbf{Z}$ is Markovian, with a homogeneous semigroup which fulfills the branching property. Specifically, consider a sequence of i.i.d. copies of $\mathbf{Z}$, say $(\mathbf{Z}^{[j]})_{j\in\mathbb{N}}$. Then for $\mathbf{z} = (z_1, \ldots) \in \ell^{2\downarrow}$, it follows from Theorem 1 that

$$\mathbb{E}\left(\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left(z_j Z_i^{[j]}(t)\right)^2\right) = \exp\left(t\kappa(2)\sum_{j=1}^{\infty} z_j^2\right) < \infty, \tag{25}$$

so we can rank the elements of the family $(z_j Z_i^{[j]}(t) : i \in \mathbb{N}, j \in \mathbb{N})$ in the decreasing order. We denote by $\mathbb{P}_{\mathbf{z}}$ the law of the resulting process (in particular, $\mathbb{P} = \mathbb{P}_{\mathbf{1}}$). It is then immediately checked from the branching property of $\mathcal{Z}$ that for every $t \geq 0$, the conditional law of $(\mathbf{Z}(t+s))_{s\geq 0}$ given $(\mathbf{Z}(r))_{0\leq r\leq t}$ is $\mathbb{P}_{\mathbf{z}}$ with $\mathbf{z} = \mathbf{Z}(t)$.

We next observe the following Feller-type property of the family of laws $\{\mathbb{P}_{\mathbf{z}} : \mathbf{z} \in \ell^{2\downarrow}\}$.

Corollary 2 *Let $(\mathbf{z}_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^{2\downarrow}$ which converges in ℓ^2 to $\mathbf{z}_\infty$. Then as $n \to \infty$, there is the weak convergence*

$$\mathbb{P}_{\mathbf{z}_n} \Longrightarrow \mathbb{P}_{\mathbf{z}_\infty}$$

in the sense of finite dimensional distributions on $\ell^{2\downarrow}$.

Proof: Just as above, let $(\mathbf{Z}^{[j]})_{j\in\mathbb{N}}$ denote a sequence of i.i.d. copies of $\mathbf{Z}$, and for every $n = 1, \ldots, \infty$, consider the family of real random variables $(z_{n,j} Z_i^{[j]}(t) : i, j \in \mathbb{N})$, where $\mathbf{z}_n = (z_{n,1}, z_{n,2}, \ldots)$. From (25), we get for every $t \geq 0$ that

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left((z_{n,j} - z_{\infty,j}) Z_i^{[j]}(t)\right)^2\right) = 0.$$

Now recall the well-known fact that rearranging sequences of positive real numbers in the decreasing order decreases the ℓ^2-distance, that is if $\mathbf{x}^\downarrow$ and $\mathbf{y}^\downarrow$ denote the decreasing rearrangements of two sequences $\mathbf{x}$ and $\mathbf{y}$, then $\|\mathbf{x}^\downarrow - \mathbf{y}^\downarrow\|_{\ell^2} \leq \|\mathbf{x} - \mathbf{y}\|_{\ell^2}$, where for every countable family of real numbers

$\mathbf{x} = (x_i)_{i \in I}$, we write $\|\mathbf{x}\|^2_{\ell^2} = \sum_{i \in I} x_i^2$. See, e.g., Theorem 3.5 in [20]. As a consequence

$$\lim_{n\to\infty} \mathbb{E}\left(\|(z_{n,j} Z_i^{[j]}(t))^{\downarrow} - (z_{\infty,j} Z_i^{[j]}(t))^{\downarrow}\|^2_{\ell^2} \right) = 0,$$

which implies our claim. □

Another important consequence of Theorem 1 is that it yields a remarkable family of martingales.

Corollary 3 *For every $q \geq 2$, the process*

$$M^{(q)}(t) = \exp(-t\kappa(q)) \sum_{i=1}^{\infty} Z_i^q(t) = \exp(-t\kappa(q)) \langle \mathcal{Z}_t, \mathrm{e}^{qz} \rangle \,, \qquad t \geq 0$$

is a martingale.

Proof: This follows directly from the Markov property and the formula of Theorem 1. □

Additive martingales play a fundamental role in the analysis of the asymptotic behavior of branching Markov chains (see in particular Biggins [12, 13]), and thus also for that of homogeneous fragmentations (see [8], [11], [6]). The detailed study of the additive martingales $M^{(q)}$ and their applications for compensated fragmentations will be developed in a subsequent work. We now complete this section by establishing the regularity of the paths of compensated fragmentations; the argument relies crucially on Corollary 3.

Proposition 2 *Every compensated fragmentation $\mathbf{Z}$ possesses a càdlàg version in $\ell^{2\downarrow}$ and fulfills the strong Markov property.*

Although it is well-known that Feller processes with values on a locally compact space always possess a càdlàg version and have the strong Markov property, we cannot apply Corollary 2 to establish Proposition 2 as the state space $\ell^{2\downarrow}$ is not locally compact. However, we can use a more direct approach.

Proof: Let $\mathcal{Z}$ denote the branching Lévy process with characteristics (σ^2, c, μ) associated with $\mathbf{Z}$, and recall from Definition 2 that $\mathcal{Z}_s = \lim_{b\uparrow\infty} \uparrow \mathcal{Z}_s^{(b)}$, where $\mathcal{Z}^{(b)}$ is a branching Lévy process with characteristics $(\sigma^2, c, \mu^{(b)})$ and $\mu^{(b)}$ is the image of μ by the map $\mathbf{r} \mapsto \mathbf{r}^{(b)}$.

Because $p_k \leq 1/k$ for every mass-partition $\mathbf{p} = (p_1, \ldots)$, the measure $\mu^{(b)}$ is carried by the set

$$\{\mathbf{r} = (r_1, \ldots) \in \mathcal{R} : r_k = -\infty \text{ for all } k > \mathrm{e}^b\}.$$

Therefore at most $\lceil \mathrm{e}^b \rceil$ new particles are born at each birth event for $\mathcal{Z}^{(b)}$ and if $\mathbf{Z}^{(b)}$ is the compensated fragmentation associated to $\mathcal{Z}^{(b)}$, then it is easy to check that the process $\mathbf{Z}^{(b)}$ is càdlàg a.s. (because the set of birth-times of $\mathbf{Z}^{(b)}$ is discrete a.s. and the displacement of its components between consecutive birth events are governed by càdlàg processes). The first claim of the statement then follows from Lemma 4 below.

Once we know that there is a càdlàg version, the strong Markov property follows from Corollary 2 by a standard argument (the Markov property holds for simple stopping times and one approximates a general stopping time by a decreasing sequence of simple stopping times). $\square$

Lemma 4 *For every $t \geq 0$, we have*

$$\lim_{b\to\infty} \sup_{0\leq s\leq t} \|\mathbf{Z}(s) - \mathbf{Z}^{(b)}(s)\|_{\ell^2}^2 = 0 \qquad \textit{in probability.}$$

Proof: By construction, the components of $\mathbf{Z}^{(b)}(s)$ form a (random) subsequence of those of $\mathbf{Z}(s)$, say $\mathbf{Z}^{(b)}(s) = (Z_{\varphi(i)}(s))_{i\in\mathbb{N}}$ for some strictly increasing map $\varphi : \mathbb{N} \to \mathbb{N}$. Define $\tilde{Z}_j^{(b)}(s) = Z_j(s)$ for j in the range $\varphi(\mathbb{N})$ of φ and $\tilde{Z}_j^{(b)}(s) = 0$ otherwise, so that $\mathbf{Z}^{(b)}(s)$ is the sequence of the components of $\tilde{\mathbf{Z}}^{(b)}(s)$ rearranged in the decreasing order.

Since $Z_i(s) - \tilde{Z}_i^{(b)}(s)$ equals either 0 or $Z_i(s)$ depending on whether $i \in \varphi(\mathbb{N})$ or not, we have

$$\|\mathbf{Z}(s) - \tilde{\mathbf{Z}}^{(b)}(s)\|_{\ell^2}^2 = \|\mathbf{Z}(s)\|_{\ell^2}^2 - \|\tilde{\mathbf{Z}}^{(b)}(s)\|_{\ell^2}^2 = \|\mathbf{Z}(s)\|_{\ell^2}^2 - \|\mathbf{Z}^{(b)}(s)\|_{\ell^2}^2.$$

We use again the fact that rearranging sequences of positive real numbers in the decreasing order decreases the ℓ^2-distance to deduce

$$\|\mathbf{Z}(s) - \mathbf{Z}^{(b)}(s)\|_{\ell^2}^2 \leq \|\mathbf{Z}(s)\|_{\ell^2}^2 - \|\mathbf{Z}^{(b)}(s)\|_{\ell^2}^2.$$

In turn, this yields the bounds

$$\begin{aligned}\sup_{0\leq s\leq t} \|\mathbf{Z}(s) - \mathbf{Z}^{(b)}(s)\|_{\ell^2}^2 &\leq \sup_{0\leq s\leq t} \mathrm{e}^{(t-s)\kappa(2)}(\|\mathbf{Z}(s)\|_{\ell^2}^2 - \|\mathbf{Z}^{(b)}(s)\|_{\ell^2}^2) \\ &\leq \mathrm{e}^{t\kappa(2)} \sup_{0\leq s\leq t} \left|M^{(2)}(s) - M^{(b,2)}(s)\right|,\end{aligned}$$

where

$$M^{(2)}(s) = \exp(-s\kappa(2))\|\mathbf{Z}(s)\|_{\ell^2}^2 \ , \ M^{(b,2)}(s) = \exp(-s\kappa^{(b)}(2))\|\mathbf{Z}^{(b)}(s)\|_{\ell^2}^2$$

and $\kappa^{(b)} \leq \kappa$ is given by (23).

Now we recall from Corollary 3 that $M^{(2)}$ and $M^{(b,2)}$ are two (nonnegative) martingales. Plainly $\lim_{b\to\infty} \uparrow \kappa^{(b)}(2) = \kappa(2)$ and, further, by Definitions 1 and 2 and by monotone convergence, $\lim_{b\to\infty} \uparrow \|\mathbf{Z}^{(b)}(t)\|_{\ell^2}^2 = \|\mathbf{Z}(t)\|_{\ell^2}^2$. Since, by Theorem 1, $\|\mathbf{Z}(t)\|_{\ell^2}^2 \in L^1(\mathbb{P})$, we deduce that

$$\lim_{b\to\infty} \mathbb{E}\left(\left|M^{(2)}(t) - M^{(b,2)}(t)\right|\right) = 0$$

and the proof is completed by an appeal to Doob's maximal inequality. $\square$

5 Convergence of compensated fragmentations

In this section, we shall establish the convergence of homogeneous fragmentations with dilations to compensated fragmentations, which is the main motivation for this work; see the Introduction. We shall need to enlarge the state-space and view here compensated fragmentations as random processes with values in

$$\ell^{q\downarrow} = \left\{ \mathbf{z} = (z_1, \dots) : z_1 \geq z_2 \geq \cdots \geq 0 \text{ and } \sum_{i=1}^{\infty} z_i^q < \infty \right\}$$

endowed with the ℓ^q-distance, for some fixed $q > 2$. The embedding $\ell^2 \subset \ell^q$ being a contraction of Banach spaces, Proposition 2 entails that every compensated fragmentation can still be viewed as a càdlàg process in $\ell^{q\downarrow}$. The purpose of this change is that sets in $\ell^{2\downarrow}$ which are bounded for the ℓ^2-distance are relatively compact for the ℓ^q-distance, but in general not for the ℓ^2-distance.

We shall use the notation $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and consider for each $n \in \bar{\mathbb{N}}$, $\sigma_n^2 \geq 0$, $c_n \in \mathbb{R}$ and ν_n a measure on $\mathcal{P}$ such that (6) holds. We shall often further assume that

$$\nu_\infty(\{\mathbf{0}\}) = 0. \tag{26}$$

We write $\mathbf{Z}_n = (\mathbf{Z}_n(t))_{t\geq 0}$ for a compensated fragmentation process with characteristics (σ_n^2, c_n, ν_n). The condition (26) means that in any dislocation event for $\mathbf{Z}_\infty$, the component that dislocates is never entirely shattered to dust. This is very minor technical assumption which we make to ease some arguments. It could easily be dropped, essentially at the cost of a heavier notation; details shall be left to interested readers.

We now state the main result of this work.

Theorem 2 *Suppose that* (26) *holds, that*

$$\lim_{n\to\infty} c_n = c_\infty, \tag{27}$$

and that there is the weak convergence of finite measures on $\mathcal{P}$

$$\sigma_n^2 \delta_{\mathbf{1}}(\mathrm{d}\mathbf{p}) + (1-p_1)^2 \nu_n(\mathrm{d}\mathbf{p}) \Longrightarrow \sigma_\infty^2 \delta_{\mathbf{1}}(\mathrm{d}\mathbf{p}) + (1-p_1)^2 \nu_\infty(\mathrm{d}\mathbf{p}). \tag{28}$$

Then, for every $q > 2$, *there is the convergence in distribution as* $n \to \infty$

$$\mathbf{Z}_n \Longrightarrow \mathbf{Z}_\infty \tag{29}$$

in the sense of weak convergence on the space $D(\mathbb{R}_+, \ell^{q\downarrow})$ *of càdlàg functions with values in* $\ell^{q\downarrow}$ *endowed* *with the Skorohod J_1-topology.*

It is interesting to point out that under the hypotheses of Theorem 2, we have always, in the notation (21), that

$$\lim_{n\to\infty} \kappa_n(q) = \kappa_\infty(q) \qquad \text{for all } q > 2.$$

However, it is easy to provide examples such that $\kappa_n(2)$ *does not converge* to $\kappa_\infty(2)$. Recalling Theorem 1, this a strong evidence that the convergence (29) shall not hold in full generality for $q = 2$.

Before tackling the proof of Theorem 2, we point out that it encompasses (8), which motivated the present work.

Corollary 4 *For each $n \in \mathbb{N}$, consider a measure ν_n on $\mathcal{P}$ which fulfills the integral condition* (2)*, and let $\mathbf{X}_n$ denote a homogeneous fragmentation with dislocation measure ν_n and no erosion. Suppose that as $n \to \infty$, there is the weak convergence of finite measures on $\mathcal{P}$*

$$(1-p_1)^2 \nu_n(\mathrm{d}\mathbf{p}) \Longrightarrow \sigma_\infty^2 \delta_{\mathbf{1}}(\mathrm{d}\mathbf{p}) + (1-p_1)^2 \nu_\infty(\mathrm{d}\mathbf{p}),$$

where $\sigma_\infty^2 \geq 0$ and ν_∞ fulfills (6) *and* (26)*, and set*

$$c_n = \int_{\mathcal{P}} (1-p_1)\nu_n(\mathrm{d}\mathbf{p}) \,.$$

Then for every $q > 2$, there is the weak convergence as $n \to \infty$ in the sense of Skorohod J_1-topology on the space $D(\mathbb{R}_+, \ell^{q\downarrow})$

$$(\exp(c_n t)\mathbf{X}_n(t))_{t\geq 0} \Longrightarrow (\mathbf{Z}(t))_{t\geq 0},$$

where $\mathbf{Z}$ is a compensated fragmentation process with characteristics $(\sigma_\infty^2, 0, \nu_\infty)$.

Proof: It is easy to check that for each n, the homogeneous dilated fragmentation

$$\mathbf{Z}_n(t) = \exp(c_n t)\mathbf{X}_n(t), \qquad t \geq 0$$

can be viewed as a compensated fragmentation with characteristics $(0, 0, \nu_n)$ (the case when ν_n is finite has been explained in Section 2, and the general case is then readily deduced). The statement thus follows directly from Theorem 2. $\square$

Remark. We stress that the parameter c_n of the rescaling of the fragmentation $\mathbf{X}_n$ in Corollary 4 has merely been chosen to ensure the weak convergence of the selected fragment (cf. Lemma 1 and Corollary 1), and it is not clear *a priori* that this should be sufficient for the convergence of the whole process. In particular, recall that the selected fragment may well not be the largest of all fragments, and one might have thought that, as a matter of fact, a stronger rescaling would be needed to control the largest component of $\mathbf{X}_n$. In this direction, it is interesting to recall that for a branching random walk on $\mathbb{R}$, if at each birth event we select the right-most atom, then for large times, this selected atom is in fact far away from the right-most atom of the branching random walk (see [6] for a precise study in the setting of homogeneous fragmentations). We stress however that the framework of large-time asymptotics for branching random walks differs from the present one, as we rather work at fixed times, with a sequence of different branching dynamics. In our setting, the total rate of branching events increases, but the distribution of these events also vary with the parameter n, and it turns out that then the selected fragment of $\mathbf{X}_n$ remains relatively close to the largest fragment as $n \to \infty$.

The rest of this section is devoted to the proof of Theorem 2; our strategy is the following. Rather than working directly with the processes $\mathbf{Z}_n$, we shall consider their approximations $\mathbf{Z}_n^{(b)}$. Specifically, we write as usual μ_n for the image of ν_n by the map $\mathbf{p} \mapsto \ln \mathbf{p}$ and consider the branching Lévy process $\mathcal{Z}_n$ associated to $\mathbf{Z}_n$ by Definition 3. Recall from Definition 2 that $\mathcal{Z}_n = \lim_{b\uparrow\infty} \uparrow \mathcal{Z}_n^{(b)}$, where $\mathcal{Z}_n^{(b)}$ is a branching Lévy process with characteristics $(\sigma_n^2, c_n, \mu_n^{(b)})$ and $\mu_n^{(b)}$ is the image of μ_n by the map $\mathbf{r} \mapsto \mathbf{r}^{(b)}$ (equivalently, $\mu_n^{(b)}$ is the image of $\nu_n^{(b)}$ by the map $\mathbf{p} \mapsto \ln \mathbf{p}$). Then $\mathbf{Z}_n^{(b)}$ denotes the compensated fragmentation associated to $\mathcal{Z}_n^{(b)}$ by Definition 3. We shall check in Corollary 5 below that for b fixed, $\mathbf{Z}_n^{(b)}$ converges weakly as $n \to \infty$ in the sense of finite-dimensional distributions to $\mathbf{Z}_\infty^{(b)}$. Finally, we will then deduce Theorem 2 from the following explicit bounds and Aldous' tightness criterion.

For the sake of simplicity, we drop the index $n \in \bar{\mathbb{N}}$ from the notation in the next statement.

Lemma 5 *For every $t \geq 0$, $b \geq 0$ and $q \geq 2$, we have*

$$\mathbb{E}\left(\|\mathbf{Z}(t) - \mathbf{Z}^{(b)}(t)\|_{\ell^q}^q\right) \leq \exp(t\kappa(q)) \left(1 - \exp\left(-t \int_{\mathcal{P}} \sum_{i=2}^{\infty} \mathbb{1}_{\{p_i \leq \mathrm{e}^{-b}\}} p_i^q \nu(\mathrm{d}\mathbf{p})\right)\right).$$

Proof: Because the operation of rearranging sequences of positive real numbers in the decreasing order decreases the ℓ^q-distance, the same argument as in Lemma 4 shows that

$$\|\mathbf{Z}(s) - \mathbf{Z}^{(b)}(s)\|_{\ell^q}^q \leq \|\mathbf{Z}(s)\|_{\ell^q}^q - \|\mathbf{Z}^{(b)}(s)\|_{\ell^q}^q.$$

We deduce from Theorem 1 that

$$\begin{aligned}
\mathbb{E}\left(\|\mathbf{Z}(s) - \mathbf{Z}^{(b)}(s)\|_{\ell^q}^q\right) &\leq \exp(t\kappa(q)) - \exp(t\kappa^{(b)}(q)) \\
&= \exp(t\kappa(q)) \left(1 - \exp\left(-t \int_{\mathcal{P}} \sum_{i=2}^{\infty} \mathbb{1}_{\{p_i \leq \mathrm{e}^{-b}\}} p_i^q \nu(\mathrm{d}\mathbf{p})\right)\right),
\end{aligned}$$

where $\kappa^{(b)}$ is given by (23) and the last line results from the expressions of κ and $\kappa^{(b)}$. $\square$

We next point out that

$$\nu_\infty\left(\left\{p \in \mathcal{P} : \exists i \geq 2 \text{ such that } p_i = \mathrm{e}^{-b}\right\}\right) = 0 \tag{30}$$

except for at most countably many $b's$. In the sequel, we will implicitly assume that the parameter b fulfills (30) (in particular expressions such as $\lim_{b\to\infty}$ shall be understood in this setting). We recall that the map $\mathbf{p} \mapsto \mathbf{p}^{(b)}$ from $\mathcal{P}$ to itself has been defined in (22) and that we write $\nu_n^{(b)}$ for the image of ν_n by this map.

Our next goal is to establish a finite-dimensional version of Theorem 2 with $\nu_n^{(b)}$ replacing ν_n. In this direction, we first point out the following elementary fact.

Lemma 6 *If (26) and (28) hold, then for $b > 0$ fixed, there is also the weak convergence of finite measures on $\mathcal{P}$ as $n \to \infty$*

$$\sigma_n^2 \delta_{\mathbf{1}}(\mathrm{d}\mathbf{p}) + (1-p_1)^2 \nu_n^{(b)}(\mathrm{d}\mathbf{p}) \Longrightarrow \sigma_\infty^2 \delta_{\mathbf{1}}(\mathrm{d}\mathbf{p}) + (1-p_1)^2 \nu_\infty^{(b)}(\mathrm{d}\mathbf{p}) \tag{31}$$

and

$$\nu_n^{(b)}(\cdot, \mathcal{P}\backslash\mathcal{P}_1) \Longrightarrow \nu_\infty^{(b)}(\cdot, \mathcal{P}\backslash\mathcal{P}_1).$$

Proof: Indeed, if $f : \mathcal{P} \to \mathbb{R}$ is a continuous function on $\mathcal{P}$, then $\mathbf{p} \to f(\mathbf{p}^{(b)})$ is continuous at every mass-partition $\mathbf{p} = (p_1, \ldots)$ such that $p_i \neq \mathrm{e}^{-b}$ for all $i \geq 2$. Our first assertion thus stems from (30) and the continuous mapping theorem.

Next, we observe that

$$\int_{\mathcal{P}\backslash\mathcal{P}_1} f(\mathbf{p}) \nu_n^{(b)}(\mathrm{d}\mathbf{p}) = \int_{\mathcal{P}} f(\mathbf{p}) \mathbb{1}_{\{p_2 > \mathrm{e}^{-b}\}} \nu_n^{(b)}(\mathrm{d}\mathbf{p}) + f(\mathbf{0}) \nu_n^{(b)}(\{\mathbf{0}\}).$$

On the one hand, we have thanks to (26) and the portmanteau theorem that

$$\limsup_{n\to\infty} \nu_n^{(b)}(\{\mathbf{0}\}) \leq \nu_\infty^{(b)}(\{\mathbf{0}\}) = 0,$$

since the singleton $\{\mathbf{0}\}$ is closed in $\mathcal{P}$.

On the other hand, it is easy to construct a bounded continuous function $g : \mathcal{P} \to \mathbb{R}$ such that $f(\mathbf{p}) = (1-p_1)^2 g(\mathbf{p})$ for all $\mathbf{p} \in \mathcal{P}$ with $1 - p_1 > \mathrm{e}^{-b}$, and *a fortiori* with $p_2 > \mathrm{e}^{-b}$. It thus follows from the first assertion that

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathcal{P}} f(\mathbf{p}) \mathbb{1}_{\{p_2 > \mathrm{e}^{-b}\}} \nu_n^{(b)}(\mathrm{d}\mathbf{p}) &= \lim_{n\to\infty} \int_{\mathcal{P}} g(\mathbf{p}) \mathbb{1}_{\{p_2 > \mathrm{e}^{-b}\}} \left(\sigma_n^2 \delta_{\mathbf{1}}(\mathrm{d}\mathbf{p}) + (1-p_1)^2 \nu_n^{(b)}(\mathrm{d}\mathbf{p}) \right) \\
&= \int_{\mathcal{P}} g(\mathbf{p}) \mathbb{1}_{\{p_2 > \mathrm{e}^{-b}\}} \left(\sigma_\infty^2 \delta_{\mathbf{1}}(\mathrm{d}\mathbf{p}) + (1-p_1)^2 \nu_\infty^{(b)}(\mathrm{d}\mathbf{p}) \right) \\
&= \int_{\mathcal{P}} f(\mathbf{p}) \mathbb{1}_{\{p_2 > \mathrm{e}^{-b}\}} \nu_\infty^{(b)}(\mathrm{d}\mathbf{p}).
\end{aligned}$$

This completes the proof of the statement. $\square$

Lemma 6 enables us to prove the following limit result for the sequence of processes $\mathbf{Z}_n^{(b)}$.

Corollary 5 *Suppose* (26), (27) *and* (28) *hold, and fix* $b > 0$. *We can construct on the same probability space compensated fragmentation processes* $\mathbf{Z}_n^{(b)}$ *with characteristics* $(\sigma_n^2, c_n, \nu_n^{(b)})$ *for* $n \in \bar{\mathbb{N}}$ *such that for every* $t \geq 0$,

$$\lim_{n\to\infty} \|\mathbf{Z}_n^{(b)}(t) - \mathbf{Z}_\infty^{(b)}(t)\|_{\ell^2} = 0 \qquad \textit{in probability.}$$

Proof: We consider the setting of branching Lévy processes and use notation that should be obvious. Recall from the proof of Proposition 2 that at most $\lceil e^b \rceil$ new particles are born at each birth event for $\mathcal{Z}_n^{(b)}$. So for each $n \in \bar{\mathbb{N}}$, in the construction of $\mathcal{Z}_n^{(b)}$ provided by Definition 1, we may restrict the processes $(\lambda_{n,u}^{(b)})_{u\in\mathbb{U}}$, $(\mathbf{\Delta} a_{n,u}^{(b)})_{u\in\mathbb{U}}$ and $(\xi_{n,u}^{(b)})_{u\in\mathbb{U}}$ to nodes u in the $\lceil e^b \rceil$-regular tree, say $\mathbb{U}^{(b)}$. More precisely, let $\mathcal{N}_{n,t}^{(b)}$ denote the a.s. finite set of nodes of $\mathbb{U}^{(b)}$ such that $b_{n,u}^{(b)} \leq t < b_{n,u}^{(b)} + \lambda_{n,u}^{(b)}$. Then the atoms of $\mathcal{Z}_n^{(b)}$ present at time t are located at $a_{n,u}^{(b)} + \xi_{n,u}^{(b)}(t - b_{n,u}^{(b)})$ for $u \in \mathcal{N}_{n,t}^{(b)}$, where we implicitly agree to discard atoms at $-\infty$. In other words, $\mathbf{Z}_n^{(b)}(t)$ is given by the decreasing rearrangement of the finite family

$$\exp(a_{n,u}^{(b)} + \xi_{n,u}^{(b)}(t - b_{n,u}^{(b)})) \qquad \text{for } u \in \mathcal{N}_{n,t}^{(b)}.$$

We stress that the random set of nodes $\mathcal{N}_{n,t}^{(b)}$ only depends on the $(\lambda_{n,u}^{(b)})$, and is thus independent of the processes $(a_{n,u}^{(b)})$ and $(\xi_{n,u}^{(b)})$.

Recall from Definition 1 that $(\lambda_{n,u}^{(b)})$ is a sequence of i.i.d. exponential variables with parameter $\nu_n^{(b)}(\mathcal{P}\backslash\mathcal{P}_1)$. By Lemma 6, we may thus couple the process $(\lambda_{n,u}^{(b)})$ for $n \in \mathbb{N}$ with $(\lambda_{\infty,u}^{(b)})$ such that $\lambda_{n,u}^{(b)} \to \lambda_{\infty,u}^{(b)}$ a.s. as $n \to \infty$, and it then follows that the random sets $\mathcal{N}_{n,t}^{(b)}$ and $\mathcal{N}_{\infty,t}^{(b)}$ coincide with high probability as $n \to \infty$.

Lemma 6 and Skorohod's coupling also enables us to couple the processes $(\mathbf{\Delta} a_{n,u}^{(b)})$ and $(\xi_{n,u}^{(b)})$ for $n \in \mathbb{N}$ with $(\mathbf{\Delta} a_{\infty,u}^{(b)})$ and $(\xi_{\infty,u}^{(b)})$ such that for every node $u \in \mathbb{U}^{(b)}$

$$\lim_{n\to\infty} \Delta a_{n,ui}^{(b)} = \Delta a_{\infty,ui}^{(b)} \qquad \text{for all } i \in \mathbb{N}, \text{ a.s.,}$$

and

$$\lim_{n\to\infty} \xi_{n,u}^{(b)}(s) = \xi_{\infty,u}^{(b)}(s) \qquad \text{for all } s \geq 0, \text{ a.s.}$$

See, for instance, Theorems 15.14 and 15.17 in [18] for the last assertion. It follows that

$$\lim_{n\to\infty} \exp(a_{n,u}^{(b)} + \xi_{n,u}^{(b)}(t - b_{n,u}^{(b)})) = \exp(a_{\infty,u}^{(b)} + \xi_{\infty,u}^{(b)}(t - b_{\infty,u}^{(b)})) \qquad \text{a.s.}$$

for every $u \in \mathbb{U}$. Since $\mathcal{N}_{n,t}^{(b)}$ and $\mathcal{N}_{\infty,t}^{(b)}$ are finite and coincide with high probability as $n \to \infty$, and since rearranging sequences of positive real numbers in the decreasing order decreases the ℓ^2-distance, this yields our claim. $\square$

We may now establish a weaker version of Theorem 2.

Lemma 7 *Theorem 2 holds if the weak convergence* (29) *there is taken in the sense of finite-dimensional distributions on* $\ell^{q\downarrow}$.

Proof: For the sake of simplicity, we shall only establish the one-dimensional convergence. The argument for the multi-dimensional case is the same with heavier notation. We first establish tightness.

We start by observing that for any $r > 0$, the set

$$K_r = \left\{ \mathbf{z} \in \ell^{2\downarrow} : \|\mathbf{z}\|_{\ell^2} \leq r \right\}$$

is a compact subset of $\ell^{q\downarrow}$. Indeed, from any sequence in K_r, the diagonal procedure enables us to extract a subsequence which converges pointwise, and its limit belongs to K_r due to Fatou's lemma. By an argument of equi-summability, the convergence also holds in ℓ^q. Next, for any $n \in \bar{\mathbb{N}}$, we have from Theorem 1 and Markov's inequality

$$\mathbb{P}(\mathbf{Z}_n(t) \notin K_r) \leq r^{-2}\mathbb{E}(\|\mathbf{Z}_n(t)\|^2_{\ell^2}) = r^{-2}\exp(t\kappa_n(2)). \tag{32}$$

Since (28) easily entails that $\sup_{n\in\mathbb{N}} \kappa_n(2) < \infty$, (32) shows that the sequence of random variables $\mathbf{Z}_n(t)$ is tight in $\ell^{q\downarrow}$. Hence, we only need to verify uniqueness of the weak limit of a converging sub-sequence.

Fix $k \geq 1$ arbitrarily and consider a continuous function $F : \mathbb{R}^k_+ \to [0,1]$. For every sequence $\mathbf{z} = (z_1, \ldots)$, we write $F(\mathbf{z}) = F(z_1, \ldots, z_k)$, and recall that two probability measures ρ and ρ' on ℓ^q with $\int F(\mathbf{z})\rho(\mathrm{d}\mathbf{z}) = \int F(\mathbf{z})\rho'(\mathrm{d}\mathbf{z})$ for all such functions F and $k \geq 1$ are necessarily identical. So we need to establish that

$$\lim_{n\to\infty} \mathbb{E}(F(Z_n(t))) = \mathbb{E}(F(\mathbf{Z}_\infty(t))). \tag{33}$$

Fix $\varepsilon > 0$ arbitrarily small; F is uniformly continuous on K_r and there exists $\eta > 0$ such that

$$|F(\mathbf{z}) - F(\mathbf{z}')| < \varepsilon \qquad \text{for all } \mathbf{z}, \mathbf{z}' \in K_r \text{ with } \|\mathbf{z} - \mathbf{z}'\|_{\ell^q} \leq \eta.$$

Observe also that if $\mathbf{Z}_n(t) \in K_r$, then *a fortiori* $\mathbf{Z}^{(b)}_n(t) \in K_r$ for any $b > 0$. This yields

$$\begin{aligned}
& |\mathbb{E}(F(\mathbf{Z}_n(t))) - \mathbb{E}(F(\mathbf{Z}_\infty(t)))| \\
\leq\ & |\mathbb{E}(F(\mathbf{Z}^{(b)}_n(t))) - \mathbb{E}(F(\mathbf{Z}^{(b)}_\infty(t)))| + \mathbb{E}(|F(\mathbf{Z}_n(t)) - F(\mathbf{Z}^{(b)}_n(t))|) + \mathbb{E}(|F(\mathbf{Z}_\infty(t)) - F(\mathbf{Z}^{(b)}_\infty(t))|) \\
\leq\ & |\mathbb{E}(F(\mathbf{Z}^{(b)}_n(t))) - \mathbb{E}(F(\mathbf{Z}^{(b)}_\infty(t)))| + \varepsilon + \mathbb{P}(\mathbf{Z}_n(t) \notin K_r) + \mathbb{P}(\|\mathbf{Z}_n(t) - \mathbf{Z}^{(b)}_n(t)\|_{\ell^q} > \eta) \\
& + \varepsilon + \mathbb{P}(\mathbf{Z}_\infty(t) \notin K_r) + \mathbb{P}(\|\mathbf{Z}_\infty(t) - \mathbf{Z}^{(b)}_\infty(t)\|_{\ell^q} > \eta).
\end{aligned}$$

Recall from (32) that we may pick $r > 0$ sufficiently large such that

$$\mathbb{P}(\mathbf{Z}_n(t) \notin K_r) < \varepsilon \qquad \text{for all } n \in \bar{\mathbb{N}}.$$

Then define a function $f_b : \mathcal{P} \to \mathbb{R}_+$ by $f_b(\mathbf{1}) = 0$ and

$$f_b(\mathbf{p}) = (1-p_1)^{-2} \sum_{i=2}^{\infty} \mathbb{1}_{\{p_i \le \mathrm{e}^{-b}\}} p_i^q, \qquad \mathbf{p} \neq \mathbf{1}.$$

Observe that

$$\sum_{i=2}^{\infty} \mathbb{1}_{\{p_i \le \mathrm{e}^{-b}\}} p_i^q \le \mathrm{e}^{-b(q-2)} \sum_{i=2}^{\infty} p_i^2 \le \mathrm{e}^{-b(q-2)}(1-p_1)^2, \qquad \text{for all } \mathbf{p} \in \mathcal{P},$$

and therefore $f_b(\mathbf{p}) \le \mathrm{e}^{-b(q-2)}$. It then follows from (28) that

$$\lim_{b\to\infty} \sup_{n\in\bar{\mathbb{N}}} \int_{\mathcal{P}} \sum_{i=2}^{\infty} \mathbb{1}_{\{p_i \le \mathrm{e}^{-b}\}} p_i^q \nu_n(\mathrm{d}\mathbf{p}) = 0.$$

Recall that $\sup_{n\in\bar{\mathbb{N}}} \kappa_n(q) < \infty$. We deduce from Lemma 5 and the observation above that we may choose b sufficiently large such that

$$\sup_{n\in\bar{\mathbb{N}}} \mathbb{E}\left(\|\mathbf{Z}_n(t) - \mathbf{Z}_n^{(b)}(t)\|_{\ell^q}^q \right) \le \varepsilon \eta^q;$$

and then, from Markov's inequality,

$$\sup_{n\in\bar{\mathbb{N}}} \mathbb{P}(\|\mathbf{Z}_n(t) - \mathbf{Z}_n^{(b)}(t)\|_{\ell^q} > \eta) \le \varepsilon.$$

Putting the pieces together, we have shown that

$$|\mathbb{E}(F(\mathbf{Z}_n(t))) - \mathbb{E}(F(\mathbf{Z}_\infty(t)))| \le 6\varepsilon + |\mathbb{E}(F(\mathbf{Z}_n^{(b)}(t))) - \mathbb{E}(F(\mathbf{Z}_\infty^{(b)}(t)))|,$$

and we can now complete the proof with an appeal to Corollary 5. Specifically, we work with a version of $\mathbf{Z}_n^{(b)}(t)$ for $n \in \bar{\mathbb{N}}$ such that the conclusion of Corollary 5 is fulfilled. In particular there exists $n_\varepsilon < \infty$ such that

$$\mathbb{P}(\|\mathbf{Z}_n^{(b)}(t) - \mathbf{Z}_\infty^{(b)}(t)\|_{\ell^2} > \eta) \le \varepsilon \qquad \text{for all } n \ge n_\varepsilon.$$

Since

$$\begin{aligned} & |\mathbb{E}(F(\mathbf{Z}_n^{(b)}(t))) - \mathbb{E}(F(\mathbf{Z}_\infty^{(b)}(t)))| \\ \le\ & \varepsilon + \mathbb{P}(\|\mathbf{Z}_n^{(b)}(t) - \mathbf{Z}_\infty^{(b)}(t)\|_{\ell^2} > \eta) + \mathbb{P}(\mathbf{Z}_n^{(b)}(t) \notin K_r) + \mathbb{P}(\mathbf{Z}_\infty^{(b)}(t) \notin K_r) \\ \le\ & 4\varepsilon, \end{aligned}$$

we have found for every $\varepsilon > 0$ an integer n_ε such that $|\mathbb{E}(F(Z_n(t))) - \mathbb{E}(F(\mathbf{Z}_\infty(t)))| \le 10\varepsilon$ whenever

$n \geq n_\varepsilon$, and (33) is proven. $\square$

To complete the proof of Theorem 2, it now suffices to verify Aldous' tightness criterion (see, e.g. Theorem 16.11 in [18]), which is our final lemma.

Lemma 8 *Assume* (26)*,* (27) *and* (28) *hold. Then for any sequence $h_n > 0$ with $\lim_{n\to\infty} h_n = 0$ and any bounded sequence τ_n of $\mathbf{Z}_n$-stopping times, we have for every $q \geq 2$*

$$\lim_{n\to\infty} \|\mathbf{Z}_n(\tau_n) - \mathbf{Z}_n(\tau_n + h_n)\|_{\ell^q} = 0 \qquad \textit{in probability.}$$

Proof: Since $\|\cdot\|_{\ell^2} \geq \|\cdot\|_{\ell^q}$ for $q \geq 2$, it suffices to prove the statement for $q = 2$. We shall check that the limit in the statement holds in fact in $L^2(\mathbb{P})$. Firstly, from the strong Markov property (see Proposition 2) and Theorem 1, we have

$$\mathbb{E}\left(\|\mathbf{Z}_n(\tau_n) - \mathbf{Z}_n(\tau_n + h_n)\|_{\ell^2}\right) = \mathbb{E}\left(\|\mathbf{Z}_n(\tau_n)\|^2_{\ell^2}\right) \mathbb{E}\left(\|\mathbf{Z}_n(h_n) - \mathbf{1}\|^2_{\ell^2}\right).$$

Assuming that $\tau_n \leq t$ a.s., we get from Corollary 3 and the optional sampling theorem the bound

$$\mathbb{E}\left(\|\mathbf{Z}_n(\tau_n)\|^2_{\ell^2}\right) \leq \exp(t\kappa_n(2)),$$

and this quantity remains bounded as $n \to \infty$.

In order to evaluate the second term in the product, we consider the transposition of $\mathbb{N}$ which permutes 1 and j, where j is the rank of the selected fragment in $\mathbf{Z}_n(h_n)$ (so this transposition is simply the identity when the selected fragment coincides with the largest fragment), and denote the transposed version of $\mathbf{Z}_n(h_n)$ by $\tilde{\mathbf{Z}}_n(h_n)$. Since rearranging sequences of positive real numbers in the decreasing order decreases the ℓ^2-distance, we have $\|\mathbf{Z}_n(h_n) - \mathbf{1}\|^2_{\ell^2} \leq \|\tilde{\mathbf{Z}}_n(h_n) - \mathbf{1}\|^2_{\ell^2}$. In view of (24) and Corollary 1, this yields in the obvious notation

$$\begin{aligned}
& \mathbb{E}(\|\mathbf{Z}_n(h_n) - \mathbf{1}\|^2_{\ell^2}) \\
\leq\ & \mathbb{E}\left(|\exp(\xi_{*,n}(h_n)) - 1|^2\right) + \exp\left(h_n\kappa_n(2)\right) - \exp\left(h_n\Psi_{*n}(2)\right) \\
\leq\ & \mathbb{E}\left(|\exp(\xi_{*,n}(h_n)) - 1|^2\right) + \exp\left(h_n\kappa_n(2)\right)\left(1 - \exp\left(-h_n \int_{\mathcal{P}} \sum_{i=2}^{\infty} p_i^2 \nu_n(\mathrm{d}\mathbf{p})\right)\right).
\end{aligned}$$

On the one hand, using $\sum_{i=2}^{\infty} p_i^2 \leq (1-p_1)^2$, we see from (28) that

$$\lim_{n\to\infty} \exp\left(h_n\kappa_n(2)\right)\left(1 - \exp\left(-h_n \int_{\mathcal{P}} \sum_{i=2}^{\infty} p_i^2 \nu_n(\mathrm{d}\mathbf{p})\right)\right) = 0.$$

On the other hand, we get from Corollary 1

$$\begin{aligned}\mathbb{E}\left(|\exp(\xi_{*,n}(h_n))-1|^2\right) &= \mathbb{E}\left(\exp(2\xi_{*,n}(h_n))\right)-2\mathbb{E}\left(\exp(\xi_{*,n}(h_n))\right)+1\\ &= \exp(h_n\Psi_{*n}(2))-2\exp(h_n\Psi_{*n}(1))+1.\end{aligned}$$

Since it follows readily from (20) and (28) that $\lim_{n\to\infty}\Psi_{*n}(r)=\Psi_{*\infty}(r)$ for $r\geq 0$, we have

$$\lim_{n\to\infty}\mathbb{E}\left(|\exp(\xi_{*,n}(h_n))-1|^2\right)=0,$$

which completes the proof. $\square$

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