# On the analysis of thin walled members in the framework of the Generalized Beam Theory

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# On the analysis of thin walled members in the framework of the Generalized Beam Theory

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**Abstract.** *The analysis of thin walled members in the framework of the Generalized Beam Theory (GBT) is critically revised. Firstly, the classic form of the GBT is briefly explained along with its main advantages and problems. Subsequently, a new formulation that coherently accounts for shear deformation is presented along with a unique modal decomposition, the Cross Section analysis, that allows to recover classical shear deformable beam theories exactly. Furthermore, a stress recovery procedure for the finite element analysis of GBT beams is proposed as an improvement on the traditional elasto-kinematic approach. Performance is shown in numerical tests.*

**Keywords:** GBT; Thin-Walled Beams; Shear deformability; Stress Recovery.

## 1 INTRODUCTION

Much research has been devoted to the creation of computational tools that allow describing the mechanics of thin-walled structural systems. The Generalized Beam Theory (GBT) originally proposed by Schardt [1] is one of such tools and has been proven to consistently account for cross-section distortion along with the more classical kinematics of axial displacement, bending and torsional rotation in a comprehensive fashion.

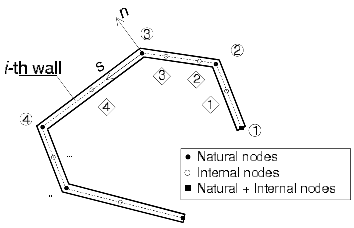



Figure 1: Thin-walled cross-section

The fundamental idea of the GBT is to consider a thin-walled beam as an "assembly" of thin plates (see Figure 1) and to assume the displacement field of the beam as a linear combination of predefined cross-section *deformation modes* (which are known beforehand) multiplied by unknown functions depending on the beam axial coordinate, that can be called *kinematic parameters* or *generalized displacements*. In the original GBT, the following displacement field is assumed for the generic $i$-th wall of the cross-section:

$$d_n(n, s, z) = \psi(s)\mathbf{v}(z), \tag{1}$$

$$d_s(n, s, z) = [\mu(s) - n\partial_s\psi(s)]\,\mathbf{v}(z), \tag{2}$$

$$d_z(n, s, z) = [\varphi(s) - n\psi(s)]\,\partial_z\mathbf{v}(z), \tag{3}$$

where $d_n$ is the displacement orthogonal to the wall midline, $d_s$ is the displacement tangent to the wall midline, $d_z$ is the displacement in the beam axial direction, $\psi$, $\mu$ and $\varphi$ are row matrices collecting the assumed cross-section defor-

mation modes (depending only on $s$) and $\mathbf{v}$ is the vector that collects the unknown kinematic parameters (depending only on $z$). Moreover, $\partial_s$ and $\partial_z$ denote the derivative with respect to the $s$ coordinate and to the $z$ coordinate, respectively. Several contributions [2] have gone to extend the original model to include open unbranched sections, non-linear effects for the analysis of buckling problems and especially shear deformation [3]. However, due to the constant warping displacement along the wall thickness, the traditional way to account for shear deformability leads to null shear strain $\gamma_{zn}$ between the direction of the beam axis and that orthogonal to the wall midline. From the point of view of the global beam behaviour, this engenders a non-perfect coherence between the bending and shear strain components of the beam. As a consequence, some drawbacks on both theoretical and practical sides arise. An *ad hoc* modal decomposition procedure for shear modes should be used, different from the one used for flexural modes. Two generalized strain components turn out to be associated to each shear mode [2], instead of one as should be natural to expect. Classical shear deformable beam theories are not recovered exactly: bending deflections (Figure 2 (b)) are not clearly distinguishable from deflections due to shearing strains (Figure 2 (c)) and, as a consequence, pure cross-section flexural rotations do not result as natural kinematic parameters.
Additionally, recovery of three-dimensional stresses in the context of GBT has generally been kept at the most basic level of elasto-kinematic relations which, by the very nature of the GBT kinematics, paint an incomplete picture of the stress profiles.

## 2 A REVISED GBT FORMULATION

In this section, a new formulation of the kinematics is proposed, which preserves the general format of the original GBT for flexural modes, and introduces the variability of the warping displacement along the wall thickness besides that along the wall midline. This engenders non-null shear strains $\gamma_{zn}$ and $\gamma_{zs}$, and guarantees that the bending and the shear strain components of the beam perfectly match through the wall thickness too.
The following expressions are assumed for the displacement field of the $i$-th wall (Figure 1):

$$d_n(n,s,z) = \psi(s)\mathbf{v}(z), \tag{4}$$
$$d_s(n,s,z) = [\mu(s) - n\partial_s\psi(s)]\,\mathbf{v}(z), \tag{5}$$
$$d_z(n,s,z) = [\varphi(s) - n\psi(s)]\,[\partial_z\mathbf{v}(z) + \delta(z)] + \varphi^h(s)\delta^h(z). \tag{6}$$

In the above expressions, the unknown kinematic parameters are divided between those corresponding to flexural modes, $\mathbf{v}$, and those to shear modes, $\delta$ and $\delta^h$. In turn, these last parameters are divided between those corresponding to the basic shear modes, $\delta$, in the same number of flexural modes (both fundamental and local), and those corresponding to the additional shear modes, $\delta^h$. As it can be noted, if $\delta$ and $\delta^h$ are disregarded, then the new kinematics coincides with the conventional one, Equation (1)-(3). Regarding the shear parameters, $\delta$ is the typical term of shear deformable theories where the longitudinal displacement is not governed by the derivative of the transverse displacement $\mathbf{v}$. Note that the relevant contribution is different from the classical shear deformable GBT [2]. Specifically, because of the term $-n\psi\delta$, in the present formulation the warping displacement associated to parameters $\delta$ varies along the wall thickness $n$, in the same way of the flexural one as shown in Figure 2 (d). Parameters $\delta^h$ serve to further enrich the warping description along the wall direction $s$. Note that the associated warping displacement is constant along the wall thickness $n$. Indeed, the corresponding contribution is typical of beam theories with enriched warping description.

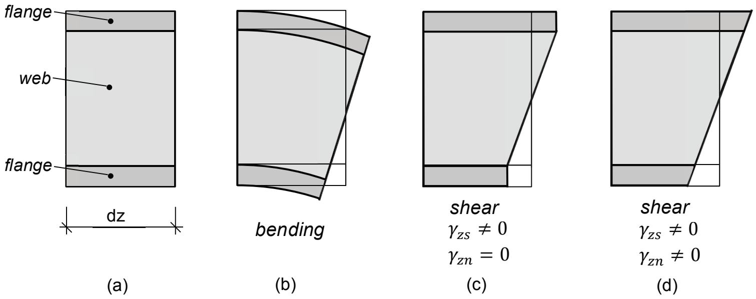



Figure 2: Sketch of the main difference between classical shear deformable GBT and the present formulation: (a) undeformed elementary beam, (b) bending strain component, (c) corresponding shear strain component in the classical GBT, (d) corresponding shear strain component in the Timoshenko beam theory and in the present theory.

## 3 STRESS RECOVERY

This section presents a stress recovery procedure conceived to improve on the elasto-kinematic approach generally used. This improvement is necessary since the kinematic constraints inherent to the GBT formulation imply an unbalanced internal stress profile. Stresses are divided first into their membrane $(M)$ and bending $(B)$ parts as follows

$$\sigma_{zz} = \sigma_{zz}^{M(A)} + n\sigma_{zz}^{B(A)} \tag{7}$$
$$\sigma_{ss} = \sigma_{ss}^{M(R)} + n\sigma_{ss}^{B(A)} \tag{8}$$
$$\tau_{zs} = \tau_{zs}^{M(R)} + n\tau_{zs}^{B(A)} \tag{9}$$
$$\tau_{zn} = g(n)\tau_{zn}^{B(R)} \tag{10}$$
(11)

where $A$ stands for "active" and represents the stress component related to deformation by a constitutive equation, while $R$ stands for "reactive" and represents the stress component that does no work on the prescribed deformations and acts instead as a reaction to the kinematic constraints. In keeping with the mechanics of thick plates, $g$ is assumed to be a parabola with unitary area. Following an approach analogous to the one used for laminated plates, the reactive part is calculated by the integration of the indefinite equilibrium equations [4]. Indeed, the accuracy of this reconstruction depends on the accuracy of the first and second derivatives of the beam stress resultants, whose convergence everywhere within the element is not a priori guaranteed. Therefore, the Recovery by Compatibility in Patches procedure [5, 6], suitably extended to the present case, is used to improve the approximation of the beam stress resultants.

## 4 NUMERICAL TESTS

To test the performance of this approach, a Z-section cantilever beam with a transversal distributed load applied at the end of the web is considered as shown in Figure 3. The material is steel and the dimensions are: $b = 40$ mm, $h = 120$ mm, $w = 15$ mm, $t = 1.5$ mm, $L = 500$ mm. The results obtained with the present formulation with and without shear modes $-$ denoted respectively by "GBT w S modes" and "GBT wo S modes" $-$ are compared with those obtained with a 3D shear deformable shell model and with the Vlasov theory.

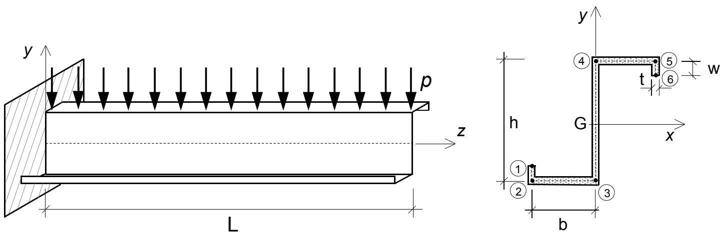



Figure 3: Z section cantilever beam under transversal load

The $y$-displacement of node 2, shown in Figure 4 normalized by the factor $\frac{pL^4}{EI}$, is influenced by both the shear deformation and the cross-section in-plane deformation. In particular, the difference between the results of the present formulation with and without shear modes allows to evaluate the effect of the shear deformation. The results of the present formulation with shear modes are in very good agreement with those predicted by the 3D model.
Additionally, local stresses were recovered for the same profile using the proposed procedure. Figure 5 shows the distribution of $\tau_{zs}$ along the midline for a section at $z = 100$mm as well as $\tau_{zn}$ at the midpoint of wall 3 along the wall thickness for $z = 160$mm. Comparison for this example was done versus a rich brick model that allows to recover stress distribution along the wall thickness. In both cases, the proposed approach constitutes a clear improvement on the recovery of stresses with respect to the elasto-kinematic formulation and is in very good agreement with the 3D model.

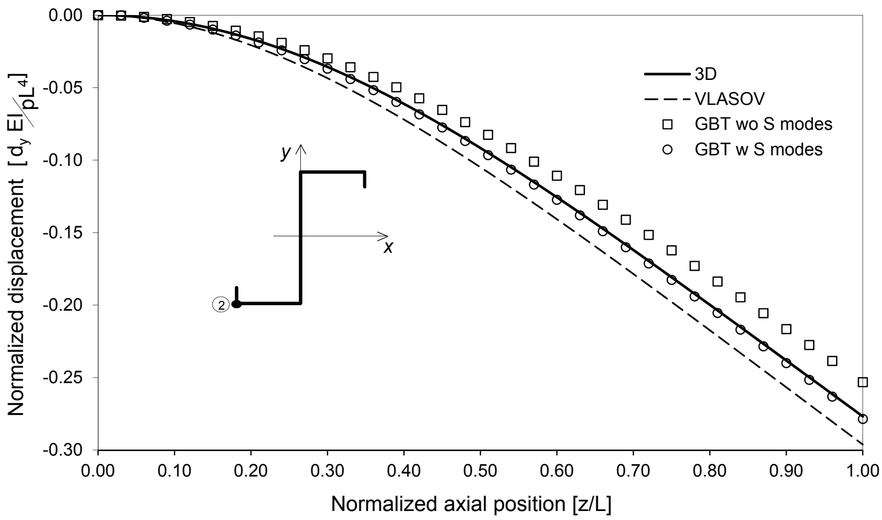
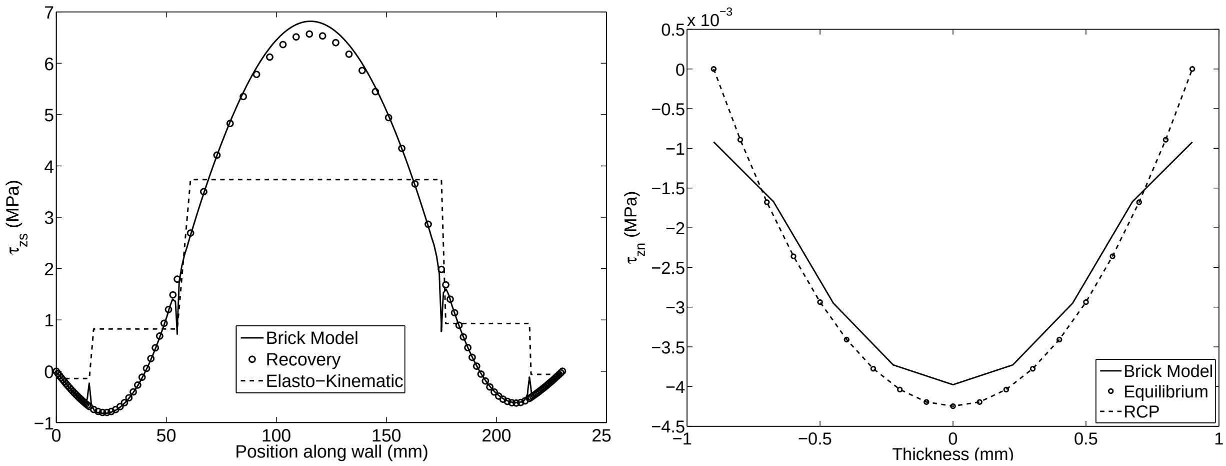



Figure 4: Comparison of displacements along the axial coordinate




(a) Shear stresses along midline

(b) Shear stresses along thickness

Figure 5: Recovery of local stresses

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