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Abstract

We develop a model for the evolution of wealth in a non-conservative economic environment, extending a theory developed in [13]. The model considers a system of rational agents interacting in a game theoretical framework. This evolution drives the dynamic of the agents in both wealth and economic configuration variables. The cost function is chosen to represent a risk averse strategy of each agent. That is, the agent is more likely to interact with the market, the more predictable the market, and therefore the smaller its individual risk. This yields a kinetic equation for an effective single particle agent density with a Nash equilibrium serving as the local thermodynamic equilibrium. We consider a regime of scale separation where the large scale dynamics is given by a hydrodynamic closure with this local equilibrium. A class of generalized collision invariants (GCIs) is developed to overcome the difficulty of the non-conservative property in the hydrodynamic closure derivation of the large scale dynamics for the evolution of wealth distribution. The result is a system of gas dynamics-type equations for the density and average wealth of the agents on large scales. We recover the inverse Gamma distribution, which has been previously considered in the literature, as a local equilibrium for particular choices of the cost function.

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1 Introduction

1.1 Framework

A theory on the evolution of wealth distribution driven by local Nash equilibria in a conservative economy was developed by the authors in [13] in the framework set up by [12], which is closely related to Mean-Field Games [8, 20]. By conservative, we meant that the total wealth is preserved in the time evolution. This assumption enabled us to derive a large scale dynamics for the evolution of the wealth distribution by using a hydrodynamic closure with a Nash equilibrium serving as the local thermodynamic equilibrium. This resulted in a system of gas dynamics-type equations for the density and average wealth of the agents on large scales. The goal of this paper is to extend this theory to some more realistic models in non-conservative economies, where global wealth is gained or lost at a certain rate due to either productivity or inflation. To overcome the difficulty of the non-conservative property in the hydrodynamic closure, we adapt and develop a concept of Generalized Collision Invariant (GCI) developed by Degond and Motsch in [14] for flocking dynamics.

We consider an economy modeled as a closed ensemble of agents. The state of each agent is described by two variables. The variable x, describes its location in the economic configuration space $\mathcal{X}$ [15]. In addition, the state is described by the wealth $y \geq 0$ of the agent. The dynamic of these attributes is given by some motion mechanism in the economic configuration variable x and by the exchange of wealth (trading) in the wealth variable y.

The subject of understanding the wealth distribution has a long history since Pareto in 1896 [30]. Amoroso in 1925 [1] developed a dynamic equilibrium theory and re-wrote the Pareto distribution in terms of inverse Gamma distribution. The wealth distribution results from the combination of two important mechanisms: the first one is the geometric Brownian motion of finance which has first been proposed by Bachelier in 1900 [3] and the second one is the trading model, one the earlier ones being that of Edgeworth, dating back to 1881 [16]. These pioneering works have been followed by numerous authors and have given rise to the field of econophysics. Recent references on this problem can be found e.g. in the books [9, 26, 35, 36] and e.g. in the references [21, 34, 29, 37, 39]. The large-scale dynamics of spatially heterogeneous social models is currently the subject of an intense research (see e.g. [6], where the authors investigate a spatially heterogeneous version of Deffuant-Weisbuch opinion model of interacting agents that exhibits a transition between a socially cohesive phase and a socially disconnected phase).

The basic equation considered in this paper is of the form

$$\partial_t f(x,y,t) + \partial_x (f\, V(x,y)) = -\partial_y (f\, \mathcal{F}_f) + d\partial_y (\partial_y (y^2 f)) \equiv Q(f), \tag{1.1}$$

where $f(x, y, t)$ is the density of agents in economic configuration space x having wealth y at time t. The second term at the right hand side of (1.1) models the uncertainty and has the form of a diffusion operator corresponding to the geometric Brownian motion of economy and finance, with variance $2dy^2$ quadratic in y. The justification of this operator can be found in [28].

Here $\mathcal{F}_f$ describes the control, action or strategy. In [13], the authors take the action as the negative gradient of the cost function Φ_f, i.e., $\mathcal{F}_f = -\partial_y \Phi_f$. A quadratic cost function with coefficients depending functionally on the density f was used to describe trading behavior between agents. We write this cost function in general form as

$$\Phi_f(y) = \frac{1}{2} a_f y^2 + b_f y + c_f \ , \tag{1.2}$$

with coefficients a_f, b_f and c_f functionally dependent on the density f.

In the framework of a non-atomic non-cooperative anonymous game with a continuum of players [2, 22, 32, 33], also known as a Mean-Field Game [8, 20], players interact with each other to minimize their own cost function. In this paper we consider a more realistic model, where each player interacts with the ensemble of players, i.e. the market. For each player, the equilibrium reached under this interaction corresponds to the wealth difference between him/her and the market average being at one of the minima of this cost function.

We note that this model only considers the exchange of money and does not keep track of the goods and services traded. Therefore, this game does not mean that each players wishes to share some of its wealth with the trading partner. Rather, the utility of the exchange is to maximize the economic action resulting in the optimal exchange of goods and services. Within this framework, the dynamic of agents following these strategies can be viewed as given by the following game: each agent follows what is known as the best-reply strategy, that is, it tries to minimize the cost function with respect to its wealth variable, assuming that the other agents do not change theirs.

This gives for the control action $\mathcal{F}_f$ in (1.1) $\mathcal{F}_f(y) = -\partial_y \Phi_f = -a_f y - b_f$, and for the operator Q in (1.1), including effects of uncertainty, given by the geometric Brownian motion,

$$Q(f) = \partial_y \big(d\partial_y (y^2 f) + (a_f y + b_f) f \big)$$

We consider a closed system, where the number of agents in the market is conserved. So, equation (1.1) is supplemented by the boundary condition $d\partial_y (y^2 f) + (a_f y + b_f) f|_{y=0} = 0$.

In [13, 7], a model resulting from pairwise interactions, proportional to the quadratic distance between the wealth of the two agents is derived. The goal of the present paper is to extend this framework to general potentials, particularly to remove the conservation constraint for the the total wealth $\int_0^\infty y f(y, t)\, dy$. In the following, we refer to this scenario as a "non-conservative economy". In addition, we consider an alternative (and, in some sense, more realistic) model, where players do not interact with each other in the form of binary interactions, but with the whole ensemble of players (the market). That is, we do not consider the mean field limit of a binary interaction model, but start from an inherent mean field model.

Naturally, one takes moments of the wealth distribution function f with respect to the wealth variable y. We define the density of agents $\rho(x, t)$ and the density of higher order moments of the wealth variable $\rho\Upsilon_k(x,t)$, by:

$$\rho(x,t) = \int f(x,y,t)\ dy, \qquad \rho\Upsilon_k(x,t) = \int y^k f(x,y,t)\ dy\ , k = 1, 2, \ldots \tag{1.3}$$

So, $\rho(x,t)$ is the density of agents in the economic configuration space, $\rho\Upsilon_1(x,t)$ is the density of the mean wealth, $\rho(\Upsilon_2 - \Upsilon_1^2)$ is the density of the variance of the wealth, and so on. We will restrict the dependence of a_f, b_f, c_f in the cost functional Φ_f to a dependence on the above defined mean densities $\Upsilon_1, \Upsilon_2, \ldots$.

1.2 Conservative vs. non-conservative economies

Computing the first three moments of the operator Q in (1.1) gives, using integration by parts

$$\int \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} Q(f)(x,y,t)\ dy = \begin{pmatrix} 0 \\ -a_f\Upsilon_1 - b_f \\ 2(d-a_f)\Upsilon_2 - 2b_f\Upsilon_1 \end{pmatrix} \rho(x,t)$$

Consequently, we obtain a hierarchy for the moments of the density function $f(x,y,t)$ with respect to the wealth variable y. The first three term of the hierarchies are of the form

$$\partial_t \begin{pmatrix} \rho \\ \rho\Upsilon_1 \\ \rho\Upsilon_2 \\ \ldots \end{pmatrix} + \partial_x \int V(x,y) f(x,y,t) \begin{pmatrix} 1 \\ y \\ y^2 \\ \ldots \end{pmatrix}\ dy = \begin{pmatrix} 0 \\ -a_f\Upsilon_1 - b_f \\ 2(d-a_f)\Upsilon_2 - 2b_f\Upsilon_1 \\ \ldots \end{pmatrix} \rho(x,t)\ . \tag{1.4}$$

The system (1.4) is of course not closed, since the flux terms on the left hand side of (1.4) are in general unknown for an arbitrary density function f. The closure of the hierarchy (1.4) at a certain level has to be performed by some asymptotic analysis and scaling arguments, which are the subject of this paper. We are faced with a conservative economy if the dependence of the coefficients in the quadratic cost functional Φ_f on the density f are such that $a_f\Upsilon_1 + b_f = 0$ holds for any density f. In this case, the total wealth $\rho\Upsilon_1$ is preserved, when integrated over the configuration variable x. So, we consider a conservative economy, for $a_f\Upsilon_1 + b_f = 0$. In this case, we would have , considering equn. (1.1), $\frac{d}{dt}\iint yf(x,y,t)\,dxdy = 0$, and the total wealth in the economy would be conserved in time.

The case of a conservative economy ($a_f\Upsilon_1 + b_f = 0,\ \forall f,$), i.e. the cost functional Φ_f in (1.2) being a parabola, centered around Υ_1, has been considered in [15] and, in a game theoretical framework, in [13]. In this paper, we consider a non - conservative economy ($a_f\Upsilon_1 + b_f \neq 0$, except in equilibrium) where wealth is generated or lost due to productivity of the agents or inflation.

1.3 Frequent trading

In this paper we will consider an asymptotic regime, where the dynamics is dominated by the trading interaction of the agents, i.e. where the operator Q is the dominant term in equation (1.1). In the case of a conservative economy (preserving wealth with $a_f\Upsilon_1 + b_f = 0,\ \forall f$), this leads to a closed macroscopic system for the variables ρ and Υ_1. This system has been treated in [13] and [15]. The more general form of the collision operator, with a general potential Φ_f in (1.2), still preserves the density of agents, so 1 is a collision invariant. (For simplicity we disregard the birth and death of the agents.) However, the total wealth in the system is no longer necessarily conserved if $a_f\Upsilon_1 + b_f \neq 0$ holds, although wealth is conserved in each individual transactions. This is indeed the main driving force behind the economy and results in non - conservative economy. The non - conservative case considerably complicates the derivation of a macroscopic evolution equation for the density $\rho(x,t)$, since it is not possible to use a local conservation law for the mean wealth density $\rho\Upsilon_1$ in the frequent trading limit, as done in [13] and [15]. We address this problem by using the concept of a general collision invariant (GCI), as introduced in [14]. This yields a macroscopic balance law (which is not conservative) for the mean wealth density $\rho(x,t)\Upsilon_1(x,t)$ in the limit of frequent trading.

The local equilibrium wealth distribution is also a Nash equilibrium for the non-conservative economy. It is in general computed by solving an infinite dimensional fixed point problem. However, the fixed point solution cannot be given explicitly for general coefficients a_f, b_f and c_f, in contrast to the previous literature where they could be expressed in terms of an inverse Gamma distribution [13]. Rather, they are found by solving a linear partial differential equation together with a finite dimensional fixed point equation. If multiple solutions to this fixed point equation exist, corresponding to multiple stable equilibria, this indicates that phase transitions in the wealth distribution are possible. However, we leave the question of the existence and enumeration of the solutions to the fixed point equation to future work.

In Section 4 we make a particular modeling choice for the coefficients a_f and b_f in the cost functional Φ. This choice corresponds to each player interacting with the market ("trading") with a frequency which is inverse proportional to the uncertainty of the market, i.e. to the variation coefficient of the probability distribution f in (1.1). We refer to this assumption as the "risk averse" scenario, which means that traders are more likely to trade, the better they can predict the development of the market. In addition, each player tries to achieve an acceptable risk level (given by a constant κ which has to be matched to actual market data). These choices allow us to express the macroscopic large time average equations of the distribution of players and their wealth explicitly in equation (1.4)

This paper is organized as follows. In Section 2, we present the multi-agent model for the dynamics of N agents, each interacting with the market (the ensemble of all agents). This gives the Fokker-Planck equation (1.1) for the effective single agent density $f(x,y,t)$. In Section 3, the equations are put in dimensionless form and the Gibbs measure in the frequent trading limit is introduced. We show that the Gibbs measure expresses a Nash equilibrium, i.e. no player can improve on the cost function by choosing a different direction in y. In Section 4 we consider the inhomogeneous case. We introduce the GCI concept in a general setting and then, specify a simplified yet economically relevant setting where the GCI concept leads to explicit calculations. This leads to an explicit closure of the moments of the kinetic equation (1.1). The final macroscopic model is summarized in Section 5. Finally, we conclude by drawing some perspectives in section 6.

2 Game theoretical framework

We consider a set of N market agents. Each agent, labeled j, is endowed with two variables: its wealth $Y_j \in \mathbb{R}_+$ and a variable $X_j \in \mathcal{X}$, where $\mathcal{X}$ is an interval of $\mathbb{R}$. The variable X_j characterizes the agent's economic configuration, i.e. the category of agents it usually interacts with. We ignore the possibility of debts so that we take $Y_j \geq 0$. We use notations $\vec{X}(t) = (X_1, \dots, X_N)$, $\vec{Y}(t) = (Y_1, \dots, Y_N)$ to describes the ensemble of all agents. To single out the market environment for the j-th agent, we denote $\hat{X}_j = (X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_N)$ and $\hat{Y}_j = (Y_1, \dots, Y_{j-1}, Y_{j+1}, \dots, Y_N)$ for the ensemble of all agents other than his/her self (note that in game theory, $\hat{Y}_j$ is often denoted Y_{-j}). We also write $\vec{X} = (X_j, \hat{X}_j)$ and $\vec{Y} = (Y_j, \hat{Y}_j)$ to represent the agent j in the market environment $(X_j, \hat{X}_j, Y_j, \hat{Y}_j)$. We denote the cost function for the j-th agent in this market environment as $\Phi^N(X_j, \hat{X}_j, Y_j, \hat{Y}_j, t)$ or $\Phi^N(\vec{X}, Y_j, \hat{Y}_j, t)$. The best-reply strategy is mostly used in economy. Each agent tries to minimize the cost function with respect to its wealth variable, assuming that the other agents do not change theirs. The agents choose the steepest descent direction of their cost function $Y_j \to \Phi^N(\vec{X}, Y_j, \hat{Y}_j)$ as their action in wealth space, i.e.,

$$\mathcal{F}^N(\vec{X}, Y_j, \hat{Y}_j, t) = -\partial_{Y_j} \Phi^N(\vec{X}, Y_j, \hat{Y}_j, t)$$

This action is supplemented with a geometric Brownian noise which models volatility. The resulting dynamics of the j-th agent is described below

$$\dot{X}_j = V(X_j(t), Y_j(t)), \tag{2.5}$$

$$dY_j = \mathcal{F}^N(\vec{X}, Y_j, \hat{Y}_j, t)\, dt + \sqrt{2d}\; Y_j\, dB_t^j. \tag{2.6}$$

The stochastic geometric Brownian noise is understood in the Itò sense and the quantity $\sqrt{2d}$ is the volatility while the notation B_t^j denote independent Brownian motions. The first equation above describes how fast the agent evolves in the economic configuration space as a function of its current wealth and current economic configuration and $V(x, y)$ is a measure of the speed of this motion. We assume that the function V decays to zero at far field if the domain is unbounded, and that $V = 0$ holds on the boundary $\partial\mathcal{X}$ if the domain is bounded, i.e.

$$V \to 0 \quad \text{as} \quad x \to \partial\mathcal{X}, \tag{2.7}$$

holds.

In this dynamics, the agents would eventually, at large times, reach a point of minimum of their cost function. This minimum would then be written

$$Y_j^N(\vec{X}, \hat{Y}_j, t) = \arg \min_{Y_j \in \mathbb{R}_+} \Phi^N(\vec{X}, Y_j, \hat{Y}_j, t), \quad \forall j \in \{1, \dots, N\}. \tag{2.8}$$

and corresponds to a Nash equilibrium of the agents. Therefore, the dynamics correspond to a non-cooperative non-atomic anonymous game [2, 22, 32, 33], also known as a Mean-Field Game [8, 20], where the equilibrium assumption is replaced by a time dynamics describing the march towards a Nash equilibrium. A game theoretical framework for this general setting was developed by the authors in [12] and applied to study conservative economies in [13].

In this paper we consider a modified, and in some sense more realistic, model where the cost functional Φ does not depend on the individual values $\hat{Y}_j$ of the other agents, but depends instead on average quantities of the ensemble. This means that agents are not trading with each other individually, but trade with a market (i.e. the ensemble of all other agents), still trying to optimize their individual costs. So we consider a cost functional of the form

$$\Phi^N = \Phi^N(\vec{X}, Y_j, \Upsilon), \quad \mathcal{F}^N = -\partial_{Y_j}\Phi^n$$

with Υ given by the averaged properties of the ensemble of all agents (the market). (In this paper, we will take Υ to be the given by the first two moments, corresponding to the mean and the variance, of the wealth in the whole market. So, $\Upsilon = (\Upsilon_1, \Upsilon_2) = (\sum_k Y_k, \sum_k Y_k^2)$ holds.) In the limit $N \to \infty$, the one-particle distribution function f is then a solution of the Fokker-Planck equation :

$$\partial_t f + \partial_x(V(x,y)f) + \partial_y(F_f\, f) = d\,\partial_y^2\big(y^2 f\big), \tag{2.9}$$

where $F_f = F_f(x,y,t)$ is given by

$$F_f(x,y,t) = -\partial_y \Phi_{f(t)}(x,y)\ , \tag{2.10}$$

and Φ_f depends on the density f only through $\Upsilon(f)$. This equation is posed for $(x,y) \in \mathcal{X} \times [0,\infty[$. We supplement this equation with the no flux boundary condition at $y=0$:

$$d\partial_y(y^2 f) - F_f\, f|_{y=0} = 0, \quad \forall x \in \mathcal{X}, \quad \forall t \in \mathbb{R}_+. \tag{2.11}$$

With the assumption (2.7) on V, there is no need for any boundary condition on f on $\partial\mathcal{X}$. These conditions imply that the number of agents is conserved in time for the kinetic system, i.e. $\int_{x\in\mathcal{X}}\int_{y\in[0,\infty)} f(x,y,t)\,dx\,dy =$ Constant. We also provide an initial condition $f(x,y,0) = f_0(x,y)$.

In this paper we consider a specific trading model with the market and take a the following quadratic cost function with coefficients depending functionally on the ensemble of agents

$$\Phi_f(x,y) = \frac{1}{2}a_f(y + \frac{b_f}{a_f})^2 + c_f - \frac{1}{2}\frac{b_f^2}{a_f} = \frac{1}{2}a_f y^2 + b_f y + c_f\ , \tag{2.12}$$

a_f represents the trading frequency with the market and $y = -b_f/a_f$ represents the optimum the agent tries to achieve. Note that constant c_f plays no role in strategy $\mathcal{F}_f$ and we can set it as $b_f^2/(2a_f)$. The cost function (2.12) resembles the structure of the cost function used in [13], but contains now arbitrary coefficients a_f and b_f. The trading frequency now is taken to be uniform and depends on the market environment. The coefficient a_f will be given an interpretation in the example of the risk-adverse strategy below. The flexibility in the choice of a_f and b_f in the functional enables us to model market strategies. Specifically, in Section 4, a risk averse strategy will be taken for a_f

$$a_f = \frac{d\Upsilon_2}{\Upsilon_2 - \Upsilon_1^2}$$

where Υ_1 and Υ_2 are the first and second moments of the agent ensemble defined as above. a_f/d represents the ratio between strategy action and the volatility and is given by $\Upsilon_2/(\Upsilon_2 - \Upsilon_1^2)$, the reciprocal of the variation coefficient of the $\vec{Y}$. In completely deterministic market, with no variation, the trading frequency of the agent would be infinite. On the other hand, in an extremely uncertain market, with an infinite variance, trading frequency would be given just by the uncertainty introduced by the Brownian motion, and $a_f = d$ holds.

3 Dimensionless formulation and the frequent trading limit

3.1 Dimensionless formulation

One of the main characterization in the evolution of wealth distribution is spatio-temporal scale separation. The economic interaction (the dynamic in the y-direction) is fast compared to the spatiotemporal scale of the motion in the economic configuration space (i.e. the x variable). In order to manage the various scales in a proper way, we change the variables to dimensionless ones. Following the procedure developed in [12], we introduce the macroscopic scale. We assume that the changes in economic configuration x are slow compared to the exchanges of wealth between agents. We introduce t_0 and $x_0 = v_0 t_0$ the time and economic configuration space units, with v_0 the typical magnitude of V. We scale the wealth variable y, by a monetary unit y_0. Defining $x_s = \frac{x}{x_0}$, $y_s = \frac{y}{y_0}$, $t_s = \frac{t}{t_0}$ and $f_s(x_s, y_s, t_s) = x_0 y_0 f(x, y, t)$. Correspondingly, we scale the mean wealth density Υ_1 and the velocity $V(x,t)$ by $\Upsilon_1(x,t) = y_0 \Upsilon_{1s}(x_s, t_s)$ and $V(x,t) = \frac{x_0}{t_0} V_s(x_s, t_s)$. We scale the trading frequency parameters a_f and b_f in (1.2) by $a_f = \frac{1}{\varepsilon t_0} a_{f_s}$ and $b_f = \frac{y_0}{\varepsilon t_0} b_{f_s}$ and the variance d in the geometric Brownian motion by $d = \frac{1}{\varepsilon t_0} d_s$, with $\varepsilon \ll 1$ a small dimensionless parameter. This means, that we consider the frequency of the trading activity, given by the parameters d, a_f, b_f to be large compared to the frequency of movement in the economic configuration space, given by the average size v_0 of V. This gives the dimensionless formulation of equation (1.1) as (dropping the subscript s for notational convenience):

$$\partial_t f^\varepsilon + \partial_x (f^\varepsilon V(x,y)) = \frac{1}{\varepsilon} Q(f^\varepsilon), \tag{3.13}$$

$$Q(f) = \partial_y \left[d\, \partial_y (y^2 f) + (a_f\, y + b_f)\, f \right]. \tag{3.14}$$

In the dimensionless formulation the moment hierarchy (1.4) is still given by

$$\partial_t \begin{pmatrix} \rho^\varepsilon \\ \rho^\varepsilon \Upsilon_1^\varepsilon \\ \rho^\varepsilon \Upsilon_2^\varepsilon \\ \dots \end{pmatrix} + \partial_x \int V(x,y)\, f^\varepsilon(x,y,t) \begin{pmatrix} 1 \\ y \\ y^2 \\ \dots \end{pmatrix} dy = \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ -(a_{f^\varepsilon} \Upsilon_1^\varepsilon + b_{f^\varepsilon}) \\ 2(d - a_{f^\varepsilon}) \Upsilon_2^\varepsilon - 2 b_{f^\varepsilon} \Upsilon_1^\varepsilon \\ \dots \end{pmatrix} \rho^\varepsilon(x,t)\ . \tag{3.15}$$

The left-hand side of (3.15) describes the slow dynamics of the moments of distribution in the economy configuration variable x and time t. This evolution is driven by the fast, local evolution of this distribution as a function of the individual decision variables y described by the right-hand side. The parameter ε at the denominator highlights that fact that the internal decision variables evolve on a faster time scale than the external economy configuration variables. According to [13], the fast evolution of the internal decision variables drives agents performing a "rapid march", i.e. on a $O(\frac{1}{\varepsilon})$ time scale, towards a Nash equilibrium, defined by the game of minimizing the functional Φ_f in (1.2), up to a diffusion.

3.2 The frequent trading limit and the Gibbs measure

In the limit of frequent trading interaction (when ε in the previous section is small compared to 1), the macroscopic dynamics are given by the shape of the solution of $Q(f) = 0$. In the following we will restrict the form of the nonlinear operator Q such that the coefficients a_f and b_f in (3.13) depend only on the means of the first 2 moments of the wealth variable. We define the vector valued functional $\underline{\Upsilon}(f)$ acting from the space of distribution functions into $\mathbb{R}^2$ via the definition

$$\underline{\Upsilon}(f) = (\underline{\Upsilon}_1(f), \underline{\Upsilon}_2(f)), \quad \underline{\Upsilon}_k(f) = \frac{\int y^k f(y)\, dy}{\int f(y)\, dy},\ k = 1, 2\ .$$

So, the scaled trading operator Q in (3.13) takes the form $Q(f) = C[f, \underline{\Upsilon}(f)]$, with the operator C given by

$$Q(f) = C[f, \underline{\Upsilon}(f)] = \partial_y[d\partial_y(y^2 f) + (a_{\underline{\Upsilon}(f)} y + b_{\underline{\Upsilon}(f)})f].$$

We note that, although Q is a nonlinear operator, the nonlinearity is restricted to the dependence of Q on the mean moments $\underline{\Upsilon}(f)$. In other words, for a given vector Υ the operator $C[f, \Upsilon]$ is linear in f. This allows for the definition of a normalized Gibbs measure $G_\Upsilon(y)$ satisfying (for a given vector Υ) the linear problem

$$C[G_\Upsilon, \Upsilon] = \partial_y[d\partial_y(y^2 G_\Upsilon) + (a_\Upsilon\, y + b_\Upsilon)\, G_\Upsilon] = 0, \quad \int_0^\infty G_\Upsilon(y)\ dy = 1 \tag{3.16}$$

We reformulate the solution of $Q(f) = 0$ as the combination of a linear infinite dimensional problem (solving the linear PDE (3.16) for a given vector Υ), and a two dimensional fixed point problem. The computation of the local thermodynamic equilibrium, the solution of $Q(f) = 0$, $\int f\ dy = 1$, is then given by the solution G_Υ of (3.16) where the two-dimensional vector Υ is a solution of the fixed point problem:

$$\underline{\Upsilon}(G_\Upsilon) = \Upsilon\ . \tag{3.17}$$

The shape of the probability distribution $f(x, y, t)$ in the frequent trading limit $\varepsilon \to 0$ is then given by $f^{\text{equ}}(x, y, t) = \rho(x, t) G_\Upsilon(y)$, with G_Υ satisfying (3.16) and Υ satisfying the fixed point problem (3.17), since multiplying G_Υ by a y-independent density $\rho(x, t)$ does not change the mean moments Υ.

The form (3.16) of the trading operator $C[G_\Upsilon, \Upsilon]$ allows for the computation of the mean moment vector $\underline{\Upsilon}(G_\Upsilon)$ via a recursion formula which is obtained by a simple integration by parts argument. Integrating equation (3.16) against y^k gives, using the zero flux boundary condition at $y = 0$

$$\int_0^\infty [(a_\Upsilon - d(k-1))\, y^k + b_\Upsilon\, y^{k-1}]\, G_\Upsilon\; dy = 0, \quad \int_0^\infty G_\Upsilon(y)\; dy = 1\ ,$$

and, in particular for the first two moments $\underline{\Upsilon}(G_\Upsilon)$ with $k = 1, 2$:

$$a_\Upsilon \underline{\Upsilon}_1(G_\Upsilon) + b_\Upsilon = 0, \quad (a_\Upsilon - d)\, \underline{\Upsilon}_2(G_\Upsilon) + b_\Upsilon\, \underline{\Upsilon}_1(G_\Upsilon) = 0\ . \tag{3.18}$$

The fixed point equations (3.17) take then the form

$$a_\Upsilon \Upsilon_1 + b_\Upsilon = 0, \qquad (a_\Upsilon - d)\Upsilon_2 + b_\Upsilon \Upsilon_1 = 0\ . \tag{3.19}$$

- So, the equilibrium solution is computed by first finding all solutions to the fixed point equation (3.19), i.e. (3.19) plays the role of a constitutive relation for the moments in local equilibrium.

- For any vector $\Upsilon = (\Upsilon_1, \Upsilon_2)$ satisfying the constitutive relations (3.19) there exists a local equilibrium $f^{\text{equ}}(x, y, t)$ given by $f^{\text{equ}}(x, y, t) = \rho(x, t) G_\Upsilon(y)$ with a local agent density $\rho(x, t)$ and G_Υ the solution of problem (3.16).

- The shape of the local equilibrium solution $f^{\text{equ}} = \rho G_\Upsilon$ determines of course the large time average of the solution, and in turn this shape depends on modeling the coefficients a_Υ and b_Υ. So, modeling a_Υ and b_Υ determines the form of the macroscopic equations given in Section 4. To obtain macroscopic balance laws, in addition to the trivial conservation law for the number of agents, the coefficients a_Y, b_Y have to be such that the constitutive relations (3.19) have multiple solutions.

- In [13] and [15] the special case, when a_Υ and b_Υ depend only on the first moment Υ_1, has been treated. In this case finding the Gibbs measure by solving (3.16), (3.17) reduces to a linear problem and solutions can be computed explicitly in terms of inverse Gamma distributions, recovering well known results given c.f. in [1].

- Unfortunately, it turns out that this makes the macroscopic equations trivial, except in the case of a conservative economy when the coefficients a_Υ and b_Υ satisfy $a_\Upsilon\, \Upsilon_1 + b_\Upsilon = 0$.

- In this paper, we therefore consider a more refined model, where the coefficients a_Υ and b_Υ depend on Υ_1 and Υ_2, i.e. on the mean and the variance of the wealth of the market, which allows for the consideration of non - conservative economies with $a_\Upsilon\, \Upsilon_1 + b_\Upsilon \neq 0$.

4 Large time averages and hydrodynamic hierarchy closures using the Gibbs measure

The goal of this section is to close the hierarchy (3.15) in Section 3 by a local equilibrium, i.e. by a probability density function f of the form $f(x, y, t) = \rho(x, t) G_{\Upsilon(x,t)}(y)$ with the

Gibbs measure $G_\Upsilon(y)$ computed from the results in Subsection 4.2. For a conservative economy, where the coefficients $a_\Upsilon,\ b_\Upsilon$ are such that $a_\Upsilon \Upsilon_1 + b_\Upsilon = 0$ holds $\forall f$ in equation (1.4), this is rather straight forward since we immediately obtain two conservation laws for the density of agents and the mean wealth on large $O(\frac{1}{\varepsilon})$ time scales. These can be closed by replacing $f(x,y,t)$ by the local equilibrium density $\rho(x,t)G_{\Upsilon(x,t)}(y)$ in (3.15). This has been done in the papers [15] and, in a game theoretical framework, in [13]. *In the case of a non - conservative economy $a_\Upsilon \Upsilon_1 + b_\Upsilon \neq 0$, just taking the first moment of the transport equation 3.13 with respect to y does not yield a macroscopic conservation law on large time scales, i.e. an equation which is independent of ε.* We therefore need to integrate the transport equation 3.13 against a more sophisticated test function, called a generalized collision invariant (GCI), proposed in [14].

4.1 The GCI concept

We consider a kinetic equation of the form

$$\partial_t f^\varepsilon + \partial_x (V\, f^\varepsilon) = \frac{1}{\varepsilon} Q(f^\varepsilon) \tag{4.20}$$

with $Q(f)$ a nonlinear operator of the form $Q(f) = C[f, \underline{\Upsilon}(f)]$. The mean moment operator $\underline{\Upsilon}(f) = (\underline{\Upsilon}_1(f), \dots, \underline{\Upsilon}_K(f))$ is defined as in Section 1 by $\int y^k f\ dy = \underline{\Upsilon}_k \int f\ dy$, $k = 1, \dots, K$. The operator $f \mapsto C[f, \Upsilon]$ is linear for a given vector $\Upsilon \in \mathbb{R}_+^K$. So, the nonlinear dependence of $Q(f)$ on f is restricted to the nonlinear dependence of $C[f, \underline{\Upsilon}(f)]$ on $\underline{\Upsilon}(f)$. Integrating (4.20) against any test function $z(x,y)$ w.r.t. y gives

$$\int z\{\partial_t f^\varepsilon + \partial_x (V\, f^\varepsilon)\}\ dy = \frac{1}{\varepsilon}\int z Q(f^\varepsilon)\ dy\ , \tag{4.21}$$

A macroscopic balance law results if $\int z Q(f)\ dy = 0$. One obvious choice is $z = 1$, giving the conservation of the number of agents. In the case of a conservative economy, with $\int y Q(f)\ dy = 0,\ \forall f$, treated in [15] and [13], the other choice is $z = y$, giving a set of hydrodynamic type equations on the macroscopic level. The basic idea of a GCI, developed in [14], is to make the function z dependent on the moments $\underline{\Upsilon}(f)$ of the kinetic solution f, such that the right hand side in (4.21) vanishes. This yields a macroscopic balance law of the form

$$\int \chi_{\underline{\Upsilon}(f^\varepsilon)} \{\partial_t f^\varepsilon + \partial_x (V\, f^\varepsilon)\}\ dy = 0\ , \tag{4.22}$$

if, for any $\Upsilon \in \mathbb{R}_+^K$, we can find $z = \chi_\Upsilon$ such that

$$\int \chi_\Upsilon\, C[f, \Upsilon]\ dy = 0,\ \forall f \text{ such that } \underline{\Upsilon}(f) = \Upsilon \text{ holds.} \tag{4.23}$$

Using the special structure of $Q(f) = C[f, \underline{\Upsilon}(f)]$, this can be achieved by using the L^2-adjoint of the operator $f \mapsto C[f, \Upsilon]$. Let $C^{\mathrm{adj}}[g, \Upsilon]$ be defined by

$$\int g\, C[f, \Upsilon]\ dy = \int f\, C^{\mathrm{adj}}[g, \Upsilon]\ dy\ .$$

That χ_Υ satisfies (4.23) is equivalent to saying that

$$\exists(\lambda_1, \dots, \lambda_K) \in \mathbb{R}^K \ \text{ such that } \ C^{\text{adj}}[\chi_\Upsilon, \Upsilon] = \sum_{k=1}^{K} \lambda_k (\Upsilon_k - y^k) \ . \tag{4.24}$$

Then we have

$$\begin{aligned} \int \chi_{\underline{\Upsilon}(f)}\, Q(f)\, dy &= \int \chi_{\underline{\Upsilon}(f)}\, C[f, \underline{\Upsilon}(f)]\, dy \\ &= \int f\, C^{\text{adj}}[\chi_{\underline{\Upsilon}(f)}, \underline{\Upsilon}(f)]\, dy \\ &= \sum_{k=1}^{K} \lambda_k \int f\, (\underline{\Upsilon}_k(f) - y^k)\, dy = 0 \ , \end{aligned}$$

by the definition of $\underline{\Upsilon}_k(f)$. So the problem of finding the macroscopic balance laws for equation (4.20) reduces to finding all the GCI's i.e. all the solutions of (4.24). For any given vector Υ, the set of associated GCI forms a linear manifold of dimension $M+1$, with $M \le K$: indeed, the constants are solutions and form a linear space of dimension 1 and the non-constant GCI's form a linear vector space of dimension M. We can have $M < K$ since some compatibility conditions between the λ_k may be required. From now on, χ_Υ denotes a vector of M independent non-constant GCI.

If we can prove that the solution of the kinetic equation (4.20) is really given up to order $O(\varepsilon)$ by the equilibrium solution, i.e. if $f^\varepsilon = \rho G_\Upsilon + \varepsilon f_1$ holds, then

$$\partial_t(\rho\, G_\Upsilon) + \partial_x(V \rho\, G_\Upsilon) = \frac{1}{\varepsilon} \rho C[G_{\underline{\Upsilon}(G_\Upsilon + \varepsilon f_1)}, \quad \underline{\Upsilon}(G_\Upsilon + \varepsilon f_1)] + O(\varepsilon) \tag{4.25}$$

holds. Letting $\varepsilon \to 0$ gives an indefinite limit of the form $\frac{0}{0}$ on the right hand side of equation (4.25), since Υ satisfies the constitutive equations $\underline{\Upsilon}(G_\Upsilon) = \Upsilon$, and $C[G_\Upsilon, \Upsilon] = 0$ holds. Integrating (4.25) against $\chi_{\underline{\Upsilon}(\rho G_\Upsilon + \varepsilon f_1)}$ gives

$$\int \chi_{\underline{\Upsilon}(\rho G_\Upsilon + \varepsilon f_1)} [\partial_t(\rho\, G_\Upsilon) + \partial_x(V \rho\, G_\Upsilon)]\, dy = O(\varepsilon) \ ,$$

and, in the limit $\varepsilon \to 0$ the closed macroscopic equations

$$\partial_t \rho + \partial_x(\rho \int V(x,y)\, G_\Upsilon\, dy) = 0, \qquad \int \chi_\Upsilon [\partial_t(\rho\, G_\Upsilon) + \partial_x(V \rho\, G_\Upsilon)]\, dy = 0 \ , \tag{4.26}$$

with Υ satisfying the constitutive relations $\underline{\Upsilon}(G_\Upsilon) = \Upsilon$.

This leads to the following recipe for computing macroscopic balance laws for a kinetic equation of the form (4.20) with a collision operator $Q(f)$, only conserving the number of agents, i.e. only satisfying $\int Q(f)\, dy = 0$, $\forall f$, but not conserving any additional moments.

- For a general vector Υ, find the solution of (4.24). Unfortunately, this will have to be done, in practice, numerically for nontrivial operators C^{adj}.

- As pointed out earlier, the Lagrange multipliers λ_k, $k = 1, \dots, K$ may not be chosen arbitrarily. Indeed, they have to satisfy certain conditions, depending on the structure of the operator C^{adj}, such that the GCI equation (4.24) is solvable. We also repeat that the GCI's form a linear vector space and that we denote by χ_Υ a vector of independent non-constant GCI spanning the space of non-constant GCI.

- This gives in the limit $\varepsilon \to 0$ the macroscopic equations, which are independent of the microscopic variable y and the parameter ε:

$$\partial_t \rho + \partial_x \Big(\int f\, V(x, y)\, dy \Big) = 0, \qquad \int \chi_{\underline{\Upsilon}(f)} \left\{ \partial_t f + \partial_x (f\, V(x, y)) \right\} dy = 0, \quad (4.27)$$

  with ρ defined as $\rho(x, t) = \int f(x, y, t)\, dy$. The system (4.27) still has to be closed by choosing an approximate solution f for the kinetic equation (4.20).

- The system (4.27) is closed by choosing $f = f^{\mathrm{equ}} = \rho G_\Upsilon$, with G_Υ being the Gibbs measure from Section 4.2 in our case, this choice being justified by the formal limit $\varepsilon \to 0$ in (4.20).

- To compute the Gibbs measure G_Υ in Subsection 4.2, we have to solve the infinite dimensional problem $C[G_\Upsilon, \Upsilon] = 0$, $\int G_\Upsilon\, dy = 1$, for a general vector Υ, and then solve the, finite dimensional, fixed point problem $\underline{\Upsilon}(G_\Upsilon) = \Upsilon$ for the vector Υ.

- The final macroscopic equations (4.27) will be of the form

$$\partial_t \rho + \partial_x \Big(\int \rho G_\Upsilon\, V(x, y)\, dy \Big) = 0, \qquad \int \chi_\Upsilon \left\{ \partial_t (\rho G_\Upsilon) + \partial_x (\rho G_\Upsilon\, V) \right\} dy = 0, \quad (4.28)$$

  with Υ satisfying the constitutive relation $\underline{\Upsilon}(G_\Upsilon) = \Upsilon$.

- For the system (4.28) to be closed, the fixed point equation $\underline{\Upsilon}(G_\Upsilon) = \Upsilon$ should have a manifold structure, parametrized by as many independent parameters as independent non-constant GCI. The free parameters in the fixed point equation $\underline{\Upsilon}(G_\Upsilon) = \Upsilon$ are essentially the other dependent variable (besides ρ) in the system (4.28), although it it might never be explicitly expressed, but given implicitly by the constitutive equations. In the example of the risk-adverse strategy below, the variables are the density and the mean wealth (meaning that the constitutive relation has only a one-parameter family of solutions, parametrized by the mean wealth) and the macroscopic system consists of the density conservation equation and a non-conservative balance equation for the mean-wealth.

4.2 Non-conservative economies with risk averse trading strategies

In the model, considered in this paper, individual agents try to minimize the cost functional $\Phi_{\underline{\Upsilon}(f)}(y)$ with

$$\Phi_\Upsilon(y) = \frac{1}{2} a_\Upsilon y^2 + b_\Upsilon y + c_\Upsilon = \frac{1}{2} a_\Upsilon \Big(y + \frac{b_\Upsilon}{a_\Upsilon}\Big)^2 + c_\Upsilon - \frac{1}{2} \frac{b_\Upsilon^2}{a_\Upsilon}\ ,$$

given market conditions represented by the density f. So, a_Υ represents (in dimensionless variables) the frequency of the trades with the market, i.e. the strategy of an agent to trade or not to trade, and $y = -\frac{b_\Upsilon}{a_\Upsilon}$ represents the (market dependent) optimum, the agent tries to achieve. We consider a risk averse strategy of the form

$$a_\Upsilon = \frac{d\,\Upsilon_2}{\Upsilon_2 - \Upsilon_1^2}, \tag{4.29}$$

and refer to the end of Section 2 for its interpretation. The constant in the potential does not influence the dynamics and we can take $c_\Upsilon - \frac{1}{2}\frac{b_\Upsilon^2}{a_\Upsilon} = 0$. We choose the coefficient b_Υ such that,

$$b_\Upsilon = -(1+\kappa)\, d\,\Upsilon_1, \tag{4.30}$$

with a fixed constant $\kappa > 0$. This choice is motivated by the consideration of the Nash equilibrium below.

Using the choice (4.29) for a_Υ, we compute the Gibbs measure introduced in Section 3 3.2 from $C[G_\Upsilon, \Upsilon] = 0$, $\int G_\Upsilon \; dy = 1$, i.e. from equation (3.16). It yields the constitutive relations for the vector $\Upsilon = (\Upsilon_1, \Upsilon_2)$ from the recursion formula (3.18) as

$$\frac{d\,\Upsilon_2}{\Upsilon_2 - \Upsilon_1^2}\,\Upsilon_1 + b_\Upsilon = 0, \qquad (\frac{d\,\Upsilon_2}{\Upsilon_2 - \Upsilon_1^2} - d)\,\Upsilon_2 + b_\Upsilon\,\Upsilon_1 = \Upsilon_1\,(\frac{d\,\Upsilon_1\,\Upsilon_2}{\Upsilon_2 - \Upsilon_1^2} + b_\Upsilon) = 0\ . \tag{4.31}$$

Since the two equations involved in (4.31) are the same, up to a multiplicative factor Υ_1, the first Eq. (4.31) yields the constitutive relation. For any choice of b_Υ (and in particular, for the choice given by (4.30)), this equation is one equation in two unknowns Υ_1, Υ_2 and has a one parameter family of solutions.

Now, using the first equation (4.31) together with (4.30), we obtain

$$\frac{\Upsilon_2}{\Upsilon_2 - \Upsilon_1^2} = -\frac{b_\Upsilon}{d\Upsilon_1} = 1 + \kappa, \qquad \text{or equivalently} \qquad \Upsilon_2 - \Upsilon_1^2 = \frac{1}{\kappa}\Upsilon_1^2. \tag{4.32}$$

This means that, at the Nash equilibrium when every player has optimized its cost functional, there exists a finite amount of risk in the market, measured by the fraction $\frac{1}{\kappa}$ of the squared mean wealth Υ_1^2. So, the choice (4.30) is equivalent to choosing some desired global risk, i.e. a global variation coefficient $\frac{1}{\kappa}$ in the equilibrium market. The first Eq. (4.32) leads to the following relation between Υ_1 and Υ_2 at equilibrium:

$$\Upsilon_2 = \frac{1+\kappa}{\kappa}\Upsilon_1^2\ . \tag{4.33}$$

which is the form taken by the constitutive relation (3.19) in the present example.

To arrive at the closed macroscopic system (4.28) we still have to compute the Gibbs measure G_Υ and the GCI χ_Υ for a general vector $\Upsilon = (\Upsilon_1, \Upsilon_2)$, satisfying the constitutive relations (4.33). The Gibbs measure is given, according to equation (3.16) by the solution of

$$\partial_y[d\,\partial_y(y^2\,G_\Upsilon) + (\frac{d\,\Upsilon_2}{\Upsilon_2 - \Upsilon_1^2}\,y - d\,\Upsilon_1\,(1+\kappa))\,G_\Upsilon] = 0, \qquad \int_0^\infty G_\Upsilon(y)\;dy = 1\ , \tag{4.34}$$

with Υ satisfying (4.33). Using the constitutive relations (4.33) this gives

$$\partial_y[d\,\partial_y\,(y^2\,G_\Upsilon) + d\,(1+\kappa)\,(y-\Upsilon_1)\,G_\Upsilon] = 0, \quad \int_0^\infty G_\Upsilon(y)\;dy = 1\;, \tag{4.35}$$

together with the zero flux boundary condition $d\,\partial_y\,(y^2\,G_\Upsilon)+d\,(1+\kappa)\,(y-\Upsilon_1)\,G_\Upsilon|_{y=0} = 0$, which guarantees the conservation of the number of agents in the system. The solution of (4.35) is given by

$$G_\Upsilon(y) = \frac{1}{c_\Upsilon}\,y^{-\kappa-3}\,e^{-\frac{(1+\kappa)\,\Upsilon_1}{y}}, \quad c_\Upsilon = \int_0^\infty y^{-\kappa-3}\,e^{-\frac{(1+\kappa)\,\Upsilon_1}{y}}\;dy\;. \tag{4.36}$$

G_Υ is therefore given by an inverse Gamma distribution, i.e.

$$G_\Upsilon(y) = g_{\kappa+2,(1+\kappa)\Upsilon_1}(y)$$

where the inverse Gamma distribution $g_{\alpha,\beta}$ is defined as $g_{\alpha,\beta} = \frac{\beta^\alpha}{\Gamma(\alpha)}y^{-1-\alpha}e^{-\frac{\beta}{y}}$ with shape parameter α and scale parameter β and $\Gamma(\alpha)$ denoting the Euler Gamma function evaluated at α. It is related to the usual Gamma function Γ by: $\gamma_{\alpha,\beta}(z) = \frac{\beta^\alpha}{\Gamma(\alpha)}z^{\alpha-1}e^{-\beta z}$ by the change of variables $z = \frac{1}{y}$. This distribution has been previously found in [7]. When y is large, the distribution becomes the Pareto power law distribution, which has a very strong agreement with economic data (see e.g. the review in [39]). $G_\Upsilon(y) = g_{\kappa+2,(1+\kappa)\Upsilon_1}(y)$ represent the large time average (i.e. the Nash equilibrium) of a game of players, where each player tries to play the market to achieve a desired risk, given by the constitutive relation (4.33), which is a dimensionless measure of the uncertainty of the market. We also note that, in order for the local equilibrium distribution G_Υ to have a finite variance, i.e. $\int_0^\infty y^2\,G_\Upsilon\;dy < \infty$, the value of κ in (4.36) should be positive ($\kappa > 0$).

4.3 The GCI for risk-adverse trading strategies

Let $\Upsilon = (\Upsilon_1, \Upsilon_2)$ be given, not necessarily related by the constitutive relation (4.33). With the choices (4.29), (4.30), Eq. (4.24) is written:

$$\partial_y\big(y^2 G_\Upsilon \partial_y \psi\big) = \lambda_1(y-\Upsilon_1)G_\Upsilon + \lambda_2(y^2-\Upsilon_2)G_\Upsilon. \tag{4.37}$$

The weak formulation of this equation is

$$\int_0^\infty y^2 G_\Upsilon\,\partial_y\psi\,\partial_y\sigma = -\int_0^\infty \lambda_1(y-\Upsilon_1)\,G_\Upsilon\,\sigma\,dy - \int_0^\infty \lambda_2(y^2-\Upsilon_2)\,G_\Upsilon\,\sigma\,dy, \tag{4.38}$$

for all σ. We note that the formalism of the paper [13] and particularly of its Lemma 3.5 applies. It uses an appropriate functional setting, and we refer the reader to[13] for the details. In [13], it is shown that a solution to (4.38) exists if and only if the following solvability condition (whose necessity is easily found by taking $\sigma = 1$) is satisfied:

$$\int_0^\infty \lambda_1(y-\Upsilon_1)\,G_\Upsilon\,dy + \int_0^\infty \lambda_2(y^2-\Upsilon_2)\,G_\Upsilon\,dy = 0,$$

or in other words:

$$\lambda_1\,(\underline{\Upsilon}_1(G_\Upsilon)-\Upsilon_1)+\lambda_2(\underline{\Upsilon}_2(G_\Upsilon)-\Upsilon_2)=0. \tag{4.39}$$

Now, we define

$$\chi_\Upsilon=\frac{y^2}{2}-\Upsilon_1\,y, \tag{4.40}$$

Using (4.34) (and not (4.35) because we do not suppose the constitutive relation (4.33) to be satisfied), we get

$$\partial_y\big(y^2G_\Upsilon\partial_y\chi_\Upsilon\big)=\frac{\Upsilon_1}{\Upsilon_2-\Upsilon_1^2}\Big\{-\Upsilon_1(y^2-\Upsilon_2)+\Upsilon_2\Big(1+(1+\kappa)\big(1-\frac{\Upsilon_1^2}{\Upsilon_2}\big)\Big)\,(y-\Upsilon_1)\Big\}\,G_\Upsilon. \tag{4.41}$$

This equation is of the form (4.37) with

$$\lambda_1=\frac{\Upsilon_1}{\Upsilon_2-\Upsilon_1^2}\,\Upsilon_2\Big(1+(1+\kappa)\big(1-\frac{\Upsilon_1^2}{\Upsilon_2}\big)\Big),\qquad \lambda_2=-\frac{\Upsilon_1}{\Upsilon_2-\Upsilon_1^2}\,\Upsilon_1.$$

With the help of (3.18) to compute $\underline{\Upsilon}_k(G_\Upsilon)$, $k=1,2$, we immediately verify that the constraint (4.39) is satisfied. From (4.39), it follows that the space of non-constant GCI is of dimension 1 and since χ_Υ is a non-constant GCI, all non-constant GCi are proportional to χ_Υ.

4.4 The equation for the mean wealth

Thanks to (4.40), the second Eq. (4.28) is given by:

$$\int_0^\infty\big(\frac{y^2}{2}-\Upsilon_1(x,t)\,y\big)\,\partial_t(\rho G_\Upsilon)\,dy+\int_0^\infty\big(\frac{y^2}{2}-\Upsilon_1(x,t)\,y\big)\,\partial_x(V(x,y)\,\rho G_\Upsilon)\,dy=0. \tag{4.42}$$

This gives

$$\begin{aligned}\partial_t\int_0^\infty\big(\frac{y^2}{2}-\Upsilon_1\,y\big)\,\rho G_\Upsilon\,dy+\partial_t\Upsilon_1\int_0^\infty y\,\rho G_\Upsilon\,dy\\ +\,\partial_x\int_0^\infty\big(\frac{y^2}{2}-\Upsilon_1\,y\big)\,V\,\rho G_\Upsilon\,dy+\partial_x\Upsilon_1\int_0^\infty y\,V\,\rho G_\Upsilon\,dy=0.\end{aligned} \tag{4.43}$$

We also remind the mass conservation equation (the first Eq. (4.28)). We define:

$$U_k(x;\Upsilon_1)=\Big(\int_0^\infty V(x,y)\,G_\Upsilon(y)\,y^k\,dy\Big)\Big|_{\Upsilon_2=\frac{1+\kappa}{\kappa}\Upsilon_1^2},\quad k\in\mathbb{N}, \tag{4.44}$$

and we get

$$\partial_t\rho+\partial_x(\rho U_0)=0\ . \tag{4.45}$$

Now, we have, thanks to (4.33),

$$\partial_t\int_0^\infty\big(\frac{y^2}{2}-\Upsilon_1\,y\big)\,\rho G_\Upsilon\,dy+\partial_t\Upsilon_1\int_0^\infty y\,\rho G_\Upsilon\,dy=-\frac{1-\kappa}{2\kappa}\,\Upsilon_1^2\,\partial_x(\rho U_0)+\frac{1}{\kappa}\rho\Upsilon_1\partial_t\Upsilon_1. \tag{4.46}$$

and

$$\partial_x \int_0^\infty \big(\frac{y^2}{2} - \Upsilon_1\, y\big)\, V\, \rho G_\Upsilon\, dy + \partial_x \Upsilon_1 \int_0^\infty y\, V\, \rho G_\Upsilon\, dy = \partial_x\big(\rho \frac{U_2}{2}\big) - \Upsilon_1\, \partial_x(\rho U_1). \quad (4.47)$$

Inserting (4.46), (4.47), into (4.43), we finally get the equation for the mean wealth Υ_1:

$$\rho \partial_t \Upsilon_1 + \frac{\kappa}{2\Upsilon_1} \partial_x(\rho U_2) - \big[\kappa\, \partial_x(\rho U_1) + \frac{1-\kappa}{2}\, \Upsilon_1\, \partial_x(\rho U_0)\big] = 0. \quad (4.48)$$

5 The macroscopic model

To summarize, the macroscopic model is the following system for the agent density $\rho(x,t)$ and the local mean wealth $\Upsilon_1(x,t)$:

$$\partial_t \rho + \partial_x(\rho U_0) = 0, \quad (5.49)$$

$$\rho \partial_t \Upsilon_1 + \frac{\kappa}{2\Upsilon_1} \partial_x(\rho U_2) - \big[\kappa\, \partial_x(\rho U_1) + \frac{1-\kappa}{2}\, \Upsilon_1\, \partial_x(\rho U_0)\big] = 0. \quad (5.50)$$

with

$$U_k = U_k(x; \Upsilon_1) = \Big(\int_0^\infty V(x,y)\, G_\Upsilon(y)\, y^k\, dy \Big)\Big|_{\Upsilon_2 = \frac{1+\kappa}{\kappa} \Upsilon_1^2}, \quad k = 0,\, 1,\, 2. \quad (5.51)$$

It could be further simplified by assuming specific values of $V(x,y)$. We leave this to future work.

6 Conclusions

We have derived a model for the large time averages of a set of agents, interacting with each other through a market, and moving around in an abstract configuration space. Each player interacts with the market (”trades”) with a frequency which is inverse proportional to the uncertainty of the market, and tries to achieve an acceptable risk (given by a constant κ which has to be matched to actual market data). The model does not rely on the assumption of conservation of the total wealth in the system, but instead uses the concept of generalized collision invariants to derive macroscopic equations for the large time averages. In this sense, this paper is a generalization, as well as an alternative, to previously considered models in [7], [12], [15], where only binary trading interactions between individual agents have been considered under the assumption of conservation of the total wealth in the system. The final macroscopic model consists of a conservation law for the number of agents in the system and a balance law for the mean and the variance of the total wealth, supplemented by a constitutive relation for mean and variance. So, in the large time limit, agents move in configuration space (which is assumed to be one dimensional in this paper for the sake of notational simplicity) according to two partial differential equations (5.49), (5.50) in time and one spatial variable.

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