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Variational Formulations and Functional Approximation Algorithms in Stochastic Plasticity of Materials

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Abstract. *Within the framework of elastoplastic analysis the focus is set on the rate-independent evolutionary problem with general hardening whose material characteristics are assumed to be uncertain. These processes are described via a stochastic convex energy function and evolution equations for internal variables. Computationally such posed problem reduces to the stochastic minimisation of smooth convex energy functional on discrete tensor product subspaces whose unique minimizer is obtained via a stochastic closest point projection algorithm.*

Keywords: stochastic elastoplasticity, stochastic closest point projection algorithm, stochastic Galerkin

1 INTRODUCTION

Due to their widespread applications in industry, the descriptions of nonlinear elastic and elastoplastic phenomena have found an important place in the field of computational mechanics. Presently existing models rely on the assumption of complete knowledge of the system, i.e. one assumes that the material characteristics as well as the external loadings applied on the structure are entirely known. However, uncertainty is widely present, from the mathematical model which cannot perfectly match the data to the poorly known external loadings.

Including parametric uncertainties into the model, this work provides an extended formulation of nonlinear irreversible phenomena in which existing uncertainties are modelled in a probabilistic manner. Furthermore, this paper investigates the impact of the uncertainty on the system response, as well as the accuracy of the mathematical models describing the true physics. The focus is set on the rate-independent evolutionary problem with general hardening whose material characteristics are assumed to be uncertain. In this regard, within the framework of infinitesimal and large displacement elastoplasticity theory a class of stochastic mixed variational inequalities described by uncertain parameters is considered, both theoretically and numerically.

To the author's knowledge there are two main streams in modelling stochastic infinitesimal elastoplastic behaviour: the approximate plasticity theory by Anders and Hori [1], and the moment equations method by Jermić and Sett [2]. In contrast to the infinitesimal, the finite deformation theory is not yet completely understood in the deterministic sense. Namely, the convexity of the total potential energy functional is broken and thus the symmetry between the potential energy principle and complementary energy is lost [3]. These problems lead to the complicated phenomena followed by high degrees of nonlinearity, which are very difficult to handle especially in a stochastic setting. Therefore, the stochastic description of the large deformation plasticity seems not to be an easy task; and the only work pursued so far in this direction can be found in [4]. These papers consider the simple plasticity problem described by a power plastic flow law with uncertain isotropic resistance or the fiber orientation of the hyperelastic material.

Uncertainty propagation with the help of direct integration techniques such as Monte Carlo methods requires a huge computational effort. Therefore, this paper investigates and develops more efficient methods for the computation of the unique solution of the stochastic convex optimisation problem. These methods employ the stochastic Galerkin projection in its fully intrusive or non-intrusive variant. The former method represents the direct, purely algebraic way of computing the response in each iteration of Newton-like methods, while the latter evaluates the residuum in each iteration via high-dimensional integration rules based on random or deterministic sampling, e.g. Monte Carlo

and related techniques.

2 MODEL PROBLEM

To describe quasi-static elastoplasticity with hardening one has to consider the non-smooth evolution of the internal parameters [5], which gives a variational inequality as a generalisation of the differential equation. The state variable is $w = (u, \varepsilon_p, \nu)$, where u denotes the displacement field, ε_p is the plastic deformation, and ν the appropriate internal hardening variable. In a mixed formulation [5] one has to consider at the same time a dual variable w^* of stress-like quantities. These quantities have to stay inside a non-empty closed convex set \mathcal{K} , the so-called elastic domain. The abstract mixed formulation then is to find functions $w(t)$ and $w^*(t)$ such that $w^*(t) \in \mathcal{K}$ and

$$A(w(t)) + w^*(t) = f(t), \quad (1)$$

$$\forall z^* \in \mathcal{K} \quad \langle \dot{w}(t), z^* - w^*(t) \rangle \leq 0. \quad (2)$$

In this description, the first equation represents the equilibrium condition, while the second is the so-called flow rule, stating that the rate of change $\dot{w}(t) = (\partial/\partial t)w(t)$ is in the normal cone of \mathcal{K} at $w^*(t)$; for more details see [5]. Note that all variables of consideration are of stochastic nature, i.e. described by random fields/ processes in a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, where Ω represents the space of all events ω , \mathcal{B} is the σ -algebra and \mathbb{P} is the probability measure. The reason for the probabilistic modelling is possible presence of uncertainty in material parameters (e.g. constitutive tensor, yield stress), boundary conditions, geometry etc. Following this, one may formally define w as a mapping $\Omega \rightarrow \mathcal{W}$ and state that $w \in \mathcal{W} = L_2(\Omega, \mathcal{B}, \mathbb{P}; \mathcal{W}) \simeq \mathcal{W} \otimes (S)$. Similar is valid for w^* . Note that the choice of (S) and the stochastic regularity of the solution depend on the stochastic regularity of the right hand side and parameters [6]. For the sake of simplicity, (S) is taken to be $L_2(\Omega)$ such that \mathcal{W} is the Hilbert space induced by the inner product: $\langle w_1 | w_2 \rangle = \mathbb{E}(\langle w_1 | w_2 \rangle_{\mathcal{W}})$, and duality pairing:

$$\langle w_1, w_2 \rangle = \mathbb{E}(\langle w_1, w_2 \rangle_{\mathcal{W}}). \quad (3)$$

Here, $\mathbb{E}(\cdot) = \int_{\Omega} (\cdot) \mathbb{P}(d\omega)$ is the mathematical expectation with respect to the probability measure \mathbb{P} .

3 STOCHASTIC CLOSEST POINT PROJECTION

Time discretised as usual with backward Euler, the inequality describing the associative plastic flow rule (see Eq. 2) in case of general hardening plasticity collapses to the (iterative) solution of a convex mathematical programming problem. More special cases such as perfect plasticity and linear hardening can be easily derived by taking into account appropriate restrictions (see [5]).

With the previous assumptions the inequality in Eq. (2) reduces to the optimization problem [7, 8] of the following form:

$$\begin{aligned} \text{minimise} \quad & \Phi_n(\omega) := \frac{1}{2} \langle \Sigma_n^{trial} - \Sigma_n, A^{-1}(\Sigma_n^{trial} - \Sigma_n) \rangle, \\ \text{subjected to} \quad & \Sigma_n \in \mathcal{R} : \quad \phi_n(\Sigma_n) \leq 0, \end{aligned} \quad (4)$$

where Σ_n is the unknown current value of the generalised stress (Cauchy stress plus conjugate stresses), Σ_n^{trial} is the trial generalised stress, Φ_n is the convex functional in time n and $\phi_n(\Sigma_n)$ is the convex mapping from the stress space \mathcal{R} to \mathcal{K} called the yield function. In deterministic theory [9] the optimization problem in Eq. (4) is generally solved by the so-called *closest point projection* or *radial return map* algorithm¹. The algorithm consists of two steps: *reversible* (non-dissipative) and *irreversible* (dissipative), often called *elastic predictor* and *plastic corrector*. The non-dissipative step predicts the trial stress state Σ_n^{trial} from the known displacement increment (obtained from the residual equation) via constitutive laws, while the actual projection happens in the corrector step if the yield condition is not fulfilled (the stress lies outside of the elastic domain). In such a case the dissipative step searches for the closest distance in the energy norm of a trial state to a convex set of elastic domain, and then projects the stress back to the yield surface, see Fig. 1.

The solution of the minimisation problem is determined by the Kuhn-Tucker theorem [7] extended to the stochastic space. Namely, by assuming that $\Phi_n(\omega)$ and ϕ_n are Gâteaux differentiable and that Σ_n is a regular point of the inequality $\phi_n(\Sigma_n) \leq 0$ almost surely, one may show that there exists a Lagrange multiplier $\lambda \in \mathcal{R}^*$, $\lambda \geq 0$ such that

$$\mathcal{L}_n(\omega) = \Phi_n(\omega) + \langle \phi_n, \lambda \rangle \quad (5)$$

¹ the second name is more appropriate for perfect plastic behaviour

Predictor: Solve equilibrium equation \rightarrow Compute Σ_n^{trial} and then $\Sigma_n \rightarrow$ check yield criterion $\phi_n(\Sigma_n) \leq 0$ a.s.

Corrector: if $\perp \phi_n(\Sigma_n) \leq 0$ a.s. \rightarrow project stress back to \mathcal{K} (i.e. solve Eq. (6)) \rightarrow update variables

Figure 1: Algorithmic representation of operator split to predictor and corrector steps.

is stationary at Σ_n almost surely; furthermore $\langle \phi_n, \lambda \rangle = 0$ almost surely. Then, the corresponding minimiser of Eq. (4) follows from the optimality conditions:

$$\partial_{\Sigma} \mathcal{L}_n = 0 \text{ and } \partial_{\lambda} \mathcal{L}_n = 0 \quad \text{a.s.} \quad (6)$$

Note that the second condition may be rewritten in a Kuhn-Tucker form:

$$\lambda \geq 0 \text{ and } \lambda \phi_n(\Sigma_n) = 0 \text{ a.s.} \quad (7)$$

The system in Eq. (6) can be solved with the help of the local stochastic Newton method for the generalised plastic strain (i.e. stress) and λ . Once the system is solved, the state of the material can be easily updated to the next time step.

4 NUMERICAL COMPUTATION

Before numerical integration has been performed, one has to fully discretise the problem. Since the solution of the elastoplastic problem belongs to the tensor product space $\mathcal{W} = \mathcal{W} \otimes (S)$, its further discretisation is quite easy to perform by discretisation of each of the components separately. In this paper, the space \mathcal{W} is approximated by the finite element subspace \mathcal{W}_h , and (S) by the finite subspace $(S)_{\mathcal{J}} := \text{span} \{X_{\alpha}(\theta(\omega))\}_{\alpha \in \mathcal{J}} \subset (S)$ where $X_{\alpha}(\theta(\omega))$ denotes the multivariate polynomial in the mutually independent RVs $\theta(\omega)$, and \mathcal{J} represents the set of multi-indices. Following this, the global phase of the numerical algorithm searches for the increment of displacement

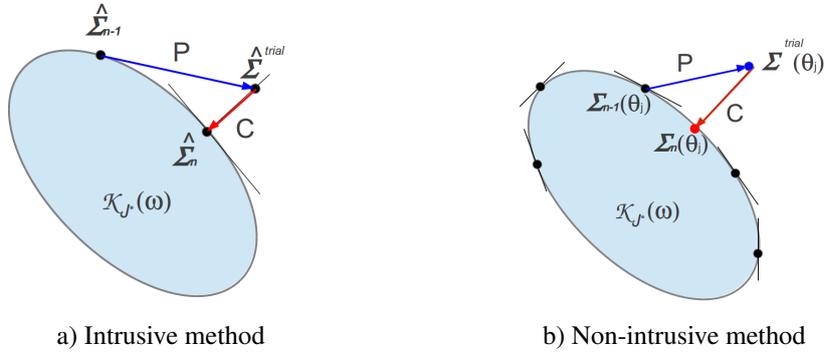


Figure 2: Schematic representation of stochastic closest point projection algorithm

by projecting the residual Q in Eq. (1) in a stochastic Galerkin manner. In other words one solves the system of equations:

$$\forall \beta \in \mathcal{J} : \quad Q^{(\beta)} := \mathbb{E} \left(Q(\omega) \left[\left(\sum_{\alpha \in \mathcal{J}} \Delta u_n^{(\alpha)} X_{\alpha}(\theta) \right) \right] X_{\beta}(\theta) \right) = 0 \quad (8)$$

for $\Delta \hat{u}_n := \Delta u_n^{(\alpha)} X_{\alpha}(\theta)$ in a stochastic Newton style as described in [5]. Once the increment of the displacement is computed, the local stage of the algorithm searches for the next state of the system in each Gauss FEM (Finite Element Method) integration point with the help of the stochastic closest point projection procedure. For this one requires the weak approximation \mathcal{K}^{\star} of the convex set \mathcal{K} . Depending on the type of the approximation the convex domain \mathcal{K} takes, one may build two different kinds of solution procedures: intrusive and non-intrusive. The former one approximates the convex domain by the set $\mathcal{K}_{\mathcal{J}}^{\star} = \{\Sigma \in \mathcal{Y}_h \otimes (S)_{\mathcal{J}} \mid \Pr(\hat{\phi}_n \leq 0) \geq p_r\}$, i.e. one projects the yield function onto the space $(S)_{\mathcal{J}}$ to $\hat{\phi}_n$ and investigates the yield condition with the help of the probability criteria. This corresponds to the process of determination of the elastic state of the FEM integration point with

the probability p_r . In a computational sense the intrusive method uses linear algebra operations on the polynomial chaos approximations of RVs w and w^* to perform both predictor and corrector steps, and hence the FEM code can not be used in a black-boxed manner. On the other side, the non-intrusive method uses something similar to the “classical” approximation of the convex domain, i.e. \mathcal{K} is represented by the finite number of half-planes such that $\mathcal{K}_{\mathcal{J}}^* = \{\boldsymbol{\Sigma}_h \in \mathcal{Y}_h \otimes (S)_{\mathcal{J}} : \phi_n(\boldsymbol{\Sigma}_h(\boldsymbol{\theta}_z)) \leq 0, \quad \forall \boldsymbol{\theta}_z \in \Xi\}$. Here, Ξ represents the set of points $\{\boldsymbol{\theta}_i\}_{i=1}^N$, i.e. samples. Such construction allows the decoupling of the problem into N smaller problems, which may be solved independently. Note that each of them corresponds to the normal deterministic optimization problem as presented in [9]. This is, however, not valid for the intrusive method.

5 CONCLUSIONS

The idea of random variables as functions in an infinite dimensional space approximated by the elements of finite dimensional spaces has brought a new view to the field of stochastic elastoplasticity. In this paper, an extension of the stochastic finite element method and related numerical procedures to the resolution of inelastic stochastic problems in the context of Galerkin methods has been proposed. Namely, the paper provides stochastic mathematical formulation of irreversible phenomena in a form of abstract mixed variational inequality which better addresses numerical algorithms actually used in industry. In this regard the author has developed the stochastic closest point algorithm for the computation of the state of the system in two different forms: intrusive and non-intrusive. These methods are based on functional approximations of variables of consideration and “white noise analysis”.

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