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Abstract. We investigate the tail distribution of the first exit time of Brownian motion with drift from a cone and find its exact asymptotics for a large class of cones. Our results show in particular that its exponential decreasing rate is a function of the distance between the drift and the cone, whereas the polynomial part in the asymptotics depends on the position of the drift with respect to the cone and its polar cone, and reflects the local geometry of the cone at the point where the drift is orthogonally projected.

1. Introduction and results

General context. Let B_t be a d-dimensional Brownian motion with drift $a \in \mathbb{R}^d$. For any cone $C \subset \mathbb{R}^d$, define the first exit time

$$\tau_C = \inf\{t > 0 : B_t \notin C\}.$$

In this article we study the probability for the Brownian motion started at x not to exit C before time t, namely,

$$\mathbb{P}_x[\tau_C > t], \tag{1}$$

and its asymptotics

$$\kappa h(x) t^{-\alpha} e^{-\gamma t}(1 + o(1)), \quad t \to \infty. \tag{2}$$

Zero drift case. In the literature, these problems have first been considered for Brownian motion with no drift ($a = 0$). In [25], Spitzer considered the case $d = 2$ and obtained an explicit expression for the probability (1) for any two-dimensional cone. He also introduced the winding number process $\theta_t = \arg B_t$ (in dimension $d = 2$, the Brownian motion does not exit a given cone before time t if and only if θ_t stays in some interval). He proved a weak limit theorem for θ_t as $t \to \infty$. Later on, this result has been extended by many authors in several directions (e.g., strong limit theorems, winding numbers not only around points but also around certain curves, winding numbers for other processes), see for instance [22].

In [12], motivated by studying the eigenvalues of matrices from the Gaussian Unitary Ensemble, Dyson analyzed the Brownian motion in the cone formed by the Weyl chamber of type A, namely,

$$\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 < \cdots < x_d\}. \tag{3}$$

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He also defined the Brownian motion conditioned never to exit the chamber. These results have been extended by Biane [3] and Grabiner [17]. In [2], Biane studied some further properties of the Brownian motion conditioned to stay in cones, and in particular generalized the famous Pitman's theorem to that context. In [21] König and Schmid analyzed the non-exit probability (1) of Brownian motion from a growing truncated Weyl chamber.

In [5], Burkholder considered open right circular cones in any dimension and computed the values of $p > 0$ such that

$$\mathbb{E}_x[\tau_C^p] < \infty.$$

In [9, 10], for a fairly general class of cones, DeBlassie obtained an explicit expression for the probability (1) in terms of the eigenfunctions of the Dirichlet problem for the Laplace-Beltrami operator on

$$\Theta = \mathbb{S}^{d-1} \cap C,$$

see [9, Theorem 1.2]. DeBlassie also derived the asymptotics (2), see [9, Corollary 1.3]: he found $\gamma = 0$ (indeed, the drift is zero), while α is related to the first eigenvalue and $h(x)$ to the first eigenfunction. The basic strategy in [9, 10] was to show that the probability (1) is solution to the heat equation and to solve the latter. In [1], Bañuelos and Smits refined the results of DeBlassie [9, 10]: they considered more general cones, and obtained a quite tractable expression for the heat kernel (the transition densities for the Brownian motion in C killed on the boundary), and thus for (1).

We conclude this part by mentioning the work [11], in which Doumerc and O'Connell found a formula for the distribution of the first exit time of Brownian motion from a fundamental region associated with a finite reflection group.

Non-zero drift case. For Brownian motion with non-zero drift, much less is known. Only the case of Weyl chambers (of type A) has been investigated. In [4], Biane, Bougerol and O'Connell obtained an expression for the probability $\mathbb{P}_x[\tau_C = \infty] = \lim_{t\to\infty} \mathbb{P}_x[\tau_C > t]$ in the case where the drift is inside of the Weyl chamber (and hence the latter probability is positive). In [24], Puchała and Rolski gave, for any drift a, the exact asymptotics (2) of the tail distribution of the exit time, in the context of Weyl chambers too. The different quantities in (2) were determined explicitly in terms of the drift a and of a vector obtained by a procedure involving the construction of a stable partition of the drift vector.

Aim of this paper. In this article, we find the asymptotics (2) for a very general class of cones C, and we identify κ, $h(x)$, α and γ in terms of the cone C and the drift a. This way, we extend the results of [4] and partially those of [24]. We shall consider six different cases corresponding to a partition of the Euclidian space $\mathbb{R}^d$ with respect to (w.r.t.) the cone. To do this, we introduce the polar cone (which is a closed set)

$$C^\sharp = \{x \in \mathbb{R}^d : \langle x, y\rangle \leqslant 0, \forall y \in C\}.$$

See Figure 3 for an example of polar cone. Below and throughout, we shall denote by D^o (resp. $\overline{D}$) the interior (resp. the adherence) of a set $D \subset \mathbb{R}^d$. The six cases are, in order of appearance,

A. polar interior drift: $a \in (C^\sharp)^o$;

B. zero drift: $a = 0$;
C. interior drift: $a \in C$;
D. boundary drift: $a \in \partial C \setminus \{0\}$;
E. non-polar exterior drift: $a \in \mathbb{R}^d \setminus (\overline{C} \cup C^\sharp)$;
F. polar boundary drift: $a \in \partial C^\sharp \setminus \{0\}$.

These cases will be analyzed in Theorems A, B, C, D, E and F, respectively. Our results show in particular that the exponential decreasing rate $e^{-\gamma}$ in (2) is related to the distance between the drift and the cone by the formula

$$\gamma = \min_{y \in \overline{C}} |a - y|^2/2.$$

As for the polynomial part $t^{-\alpha}$ in (2), it depends on the case under consideration and reflects the local geometry of the cone at the point(s) where the drift projects orthogonally, plus the local geometry at the *contact points* in case F.

Our results extend those of Puchała and Rolski in [24] about Weyl chambers of type A in cases A, B and C only. (Note that case A does not concern Weyl chambers since the polar cone has then an empty interior, whereas case B has already been settled in [1] but is presented here for the sake of completeness.) Indeed, we will treat cases D, E and F under a smoothness assumption on the cone that excludes Weyl chambers from our analysis. The reason is that we will need estimates for the heat kernel of the cone at boundary points, and those are only available (to our knowledge) in the case of smooth cones or, on the other hand, in the case of Weyl chambers.

However, it is worth pointing out the fact that the formula in [24] for γ is exactly the same as ours. Indeed, though it is not mentioned, the vector f obtained in [24] via the construction of a *stable partition* of the drift is exactly the orthogonal projection of the drift on the Weyl chamber, and their formula (4.10) reads $\gamma = |a - f|^2/2$, as the reader can easily check.

2. Two-dimensional Brownian motion in cones

For the one-dimensional Brownian motion and the cone $C = (0, \infty)$, there are three regimes for the asymptotics of the non-exit probability, according to the sign of the drift $a \in \mathbb{R}$. Precisely, for any $x > 0$, as $t \to \infty$ one has, with obvious notations (see [19, section 2.8]),

$$\mathbb{P}_x[\tau_{(0,\infty)} > t] = (1 + o(1)) \begin{cases} \dfrac{x e^{-ax} e^{-ta^2/2}}{\sqrt{2\pi} a^2 t^{3/2}} & \text{if} \quad a < 0, \\ \dfrac{\sqrt{2}x}{\sqrt{\pi t}} & \text{if} \quad a = 0, \\ 1 - e^{-2ax} & \text{if} \quad a > 0. \end{cases} \tag{4}$$

In dimension 2, any (connected and proper) open cone is a rotation of

$$\{\rho e^{i\theta} : \rho > 0, 0 < \theta < \beta\}$$

for some $\beta \in (0, 2\pi]$, see Figure 3. For some specific cones, the asymptotics of the non-exit probability is easy to determine. This is for example the case of the upper half-plane ($\beta = \pi$), since this is essentially a one-dimensional case. It is also an easy task to deal with the quarter plane Q ($\beta = \pi/2$). Indeed, by independence of the coordinates $(B_t^{(1)}, B_t^{(2)})$ of the Brownian motion B_t and noting $x = (x_1, x_2)$ the starting point, the non-exit probability can be written as

$$\mathbb{P}_x[\tau_Q > t] = \mathbb{P}_{x_1}[\tau_{(0,\infty)}(B^{(1)}) > t] \cdot \mathbb{P}_{x_2}[\tau_{(0,\infty)}(B^{(2)}) > t].$$

Denoting by $a = (a_1, a_2)$ the coordinates of the drift and making use of (4), one readily deduces the asymptotics $\mathbb{P}_x[\tau_Q > t] = \kappa h(x) t^{-\alpha} e^{-\gamma t}(1 + o(1))$, as summarized in Figure 1, where the value of α is given, according to the position of the drift (a_1, a_2) in the quarter plane. We focus on α and not on γ, since the value of γ is always obtained in the same way, as we shall see in section 3.

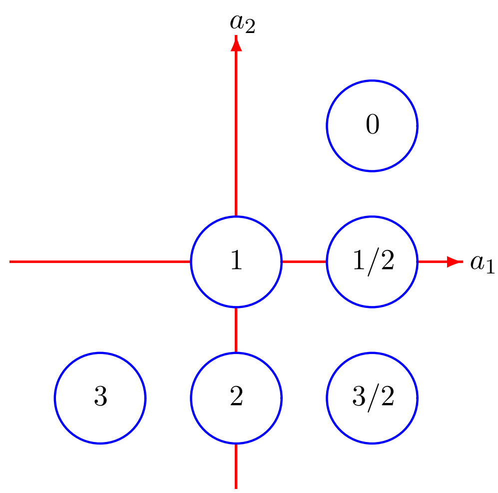

Figure 1. Value of α in terms of the position of the drift (a_1, a_2) in the plane (case of the quarter plane)

More generally, we shall prove in this article that the value of α for any two-dimensional cone is given as in Figure 2. This result can be understood as follows: when the drift is negative (i.e., when it belongs to the polar cone $C^\sharp$), one sees the influence of the vertex of the cone (α is expressed with the opening angle β) since the trajectories that do not leave the cone will typically stay close to the origin. In all other cases, the Brownian motion will move away from the vertex, and see the cone as a half-space (boundary drift and non-polar exterior drift) or as a whole-space (interior drift).

3. Heat kernel of the cone and non-exit probability

In this section we introduce all necessary tools for our study. We first give the expression of the non-exit probability (1) in terms of the heat kernel of the cone C (see Lemmas 1 and 3). Then we guess the value of the exponential decreasing rate of this probability, by simple considerations on its integral expression. Finally we present our general strategy to calculate the asymptotics of the non-exit probability.

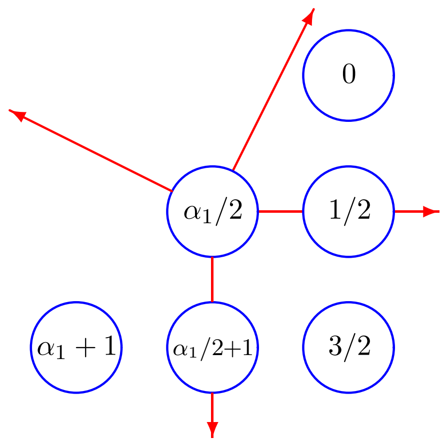

FIGURE 2. Value of α in terms of the position of the drift (a_1, a_2) in the plane (case of a general cone of opening angle β, for which $\alpha_1 = \pi/\beta$, see Figure 3)

Expression of the non-exit probability. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}_x)$ be a filtered probability space on which is defined a process $(B_t)_{t\geqslant 0}$ which is, under $\mathbb{P}_x$, a d-dimensional Brownian motion started at x with drift a and identity covariance matrix. The filtration $(\mathcal{F}_t)_{t\geqslant 0}$ is the natural Brownian filtration.

The lemma hereafter gives an expression of the non-exit probability for Brownian motion with drift a in terms of an integral involving the transition probabilities of the Brownian motion with zero drift killed at the boundary of the cone. This is a quite standard result (see [24, Proposition 2.2] for example) and an easy consequence of Girsanov theorem. We sketch the proof for the reader's convenience. Notice that this result is not at all specific to cones and would be valid for any domain in $\mathbb{R}^d$.

Lemma 1. *Let $p^C(t,x,y)$ denote the transition probabilities of the Brownian motion with zero drift killed at the boundary of the cone C. We have*

$$\mathbb{P}_x[\tau_C > t] = e^{\langle -a,x\rangle - t|a|^2/2} \int_C e^{\langle a,y\rangle} p^C(t,x,y)\mathrm{d}y, \quad \forall t \geqslant 0. \tag{5}$$

Proof. The Laplace transform of the Brownian motion with drift a started at x is given by

$$L_x(t,\lambda) = \mathbb{E}_x[e^{\langle \lambda, B_t\rangle}] = e^{t|\lambda|^2/2 + t\langle \lambda, a\rangle + \langle \lambda, x\rangle}. \tag{6}$$

We define a martingale $(Z_t)_{t\geqslant 0}$ by setting

$$Z_t = \frac{e^{\langle -a, B_t\rangle}}{L_x(t,-a)}.$$

It follows then from Girsanov theorem [19, page 192] that there exists a unique probability measure $\mathbb{P}_x^*$ on $(\Omega, \mathcal{F})$ such that

$$\mathbb{P}_x^*[A] = \mathbb{E}_x[1_A Z_t], \quad \forall A \in \mathcal{F}_t, \quad \forall t \geqslant 0. \tag{7}$$

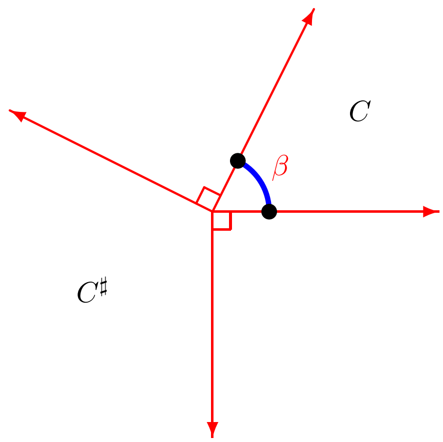

FIGURE 3. Cones C with opening angle β and polar cones $C^\sharp$ in dimension 2. The set Θ (in blue above) and its boundary (in black) are particularly important in our analysis.

In addition, under $\mathbb{P}_x^*$, the process $(B_t)_{t\geqslant 0}$ is a zero drift Brownian motion. By equation (7) we have

$$\mathbb{E}_x[h(B_t), \tau_C > t] = L_x(t, -a)\mathbb{E}_x^*[h(B_t)e^{\langle a, B_t\rangle}, \tau_C > t],$$

for all bounded and measurable h. Therefore, the transition probabilities of the Brownian motion with drift a killed at the boundary of C are given by (see [18, section 4], in particular equation (17) or [7, section 2.2])

$$\mathbb{P}_x[B_t \in \mathrm{d}y, \tau_C > t] = e^{\langle -a, x\rangle - t|a|^2/2} e^{\langle a, y\rangle} p^C(t, x, y)\mathrm{d}y.$$

This completes the proof of Lemma 1. $\square$

In this work we consider general cones as defined by Bañuelos and Smits in [1]. Namely, given a proper open and connected subset Θ of the unit sphere $\mathbb{S}^{d-1}$, we consider the cone C generated by Θ, that is, the set of all rays emanating from the origin and passing through Θ. We refer to Figure 3 for the two-dimensional case.

We now write down a series expansion for the transition probabilities of the Brownian motion killed at the boundary of C (or equivalently, see [18, section 4], for the heat kernel $p^C(t, x, y)$ of the cone C), as given in [1]. Denote by $L_{\mathbb{S}^{d-1}}$ the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. Our first assumption on the cones studied here is the following:

(C1) The set $\Theta = \mathbb{S}^{d-1} \cap C$ is regular w.r.t. the operator $L_{\mathbb{S}^{d-1}}$.

By regular we mean that there exists a complete set of eigenfunctions $(m_j)_{j\geqslant 1}$ orthonormal w.r.t. the surface measure on Θ with corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$, satisfying for any $j \geqslant 1$

$$\begin{cases} L_{\mathbb{S}^{d-1}} m_j = -\lambda_j m_j & \text{on} \quad \Theta, \\ m_j = 0 & \text{on} \quad \partial\Theta. \end{cases} \tag{8}$$

It is proved in [6] (see in particular [6, page 169]) that if the domain Θ is normal, that is, piecewise infinitely differentiable, then Θ is regular w.r.t. $L_{\mathbb{S}^{d-1}}$. For any $j \geqslant 1$, we set $\alpha_j = \sqrt{\lambda_j + (d/2-1)^2}$ and $p_j = \alpha_j - (d/2-1)$.

Example 1. In dimension 2, a direct computation starting from (8) yields $\lambda_j = (j\pi/\beta)^2$, and thus $p_j = \alpha_j = j\pi/\beta$, for any $j \geqslant 1$. Further, the eigenfunctions (8) are given in polar coordinates by

$$m_j(\theta) = \frac{2}{\beta}\sin\left(\frac{j\pi\theta}{\beta}\right), \quad \forall j \geqslant 1, \tag{9}$$

where the term $2/\beta$ comes from the normalization $\int_0^\beta m_j(\theta)^2 \mathrm{d}\theta = 1$.

Going back to the general case ($d \geqslant 2$), we denote by I_ν the modified Bessel function of order ν:

$$I_\nu(x) = \frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)}\int_0^{\frac{\pi}{2}} (\sin t)^{2\nu}\cosh(x\cos t)\mathrm{d}t = \sum_{m=0}^{\infty}\frac{(x/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{10}$$

It satisfies the second order differential equation

$$I_\nu''(x) + \frac{1}{x}I_\nu'(x) = \left(1+\frac{\nu^2}{x^2}\right)I_\nu(x).$$

In the neighborhood of 0, it has the equivalent

$$I_\nu(x) = \frac{x^\nu}{2^\nu\Gamma(\nu+1)}(1+o(1)), \quad x \to 0. \tag{11}$$

We refer to [26] for proofs of the facts above and for any further result.

In what follows, for any $x \neq 0$, we denote by $\vec{x} = x/|x|$ its projection on the unit sphere $\mathbb{S}^{d-1}$.

Lemma 2 ([1])**.** *The heat kernel of the cone C has the series expansion*

$$p^C(t,x,y) = \frac{e^{-\frac{|x|^2+|y|^2}{2t}}}{t(|x||y|)^{d/2-1}}\sum_{j=1}^{\infty} I_{\alpha_j}\left(\frac{|x||y|}{t}\right)m_j(\vec{x})m_j(\vec{y}), \tag{12}$$

where the convergence is uniform for $(t,x,y) \in [T,\infty) \times \{x \in C : |x| \leqslant R\} \times C$, for any positive constants T and R.

Making the change of variables $y \mapsto ty$ in (5) and using (12), we easily obtain the following lemma, where the expression of the non-exit probability now involves an integral of Laplace's type.

Lemma 3. *Let C be a cone satisfying to (C1). For Brownian motion with drift a, the non-exit probability is given by*

$$\mathbb{P}_x[\tau_C > t] = e^{\langle -a,x\rangle - |x|^2/(2t) + |x|^2/2}t^{d/2}\int_C e^{|y|^2/2}p^C(1,x,y)e^{-t|a-y|^2/2}\mathrm{d}y, \quad \forall t \geqslant 0. \tag{13}$$

Exponential decreasing rate of the non-exit probability. The aim now is to understand the asymptotic behavior as $t \to \infty$ of the integral in the right-hand side of (13). To do this, we shall use Laplace's method [8, Chapter 5]. The first question when applying this method is to locate the points $y \in \overline{C}$ where the function

$$|a-y|^2/2 \tag{14}$$

in the exponential reaches its minimum value, for it is expected that only a neighborhood of these points will contribute to the asymptotics. And indeed, we shall prove that the exponential decreasing rate $e^{-\gamma}$ of the non-exit probability in (2) is given, for the six cases A–F, by

$$\gamma = \min_{y\in\overline{C}} |a-y|^2/2.$$

For a convex cone (or any convex set), the minimum of the function (14) on $\overline{C}$ is reached at a unique point, namely the orthogonal projection $p_C^\perp(a)$ of a on $\overline{C}$. Though the orthogonal projection might not be well defined everywhere in general (that is, when the cone is not convex), it is still true in cases A, B and F (resp. C and D), that the minimum of the function (14) on $\overline{C}$ is attained at a unique point, namely $p=0$ (resp. $p=a$) and that this point satisfies the usual property $\langle a-p, y-p\rangle \leqslant 0$ for all $y\in\overline{C}$. Therefore, we still call this point the orthogonal projection and write $p_C^\perp(a)$ for it. The case E presents more complexity: according to the cone, the minimum can be reached at different points (possibly infinitely many), and for that reason, we shall only treat this case for convex cones (though the arguments would adapt to treat the case where there is a finite number of minimum points). With this convention, the exponential decreasing rate of the non-exit probability is given by

$$\gamma = |a-p_C^\perp(a)|^2/2. \tag{15}$$

Surprisingly, the case F is the most difficult. It is a mixture between cases A and B, and its analysis involves a second application of Laplace's method. This explains why it will be treated at the end.

To conclude this part, let us compare the exponential decreasing rates of the non-exit probability $\mathbb{P}_x[\tau_C > t]$ from convex cones for Brownian motion and random walks. For random walks, Theorem 1 in [15] asserts that the exponential rate is the minimum on $-C^\sharp$ of the Laplace transform of the law of the random walk increments.[1] The analogy with the Brownian motion exponential rate given in (15) can be seen as follows: it is the minimum value on $-C^\sharp$ of the Laplace transform of the one-unit-of-time increment $B_{t+1}-B_t$. Indeed, the latter transform is given by (see equation (6))

$$L_0(1,\lambda) = e^{|\lambda|^2/2+\langle\lambda,a\rangle},$$

and the minimum on $-C^\sharp$ of $|\lambda|^2/2+\langle\lambda,a\rangle$ is obviously the minimum on $C^\sharp$ of

$$|\lambda|^2/2-\langle\lambda,a\rangle = |\lambda-a|^2/2-|a|^2/2.$$

[1]This is proved in [15] for cones which are intersections of half-spaces only, but the result is very likely quite more general, and should be valid for convex cones.

It is reached at $\lambda = p_{C^\sharp}^\perp(a)$, and an easy computation shows that

$$|p_{C^\sharp}^\perp(a) - a|^2/2 - |a|^2/2 = -|a - p_C^\perp(a)|^2/2,$$

using the fact that, for any convex cone C, a is the orthogonal sum of $p_C^\perp(a)$ and $p_{C^\sharp}^\perp(a)$.

General strategy. According to formula (13), we only need to analyze the asymptotic behavior of

$$I(t) = t^{d/2} \int_C e^{|y|^2/2} p^C(1, x, y) e^{-t|a-y|^2/2} \mathrm{d}y.$$

We recall that the function (14) in the exponential reaches its minimum at a unique point, namely $p_C^\perp(a)$.

The lemma below shows that if the domain of integration is restricted to the complement of any neighborhood of $p_C^\perp(a)$, then the integral above becomes negligible with respect to the (expected) exponential rate $e^{-t\gamma}$.

Lemma 4. *For any $\delta > 0$, there exists $\eta > 0$ such that*

$$\int_{\{y\in C:|y-p_C^\perp(a)|\geqslant\delta\}} e^{|y|^2/2} p^C(1, x, y) e^{-t|a-y|^2/2} \mathrm{d}y = O(e^{-t(\gamma+\eta)}), \quad t \to \infty,$$

where γ is the quantity defined in (15).

Proof. Let $\delta > 0$ be given and define

$$J(t) = \int_{\{y\in C:|y-p_C^\perp(a)|\geqslant\delta\}} e^{|y|^2/2} p^C(1, x, y) e^{-t|a-y|^2/2} \mathrm{d}y.$$

From the inequality $|y|^2 \leqslant (|y - a| + |a|)^2 \leqslant 2|y - a|^2 + 2|a|^2$, we obtain the upper bound $e^{|y|^2/2} \leqslant ce^{2|y-a|^2}$, from which we deduce that

$$0 \leqslant J(t) \leqslant c \int_{\{y\in C:|y-p_C^\perp(a)|\geqslant\delta\}} p^C(1, x, y) e^{-s|a-y|^2/2} \mathrm{d}y,$$

where $s = t - 2$. Since $y \mapsto |a - y|^2/2$ is coercive, continuous and reaches its minimum γ on $\overline{C}$ at the unique point $y = p_C^\perp(a)$, there exists $\eta > 0$ such that $|a - y|^2/2 \geqslant \gamma + \eta$ on $\{y \in C : |y - p_C^\perp(a)| \geqslant \delta\}$. Hence, for all $s \geqslant 0$, we have

$$0 \leqslant J(t) \leqslant ce^{-s(\gamma+\eta)} \int_{\{y\in C:|y-p_C^\perp(a)|\geqslant\delta\}} p^C(1, x, y) \mathrm{d}y \leqslant ce^{-s(\gamma+\eta)}.$$

This concludes the proof of the lemma. □

It is now clear that the strategy for analyzing the non-exit probability is to determine the asymptotic behavior of the integral $I_\delta(t)$, which is defined by

$$I_\delta(t) = t^{d/2} \int_{\{y\in C:|y-p_C^\perp(a)|\leqslant\delta\}} e^{|y|^2/2} p^C(1, x, y) e^{-t|a-y|^2/2} \mathrm{d}y, \tag{16}$$

and to check in particular that it has the right exponential decreasing rate $e^{-\gamma}$, as expected. Indeed, in this case, the asymptotic behavior of $I(t)$, and consequently that of the non-exit probability, can be derived from the asymptotics of $I_\delta(t)$, as explained in the next lemma, which will constitute our general proof strategy.

Lemma 5. *Suppose that $g(t)$ is a function satisfying to conditions (i) and (ii) below:*

(i) *$g(t) = \kappa t^{-\alpha} e^{-t\gamma}$ for some $\kappa > 0$ and $\alpha \in \mathbb{R}$;*

(ii) *For all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$1 - \epsilon \leqslant \liminf_{t\to\infty} \frac{I_\delta(t)}{g(t)} \leqslant \limsup_{t\to\infty} \frac{I_\delta(t)}{g(t)} \leqslant 1 + \epsilon.$$

Then $I(t) = g(t)(1 + o(1))$ as $t \to \infty$.

Proof. It follows from Lemma 4 as an easy exercise. □

4. Case A (polar interior drift)

In this section, we study the case where the drift a belongs to the interior of the polar cone $C^\sharp$. It might be thought of as the natural generalization of the one-dimensional negative drift case. We shall note

$$u(x) = |x|^{p_1} m_1(\vec{x}). \tag{17}$$

The function u is the unique (up to multiplicative constants) positive harmonic function of Brownian motion killed at the boundary of C. We also define

$$\kappa_A = \frac{1}{2^{\alpha_1}\Gamma(\alpha_1 + 1)} \int_C e^{\langle a,y\rangle} u(y)\mathrm{d}y,$$

as well as

$$h_A(x) = e^{\langle -a,x\rangle} u(x).$$

Note that κ_A is finite since $a \in (C^\sharp)^o$ (see Lemma 8). Our main result in this section is the following:

Theorem A (Case A)**.** *Let C be a cone satisfying to (C1). If the drift a belongs to the interior of the polar cone $C^\sharp$, then*

$$\mathbb{P}_x[\tau_C > t] = \kappa_A h_A(x) t^{-(\alpha_1+1)} e^{-t|a|^2/2}(1 + o(1)), \quad t \to \infty.$$

Proof. Since $a \in (C^\sharp)^o$, the projection $p_C^\perp(a)$ is 0 and $\gamma = |a|^2/2$. According to our general strategy, we focus our attention on

$$I_\delta(t) = t^{d/2} \int_{\{y\in C:|y|\leqslant\delta\}} e^{|y|^2/2} p^C(1, x, y) e^{-t|a-y|^2/2}\mathrm{d}y.$$

Let $\epsilon > 0$ be given. It follows from Lemma 6 below that there exists $\delta > 0$ such that $p^C(1, x, y)$ is bounded from above and below on $\{y \in C : |y| \leqslant \delta\}$ by

$$(1 \pm \epsilon) b u(x) u(y) e^{-(|x|^2+|y|^2)/2},$$

where $b = (2^{\alpha_1}\Gamma(\alpha_1 + 1))^{-1}$. Therefore, $I_\delta(t)$ is bounded from above and below by

$$(1 \pm \epsilon) b u(x) e^{-|x|^2/2} t^{d/2} \int_{\{y\in C:|y|\leqslant\delta\}} u(y) e^{-t|a-y|^2/2}\mathrm{d}y. \tag{18}$$

By making the change of variables $v = ty$ and using the homogeneity of u, this expression becomes

$$(1 \pm \epsilon) b u(x) e^{-|x|^2/2} t^{-(\alpha_1+1)} e^{-t|a|^2/2} \int_{\{v\in C:|v|\leqslant t\delta\}} u(v) e^{\langle a,v\rangle - |v|^2/(2t)}\mathrm{d}v.$$

Now, since $a \in (C^\sharp)^o$ implies that $\langle a, v\rangle \leqslant -c|v|$ for all $v \in C$, for some $c > 0$ (see Lemma 8 below), the function $u(v)e^{\langle a,v\rangle}$ is integrable on C. Therefore, we can apply the dominated convergence theorem to obtain

$$\int_{\{v\in C:|v|\leqslant t\delta\}} u(v)e^{\langle a,v\rangle-|v|^2/(2t)}\mathrm{d}v = (1+o_\delta(1))\int_C u(v)e^{\langle a,v\rangle}\mathrm{d}v, \quad t\to\infty.$$

Hence, the bound for $I_\delta(t)$ can finally be written as

$$(1\pm\epsilon)\kappa_A u(x)e^{-|x|^2/2}t^{-(\alpha_1+1)}e^{-t|a|^2/2}(1+o_\delta(1)), \quad t\to\infty,$$

and a direct application of Lemma 5 gives

$$I(t) = \kappa_A u(x)e^{-|x|^2/2}t^{-(\alpha_1+1)}e^{-t|a|^2/2}(1+o(1)), \quad t\to\infty.$$

The theorem then follows thanks to the expression (13) of the non-exit probability. □

We now state and prove a lemma, that was used (in a crucial way) in the proof of Theorem A. Similar estimates can be found in [14, section 5].

Lemma 6. *We have*

$$\lim_{|y|\to 0}\frac{p^C(1,x,y)e^{(|x|^2+|y|^2)/2}}{u(x)u(y)} = (2^{\alpha_1}\Gamma(\alpha_1+1))^{-1}$$

uniformly on $\{x\in C : |x|\leqslant R\}$, for any positive constant R.

Proof. For brevity, let us write $x=\rho\theta$ and $y=r\eta$, with $\rho, r>0$ and $\theta,\eta\in\Theta$, and set $M=\rho r$. It follows from the expression of the heat kernel (12) that

$$\frac{p^C(1,\rho\theta,r\eta)e^{(\rho^2+r^2)/2}}{u(\rho\theta)u(r\eta)} = \sum_{j=1}^\infty \frac{I_{\alpha_j}(M)}{M^{\alpha_1}}\frac{m_j(\theta)}{m_1(\theta)}\frac{m_j(\eta)}{m_1(\eta)}.$$

Using then equation (21) from Lemma 7 below, we find the upper bound (below and throughout, c will denote a positive constant, possibly depending on the dimension d, which can take different values from line to line)

$$\left|\frac{I_{\alpha_j}(M)}{M^{\alpha_1}}\frac{m_j(\theta)}{m_1(\theta)}\frac{m_j(\eta)}{m_1(\eta)}\right| \leqslant \frac{c}{M^{\alpha_1}}\frac{I_{\alpha_j}(M)}{I_{\alpha_j}(1)}. \tag{19}$$

Now, using the integral expression (10) for I_{α_j}, we obtain

$$\begin{aligned} I_{\alpha_j}(M) \leqslant& \frac{2\left(\frac{M}{2}\right)^{\alpha_j}}{\sqrt{\pi}\Gamma(\alpha_j+1/2)}\cosh(M)\int_0^{\frac{\pi}{2}}(\sin t)^{2\alpha_j}\mathrm{d}t,\\ I_{\alpha_j}(1) \geqslant& \frac{2\left(\frac{1}{2}\right)^{\alpha_j}}{\sqrt{\pi}\Gamma(\alpha_j+1/2)}\int_0^{\frac{\pi}{2}}(\sin t)^{2\alpha_j}\mathrm{d}t. \end{aligned}$$

We conclude that

$$\frac{I_{\alpha_j}(M)}{I_{\alpha_j}(1)} \leqslant M^{\alpha_j}\cosh(M).$$

Using the latter estimation in (19), we deduce that

$$\left|\frac{I_{\alpha_j}(M)}{M^{\alpha_1}}\frac{m_j(\theta)}{m_1(\theta)}\frac{m_j(\eta)}{m_1(\eta)}\right| \leqslant cM^{\alpha_j-\alpha_1}\cosh(M).$$

It is easily seen from equation (20) in Lemma 7 below that $\sum_{j=1}^{\infty} M^{\alpha_j-\alpha_1}\cosh(M)$ is a uniformly convergent series for $M\in[0,1-\epsilon]$, for any $\epsilon\in(0,1]$. This immediately implies that the series

$$\sum_{j=1}^{\infty}\frac{I_{\alpha_j}(M)}{M^{\alpha_1}}\frac{m_j(\theta)}{m_1(\theta)}\frac{m_j(\eta)}{m_1(\eta)}$$

is uniformly convergent for $(M,\theta,\eta)\in[0,1-\epsilon]\times\Theta\times\Theta$, for any $\epsilon\in(0,1]$. Therefore we can take the limit term by term. Since

$$\lim_{M\to 0}\frac{I_{\alpha_j}(M)}{M^{\alpha_1}}\frac{m_j(\theta)}{m_1(\theta)}\frac{m_j(\eta)}{m_1(\eta)}=\begin{cases}\dfrac{1}{2^{\alpha_1}\Gamma(\alpha_1+1)} & \text{if} \quad j=1,\\ 0 & \text{if} \quad j\geqslant 2,\end{cases}$$

uniformly in $(\theta,\eta)\in\Theta\times\Theta$ (see (11) and Lemma 7 below), we reach the conclusion that

$$\lim_{M\to 0}\sum_{j=1}^{\infty}\frac{I_{\alpha_j}(M)}{M^{\alpha_1}}\frac{m_j(\theta)}{m_1(\theta)}\frac{m_j(\eta)}{m_1(\eta)}=\frac{1}{2^{\alpha_1}\Gamma(\alpha_1+1)},$$

where the convergence is uniform for $(\theta,\eta)\in\Theta\times\Theta$. The proof of Lemma 6 is complete. □

The following facts in the lemma below, concerning the eigenfunctions (8), are proved in [1].

Lemma 7 ([1])**.** *There exist two constants $0<c_1<c_2$ such that*

$$c_1 j^{1/(d-1)}\leqslant\alpha_j\leqslant c_2 j^{1/(d-1)},\quad \forall j\geqslant 1. \tag{20}$$

If C is a Lipschitz cone,[2] *then there exists a constant c such that*

$$m_j^2(\eta)\leqslant\frac{c m_1^2(\eta)}{I_{\alpha_j}(1)},\quad \forall j\geqslant 1,\quad \forall\eta\in\Theta. \tag{21}$$

We conclude this section with a useful characterization of the interior of the polar cone, which was used in the proof of Theorem A:

Lemma 8. *The drift vector a belongs to $(C^\sharp)^o$ if and only if there exists $\delta>0$ such that $\langle a,y\rangle\leqslant -\delta|y|$ for all $y\in\overline{C}$.*

Proof. Assume first that a satisfies the above condition. For all x such that $|a-x|<\delta$ and all $y\in C$, we have by Cauchy-Schwarz inequality

$$\langle x,y\rangle=\langle a,y\rangle+\langle x-a,y\rangle<-\delta|y|+\delta|y|=0,$$

hence $C^\sharp$ contains the open ball $B(a,\delta)$, and a is an interior point. Conversely, suppose that there exists $r>0$ such that the closed ball $\overline{B(a,r)}$ is included in $C^\sharp$. It is easily seen that

$$C^\sharp=\{x\in\mathbb{R}^d:\langle x,u\rangle\leqslant 0,\forall x\in\overline{C}\cap\mathbb{S}^{d-1}\}.$$

Since $\overline{C}\cap\mathbb{S}^{d-1}$ is a compact set, there exists a vector u_0 in this set such that

$$\gamma=\langle a,u_0\rangle=\max_{u\in\overline{C}\cap\mathbb{S}^{d-1}}\langle a,u\rangle.$$

[2] See the footnote 3 for the definition of a real-analytic cone (the definition of a Lipschitz cone is the same, replacing the real-analyticity by a Lipschitz condition).

Hence it remains to prove that $\gamma < 0$. To that aim, we select a family $\{x_1, \ldots, x_d\}$ of vectors of $\partial B(a, r)$ which forms a basis of $\mathbb{R}^d$. One of them, say x_1, must satisfy $\langle x_1, u_0 \rangle < 0$, since else we would have $\langle x_i, u_0 \rangle = 0$ for all $i \in \{1, \ldots, d\}$, and therefore $u_0 = 0$. Let $\overline{x}_1 = 2a - x_1$ be the opposite of x_1 on $\partial B(a, r)$. Since $\langle x_1, u_0 \rangle < 0$ and $\langle \overline{x}_1, u_0 \rangle \leqslant 0$, it follows that $\gamma = \langle a, u_0 \rangle = (\langle x_1, u_0 \rangle + \langle \overline{x}_1, u_0 \rangle)/2 < 0$. □

5. Case B (zero drift)

The case of a driftless Brownian motion, that we consider in the present section, has already been investigated by many authors, see [25, 9, 10, 1]. Define

$$\kappa_B = \frac{1}{2^{\alpha_1} \Gamma(\alpha_1 + 1)} \int_C u(y) e^{-|y|^2/2} \mathrm{d}y.$$

Theorem B (Case B)**.** *Let C be a cone satisfying to (C1). If the drift a is zero, then*

$$\mathbb{P}_x[\tau_C > t] = \kappa_B u(x) t^{-p_1/2}(1 + o(1)), \quad t \to \infty.$$

Although a proof of Theorem B can be found in [9, 1], for the sake of completeness we wish to write down some of the details below. As we shall see, the arguments are very similar to those used for proving Theorem A.

Proof of Theorem B. We have $a = 0$ and $\gamma = 0$. Thus, the lower and upper bounds (18) for $I_\delta(t)$ write

$$(1 \pm \epsilon) b u(x) e^{-|x|^2/2} t^{d/2} \int_{\{y \in C : |y| \leqslant \delta\}} u(y) e^{-t|y|^2/2} \mathrm{d}y.$$

This time, we make the change of variables $v = \sqrt{t} y$ and use the homogeneity of u in order to transform this expression into

$$(1 \pm \epsilon) b u(x) e^{-|x|^2/2} e^{-t|a|^2/2} t^{-p_1/2} \int_{\{v \in C : |v| \leqslant \sqrt{t}\delta\}} u(v) e^{-|v|^2/2} \mathrm{d}v.$$

Since the function $u(v)e^{-|v|^2/2}$ is integrable on C, it comes from the dominated convergence theorem that

$$\int_{\{v \in C : |v| \leqslant \sqrt{t}\delta\}} u(v) e^{-|v|^2/2} \mathrm{d}v = (1 + o_\delta(1)) \int_C u(v) e^{-|v|^2/2} \mathrm{d}v, \quad t \to \infty.$$

Hence, the bounds for $I_\delta(t)$ can finally be written as

$$(1 \pm \epsilon) \kappa_B u(x) e^{-|x|^2/2} t^{-p_1/2} (1 + o_\delta(1)), \quad t \to \infty.$$

The theorem then follows by an application of Lemma 5 and formula (13). □

6. Case C (interior drift)

Now we turn to the case when the drift a is inside the cone C.

Theorem C (Case C)**.** *Let C be a cone satisfying to (C1). If a belongs to C, then*

$$\mathbb{P}_x[\tau_C = \infty] = \lim_{t \to \infty} \mathbb{P}[\tau_C > t] = (2\pi)^{d/2} e^{|x-a|^2} p^C(1, x, a).$$

Proof. Since $a \in C$, one has $p_C^\perp(a) = a$ and $\gamma = 0$. As in the previous cases, we focus our attention on

$$I_\delta(t) = t^{d/2} \int_{\{y\in C:|y-a|\leqslant\delta\}} e^{|y|^2/2} p^C(1,x,y) e^{-t|a-y|^2/2} \mathrm{d}y.$$

For any given $\epsilon > 0$, we choose $\delta > 0$ so small that $\overline{B(a,\delta)} \subset C$ and

$$f(y) = e^{|y|^2/2} p^C(1,x,y)$$

is bounded from above and below by $f(a) \pm \epsilon$ for all $y \in \overline{B(a,\delta)}$. With this choice, $I_\delta(t)$ is then bounded from above and below by

$$(f(a) \pm \epsilon) t^{d/2} \int_{\{y\in\mathbb{R}^d:|y-a|\leqslant\delta\}} e^{-t|y-a|^2/2} \mathrm{d}y.$$

By the change of variables $v = \sqrt{t}(y-a)$, this expression becomes

$$(f(a) \pm \epsilon) \int_{\{v\in\mathbb{R}^d:|v|\leqslant\sqrt{t}\delta\}} e^{-|v|^2/2} \mathrm{d}v = (f(a) \pm \epsilon)(2\pi)^{d/2}(1 + o_\delta(1)), \quad t \to \infty.$$

Hence, the theorem follows from Lemma 5 and formula (13). □

Example 2. In the case where C is the Weyl chamber of type A, see (3), the heat kernel is given by the Karlin-McGregor formula (see [20, Theorem 1]):

$$p^C(t,x,y) = \det(p(t,x_i,y_j))_{1\leqslant i,j\leqslant d},$$

with $p(t,x_i,y_j) = (2\pi t)^{-1/2} e^{-(x_i-y_j)^2/2t}$. An easy computation then shows that $p^C(1,x,a)$ is equal to

$$p^C(1,x,a) = (2\pi)^{-d/2} e^{-(|x|^2+|a|^2)/2} \det(e^{x_i a_j})_{1\leqslant i,j\leqslant d}.$$

Hence

$$\lim_{t\to\infty} \mathbb{P}_x[\tau_C > t] = e^{\langle -a,x\rangle} \det(e^{x_i a_j})_{1\leqslant i,j\leqslant d}.$$

This result was derived earlier by Biane, Bougerol and O'Connell in [4, section 5].

7. Case D (boundary drift)

In this section we make the following hypothesis on the cone:

(C2) The set $\Theta = \mathbb{S}^{d-1} \cap C$ is real-analytic.[3]

This assumption ensures that the heat kernel can be extended to a bigger cone, and thus admits a Taylor expansion at any boundary point. To our knowledge, for more general cones like those which are intersections of smooth deformations of half-spaces, the boundary behavior of the heat kernel at a corner point (i.e., at a point located at the intersection of two, or more, half-spaces) is not known, except in the particular case of Weyl chambers [20, 4]. This behavior will determine the polynomial part $t^{-\alpha}$ in the

[3] A bounded domain $\Omega \subset \mathbb{R}^d$ and its boundary $\partial\Omega$ are real-analytic if at each point $x \in \partial\Omega$ there is a ball $B(x,r)$ with $r > 0$ and a one-to-one mapping ψ of $B(x,r)$ onto a certain domain $D \subset \mathbb{R}^d$ such that (i) $\phi(B(x,r) \cap \Omega) \subset [0,\infty)^d$, (ii) $\phi(B(x,r) \cap \partial\Omega) \subset \partial([0,\infty)^d)$, (iii) ψ and ψ^{-1} are real-analytic functions on $B(x,r)$ and D, respectively. This is equivalent to the fact that each point of $\partial\Omega$ has a neighborhood in which $\partial\Omega$ is the graph of a real-analytic function of $n-1$ coordinates. We refer to [16, section 6.2] for more details.

asymptotics of the non-exit probability. The case of Weyl chambers is treated in [24]. Here, we deal with the opposite (i.e., smooth) setting.

Define the function

$$h_D(x) = e^{|x-a|^2/2}\partial_n p^C(1, x, a),$$

where n stands for the (inner-pointing) unit vector normal to C at a, and $\partial_n p^C(1, x, a)$ denotes the normal derivative of the function $y \mapsto p^C(1, x, y)$ at $y = a$. Function $h_D(x)$ is non-zero thanks to Lemma 10 below. Define also the constant

$$\kappa_D = (2\pi)^{(d-1)/2}.$$

Theorem D (Case D)**.** *Let C be a cone satisfying to (C1) and (C2).*[4] *If $a \neq 0$ belongs to ∂C, then*

$$\mathbb{P}_x[\tau_C > t] = \kappa_D h_D(x) t^{-1/2}(1 + o(1)), \quad t \to \infty.$$

Proof. As in case C, we have $p_C^{\perp}(a) = a$ and $\gamma = 0$, and the formula (16) for $I_\delta(t)$ writes

$$I_\delta(t) = t^{d/2} \int_{\{y \in C : |y-a| \leqslant \delta\}} f(y) e^{-t|a-y|^2/2} \mathrm{d}y,$$

where $f(y) = e^{|y|^2/2} p^C(1, x, y)$. In the present case, $f(y)$ vanishes at $y = a$, contrary to case C. Since the function $y \mapsto p^C(1, x, y)$ is assumed to be infinitely differentiable in a neighborhood of a (see Lemma 9), it follows from Taylor's formula that, for any (sufficiently small) $\delta > 0$, there exists $M > 0$ such that

$$|f(y) - \langle y - a, \nabla f(a)\rangle| \leqslant M|y - a|^2, \quad \forall |y - a| \leqslant \delta.$$

Therefore, for any fixed $\delta > 0$, one has

$$I_\delta(t) = t^{d/2} \int_{\{y \in C : |y-a| \leqslant \delta\}} (\langle y - a, \nabla f(a)\rangle + O(|y - a|^2)) e^{-t|y-a|^2/2} \mathrm{d}y.$$

Making the change of variables $v = \sqrt{t}(y - a)$ implies that the above equation is the same as

$$t^{-1/2} \int_{(C - \sqrt{t}a) \cap \{v \in \mathbb{R}^d : |v| \leqslant \sqrt{t}\delta\}} \langle v, \nabla f(a)\rangle e^{-|v|^2/2} \mathrm{d}v + O(t^{-1}).$$

Now, due to the regularity of ∂C at a (see hypothesis (C2)—in fact it would be enough for ∂C to be continuously differentiable at a), the set

$$(C - \sqrt{t}a) \cap \{v \in \mathbb{R}^d : |v| \leqslant \sqrt{t}\delta\}$$

goes to $\{v \in \mathbb{R}^d : \langle v, n\rangle > 0\}$ as $t \to \infty$. Furthermore, an easy computation shows that

$$\int_{\{v \in \mathbb{R}^d : \langle v, n\rangle > 0\}} v e^{-|v|^2/2} \mathrm{d}v = (2\pi)^{(d-1)/2} n.$$

Hence, we deduce that

$$I_\delta(t) = t^{-1/2}(2\pi)^{(d-1)/2} \partial_n f(a) + o_\delta(t^{-1/2}), \quad t \to \infty.$$

Since $\partial_n f(a) = e^{|a|^2/2} \partial_n p^C(1, x, a) \neq 0$ by Lemma 10, Theorem D follows from Lemma 5 and formula (13). □

[4] Note that the hypothesis (C2) implies (C1).

The two following lemmas have been used in the proof of Theorem D. The first of the two lemmas is quite standard. As for the second one, it is an immediate consequence of [13, Theorem 2].

Lemma 9. *Under (C2), the heat kernel $y \mapsto p^C(1,x,y)$ is infinitely differentiable and is extendable in a neighborhood of $y=a$ (with the same regularity, and as a solution of the heat equation).*

Lemma 10. *Under (C2), the normal derivative of the function $y \mapsto p^C(1,x,y)$ at $y=a$ exists and is non-zero (for any $a \in \partial C \setminus \{0\}$).*

Example 1 (continued)**.** In the particular case of the dimension 2, with a cone of opening angle β (see Figure 3), one has the following expression for the normal derivative at a:

$$\partial_n f(a) = \frac{2\pi}{|a|\beta^2} e^{-|x|^2/2} \sum_{j=1}^{\infty} I_{\alpha_j}(|x||a|) m_j(x) j,$$

which gives a simplified expression for function $h_D(x)$. The above identity is elementary: it follows from the expression (9) of the eigenfunctions together with the definition of function f and some uniform estimates (to be able to exchange the summation and the derivation in the series defining the heat kernel).

8. Case E (non-polar exterior drift)

In that section in addition to (C1) and (C2) we shall assume that:

(C3) The cone C is convex.

A consequence of this assumption is that the orthogonal projection $p_C^\perp(a)$ of a on $\overline{C}$ is well and uniquely defined. Theorem E below can be extended without difficulty to the case of cones with finitely many points where the maximum of the function (14) is reached. However, we leave the case of infinitely many maximum points as an open problem.

Define the function

$$h_E(x) = e^{|x-a|^2/2} \partial_n p^C(1,x,p_C^\perp(a)).$$

We shall prove the following:

Theorem E (Case E)**.** *Let C be a cone satisfying to (C1), (C2) and (C3). If a belongs to $\mathbb{R}^d \setminus (\overline{C} \cup C^\sharp)$, then*

$$\mathbb{P}_x[\tau_C > t] = \kappa_E h_E(x) t^{-3/2} e^{-t|a-p_C^\perp(a)|^2/2}(1+o(1)), \quad t \to \infty,$$

where the constant $\kappa_E > 0$ will be made explicit in the proof.

Proof. The beginning of the proof is similar to that of Theorem D, except that we have to make a Taylor expansion with three (and not two) terms, for reasons that will be clear later. In case E, the projection p (for brevity, we set here $p = p_C^\perp(a)$) belongs to ∂C and is different from 0 and a, and $\gamma = |p-a|^2/2$.

For the same reasons as in case D, for any $\delta > 0$ small enough, we have

(22)
$$I_\delta(t) = t^{d/2} \int \left(\langle y-p, \nabla f(p) \rangle + \frac{1}{2}(y-p)^\top \nabla^2 f(p)(y-p) + O(|y-p|^3) \right) e^{-t|y-a|^2/2} \mathrm{d}y,$$

where $f(y)=e^{|y|^2/2}p^C(1,x,y)$, $(y-p)^\top$ is the transpose of the vector $y-p$, $\nabla^2 f(p)$ denotes the Hessian matrix of f at p, and the domain of integration is $\{y\in C : |y-p|\leqslant\delta\}$. To compute the asymptotics of the integral $I_\delta(t)$ as $t\to\infty$, we shall make a series of two changes of variables. First, the change of variables $u=y-p$ and the use of the identity

$$e^{-t|y-a|^2/2}=e^{-t\gamma}e^{-t|y-p|^2/2-t\langle y-p,p-a\rangle}$$

give the following alternative expression

$$I_\delta(t)=t^{d/2}e^{-t\gamma}\int_D\left(\langle u,\nabla f(p)\rangle+\frac{1}{2}u^\top\nabla^2 f(p)u+O(|u|^3)\right)e^{-t|u|^2/2}e^{-t\langle u,p-a\rangle}\mathrm{d}u, \tag{23}$$

where the domain of integration D equals $(C-p)\cap\{u\in\mathbb{R}^d : |u|\leqslant\delta\}$.

In the sequel, we will assume (without loss of generality) that the inner-pointing unit normal to ∂C at p is equal to e_1, the first vector of the canonical basis. With this convention $p-a=|p-a|e_1$, and the only non-zero component of $\nabla f(p)$ is in the e_1-direction. Indeed, since $f(y)=0$ for $y\in\partial C$, the boundary of the cone is a level curve for the function f, and it is well known that the gradient is orthogonal to the level curves. Therefore, the quantity $\langle u,\nabla f(p)\rangle$ is equal to $u_1\partial_1 f(p)$.

Our last change is $v=\phi_t(u)$; it sends $(u_1,u_2,\ldots,u_d)$ onto $(tu_1,\sqrt{t}u_2,\ldots,\sqrt{t}u_d)$. Note that the scalings in the normal and tangential directions are not the same; this entails that in (22) the second term in the integrand is not negligible w.r.t. the first one, and this is the reason why we have to make a Taylor expansion with three terms and not two. Note also that the Jacobian of this transformation is $t^{(d+1)/2}$. From this and (23) we deduce that as $t\to\infty$,

$$\begin{aligned}t^{3/2}e^{t\gamma}I_\delta(t)=\int_{\phi_t(D)}&\left(v_1\partial_1 f(p)+\frac{1}{2}(0,v_2,\ldots,v_d)^\top\nabla^2 f(p)(0,v_2,\ldots,v_d)\right)\\&\times e^{-v_1|p-a|}e^{-(v_2^2+\cdots+v_d^2)/2}e^{-v_1^2/(2t)}\mathrm{d}v+O(t^{-1/2}).\end{aligned} \tag{24}$$

Our aim now is to understand the behavior of the domain $\phi_t(D)$ as $t\to\infty$. Since the cone C is tangent to the hyperplane $\{u\in\mathbb{R}^d : u_1=0\}$ at p and its boundary is real-analytic, there exists a real-analytic function g with $g(0)=0$ and $\nabla g(0)=0$, such that, for δ small enough, the domain D coincides with

$$\{u\in\mathbb{R}^d : u_1>g(u_2,\ldots,u_d),|u|\leqslant\delta\}.$$

An application of Taylor formula then gives that (up to a set of Lebesgue measure zero),

$$\lim_{t\to\infty}\phi_t(D)=\phi_\infty(D)=\{v\in\mathbb{R}^d : v_1>\frac{1}{2}(v_2,\ldots,v_d)^\top\nabla^2 g(0)(v_2,\ldots,v_d)\}.$$

Let us now compare the limit domain $\phi_\infty(D)$ and the integrand in equation (24). Since f vanishes on the boundary of the cone, we have for u in a neighborhood of 0 that

$$f(p_1+g(u_2,\ldots,u_d),p_2+u_2,\ldots,p_d+u_d)=0.$$

Differentiating twice this identity, we obtain that

$$(0,v_2,\ldots,v_d)^\top\nabla^2 f(p)(0,v_2,\ldots,v_d)=-\partial_1 f(p)(v_2,\ldots,v_d)^\top\nabla^2 g(0)(v_2,\ldots,v_d).$$

Therefore, equation (24) can be rewritten as

$$t^{3/2}e^{t\gamma}I_\delta(t) = \partial_1 f(p)\int_{\phi_t(D)}\left(v_1 - \frac{1}{2}(v_2,\ldots,v_d)^\top\nabla^2 g(0)(v_2,\ldots,v_d)\right) \\ \times e^{-v_1|p-a|}e^{-(v_2^2+\cdots+v_d^2)/2}e^{-v_1^2/(2t)}\mathrm{d}v + O(t^{-1/2})$$

as $t\to\infty$. Note that the limit domain $\phi_\infty(D)$ is exactly equal to the subset of $\mathbb{R}^d$ where the integrand is positive. Thus, the constant

$$\kappa_E = e^{(|p|^2-|a|^2)/2} \\ \times\int_{\phi_\infty(D)}\Big(v_1-(v_2,\ldots,v_d)^\top\nabla^2 g(0)(v_2,\ldots,v_d)\Big)\,e^{-v_1|p-a|}e^{-(v_2^2+\cdots+v_d^2)/2}\mathrm{d}v$$

is positive. Noting that $\partial_1 f(p) = e^{|p|^2/2}\partial_1 p^C(1,x,p)\neq 0$ by Lemma 10, we obtain that

$$I_\delta(t) = \kappa_E e^{|a|^2/2}\partial_1 p^C(1,x,p)t^{-3/2}e^{-t\gamma}(1+o(1)),\quad t\to\infty,$$

and we conclude the proof of Theorem E by using Lemma 5 and equation (13). $\square$

Example 1 (continued)**.** In the particular case of two-dimensional cones, $\nabla^2 g(0)=0$ and the limit domain of integration $\phi_\infty(D)$ is the half-space $\{v\in\mathbb{R}^2 : v_1\geqslant 0\}$. The constant κ_E can then be computed:

$$\kappa_E = \frac{e^{(|p|^2-|a|^2)/2}}{|p-a|^2}\sqrt{2\pi}.$$

9. Case F (polar boundary drift)

We finally consider the case where the drift $a\neq 0$ belongs to $\partial C^\sharp$. Let us first notice that the existence of such a vector a implies that the cone C is entirely included in some half-space. More precisely, by definition of the polar cone, the inner product of a with any $y\in C$ is non-positive, so that C is included in the half-space $\{y\in\mathbb{R}^d : \langle a,y\rangle\leqslant 0\}$. Moreover, there must exist some $\theta_c\in\partial\Theta=\partial(C\cap\mathbb{S}^{d-1})$ such that $\langle a,\theta_c\rangle = 0$, for else a would belong to the interior of $C^\sharp$, as seen in Lemma 8. We call Θ_c the set of all these *contact points* θ_c between $\partial\Theta$ and the hyperplane $a^\perp=\{y\in\mathbb{R}^d : \langle a,y\rangle = 0\}$. As we shall see, the asymptotics of $\mathbb{P}_x[\tau_C>t]$ is determined by the local geometry of the cone C near these points.

We first present some general aspects of our approach, and then we will treat the case $d=2$ for cones with opening angle $\beta\in(0,\pi)$, and the case $d=3$ for cones with a real-analytic boundary and a finite number of contact points. Other cases are left as open problems. In the sequel, we will assume (without loss of generality) that $a=-|a|e_d$, where e_d stands for the last vector of the canonical basis.

As in case A, we have $p_C^\perp(a)=0$ and $\gamma=|a|^2/2$, so that the formula (16) for $I_\delta(t)$ can be written as

$$I_\delta(t) = t^{d/2}\int_{\{y\in C:|y|\leqslant\delta\}} e^{|y|^2/2}p^C(1,x,y)e^{-t|a-y|^2/2}\mathrm{d}y.$$

Let $\epsilon > 0$ be given. Arguing as in case A, we can pick $\delta > 0$ small enough so that $I_\delta(t)$ is bounded from above and below by

$$(1 \pm \epsilon) b u(x) e^{-|x|^2/2} t^{d/2} \int_{\{y \in C : |y| \leqslant \delta\}} u(y) e^{-t|a-y|^2/2} \mathrm{d}y, \tag{25}$$

where $b = (2^{\alpha_1}\Gamma(\alpha_1 + 1))^{-1}$. Thus, we are led to study the asymptotic behavior of

$$\begin{aligned} J_\delta(t) &= t^{d/2} \int_{\{y \in C : |y| \leqslant \delta\}} u(y) e^{-t|a-y|^2/2} \mathrm{d}y, \\ &= e^{-t\gamma} t^{d/2} \int_{\{y \in C : |y| \leqslant \delta\}} u(y) e^{-t|y|^2/2} e^{-t|a|y_d} \mathrm{d}y. \end{aligned}$$

Making the change of variables $z = \sqrt{t}y$ and using the homogeneity property of u (see (17)), we obtain

$$J_\delta(t) = e^{-t\gamma} t^{-p_1/2} \int_{\{z \in C : |z| \leqslant \sqrt{t}\delta\}} u(z) e^{-|z|^2/2} e^{-\sqrt{t}|a|z_d} \mathrm{d}z. \tag{26}$$

Laplace's method suggests that only some neighborhood of the hyperplane $\{z \in \mathbb{R}^d : z_d = 0\}$ will contribute to the asymptotics. More precisely, we have the following result:

Lemma 11. *For any $\eta > 0$, we have*

$$\int_{\{z \in C : z_d > \eta |z|\}} u(z) e^{-|z|^2/2} e^{-\sqrt{t}|a|z_d} \mathrm{d}z = o(t^{-d/2}), \quad t \to \infty.$$

Proof. Since $|u(z)| \leqslant M|z|^{p_1}$, the integral above is bounded from above by

$$M \int_{\mathbb{R}^d} |z|^{p_1} e^{-\eta\sqrt{t}|a||z|} \mathrm{d}z = M t^{-(p_1+d)/2} \int_{\mathbb{R}^d} |w|^{p_1} e^{-\eta|a||w|} \mathrm{d}w,$$

which is equal to $O(t^{-(p_1+d)/2})$. Lemma 11 follows since $p_1 > 0$. □

From now on, we shall assume that:

(C4) The set of contact points Θ_c is finite.

Let $\eta > 0$ be so small that the d-dimensional balls $\{B(\theta_c, \eta), \theta_c \in \Theta_c\}$ are disjoints. Since the set of all $\theta \in \overline{\Theta}$ that do not belong to any of these open balls is compact and does not contain any contact point, there exists some $\eta' > 0$ such that $\theta_d > \eta'$ for all such θ. For $\theta_c \in \Theta_c$, we define the cones

$$C(\theta_c, \eta) = \{z \in C : z/|z| \in B(\theta_c, \eta)\}. \tag{27}$$

Then C can be written as the disjoint union of these (thin) cones and of a (big) remaining cone whose points z all satisfy the inequality $z_d/|z| > \eta'$. Thus, according to formula (26) and Lemma 11, we have

$$J_\delta(t) = e^{-t\gamma} t^{-p_1/2} \left(\sum_{\theta_c \in \Theta_c} K_{\delta,\eta,\theta_c}(t) + o(t^{-d/2}) \right), \tag{28}$$

where

$$K_{\delta,\eta,\theta_c}(t) = \int_{\{z \in C(\theta_c,\eta) : |z| \leqslant \sqrt{t}\delta\}} u(z) e^{-|z|^2/2} e^{-\sqrt{t}|a|z_d} \mathrm{d}z \tag{29}$$

represents the contribution of each contact point.

Two-dimensional cones. Here the cone is $C = \{\rho e^{i\theta} : \rho > 0, \theta \in (0, \beta)\}$ with $\beta \in (0, \pi)$. Define

$$h_F(x) = e^{\langle -a,x\rangle} u(x)$$

and the constant

$$\kappa_F = \frac{\pi 2^{p_1/2}\Gamma(p_1/2)}{2^{\alpha_1}\Gamma(\alpha_1+1)\beta^2|a|^2}.$$

Theorem F (Case of the dimension 2)**.** *Let C be any two-dimensional cone with $\beta \in (0, \pi)$. If $a \neq 0$ belongs to $\partial C^\sharp$, then*

$$\mathbb{P}_x[\tau_C > t] = \kappa_F h_F(x) e^{-t|a|^2/2} t^{-(p_1/2+1)}(1 + o(1)), \quad t \to \infty.$$

Proof. Since $\beta < \pi$, there is only one contact point, namely $\theta_c = (1, 0)$. Let us analyze its contribution. According to (29), we have

$$K_{\delta,\eta,\theta_c}(t) = \int_{\{z\in\mathbb{R}^2 : 0<z_2<\eta z_1, |z|\leqslant\sqrt{t}\delta\}} u(z) e^{-|z|^2/2} e^{-\sqrt{t}|a|z_2} \mathrm{d}z,$$

as soon as η is small enough. (In fact, the condition is $\arcsin \eta < \beta$, and η in the integral should be $\tan(\arcsin \eta)$.)

We now proceed to the change of variables $v = \phi_t(z) = (z_1, \sqrt{t}z_2)$, which leads to

$$K_{\delta,\eta,\theta_c}(t) = t^{-1/2}\int_{D_t} u\left(v_1, \frac{v_2}{\sqrt{t}}\right) e^{-|v_1|^2/2} e^{-|v_2|/2t} e^{-|a|v_2} \mathrm{d}v,$$

where $D_t = \phi_t(\{z \in \mathbb{R}^2 : 0 < z_2 < \eta z_1, |z| \leqslant \sqrt{t}\delta\})$. Notice that $(v_1, v_2) \in D_t$ implies that $|v_2/(v_1\sqrt{t})| < \eta$. It follows from the Taylor-Lagrange inequality that (if η is small enough) there exists M such that

$$u(1, h) = \partial_2 u(1, 0)h + h^2 R(h),$$

with $|R(h)| \leqslant M$ for all $|h| \leqslant \eta$. Therefore, using the homogeneity of u, we obtain

$$\sqrt{t}u\left(v_1, \frac{v_2}{\sqrt{t}}\right) = \sqrt{t}v_1^{p_1} u\left(1, \frac{v_2}{v_1\sqrt{t}}\right) = v_1^{p_1-1} v_2(\partial_2 u(1, 0) + hR(h)),$$

with $h = v_2/(v_1\sqrt{t})$ and $|hR(h)| \leqslant \eta M$ for all $(v_1, v_2) \in D_t$. As $t \to \infty$, the domain D_t converges to the quarter plane $\mathbb{R}^2_+$, and it follows from the dominated convergence theorem that as $t \to \infty$,

$$\begin{aligned} K_{\delta,\eta,\theta_c}(t) &= t^{-1}\partial_2 u(1, 0)\int_{\mathbb{R}^2_+} v_1^{p_1-1} v_2 e^{-v_1^2/2} e^{-|a|v_2} \mathrm{d}v + o(t^{-1}) \\ &= t^{-1}\frac{\pi 2^{p_1/2}\Gamma(p_1/2)}{\beta^2|a|^2}(1 + o(1)), \end{aligned} \tag{30}$$

where we have used the fact that $\partial_2 u(1, 0) = 2\pi/\beta^2$ (see (9) for $j = 1$). For $\beta < \pi$, there is no other contribution and, therefore, combining equations (30), (28) and (25) shows that bounds for $I_\delta(t)$ are given by

$$(1 \pm \epsilon)\kappa_F u(x) e^{-|x|^2/2} e^{-t\gamma} t^{-(p_1/2+1)}(1 + o(1)), \quad t \to \infty.$$

Hence, as in the other cases, the result follows from Lemma 5 and formula (13). □

When $\beta = \pi$, the point $(-1, 0)$ is a second contact point. By symmetry, its contribution is exactly the same as that of $(1, 0)$. Hence the result of Theorem F is still valid if κ_F is replaced with $2\kappa_F$.

Three-dimensional cones with real-analytic boundary. Recall that (by convention) $a = -|a|e_3$ and the cone C is contained in the half space $\{z_3 > 0\}$, see Figure 4. Thanks to (28), the asymptotic behavior of $\mathbb{P}_x[\tau_C > t]$ will follow from the study of the contributions

$$K_{\delta,\eta,\theta_c}(t) = \int_{\{z\in C(\theta_c,\eta):|z|\leqslant\sqrt{t}\delta\}} u(z)e^{-|z|^2/2}e^{-\sqrt{t}|a|z_3}\mathrm{d}z$$

of the contact points $\theta_c \in \Theta_c$ between $\partial\Theta$ and the hyperplane $a^\perp = \{z \in \mathbb{R}^3 : z_3 = 0\}$. As we shall see, the behavior of the integral above will depend on the geometry of Θ at the point θ_c.

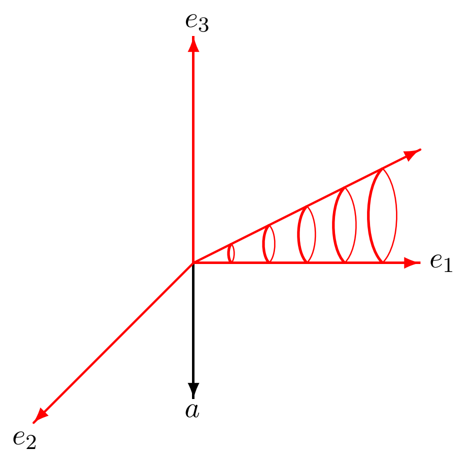

FIGURE 4. Three-dimensional cones in the proof of Theorem F

Contribution of each particular contact point. Without loss of generality, let us assume that $\theta_c = e_1$. Since the cone is tangent to the plane $\{z \in \mathbb{R}^3 : z_3 = 0\}$ at the point θ_c and since its boundary is assumed to be real-analytic, there exists a real-analytic function $g(z_2)$ with $g(0) = 0$ and $g'(0) = 0$, such that the intersection of C with $\{z \in \mathbb{R}^3 : z_1 = 1\}$ coincides (in a neighborhood of θ_c) with the set

$$g^+ = \{z \in \mathbb{R}^3 : z_1 = 1, z_3 > g(z_2)\}.$$

Define

$$q = q(\theta_c) = \inf\{n \geqslant 2 : g^{(n)}(0) \neq 0\},$$

and

$$c = c(\theta_c) = \frac{g^{(q)}(0)}{q!}.$$

Since θ_c is isolated from the other contact points (recall that Θ_c is assumed to be finite), the function $g(z_2)$ must be positive for all $z_2 \neq 0$ in a neighborhood of 0. Thus, by real-analyticity, q must be finite, even, and such that $g^{(q)}(0)>0$. Finally, set

$$\kappa(q)=\frac{2^{(p_1+1-1/q)/2}(1-\frac{1}{q+1})}{|a|^{2+1/q}}\Gamma\left(\frac{p_1+1-1/q}{2}\right)\Gamma\left(2+\frac{1}{q}\right).$$

Then we have:

Lemma 12. *For any $\delta>0$ and $\eta>0$ small enough, the contribution of each contact point θ_c to the asymptotics of the non-exit probability is given by*

$$K_{\delta,\eta,\theta_c}(t)=\frac{\kappa(q)\partial_n u(\theta_c)}{c(\theta_c)^{1+1/q}}t^{-(1+1/(2q))}(1+o(1)),\quad t\to\infty,$$

where $\partial_n u(\theta_c)$ stands for the (inner-pointing) normal derivative of the function u at θ_c.

We postpone the proof of Lemma 12 after the statement and the proof of Theorem F.

Statement of Theorem F. Let q_1 be the maximum value of $q(\theta_c)$ for $\theta_c\in\Theta_c$. We define

$$h_F(x)=u(x)e^{-\langle a,x\rangle}$$

as well as

$$\kappa_F=b\kappa(q_1)\sum_{q(\theta_c)=q_1}\frac{\partial_n u(\theta_c)}{c(\theta_c)^{1+1/q}},$$

where $b=(2^{\alpha_1}\Gamma(\alpha_1+1))^{-1}$. Then we have:

Theorem F (Case of the dimension 3)**.** *Let C be a three-dimensional cone satisfying to (C1), (C2) and (C4). If $a\neq 0$ belongs to $\partial C^\sharp$, then*

$$\mathbb{P}_x[\tau_C>t]=\kappa_F h_F(x)t^{-(p_1/2+1+1/(2q_1))}e^{-t|a|^2/2}(1+o(1)),\quad t\to\infty.$$

Proof. Since $K_{\delta,\eta,\theta_c}(t)$ is of order $t^{-(1+1/(2q))}$ by Lemma 12, only those θ_c with $q(\theta_c)=q_1$ will contribute in (28) to the asymptotics of $J_\delta(t)$. Thus, we get that

$$J_\delta(t)=e^{-t\gamma}t^{-(p_1/2+1+1/(2q_1))}\kappa(q_1)\sum_{q(\theta_c)=q_1}\frac{\partial_n u(\theta_c)}{c(\theta_c)^{1+1/q_1}}(1+o(1)),\quad t\to\infty.$$

Now, equation (25) shows that bounds for $I_\delta(t)$ are given by

$$(1\pm\epsilon)\kappa_F u(x)e^{-|x|^2/2}e^{-t\gamma}t^{-(p_1/2+1+1/(2q_1))}(1+o(1)),\quad t\to\infty.$$

Hence, the result follows from Lemma 5 and formula (13). □

Proof of Lemma 12. With the conventions made just above, we analyze the contribution of $\theta_c=(1,0,0)$, namely,

$$K_{\delta,\eta,\theta_c}(t)=\int_{\{z\in C(\theta_c,\eta):|z|\leqslant\sqrt{t}\delta\}}u(z)e^{-|z|^2/2}e^{-\sqrt{t}|a|z_3}\mathrm{d}z.$$

By making the linear change of variables $v=\phi_t(z)$, with

$$\phi_t(z_1,z_2,z_3)=(z_1,t^{1/(2q)}z_2,\sqrt{t}z_3),$$

we obtain

$$K_{\delta,\eta,\theta_c}(t)=t^{-1/2-1/(2q)}\int_{D_t}u\left(v_1,\frac{v_2}{t^{1/(2q)}},\frac{v_3}{\sqrt{t}}\right)e^{-v_1^2/2}e^{-|a|v_3}(1+o(1))\mathrm{d}v,\tag{31}$$

where $D_t = \phi_t(\{z \in C(\theta_c, \eta) : |z| \leqslant \sqrt{t}\delta\})$, and $1 + o(1)$ goes increasingly to 1 as $t \to \infty$.

In order to understand the behavior of D_t as $t \to \infty$, we first notice that

$$\lim_{t\to\infty} D_t = \lim_{t\to\infty} \phi_t(C(\theta_c, \eta)).$$

Then, since the first coordinate is left invariant by ϕ_t, we shall look at what happens in the plane $\{z_1 = 1\}$. It follows from the definition of q that

$$g^+ = \{z \in \mathbb{R}^3 : z_1 = 1, z_3 > cz_2^q + o(z_2^q)\},$$

with $c = g^{(q)}(0)/q! > 0$. From this and the definition of ϕ_t, it is easily seen that

$$\lim_{t\to\infty} \phi_t(C(\theta_c, \eta) \cap \{z \in \mathbb{R}^3 : z_1 = 1\}) = \{v \in \mathbb{R}^3 : v_1 = 1, v_3 > cv_2^q\}.$$

Further, the homogeneity of the cone and the linearity of ϕ_t immediately imply that

$$\lim_{t\to\infty} \phi_t(C(\theta_c, \eta) \cap \{z \in \mathbb{R}^3 : z_1 = \lambda\}) = \{v \in \mathbb{R}^3 : v_1 = \lambda, \lambda^{q-1} v_3 > cv_2^q\},$$

for all $\lambda > 0$. Now, if $\eta > 0$ is small enough, the cone $C(\theta_c, \eta)$ does not contain any z with $z_1 \leqslant 0$. Therefore,

$$\lim_{t\to\infty} \phi_t(C(\theta_c, \eta)) = \{v \in \mathbb{R}^3 : v_1 > 0, v_3 > 0, v_1^{q-1} v_3 > cv_2^q\}. \tag{32}$$

We call D the limit domain in (32).

It remains to analyze the behavior of the integrand in (31), i.e., to find the asymptotics of

$$u\left(v_1, \frac{v_2}{t^{1/(2q)}}, \frac{v_3}{\sqrt{t}}\right) = v_1^{p_1} u\left(1, \frac{v_2}{v_1 t^{1/(2q)}}, \frac{v_3}{v_1\sqrt{t}}\right)$$

for $v_1 > 0$, as $t \to \infty$. To this end, we shall use a Taylor expansion of $u(1, x, y)$ in a neighborhood of $(0, 0)$. This can be done since it is known that the real-analyticity of Θ ensures that u can be extended to a strictly bigger cone, inside of which u is (still) harmonic, see [23, Theorem A]. Since u is equal to zero on the boundary of C, the relation

$$u(1, z_2, g(z_2)) = 0$$

holds for all z_2 in a neighborhood of 0, and a direct application of Lemma 15 below for $n = 1$ and $k \in \{0, \ldots, q-1\}$ shows that

$$\partial^{(j)}_{2,2,\ldots,2} u(1, 0, 0) = \begin{cases} 0 & \text{if } 1 \leqslant j \leqslant q-1, \\ -\partial_3 u(1, 0, 0) g^{(q)}(0) & \text{if } j = q. \end{cases} \tag{33}$$

Hence, the Taylor expansion of $u(1, z_2, z_3)$ leads to

$$\lim_{t\to\infty} \sqrt{t} u\left(1, \frac{v_2}{v_1 t^{1/(2q)}}, \frac{v_3}{v_1\sqrt{t}}\right) = \partial_3 u(1, 0, 0)\left(\frac{v_3}{v_1} - \frac{g^{(q)}(0)}{q!}\frac{v_2^q}{v_1^q}\right).$$

The proof that this convergence is dominated is deferred to Lemma 13 below, where the role—crucial—of $C(\theta_c, \eta)$ will appear clearly. Therefore, as $t \to \infty$,

$$\begin{aligned} K_{\delta,\eta,\theta_c}(t) = {} & t^{-1-1/(2q)} \partial_3 u(1, 0, 0) \\ & \times \int_D v_1^{p_1 - q}(v_1^{q-1} v_3 - cv_2^q) e^{-v_1^2/2} e^{-|a|v_3} \mathrm{d}v + o(t^{-1-1/(2q)}). \end{aligned} \tag{34}$$

Notice that the last integral is positive since D has positive (infinite) Lebesgue measure and is exactly the domain where the integrand is positive. We now compute its value. Since q is even, for any fixed $v_1 > 0$ and $v_3 > 0$, we have

$$\int_{\{v_2\in\mathbb{R}:v_1^{q-1}v_3>cv_2^q\}} (v_1^{q-1}v_3 - cv_2^q)\mathrm{d}v_2 = 2\left(1-\frac{1}{q+1}\right)(c^{-1}v_1^{q-1}v_3)^{1+1/q}.$$

Thus, by an application of Fubini's theorem, the integral in (34) becomes

$$2\left(1-\frac{1}{q+1}\right)c^{-1-1/q}\int_0^\infty v_1^{p_1-1/q}e^{-v_1^2/2}\mathrm{d}v_1\int_0^\infty v_3^{1+1/q}e^{-|a|v_3}\mathrm{d}v_3,$$

and can be expressed in terms of the Gamma function as

$$\frac{2^{(p_1+1-1/q)/2}(1-\frac{1}{q+1})}{|a|^{2+1/q}c^{1+1/q}}\Gamma\left(\frac{p_1+1-1/q}{2}\right)\Gamma\left(2+\frac{1}{q}\right) = \kappa(q)c^{-(1+1/q)}.$$

This concludes the proof of Lemma 12. □

Lemma 13. *Let $a_{i,j}$ denote the coefficient of $z_2^i z_3^j$ in the Taylor expansion of $u(1,z_2,z_3)$ at $(0,0)$. If $\eta>0$ in the definition (27) of $C(\theta_c,\eta)$ is small enough, then*

$$\int_{D_t} v_1^{p_1}\left|\sqrt{t}u\left(1,\frac{v_2}{v_1t^{1/(2q)}},\frac{v_3}{v_1\sqrt{t}}\right)-\left(a_{0,1}\frac{v_3}{v_1}+a_{q,0}\frac{v_2^q}{v_1^q}\right)\right|e^{-v_1^2/2}e^{-|a|v_3}\mathrm{d}v = o(1),\quad t\to\infty.$$

Proof. Since the function $u(1,z_2,z_3)$ can be extended to a function infinitely differentiable in a neighborhood of $(0,0)$, see [23, Theorem A], there exists $M>0$ such that, for $\eta_0>0$ small enough,

$$u(1,z_2,z_3) = \sum_{i+j\leqslant q} a_{i,j}z_2^iz_3^j + |(z_2,z_3)|^{q+1}R(z_2,z_3),$$

where $|R(z_2,z_3)|\leqslant M$ for all $(z_2,z_3)\in B(0,\eta_0)$. We already know (see (33) in the proof of Theorem F) that $a_{i,0}=0$ for all $i\in\{0,\ldots,q-1\}$, hence

$$|u(1,z_2,z_3)-(a_{0,1}z_3+a_{q,0}z_2^q)| \leqslant \sum_{2\leqslant j\leqslant q}|a_{0,j}z_3^j| + \sum_{\substack{i,j\geqslant 1\\ i+j\leqslant q}}|a_{i,j}z_2^iz_3^j| + |(z_2,z_3)|^{q+1}M.$$

Let $0<\epsilon<1$ be fixed. For $(z_2,z_3)\in B(0,\eta_0)$, we use the upper bound

$$|a_{0,j}||z_3|^{1+\epsilon}\eta_0^{j-(1+\epsilon)},\quad \forall j\geqslant 2,$$

for the terms inside of the first sum, and the upper bound

$$|a_{i,j}||z_2||z_3|\eta_0^{i+j-2},\quad \forall i+j\geqslant 2,$$

for the terms inside of the second sum. For the last term, we write

$$|(z_2,z_3)|^{q+1}\leqslant C(|z_2|^{q+1}+|z_3|^{q+1})\leqslant C(|z_2|^{q+1}+|z_3|^{1+\epsilon}\eta_0^{q-\epsilon}),$$

and we finally obtain the upper bound

$$|u(1,z_2,z_3)-(a_{0,1}z_3+a_{q,0}z_2^q)|\leqslant C_1|z_3|^{1+\epsilon}+C_2|z_2||z_3|+C_3|z_2|^{q+1}, \tag{35}$$

where $C_1,C_2,C_3>0$ are positive constants (depending on η_0 and ϵ only).

On the other hand, the definition of $C(\theta_c, \eta)$ ensures that

$$\left|\left(\frac{v_2}{v_1 t^{1/(2q)}}, \frac{v_3}{v_1\sqrt{t}}\right)\right| \leqslant \eta + o(\eta), \quad \eta \to 0,$$

for all $(v_1, v_2, v_3) \in D_t$. Therefore, if $\eta > 0$ is small enough so that $\eta + o(\eta) \leqslant \eta_0$, then according to (35) we have

$$\left|\sqrt{t}u\left(1, \frac{v_2}{v_1 t^{1/(2q)}}, \frac{v_3}{v_1\sqrt{t}}\right) - \left(a_{0,1}\frac{v_3}{v_1} + a_{q,0}\frac{v_2^q}{v_1^q}\right)\right| \\ \leqslant o(1)\left(C_1\left|\frac{v_3}{v_1}\right|^{1+\epsilon} + C_2\left|\frac{v_2}{v_1}\right|\left|\frac{v_3}{v_1}\right| + C_3\left|\frac{v_2}{v_1}\right|^{q+1}\right),$$

(where $o(1)$ is a function of t alone) for all $(v_1, v_2, v_3) \in D_t$, and the result follows from Lemma 14 below, provided that ϵ has been chosen so small that $1 + \epsilon + 1/q \leqslant 2$. □

Lemma 14. *The integral*

$$\int_D v_1^{p_1}\left|\frac{v_2}{v_1}\right|^{\alpha}\left|\frac{v_3}{v_1}\right|^{\beta} e^{-v_1^2/2}e^{-|a|v_3}\mathrm{d}v$$

is finite for all $\alpha, \beta \geqslant 0$ such that $\beta + (\alpha+1)/q \leqslant 2$.

Proof. Using Fubini's theorem, this integral can be shown to be equal to

$$\int_0^\infty v_1^{p_1+1-\beta-(\alpha+1)/q}e^{-v_1^2/2}\mathrm{d}v_1 \int_0^\infty v_3^{\beta+(\alpha+1)/q}e^{-|a|v_3}\mathrm{d}v_3,$$

up to some positive multiplicative constant. The result follows since $p_1 > 0$. □

Lemma 15. *Let $n \geqslant 1$ and $k \geqslant 0$, and assume that $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, with $g(0) = 0$, are two functions infinitely differentiable such that for some constant c,*

$$f(x, g(x), g'(x), \ldots, g^{(n-1)}(x)) = c, \tag{36}$$

for all x in some neighborhood of $x = 0$, and

$$g'(0) = g^{(2)}(0) = \cdots = g^{(n-1+k)}(0) = 0. \tag{37}$$

Then

$$\partial^{(k+1)}_{1,1,\ldots,1}f(0) = -\partial_{n+1}f(0)g^{(n+k)}(0).$$

Proof. Let $H(n,k)$ denote the statement that the conclusion of the lemma is true for the pair (n,k). We shall prove that

- $H(n,0)$ holds for all $n \geqslant 1$;
- For all $n \geqslant 1$ and $k \geqslant 1$, $H(n+1,k-1)$ implies $H(n,k)$.

The lemma will clearly follow by induction.

Let f and g be two functions satisfying the hypotheses of Lemma 15 for some $n \geqslant 1$ and $k \geqslant 0$, and set $\gamma(x) = (x, g(x), g'(x), \ldots, g^{(n-1)}(x))$. First, differentiating relation (36) with respect to x shows that

$$\partial_1 f(\gamma(x)) + \sum_{j=2}^n \partial_j f(\gamma(x))g^{j-1}(x) + \partial_{n+1}f(\gamma(x))g^{(n)}(x) = 0, \tag{38}$$

for all x in some neighborhood of $x = 0$. Hence, according to (37), we get that

$$\partial_1 f(0) + \partial_{n+1} f(0) g^{(n)}(0) = 0,$$

thereby proving $H(n, 0)$. Furthermore, equation (38) can be rewritten as

$$h(x, g(x), g'(x), \ldots, g^{(n)}(x)) = 0,$$

in some neighborhood of $x = 0$, where $h : \mathbb{R}^{n+2} \to \mathbb{R}$ is defined by

$$h(x_1, x_2, x_3, \ldots, x_{n+2}) = \partial_1 f(\gamma(x)) + \sum_{j=2}^{n} \partial_j f(\gamma(x)) x_{j+1} + \partial_{n+1} f(\gamma(x)) x_{n+2}. \tag{39}$$

Since equation (37) is left invariant when replacing n by $n+1$ and k by $k-1$, functions h and g fulfill the hypotheses of the lemma for the pair $(n+1, k-1)$. Therefore, if $H(n+1, k-1)$ holds, then

$$\partial^{(k)}_{1,1,\ldots,1} h(0) = -\partial_{n+2} h(0) g^{(n+k)}(0).$$

But it is clear from the definition (39) of h that

$$\partial^{(k)}_{1,1,\ldots,1} h(0) = \partial^{(k+1)}_{1,1,\ldots,1} f(0),$$

and

$$\partial_{n+2} h(0) = \partial_{n+1} f(0).$$

Hence $H(n, k)$ holds. $\square$

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