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Abstract

A numerical method to approximate ruin probabilities is proposed within the frame of a compound Poisson ruin model. The defective density function associated to the ruin probability is projected in an orthogonal polynomial system. These polynomials are orthogonal with respect to a probability measure that belongs to Natural Exponential Family with Quadratic Variance Function (NEF-QVF). The method is convenient in at least four ways. Firstly, it leads to a simple analytical expression of the ultimate ruin probability. Secondly, the implementation does not require strong computer skills. Thirdly, our approximation method does not necessitate any preliminary discretisation step of the claim sizes distribution. Finally, the coefficients of our formula do not depend on initial reserves.

Keywords: compound Poisson model, ultimate ruin probability, natural exponential families with quadratic variance functions, orthogonal polynomials, gamma series expansion, Laplace transform inversion.

1 Introduction

A non-life insurance company is assumed to be able to follow the financial reserves' evolution associated with one of its portfolios in continuous time. The number of claims until time t is assumed to be an homogeneous Poisson process $\{N_t\}_{t\geq 0}$, with intensity β. The successive claim amounts $(U_i)_{i\in\mathbb{N}^*}$, form a sequence of positive i.i.d. continuous random variables and independent of $\{N_t\}_{t\geq 0}$, with density function f_U and mean μ. The initial reserves are of amounts $u \geq 0$, and the premium rate is constant and equal to $p \geq 0$. The risk reserve process is therefore defined as

$$R_t = u + pt - \sum_{i=1}^{N(t)} U_i,$$

the associated claims surplus process is defined as $S_t = u - R_t$. In this work, we focus on the evaluation of ultimate ruin probabilities (or infinite-time ruin probabilities) defined as

$$\psi(u) = P\left(\inf_{t\geq 0} R_t < 0 | R_0 = u\right) = P\left(\sup_{t\geq 0} S_t > u | S_0 = 0\right). \tag{1.1}$$

This model is called a compound Poisson model (also known as Cramer-Lundberg ruin model) and has been widely studied in the risk theory literature. For general background about ruin theory, we refer to [16], and [4].

There are several usual techniques for calculation of ultimate ruin probabilities. First, iterative methods with the so called Panjer's algorithm derived in [14] and applied to the ultimate ruin probability computation in [8]. Then, we have numerical inversion of the Laplace transform used for probability distributions recovery. In a few particular cases, the inversion is manageable analytically and leads to closed formula but in most cases numerical methods are needed. The numerical inversion via Fourier-series techniques (Fast Fourier Transform) received a great deal of interest. These techniques have been presented in [2] in a queuing theory setting. For an application within the actuarial framework, we refer to [10]. Recently, inversion techniques via the scaled Laplace transform and exponential moments recovery has been performed in [12] for ruin probabilities computations. We also mention the numerical inversion of Laplace transform using Laguerre method described in [19] and [1]. The recovered function takes the form of a weighted sum of Laguerre functions derived though orthogonal projections, it can be viewed as both a polynomials and a gamma series expansion that have been commonly used in the actuarial litterature. The expansion of probability density function as a sum of gamma densities with actuarial applications has been first proposed in [7] and gave rise to the so-called Beekman-Bowers approximation for the ultimate ruin probability, derived in [6]. The idea is to approximate the ultimate ruin probability by the survival function of a gamma distribution using moments fitting. Gamma series expansion has been employed in [18] and later in [3]. The authors highlight that it is useful to carry out both analytical calculations and numerical approximations. They focus on the finite-time ruin probability, injecting directly the gamma series expression into integro-differential equations leading to reccurence relations between the expansion's coefficients. The results are valid in the infinite-time case by letting the time t tend to infinity. In addition, we can mention [15], in which ruin probabilities expressions are derived using generalized Appell polynomials.

Our method is an expansion using orthogonal polynomials and can be viewed as an extension of [7], based on Laguerre's polynomials. It can also be related to gamma series expansions and numerical inversions of Laplace transform. In this paper, we provide a new way to both construct and justify expansions with orthogonal polynomials that leads to an approximation with good numerical behaviour. Our results rely on some properties of orthogonal polynomials with respect to probability measures in NEF-QVF. From a computational point of view, no discretization of the claim sizes distribution is needed and the coefficients that require a large part of the CPU time are the same for any value of u. Moreover, the accuracy is not much sensitive to initial reserves, even for large value.

In Section 2, we introduce a density expansion formula based on orthogonal projection within the frame of NEF-QVF. Our main results are developped in Section 3: the expansion for ultimate ruin probabilities is derived and a sufficient condition of applicability is given. Section 4 is devoted to numerical illustrations.

2 Polynomial expansions of a probability density function

Let $F = \{P_\theta, \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}$ be a Natural Exponential Family (NEF), see [5], generated by a probability measure ν on $\mathbb{R}$ such that

$$\begin{aligned} P_\theta(X \in A) &= \int_A exp\{x\theta - \kappa(\theta)\} d\nu(x) \\ &= \int_A f(x, \theta) d\nu(x), \end{aligned}$$

where

- $A \subset \mathbb{R}$,
- $\kappa(\theta) = log\left(\int_{\mathbb{R}} e^{\theta x} d\nu(x)\right)$ is the Cumulant Generating Function (CGF),
- $f(x, \theta)$ is the density of P_θ with respect to ν.

Let X be a random variable P_θ distributed. We have

$$\begin{array}{ccccccc} \mu & = & \mathrm{E}_\theta(X) & = & \int x dF_\theta(x) & = & \kappa'(\theta), \\ \mathrm{V}(\mu) & = & \mathrm{Var}_\theta(X) & = & \int (x-\mu)^2 dF_\theta(x) & = & \kappa''(\theta). \end{array}$$

The application $\theta \to \kappa'(\theta)$ is one to one. Its inverse function $\mu \to h(\mu)$ is defined on $\mathcal{M}$ $= \kappa'(\Theta)$. With a slight change of notation, we can rewrite $F = \{P_\mu, \mu \in \mathcal{M}\}$, where P_μ has mean μ and density $f(x, \mu) = exp\{h(\mu)x - \kappa(h(\mu))\}$ with respect to ν. A NEF has a Quadratic Variance Function (QVF) if there exists reals v_0, v_1, v_2 such that

$$V(\mu) = v_0 + v_1\mu + v_2\mu^2. \tag{2.1}$$

The Natural Exponential Families with Quadratic Variance Function (NEF-QVF) include the normal, gamma, hyperbolic, Poisson, binomial and negative binomial distributions. Define

$$Q_n(x, \mu) = V^n(\mu) \left\{ \frac{\partial^n}{\partial \mu^n} f(x, \mu) \right\} / f(x, \mu), \tag{2.2}$$

for $n \in \mathbb{N}$. Each $Q_n(x, \mu)$ is a polynomial of degree n in both μ and x. Moreover, $\{Q_n\}_{n \in \mathbb{N}}$ is a family of orthogonal polynomials with respect to P_μ in the sense that

$$< Q_n, Q_m >= \int Q_n(x, \mu) Q_m(x, \mu) dP_\mu(x) = \delta_{nm} ||Q_n||^2, \qquad m, n \in \mathbb{N},$$

where δ_{mn} is the Kronecker symbol equal to 1 if $n = m$ and 0 otherwise. For the sake of simplicity, we choose $\nu = P_{\mu_0}$. Then, $f(x, \mu_0) = 1$ and we write

$$Q_n(x) = Q_n(x, \mu_0) = V^n(\mu_0) \left\{ \frac{\partial^n}{\partial \mu^n} f(x, \mu) \right\}_{\mu = \mu_0}. \tag{2.3}$$

For an exhaustive review regarding NEF-QVF and their properties, we refer to [13].
We will denote by $L^2(\nu)$ the space of functions square integrable with respect to ν.

Proposition 1. *Let $\nu = P_{\mu_0}$ be a probability measure that generates a NEF-QVF, with associated orthogonal polynomials $\{Q_n, n \in \mathbb{N}\}$ given by (2.3). Let X be a random variable with density function $\frac{dP_X}{d\nu}$ with respect to ν. If $\frac{dP_X}{d\nu} \in L^2(\nu)$ then we have the following expansion*

$$\frac{dP_X}{d\nu}(x) = \sum_{n=0}^{+\infty} E(Q_n(X)) \frac{Q_n(x)}{||Q_n||^2}. \tag{2.4}$$

Proof. By construction $\{Q_n\}_{n\in\mathbb{N}}$ forms an orthogonal basis of $L^2(\nu)$, and by orthogonal projection we get

$$\frac{dP_X}{d\nu}(x) \;=\; \sum_{n=0}^{+\infty} < \frac{Q_n}{||Q_n||}, \frac{dP_X}{d\nu} > \frac{Q_n(x)}{||Q_n||}.$$

It follows that

$$\begin{aligned} < \frac{Q_n}{||Q_n||}, \frac{dP_X}{d\nu} > \frac{Q_n(x)}{||Q_n||} &= \int \frac{Q_n(y)}{||Q_n||}\frac{dP_X}{d\nu}(y)d\nu(y) \times \frac{Q_n(x)}{||Q_n||} \\ &= \int Q_n(y)dP_X(y) \times \frac{Q_n(x)}{||Q_n||^2} \\ &= E(Q_n(X))\frac{Q_n(x)}{||Q_n||^2}. \end{aligned}$$

□

3 Application to the ruin problem

3.1 General formula

The ultimate ruin probability in the Cramer Lundberg ruin model is the survival function of a geometric compound distributed random variable

$$M = \sum_{i=1}^{N} U_i^I,$$

where

- N is an integer valued random variable having a geometric distribution with parameter $\rho = \frac{\beta\mu}{p}$,
- $(U_i^I)_{i\in\mathbb{N}^*}$ is a sequence of independent and identically distributed nonnegative random variables having CDF $F_{U^I}(x) = \frac{1}{\mu}\int_0^x \overline{F_U}(y)dy$.

The distribution of M has an atom at 0 with probability mass $P(N = 0) = 1 - \rho$. The probability measure governing the values of M is

$$dP_M(x) = (1-\rho)\,\delta_0(x) + dG_M(x), \tag{3.1}$$

where dG_M is the continuous part of the probability measure associated to M which admits a defective probability density function with respect to the Lebesgue measure. The ultimate ruin probability is then obtained by integrating the continuous part as the discrete part vanishes

$$\psi(u) = P(M > u) = \int_u^{+\infty} dG_M(x).$$

Theorem 1. *Let ν be an univariate distribution having a probability density function with respect to the Lebesgue measure, and that generates a NEF-QVF. If $\frac{dG_M}{d\nu} \in L^2(\nu)$ then*

$$\psi(u) = \sum_{n=0}^{+\infty} V(\mu_0)^n \left\{\frac{\partial^n}{\partial\mu^n} e^{-\kappa(h(\mu))}\widehat{G_M}(h(\mu))\right\}_{\mu=\mu_0} \frac{\int_u^{+\infty} Q_n(x)d\nu(x)}{||Q_n||^2}, \tag{3.2}$$

where $\widehat{G_M}$ is the Laplace-Stieltjes Transform of G_M defined by $\widehat{G_M}(s) = \int e^{sx}dG_M(x)$.

Proof. We start by applying Proposition 1 to $\frac{dG_M}{d\nu}$ which leads to

$$\begin{aligned}\frac{dG_M}{d\nu}(x) &= \sum_{n=0}^{+\infty} < \frac{Q_n}{||Q_n||}, \frac{dG_M}{d\nu} > \frac{Q_n(x)}{||Q_n||} &(3.3)\\ &= \sum_{n=0}^{+\infty} \int Q_n(x) dG_M(x) \frac{Q_n(x)}{||Q_n||^2}. &(3.4)\end{aligned}$$

By the definition of $Q_n(x)$ as defined in (2.3), we obtain

$$\begin{aligned}\frac{dG_M}{d\nu}(x) &= \sum_{n=0}^{+\infty} V^n(\mu_0) \left\{ \frac{\partial^n}{\partial \mu^n} e^{-\kappa(h(\mu))} \int e^{h(\mu)x} dG_M(x) \right\}_{\mu=\mu_0} \frac{Q_n(x)}{||Q_n||^2} &(3.5)\\ &= \sum_{n=0}^{+\infty} V^n(\mu_0) \left\{ \frac{\partial^n}{\partial \mu^n} e^{-\kappa(h(\mu))} \widehat{G_M}(h(\mu)) \right\}_{\mu=\mu_0} \frac{Q_n(x)}{||Q_n||^2}. &(3.6)\end{aligned}$$

Integration of (3.6) between u and $+\infty$ gives the expression (3.2) for the ultimate ruin probability. □

Remark 1. *Equation* (3.2) *involves the Laplace-Stieljes transform of* G_M. *The presented method could also be related to Laplace transform inversion techniques.*

3.2 Approximation with Laguerre polynomials

We derive an approximation for the ultimate ruin probability, using Theorem 1, combined with truncations of the infinite series (3.2). For $K \in \mathbb{N}$, we will denote by

$$\psi_K(u) = \sum_{n=0}^{K} V^n(\mu_0) \left\{ \frac{\partial^n}{\partial \mu^n} e^{-\kappa(h(\mu))} \widehat{G_M}(h(\mu)) \right\}_{\mu=\mu_0} \frac{\int_u^{+\infty} Q_n(x) d\nu(x)}{||Q_n||^2} \tag{3.7}$$

the approximated ruin probability with truncation order K.

Remark 2. *We can write* (3.7) *as*

$$\psi_K(u) = \sum_{n=0}^{K} a_n \frac{\int_u^{+\infty} Q_n(x) d\nu(x)}{||Q_n||^2},$$

where a_n *are independent of* u. *Once the evaluation of the* a_n *for all* $n \leq K$ *is done, estimating the ruin probability requires one integral calculation.*

In practice, as the distribution of M is supported on $\mathbb{R}^+$, we will choose the exponential distribution with parameter ξ for ν, that is:

$$d\nu(x) = \xi e^{-\xi x} \mathbf{1}_{\mathbb{R}^+}(x) d\lambda(x).$$

The associated orthogonal polynomials are the Laguerre ones ,see [17], satisfying

$$\int_0^{+\infty} L_n(x) L_m(x) e^{-x} dx = \delta_{nm}.$$

The polynomials defined in (2.3) are the Laguerre polynomials with a slight change in comparison to the definition given in [17]: $Q_n(x) = \left(-\frac{1}{\xi}\right)^n n! L_n(\xi x)$ and their norm is

$||Q_n|| = \frac{n!}{\xi^n}$. As ν is the exponential probability measure with parameter ξ, the mean of ν is $\mu_0 = \frac{1}{\xi}$, the variance function is $V(\mu_0) = \frac{1}{\xi^2}$, the cumulant generating function is $\kappa(\theta) = log\left(\frac{\xi}{\xi-\theta}\right)$ and the inverse function of the first derivative of $\kappa(.)$ is $h(\mu) = \frac{\xi\mu-1}{\mu}$. We can write the expression of the ruin probability (3.2) in a more tractable way, that is:

$$\psi(u) = \sum_{n=0}^{+\infty}\left\{\frac{\partial^n}{\partial\mu^n}\frac{1}{\xi\mu}\widehat{G_M}\left(\frac{\xi\mu-1}{\mu}\right)\right\}_{\mu=\mu_0}\frac{(-\xi)^n}{n!}\int_u^{+\infty}L_n(\xi x)d\nu(x). \tag{3.8}$$

Remark 3. *The choice of ν is arbitrary in the sense that we could have chosen a more general gamma distribution $d\nu(x) = \frac{\xi^\alpha x^{\alpha-1}e^{-\xi x}}{\alpha)}\mathbf{1}_{\mathbb{R}^+}(x)$. The orthogonal polynomials would have been the generalized Laguerre polynomials and, with $\xi = 1$, our expansion would have been the same as in [7]. By taking a gamma distribution with its two first moments equal to those of M, the first term of the obtained expansion gives the Beekman-Bowers approximation. The use of a Normal distribution would have implied an expansion involving Hermite polynomials, but it seems less intuitive to approximate a probability density function supported on $\mathbb{R}^+$ by a sum of probability density function supported on $\mathbb{R}$.*

Remark 4. *Laguerre polynomials analytical expression is*

$$L_n(x) = \sum_{k=0}^{n}\binom{n}{n-k}\frac{(-x)^k}{k!}. \tag{3.9}$$

Denoting by $a_n = \left\{\frac{\partial^n}{\partial\mu^n}\frac{1}{\xi\mu}\widehat{G_M}\left(\frac{\xi\mu-1}{\mu}\right)\right\}_{\mu=\mu_0}\frac{(-\xi)^n}{n!}$ for $n \in \mathbb{N}$, the injection of Laguerre polynomials expression (3.9) *into the ruin probability expansion* (3.8) *gives*

$$\begin{aligned}\psi(u) &= \sum_{n=0}^{+\infty}a_n\sum_{k=0}^{n}\binom{n}{n-k}\int_u^{+\infty}\frac{\xi^{k+1}x^ke^{-\xi x}}{k!}dx\\ &= \sum_{k=1}^{+\infty}\sum_{n=k-1}^{+\infty}a_n\binom{n}{n-k+1}\int_u^{+\infty}\frac{\xi^kx^{k-1}e^{-\xi x}}{\Gamma(k)}dx\\ &= \sum_{k=1}^{+\infty}b_k\int_u^{+\infty}\frac{\xi^kx^{k-1}e^{-\xi x}}{\Gamma(k)}dx.\end{aligned} \tag{3.10}$$

The right hand side of (3.10) *is exactly a gamma series expansion as defined in [18].*

The defective probability density function associated to G_M has the following expression

$$g_M(x) = \sum_{n=1}^{+\infty}(1-\rho)\rho^n f_{U^I}^{*n}(x). \tag{3.11}$$

Remark 5. *The Laguerre functions are defined in [1] as*

$$l_n(x) = e^{-x/2}L_n(x),\ x \geq 0. \tag{3.12}$$

The application of the Laguerre method consists in representing g_M as a Laguerre serie

$$g_M(x) = \sum_{n=0}^{+\infty}q_nl_n(t). \tag{3.13}$$

One can note that the representation (3.13) *is really close to the expansion proposed in this paper.*

By Taking the Laplace transform of (3.11), we get

$$\widehat{G_M}(s) = \frac{(1-\rho)\rho\widehat{F_{U^I}}(s)}{1-\rho\widehat{F_{U^I}}(s)}, \tag{3.14}$$

with $\widehat{F_{U^I}}(s) = \int e^{sx} f_{U^I}(x)dx$ the Laplace Stieljes transform of F_{U^I}. The moment generating function of the claim size distribution appears in the formula. This fact limits the application to claim sizes distributions that admit a well defined moment generating function, namely light-tailed distributions.

3.3 Integrability condition

There exists a link between the choice of ξ and the adjustment coefficient γ. The adjustment coefficient is the only positive solution of the so-called Cramer-Lundberg equation,

$$\widehat{F_{U^I}}(s) = \frac{1}{\rho}. \tag{3.15}$$

The integrability condition $\frac{dG_M}{d\nu} \in L^2(d\nu)$ is equivalent to

$$\int_0^{+\infty} g_M(x)^2 e^{\xi x} dx < \infty. \tag{3.16}$$

In order to ensure this condition, we need the following results.

Theorem 2. *Assume that U^I admits a bounded density function and that the equation* (3.15) *admits a positive solution, then for all $x \geq 0$*

$$g_M(x) \leq C(s_0)e^{-s_0 x}, \tag{3.17}$$

with $s_0 \in [0, \gamma)$ and $C(s_0) \geq 0$, where γ is the adjustment coefficient.

Proof. In order to prove the theorem we need the following lemma regarding the survival function $\overline{F_U}$ of the claim sizes distribution.

Lemma 1. *Let U be a non-negative random variable with bounded density function f_U. Assume there exists $s_0 > 0$ such that $\widehat{F_U}(s_0) < +\infty$. Then there exists $A(s_0) > 0$ such that for all $x \geq 0$*

$$\overline{F_U}(x) \leq A(s_0)e^{-s_0 x}. \tag{3.18}$$

Proof. As $\widehat{F_U}(s_0) < +\infty$, we have

$$\begin{aligned}
\widehat{F_U}(s_0) - 1 &= \int_0^{+\infty} (e^{s_0 x} - 1) f_U(x)dx \\
&= s_0 \int_0^{+\infty} \int_0^{x} e^{s_0 y} f_U(x)dydx \\
&= s_0 \int_0^{+\infty} e^{s_0 y}\overline{F_U}(y)dy \\
&\geq s_0 \int_0^{x} e^{s_0 y}\overline{F_U}(y)dy \\
&\geq \overline{F_U}(x)(e^{s_0 x} - 1).
\end{aligned}$$

thus, we deduce that $\forall x \geq 0$

$$\overline{F_U}(x) \leq (\widehat{F_U}(s_0) - 1 + \overline{F_U}(x))e^{-s_0 x}. \tag{3.19}$$

□

The equation (3.15) is equivalent to

$$\rho\widehat{F_U}(s) = 1 + s\mu. \tag{3.20}$$

The fact that γ is a solution of the equation (3.15) implies that $\widehat{F_U}(s) < +\infty$, $\forall s_0 \in [0, \gamma)$ and by application of Lemma 1, we get the following inequality upon the density function of U^I

$$f_{U^I}(x) = \frac{\overline{F_U}(x)}{\mu} \leq B(s_0)e^{-s_0 x}. \tag{3.21}$$

In view of (3.11), it is easily checked that g_M satisfies the following defective renewal equation,

$$g_M(x) = \rho(1-\rho)f_{U^I}(x) + \rho\int_0^x f_{U^I}(x-y)g_M(y)dy. \tag{3.22}$$

We can therefore bound g_M as in (3.17),

$$\begin{aligned}
g_M(x) &\leq \rho(1-\rho)f_{U^I}(x) + \int_0^{+\infty} f_{U^I}(x-y)g_M(y)dy \\
&\leq \rho(1-\rho)B(s_0)e^{-s_0 x} + B(s_0)e^{-s_0 x}\int_0^{+\infty} e^{s_0 y}g_M(y)dy \\
&= (\rho(1-\rho) + \widehat{G_M}(s_0))B(s_0)e^{-s_0 x} \\
&= C(s_0)e^{-s_0 x}.
\end{aligned}$$

□

The application of Theorem 2 yields a sufficient condition in order to use the polynomial expansion.

Corollary 1. *For $\xi < 2\gamma$, the integrability condition* (3.16) *is satisfied.*

We note the importance of the choice of the parameter ξ. The Laguerre method, briefly described in Remark 5, does not offer the possibility of changing some parameter. The expansion is still based on orthogonal projection permitted under an integrability condition. However, if the function does not satisfy the integrability condition then a damped version of it is expanded.

4 Numerical illustrations

First, we analyse the convergence of the sum in our method toward known exact values of ruin probabilities with exponential, gamma and phase-type cases. For those claim sizes distribution we have explicit formulas that allow us to assess the acuracy of our approximated ruin probabilities. The goodness of the approximation depends on the order of truncation K, and results show also a dependence on ξ. Our method also enables us to approximate ruin probabilities in cases that are relevant for applications but where no formulas are currently available. We compare the results with Monte-Carlo simulations and discuss the interest of our method in comparison to the widely used Panjer's algorithm. First, we plot the difference between the exact ruin probability value and its approximation

$$\Delta\psi(u) = \psi(u) - \psi_K(u), \tag{4.1}$$

then we study the behaviour of our approximation when changing the value of ξ and finally we compare our approximations with the ones of Panjer's algorithm. In order to use

Panjer's algorithm, we need to discretize the integrated tail distribution of the severities and choose a bandwidth h. The Rounded Method is employed to do the discretization as it seems to be the best way according to [9]. It consists in rounding the severities to the closest integer multiple of h. We choose a bandwith arbitrary equal to 0.01 as there is no tractable formula available for the discretization error.
To produce simulations of the compound geometric sum, we use the procedure described in [11]. An iterative method is given to simulate random values from integrated tail distribution. The number of simulations needed, 10 000, and the iterative component in the simulation procedure imply a significant CPU time that already justify the use of numerical techniques. Confidence intervals are given in addition to the estimation.
Regarding the ruin model settings, we fix a safety loading at 20%.

4.1 Exponentially distributed claim sizes

In the case of exponentially distributed claim sizes with parameter δ, the ultimate ruin probability is

$$\psi(u) = \rho e^{-\delta(1-\rho)u}, \tag{4.2}$$

where $\rho = \beta/\delta p$, β is the Poisson process intensity, p the premium rate and u is the initial reserves. In this particular case, results (3.2) or (3.8) can be used as a tool for computations. After some tedious calculus, we get

$$\psi(u) = \int_u^{+\infty} \rho \sum_{n=0}^{+\infty} \left(\frac{\delta(1-\rho) - \xi}{\delta(1-\rho)} \right)^n L_n(\xi x) d\nu(x).$$

We use a property upon the generating function of Laguerre polynomials, $\sum_{n=0}^{+\infty} w^n L_n(x) = (1-w)^{-1} exp\left(-\frac{xw}{1-w}\right)$, which gives after straightforward integration

$$\psi(u) = \rho e^{-\delta(1-\rho)u}. \tag{4.3}$$

For numerical illustrations, we set $\delta = 1$. Results are displayed in Figure 1 and Table 1.

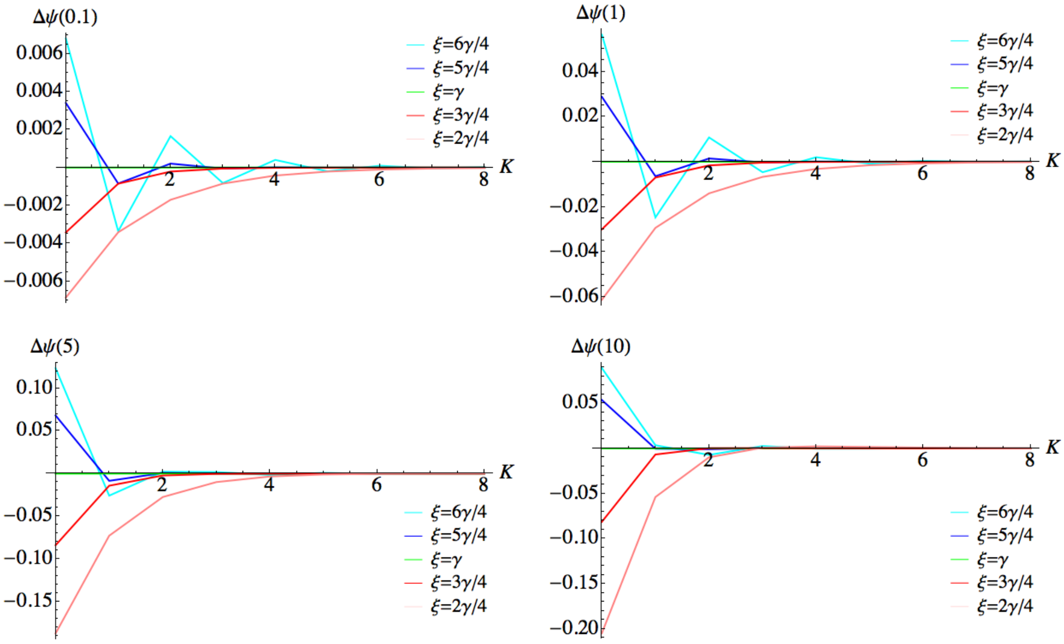

Figure 1: Difference between exact and approximated ruin probabilities for exponentially distributed claim sizes

		K				
		0	5	10	15	20
ξ	$6\gamma/4$	0.0564	-0.00085	0.00001	0.	0.
	$5\gamma/4$	0.02879	-0.00002	0.	0.	0.
	γ	0.	0.	0.	0.	0.
	$3\gamma/4$	-0.03001	-0.00002	0.	0.	0.
	$2\gamma/4$	-0.0613	-0.00153	-0.00004	0.	0.

Table 1: Difference between exact and approximated ruin probabilities for exponentially distributed claim sizes, with $u = 1$

4.2 Phase-type distributed claim sizes

A phase-type distribution is the distribution of the absorbtion time of some continuous-time absorbing Markov process with a finite states space. Many common distributions are of phase-type, for instance exponential, hyperexponential or Erlang distributions admit a phase-type representation. The exact ruin probability is then given in the form of a Matrix-Exponential, see the book [4] for details. The entire chapter VIII is dedicated to phase-type distributions. In this second example, we assume that the claim sizes distribution is a mixture of two Erlang distributions. The associated density function is

$$f(x) = qErlang(k_1, \delta_1) + (1-q)Erlang(k_2, \delta_2), \tag{4.4}$$

where $Erlang(k, \delta) = \frac{\delta^k x^{k-1} e^{-\delta x}}{(n-1)!}\mathbf{1}_{\mathbb{R}^+}$ and $q \in [0, 1]$. We set $k_1 = 3$, $k_2 = 2$, $\delta_1 = 1$, $\delta_2 = 2/3$, and $q = 2/5$. Results are displayed in Figure 2, and Tables 2, 3.

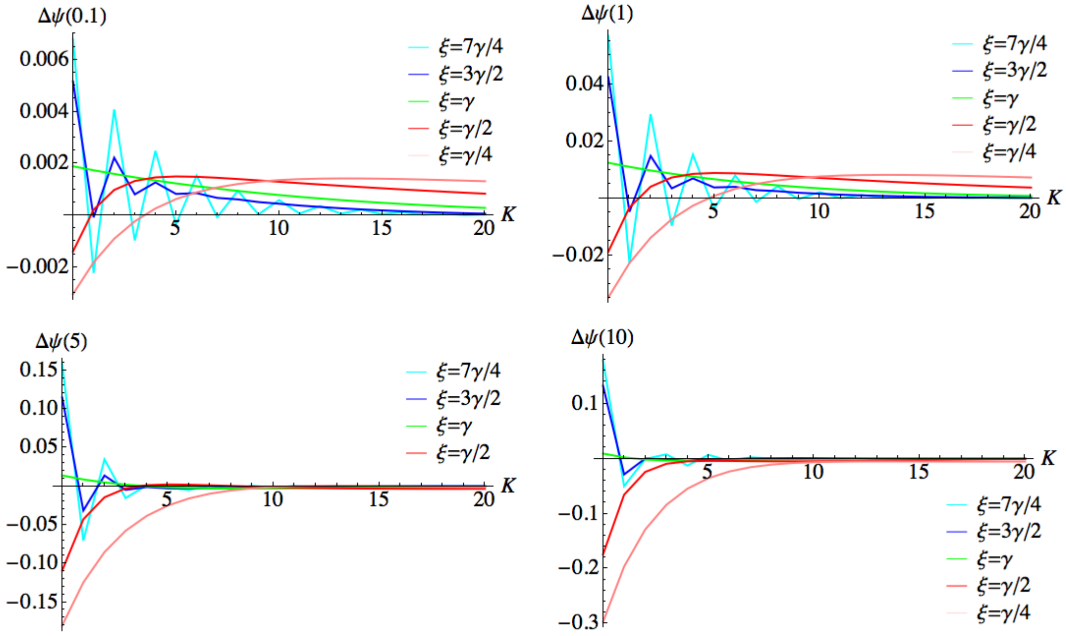

Figure 2: Difference between exact and approximated ruin probabilities for phase-type distributed claim sizes

		K					
		0	20	40	60	80	100
ξ	$7\gamma/4$	0.05686	0.00007	0.00002	0.	0.	0.
	$3\gamma/2$	0.04233	0.00013	0.00003	0.00001	0.	0.
	γ	0.0124	0.0008	0.00004	0.00003	0.00002	0.00001
	$\gamma/2$	-0.01874	0.00372	0.00088	0.00016	0.00004	0.00003
	$\gamma/4$	-0.03478	0.00725	0.00387	0.00194	0.00092	0.00041

Table 2: Difference between exact and approximated ruin probabilities for phase-type distributed claim sizes for phase-type distributed claim sizes, with $u = 1$

u	Exact Value	Polynomials expansion $\xi = \gamma$, K=120	Panjer's algorithm h=0.01
0.1	0.828641	0.828642	0.828647
1	0.782188	0.782187	0.782266
5	0.574502	0.574504	0.574904
10	0.386405	0.386406	0.386987
50	0.016181	0.016181	0.016310

Table 3: Ruin probabilities for phase-type distributed claims amounts approximated with polynomials expansions and Panjer's algorithm

4.3 Gamma distributed claim sizes

We assume that the claim sizes are gamma distributed with a scale parameter which is not integer. The exact form of the ruin probability has been derived in [2] for the $\Gamma(1/2, 1/2)$ special case, numerical results are displayed in Figure 3 and Tables 4, 5. We finally compare approximations for the $\Gamma(1/3, 1)$ case to results obtained though Monte Carlo simulations, see Figure 4 and Table 6.

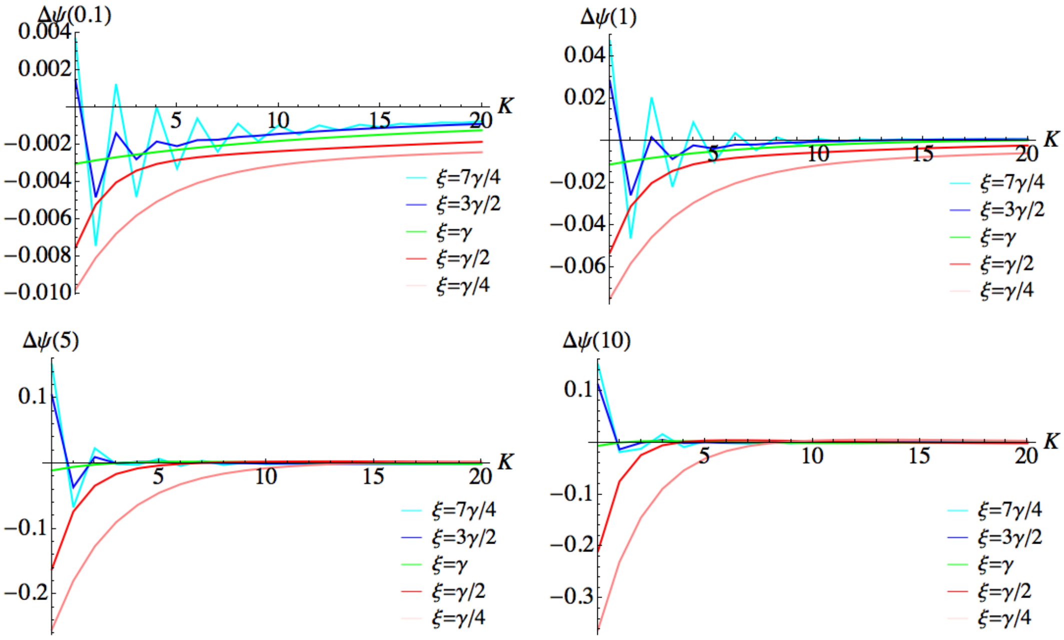

Figure 3: Difference between exact and approximated ruin probabilities for $\Gamma(1/2, 1/2)$ distributed claim sizes

		K						
		0	20	40	60	80	100	120
ξ	$7\gamma/4$	0.04726	0.00061	0.00009	-0.00012	-0.00009	-0.00001	0.00003
	$3\gamma/2$	0.02826	0.00062	0.00023	-0.00008	-0.00012	-0.00007	0.
	γ	-0.01131	0.00016	0.0006	0.00025	-0.00001	-0.00011	-0.00012
	$\gamma/2$	-0.0531	-0.00231	0.00013	0.00066	0.00063	0.00045	0.00026
	$\gamma/4$	-0.07486	-0.00594	-0.00242	-0.00073	0.00012	0.00052	0.00067

Table 4: Difference between exact and approximated ruin probabilities for $\Gamma(1/2, 1/2)$ distributed claim sizes, and $u = 1$

u	Exact Value	Polynomials expansion $\xi = \gamma$, K=120	Panjer's algorithm h=0.01
0.1	0.821313	0.821424	0.821356
1	0.736114	0.736238	0.736395
5	0.47301	0.472944	0.473757
10	0.274299	0.274252	0.275131
50	0.00352109	0.00352476	0.00357292

Table 5: Ruin probabilities for $\Gamma(1/2, 1/2)$ distributed claims amounts approximated with polynomials expansion and Panjer's algorithm

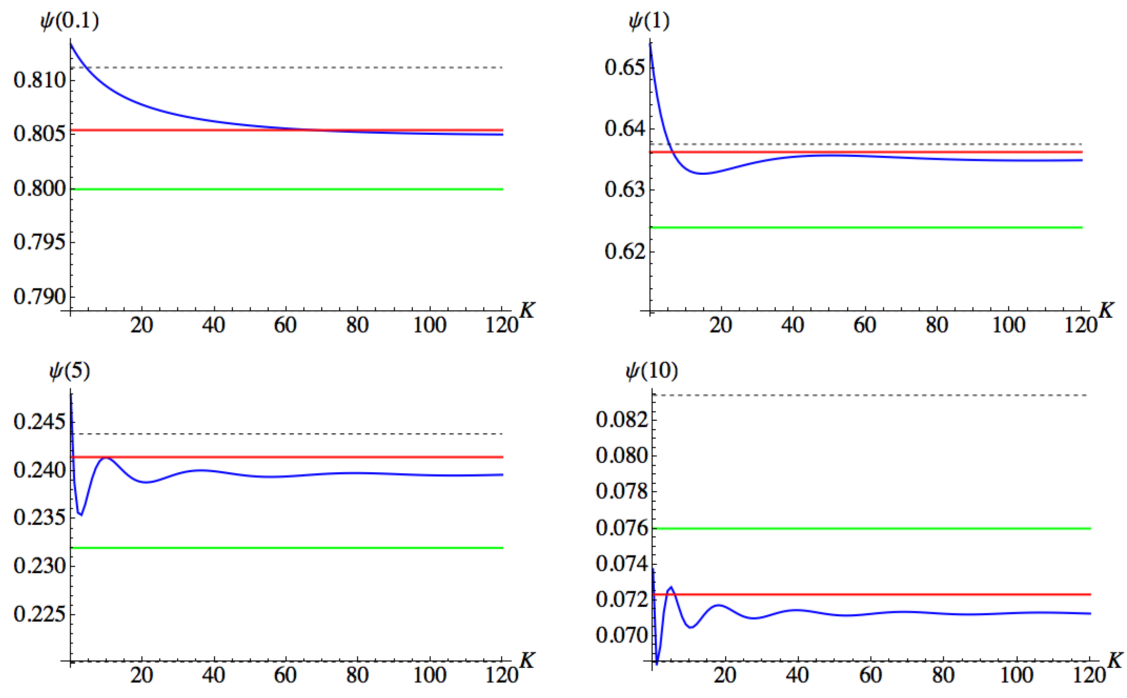

Figure 4: Ruin probabilities for $\Gamma(1/3, 1)$ distributed claim sizes approximated with polynomials expansion (blue line), Panjer's algorithm (red line), and Monte-Carlo simulations: estimation (green line) and 99% confidence interval (black dashed line)

u	Monte-Carlo simulations	Polynomials expansion $\xi = \gamma$, K=120	Panjer's algorithm h=0.01
0.1	0.8	0.80505	0.805454
1	0.624	0.634979	0.636315
5	0.232	0.239601	0.241442
10	0.076	0.0712518	0.0723159
50	0	4.569555×10^{-6}	4.686×10^{-6}

Table 6: Ruin probabilities for $\Gamma(1/3, 1)$ distributed claims amounts estimated with polynomials expansion and Panjer's algorithm

4.4 Discussion of the numerical results

- Approximations seem to behave very well for every value of the initial reserves. We did not put the results here, but when ξ is chosen greater than 2γ approximations seems to diverge quickly.

- The order of truncation needed to reach a certain level of accuracy depends on the complexity of the claim sizes distribution. This fact is clearly observed through simulations within the $\Gamma(1/2, 1/2)$ case, in which a greater order a truncation is needed to reach an equal level of accuracy.

- In the exponential case, there is a symmetric pattern. ξ equal to γ is the optimal choice in terms of order of truncation needed. In the other cases studied, there might exist an optimal choice for ξ in the range $[\gamma, 2\gamma)$.

- Panjer's algorithm performs well for small initial reserves, sometimes better than our method. But the ruin probability approximation for large initial reserves, that are relevant for applications, is problematical in our opinion because the computation time is clearly increasing and the accuracy is worsening. Our method produces, in a reasonable time, an acceptable approximated ruin probability for every value of u.

5 Conclusion

Our proposed method provides a very good approximation of the ruin probability when the claim sizes distribution is light-tailed. We obtained a theoretical result that allows us to ensure the validity of our expansions. In addition, the repeatability of the coefficients a_n (see Remark 2) makes our method very fast when changing the initial reserves u, which makes it very convenient compared to Panjer's algorithm. The problem we dealt with is the approximation of a geometric compound distribution density function. The results are quite promising and allow us to envisage an extension to more general compound distributions or even finite-time horizon ruin probabilities. In forthcoming research work, we would like to study the possibility of a statistical extension when data are available. The explicit formula obtained involves quantities that can be estimated empirically and plugged in.

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