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GENERIC ORBITS AND TYPE ISOLATION IN THE GURARIJ SPACE

ITAI BEN YAACOV AND C. WARD HENSON

ABSTRACT. This version is incomplete, and was uploaded merely to supersede an earlier version which contained some mistakes.

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INTRODUCTION

In 1966, Gurarij [Gur66] defined what came to be known as the (separable) Gurarij space, and proved that it almost isometrically unique. The isometric uniqueness of the Gurarij space was only proved in 1976 by Lusky [Lus76]. In the same paper, Lusky points out that the arguments could be modified to prove also the isometric uniqueness of the separable Gurarij space equipped with a distinguished smooth unit vector. In other words, if \mathbf{G} denotes the separable Gurarij space, then the set of smooth unit vectors in \mathbf{G} forms an orbit under the action of the linear isometry group $\text{Aut}(\mathbf{G})$. By Mazur [Maz33], this orbit is moreover a dense G_δ subset of the unit sphere.

These facts are strongly reminiscent of model theoretic phenomena, and indeed turn out to be special cases of such. It was observed some time ago by the second author that the uniqueness of the Gurarij space can be accounted for as it being the unique separable model of an \aleph_0 -categorical theory, which moreover eliminates quantifiers. Similarly, the Gurarij space is atomic over a vector if and only if the latter is smooth, so Lusky's second uniqueness result is a special case of the uniqueness of the prime model (namely, separable atomic model) over a vector of norm one (by quantifier elimination, the type of a single vector is entirely determined by its norm).

These observations serve as a starting point for the present paper, whose goals are threefold:

- Our primary goal is to make the observations above precise, and generalise them to uniqueness results over a subset other than the empty set or a singleton – in other words, we study uniqueness and primeness of the Gurarij space over a subspace E of dimension possibly greater than one.
- A secondary goal is to present a subset of the toolbox of model theory in a manner accessible non logicians. Starting with a definition of types and type spaces which does not make any use of formal logic, we discuss general topics such as type isolation, the Tarski-Vaught Criterion, the Omitting Types Theorem, and the primeness and uniqueness of atomic models. While we do this

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in a fairly specific context, we attempt to present arguments which would be valid in the general case (possibly with separate follow-up results which improve the general ones in a manner specific to the context of the Gurarij space). There are few results which make explicit use of formal logic (essentially, Proposition 1.19 and Theorem 2.3), which serve mostly as parenthetical remarks required for completeness, and are not used in any way in other parts of the paper.

- A minor tertiary goal is to present to model theorists, who are familiar with the tools mentioned in the previous item *in the context of classical logic*, how these tools adapt to the metric setting.

In Section 1 we define (quantifier-free) types and type spaces over a Banach space E , and study their properties. The topometric structure of the type space, a fundamental notion of metric model theory, is defined there, as well as (topometrically) isolated types, which are one of the main objects of study of this paper.

In Section 2 we start studying Gurarij spaces. At the technical level, we define and study Gurarij (and other) spaces which are atomic over a fixed separable parameter space E , and prove the Omitting Types Theorem (Theorem 2.12). We prove appropriate generalisations of the homogeneity and universality properties of the Gurarij space to homogeneity and universality over E . In particular, we show that the prime Gurarij spaces over E (see Corollary 2.13) are those Gurarij space which are separable and atomic over E , and that they are all isometrically isomorphic over E , denoted $\mathbf{G}[E]$. We also give the standard model theoretic criterion for the existence of $\mathbf{G}[E]$.

At this point we move on to the question of when $\mathbf{G}[E]$ exists, i.e., when the isolated types over E are dense, and how to characterise them in a fashion more intrinsic to the Banach space context. In Section 3 we consider the particularly easy case where $\dim E = 1$, which serves as a good indication for where to look later on.

We conclude in Section 6 with a “counting types” result, showing that the space of types over E is metrically separable if and only if E is finite-dimensional and polyhedral. This allows us to answer a question of Avilés et al. [ACC⁺11].

Throughout, E, F and so on denote normed spaces over the real numbers. An embedding (or isomorphism, automorphism) of normed spaces is always isometric.

The topological dual of a normed space E will be denoted E^* . We shall often use the notation $E_{\leq 1}$ for the closed unit ball of E , $E_{=1}$ for the unit sphere, and so on.

1. QUANTIFIER-FREE TYPES IN BANACH SPACES

Before we start, let us state the following basic amalgamation result which we shall use many times, quite often implicitly.

Fact 1.1. *For any three Banach spaces E, F_0 and F_1 , and isometric embeddings $f_i: E \rightarrow F_i$, there is a third Banach space G and isometric embeddings $g_i: F_i \rightarrow G$ such that $g_0 f_0 = g_1 f_1$.*

Proof. Equip the direct sum $F_0 \oplus F_1$ with the semi-norm $\|v + u\| = \inf_{w \in E} \|v + w\| + \|u - w\|$, divide by the kernel and complete. ■_{1.1}

We can now define the fundamental objects of study of this section, and, to a large extent, the entire paper.

Definition 1.2. Let E be a Banach space and X a sequence of symbols which we call *variables*. We let $E(X) = E \oplus \bigoplus_{x \in X} \mathbf{R}x$, and define $S_X(E)$ to consist of all semi-norms on $E(X)$ which extend the norm on E , calling it the *space of types in X over E* . We shall denote members of $S_X(E)$ by ξ, ζ and so on, and the corresponding semi-norms by $\|\cdot\|^\xi, \|\cdot\|^\zeta$ and so on (model-theoretic tradition would have us denote types by p, q and so on, but an expression such as $\|\cdot\|^p$ may be disastrously suggestive of a meaning other than the intended one).

Quite often X will be of the form $\{x_i\}_{i \in I}$ for some index set I , in which case we write $E(I) = E \oplus \bigoplus_{i \in I} \mathbf{R}x_i$ instead of $E(X)$, and similarly $S_I(E)$, whose members are called *I -types*.

Definition 1.3. Given a Banach space extension $E \subseteq F$ and an I -sequence $\bar{a} = \{a_i\}_{i \in I} \subseteq F$, we define its *type over E* , in symbols $\xi = \text{tp}(\bar{a}/E) \in S_I(E)$, to be the semi-norm $\|b + \sum \lambda_i x_i\|^\xi = \|b + \sum \lambda_i a_i\|$, and say that \bar{a} *realises* ξ . When a sequence \bar{b} generates E , we may also write $\text{tp}(\bar{a}/\bar{b})$ for $\text{tp}(\bar{a}/E)$.

Conversely, given a type $\xi \in S_I(E)$, we define the Banach space *generated* by ξ , in symbols $E[\xi]$, as the space obtained from $(E(I), \|\cdot\|^\xi)$ by dividing by the kernel and completing, together with the distinguished generators $\{x_i\}_{i \in I} \subseteq E[\xi]$.

Definition 1.4. We equip $S_I(E)$ with a topological structure as well as with a metric structure *which are often distinct*. The *topology* on $S_I(E)$ is the least one in which, for every member $x \in E(I)$, the map $\hat{x}: \xi \mapsto \|x\|^\xi$ is continuous. Given $\xi, \zeta \in S_I(E)$, we define the *distance* $d(\xi, \zeta)$ to be the infimum, over all F extending E and over all realisations \bar{a} and \bar{b} of ξ and ζ , respectively, of $\sup_i \|a_i - b_i\|$.

Remark 1.5. A model-theorist will recognise types as we define them here as *quantifier-free* types, which do not, in general, capture “all the pertinent information”. However, by Fact 1.1, they do capture a maximal existential type. Moreover, it follows from Lemma 1.14 below (and more specifically, from the assertion that $\pi_{\bar{x}}: S_{\bar{x},y}(0) \rightarrow S_{\bar{x}}(0)$ is open) that being an existentially closed Banach space is an elementary property, so the theory of Banach spaces admits a model companion. Then Fact 1.1 can be understood to say that the model companion eliminates quantifiers, so quantifier-free types and types are in practice the same. As we shall see later, the model companion is separably categorical, and its unique separable model is \mathbf{G} , the separable Gurarij space.

It is fairly clear that the distance refines the topology, and we shall see that unless the parameter space E is trivial and I is finite, they are in fact distinct. In a sense, the distance as defined on $S_I(E)$ is “incorrect” when I is infinite (for more reasons than the mere fact that this distance can be infinite), and we should never have defined it for such I if not for Proposition 1.7 below holding for infinite I as well.

Lemma 1.6. *Let E, F be Banach spaces, I an index set, and consider tuples $\bar{a} = (a_i)_{i \in I} \in E^I$, $\bar{b} \in F^I$ and $\bar{\varepsilon} \in \mathbf{R}^I$. Let also $\mathbf{R}^{(I)}$ denote the set of all I -tuples in which all but finitely many positions are zero. The following conditions are equivalent.*

- (i) *There exists a semi-norm $\|\cdot\|$ on $E \oplus F$ extending the respective norms of E and F , such that for each $i \in I$ one has $\|a_i - b_i\| \leq \varepsilon_i$.*
- (ii) *For all $\bar{r} \in \mathbf{R}^{(I)}$, one has*

$$\left\| \sum r_i a_i \right\| - \left\| \sum r_i b_i \right\| \leq \sum |r_i| \varepsilon_i.$$

Proof. One direction being trivial, we prove the other. For $c + d \in E \oplus F$ define

$$\|c + d\|' = \inf_{\bar{r} \in \mathbf{R}^{(I)}} \left\| c - \sum r_i a_i \right\| + \left\| d + \sum r_i b_i \right\| + \sum |r_i| \varepsilon_i.$$

This is easily checked to be a semi-norm, with $\|c\|' \leq \|c\|$ for $c \in E$. Now, for $c \in E$ and $\bar{r} \in \mathbf{R}^{(I)}$ we have

$$\left\| c - \sum r_i a_i \right\| + \left\| \sum r_i b_i \right\| + \sum |r_i| \varepsilon_i \geq \left\| c - \sum r_i a_i \right\| + \left\| \sum r_i a_i \right\| \geq \|c\|.$$

Therefore $\|c\|' = \|c\|$, and similarly $\|d\|' = \|d\|$ for $d \in F$, concluding the proof. ■_{1.6}

Proposition 1.7. *Let $\xi, \zeta \in S_I(E)$, and let $E(I)_1$ consist of all $a + \sum \lambda_i x_i \in E(I)$ (so $a \in E$ and all but finitely many of the λ_i vanish) such that $\sum |\lambda_i| = 1$. Then*

$$d(\xi, \zeta) = \sup_{x \in E(I)_1} \left| \|x\|^\xi - \|x\|^\zeta \right|.$$

Moreover, the infimum in the definition of distance between types is attained.

Proof. Immediate from Lemma 1.6. ■_{1.7}

Convention 1.8. When referring to the topological or metric structure of $S_I(E)$, we shall follow the convention that unqualified terms taken from the vocabulary of general topology (open, compact and so on) apply to the topological structure, while terms specific to metric spaces (bounded, complete and so on) refer to the metric structure.

Excluded from this convention is the notion of isolation which will be defined in a manner which takes into account both the topology and the metric.

While this convention may seem confusing at first, it is quite convenient, as in the following.

- Lemma 1.9.** (i) The space $S_I(E)$ is Hausdorff, and every closed and bounded set thereof is compact.
(ii) The distance on $S_I(E)$ is lower semi-continuous. In particular, the closure of a bounded set is bounded.
(iii) Assume that I is finite, say $I = n = \{0, 1, \dots, n-1\} \in \mathbf{N}$. Then every bounded set is contained in an open bounded set. It follows that the space $S_n(E)$ is locally compact, and that a compact subset of $S_n(E)$ is necessarily (closed and) bounded.
(iv) A subset $X \subseteq S_n(E)$ is closed if and only if its intersection with every compact set is compact.
(v) Let $m \leq n$, and let $\pi: S_n(E) \rightarrow S_m(E)$ denote the obvious variable restriction map. Then for every $\xi \in S_n(E)$ and $\zeta \in S_m(E)$ we have $d(\pi\xi, \zeta) = d(\xi, \pi^{-1}\zeta)$. Moreover there exists $\rho \in \pi^{-1}\zeta$ such that $d(\pi\xi, \zeta) = d(\xi, \rho)$ and $\|x_i\|^\rho = \|x_i\|^\xi$ for all $m \leq i < n$.
In particular, the map π is metrically open.

Proof. For the first item, clearly $S_I(E)$ is Hausdorff. If $X \subseteq S_I(E)$ is bounded, then for every $x \in E(I)$ there exists M_x such that $\|x\|^\xi \leq M_x$ for all $\xi \in X$. We can therefore identify X with a subset of $Y = \prod_x [0, M_x]$, and if X is closed in $S_I(E)$ then it is closed in Y and therefore compact.

The second item follows from Proposition 1.7, and the third is immediate.

For the fourth item, assume that $X \subseteq S_n(E)$ is not closed, let $\xi \in \overline{X} \setminus X$ and let U be a bounded neighbourhood of ξ , in which case $\overline{U} \cap X$ is not compact.

For (v), the inequality \leq is immediate. For the opposite inequality, there exists an extension $F \supseteq E$ and realisations \bar{a} of ξ and \bar{b} of ζ in F such that $\|a_i - b_i\| < r$ for $i < m$. Letting $c_i = b_i$ for $i < m$, $c_i = a_i$ for $m \leq i < n$, we see that $\rho = \text{tp}(\bar{c}/E)$ is as desired for both the main assertion and the moreover part. It follows that $\pi B(\xi, r) \supseteq B(\pi\xi, r)$, so π is metrically open. ■_{1.9}

This double structure makes $S_I(E)$ a *topometric space*, in the sense of [Ben08b].

Definition 1.10. We say that a type $\xi \in S_n(E)$ is *isolated* if the distance and the topology agree at ξ , i.e., if every metric neighbourhood of ξ is also a topological one.

This is the definition of isolation in a topometric space, taking into account both the metric and the topological structure. Ordinary topological spaces can be viewed as topometric spaces by equipping them with the discrete 0/1 distance, in which case the notion of isolation as defined here coincides with the usual one.

Many results regarding ordinary topological spaces still hold, when translated correctly, with the topometric definitions. For example, the fact that a dense set must contain all isolated points becomes the following. Notice that in Lemma 1.16 below we prove that the set of isolated types is itself metrically closed.

Lemma 1.11. *Let E be a Banach space, $D \subseteq S_n(E)$ a dense, metrically closed set. Then D contains all isolated types.*

Proof. If ξ is isolated then all metric neighbourhoods of ξ , which are also topological neighbourhoods, must intersect D . ■_{1.11}

One of our aims in this paper is to characterise isolated types. We start with the easiest situation.

Proposition 1.12. *Let 0 denote the trivial Banach space. Then every type in $S_n(0)$ is isolated. In other words, the distance on $S_n(0)$ is compatible with the topology.*

Proof. Given $N \in \mathbf{N}$, let $X_N \subseteq 0(n)_1$ be the finite set consisting of all $\sum \lambda_i x_i$ where $\sum |\lambda_i| = 1$ and each λ_i is of the form $\frac{k}{N}$. For $\xi \in S_n(0)$, let $U_{\xi, N}$ be its neighbourhood consisting of all ζ such that

$$\forall x \in X_N \quad \|x\|^\xi - 1/N < \|x\|^\zeta < \|x\|^\xi + 1/N.$$

This means in particular that $\|x_i\|^\zeta < \|x_i\|^\xi + 1$ for all $i < n$, and now an easy calculation together with Proposition 1.7 yields that there exists a constant $C(\xi)$ such that for all N , $U_{\xi, N}$ is contained in the ball of radius $C(\xi)/N$ around ξ , which is what we had to show. ■_{1.12}

This already allows us to construct the following useful tool of variable change in a type.

Definition 1.13. Given a linear map $\varphi: E(\bar{y}) \rightarrow E(\bar{x})$ extending id_E , we define a pull-back map $\varphi^*: S_{\bar{x}}(E) \rightarrow S_{\bar{y}}(E)$ by $\|z\|^{\varphi^*\xi} = \|\varphi z\|^\xi$ (for $z \in E(\bar{y})$). For $A \subseteq S_{\bar{y}}(E)$, we define $\varphi_* A = (\varphi^*)^{-1}(A) \subseteq S_{\bar{x}}(E)$ (this will be particularly convenient in the proof of Lemma 2.11).

Of course, φ is entirely determined by the image $\varphi\bar{y}$. Thus, when the variables \bar{y} are known from the context, we may write $\xi \upharpoonright_{\varphi\bar{y}}$ for $\varphi^*\xi$, so

$$\left\| a + \sum \lambda_i y_i \right\|^{\xi \upharpoonright_{\bar{z}}} = \left\| a + \sum \lambda_i z_i \right\|^\xi.$$

In fact, we shall often use this latter notation with $\bar{z} = \bar{y}$.

Lemma 1.14. For a fixed tuple $\bar{y} \in E(\bar{x})^m$, the map $\xi \mapsto \xi \upharpoonright_{\bar{y}}$ is continuous and Lipschitz. If \bar{y} are linearly independent over E then this map is also topologically and metrically open. Moreover, the metric openness is “Lipschitz” as well, in the sense that there exists a constant $C = C(\bar{y})$ such that for all ξ and all $r > 0$ we have

$$B(\xi, r) \upharpoonright_{\bar{y}} \supseteq B(\xi \upharpoonright_{\bar{y}}, Cr).$$

Proof. Continuity and the Lipschitz condition are easy. We therefore assume that \bar{y} are linearly independent over E , and we first prove the moreover part. In the special case where \bar{y} generate $E(\bar{x})$ over E , this is since $(\cdot \upharpoonright_{\bar{y}})^{-1} = \cdot \upharpoonright_{\bar{x}}: S_{\bar{y}}(E) \rightarrow S_{\bar{x}}(E)$ is Lipschitz. In the general case, we may complete \bar{y} into a basis for $E(\bar{x})$ over E , and using the special case above we reduce to the case where $y_i = x_i$ for $i < m$, which is just Lemma 1.9(v).

For topological openness, we proceed as follows. In the case where $E = 0$, this follows from metric openness and Proposition 1.12. Let us consider now the case where E is finite-dimensional. We fix a basis \bar{b} for E and a corresponding tuple of variables \bar{w} . We may then identify $E(\bar{x})$ with $0(\bar{w}, \bar{x})$, and thus \bar{y} with its image in $0(\bar{w}, \bar{x})$. We already know that $\cdot \upharpoonright_{\bar{w}, \bar{y}}: S_{\bar{w}, \bar{x}}(0) \rightarrow S_{\bar{w}, \bar{y}}(0)$ is open. In addition, we have a commutative diagram

$$\begin{array}{ccc} S_{\bar{w}, \bar{x}}(0) & \xrightarrow{\cdot \upharpoonright_{\bar{w}, \bar{y}}} & S_{\bar{w}, \bar{y}}(0) \\ & \searrow \cdot \upharpoonright_{\bar{w}} & \swarrow \cdot \upharpoonright_{\bar{w}} \\ & & S_{\bar{w}}(0) \end{array}$$

and the map $\cdot \upharpoonright_{\bar{y}}: S_{\bar{x}}(E) \rightarrow S_{\bar{y}}(E)$ is homeomorphic to the fibre of the horizontal arrow over $\text{tp}(\bar{b}) \in S_{\bar{w}}(0)$, so it is open as well. The infinite-dimensional case follows from the finite-dimensional one, since any basic open set in $S_{\bar{x}}(E)$ can be defined using finitely many parameters in E . ■_{1.14}

We leave it to the reader to check that if \bar{y} are not linearly independent over E then $\cdot \upharpoonright_{\bar{y}}$ is not metrically open, and *a fortiori* not topologically so (consider for example $\cdot \upharpoonright_{x, x}: S_1(0) \rightarrow S_2(0)$).

Lemma 1.15. Let $U \subseteq S_n(E)$ be open and $r > 0$. Then $B(U, r)$ is open as well.

Proof. Let \bar{x} and \bar{y} be two n -tuples of variables. Let us identify $S_n(E)$ with $S_{\bar{x}}(E)$, and let $W \subseteq S_{\bar{x}, \bar{y}}(E)$ consist of all ξ such that $\|x_i - y_i\|^\xi < r$ for $i < n$. Then W is open, and by Lemma 1.14 so is $V = (W \cap (\cdot \upharpoonright_{\bar{x}})^{-1}(U)) \upharpoonright_{\bar{y}} \subseteq S_{\bar{y}}(E)$. Identifying $S_{\bar{y}}(E)$ with $S_n(E)$ as well, $V = B(U, r)$. ■_{1.15}

Lemma 1.16. Let E be a Banach space.

- (i) A type in $S_n(E)$ is isolated if and only if all its metric neighbourhoods have non empty interior.
- (ii) The set of isolated types in $S_n(E)$ is metrically closed.

Proof. The first assertion follows easily from Lemma 1.15, and the second from the first. ■_{1.16}

Another basic operation one can consider on types is the *restriction of parameters* $S_n(F) \rightarrow S_n(E)$ when $E \subseteq F$.

Lemma 1.17. Let $E \subseteq F$ be an isometric inclusion of Banach spaces. Then the natural type restriction map $\pi: S_n(F) \rightarrow S_n(E)$ is continuous, closed, and satisfies $\pi B(\xi, r) = B(\pi\xi, r)$.

In particular, π is both topologically and metrically a quotient map.

Proof. It is clear that π is continuous. To see that it is closed we use Lemma 1.9. Indeed, since closed sets are exactly those which intersect compact sets on compact sets, it will be enough to show that if $K \subseteq S_n(E)$ is compact then so is $\pi^{-1}K$, which follows from the characterisation of compact sets as closed and bounded. Finally, it is clear that $d(\xi, \zeta) \geq d(\pi\xi, \pi\zeta)$ for $\xi, \zeta \in S_n(F)$. Conversely, if $\zeta_0 \in S_n(E)$ then using Fact 1.1, there exists $\zeta \in \pi^{-1}\zeta_0$ with $d(\xi, \zeta) \geq d(\pi\xi, \zeta_0)$, which proves that $\pi B(\xi, r) = B(\pi\xi, r)$. $\blacksquare_{1.17}$

We also obtain the following result, which is somewhat of an aside with respect to the rest of this paper. We shall therefore allow ourselves to be brief, and assume that the reader is familiar with continuous first order logic (see [BU10, BBHU08]), and, for the part regarding Banach spaces as unbounded metric structures, also with unbounded continuous logic (see [Ben08a]).

Lemma 1.18. *Let T be an inductive theory, and for $n \in \mathbf{N}$ let $S_n^{\text{qf}}(T)$ denote the space of quantifier-free types consistent with T , equipped with the natural logic topology. Assume that, first, every two models of T amalgamate over a common substructure, and, second, for every n , the variable restriction map $S_{n+1}^{\text{qf}}(T) \rightarrow S_n^{\text{qf}}(T)$ is open. Then T admits a model completion, namely a companion which eliminates quantifiers.*

(In fact, an approximate amalgamation property for models of T over a common finitely generated substructure suffices.)

Proof. Let $\varphi(\bar{x}, y)$ be a quantifier-free formula, inducing a continuous function $\hat{\varphi}: S_{n+1}^{\text{qf}}(T) \rightarrow \mathbf{R}$ (which has compact range, by compactness of $S_{n+1}^{\text{qf}}(T)$). Let $\pi: S_{n+1}^{\text{qf}}(T) \rightarrow S_n^{\text{qf}}(T)$ denote the variable restriction map, and define $\rho: S_n^{\text{qf}}(T) \rightarrow \mathbf{R}$ as the infimum over the fibre:

$$\rho(q) = \inf \{ \hat{\varphi}(p) : \pi p = q \}.$$

Since π is continuous (automatically) and open (by hypothesis), ρ is continuous as well, and can therefore be expressed as a uniform limit of $\hat{\psi}_n: S_n^{\text{qf}}(T) \rightarrow \mathbf{R}$, where $\psi_n(\bar{x})$ are quantifier-free formulae, say $\|\rho - \hat{\psi}_n\| \leq 2^{-n}$. One can now express that $\sup_{\bar{x}} |\psi_n(\bar{x}) - \inf_y \varphi(\bar{x}, y)| \leq 2^{-n}$ for all n by a set of sentences.

Let T^* consist of T together with all sentences constructed as above, for all possible quantifier-free formulae $\varphi(\bar{x}, y)$. Then, first, every existentially closed model of T is easily checked to be a model of T^* (using our amalgamation hypothesis), so T and T^* are companions. Moreover, by induction on quantifiers, every formula is equivalent modulo T^* to a uniform limit of quantifier-free formulae, so T^* eliminates quantifiers. $\blacksquare_{1.18}$

Proposition 1.19. *Consider Banach spaces either as metric structures in unbounded continuous logic, or as bounded metric structures via their closed unit balls, as explained, say, in [Ben09]. Then (in either approach) the theory of the class of Banach spaces is inductive, and admits a model completion T^* which is moreover complete and \aleph_0 -categorical.*

When the entire Banach space is viewed as a structure then the types over a subspace are as per Definition 1.2 and Definition 1.3, and if one only considers the unit ball then the space of I -types over $E_{\leq 1}$ is $S_I^{\leq 1}(E) = \{ \xi \in S_I(E) : \|x_i\|^\xi \leq 1 \text{ for all } i \in I \}$.

Proof. Let us consider the theory T of unit balls of Banach spaces. It is clearly inductive, and it is fairly easy to check that the space of quantifier-free I -types over a unit ball $E_{\leq 1}$ is the space $S_I^{\leq 1}(E)$ defined in the statement. By the moreover part of Lemma 1.9(v), variable restriction $S_{n+1}^{\leq 1}(E) \rightarrow S_n^{\leq 1}(E)$ is metrically open. For $E = 0$ this implies in particular that $S_{n+1}^{\leq 1}(0) \rightarrow S_n^{\leq 1}(0)$ is topologically open, but this latter is just $S_{n+1}^{\text{qf}}(T) \rightarrow S_n^{\text{qf}}(T)$. This, together with Fact 1.1, fulfils the hypotheses of Lemma 1.18. By quantifier elimination, $S_n(T^*) = S_n^{\text{qf}}(T) = S_n^{\leq 1}(0)$, so in particular, $S_0(T^*)$ is a singleton, whereby T^* is complete. Finally, T^* is \aleph_0 -categorical by the Ryll-Nardzewski Theorem (see [BU07]).

The case of Banach spaces as unbounded structures follows via the bi-interpretability of the whole Banach space with its unit ball. $\blacksquare_{1.19}$

2. THE GURARIJ SPACE

Definition 2.1. We recall from, say, Lusky [Lus76] that a *Gurarij space* is a Banach space \mathbf{G} having the property that for any $\varepsilon > 0$, finite-dimensional Banach space $E \subseteq F$, and isometric embedding $\varphi: E \rightarrow \mathbf{G}$, there is a linear map $\psi: F \rightarrow \mathbf{G}$ extending φ such that in addition, for all $x \in F$, $(1 - \varepsilon)\|x\| \leq \|\psi x\| \leq (1 + \varepsilon)\|x\|$.

Some authors add the requirement that a Gurarij space be separable, but from our point of view it seems more elegant to consider separability as a separate property.

Lemma 2.2. *Let F be a Banach space. Then the following are equivalent:*

- (i) *The space F is a Gurarij space.*
- (ii) *For every n , the set of realised types $\text{tp}(\bar{a}/F)$, as \bar{a} varies over F^n , is dense in $S_n(F)$.*
- (iii) *Same for $n = 1$.*

Proof. (i) \implies (iii). Let $U \subseteq S_1(F)$ be open and $\xi \in U$. We may assume that U is defined by a finite set of conditions of the form $|\|a_i + r_i x\| - 1| < \varepsilon$, where $\|a_i + r_i x\|^\xi = 1$. Let $E \subseteq F$ be the subspace generated by the a_i , and let $E' = E + \mathbf{R}x$ be the extension of E generated by the restriction of ξ to E . By hypothesis, there is a linear embedding $\psi: E' \rightarrow F$ extending the identity such that $(1 - \varepsilon)\|y\| < \|\psi y\| < (1 + \varepsilon)\|y\|$ for all $y \in E'$, and in particular for $y = a_i + r_i x$, so $\text{tp}(\psi x/F) \in U$.

(iii) \implies (ii). We prove by induction on n , the case $n = 0$ being tautologically true. For the induction step, let $\emptyset \neq U \subseteq S_{\bar{x},y}(F)$ be open, and let $V = U \upharpoonright_{\bar{x}} \subseteq S_{\bar{x}}(F)$. By Lemma 1.14, V is open, and by the induction hypothesis there are $\bar{b} \in F^n$ such that $\text{tp}(\bar{b}/F) \in V$. Now, consider the map $\theta: S_y(F) \rightarrow S_{\bar{x},y}(F)$, sending $\text{tp}(a/F) \mapsto \text{tp}(\bar{b}, a/F)$. It is continuous (in fact, it is a topological embedding), so $\emptyset \neq \theta^{-1}U \subseteq S_1(F)$ is open. By hypothesis, there is $c \in F$ such that $\text{tp}(c/F) \in \theta^{-1}U$, i.e., such that $\text{tp}(\bar{b}, c/F) \in U$, as desired.

(ii) \implies (i). Let $E \subseteq E'$ be finite-dimensional, with $E \subseteq F$, and let $\varepsilon > 0$. Let \bar{a} be a basis for E , and let \bar{a}, \bar{b} be a basis for E' , say $|\bar{a}| = n$ and $|\bar{b}| = m$. For $N \in \mathbf{N}$, let $U_N \subseteq S_m(F)$ be defined by the (finitely many) conditions of the form $\|\sum s_i a_i + \sum r_j x_j\| \in (1 - \varepsilon, 1 + \varepsilon)$, where s_i and r_j are of the form $\frac{k}{N}$ and $\|\sum s_i a_i + \sum r_j b_j\| \in (1 - \varepsilon, 1 + \varepsilon)$. By hypothesis there is a tuple $\bar{c} \in F^m$ such that $\text{tp}(\bar{c}/F) \in U_N$, and we may define $\psi: E' \rightarrow F$ being the identity on E and sending $\bar{b} \mapsto \bar{c}$. For N big enough, it follows from the construction that if $y \in E'$, $\|y\| = 1$ then $|\|\psi y\| - 1| < 2\varepsilon$, which is good enough. $\blacksquare_{2.2}$

Model theorists may find the second and third conditions of Lemma 2.2 reminiscent of a topological formulation of the Tarski-Vaught Criterion: a metrically closed subset A of a structure is an elementary substructure if and only if the set of types over A realised in A is dense. Indeed,

Theorem 2.3. *Let T^* be the model completion of the theory of Banach spaces, as per Proposition 1.19. Then its models are exactly the Gurarij spaces. In particular, since T^* is \aleph_0 -categorical, there exists a unique separable Gurarij space (up to isometric isomorphism).*

Proof. Let E be a Banach space, and embed it in a model $F \models T^*$. Then, first, by quantifier elimination, E is a model of T^* if and only if $E \preceq F$. Second, by the topological Tarski-Vaught Criterion evoked above, $E \preceq F$ if and only if the set of types over E , in the sense of $\text{Th}(F) = T^*$, realised in E , is dense.

By Proposition 1.19 the space of types over E (in the sense of $T^* = \text{Th}(E)$) is $S_n^{\leq 1}(E)$ as defined there. By a dilation argument, the set of types realised in E is dense in $S_1(E)$ if and only if the set of types realised in $E_{\leq 1}$ is dense in $S_1^{\leq 1}(E)$, and we conclude by Lemma 2.2 (or, if one works with the whole space as an unbounded structure, the same holds without the dilation argument). $\blacksquare_{2.3}$

As mentioned in the introduction, the isometric uniqueness of the separable Gurarij space was originally proved by Lusky [Lus76] using the Lazar-Lindenstrauss matrix representation of L^1 pre-duals. The same was recently re-proved by Kubiś and Solecki [KS] using more elementary methods. Upon careful reading, their argument essentially consist of showing that the separable Gurarij space is the Fraïssé limit of the class of finite-dimensional Banach spaces, as is pointed out, alongside a general development of Fraïssé theory for metric structures (yielding yet another proof of the same result) by the first author [Bena]. From this point onward we shall leave continuous logic aside, and work entirely within the formalism of type spaces as introduced in Section 1. As we shall see, the uniqueness and existence also follow as easy corollaries from later results which do not depend explicitly on any form of formal logic (Corollary 2.7 and Lemma 2.11).

Definition 2.4. Let E be a Banach space. We say that a Banach space F is *atomic* over E if $E \subseteq F$ and the type over E of every finite tuple in F is isolated.

By Proposition 1.12, every Banach space is atomic over 0.

Theorem 2.5. *Let $E \subseteq F_0 \subseteq F_1$ be Banach spaces with $\dim F_0/E$ finite and F_1 separable and atomic over E , let $\mathbf{G} \supseteq E$ be a Gurarij space, and let $\varphi: F_0 \rightarrow \mathbf{G}$ be an isometric embedding extending id_E . Then there exist isometric embeddings $\psi: F_1 \rightarrow \mathbf{G}$ extending id_E with $\|\psi|_{F_0} - \varphi\|$ arbitrarily small.*

In particular, any separable Banach space atomic over E embeds isometrically over E in any Gurarij space containing E .

Proof. It is enough to prove this in the case where $\dim F_1/F_0 = 1$. We may then choose a basis $\bar{a} \in F_1^{n+1}$ for F_1 over E , such that in addition a_0, \dots, a_{n-1} generate F_0 . By hypothesis, $\xi = \text{tp}(\bar{a}/E) \in S_{n+1}(E)$ is isolated. Let $\rho: S_{n+1}(\mathbf{G}) \rightarrow S_{n+1}(E)$ be the parameter restriction map, and let $K = \rho^{-1}(\xi)$, observing that for any $\varepsilon > 0$, $B(K, \varepsilon) = \rho^{-1}B(\xi, \varepsilon)$ is a neighbourhood of K . We construct a sequence of tuples $\bar{c}_k \in \mathbf{G}^{n+1}$, each of which realising a type in $B(\xi, 2^{-k}r)$, as follows.

For $k = 0$, we let $V \subseteq S_{n+1}(\mathbf{G})$ be the set defined by $\|x_i - \varphi a_i\| < r$ for $i < n$, which is open and intersects K . Then $V \cap B(K, r)^\circ \neq \emptyset$ (where \cdot° denotes topological interior), and we choose \bar{c}_0 to realise some type there. Given \bar{c}_k , we let $U_k \subseteq S_{n+1}(\mathbf{G})$ be the set defined by $\|x_i - c_{k,i}\| < 2^{-k}r$ for $i \leq n$, which is again open intersecting K , and we choose \bar{c}_{k+1} to realise a type in $U_k \cap B(K, 2^{-n-1}r)^\circ$.

We obtain a Cauchy sequence (\bar{c}_k) converging to some $\bar{c} \in \mathbf{G}^{n+1}$, whose type $\text{tp}(\bar{c}/E)$, being the metric limit of $\text{tp}(\bar{c}_k/E)$, must be ξ . Then the linear map $\psi: F_1 \rightarrow \mathbf{G}$ which extends id_E by $a_i \mapsto c_i$ is an isometric embedding.

Finally, reading through our construction, we have $\|\varphi a_i - c_i\| < 3r$ for all $i < n$, and choosing r small enough, $\|\psi|_{F_0} - \varphi\|$ is as small as desired. ■_{2.5}

In particular, any two separable Gurarij spaces atomic over E embed in one another, but we can do better.

Theorem 2.6. *Let \mathbf{G}_i be separable Gurarij spaces atomic over E for $i = 0, 1$, let $E \subseteq F \subseteq \mathbf{G}_0$ with $\dim F/E$ finite, and let $\varphi: F \rightarrow \mathbf{G}_1$ be an isometric embedding extending id_E . Then there exist isometric isomorphisms $\psi: \mathbf{G}_0 \cong \mathbf{G}_1$ extending id_E with $\|\psi|_F - \varphi\|$ arbitrarily small.*

In particular, any two separable Gurarij spaces atomic over E are isometrically isomorphic over E .

Proof. Follows from Theorem 2.5 by a back-and-forth argument. ■_{2.6}

Since every Banach space is atomic over 0, we obtain the uniqueness and universality of the separable Gurarij space.

Corollary 2.7. *Every two separable Gurarij spaces are isometrically isomorphic, and every separable Banach space embeds isometrically in any Gurarij space (separable or not).*

Similarly, the Gurarij space is *approximately homogeneous*:

Corollary 2.8. *Let \mathbf{G} be a separable Gurarij space, let $F \subseteq \mathbf{G}$ be finite-dimensional, and let $\varphi: F \rightarrow \mathbf{G}$ is an isometric embedding. Then there exist isometric automorphisms $\psi \in \text{Aut}(\mathbf{G})$ such that $\|\psi|_F - \varphi\|$ is arbitrarily small.*

Moreover, if $E \subseteq F$ is such that \mathbf{G} is atomic over E , and $\varphi|_E = \text{id}$, then we may require that $\psi|_E = \text{id}$ as well.

Notation 2.9. We shall denote by \mathbf{G} the unique separable Gurarij space. Similarly, for a separable Banach space E , we let $\mathbf{G}[E]$ denote the unique atomic separable Gurarij space over E , if such exists, observing that since all types over 0 are isolated, $\mathbf{G} = \mathbf{G}[0]$.

Corollary 2.10. *Let E be a separable Banach space, and let $H = \text{Aut}(\mathbf{G})$ act by composition on the space of linear isometric embeddings $X = \text{Emb}(E, \mathbf{G})$, where both are equipped with the topology of point-wise convergence (the strong operator topology).*

- (i) *The space X is Polish, the action $H \curvearrowright X$ is continuous and all its orbits are dense.*
- (ii) *If $\mathbf{G}[E]$ exists, then the set of $\varphi \in X$ such that \mathbf{G} is atomic over φE (call these atomic embeddings) is a dense G_δ orbit under this action.*
- (iii) *If $\mathbf{G}[E]$ does not exist then there are no atomic embeddings and all orbits are meagre.*

Proof. The first item is easy and left to the reader (density is by Corollary 2.8).

It follows from Theorem 2.6 that the set $Z \subseteq \text{Emb}(E, \mathbf{G})$ of atomic embeddings forms a single orbit under $\text{Aut}(\mathbf{G})$. By definition, $Z \neq \emptyset$ if and only if $\mathbf{G}[E]$ exists. Let $\mathcal{I}_n \subseteq S_n(E)$ denote the set of isolated types.

For $r > 0$, we know that $B(\mathcal{I}_n, r)$ is a neighbourhood of \mathcal{I}_n , so there exists an open set $U_{n,r}$ such that $\mathcal{I}_n \subseteq U_{n,r} \subseteq B(\mathcal{I}_n, r)$ (in fact one can show that $B(\mathcal{I}_n, r)$ is open, but we shall not require this). For each $\bar{b} \in \mathbf{G}^n$, we define $V_{\bar{b},r} \subseteq \text{Emb}(E, \mathbf{G})$ to consist of all φ such that $\text{tp}(\bar{b}/\varphi E) \in \varphi U_{n,r}$. It is easy to see that since $U_{n,r}$ is open, so is $V_{\bar{b},r}$. Since the set of isolated types is metrically closed, we have

$$Z = \bigcap_{n, \bar{b} \in \mathbf{G}^n, r > 0} V_{\bar{b},r} = \bigcap_{n, \bar{b} \in \mathbf{G}_0^n, k} V_{\bar{b}, 2^{-k}},$$

where $\mathbf{G}_0 \subseteq \mathbf{G}$ is any countable dense subset. Thus, if $Z \neq \emptyset$ it is a dense G_δ orbit.

Assume now that $\mathbf{G}[E]$ does not exist, namely, that isolated types are not dense, and let $\psi \in X$. Then \mathbf{G} necessarily realises some type $\psi\xi \in S_n(\psi E)$ where $\xi \in S_n(E)$ is non isolated. By Lemma 1.16, for $r > 0$ small enough, the closed metric ball $\overline{B}(\xi, r)$ is (topologically) closed of empty interior. For $\bar{b} \in \mathbf{G}^n$, let $V_{\bar{b}} \subseteq \text{Emb}(E, \mathbf{G})$ consist of all φ such that $\text{tp}(\bar{b}/\varphi E) \notin \overline{B}(\varphi\xi, r)$. Reasoning as above, each $V_{\bar{b}}$ is a dense open set, and the set of $\varphi \in X$ such that \mathbf{G} omits $\varphi\xi$ is co-meagre. Since this set is also disjoint from the orbit of ψ , we are done. \blacksquare 2.10

We now turn to a criterion for the existence of $\mathbf{G}[E]$.

Lemma 2.11. *Let E be a separable Banach space, and say that a type $\xi \in S_{\mathbf{N}}(E)$ is a Gurarij type if it generates a Gurarij space. Then the set of Gurarij types over E is co-meagre in $S_{\mathbf{N}}(E)$. Moreover, there exists a dense G_δ set $Z \subseteq S_{\mathbf{N}}(E)$ such that if some $\xi \in Z$ generates F then F is Gurarij and $\{x_i\}_{i \in \mathbf{N}} \subseteq F$ is dense.*

In particular, the separable Gurarij space \mathbf{G} exists.

Proof. Let $X = \{x_i\}_{i \in \mathbf{N}}$, so $S_X(E) = S_{\mathbf{N}}(E)$. Let y be a new variable symbol. For $k \in \mathbf{N}$, let $[\emptyset \rightarrow y]: E(X) \rightarrow E(X, y)$ denote $\text{id}_{E(X)}$, let $[x_k \rightarrow y]: E(X) \rightarrow E(X, y)$ be defined as $\text{id}_{E(X \setminus \{x_k\})}$ together with $x_k \mapsto y$, and let $[y \rightarrow x_k]: E(X, y) \rightarrow E(X)$ be defined as $\text{id}_{E(X)}$ together with $y \mapsto x_k$.

The space $S_{X,y}(E)$ has a countable base of open sets $\{U_n\}_{n \in \mathbf{N}}$, and we may assume furthermore that each $U_n \neq \emptyset$ can be defined using only variables from among $\{x_0, \dots, x_{n-1}, y\}$, so $[y \rightarrow x_n]_* U_n = [x_n \rightarrow y]_* U_n \neq \emptyset$. For each n we define $Z_n \subseteq S_X(E) = S_{\mathbf{N}}(E)$ to consist of all types ξ such that either

- $\xi \notin [\emptyset \rightarrow y]_* U_n$, which defines a closed set, since $[\emptyset \rightarrow y]^*$ is open, or
- there exists some k such that $\xi \in [y \rightarrow x_k]_* U_n$, which defines an open set.

Then Z_n is a G_δ set, and we claim that it is dense. Indeed, let $\emptyset \neq W \subseteq S_X(E)$ be open. We may assume that $U_n \cap [\emptyset \rightarrow y]_* W \neq \emptyset$, since otherwise $W \cap Z_n = \emptyset$. Then there exist k such that $U_k \subseteq U_n \cap [\emptyset \rightarrow y]_* W$, so

$$\begin{aligned} \emptyset \neq [y \rightarrow x_k]_* U_k &\subseteq [y \rightarrow x_k]_* U_n \cap [y \rightarrow x_k]_* [\emptyset \rightarrow y]_* W \\ &= [y \rightarrow x_k]_* U_n \cap W \\ &\subseteq Z_n \cap W, \end{aligned}$$

proving our claim.

Now let $Z = \bigcap Z_n$, a dense G_δ set, and we claim that every $\xi \in Z$ is Gurarij. Indeed, let ξ generate F , and let $\emptyset \neq U \subseteq S_1(F)$ be open. We define $\theta: S_1(F) \rightarrow S_{X,y}(E)$ as in the proof of Lemma 1.14 (working over E , whereas there we worked over 0), and there exists n such that $U \supseteq \theta^{-1}(U_n) \neq \emptyset$. Since $\theta^{-1}(U_n) \neq \emptyset$, we have $\xi \in [\emptyset \rightarrow y]_* U_n \cap Z_n$, and so for some k we have $[y \rightarrow x_k]^* \xi \in U_n$. This means exactly that $\text{tp}(x_k/F) \in \theta^{-1}(U_n) \subseteq U$, showing that ξ is indeed Gurarij. Moreover, we have shown that every open set $\emptyset \neq U \subseteq S_1(F)$ is realised in F by some x_i , from which it follows that $\{x_i\}_{i \in \mathbf{N}}$ is dense in F . \blacksquare 2.11

Notice that since a Banach space has no isolated points, if a sequence is dense there then every tail of the sequence is dense there as well.

Theorem 2.12 (Omitting Types Theorem for Gurarij spaces). *Let E be a separable Banach space, and for each n , let $X_n \subseteq S_n(E)$ be metrically open and topologically meagre. Then there exists a separable Gurarij space $\mathbf{G} \supseteq E$ such that in addition, for every n , no type in X_n is realised in \mathbf{G} (we then say that \mathbf{G} omits all X_n). Moreover, the set of Gurarij types which generate such spaces is co-meagre.*

Proof. Let $Z \subseteq S_{\mathbf{N}}(E)$ be the set produced by Lemma 2.11. For each n , let $[\mathbf{N}]^n = \{s \subseteq \mathbf{N} : |s| = n\}$. For $s \in [\mathbf{N}]^n$ can be enumerated uniquely as an increasing sequence $\{k_0, \dots, k_{n-1}\}$, and we then define $[s] : E(n) \rightarrow E(\mathbf{N})$ by $x_i \mapsto x_{k_i}$ for $i < n$. Then $[s]^* : S_{\mathbf{N}}(E) \rightarrow S_n(E)$ is continuous and open, so $[s]_* X_n \subseteq S_{\mathbf{N}}(E)$ is meagre. Since everything is countable,

$$Z_1 = Z \setminus \bigcap_{n, s \in [\mathbf{N}]^n} [s]_* X_n$$

is co-meagre as well. All we need to show is that if $\xi \in Z_1$ generates \mathbf{G} then \mathbf{G} omits X_n . Indeed, assume that some $\xi \in X_n$ is realised in \mathbf{G} , say by \bar{a} . Since X_n is metrically open, there exists $r > 0$ such that $B(\xi, r) \subseteq X_n$. Since the sequence $\{x_i\}$ is dense in \mathbf{G} , there is an increasing sequence $k_0 < \dots < k_{n-1}$ such that $\|x_{k_j} - a_j\| < r$. But then $\text{tp}(x_{\bar{k}}/E) \in X_n$, so $\xi \in [\bar{k}]_* X_n$, contradicting the choice of ξ and completing the proof. $\blacksquare_{2.12}$

Corollary 2.13 (Criterion for primeness over E). *Let G be a Gurarij space, and let $E \subseteq G$ be a separable subspace. Then the following are equivalent:*

- (i) *The space G is prime over E , that is to say that it embeds isometrically over E in every Gurarij space containing an isometric copy of E .*
- (ii) *The space G is separable and atomic over E , namely, $G = \mathbf{G}[E]$.*

Proof. It is immediate from Theorem 2.5 that $\mathbf{G}[E]$ is prime over E . For the other direction, assume that \mathbf{G} is prime over E . Since E is separable, it embeds (by Theorem 2.5) in a separable Gurarij space, so \mathbf{G} must be separable as well. Finally, assume toward a contradiction that \mathbf{G} realises some non isolated type ξ . By Lemma 1.16 there exists $r > 0$ such that the closed metric ball $\overline{B}(\xi, r)$ has empty interior. Since the metric is lower semi-continuous, the closed metric ball is topologically closed, and is therefore meagre, as is the open ball $B(\xi, r)$. By Theorem 2.12, there exists a separable Gurarij space $\mathbf{G} \supseteq E$ which omits $B(\xi, r)$. Thus G cannot embed over E in \mathbf{G} , a contradiction. $\blacksquare_{2.13}$

Proposition 2.14. *Let E be a separable Banach space. Then $\mathbf{G}[E]$ exists if and only if, for each n , the set of isolated types in $S_n(E)$ is dense.*

Proof. Assume first that the sets of isolated types are dense. For a given n , let \mathcal{I}_n be the set of isolated types in $S_n(E)$, and assume that it is dense. Then $B(\mathcal{I}_n, r)$ contains a dense open set, and $\bigcap_{r>0} B(\mathcal{I}_n, r) = \overline{\mathcal{I}_n}$ is co-meagre. By Lemma 1.16 we have $\mathcal{I}_n = \overline{\mathcal{I}_n}$, so $S_n(E) \setminus \mathcal{I}_n$ is meagre and metrically open. By Theorem 2.12, if \mathcal{I}_n is dense for all n then an atomic separable Gurarij space over E exists.

Conversely, assume that $\mathbf{G}[E]$ exists. Then the set of n -types over E realised in $\mathbf{G}[E]$ is dense (by Lemma 2.2), and they are all isolated. $\blacksquare_{2.14}$

Model theorists will recognise Proposition 2.14 as the usual criterion for the existence of an atomic model, and as such it is in no way particular to Banach spaces. In the specific context of Banach spaces, however, it can be improves as follows.

Lemma 2.15. *For a type $\xi \in S_{\bar{x}}(E)$ the following are equivalent*

- (i) *The type ξ is isolated.*
- (ii) *The type $\xi \upharpoonright_{\bar{y}}$ is isolated for $\bar{y} \in E(\bar{x})^m$ (and every m).*
- (iii) *The type $\xi \upharpoonright_{\bar{y}}$ is isolated for every $\bar{y} \in E(\bar{x})$.*

Proof. (i) \implies (ii). When \bar{y} are linearly independent over E , this follows from Lemma 1.14. For the general case, it will now be enough to consider the case where \bar{y} , of length m , extends the original tuple of variables \bar{x} , and for $j < m$ let us write $y_j = a_j + \sum \lambda_{ij} x_i$. Given $r > 0$, there exists by hypothesis an open set U such that $\xi \in U \subseteq B(\xi, r)$, and let $V = (\cdot \upharpoonright_{\bar{x}})^{-1} U \subseteq S_{\bar{y}}(E)$. Intersecting V with the open sets defined by $\|y_j - \sum_{i < m} \lambda_{ij} y_i\| < r$ we obtain an open set V' with $\xi \upharpoonright_{\bar{y}} \rightarrow V' \subseteq B(\xi \upharpoonright_{\bar{y}}, r')$ for some $r' = r'(r, \bar{y})$ which goes to zero with r .

(ii) \implies (iii). Immediate.

(iii) \implies (i). We repeat the proof of Proposition 1.12 (in fact, that result is merely a special case of the present, alongside the fact that types in $S_1(0)$ are trivially isolated). Indeed, for each N there exists by

hypothesis a neighbourhood $U_N \ni \xi$ consisting of ζ such that

$$\forall y \in X_N \quad d(\zeta \upharpoonright_y, \xi \upharpoonright_y) < 1/N.$$

Using Proposition 1.7 we conclude as for Proposition 1.12. ■_{2.15}

Theorem 2.16. *The following are equivalent for a separable Banach space E :*

- (i) *The space $\mathbf{G}[E]$ exists.*
- (ii) *For each n , the set of isolated types in $S_n(E)$ is dense.*
- (iii) *The set of isolated types in $S_1(E)$ is dense.*

Proof. We only need to show that if the set of isolated 1-types is dense then $\mathbf{G}[E]$ exists. Indeed, proceeding as in the proof of Proposition 2.14 there exists a separable Gurarij space $\mathbf{G} \supseteq E$ which only realises isolated 1-types over E . By Lemma 2.15, \mathbf{G} is atomic over E . ■_{2.16}

3. ISOLATED TYPES OVER ONE-DIMENSIONAL SPACES

In this paper we shall attempt to characterise isolated types over arbitrary E . We start with the next-easiest case after $E = 0$, namely when $\dim E = 1$. Even though this case will be fully subsumed in the general finite-dimensional case, it is technically significantly simpler and deserves some specific comments, so we chose to treat it separately.

Definition 3.1. A *norming linear functional* for $v \in E \setminus \{0\}$ is a continuous linear functional $\lambda \in E^*_{=1}$ such that $\lambda v = \|v\|$.

By the Hahn-Banach Theorem, a norming linear functional always exists. We say that v is *smooth* in E if the norming linear functional is unique.

Proposition 3.2. *Let E be a Banach space, and let $v \in E \setminus \{0\}$. Then E is atomic over v if and only if v is smooth in E .*

Proof. By Lemma 2.15, we may assume that $E = \langle v, u \rangle$ and show that $\text{tp}(u/v)$ is isolated if and only if v is smooth in E . Assume first that for some $s, \varepsilon > 0$ and $D \in \mathbf{R}$ we have

$$\|v \pm su\| < \|v\| \pm sD + s\varepsilon.$$

It follows by the triangle inequality that

$$\|v\| \pm tD - t\varepsilon \leq \|v \pm tu\| < \|v\| \pm tD + t\varepsilon, \quad 0 < t \leq s,$$

or equivalently,

$$\left| \| \pm rv + u \| - r\|v\| \mp D \right| < \varepsilon, \quad r \geq s^{-1}.$$

If v is smooth, let λ be the unique norming functional, and let $D = \lambda u$. Then for any $\varepsilon > 0$ there exists s as above. Then $\xi = \text{tp}(u/v)$ satisfies the open condition $\|v \pm sx\| < \|v\| \pm sD + s\varepsilon$, which in turn implies that $\left| \|rv - u\| - \|rv - x\| \right| \leq 2\varepsilon$ for all $|r| \geq s^{-1}$. Finitely many additional open condition can ensure that that the same holds for all r yielding an open set $\xi \in U \subseteq B(\xi, 3\varepsilon)$, showing that ξ is isolated.

Conversely, if v is not smooth then there are norming functionals λ^\pm , where $D^- = \lambda^- u < D^+ = \lambda^+ u$. Any neighbourhood of ξ contains one U which is defined by finitely many conditions of the form $\left| \|r_i v + x\| - \|r_i v + u\| \right| < \varepsilon$. We can construct a Banach space E' generated by $\{v, w\}$, with $\|v\|$ as in E , such that $\zeta = \text{tp}(w/v) \in U$ and v is smooth in E' , with unique norming functional being defined by $\mu w = D^-$. This means that for r big enough we have

$$\|rv + w\| \approx r\|v\| + D^- \leq \|rv + u\| + D^- - D^+,$$

so $d(\xi, \zeta) \geq D^+ - D^-$. Therefore $B(\xi, D^+ - D^-)$ is *not* a topological neighbourhood of ξ , and ξ is not isolated. ■_{3.2}

We provided a fairly elementary argument to the “only if” part of Proposition 3.2. The machinery developed above provides us with a conceptually different argument, which in a sense we find preferable. First, let us recall that by Mazur [Maz33, Satz 2], the set of smooth points in the unit sphere of a separable Banach space is a dense G_δ . Assume now that E is atomic over v , and without loss of generality, say that

$\|v\| = 1$, and let $u \in \mathbf{G}$ be smooth of norm one. By Theorem 2.5 there exists an isometric embedding of E in \mathbf{G} sending v to u , so v must be smooth.

Yet a third way to prove that if E is atomic over v then v is smooth in E is via Lemma 1.11. We follow this path in a more general case below.

It follows from Lemma 1.17 that if $E \subseteq F$, and the topology and metric coincided on $S_n(F)$, then they would also coincide on $S_n(E)$, which would mean that every type in $S_n(E)$ is isolated. Given Proposition 3.2, it follows that the metric strictly refines the topology on $S_n(E)$ for every $E \neq 0$.

We obtain the following result, stated by Lusky [Lus76] (the proof is not spelled out explicitly, but given as ‘‘apply the following modifications to the proof of the uniqueness of the Gurarij space’’).

Corollary 3.3. *The smooth points in the unit sphere of \mathbf{G} form a single dense G_δ orbit under isometric automorphisms.*

Proof. Immediate from Proposition 3.2 and Theorem 2.6. ■_{3.3}

4. THE LEGENDRE-FENCHEL TRANSFORMATION OF 1-TYPES

In this section we recall and develop a few technical tools which will be used later in order to characterise isolated types over arbitrary E . We start the *Legendre-Fenchel transformation*.

Fact 4.1 (Hahn-Banach Theorem, see Brezis [Bre83]). *A closed convex subset of a locally convex topological vector space is the intersection of the closed half-spaces which contain it. In particular, it is weakly closed.*

Definition 4.2. Following Rockafellar [Roc70], we shall say that a convex function $f: E \rightarrow \mathbf{R} \cup \{\infty\}$ is *proper* if it is not identically ∞ , and that it is *closed* if it is lower semi-continuous (in norm, or equivalently, in the weak topology). We shall say that a function on E^* is **-closed* if it is closed in the weak* topology.

Fact 4.3. *Let $f: E \rightarrow \mathbf{R} \cup \{\infty\}$ be a closed proper convex function. Call $\text{dom } f = \{v \in E: f(v) < \infty\}$, the domain of f , itself a non empty convex set, and define the conjugate $f^*: E^* \rightarrow \mathbf{R} \cup \{\infty\}$ by*

$$f^*(\lambda) = \sup_{v \in E} \lambda v - f(v) = \sup_{v \in \text{dom } f} \lambda v - f(v) \in \mathbf{R} \cup \{\infty\}.$$

Then f^ is a *-closed (and in particular closed) proper convex function on E^* , and $f = f^{**}|_E$ under the canonical identification $E \subseteq E^{**}$. When $E = F^*$ is a dual space and f is *-closed then $f = (f^*|_F)^*$.*

Moreover, if g is another such function, then $\|f - g\| = \|f^ - g^*\|$, where $\|\cdot\|$ denotes the supremum norm, possibly infinite, and we agree that $|\infty - \infty| = 0$.*

Proof. For the finite-dimensional case, see Rockafellar [Roc70, Section 12]. The general case is proved essentially in the same fashion, using Fact 4.1, and for the dual case, the observation that a convex weak*-closed set in F^* is the intersection of half-spaces containing it defined by vectors in F . The moreover part is easy to check directly. ■_{4.3}

Clearly, $\lambda v \leq f(v) + f^*(\lambda)$ for any v, λ . Equality holds if and only if $f \geq \lambda - \lambda v + f(v)$ (if and only if the graph of $\lambda - \lambda v + f(v)$ is tangent in a weak sense to the graph of f at $(v, f(v))$).

Corollary 4.4. *Let $f: E \rightarrow \mathbf{R} \cup \{\infty\}$ be a closed proper convex function. Let $R \geq 0$, and let $v \in \text{dom } f$ be such that f is (finite and) R -Lipschitz in some neighbourhood of v . Then there exists $\lambda \in E^*$ such that $\lambda v = f(v) + f^*(\lambda)$, and any such λ satisfies $\|\lambda\| \leq R$.*

Proof. Let $r > 0$ be such that f is R -Lipschitz on $B(v, r)$, and for $t > 0$ let $S_t \subseteq E^*$ be the set of λ such that $\lambda v \geq f(v) + f^*(\lambda) - t$. This set is non empty (since $f^{**}(v) = f(v)$ is finite) and weak*-closed (since f^* is lower semi-weak*-continuous). Let $\lambda \in S_t$, and let $u \in E_{=1}$. Then

$$f(v) + rR \geq f(v + ru) \geq \lambda(v + ru) - f^*(\lambda) \geq f(v) + r\lambda u - t.$$

Taking the supremum over all $u \in E_{=1}$ we have $\|\lambda\| \leq R + \frac{t}{r}$, so S_t is moreover bounded. It follows that $\bigcap_{t>0} S_t$ is non empty and contained in $E_{\leq R}^*$, as desired. ■_{4.4}

Lemma 4.5. *Let $f, g: E \rightarrow \mathbf{R} \cup \{\infty\}$ be closed proper convex functions. Let $X \subseteq E$ be convex and open, and let $Y \subseteq E^*$ be the set of λ for which there exists $v \in X$ with $f(v) + f^*(\lambda) = \lambda v$. If g agrees with f on X , then g^* agrees with f^* on Y .*

Proof. Let $\lambda \in Y$, and let $v \in X$ be such that $f(v) + f^*(\lambda) = \lambda v$, i.e., $f \geq \lambda - \lambda v + f(v)$. Therefore $g \geq \lambda - \lambda v + g(v)$ on X , and since X is open and g convex, throughout, i.e., $g(v) + g^*(\lambda) = \lambda v$. Then $g^*(\lambda) = \lambda v - g(v) = \lambda v - f(v) = f^*(v)$, as desired. $\blacksquare_{4.5}$

The relevance of convex conjugation to our context comes from the following alternative characterisation of 1-types over a normed space E , introduced in [Benb] (see also Uspenskij [Usp08]).

Definition 4.6. Let X be an arbitrary metric space. A *Katětov function* on X is a function $f: X \rightarrow \mathbf{R}$ satisfying $f(x) \leq f(y) + d(x, y)$ and $d(x, y) \leq f(x) + f(y)$ for all $x, y \in X$. The space of Katětov functions on X is denoted $K(X)$. As with type spaces, we equip $K(X)$ with a double structure, the topology of point-wise convergence and the metric of uniform convergence (i.e., the supremum metric).

If X is a normed space, or a convex subset thereof, we let $K_C(X)$ denote the space of *convex* Katětov functions on X , with the induced topometric structure.

Fact 4.7. Let ξ be a 1-type over a normed space E , and let $f_\xi(a) = \|a\|^\xi$ for $a \in E$. Then

(i) The map $\xi \mapsto f_\xi$ defines a bijection between $S_1(E)$ and $K_C(E)$, whose inverse is given by

$$\|\alpha x - a\|^\xi = \begin{cases} \|a\| & \alpha = 0 \\ |\alpha| \xi(a/\alpha) & \alpha \neq 0. \end{cases}$$

(ii) This bijection is a topological homeomorphism and a metric isometry.

Proof. The first item is [Benb, Lemma 1.2]. For the second, that the bijection is homeomorphic (in the respective topologies of point-wise convergence) follows easily from the characterisation of the inverse, while the isometry is exactly Proposition 1.7 for 1-types. $\blacksquare_{4.7}$

Consequently, from now on we shall identify $K_C(E)$ with $S_1(E)$.

Fact 4.8. Let $X \subseteq Y$ be metric spaces, and for $f \in K(X)$ and $y \in Y$ define

$$\tilde{f}(y) = \inf_{x \in X} f(x) + d(x, y).$$

Then $\tilde{f} \in K(Y)$ extends f , and the induced embedding $K(X) \subseteq K(Y)$ is isometric. When $Y = E$ is a normed space, $X \subseteq E$ is convex and $f \in K_C(X)$, the extension \tilde{f} is convex as well, inducing an isometric embedding $K_C(X) \subseteq K_C(E)$.

Proof. The first assertion goes back to Katětov [Kat88], and the second is [Benb, Lemma 1.3(i)]. $\blacksquare_{4.8}$

Question 4.9. If $X \subseteq E$ is convex and compact (or totally bounded) then the topology and metric agree on $K_C(X)$, and it follows that the inclusion $K_C(X) \subseteq K_C(E)$ is also continuous, and therefore homeomorphic (since the restriction map is always continuous). At the other extremity, if $X = E$ then the inclusion is homeomorphic as well. What about general convex $X \subseteq E$?

A closed proper convex function $f: E \rightarrow \mathbf{R} \cup \{\infty\}$ is essentially the same thing as a closed convex function $f: X \rightarrow \mathbf{R}$, with convex domain X , such that $\liminf_{v \rightarrow u} f(v) = \infty$ for all $u \in \overline{X} \setminus X$. Indeed, we can get one from the other by restricting to the finite domain in one direction, or by extending by ∞ in the other. A special case of the second form is when $X \subseteq E$ is closed and convex and $f \in K_C(X)$. If X is merely convex, every $f \in K_C(X)$, being 1-Lipschitz, admits a unique extension to $\overline{f} \in K_C(\overline{X})$, so requiring X to be closed is not truly a constraint.

Lemma 4.10. Let $X \subseteq E$ be closed and convex and let $f \in K_C(X)$. Then

- (i) The domain $\text{dom } f^*$ contains the closed unit ball of E^* , and if $\lambda \in \text{dom } f^*$, $\|\lambda\| > 1$, then $f^*(\lambda) = \sup_{v \in \partial X} \lambda v - f(v)$. In particular, if $X = E$ (so $\partial X = \emptyset$) then $\text{dom } f^*$ is exactly the closed unit ball.
- (ii) If X is bounded and $\|\lambda\| = 1$ then $f^*(\lambda) = \sup_{v \in \partial X} \lambda v - f(v)$.
- (iii) Let $\tilde{f} \in K_C(E)$ be as per Fact 4.8. Then

$$\tilde{f}^*(\lambda) = \begin{cases} f^*(\lambda) & \|\lambda\| \leq 1 \\ \infty & \|\lambda\| > 1 \end{cases} \quad \text{and} \quad \tilde{f}(v) = \sup_{\|\lambda\| \leq 1} \lambda v - f^*(\lambda).$$

In addition, if $v \notin X$, then $\tilde{f}(v) = \sup_{\|\lambda\|=1} \lambda v - f^*(\lambda)$.

- (iv) When $X = E$, we have $f \in K_C(E) = S_1(E)$. For $\lambda \in E_{=1}^*$, the least possible value of a norm-preserving extension of λ at a realisation of f is $f^*(\lambda)$.
- (v) Let $g: E \rightarrow \mathbf{R} \cup \{\infty\}$ be any closed proper convex function. Then $g \in K_C(E)$ if and only if $\text{dom } g^* = E_{\leq 1}^*$ and $g^*(\lambda) + g^*(-\lambda) \leq 0$ for all $\lambda \in E_{\leq 1}^*$, or, equivalently, for all $\lambda \in E_{=1}^*$.

Proof. For (i), first let $\|\lambda\| \leq 1$, and let $u \in X$ be fixed. Then for all $v \in X$ we have $f(u) + \|u\| \geq \|v - u\| - f(v) + \|u\| \geq \lambda v - f(v)$, whereby $f^*(\lambda) \leq f(u) + \|u\| < \infty$. Now let $\|\lambda\| > 1$, say $\lambda w > \|w\|$, and assume that $\lambda \in \text{dom } f^*$. Then for each $v \in \text{dom } f$, the ray $\{v + \alpha w: \alpha \geq 0\}$ cannot be contained in X (or else $f^*(\lambda) = \infty$) and therefore intersects the boundary, say at v' . In this case $\lambda v - f(v) \leq \lambda v' - f(v')$, proving our assertion. When $\|\lambda\| = 1$ but X is assumed to be bounded, for every $v \in X$ we can find $v' \in \partial X$ with $\lambda v - f(v) \leq \lambda v' - f(v') + \varepsilon$ for ε arbitrarily small, whence (ii).

For (iii), we already know that $\text{dom } \tilde{f}^*$ is exactly the closed unit ball. In addition, $\tilde{f} \leq f$ implies $\tilde{f}^* \geq f^*$, and if $\|\lambda\| \leq 1$, then for every $v \in E$ and $u \in X$:

$$\lambda v - \tilde{f}(v) = \lambda v - \inf_{u \in X} [f(u) + \|v - u\|] \leq \sup_{u \in X} \lambda u - f(u) = f^*(\lambda),$$

whereby $\tilde{f}^*(\lambda) \leq f^*(\lambda)$. This gives us the first identity, and then Fact 4.3 gives the second one. Now assume that $v \notin X$, so by Fact 4.1 there exists $\mu \in E_{=1}^*$ such that $\mu \upharpoonright_X < \mu v$. For any $\lambda \in E_{\leq 1}^*$ there exists $\alpha \geq 0$ such that $\|\lambda + \alpha\mu\| = 1$. Then $f^*(\lambda + \alpha\mu) \leq f^*(\lambda) + \alpha\mu v$, or equivalently, $\lambda v - f^*(\lambda) \leq (\lambda + \alpha\mu)v - f^*(\lambda + \alpha\mu)$, whence it follows that $\tilde{f}(v) = \sup_{\|\lambda\|=1} \lambda v - f^*(\lambda)$.

Item (iv) is immediate. For (v), we have already seen that if $g \in K_C(E)$ then $\text{dom } g^* = E_{\leq 1}^*$, and the previous item implies that $g^*(\lambda) + g^*(-\lambda) \leq 0$ for $\lambda \in E_{=1}^*$, and therefore, by convexity, for $\lambda \in E_{\leq 1}^*$. Conversely, assume that $\text{dom } g^* = E_{\leq 1}^*$. Then $g = g^{**} \upharpoonright_E$ is necessarily 1-Lipschitz, so $\text{dom } g = E$. Finally, for distinct $v, u \in E$, let $\lambda \in E_{=1}^*$ norm $v - u$. Then

$$g(v) + g(u) \geq \lambda v - g^*(\lambda) - \lambda u - g^*(-\lambda) \geq \lambda(v - u) = \|v - u\|,$$

as desired. ■_{4.10}

Remark 4.11. Let $F \subseteq E$ be normed spaces and let $g \in K_C(F)$. Since F is convex in E , there may be some ambiguity about g^* , so let g_F^* denote the conjugate as a convex function on F and let g_E^* denote the conjugate of the extension by infinity to E^* . Let also $\tilde{g} \in K_C(E)$ denote the canonical extension of g . Then $g_E^*(\lambda) = g_F^*(\lambda \upharpoonright_F)$ for $\lambda \in E^*$, and by Lemma 4.10(iii), if $\|\lambda\| \leq 1$ then this is further equal to $\tilde{g}^*(\lambda)$. Therefore, in what interests us, this ambiguity can never lead to any form of confusion.

5. ISOLATED TYPES OVER ARBITRARY SPACES

In [Benb] we distinguished a special kind of “well behaved” convex Katětov functions. These will play a crucial role here as well, and admit a natural characterisation in terms of their conjugate.

Definition 5.1. We say that a function $f \in K_C(E)$ is *local* if there are $f_k \in K_C(X_k)$, where each $X_k \subseteq E$ is convex and compact, such that $f_k \rightarrow f$ uniformly. The set of local functions in $K_C(E)$ was denoted in [Benb] by $K_{C,0}(E)$.

Lemma 5.2. *Let E be a normed space, and let $f \in K_C(E)$. Then f is local if and only if f^* is weak*-continuous on $E_{\leq 1}^*$.*

Proof. Let first $X \subseteq E$ be compact and let $g \in K_C(X)$. If $X \subseteq \bigcup_{i < n} B(v_i, r)$ then $g^*(\lambda) - g^*(\mu) \leq 2r\|\lambda - \mu\| + \max_i (\lambda - \mu)v_i$, whence it follows that g^* is weak*-continuous on every bounded subset of E^* , and in particular on $E_{\leq 1}^*$. Since a uniform limit of continuous functions is continuous, if f is local then f^* is weak*-continuous on $E_{\leq 1}^*$.

Conversely, assume that f^* is weak*-continuous on $E_{\leq 1}^*$. It follows by general topological arguments that there exists a separable $F \subseteq E$ such that $f^*(\lambda)$ only depends on $\lambda \upharpoonright_F$. For $\|\lambda\| < 1$, this implies that $f^*(\lambda) = g^*(\lambda \upharpoonright_F)$ where $g = f \upharpoonright_F$, and using the fact that a closed convex on \mathbf{R} is continuous on its domain, the same follows for $\|\lambda\| = 1$. Therefore g^* is weak*-continuous on $F_{\leq 1}^*$, and $f = \tilde{g}$ by Remark 4.11, reducing to the case where E is separable.

We may therefore choose an increasing sequence of compact convex subsets $X_k \subseteq E$ such that $\bigcup X_k$ is dense in E (take closed balls of increasing radius, of sub-spaces of increasing finite dimension). For each k let $f_k = f|_{X_k}$. Then $\tilde{f}_k \searrow f$ point-wise, and for $\lambda \in E_{\leq 1}^*$ we have

$$f^*(\lambda) = \sup_v \lambda v - f(v) = \sup_{v,k} \lambda v - f_k(v) = \sup_k f_k^*(\lambda),$$

i.e., $f_k^* \nearrow f^*$ point-wise on $E_{\leq 1}^*$. Since each f_k^* is lower weak*-semi-continuous, f^* is weak*-continuous, and $E_{\leq 1}^*$ is weak*-compact, this implies that $f_k^* \rightarrow f^*$ uniformly on $E_{\leq 1}^*$, whereby $\tilde{f}_k \rightarrow f$ uniformly, and f is local. ■_{5.2}

In fact, we shall only require the easy direction of Lemma 5.2, but given the crucial role played by local functions both here and in [Benb] it seemed appropriate to give the full characterisation.

A second ingredient is the following.

Definition 5.3. Let E be a normed space and let $r \geq 0$. We say that $f \in K_C(E)$ is ∂ - r -maximal (where ∂ could be pronounced *boundary*) if for every $g \in K_C(E)$ whenever $g \leq f$ (i.e., $g^* \geq f^*$) we have $g^* \leq f^* + r$ on the sphere $E_{=1}^*$. If $r = 0$ we omit it and say that f is ∂ -maximal.

We observe that it does not matter whether we require that $f' \geq f$ on the sphere or on the entire ball, since, if $f' \geq f$ on the sphere, then we may take f'' to be the greatest convex function on the ball which agrees with f' on the sphere, and then $f'' \in K_C^*(E)$ as well, and $f'' \geq f$ on the entire ball.

Lemma 5.4. Let $f, g \in K_C(E)$ with f ∂ - r -maximal and $g \leq f$. Then g is ∂ - r -maximal as well.

Assume furthermore that $f = \widetilde{f|_X}$ for some convex $X \subseteq E$. Then outside X we have $g \geq f - r$.

Proof. By hypothesis we have $g^* \geq f^*$, and we clearly obtain the first assertion, as well as $g^*(\lambda) \leq f^*(\lambda) + r$ for $\|\lambda\| = 1$. By Lemma 4.10(iii), if $v \notin X$ then

$$\tilde{f}(v) = \sup_{\|\lambda\|=1} \lambda v - f^*(\lambda) \leq \sup_{\|\lambda\|=1} \lambda v - g^*(\lambda) + r \leq g(v) + r,$$

as claimed. ■_{5.4}

Fact 5.5 (Bishop-Phelps Theorem, [BP61]). If E is a Banach space, then the set of $\lambda \in E^*$ whose norm is attained is dense in norm.

Lemma 5.6. Let E be a normed space, let $f \in K_C(E)$ be ∂ - r -maximal and local, and let $\delta > 0$. Then $f + \delta$ is ∂ - $(r + 2\delta)$ -maximal.

In particular, if (by locality) $X \subseteq E$ is a compact convex set such that $\widetilde{f|_X} \leq f + \delta$ then $\widetilde{f|_X}$ is ∂ - $(r + 2\delta)$ -maximal.

Proof. Assume not, so let $g \in K_C(E)$, $g \leq f + \delta$, and let $\lambda \in E_{=1}^*$ be such that $g^*(\lambda) > f^*(\lambda) + r + \delta$. Then there exists $v \in E$ such that $\lambda v - g(v) > f^*(\lambda) + r + \delta$. It will be convenient to identify E with its image in E^{**} , and $E \oplus \mathbf{R}$ with a family of affine functions on E^* , so we may say that $g^* \geq v - g(v)$. By Fact 5.5 we may assume that $\|\lambda\|$ is attained, i.e., that there exists $u \in E_{=1}$ such that $\lambda u = 1$ (in fact, all we need is that the set of functionals whose norm is attained is weak* dense, which does not even require Bishop-Phelps).

For $\beta \geq 0$ define $h_\beta = v - g(v) + \beta(u - 1) - \delta$, and let $X = \{\mu \in E_{=1}^* : \mu u = 1\}$ be the face defined by u . Since $(v - g(v)) \vee (f^* - \delta) \leq g^*$, for each $\mu \in X$ we must have $\mu v - g(v) + f^*(-\mu) - \delta < 0$, and since f is weak*-continuous, the same must hold on a neighbourhood $V \subseteq E_{\leq 1}^*$ of X . Since $E_{\leq 1}^* \setminus V$ is weak*-compact, $u - 1$ is bounded away from zero there, so for β large enough we have $h_\beta(\mu) < -\sup f$ outside V . Thus $h_\beta(\mu) + f(-\mu) < 0$ for all $\|\mu\| \leq 1$. In addition, $v - g(v) \leq g^*$ implies that $h_\beta(\mu) + h_\beta(-\mu) < -2\delta < 0$, so $f^* \vee h_\beta$ satisfies the antipode inequality. However, $h_\beta(\lambda) = \lambda v - g(v) - \delta > f^*(\lambda) + r$, so $(f^* \vee h_\beta)^*|_E \leq$ witnesses that f is not ∂ - r -maximal. ■_{5.6}

Lemma 5.7. Let E be a normed space, let $f \in K_C(E)$ be ∂ - r -maximal and local, and let $r' > r$. Then f admits a neighbourhood $f \in W \subseteq K_C(E)$ such that $\text{diam } W < r'$ and every $g \in W$ is ∂ - r' -maximal.

Proof. Let $\delta > 0$ be fixed later. For $\lambda \in E_{\leq 1}^*$ choose $v_\lambda \in E$ such that $\lambda v_\lambda - f(v_\lambda) + \delta > f^*(\lambda)$. Let V_λ be a weak* neighbourhood of λ such that $f^*(\lambda) > f^*(\mu) - \delta$ and $\lambda v_\lambda < \mu v_\lambda + \delta$ for all $\mu \in V_\lambda$. For some finite set $\{\lambda_i\}_{i < n} \subseteq E_{\leq 1}^*$ we have $E_{\leq 1}^* \subseteq \bigcup_i V_{\lambda_i}$.

By locality, there exists a compact convex set $X \subseteq E$ such that $f + \delta > f' = \widetilde{f|_X}$, and we may assume that X contains all v_{λ_i} . Let $W \subseteq K_C(E)$ consist of all g such that $|f - g| < \delta$ on X . Since X is compact and Katětov functions are 1-Lipschitz, W is open, and we claim that it is as desired.

Indeed, let $g \in W$. If $\mu \in E_{\leq 1}^*$, say $\mu \in V_\lambda$ for one of the λ_i , then

$$g^*(\mu) \geq \mu v_\lambda - g(v_\lambda) > \lambda v_\lambda - f(v_\lambda) - 2\delta > f^*(\lambda) - 3\delta > f^*(\mu) - 4\delta,$$

i.e., $g < f + 4\delta \leq f' + 4\delta$, so already g is ∂ - r' -maximal for δ small enough. On the other hand, by Lemma 5.6, $f' + 4\delta < f + 5\delta$ are both ∂ - $(r + 10\delta)$ -maximal. By Lemma 5.4, and since $f' = \widetilde{f|_X}$, we have $g > f' - r - 6\delta \geq f - r - 6\delta$ outside X , while inside X we even have $g > f - \delta$. Thus $f - r - 6\delta < g < f + 4\delta$ throughout, for all $g \in W$, yielding the desired diameter for δ small enough. $\blacksquare_{5.7}$

Theorem 5.8. *Let $f \in K_C(E)$. Then f is isolated if and only if it is both local and ∂ -maximal.*

Proof. One direction follows directly from Lemma 5.7, so let us assume that f is isolated. We can then construct a sequence of neighbourhoods $W_k \ni f$ such that $\text{diam } W_k \rightarrow 0$, each defined using finitely many parameters. We let X_k be the (compact) convex hull of these parameters and $f_k = f|_{X_k}$. Then $\widetilde{f}_k \in W_k$, so $\widetilde{f}_k \rightarrow f$ uniformly and f is local.

Assume now that $g \in K_C(E)$, $g \leq f$. Let $W \ni f$ be a neighbourhood of small diameter, say $g \in W$ if and only if $|g(v_i) - f(v_i)| < \varepsilon$ for some v_i , $i < n$. We know that $f(v) = \sup_{\|\lambda\| \leq 1} \lambda v - f^*(\lambda)$, and since a closed convex function in dimension one is continuous on its domain, we have in fact $f(v) = \sup_{\|\lambda\| < 1} \lambda v - f^*(\lambda)$. Therefore, for each $i < n$ we may choose $\lambda_i \in E_{< 1}^*$ such that $f(v_i) - \varepsilon < \lambda_i v_i - f^*(\lambda_i) \leq f(v_i)$. Let $g' = g \vee \bigvee_{i < n} \lambda_i - f^*(\lambda_i)$. Since $\|\lambda_i\| < 1$ for each i , g' agrees with g outside some ball. On the other hand, we have $g'(v_i) > f(v_i) - \varepsilon$ and $g' \leq f$, so $g' \in W$. Therefore $|f - g| \leq \text{diam } W$ outside some ball. Since $\text{diam } W$ can be taken arbitrarily small, $f = g$ asymptotically. $\blacksquare_{5.8}$

6. COUNTING TYPES

We conclude with a calculation of the size of the type-space over a separable Banach space E . By “size” we mean here its metric density character (the cardinal $|S_n(E)|$ is the continuum as soon as $n > 0$ and $E \neq 0$).

Theorem 6.1. *Let E be a separable Banach space.*

- (i) *If E is finite-dimensional and polyhedral then $S_n(E)$ is metrically separable.*
- (ii) *Otherwise, $S_n(E)$ has metric density character equal to the continuum for every $n \geq 1$.*

Proof. Assume first that E is finite-dimensional and polyhedral. Then by Melleray [Mel07, Remarks following Corollary 4.6], the space $K(E)$ is separable, and *a fortiori* so is $S_1(E) = K_C(E)$. The passage from 1-types to n -types is done as in the proof of Lemma 2.15, and is left to the reader.

Now assume that E is not so. Then by Lindenstrauss [Lin64, Theorem 7.7] there exists a sequence $\{v_n\} \subseteq E$ such that for any $n \neq m$ and choice of signs:

$$\|v_n \pm v_m\| \leq \|v_n\| + \|v_m\| - 1.$$

Embed E (isometrically) in ℓ_∞ , and for a sequence $\bar{\varepsilon} \in \{\pm 1\}^{\mathbf{N}}$, consider the family of closed balls $\overline{B}(\varepsilon_n v_n, \|v_n\| - \frac{1}{2})$. By hypothesis every two such balls intersect at a non empty set, and therefore there exists $v \in \ell_\infty$ which belongs to them all. In other words, there exists $\xi_{\bar{\varepsilon}} = \text{tp}(v/E) \in S_1(E)$ such that $\|x - \varepsilon_n v_n\|^{\xi_{\bar{\varepsilon}}} \leq \|v_n\| - \frac{1}{2}$. If $\bar{\varepsilon} \neq \bar{\varepsilon}'$ then $d(\xi_{\bar{\varepsilon}}, \xi_{\bar{\varepsilon}'}') \geq 1$, so the density character of $S_1(E)$ is at least the continuum. The same holds *a fortiori* for $S_n(E)$, $n \geq 1$. $\blacksquare_{6.1}$

Remark 6.2. Lindenstrauss’s argument is quite elementary and yields a quick proof for Theorem 6.1(ii) which does not depend on the machinery developed in earlier sections. Arguments closer to the spirit of the present paper can also be given.

First, let X_0 be the set of extreme points in $E_{\leq 1}^*$, and let Ξ be the set of lower semi-continuous functions $f_0: X_0 \rightarrow \mathbf{R}$ which satisfy in addition $f_0(\lambda) + f_0(-\lambda) \leq 0$. Then E is not a finite-dimensional polyhedral space if and only if X_0 is infinite, in which case Ξ has density character continuum. If $f_0 \in \Xi$ and $f = f_0^{**}$ as in ?? then $f|_{X_0} = f_0$ and $f(\lambda) + f(-\lambda) \leq 0$ throughout $E_{\leq 1}^*$, so $f^* \in K_C(E)$ and we are done. Notice that this argument has the advantage of treating the two cases of “finite-dimensional, non polyhedral” and

“infinite-dimensional” in the same manner, while the proof of [Lin64, Theorem 7.7] treats them separately, with the second one being significantly more involved.

Second, in the case where E is infinite-dimensional, Theorem 6.1(ii) is a special case of a general principle which may be worth a mention as well. This principle, akin to the fact that the cardinal of a perfect set is at least the continuum, says that if isolated types are not dense in $S_n(E)$ (as is the case per ??) then the metric density character of $S_n(E)$ must be at least the continuum. The general argument is as follows. First, for $r > 0$, let $X_r \subseteq S_n(E)$ be the union of all open sets of diameter $\leq r$. If X_r is dense for every r then in every open set U one can find a sequence $\xi_n \in X_{2^{-n}}$ which converges metrically to some $\xi \in U$, and by Lemma 1.16 ξ is isolated, so the isolated types are dense after all. Therefore, for some $r > 0$, which we now fix, X_r is not dense (in our case, X_r is not dense for any $r > 0$; notice also that $S_n(E)$ need not be metrisable, so we cannot use the Baire Category Theorem). We also fix an open set U disjoint from X_r . Since every open subset of U has diameter $> r$, and since the metric on $S_n(E)$ is lower semi-continuous, one can build a binary tree of open subsets U_s , $s \in 2^{<\omega}$, such that $U_\emptyset = U$, $\overline{U_{s \smallfrown i}} \subseteq U_s$ for $i = 0, 1$, and $d(U_{s \smallfrown 0}, U_{s \smallfrown 1}) > r$. Then for $\sigma \in 2^{\mathbb{N}}$ we have $\bigcap_n U_{\sigma \upharpoonright_n} \neq \emptyset$, and these intersections all have distance $> r$ from one another, as desired.

This answers a Problem 2 of Avilés et al. [ACC⁺11, Section 4] in the negative (and we thank Wiesław KUBIŚ for having pointed this out to us). They say that a Banach space G is of *universal disposition for finite-dimensional spaces* if it satisfies a strengthening of Definition 2.1 with ψ being an isometry.

Corollary 6.3. *The density character of any space of universal disposition for finite-dimensional spaces is at least the continuum. In other words, the answer to Problem 2 of [ACC⁺11, Section 4] is negative.*

Proof. Assume that G is of universal disposition for finite-dimensional spaces. Then the Euclidean plane E embeds isometrically in G , and all types over E are realised in G , so the density character of G must be at least the metric density character of $S_1(E)$, namely the continuum. ■_{6.3}

On the other hand, say that a Gurarij space G is *strongly \aleph_1 -homogeneous* if the following stronger version of Corollary 2.8 holds in G :

For every separable $F \subseteq G$ and isometric embedding $\varphi: F \rightarrow G$ there exists an isometric automorphism $\psi \in \text{Aut}(G)$ extending φ .

Clearly, a strongly \aleph_1 -homogeneous Gurarij space is of universal disposition for finite-dimensional (and even separable) spaces. Moreover, there does exist such a space of density character the continuum. This is merely a special case of a general model theoretic result: for any cardinal κ and structure \mathbf{M} of density character $\leq 2^\kappa$, in a language of cardinal $\leq \kappa$, there exists an elementary extension $\mathbf{M}' \succeq \mathbf{M}$ of density character still $\leq 2^\kappa$, which is moreover κ^+ -saturated and strongly κ^+ -homogeneous. Apply this to $\mathbf{M} = \mathbf{G}$ and $\kappa = \aleph_0$.

REFERENCES

- [ACC⁺11] Antonio AVILÉS, Félix CABELLO SÁNCHEZ, Jesús M. F. CASTILLO, Manuel GONZÁLEZ, and Yolanda MORENO, *Banach spaces of universal disposition*, Journal of Functional Analysis **261** (2011), no. 9, 2347–2361, doi:10.1016/j.jfa.2011.06.011.
- [BBHU08] Itai BEN YAACOV, Alexander BERENSTEIN, C. Ward HENSON, and Alexander USVYATSOV, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis, volume 2 (Zoé CHATZIDAKIS, Dugald MACPHERSON, Anand PILLAY, and Alex WILKIE, eds.), London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315–427.
- [Bena] Itai BEN YAACOV, *Fraïssé limits of metric structures*, research notes, arXiv:1203.4459.
- [Benb] ———, *The linear isometry group of the Gurarij space is universal*, Proceedings of the American Mathematical Society, to appear, arXiv:1203.4915.
- [Ben08a] ———, *Continuous first order logic for unbounded metric structures*, Journal of Mathematical Logic **8** (2008), no. 2, 197–223, doi:10.1142/S0219061308000737, arXiv:0903.4957.
- [Ben08b] ———, *Topometric spaces and perturbations of metric structures*, Logic and Analysis **1** (2008), no. 3–4, 235–272, doi:10.1007/s11813-008-0009-x, arXiv:0802.4458.
- [Ben09] ———, *Modular functionals and perturbations of Nakano spaces*, Journal of Logic and Analysis **1** (2009), Paper 1, 42, doi:10.4115/jla.2009.1.1, arXiv:0802.4285.
- [BP61] Errett BISHOP and R. R. PHELPS, *A proof that every Banach space is subreflexive*, Bulletin of the American Mathematical Society **67** (1961), 97–98.
- [Bre83] Haïm BREZIS, *Analyse fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983, Théorie et applications.

- [BU07] Itai BEN YAACOV and Alexander USVYATSOV, *On d -finiteness in continuous structures*, *Fundamenta Mathematicae* **194** (2007), 67–88, doi:10.4064/fm194-1-4.
- [BU10] ———, *Continuous first order logic and local stability*, *Transactions of the American Mathematical Society* **362** (2010), no. 10, 5213–5259, doi:10.1090/S0002-9947-10-04837-3, arXiv:0801.4303.
- [Gur66] Vladimir I. GURARIJ, *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces*, *Sibirsk. Mat. Ž.* **7** (1966), 1002–1013.
- [Kat88] Miroslav KATĚTOV, *On universal metric spaces*, *General topology and its relations to modern analysis and algebra*, VI (Prague, 1986), *Res. Exp. Math.*, vol. 16, Heldermann, Berlin, 1988, pp. 323–330.
- [KS] Wiesław KUBIŚ and Sławomir SOLECKI, *A proof of the uniqueness of the Gurarij space*, *Israel Journal of Mathematics*, to appear, arXiv:1110.0903.
- [Lin64] Joram LINDENSTRAUSS, *Extension of compact operators*, *Memoirs of the American Mathematical Society* **48** (1964), 112.
- [Lus76] Wolfgang LUSKY, *The Gurarij spaces are unique*, *Archiv der Mathematik* **27** (1976), no. 6, 627–635.
- [Maz33] Stanisław MAZUR, *Über konvexe mengen in linearen normierten räumen*, *Studia Mathematica* **4** (1933), 70–84.
- [Mel07] Julien MELLERAY, *On the geometry of Urysohn's universal metric space*, *Topology and its Applications* **154** (2007), no. 2, 384–403, doi:10.1016/j.topol.2006.05.005.
- [Roc70] R. Tyrrell ROCKAFELLAR, *Convex analysis*, *Princeton Mathematical Series*, No. 28, Princeton University Press, Princeton, N.J., 1970.
- [Usp08] Vladimir V. USPENSKIJ, *On subgroups of minimal topological groups*, *Topology and its Applications* **155** (2008), no. 14, 1580–1606, doi:10.1016/j.topol.2008.03.001, arXiv:math/0004119.

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