On three polynomial kernels of sequences for arbitrarily partitionable graphs

Julien Bensmail

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Univ. Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France
CNRS, LaBRI, UMR 5800, F-33400 Talence, France

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Abstract

A graph G is arbitrarily partitionable if every sequence (n_1, n_2, ..., n_p) of positive integers summing up to $|V(G)|$ is realizable in G, i.e. there exists a partition $(V_1, V_2, ..., V_p)$ of $V(G)$ such that V_i induces a connected subgraph of G on n_i vertices for every $i \in \{1, 2, ..., p\}$. Given a family $\mathcal{F}(n)$ of graphs with order $n \geq 1$, a kernel of sequences for $\mathcal{F}(n)$ is a set $K_{\mathcal{F}}(n)$ of sequences summing up to n such that every member G of $\mathcal{F}(n)$ is arbitrarily partitionable if and only if every sequence of $K_{\mathcal{F}}(n)$ is realizable in G. We herein provide kernels with polynomial size for three classes of graphs, namely complete multipartite graphs, graphs with about a half universal vertices, and graphs made up of several arbitrarily partitionable components. Our kernel for complete multipartite graphs yields a polynomial-time algorithm to decide whether such a graph is arbitrarily partitionable.

Keywords: arbitrarily partitionable graph, kernel of sequences

1 Introduction

Let $n \geq 1$, and $\pi = (n_1, n_2, ..., n_p)$ be an *n-sequence*, that is a sequence of positive integers summing up to n. By the *spectrum* of π, denoted $sp(\pi)$, we refer to the set $\{s_1, s_2, ..., s_{p'}\}$ induced by π, where $p' \leq p$. In other words, the sequence π, which is a multiset, is made up of several copies of the element s_i, but at least one, for every $i \in \{1, 2, ..., p'\}$. In the very special case where each element of π is unique in π, i.e. we have $n_i \neq n_j$ for every $i \neq j \in \{1, 2, ..., p\}$, note that π and $sp(\pi)$ are the same. By the *size* of π or $sp(\pi)$, we refer to the number of elements in π or $sp(\pi)$, respectively. We denote $|\pi|$ and $|sp(\pi)|$ these parameters. We also use the notation $\|\pi\|$ to refer to the sum of the elements in π.

Now consider a graph G with order n. We say that π is *realizable* in G if there exists a partition $(V_1, V_2, ..., V_p)$ of $V(G)$ such that V_i induces a connected subgraph on n_i vertices for every $i \in \{1, 2, ..., p\}$. We further call $(V_1, V_2, ..., V_p)$ a *realization* of π in G. This notion can be directly extended to sets of n-sequences by defining such a set to be realizable in G if all of the sequences it contains are realizable in G. We say that G is *arbitrarily partitionable* (AP for short) if the set of every n-sequences is realizable in G. AP graphs were first introduced in [1] as a model to deal with a practical problem of resource sharing among an arbitrary number of users.

Let $\mathcal{F}(n)$ be a family of graphs with order n. A *kernel* (of sequences) for $\mathcal{F}(n)$ is a set of n-sequences $K_{\mathcal{F}}(n)$ such that every member G of $\mathcal{F}(n)$ is AP if and only if $K_{\mathcal{F}}(n)$ is realizable in G. In other words, the realizability of the n-sequences of $K_{\mathcal{F}}(n)$ in members of $\mathcal{F}(n)$ is representative of the realizability of all n-sequences (i.e. even of those which do not belong to $K_{\mathcal{F}}(n)$).

By definition, any graph G can be shown to be AP by showing that a kernel for G is realizable in G. The more sequences such a kernel has, the more sequences we have to try to realize in G to show it to be AP. Hence, we are rather interested in minimizing the size of a kernel regarding a given family of graphs with order n, i.e. with size polynomial in n (we say that such a kernel is *polynomial*). In particular, note that the trivial kernel

$$K_t(n) := \{\pi : \|\pi\| = n\}$$

for $\mathcal{G}(n)$, the set of all graphs with order n, is not interesting for our concern as its number of sequences is exponential in n according to the following theorem.

Theorem 1 ([9]). *The number of partitions of n tends to $\frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$ as n grows to infinite.*

Since previous investigations, it is believed that there should exist a polynomial kernel for $\mathcal{G}(n)$ for every value of $n \geq 1$, and, hence, that the property of being AP of every graph only relies on the realizability of a polynomial number of sequences.

Conjecture 2. *There exists a polynomial kernel for $\mathcal{G}(n)$ for every $n \geq 1$.*

Conjecture 2 was in particular raised independently in [2] and [6]. So far, polynomial kernels have been mainly exhibited regarding two families of graphs. In what follows, a *tripode* designates a tree obtained by subdividing the edges of a claw an arbitrary number of times (equivalently, a tripode is a tree with maximum degree 3 and exactly one node with degree 3), while a *split graph* is a graph whose vertex set admits a partition $I \cup C$ into two parts such that I is an independent set and C induces a clique. We denote by $\mathcal{T}(n)$ and $\mathcal{S}(n)$ the sets of tripodes and split graphs with order n, respectively.

Two rather distinct polynomial kernels for $\mathcal{T}(n)$ were exhibited in [1] and [11], respectively, with size $\mathcal{O}(n^3)$ and $\mathcal{O}(n^{13})$, respectively. These two kernels are

$$K_{\mathcal{T}}(n) := \{\pi : \|\pi\| = n \text{ and } (\pi = (k, k, ..., k, r) \text{ or } \\ \pi = (k, k, ..., k, k+1, k+1, ..., k+1, r))\}$$

and

$$K'_{\mathcal{T}}(n) := \{\pi : \|\pi\| = n \text{ and } |sp(\pi)| \leq 7\},$$

where we have $k \leq n-1$ and $r < k$ in the definition of $K_{\mathcal{T}}(n)$. The second kernel $K'_{\mathcal{T}}(n)$ is much more bigger than $K_{\mathcal{T}}(n)$, but its larger size makes it easier to prove that any n-sequence $\pi \notin K'_{\mathcal{T}}(n)$ is equivalent (in terms of realizability) to some n-sequence $\pi' \in K'_{\mathcal{T}}(n)$ than it is for $K_{\mathcal{T}}(n)$.

Regarding split graphs, it was shown in [6] that $\mathcal{S}(n)$ admits

$$K_{\mathcal{S}}(n) := \{\pi : \|\pi\| = n \text{ and } (\pi = (1, 3, 3, ..., 3) \text{ or } sp(\pi) = \{2, 3\})\},$$

as a polynomial kernel with size $\mathcal{O}(n^3)$.

Each $K(n)$ of the three kernels above results from the fact that any realization of any n-sequence $\pi \in K(n)$ implies realizations of n-sequences $\pi' \notin K(n)$. In the case of tripodes, this property results from the structure of these graphs, which are made up of one "central" vertex directly connected to three components which have "fair" partition properties (paths are the easiest graphs to partition). For split graphs, this property follows from the fact that these graphs are locally dense, implying that any connected subgraph resulting from a realization is likely to be partitionable itself because of its size (the more edges a graph has, the more partition possibilities there are).

Generalizing some of the arguments behind the proofs that $K_{\mathcal{T}}(n)$, $K'_{\mathcal{T}}(n)$ and $K_{\mathcal{S}}(n)$ are kernels for $\mathcal{T}(n)$ and $\mathcal{S}(n)$, respectively, we exhibit new polynomial kernels for three classes of graphs, namely complete multipartite graphs (Section 2), graphs with about a half universal vertices (Section 3), and *compound graphs*, i.e. graphs made up of several components which may be partitioned even when some vertex-membership constraints are prescribed (Section 4). One interest behind our kernel for graphs with universal vertices is that it relies on a new invariant, namely the number of occurrences of the greatest value. Concluding remarks and open questions are gathered in Section 5.

Some terminology and notation

Let G be a graph on n vertices, and $\pi = (n_1, n_2, ..., n_p)$ be an n-sequence which admits a realization $(V_1, V_2, ..., V_p)$ in G. Throughout this paper, the elements of π (and similarly for its realization in G) are uniquely determined via their subscripts. The indices of both the elements of π and the parts of the realization thus bind π and its realization. By writing $\pi - \pi'$, where $\pi' = (n_{i_1}, n_{i_2}, ..., n_{i_{p'}})$ is a sequence of distinct elements from π, we refer to the sequence obtained by removing the elements with indices $i_1, i_2, ..., i_{p'}$ off π. Conversely, by writing $\pi \cup \pi'$, we refer to the sequence obtained by adding new elements in π whose values are those in π'. It is not clear in which positions these new elements are added to π (though it might be implicit if some ordering over π is defined). The new ordering $\pi \cup \pi'$ is thus clarified with each use of this operation.

2 Complete multipartite graphs

The *complete k-partite graph* $G = M_k(p_1, p_2, ..., p_k)$, with $k \geq 2$ and $1 \leq p_1 \leq p_2 \leq ... \leq p_k$, is the graph whose vertex set $V(G)$ admits a partition $V_1 \cup V_2 \cup ... \cup V_k$ such that

- V_i is an independent set of size p_i for every $i \in \{1, 2, ..., k\}$,
- $v_1 v_2$ is an edge for every two vertices $v_1 \in V_i$ and $v_2 \in V_j$ with $i \neq j \in \{1, 2, ..., k\}$.

A lot of complete multipartite graphs are AP since they are *traceable*, i.e. have an Hamiltonian path. However, all complete multipartite graphs are not AP. To be convinced of that statement, note that any graph $M_2(1, k)$ with $k \geq 5$ odd does not admit a perfect matching, and hence any realization of the $(1 + k)$-sequence $(2, 2, ..., 2)$. We show below that checking whether a complete multipartite graph is AP can be done in polynomial time. This is done by first showing that the set

$$K_{\mathcal{M}_k}(n) := \{\pi : \|\pi\| = n \text{ and } sp(\pi) = \{1, 2\}\}$$

is a (obviously polynomial) kernel for $\mathcal{M}_k(n)$, the set of complete k-partite graphs with order n, and then proving that checking whether $K_{\mathcal{M}_k}(n)$ is realizable in a complete multipartite graph can be done in polynomial time.

The proof that $K_{\mathcal{M}_k}(n)$ is a kernel for $\mathcal{M}_k(n)$ relies on the following lemma.

Lemma 3. *Let $G = M_k(p_1, p_2, ..., p_k)$ be a complete k-partite graph with $k \geq 2$ and order $n \geq k$, and $\pi = (n_1, n_2, ..., n_p)$ be an n-sequence. If π is realizable in G, then every n-sequence $\pi' = \pi - (n_i, n_{i_1}, n_{i_2}, ..., n_{i_{p'}}) \cup (n_i + n_{i_1} + n_{i_2} + ... + n_{i_{p'}})$, where $n_i \geq 2$ and $n_{i_1}, n_{i_2}, ..., n_{i_{p'}}$ are arbitrary distinct elements of $\pi - (n_i)$, is realizable in G.*

Proof. Let $(V_1, V_2, ..., V_p)$ be a realization of π in G. Since $n_i \geq 2$, observe that $G[V_i \cup \{u\}]$ is connected for every vertex $u \notin V_i$ of G, and so is every subgraph $G[V_i \cup V_j]$ with $i \neq j$. It then follows directly that

$$(V_1, V_2, ..., V_{i-1}, V_i \cup V_{i_1}, V_{i+1}, V_{i+2}, ..., V_p) - (V_{i_1})$$

is a realization of the n-sequence $(n_1, n_2, ..., n_{i-1}, n_i + n_{i_1}, n_{i+1}, n_{i+2}, ..., n_p) - (n_{i_1})$ in G. Repeating the same argument as many times as necessary, we eventually get a realization of π' in G. □

Theorem 4. *The set $K_{\mathcal{M}_k}(n)$ is a kernel for $\mathcal{M}_k(n)$ for every $k \geq 2$ and $n \geq k$.*

Proof. Let $G \in \mathcal{M}_k(n)$ be a complete k-partite graph. We show that G is AP if and only if $K_{\mathcal{M}_k}(n)$ is realizable in G. As the necessity follows from the definition of an AP graph, let us focus on the sufficiency. Assume $K_{\mathcal{M}_k}(n)$ is realizable in G, and that $\pi = (n_1, n_2, ..., n_p)$ is an n-sequence with $\pi \notin K_{\mathcal{M}_k}(n)$. We deduce an n-sequence $\pi' \in K_{\mathcal{M}_k}(n)$ whose realizability in G implies the realizability of π in G.

Since $\pi \notin K_{\mathcal{M}_k}(n)$, there are elements of π with value at least 3. For each such element n_i, replace n_i in π' with one occurrence of the element 2 and $n_i - 2$ occurrences of the element 1. Directly transfer any other element of π, i.e. with value at most 2, to π'. By construction, we have $\pi' \in K_{\mathcal{M}_k}(n)$. Now consider a realization of π' in G, which exists by assumption, and any element n_i of π which was split into one occurrence of 2 and several occurrences of 1. Then, according to Lemma 3, we can merge exactly one part with size 2 of the realization and $n_i - 2$ parts with size 1 so that their union induces a connected subgraph of G with order n_i. Repeating the same argument for every split element of π, we eventually get a realization of π in G. □

Clearly, if the n-sequence $(2, 2, ..., 2)$ (or $(2, 2, ..., 2, 1)$ if n is odd) is not realizable in a graph $G \in \mathcal{M}_k(n)$, then any other n-sequence with spectrum $\{1, 2\}$ cannot be realizable in G. We may then restrict our concern on $(2, 2, ..., 2)$ (or $(2, 2, ..., 2, 1)$) to check whether $K_{\mathcal{M}_k}(n)$ is realizable in a complete k-partite graph with order n. Since a realization of $(2, 2, ..., 2)$ (or $(2, 2, ..., 2, 1)$) in a graph forms a perfect matching (or quasi perfect-matching), deciding whether $K_{\mathcal{M}_k}(n)$ is realizable in a complete k-partite graph G is equivalent to the problem of deciding whether G has a matching with size $\lfloor \frac{|V(G)|}{2} \rfloor$. Since this problem is handleable in polynomial time using the Blossom algorithm introduced by Edmonds [8], we get the following.

Corollary 5. *We can check whether a complete multipartite graph with order n is AP in time $\mathcal{O}(n^{\mathcal{O}(1)})$.*

3 Graphs with about a half universal vertices

A vertex u of some graph G is *universal* if u neighbours any other vertex of G, i.e. we have $d(u) = |V(G)| - 1$. In this section we exhibit a polynomial kernel for graphs with "about" a half universal vertices. We beforehand motivate the study of such a kernel by showing that a graph with "a lot", i.e. about a third, of universal vertices is not necessarily AP. This is done by showing that the decision problem

Realizable Sequence
Input: a graph G and a $|V(G)|$-sequence π.
Question: is π realizable in G?

remains NP-complete when G has about $\frac{|V(G)|}{3}$ universal vertices. We refer the reader to [5] where some properties of the NP-hardness of the Realizable Sequence problem are investigated.

Theorem 6. Realizable Sequence *is NP-complete, even when restricted to graphs with about a third universal vertices.*

Proof. Realizable Sequence is clearly in NP, so let us now show that Realizable Sequence is NP-hard when the condition of the claim is met. It was proved in [7] that Realizable Sequence remains NP-hard when restricted to instances with $\pi = (3, 3, ..., 3)$ (but with no conditions on the structure of G). We use this restriction of Realizable Sequence for the reduction. Namely, from a graph G on $3n$ vertices, we construct a graph G' with order $3n'$ ($n' > n$) and about a third universal vertices, and such that

$$\text{the } 3n\text{-sequence } (3, 3, ..., 3) \text{ is realizable in } G$$
$$\Leftrightarrow$$
$$\text{the } 3n'\text{-sequence } (3, 3, ..., 3) \text{ is realizable in } G'.$$

To obtain the graph G', proceed as follows. Let $k \geq 1$ be some integer, and add $3k$ new vertices to G. Arbitrarily partition these $3k$ newly added vertices into two parts $I \cup U$ in such a way that $|I| = 2k$ and $|U| = k$. Finally turn the vertices in U into universal vertices. The graph G' is thus made of three components $G'[I]$, $G'[U]$ and $G'[V(G)]$, which are connected in such a way that all vertices from the "central" component $G'[U]$ are neighbouring every vertex of G', the vertices from I neighbour vertices from U only, and $G'[V(G)]$ is nothing but G (with all possible edges between vertices of U and $V(G)$). Clearly $|V(G')| = 3n' = 3(n+k)$.

Notice that in any realization of the $3n'$-sequence $(3, 3, ..., 3)$ in G', any vertex from I has to belong to a same part as a vertex from U because of the structure of G'. Besides, any part containing a vertex from U can only "cover" up to two vertices in I. For these reasons, and since there are $2k$ vertices in I and k vertices in U by construction, note that, in any realization of the $3n'$-sequence $(3, 3, ..., 3)$ in G', exactly k parts necessarily consist in exactly one vertex from U and two vertices from I. Once these k parts are removed off G', what remains is G. Hence, the existence of a realization of the $3n'$-sequence $(3, 3, ..., 3)$ in G' only depends on the existence of a realization of the $3n$-sequence $(3, 3, ..., 3)$ in $G' - (I \cup U) = G$. The equivalence between the two instances of Realizable Sequence thus follows naturally.

It should be clear that the reduction above holds whatever is the value of k. In particular, note that the order of G gets irrelevant in front of the order of G', and thus that the number of universal vertices of G' tends to one third, as k grows to infinity. $\square$

We now exhibit a polynomial kernel for graphs with order n and a large number of universal vertices, namely with at least $\lceil \frac{n-\ln(n)-2}{2} \rceil$ universal vertices. It is worth mentioning that any graph with at least $\lceil \frac{n-5}{2} \rceil$ universal vertices is AP by a result from [10], where an Ore-type condition for AP graphs is exhibited. For this reason, and because $\ln(n)$ grows very slowly in front of n, our kernel is only relevant for asymptotic values of n.

We start by raising the following easy remark.

Observation 7. *Let G be a graph with $k \geq 1$ universal vertices and order $n \geq k$. Then every n-sequence $\pi = (n_1, n_2, ..., n_p)$ with size $p \leq k$ is realizable in G.*

Proof. Let $u_1, u_2, ..., u_k$ be the universal vertices of G. Under the conditions of the claim, we deduce a realization $(V_1, V_2, ..., V_p)$ of π in G as follows. Start with $V_1 = \{u_1\}$, $V_2 = \{u_2\}$, ..., $V_p = \{u_p\}$. Now consider the parts $V_1, V_2, ..., V_p$ consecutively. If V_i already has size n_i, i.e. $n_i = 1$, then consider the next part. Otherwise, add $n_i - 1$ arbitrary vertices from $V(G) - \bigcup_{j=1}^{p} V_j$ to V_i. Clearly V_i has size n_i. Besides, because $u_i \in V_i$ and u_i is a universal vertex of G, the subgraph induced by V_i is connected. $\square$

From now on, it is thus understood that all sequences have size at least $k+1$ when dealing with a graph with k universal vertices. We now introduce a result dealing with the existence of a particular realization of every sequence which is realizable in a graph having universal vertices.

Lemma 8. *Let G be a graph with $k \geq 1$ universal vertices and order $n \geq k$, and $\pi = (n_1, n_2, ..., n_p)$ be an n-sequence with $n_1 \geq n_2 \geq ... \geq n_p$. If π is realizable in G, then there exists a realization $(V_1, V_2, ..., V_p)$ of π in G such that $V_1, V_2, ..., V_k$ each contains one universal vertex.*

Proof. The claim means that if π is realizable in G, then there exists a particular realization of π in G such that the k biggest parts each includes one universal vertex. Let $u_1, u_2, ..., u_k$ denote the universal vertices of G, and assume $(V_1, V_2, ..., V_p)$ is a realization of π in G. If $(V_1, V_2, ..., V_p)$ satisfies the conditions of the claim, then we are done. Otherwise, we obtain a satisfying realization in two steps.

We first modify the realization $(V_1, V_2, ..., V_p)$ so that each of $V_1, V_2, ..., V_k$ includes at most one universal vertex of G. Suppose there is some part V_i with $i \in \{1, 2, ..., k\}$ containing at least two universal vertices, while some other part V_j with $j \in \{1, 2, ..., k\}$ and $j \neq i$ does not contain any universal vertex. We prove that we may exchange vertices between V_i and V_j in such a way that exactly one universal vertex is moved from V_i to V_j, and this without altering the sizes of V_i and V_j, nor the connectivity of the subgraphs of G they induce. Let $v_1 \in V_i$ and $v_2 \in V_j$ be arbitrary vertices of G such that v_1 is a universal vertex. Then note that $G[V_i - \{v_1\}]$ remains connected since V_i includes at least two universal vertices. Besides, the subgraph $G[V_j - \{v_2\} \cup \{v_1\}]$ is connected since v_1 is a universal vertex. It then follows that

$$\begin{gathered}(V_1\ ,\ V_2\ ,\ ...\ ,\ V_{i-1}\ ,\ V_i - \{v_1\} \cup \{v_2\}\ ,\ V_{i+1}\ ,\ V_{i+2}\ ,\ ...\ ,\ V_{j-1}\ ,\\ V_j - \{v_2\} \cup \{v_1\}\ ,\ V_{j+1}\ ,\ V_{j+2}\ ,\ ...\ ,\ V_p)\end{gathered}$$

is a satisfying realization of π in G. Repeating the same argument as many times as necessary, we eventually get a realization of π in G such that the k biggest parts each contains at most one universal vertex.

Suppose now that every part $V_1, V_2, ..., V_k$ of $(V_1, V_2, ..., V_p)$ includes at most one universal vertex. If each of $V_1, V_2, ..., V_k$ contains exactly one universal vertex, then the claim is proved. Otherwise, it means that some part different from $V_1, V_2, ..., V_k$ includes at least one universal vertex. Let V_j with $j \in \{k+1, k+2, ..., p\}$ be such a part whose set $U \subseteq \{u_1, u_2, ..., u_k\} \cap V_j$ of universal vertices is not empty, and let $u \in U$ be a universal vertex. By assumption there is another part V_i with $i \in \{1, 2, ..., k\}$ such that V_i does not include any universal vertex. We exchange vertices between V_i and V_j so that they still have size n_i and n_j, respectively, induce connected subgraphs of G, and u is the only universal vertex of U moved to the part with size n_i.

Recall that $n_i \geq n_j$. Then we can find a set $X \subseteq V_i$ such that $|X| = n_j - |U| + 1$ and $G[X]$ is connected, e.g. by applying a breadth-first search algorithm. Because u is a universal vertex, the subgraph of G induced by $V_i - X \cup (V_j - U) \cup \{u\}$ is connected. It then follows that

$$(V_1\ ,\ V_2\ ,\ \dots\ ,\ V_{i-1}\ ,\ V_i - X \cup (V_j - U) \cup \{u\}\ ,\ V_{i+1}\ ,\ V_{i+2}\ ,\ \dots\ ,\ V_{j-1}\ ,\ U - \{u\} \cup X\ ,\ V_{j+1}\ ,\ V_{j+2}\ ,\ \dots\ ,\ V_p)$$

is a realization of π in G such that the part with size n_i now includes a universal vertex, while the part with size n_j has one less universal vertex. Repeating the same arguments as many times as necessary, we eventually get a realization of π in G satisfying the conditions of the claim. □

The kernel presented below relies on the following crucial lemma.

Lemma 9. *Let G be a graph with $k \geq 1$ universal vertices and order $n \geq k$, and $\pi = (n_1, n_2, ..., n_p)$ be an n-sequence with $n_1 \geq n_2 \geq ... \geq n_p$. If π is realizable in G, then every n-sequence $\pi' = \pi - (n_i, n_{i_1}, n_{i_2}, ..., n_{i_{p'}}) \cup (n_i + n_{i_1} + n_{i_2} + ... + n_{i_{p'}})$, where $n_i \in \{1, 2, ..., k\}$ and $n_{i_1}, n_{i_2}, ..., n_{i_{p'}}$ are arbitrary distinct elements of $\pi - (n_i)$, is realizable in G.*

Proof. The claim means that from a realization of π in G, we can deduce a realization in G of any n-sequence obtained from π by combining one "big" part size with additional part sizes. Since π is realizable in G by assumption, there exists a particular realization $(V_1, V_2, ..., V_k)$ of π in G such that each of V_1, V_2, ..., V_k includes one universal vertex, see Lemma 8. By definition, there is thus one vertex in V_i neighbouring every other vertex of G, implying that the subgraph induced by $V_i \cup V_{i_1} \cup V_{i_2} \cup ... \cup V_{i_{p'}}$ is connected. It thus follows directly that

$$(V_1, V_2, ..., V_{i-1}, V_i \cup V_{i_1} \cup V_{i_2} \cup ... \cup V_{i_{p'}}, V_{i+1}, V_{i+2}, ..., V_p) - (V_{i_1}, V_{i_2}, ..., V_{i_{p'}})$$

is a realization of $\pi' = (n_1, n_2, ..., n_{i-1}, n_i + n_{i_1} + n_{i_2} + ... + n_{i_{p'}}, n_{i+1}, n_{i+2}, ..., n_p) - (n_{i_1}, n_{i_2}, ..., n_{i_{p'}})$ in G. □

Let $\mathcal{U}_k(n)$ denote the set of graphs with exactly $k \geq 1$ universal vertices and order $n \geq k$. In the upcoming results, we deal with the set

$$K_{\mathcal{U}_k}(n) := \{\pi : \|\pi\| = n \text{ and the greatest element of } \pi \text{ appears at least } k+1 \text{ times}\}.$$

We prove below that $K_{\mathcal{U}_k}(n)$ is a kernel for $\mathcal{U}_k(n)$ no matter what are the values of k and n.

Theorem 10. *The set $K_{\mathcal{U}_k}(n)$ is a kernel for $\mathcal{U}_k(n)$ for every $k \geq 1$ and $n \geq k$.*

Proof. Let G be any graph with order n and k universal vertices with $k \geq 1$ and $n \geq k$ being fixed. We prove that G is AP if and only if $K_{\mathcal{U}_k}(n)$ is realizable in G. The necessary condition is obvious by the definition of an AP graph, so let us prove the sufficient condition.

Let us assume $K_{\mathcal{U}_k}(n)$ is realizable in G. We show that for every sequence $\pi \notin K_{\mathcal{U}_k}(n)$, we can find a sequence $\pi' \in K_{\mathcal{U}_k}(n)$ such that from a realization of π' in G we can deduce a realization of π in G. Assume $\pi = (n_1, n_2, ..., n_p)$ with $n_1 \geq n_2 \geq ... \geq n_p$. Since $\pi \notin K_{\mathcal{U}_k}(n)$, there are at most k occurrences of n_1 in π. The sequence π' is obtained from π by just replacing every element n_i with $i \in \{1, 2, ..., k\}$ of π by one occurrence of the element n_{k+1} and $n_i - n_{k+1}$ occurrences of the element 1.

By construction, the biggest element of π' is n_{k+1}, this element value appearing at least $k+1$ times in π'. The sequence π' thus belongs to $K_{\mathcal{U}_k}(n)$ and is realizable in G by assumption. Then Lemma 9 implies that from some realization of π' in G we can deduce a realization of π in G. More precisely, from a realization of π' in G obtained thanks to Lemma 8, we can unify some connected parts resulting from the split of a single element of π in such a way that the resulting subgraph is also connected. $\square$

Any kernel $K_{\mathcal{U}_k}(n)$ contains n-sequences composed of one "big" element n_1 appearing $\alpha \geq k+1$ times, and a partition of $n - \alpha n_1$ whose elements have value at most k. So that $K_{\mathcal{U}_k}(n)$ has polynomial size, we need the number of partitions of $n - \alpha n_1$ to be polynomial in n. This is especially true when $n - \alpha n_1$ is logarithmic in n, see Theorem 1.

Corollary 11. *The kernel $K_{\mathcal{U}_k}(n)$ is polynomial for every $k \geq 1$ and $n \geq k$ whenever $k \geq \lceil \frac{n-\ln(n)-2}{2} \rceil$.*

Proof. Regarding the terminology above, we have $\alpha \geq k+1$ and $n_1 \geq 2$. Since $n - \alpha n_1$ must be logarithmic in n, we want $n - \alpha n_1 \leq \ln(n)$, and hence $n - 2(k+1) \leq \ln(n)$. Solving the inequality, we end up with $k \geq \lceil \frac{n-\ln(n)-2}{2} \rceil$. For such a value of k, the size of $K_{\mathcal{U}_k}(n)$ is asymptotically $\mathcal{O}(n^3)$ since n_1 and α can take up to n values. $\square$

4 Graphs made up of partitionable components

Let $k \geq 1$ be fixed, and $u_1, u_2, ..., u_k$ be k distinct vertices of some graph G with order $n \geq 1$. An n-sequence $\pi = (n_1, n_2, ..., n_p)$ with size $p \geq k$ is $(u_1, u_2, ..., u_k)$-*realizable* in G if there exists a $(u_1, u_2, ..., u_k)$-*realization* of π in G, that is a realization $(V_1, V_2, ..., V_p)$ such that $u_i \in V_i$ for every $i \in \{1, 2, ..., k\}$. In other words, not only we want to partition G into connected subgraphs whose orders agree with π, but we also want k specific vertices to each belong to one of the k resulting subgraphs. More precisely, the induced subgraphs associated with the k target vertices are those whose orders are the first k of π. We say that G is $(u_1, u_2, ..., u_k)$-*AP* if every n-sequence with size at least k is $(u_1, u_2, ..., u_k)$-realizable in G. In case G is $(u_1, u_2, ..., u_k)$-AP for every k-tuple of vertices, we say that G is *AP+k*.

AP+k graphs were first introduced in [3]. We raise some remarks to illustrate the definitions above. First, an AP+0 graph would actually correspond to an AP graph. Besides, a T-AP graph for some tuple T is also T'-AP for every tuple $T' \subset T$. This implies that any AP+k graph is also AP+k' for every $k' < k$. Every path P_n is AP and even (u, v)-AP, where u and v refer to the endvertices of P_n. However P_n is not AP+2 unless $n \leq 2$. Since adding edges to a graph can only make it more connected than it was, the previous remarks imply that every traceable graph is (u, v)-AP, where u and v are the endvertices of one of its Hamiltonian paths. By contrast, every *Hamiltonian-connected* graph, i.e. which has a Hamiltonian path joining every pair of its vertices, is AP+2.

Throughout this section, we deal with (k, ℓ)*-compound graphs* defined as follows. Let $G_1, G_2, ..., G_\ell$ be ℓ graphs such that

- for every $i \in \{1, 2, ..., \ell\}$, the graph G_i is $(u_1^i, u_2^i, ..., u_k^i)$-AP for some vertices $u_1^i, u_2^i, ..., u_k^i$,
- the mapping $f_{j,j'} : V(G_j) \to V(G_{j'})$ defined as $f_{j,j'}(u_i^j) = u_i^{j'}$ is a graph isomorphism for every $j \neq j' \in \{1, 2, ..., \ell\}$.

By $C_{k,\ell}(G_1, G_2, ..., G_\ell)$ we refer to the (k, ℓ)-compound graph obtained by identifying the vertices u_i^1, u_i^2, ..., u_i^ℓ for every $i \in \{1, 2, ..., k\}$. In other words, the graph $C_{k,\ell}(G_1, G_2, ..., G_\ell)$ is made up of ℓ components "glued" together along k of their vertices. The k vertices $u_1, u_2, ..., u_k$ resulting from the identification are called the *roots* of $C_{k,\ell}(G_1, G_2, ..., G_\ell)$. Considering the components $G_1, G_2, ..., G_j$ independently, the *projection* of any root vertex u_i to the j^{th} component G_j is denoted u_i^j.

Although a (k, ℓ)-compound graph has strong local partition properties, i.e. it is made of ℓ vertex-disjoint (more than) AP components, it does not have to be AP. To be convinced of that statement, just note that any $(1, \ell)$-compound graph $C_{1,\ell}(P_1, P_2, ..., P_\ell)$, obtained by identifying one endvertex of ℓ paths $P_1, P_2, ..., P_\ell$ (which are "more" than just AP, as pointed out above), cannot be AP whenever $\ell \geq 5$, see [2].

We exhibit below a polynomial kernel of sequences for (k, ℓ)-compound graphs with $\ell \leq k$. As in previous Section 3, we first prove some lemmas beforehand. These lemmas are more general than needed for our purpose, i.e. they hold whatever is the value of ℓ, so that they may be reused in future works.

Observation 12. *Let $G = C_{k,\ell}(G_1, G_2, ..., G_\ell)$ be a (k, ℓ)-compound graph with order $n \geq k$. Then every n-sequence $\pi = (n_1, n_2, ..., n_p)$ with size $p \leq k$ is realizable in G.*

Proof. Let $u_1, u_2, ..., u_k$ denote the root vertices of G. For every $i \in \{1, 2, ..., p\}$, let $(n_i^1, n_i^2, ..., n_i^\ell)$ be an arbitrary partition of $n_i+\ell-1$ such that $n_i^1, n_i^2, ..., n_i^\ell \geq 1$ and $n_1^j + n_2^j + ... + n_p^j = |V(G_j)|$ for every $j \in \{1, 2, ..., \ell\}$. In particular, we get

$$\sum_{i=1}^{p} \sum_{j=1}^{\ell} n_i^j = n + p(\ell - 1)$$

by construction. Now define

$$\pi_i := (n_1^i, n_2^i, ..., n_p^i).$$

for every $i \in \{1, 2, ..., \ell\}$. Since $p \leq k$, each component G_i is $(u_1^i, u_2^i, ..., u_p^i)$-AP, and the sequence π_i is a $|V(G_i)|$-sequence, there exists a $(u_1^i, u_2^i, ..., u_p^i)$-realization $(V_1^i, V_2^i, ..., V_p^i)$ of π_i in G_i. Now observe that

$$(\bigcup_{i=1}^{\ell} V_1^i, \bigcup_{i=1}^{\ell} V_2^i, ..., \bigcup_{i=1}^{\ell} V_p^i)$$

is a realization of π in G. In particular, note that every resulting part $V_i^1 \cup V_i^2 \cup ... \cup V_i^\ell$ with $i \in \{1, 2, ..., p\}$ has size n_i and that $G[V_i^1 \cup V_i^2 \cup ... \cup V_i^\ell]$ is connected since each of $G_1[V_i^1], G_2[V_i^2], ..., G_\ell[V_i^\ell]$ induces a connected subgraph and contains a "local copy" of the root vertex u_i. □

From now on, it is thus understood that every sequence has sufficiently many elements, i.e. at least $k + 1$ elements when dealing with a (k, ℓ)-compound graph. We now focus on the existence of a particular realization of every sequence which is realizable in a compound graph.

Lemma 13. *Let $G = C_{k,\ell}(G_1, G_2, ..., G_\ell)$ be a (k, ℓ)-compound graph with order $n \geq k$, and $\pi = (n_1, n_2, ..., n_p)$ be an n-sequence with $n_1 \geq n_2 \geq ... \geq n_p$. If π is realizable in G, then there exists a realization $(V_1, V_2, ..., V_p)$ of π in G such that $V_1, V_2, ..., V_k$ each contains one root vertex.*

Proof. Let $R = \{u_1, u_2, ..., u_k\}$ denote the set of root vertices of G, and assume $(V_1, V_2, ..., V_p)$ is a realization of π in G. As in the proof of Lemma 8, we successively modify the realization $(V_1, V_2, ..., V_p)$ so that it eventually respects the conditions of the claim. If these conditions are already respected, then we are done. Otherwise, for each element n_i of π, let $n_i^1, n_i^2, ..., n_i^\ell$ be possibly null elements such that

$$n_i^j = |V_i \cap V(G_j)| \text{ for every } j \in \{1, 2, ..., \ell\}.$$

If we denote by $r_i := |V_i \cap R|$, then we have $n_i = (\sum_{j=1}^{\ell} n_i^j) - r_i(\ell - 1)$. Besides, if $V_i \subset (V(G_j) - \{u_1^j, u_2^j, ..., u_k^j\})$, i.e. the part V_i only contains non-root vertices, then we have $n_i^j = n_i$ and $n_i^{j'} = 0$ for every $j' \neq j$. The original realization $(V_1, V_2, ..., V_p)$ of π in G can be then rewritten

$$(\bigcup_{i=1}^{\ell} V_1^i, \bigcup_{i=1}^{\ell} V_2^i, ..., \bigcup_{i=1}^{\ell} V_p^i),$$

where $(V_1^i, V_2^i, ..., V_p^i)$ is a realization of $(n_1^i, n_2^i, ..., n_p^i)$ in G_i for every $i \in \{1, 2, ..., \ell\}$. In particular, each of the p subgraphs induced by the realization is connected thanks to some root vertices.

We start by modifying the realization so that $r_i \leq 1$ for every $i \in \{1, 2, ..., p\}$. If this condition is already met, then consider the next step. Otherwise, let $V_{i_1}, V_{i_2}, ..., V_{i_\alpha}$ be the parts of the realization which each includes at least two root vertices, i.e. we have $r_{i_1}, r_{i_2}, ..., r_{i_\alpha} \geq 2$. For each such part V_i, let $V_{i,1}, V_{i,2}, ..., V_{i,r_i-1}$ be $r_i - 1$ distinct parts of the realization including no root vertex, i.e. $r_{i,1}, r_{i,2}, ..., r_{i,r_i-1} = 0$. These additional parts have to be chosen uniquely, i.e. the indices $(i_1, 1)$, $(i_1, 2)$, ... , $(i_1, r_{i_1} - 1)$, $(i_2, 1)$, $(i_2, 2)$, ... , $(i_2, r_{i_2} - 1)$, ... , $(i_\alpha, 1)$, $(i_\alpha, 2)$, ... , $(i_\alpha, r_{i_\alpha} - 1)$ must all be distinct. We modify the realization so that, for every $i \in \{i_1, i_2, ..., i_\alpha\}$, the parts $V_{i,1}, V_{i,2}, ..., V_{i,r_i-1}$ and V_i each contains exactly one of the r_i root vertices which originally belonged to V_i. For this purpose, we also need to refer to the parts of the original realization which includes exactly one root vertex. These are denoted $V_{j_1}, V_{j_2}, ..., V_{j_\beta}$. By definition note that

$$(V_{i_1} \cup V_{i_2} \cup ... \cup V_{i_\alpha} \cup V_{j_1} \cup V_{j_2} \cup ... \cup V_{j_\beta}) \cap R = R,$$

and also that

$$\alpha + \beta + (r_{i_1} - 1) + (r_{i_2} - 1) + ... + (r_{i_\alpha} - 1) = k.$$

We also denote by $\ell_1, \ell_2, ..., \ell_\gamma$ those $p-k$ indices not among $\{i_1, i_2, ..., i_\alpha\} \cup \{(i_1, 1)\ , (i_1, 2)\ , ...\ , (i_1, r_{i_1} - 1)\ , (i_2, 1)\ , (i_2, 2)\ , ...\ , (i_2, r_{i_2} - 1)\ , ...\ , (i_\alpha, 1)\ , (i_\alpha, 2)\ , ...\ , (i_\alpha, r_{i_\alpha} - 1)\} \cup \{j_1, j_2, ..., j_\beta\}$.

For every $i \in \{1, 2, ..., p\}$, we split $n_i \in \pi$ into ℓ elements $n_i^{1\prime}, n_i^{2\prime}, ..., n_i^{\ell\prime}$ as follows.

- If $i \in \{\ell_1, \ell_2, ..., \ell_\gamma\}$, then $n_i^{1\prime} = n_i^1, n_i^{2\prime} = n_i^2, ..., n_i^{\ell\prime} = n_i^\ell$. In particular, one of the ℓ resulting elements is equal to n_i, while all of the other elements are null.
- If $i \in \{i_1, i_2, ..., i_\alpha\}$, then $n_i^{1\prime} = n_i^1 - r_i + 1, n_i^{2\prime} = n_i^2 - r_i + 1, ..., n_i^{\ell\prime} = n_i^\ell - r_i + 1$.
- If $i \in \{(i_1, 1)\ , (i_1, 2)\ , ...\ , (i_1, r_{i_1} - 1)\ , (i_2, 1)\ , (i_2, 2)\ , ...\ , (i_2, r_{i_2} - 1)\ , ...\ , (i_\alpha, 1)\ , (i_\alpha, 2)\ , ...\ , (i_\alpha, r_{i_\alpha} - 1)\}$, then there is some c such that $V_i \subset V(G_c)$. Then let $n_i^{j\prime} = n_i$ for $j = c$, or $n_i^{j\prime} = 1$ otherwise.
- If $i \in \{j_1, j_2, ..., j_\beta\}$, then $n_i^{1\prime} = n_i^1, n_i^{2\prime} = n_i^2, ..., n_i^{\ell\prime} = n_i^\ell$.

Using the resulting elements, we build ℓ new sequences $\pi_1', \pi_2', ..., \pi_\ell'$ where each such sequence π_i' is intended to be realized independently in G_i. Formally, let us define

$$\pi_i' := ({n^i_{i_1}}', {n^i_{i_2}}', \dots, {n^i_{i_\alpha}}', {n^i_{i_1,1}}', {n^i_{i_1,2}}', \dots, {n^i_{i_1,r_{i_1}-1}}', {n^i_{i_2,1}}', {n^i_{i_2,2}}', \dots, {n^i_{i_2,r_{i_2}-1}}', \dots, {n^i_{i_\alpha,1}}', {n^i_{i_\alpha,2}}', \dots, {n^i_{i_\alpha,r_{i_\alpha}-1}}', {n^i_{j_1}}', {n^i_{j_2}}', \dots, {n^i_{j_\beta}}', {n^i_{\ell_1}}', {n^i_{\ell_2}}', \dots, {n^i_{\ell_\gamma}}').$$

Note that according to how the original elements of π have been split, each resulting sequence π_i' sums up to $V(G_i)$. Since G_i is $(u_1^i, u_2^i, ..., u_k^i)$-AP by assumption, there exists a $(u_1^i, u_2^i, ..., u_k^i)$-realization $(V_1^{i\prime}, V_2^{i\prime}, ..., V_p^{i\prime})$ of π_i' in G_i. It then follows that

$$(\bigcup_{i=1}^{\ell} V_1^{i\prime}, \bigcup_{i=1}^{\ell} V_2^{i\prime}, ..., \bigcup_{i=1}^{\ell} V_p^{i\prime})$$

is a realization of π in G. In particular, each of its k first parts induces a connected subgraph since it is made up of vertex-disjoint parts which induce connected subgraphs themselves, and each includes a local copy of a same root vertex. Besides, each resulting part has the correct size regarding π, and each root vertex belongs to exactly one resulting subgraph. The desired assumptions are then met.

Let us now assume that $r_i \leq 1$ for every $i \in \{1, 2, ..., p\}$, but some root vertex u belongs to V_x with $x \notin \{1, 2, ..., k\}$. Then there exists a $y \in \{1, 2, ..., k\}$ such that $r_y = 0$ and, hence, we have $V_y \subset V(G_c) - \{u_1^c, u_2^c, ..., u_k^c\}$ for some $c \in \{1, 2, ..., \ell\}$. Because every two roots of G belong to distinct parts of the realization, note that, because $n_y \geq n_x$, it is sufficient to modify the realization locally, i.e. restricted to G_c, by swapping V_x and V_y. This is done as follows. For every $i \in \{1, 2, ..., p\}$, let $U_i := V_i \cap V(G_c)$, and let $i_1, i_2, ..., i_k$ be the distinct indices of those parts U_i for which $r_i = 1$, and $u_{i_1}^c, u_{i_2}^c, ..., u_{i_k}^c$ the root vertices they respectively include. We additionally denote $U_{\ell_1}, U_{\ell_2}, ..., U_{\ell_\alpha}$ those parts different from U_x, and $U_{i_1}, U_{i_2}, ..., U_{i_k}$.

We may assume that $y = i_k$. Since G_c is $(u_1^c, u_2^c, ..., u_k^c)$-AP, there exists a $(u_{i_1}^c, u_{i_2}^c, ..., u_{i_{k-1}}^c, u^c)$-realization $(U_{i_1}', U_{i_2}', \dots, U_{i_{k-1}}', V_y', V_x', U_{\ell_1}', U_{\ell_2}', \dots, U_{\ell_\alpha}')$ of $(|U_{i_1}|, |U_{i_2}|, ..., |U_{i_{k-1}}|, |U_y|, |U_x|, |U_{\ell_1}|, |U_{\ell_2}|, ..., |U_{\ell_\alpha}|)$ in G_c. It then follows that

$$(V_{i_1} - U_{i_1} \cup U_{i_1}', V_{i_2} - U_{i_2} \cup U_{i_2}', \dots, V_{i_{k-1}} - U_{i_{k-1}} \cup U_{i_{k-1}}, V_x - U_x \cup V_y', V_x', U_{\ell_1}', U_{\ell_2}', \dots, U_{\ell_\alpha}')$$

is a realization of π in G in which u has been switched from the part with size n_x to the part with size n_y, and all of the other root vertices remain in the same parts. Repeating the same argument as many times as needed, we eventually get a realization of π in G satisfying the conditions of the claim. $\square$

We are now ready to exhibit our polynomial kernel for (k,ℓ)-compound graphs. We prove below that the set

$$K_{\mathcal{C}_{k,\ell}}(n) := \{\pi : \|\pi\| = n \text{ and } |sp(\pi)| \leq 2k+6\}$$

is a polynomial kernel for $\mathcal{C}_{k,\ell}(n)$ whenever $\ell \leq k$. Our proof binds several arguments used in [11] to show that $K'_{\mathcal{T}}(n)$ is a polynomial kernel for tripodes and the notion of T-AP graphs introduced above.

Theorem 14. *The set $K_{\mathcal{C}_{k,\ell}}(n)$ is a kernel for $\mathcal{C}_{k,\ell}(n)$ for every $k \geq 1$, $\ell \leq k$, and $n \geq k$.*

Proof. Let $G = C_{k,\ell}(G_1, G_2, ..., G_\ell)$ be a (k,ℓ)-compound graph with order $n \geq k$ whose parameters k and ℓ respect the conditions of the statement. We prove that G is AP if and only if $K_{\mathcal{C}_{k,\ell}}(n)$ is realizable in G.

Since every n-sequence is realizable in G under the assumption that G is AP, the necessary condition follows directly from the definitions. We thus narrow down our concern on the sufficient condition. Assume G is not AP. We need to prove that $K_{\mathcal{C}_{k,\ell}}(n)$ is not realizable in G. Since G is not AP, there is some n-sequence π not realizable in G. If $\pi \in K_{\mathcal{C}_{k,\ell}}(n)$, then we are done. Otherwise, i.e. $\pi \notin K_{\mathcal{C}_{k,\ell}}(n)$, we deduce another n-sequence $\pi' \in K_{\mathcal{C}_{k,\ell}}(n)$ which is not realizable in G, completing the proof. Equivalently, we may show that if π' is realizable in G, then so is π.

Since $\pi \notin K_{\mathcal{C}_{k,\ell}}(n)$, we have $|sp(\pi)| = \{s_1, s_2, ..., s_t\}$ with $t \geq 2k+7$ and $s_1 > s_2 > ... > s_t$. Among the $k+7$ elements in $\{s_{k+1}, s_{k+2}, ..., s_t\}$, there has to be at least four integers $s_{p_1} > s_{p_2} > s_{p_3} > s_{p_4}$ with the same parity and such that $s_{p_1} \geq k$. The sequence π' is obtained by replacing two elements with value s_{p_1} and s_{p_2}, respectively, of π by two elements with value $s_{p_m} := \frac{s_{p_1}+s_{p_2}}{2}$, which is an integer since s_{p_1} and s_{p_2} have the same parity. In doing so, note that we may have $|sp(\pi')| \geq |sp(\pi)|$, but repeating this procedure as many times as necessary, we get successive n-sequences which are all equivalent (in terms of realizability in G), converging to a sequence of $K_{\mathcal{C}_{k,\ell}}(n)$ since all these successive sequences are obtained by repeatedly dividing original elements of π, making them converge to 1.

Set $\Delta = s_{p_1} - s_{p_m} = s_{p_m} - s_{p_2}$. Let further $\pi' = (n_1, n_2, ..., n_p)$, with $n_1 \geq n_2 \geq ... \geq n_p$, be the n-sequence obtained from π by replacing two elements with value s_{p_1} and s_{p_2}, respectively, with two new occurrences n_{m_1} and n_{m_2} of s_{p_m}. By the definition of $K_{\mathcal{C}_{k,\ell}}(n)$, we have $n_1, n_2, ..., n_k > n_{m_1}, n_{m_2}$.

Assume π' is realizable in G. According to Lemma 13, there exists a $(u_1, u_2, ..., u_k)$-realization $(V_1, V_2, ..., V_p)$ of π' in G such that the root vertices $u_1, u_2, ..., u_k$ of G each distinctly belongs to one of $V_1, V_2, ..., V_k$. We may assume that $u_1 \in V_1$, $u_2 \in V_2$, ..., $u_k \in V_k$ for the sake of simplicity. As in the proof of Lemma 13, let us rewrite $(V_1, V_2, ..., V_p)$ as

$$(\bigcup_{i=1}^{\ell} V_1^i, \bigcup_{i=1}^{\ell} V_2^i, ..., \bigcup_{i=1}^{\ell} V_p^i),$$

where each subset V_i^j corresponds to $V_i \cap V(G_j)$. For each such subset, we further write $n_i^j := |V_i^j|$. By assumption, we have $V_{m_1} \subset (G_c - \{u_1^c, u_2^c, ..., u_k^c\})$ and $V_{m_2} \subset (G_{c'} - \{u_1^{c'}, u_2^{c'}, ..., u_k^{c'}\})$, with $c, c' \in \{1, 2, ..., \ell\}$ possibly equal. We modify the realization in such a way that Δ vertices are moved from V_{m_1} to V_{m_2} in either direction, and this without altering the connectivity of the subgraphs these parts induce.

Case 1 - *We have $c = c'$.*

In this situation the parts V_{m_1} and V_{m_2} are included in a same component G_c of G, say G_1. We obtain the realization of π in G as follows. Since G_1 is $(u_1^1, u_2^1, ..., u_k^1)$-AP, there exists a $(u_1^1, u_2^1, ..., u_k^1)$-realization $(V_1^{1\prime}, V_2^{1\prime}, ..., V_p^{1\prime})$ of

$$(n_1^1 \ , n_2^1 \ , \ ... \ , n_{m_1-1}^1 \ , n_{m_1}^1 + \Delta \ , n_{m_1+1}^1 \ , n_{m_1+2}^1 \ , \ ... \ , n_{m_2-1}^1 \ , n_{m_2}^1 - \Delta \ , \\ n_{m_2+1}^1 \ , n_{m_2+2}^1 \ , \ ... \ , n_p^1)$$

in G_1. It then follows that

$$((\bigcup_{i=1}^{\ell} V_1^i) - V_1^1 \cup V_1^{1\prime}, (\bigcup_{i=1}^{\ell} V_2^i) - V_2^1 \cup V_2^{1\prime}, ..., (\bigcup_{i=1}^{\ell} V_p^i) - V_p^1 \cup V_p^{1\prime})$$

is a realization of π in G. In particular, each part $(\bigcup_{j=1}^{\ell} V_i^j) - V_i^1 \cup V_i^{1\prime}$ with $i \in \{1, 2, ..., k\}$ induces a connected subgraph of G since it is made up of several connected parts each containing one local copy of a same root vertex.

Case 2 - *We have $c \neq c'$.*

From now on, we suppose that the parts V_{m_1} and V_{m_2} are included into two different components G_c and $G_{c'}$, respectively. We may assume that $c = 1$ and $c' = 2$ without loss of generality. We distinguish two cases to deduce a realization of π in G.

Case 2.1 - *We have $\sum_{i=1}^{k} (n_i^1 - 1) \geq \Delta$ or $\sum_{i=1}^{k} (n_i^2 - 1) \geq \Delta$.*

Assume $\sum_{i=1}^{k} (n_i^1 - 1) \geq \Delta$, and let $a_1, a_2, ..., a_k \geq 1$ be integers such that $a_i \leq n_i^1 - 1$ for every $i \in \{1, 2, ..., k\}$, and $\sum_{i=1}^{k} a_i = \Delta$. Note that we have both

$$(\sum_{i=1}^{p} n_i^1) - (\sum_{i=1}^{k} a_i) + \Delta = |V(G_1)|$$

and

$$(\sum_{i=1}^{p} n_i^2) + (\sum_{i=1}^{k} a_i) - \Delta = |V(G_2)|.$$

A realization of π in G is then obtained as follows. Since G_1 is $(u_1^1, u_2^1, ..., u_k^1)$-AP by assumption, there exists a $(u_1^1, u_2^1, ..., u_k^1)$-realization $(V_1^{1\prime}, V_2^{1\prime}, ..., V_p^{1\prime})$ of

$$(n_1^1 - a_1, n_2^1 - a_2, ..., n_k^1 - a_k, n_{k+1}^1, n_{k+2}^1, ..., n_{m_1-1}^1, n_{m_1}^1 + \Delta, n_{m_1+1}^1, n_{m_1+2}^1, ..., n_p^1)$$

in G_1. Similarly G_2 is $(u_1^2, u_2^2, ..., u_k^2)$-AP and then admits a $(u_1^2, u_2^2, ..., u_k^2)$-realization $(V_1^{2\prime}, V_2^{2\prime}, ..., V_p^{2\prime})$ of

$$(n_1^2 + a_1, n_2^2 + a_2, ..., n_k^2 + a_k, n_{k+1}^2, n_{k+2}^2, ..., n_{m_2-1}^2, n_{m_2}^2 - \Delta, n_{m_2+1}^2, n_{m_2+2}^2, ..., n_p^2).$$

It then follows that

$$((\bigcup_{i=1}^{\ell} V_1^i) - (V_1^1, V_1^2) \cup (V_1^{1\prime}, V_1^{2\prime}) \ , \ (\bigcup_{i=1}^{\ell} V_2^i) - (V_2^1, V_2^2) \cup (V_2^{1\prime}, V_2^{2\prime}) \ , \ ... \ , \\ (\bigcup_{i=1}^{\ell} V_p^i) - (V_p^1, V_p^2) \cup (V_p^{1\prime}, V_p^{2\prime}))$$

is a realization of π in G according to the same arguments as those used so far.

Case 2.2 - *We have* $\sum_{i=1}^{k}(n_i^1 - 1) < \Delta$ *and* $\sum_{i=1}^{k}(n_i^2 - 1) < \Delta$.

In such a situation, we have $(\sum_{i=1}^{k} n_i^1 + n_i^2) - 2k < 2\Delta - 1$, with $n_1, n_2, ..., n_k > s_{p_1}$ by assumption. Recall also that $n_i = (\sum_{j=1}^{\ell} n_i^j) - \ell + 1$ for every $i \in \{1, 2, ..., k\}$, that $k \geq \ell \geq 2$ and $s_{p_1} \geq k$, and that $\Delta = s_{p_1} - s_{p_m}$ with $s_{p_m} = \frac{s_{p_1} + s_{p_2}}{2}$. Then we get

$$\begin{aligned}
\sum_{i=1}^{k} \sum_{j=3}^{\ell} n_i^j &= (\sum_{i=1}^{k} n_i) - (\sum_{i=1}^{k} n_i^1 + n_i^2) + k\ell - k \\
&> ks_{p_1} - (2\Delta + 2k - 1) + k\ell - k \\
&> ks_{p_1} + k\ell - 3k - 2\Delta + 1 \\
&> (k-2)s_{p_1} - k + 2s_{p_m} \\
&> (k-2)s_{p_1} - s_{p_1} + s_{p_1} \\
&> (k-2)s_{p_1}. \qquad (1)
\end{aligned}$$

By assumption we have $\ell \leq k$, and hence $\sum_{i=1}^{k} n_i^\alpha > s_{p_1}$ for some $\alpha \in \{3, 4, ..., \ell\}$ since otherwise we would get $\sum_{i=1}^{k} \sum_{j=3}^{\ell} n_i^j < (k-2)s_{p_1}$, contradicting Inequality (1).

Assume $\alpha = 3$ without loss of generality. Since

$$\sum_{i=1}^{k} n_i^3 > s_{p_1} = s_{p_m} + \Delta,$$

we can "move" the part with size m_1 of the realization from G_1 to G_3 as follows. First deduce integers $a_1, a_2, ..., a_k$ such that we have $a_i \leq n_i^3 - 1$ for every $i \in \{1, 2, ..., k\}$, and $\sum_{i=1}^k a_i = s_{p_m}$. According to our terminology, since $V_{m_1} \subset V(G_1)$, recall that $n_{m_1}^1 = n_{m_1}$ and $n_{m_1}^3 = 0$. Now, since G_1 and G_3 are $(u_1^1, u_2^1, ..., u_k^1)$- and $(u_1^3, u_2^3, ..., u_k^3)$-AP, respectively, we can deduce a $(u_1^1, u_2^1, ..., u_k^1)$-realization $(V_1^{1\prime}, V_2^{1\prime}, ..., V_p^{1\prime})$ of

$$(n_1^1 + a_1, n_2^1 + a_2, ..., n_k^1 + a_k, n_{k+1}^1, n_{k+2}^1, ..., n_{m_1-1}^1, 0, n_{m_1+1}^1, n_{m_1+2}^1, ..., n_p^1)$$

in G_1, as well as a $(u_1^3, u_2^3, ..., u_k^3)$-realization $(V_1^{3\prime}, V_2^{3\prime}, ..., V_p^{3\prime})$ of

$$(n_1^3 - a_1, n_2^3 - a_2, ..., n_k^3 - a_k, n_{k+1}^3, n_{k+2}^3, ..., n_{m_1-1}^3, n_{m_1}, n_{m_1+1}^3, n_{m_1+2}^3, ..., n_p^3)$$

in G_3. We then get that

$$((\bigcup_{i=1}^{\ell} V_1^i) - (V_1^1, V_1^3) \cup (V_1^{1\prime}, V_1^{3\prime}) \ , (\bigcup_{i=1}^{\ell} V_2^i) - (V_2^1, V_2^3) \cup (V_2^{1\prime}, V_2^{3\prime}) \ , \ ... \ , \\ (\bigcup_{i=1}^{\ell} V_p^i) - (V_p^1, V_p^3) \cup (V_p^{1\prime}, V_p^{3\prime}))$$

is another realization of π' in G with

$$\sum_{i=1}^{k} (|V_i^{3\prime}| - 1) = \sum_{i=1}^{k} n_i^3 - s_{p_m} > \Delta,$$

and the parts with size m_1 and m_2 being now included in $V(G_3)$ and $V(G_2)$, respectively, hence meeting the conditions of Case 2.1. Applying the same strategy as for Case 2.1, we eventually get a realization of π in G. □

We finally point out that the size of any kernel $K_{\mathcal{C}_{k,\ell}}(n)$ with $\ell \leq k$ is polynomial regarding n.

Corollary 15. *The kernel $K_{\mathcal{C}_{k,\ell}}(n)$ is polynomial for every $k \geq 1$, $\ell \leq k$, and $n \geq k$.*

Proof. Let k, ℓ, and n be fixed. Any n-sequence π of $K_{\mathcal{C}_{k,\ell}}(n)$ is only defined by the $2k + 6$ elements of its spectrum and the number of their occurrences in π. Since these parameters are all upper bounded by n, we get that the size of $K_{\mathcal{C}_{k,\ell}}(n)$ is $\mathcal{O}(n^{2(2k+6)-1}) \sim \mathcal{O}(n^{\mathcal{O}(k)})$.

5 Discussion

Our result on complete multipartite graphs aside, two polynomial kernels for $\mathcal{U}_k(n)$ and $\mathcal{C}_{k,\ell}(n)$ with $\ell \leq k$ are presented in this work. An important thing to keep in mind is that we did not prove that checking whether graphs from $\mathcal{U}_k(n)$ and $\mathcal{C}_{k,\ell}(n)$ are AP can be done in polynomial time. Indeed, even if the property of being AP of these graphs relies on the realizability of a polynomial number of sequences only, we did not deal with the algorithmic complexity of finding a realization of these sequences in these graphs, i.e. with the complexity of REALIZABLE SEQUENCE when restricted to graphs of $\mathcal{U}_k(n)$ and $\mathcal{C}_{k,\ell}(n)$, and sequences of $K_{\mathcal{U}_k}(n)$ and $K_{\mathcal{C}_{k,\ell}}(n)$, respectively.

However, the existence of these polynomial kernels for $\mathcal{U}_k(n)$ and $\mathcal{C}_{k,\ell}(n)$ directly implies that the decision problem

ARBITRARILY PARTITIONABLE GRAPH
Input: a graph G.
Question: is G AP?

is in NP when restricted to graphs of $\mathcal{U}_k(n)$ and $\mathcal{C}_{k,\ell}(n)$, since, given a realization in such a graph of each sequence in $K_{\mathcal{U}_k}(n)$ or $K_{\mathcal{C}_{k,\ell}}(n)$, one can check in polynomial time whether all these realizations (there are a polynomial number of them) are correct. This remark is of interest as the membership of ARBITRARILY PARTITIONABLE GRAPH to NP or $\mathsf{co\text{-}NP}$ has not been established for the general case at the moment. By proving that REALIZABLE SEQUENCE is NP-hard when restricted to members of $\mathcal{U}_k(n)$ or $\mathcal{C}_{k,\ell}(n)$ and sequences from $K_{\mathcal{U}_k}(n)$ or $K_{\mathcal{C}_{k,\ell}}(n)$, respectively, one would even get that ARBITRARILY PARTITIONABLE GRAPH is NP-complete when restricted to graphs of $\mathcal{U}_k(n)$ or $\mathcal{C}_{k,\ell}(n)$, respectively. We hence address the following question.

Question 16. *Is* REALIZABLE SEQUENCE *NP-hard when restricted to graphs of $\mathcal{U}_k(n)$ or $\mathcal{C}_{k,\ell}(n)$ and sequences of $K_{\mathcal{U}_k}(n)$ or $K_{\mathcal{C}_{k,\ell}}(n)$, respectively?*

Regarding graphs with universal vertices, we proved in Theorem 6 that REALIZABLE SEQUENCE is NP-complete for graphs with k universal vertices, with k converging to one third from the bottom. In contrast, we exhibited a polynomial kernel for graphs with k universal vertices, where k converges to one half from the bottom. From the existence of such a kernel, our feeling is that deciding whether a graph with about one half universal vertices is AP could be done in polynomial time. But we have no idea about the complexity of the same problem with k lying "between" one third and one half.

Question 17. *What is the greatest ϵ such that* REALIZABLE SEQUENCE *is NP-hard when restricted to graphs of $\mathcal{U}_k(n)$ for every $k \leq \frac{n}{3} + \epsilon$?*

Although our kernel $K_{\mathcal{C}_{k,\ell}}(n)$ is a generalization of $K'_{\mathcal{T}}(n)$, the polynomial kernel for tripodes, it does not hold for tripodes (which are $(1,3)$-compound graphs) since, for $k = 1$, Theorem 14 ensures that $K_{\mathcal{C}_{k,\ell}(n)}$ is a kernel for $\ell = 1$ only (and is hence not interesting in this case as $(k,1)$-compound graphs are AP by definition). However, note that the condition

"$\ell \leq k$" in the statement of Theorem 14 cannot be strengthened as otherwise the existence of the parameter α mentioned in the proof would not be guaranteed. The proof in [11] that $K'_T(n)$ is a kernel for tripodes is quite similar to our proof of Theorem 14, except that the very special case of a compound graph with exactly three components permits to consider parts with size s_{p_3} or s_{p_4} and use them to deduce the realization.

Actually the proof from [11] could be easily generalized to show that $K'_T(n)$ is actually a polynomial kernel for $(1,3)$-compound graphs. It is however not clear what a polynomial kernel for the remaining compound graphs could look like.

Question 18. *Is there a polynomial kernel for $\mathcal{C}_{k,\ell}(n)$, where $(k,\ell) \neq (1,3)$ and $k < \ell$?*

In Section 4, we introduced the notion of compound graphs, which is closely related to the notion of T-AP graphs. It is not clear how dense a T-AP graph can be, and hence how dense a compound graph can be. We thus raise the following question.

Question 19. *What is the minimum size of a T-AP graph of order $n \geq k$, where T is some k-tuple of vertices?*

By definition, an AP+k graph is T-AP for every k-tuple T of vertices. As proved in [4], an AP+k graph on n vertices has size at least $\lceil \frac{n(k+1)}{2} \rceil$, this quantity being thus a first upper bound on the size parameter mentioned in Question 19. However this quantity is surely far from being optimal as the property of being T-AP for all possible k-tuples of vertices is way stronger than when this property is only required for only one k-tuple.

Recall that every path is (u)-AP and (u,v)-AP, with u and v denoting its endvertices. We hence get that the answer to Question 19 is $n-1$ for all $k \in \{1,2\}$.

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