Sparse Sampling in Phase Space: Theory and Applications

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Abstract—**The question that we seek to answer is: Given a continuous time sparse signal (in primal domain), is it possible to sample and reconstruct this sparse signal based on its characterization in a dual domain—the one which generalizes a number of already known transformations?**

Phase Space transformations like the Linear Canonical Transform parametrically generalize a number of interesting transformations including the Fourier, Fresnel and fractional Fourier Transform among others. In this work, we develop a sampling and reconstruction methodology for sampling a sparse signal by analyzing their properties in Phase Space which is backward compatibe with several known transformations. This problem, as will be seen, can be recast as a parameter estimation problem that has its roots in Prony's method.

Given the general scope of analysis of sparse signals in transform domain, this work has applications in areas of spread spectrum methods for imaging, phase retrieval and computational photography.

Phase Space transformations are theoretically appealing for they generalize a number of existing transformations like the Fourier transform among others. One such construction includes the case of Linear Canonical Transform (LCT). The LCT (parametrized by $\mathbf{\Lambda} \equiv \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad - bc = 1$) of a signal, say $x(t)$, is given by $\widehat{x}_{\mathbf{\Lambda}}(\omega) = \mathsf{LCT}_{\mathbf{\Lambda}}\{x\}$ or,

$$\widehat{x}_{\mathbf{\Lambda}}(\omega) \stackrel{\text{def}}{=} \begin{cases} \langle x(\cdot), \phi_{\mathbf{\Lambda}}(\cdot, \omega)\rangle & \text{for } b \neq 0, \\ \sqrt{d} e^{j\frac{1}{2}(cd\omega^2)} x(d\omega) & \text{for } b = 0, \end{cases} \quad (2)$$

where $\phi_{\mathbf{\Lambda}}(\cdot, \omega)$ is the phase space kernel defined in (1) of Table I and $\langle g, h\rangle$ denotes the inner-product in L_2–sense. For $\mathbf{\Lambda} \in \mathbb{R}^2$, the LCT is a unitary transformation and generalizes many transformations (Table I). Results applicable to the LCT domain can easily be extended to any of its special cases. The Fourier transform and the FrFT are the most notable byproducts. The LCT is equipped with an interesting composition property:

$$\mathsf{LCT}_{\mathbf{\Lambda}_2}\{\mathsf{LCT}_{\mathbf{\Lambda}_1}\{x\}\} = \mathsf{LCT}_{\mathbf{\Lambda}_3}\{x\} \Rightarrow \mathbf{\Lambda}_2 \cdot \mathbf{\Lambda}_1 = \mathbf{\Lambda}_3$$

that leads to the definition of inverse–LCT of function, which is equivalent to $x(t) = \mathsf{LCT}_{\mathbf{\Lambda}^{-1}}\{\widehat{x}_{\mathbf{\Lambda}}\}$ or,

$$x(t) = \begin{cases} \langle \widehat{x}_{\mathbf{\Lambda}}(\cdot), \phi_{\mathbf{\Lambda}^{-1}}(t, \cdot)\rangle & \text{for } b \neq 0, \\ \sqrt{a} e^{-j\frac{1}{2}(cat^2)} x(at) & \text{for } b = 0, \end{cases} \quad (3)$$

where $\mathbf{\Lambda}^{-1}$ is the inverse of $\mathbf{\Lambda}$. If $\widetilde{x}(t)$ is the approximation of $x(t)$, then $\|\widetilde{x}(t) - x(t)\|_{L_2}^2 = 0$ whenever $(\omega_m b) \leqslant \frac{\omega_s}{2}$ (Nyquist rate for LCT domain), where ω_s is the sampling frequency. When $\mathbf{\Lambda}_{\text{FT}} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, all the aforementioned results take form of Shannon's sampling theorem. In this paper, we are interested in sampling a stream of K Dirac impulses—a signal with structure:

$$x(t) \stackrel{\text{def}}{=} \sum\nolimits_{k=0}^{K-1} c_k \delta(t - t_k), \quad (4)$$

with weights and arbitrary shifts, $\{c_k, t_k\}$. By constructing an aperiodic version of Fourier Series expansion (cf. [1]) for LCT domain, it turns out that (4) has a representation of form:

$$x(t) = \frac{1}{\tau} e^{-j\frac{a}{2b}t_k^2} \underbrace{\sum_{n=-\infty}^{n=+\infty} \left(\sum_{k=0}^{K-1} \underbrace{c_k \cdot e^{j\frac{a}{2b}t_k^2}}_{a_k} \underbrace{e^{-jn\omega_0 t_k}}_{u_k^n} \right)}_{p[n]\text{—Sum of K–complex exponentials}} e^{jn\omega_0 t}. \quad (5)$$

TABLE I
LCT AS A GENERALIZATION OF OTHER TRANSFORMATIONS

$$\phi_{\mathbf{\Lambda}}(t, \omega) = \frac{1}{\sqrt{-j2\pi b}} \exp\left\{-\frac{j}{2b}\left(\left(at^2 + d\omega^2\right) - 2\omega t\right)\right\} \quad (1)$$

LCT Parameters ($\mathbf{\Lambda}$)	Corresponding Transform
$\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right] = \mathbf{\Lambda}_\theta$	Fractional Fourier Transform (FrFT)
$\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] = \mathbf{\Lambda}_{\text{FT}}$	Fourier Transform (FT)
$\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$	Fresnel Transform
$\left[\begin{smallmatrix} 1 & jb \\ j & 1 \end{smallmatrix}\right]$	Bilateral Laplace Transform
$\left[\begin{smallmatrix} 1 & -jb \\ 0 & 1 \end{smallmatrix}\right]$, $b \geq 0$	Gauss–Weierstrass Transform
$\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 0 & e^{-j\pi/2} \\ -e^{-j\pi/2} & 1 \end{smallmatrix}\right]$	Bargmann Transform

Discretizing (5) in m–points results in the following system:

$$\underbrace{x(m) e^{j\frac{a}{2b}m^2}}_{\mathbf{x}} = \frac{1}{\tau} \sum_{n=-\infty}^{n=+\infty} \underbrace{\left(\sum_{k=0}^{K-1} a_k u_k^n\right)}_{\mathbf{p}} e^{jn\omega_0 m} \Leftrightarrow \boxed{\mathbf{x} = \mathbf{D}_{\text{DFT}}^{m\times n}\mathbf{p}}$$

which is reminiscent of the problem tackled by Prony. For a full rank system, $\mathbf{D}_{\text{DFT}}^{-1}\mathbf{x}$ yields $\mathbf{p}$ and then the parameters are estimated. In what remains, we will recast this as a sampling problem [1] where we use a pre-filter to sample (4) and reconstruct it using (5).

The *finite–rate–of–innovation theory* [2] is a special case of our result (set $\Lambda = \Lambda_\theta$). Similarly, our work has close connections with Phase Retrieval problem of P. Jaming [3]. These ideas have applications in computational imaging [4]. Given that the sparse signal spectrum can be uniquely characterized in (5), the problem of source separation of of bandlimited + impulsive signal (4) can be efficiently tackled. This has applications in computational imaging [5] as well as in mitigating impulsive noise [6]. The chirp–kernel of LCT might explain the success of spread spectrum imaging method of Wiaux et al. in [7]. In context of the recent results in line of *Mathematical Theory of Super–Resolution* of Candes et al. [8], our problem can be reformulated as an optimization problem (noisy case).

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