Innovative isogeometric formulations for shear deformable beams and plates

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Abstract. *We present different innovative formulations for shear deformable beams and plates exploiting the high inter-element continuity provided by NURBS basis functions. We develop isogeometric collocation methods in standard and mixed formulations as well as Galerkin methods using an alternative set of discrete variables. All methods are free of shear locking, which is confirmed by numerical tests.*

Keywords: Isogeometric, collocation, plates, beams, locking-free, shear deformable

1 INTRODUCTION

Isogeometric analysis (IGA) [6] is an extension to finite element analysis where functions used in CAD (Computer Aided Design) are adopted as basis functions for analysis, with the goal of merging design and analysis into one model. Typically, NURBS are used in both CAD and isogeometric analysis. They are higher order functions with high inter-elements continuities. This characteristic has proved superior approximation properties compared to standard finite elements in many different applications, and it furthermore allows the development of formulations that are not possible with standard Lagrangian finite elements. Classical examples for formulations which need at least C^1-continuity are the theories of thin structures, in particular the Euler-Bernoulli beam theory and the Kirchhoff plate theory, as well as the Kirchhoff-Love shell theory. The continuity requirement inherent in these theories makes a straight-forward implementation with standard finite elements very difficult. For this reason, shear deformable theories, in particular the Timoshenko beam theory and the Reissner-Mindlin plate and shell theory, which require only C^0-continuity, are much more prominent in finite element applications. Isogeometric analysis, however, allows straight-forward implementations of the classical theories with applicability to arbitrarily shaped structures, see e.g. [7]. In the present work, we show that the continuity provided in IGA can also be used in order to develop alternative formulations based on the shear deformable theories. We present different formulations including isogeometric collocation methods and alternative discretization approaches for Galerkin methods.

2 ISOGEOMETRIC COLLOCATION METHODS

In this section, we briefly present the idea of isogeometric collocation and its application to thin structures. For the sake of brevity, we present the details only for the plate formulations, highlighting that the same approach has also been successfully applied to straight beams [3] and spatial rods [1].
The principal idea of isogeometric collocation is to discretize the strong form equations of the problem, and to collocate them on a set of suitable points such that a quadratic system of equations is obtained. Working on the strong form equations typically involves second or higher derivatives and, thus, requires at least C^1-continuity, which motivates the choice of NURBS for the discretization. The crucial point is the correct choice of collocation points. It has been shown that the so-called *Greville* abscissae provide a very reliable and stable set of collocation points, and they guarantee a quadratic system of equations. For more details, reference is made to [2].
In contrast to Galerkin methods, there are no integrals to be computed in collocation. As a consequence, there are much less evaluation points necessary than in a Galerkin formulation which makes these formulations much more efficient. In particular, in collocation only one collocation point per element is necessary, while in a Galerkin formulation typically $(p+1)^d$ Gauss points per element are used, where d denotes the dimension. In addition to that, there are also less operations per point in collocation, due to the fact that there is no multiplication with a test function. A drawback of this approach is that the system matrix is not symmetric in general, however, it is diagonally dominant.

In the following, we derive the collocation scheme for a Reissner-Mindlin plate formulation. Given the following variables and parameters: Deflection w, rotations $\boldsymbol{\varphi}$, shear strains γ, curvatures $\boldsymbol{\kappa}$, material matrices $\mathbf{D}_b$ and $\mathbf{D}_s$ for bending and shear, respectively, external load f, bending moments $\mathbf{m}$, shear forces $\mathbf{q}$, and the differential operator matrix $\mathbf{L}$ which is related to the symmetric gradient.
Then the strong form of the plate problem is given by the kinematic, constitutive and equilibrium equations:

$$\text{kinematics:} \qquad \gamma = \nabla w + \boldsymbol{\varphi} \tag{1}$$

$$\boldsymbol{\kappa} = \mathbf{L}\,\boldsymbol{\varphi} \tag{2}$$

$$\text{constitutives:} \qquad \mathbf{m} = \mathbf{D}_b\,\boldsymbol{\kappa} \tag{3}$$

$$\mathbf{q} = \mathbf{D}_s\,\gamma \tag{4}$$

$$\text{equilibrium:} \qquad \mathbf{L}^T\mathbf{m} - \mathbf{q} = \mathbf{0} \tag{5}$$

$$\nabla \cdot \mathbf{q} = -f \tag{6}$$

Using the kinematic and constitutive relations, one can rewrite the equilibrium equations in terms on the primal variables w (deflection) and $\boldsymbol{\varphi}$ (rotations):

$$\mathbf{D}_s\,(\Delta w + \nabla \cdot \boldsymbol{\varphi}) = -f \tag{7}$$

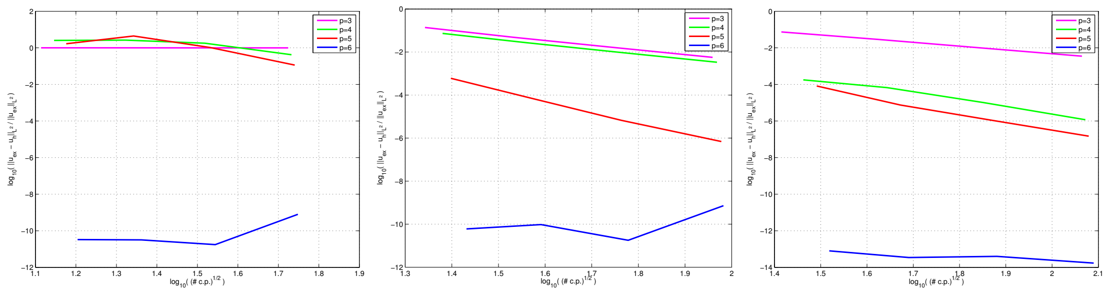
$$-\mathbf{D}_s \nabla w + (-\mathbf{D}_s + \mathbf{L}^T\mathbf{D}_b\mathbf{L})\boldsymbol{\varphi} = \mathbf{0} \tag{8}$$

Equations (7)-(8) are the collocation equations. Note that the second equation is vectorial, therefore, there are three equations in total. The first equation represents the equilibrium of forces and is collocated at the Greville abscissae related to the space of the displacements w. The second and third equation represent the equilibrium of moment in x and y, respectively, and they are collocated on the Greville abscissae related to the spaces of the rotations φ_x and φ_y, respectively.
Furthermore, a mixed formulation is developed where the shear forces $\mathbf{q}$ are discretized as independent variables. Consequently, there are five unknown fields w, φ_x, φ_y, q_x, and q_y. In addition to the equilibrium equations (5)-(6), the constitutive equations (4) are discretized and collocated,

yielding again a square system matrix. The detailed equations for the mixed formulation are established following the same rationale as above and are not reported here for the sake of brevity.

Numerical tests

We consider a problem having a known analytical solution, which consists of a unitary square plate, clamped on all four sides and subjected to a distributed polynomial load. The problem setup and the analytical solution are described in [4]. The problem is solved by both the displacement-based and the mixed collocation method. For small plate thicknesses, the displacement-based formulation with equal degree for all fields exhibits shear locking. However, the locking problem vanishes if rotated spaces are used, i.e., if the degree of φ_x is lowered by one in $x-$direction and the degree of φ_y is lowered by one in $y-$direction. The mixed formulation, on the contrary, is locking-free without specific conditions on the spaces. Figure 1 shows the convergence of the error of displacements in the L^2-norm for a plate thickness of $t = 10^{-3}$. Please note that the exact solution is of order six. Therefore, the results for $p = 6$ are always at convergence (at machine precision considering also the conditioning).



(a) (b) (c)

Figure 1: Unitary clamped quadratic plate with $t = 10^{-3}$. Convergence plots for collocation with a displacement-based formulation with equal degrees (a), rotated degrees (b), and a mixed formulation (c).

3 ALTERNATIVE DISCRETIZATIONS IN A GALERKIN APPROACH

In the Kirchhoff plate theory, shear strains are neglected and, therefore, the rotations $\boldsymbol{\varphi}$ are obtained as the negative gradient of the displacements w. The Reissner-Mindlin theory, however, accounts for shear deformability and, therefore, the relation between w and $\boldsymbol{\varphi}$ is enhanced by the shear strains γ:

$$\nabla w = -\boldsymbol{\varphi} + \boldsymbol{\gamma}. \tag{9}$$

It is the traditional approach, to discretize the displacements w and rotations $\boldsymbol{\varphi}$, and compute the shear strains γ through the relation given in (9). In our approach, we discretize the displacements w and shear strains γ instead and compute the rotations $\boldsymbol{\varphi}$ by equation (9). The element formulation is based on the standard variational formulation

$$W_{int} = W_{ext}$$
$$\int \boldsymbol{\kappa} \mathbf{D}_b \bar{\boldsymbol{\kappa}} \, \mathrm{d}A + \int \gamma \mathbf{D}_s \bar{\gamma} \, \mathrm{d}A = \int f \bar{w} \, \mathrm{d}A$$

with the only difference that γ is now a discrete variable and $\boldsymbol{\kappa}$ is a function of w and γ. As a consequence, second derivatives appear in the formulation, which necessitates C^1-continuous functions, such as NURBS. This formulation can also be interpreted as a Kirchhoff plate, enhanced by shear deformability. Rotational boundary conditions are treated in a similar manner as for Kirchhoff plates (see [7]). The great advantage of this formulation is that the Kirchhoff constraint of vanishing shear strains in the thin limit can be fulfilled without problems, since the shear strains are now a discrete variable and do not pose compatibility conditions on the different fields, and therefore, the method is locking-free. This is confirmed numerically by a the convergence study shown in Figure 2. It shows the results for the plate problem presented above, solved with the alternative Galerkin formulation. We highlight that the same approach can be applied for an

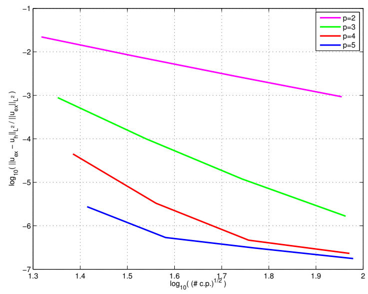


Figure 2: Unitary clamped quadratic plate with $t = 10^{-3}$. Convergence plots for Galerkin solution with alternative discretization.

alternative Timoshenko beam formulation. Furthermore, we refer to [5], where a similar approach has been presented for shells.

4 CONCLUSIONS

In this work, we have presented innovative formulations for shear deformable plates and beams, based on isogeometric discretizations. We have presented locking-free collocation methods, in displacement-based and mixed formulations, which have considerable advantages over Galerkin formulations in terms of assembly costs. Furthermore, we have shown a Galerkin method with an alternative discretization approach, where the shear strains instead of the rotations are discretized, which also yields a locking-free formulation. All presented methods have been validated numerically providing convincing results.

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