Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption

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Abstract

We obtain necessary conditions expressed in terms of Wiener type tests involving Hausdorff or Bessel capacities for the existence of large solutions to equations (1) $-\Delta_p u + e^u - 1 = 0$ or (2) $-\Delta_p u + u^q = 0$ in a bouded domain Ω when $q > p - 1 > 0$. We apply our results to equations (3) $-\Delta_p u + |\nabla u|^q + bu^{p-1} = 0$, (4) $-\Delta u + |\nabla u|^2 + u^q = 0$ and $-\Delta u + u^{-q} = 0$ with $q > 0$.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ and $1 < p \leq N$. We consider the question of existence of solutions to the problem

$$\begin{aligned} -\Delta_p u + g(u) &= 0 \qquad \text{in } \Omega \\ \lim_{\rho(x)\to 0} u(x) &= \infty \end{aligned} \tag{1.1}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $\rho(x) = \text{dist}\,(x, \partial\Omega)$ and g is a continuous nondecreasing function vanishing at 0; most often $g(u)$ is either $e^u - 1$ or $|u|^{q-1}u$ with $q > p - 1$. A solution to problem (1.1) is called a *large solution.* When the domain is regular in the sense that the Dirichlet problem with continuous boundary data ϕ

$$\begin{aligned} -\Delta_p u + g(u) &= 0 \qquad \text{in } \Omega \\ u &= \phi \qquad \text{in } \partial\Omega \end{aligned} \tag{1.2}$$

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admits a solution, it is clear that problem (1.1) admits a solution. It is known that a necessary and sufficient condition for such a result is the so called Wiener criterion (for $p = 2$ see [17]), for $p \neq 2$ see [10], [6])

$$\int_0^1 \left(\frac{C_{1,p}(B_t(x) \cap \Omega^c)}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty \qquad \forall x \in \partial\Omega, \tag{1.3}$$

where $C_{1,p}$ denotes the capacity associated to the space $W^{1,p}(\mathbb{R}^N)$: the existence of a large solution is guaranted for a large class of nondecreasing nonlinearities g satisfying the Vazquez condition [14]

$$\int_a^\infty \frac{dt}{G^{\frac{1}{p}}(t)} < \infty \qquad G(t) = \int_0^t g(s)ds, \tag{1.4}$$

a variant of the Keller-Osserman estimate [7], [12], which is the above relation when $p = 2$. If for $R > diam(\Omega)$ there exists a function v which satisfies

$$\begin{array}{rl} -\Delta_p v + g(v) = 0 & \text{in } B_R \setminus \{0\} \\ v = 0 & \text{on } \partial B_R \\ \lim_{x \to 0} v(x) = \infty, & \end{array} \tag{1.5}$$

then it is easy to see that the maximal solution of

$$-\Delta_p u + g(u) = 0 \qquad \text{in } \Omega \tag{1.6}$$

is a large solution, without any assumption on the regularity of $\partial\Omega$, provided (1.4) is satisfied. However the existence of a (radial) solution to problem (1.5) needs the fact that equation (1.6) admits solutions with isolated singularities, which is usually not true if the growth of g is too strong since Vazquez and Véron proved [15] that if

$$\liminf_{|r| \to \infty} |r|^{-\frac{N(p-1)}{N-p}} \, sign(r) g(r) > 0 \tag{1.7}$$

isolated singularities of solutions of (1.6) are removable. Conversely, if $p - 1 < q < \frac{N(p-1)}{N-p}$, Friedman and Véron [4] characterize the behavior of positive singular solutions to

$$-\Delta_p u + u^q = 0 \tag{1.8}$$

with an isolated singularities. In 2003, Labutin [8] proved that a necessary and sufficient condition in order the following problem be solvable

$$\begin{array}{rl} -\Delta u + |u|^{q-1} u = 0 & \text{in } \Omega \\ \lim_{\rho(x) \to 0} u(x) = \infty & \end{array} \tag{1.9}$$

is that

$$\int_0^1 \frac{C_{2,q'}(B_t(x) \cap \Omega^c)}{t^{N-2}} \frac{dt}{t} = \infty \qquad \forall x \in \partial\Omega, \tag{1.10}$$

where $C_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$ and $q' = q/(q-1)$. Notice that this condition is always satisfied if q is subcritical, i.e. $q < N/(N-2)$. Concerning the exponential case of problem (1.1) nothing is known, even in the case $p = 2$, besides the simple cases already mentionned.

In this article we give sufficient conditions, expressed in terms of Wiener tests, in order problem (1.1) be solvable in the two cases $g(u) = e^u - 1$ and $g(u) = |u|^{q-1} u$, $q > p-1$. For $1 < p < N$, we denote by $\mathcal{H}_1^{N-p}(E)$ the Hausdorff capacity of a set E defined by

$$\mathcal{H}_1^{N-p}(E) = \inf \left\{ \sum_j h^{N-p}(B_j) : E \subset \bigcup B_j,\, diam(B_j) \leq 1 \right\} \tag{1.11}$$

where the B_j are balls and $h^{N-p}(B_r) = c_N r^{N-p}$. Our main result concerning the exponential case is the following

Theorem 1. *Let $N \geq 2$ and $1 < p < N$. If*

$$\int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \quad \forall x \in \partial\Omega, \tag{1.12}$$

then there exists $u \in C^1_{loc}(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + e^u - 1 = 0 \qquad &\textit{in } \Omega \\ \lim_{\rho(x)\to 0} u(x) = \infty & \end{aligned} \tag{1.13}$$

As a consequence we obtained a sufficient condition for the existence of a large solution in the power case expressed in terms of some $C_{s,r}$ Bessel capacity in $\mathbb{R}^N$ associated to the Besov space $B^{s,r}(\mathbb{R}^N)$.

Theorem 2. *Let $N \geq 2$, $1 < p < N$ and $q_1 > \frac{N(p-1)}{N-p}$. If*

$$\int_0^1 \left(\frac{C_{p,\frac{q_1}{q_1-p+1}}(\Omega^c \cap B_r(x))}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = +\infty \qquad \forall x \in \partial\Omega, \tag{1.14}$$

then, for any $p - 1 < q < \frac{pq_1}{N}$, there exists $u \in C^1_{loc}(\Omega)$ satisfying

$$\begin{aligned} -\Delta_p u + u^q = 0 \qquad &\textit{in } \Omega \\ \lim_{\rho(x)\to 0} u(x) = \infty & \end{aligned} \tag{1.15}$$

In view of Labutin's theorem this last result is not optimal in the case $p = 2$, since the involved capacity is $C_{2,q_1'}$ with q_1' and thus there exists a solution to

$$\begin{aligned} -\Delta_p u + u^{q_1} = 0 \qquad &\text{in } \Omega \\ \lim_{\rho(x)\to 0} u(x) = \infty & \end{aligned} \tag{1.16}$$

with $q_1 > q$.

At end we apply the previous theorems to quasilinear viscous Hamilton-Jacobi equations. We prove that if $p - 1 < q < p$, $b < b^*$ for some $b^* > 0$ depending on p, q, Ω, and (1.12) holds, there exists a solution to

$$\begin{aligned} -\Delta_p u + |\nabla u|^q + bu^{p-1} = 0 \qquad &\text{in } \Omega \\ \lim_{\rho(x)\to 0} u(x) = \infty & \end{aligned} \tag{1.17}$$

Conversely, we prove that if for some $q > 1$, there exists a solution to

$$\begin{array}{ll} -\Delta u + |\nabla u|^2 + |u|^{q-1}u = 0 & \text{in } \Omega \\ \lim_{\rho(x)\to 0} u(x) = \infty, & \end{array} \tag{1.18}$$

then necessarily

$$\int_0^1 \frac{C_{2,s}\left(\Omega^c \cap B_r(x)\right)}{r^{N-2}} \frac{dr}{r} = +\infty \qquad \forall x \in \partial\Omega, \tag{1.19}$$

for all $s > 1$. This condition holds also if for some $p > 0$ there exists $u \in C(\overline{\Omega})$, $u > 0$ in Ω satisfying

$$\begin{array}{rl} -\Delta u + u^{-p} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{array} \tag{1.20}$$

2 Morrey classes and Wolff potential estimates

In this section Ω is a bounded domain in $\mathbb{R}^N$. We also denote by $B_r(x)$ the open ball of center x and radius r and $B_r = B_r(0)$. We also recall that a solution of (1.4) belongs to $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$, and is more regular (depending on g) on the set $\{x \in \Omega : |\nabla u(x)| \neq 0\}$.

Definition 2.1 *1- A function $f \in L^1(\Omega)$ belongs to the Morrey space $\mathcal{M}^s(\Omega)$, $1 \le s \le \infty$, if there is a constant K such that*

$$\int_{\Omega\cap B_r(x)} |f|dy \le K r^{\frac{N}{s'}} \qquad \forall r > 0, \forall x \in \mathbb{R}^N \tag{2.1}$$

The norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $||f||_{\mathcal{M}^s(\Omega)}$.
2- A function $f \in L^1(\Omega)$ belongs to the weak L^s-space $M^s(\Omega)$, $1 \le s \le \infty$, if there is a constant K such that

$$\int_E |f|dy \le K|E|^{\frac{1}{s'}} \qquad \forall E \subset \Omega,\ E \text{Borel}. \tag{2.2}$$

The quasi-norm is defined as the smallest constant K that satisfies this inequality; it is denoted by $||f||_{M^s(\Omega)}$

Clearly $L^p(\Omega) \subset M^p(\Omega) \subset \mathcal{M}^p(\Omega)$.

Definition 2.2 *Let $R \in (0,\infty]$ and $\mu \in \mathfrak{M}_+(\Omega)$, the set of positive Radon measures in Ω. If $\alpha > 0$ and $1 < p < \alpha^{-1}N$, we define the (R-truncated) Wolff potential of μ by*

$$\mathbf{W}_p^R[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \qquad \forall x \in \mathbb{R}^N, \tag{2.3}$$

and, for $1 < p < N$, the (R-truncated) fractional maximal potential of μ by

$$\mathbf{M}_{p,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-p}} \qquad \forall x \in \mathbb{R}^N, \tag{2.4}$$

where the measure is extended by 0 in Ω^c.

For $k \geq 0$, we set $T_k(u) = sign(u)\min\{k, |u|\}$.

Definition 2.3 *Assume $f \in L^1_{loc}(\Omega)$. We say that a Borel function u defined in Ω is a renormalized supersolution of*

$$-\Delta_p u + f = 0 \qquad \text{in } \Omega \tag{2.5}$$

if for any $k > 0$, $T_k(u) \in W^{1,p}_{loc}(\Omega)$, $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ and there holds

$$\int_\Omega (|\nabla T_k(u)|^{p-2}\nabla T_k(u)\nabla\varphi + f\varphi)dx \geq 0 \quad (resp. \ \leq 0) \tag{2.6}$$

for all $\varphi \in W^{1,p}(\Omega)$ with compact support in Ω and such that $0 \leq \varphi \leq k - T_k(u)$, and if $-\Delta_p u + f := \mu$ is a positive (resp. negative) distribution in Ω.

The following result is proved in [11, Theorem 4.35].

Theorem 2.4 *Let Ω be an open bounded domain in $\mathbb{R}^N$. If $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$ for some $\epsilon \in (0,p)$, u is a nonnegative supersolution of (2.5) and set $\mu := -\Delta_p u + f$. Then there holds*

$$u(x) + ||f||^{\frac{1}{p-1}}_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)} \geq c_1 W^{\frac{r}{4}}_{1,p}[\mu](x) \qquad \forall x \in \Omega \ \ s.t. \ B_r(x) \subset \Omega, \tag{2.7}$$

for some c_1 depending only on $N, p, \varepsilon, diam(\Omega)$.

Concerning renormalized solutions (see [3] for the definition) of

$$-\Delta_p u = f + \mu \qquad \text{in } \Omega \tag{2.8}$$

where $f \in L^1_{loc}(\Omega)$ and μ is a Radon measure, we have

Corollary 2.5 *Let $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$ and $\mu \in \mathfrak{M}^b_+(\Omega)$, the set of positive and bounded Radon measures in Ω. If u is a nonnegative renormalized solution to (2.8), then there exists a positive constant c_2 depending only on $N, p, \varepsilon, diam(\Omega)$ such that*

$$u(x) + ||f||^{\frac{1}{p-1}}_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)} \geq c_2 W^{\frac{d(x,\partial\Omega)}{4}}_{1,p}[\mu](x) \qquad \forall x \in \Omega. \tag{2.9}$$

Definition 2.6 *For $1 \leq s, q < \infty$, let $L^{s,q}(\Omega)$ denote the Lorentz space endowed with the norm*

$$\|f\|_{L^{s,q}} = \left(\int_0^\infty t^{\frac{q}{s}}(f^{**}(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{2.10}$$

where

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(t)dt$$

with $f^*(t) = \inf\{s > 0 : |\{x \in \Omega : |f(x)| > s\}| \leq t\}$. The dual space of $L^{s,q}(\Omega)$ is the space $L^{-s,q'}(\Omega)$ and it is naturally endowed with the dual norm.

The following result is proved in [2].

Theorem 2.7 *Let $f \in \mathcal{M}^{\frac{N}{p-\epsilon}}(\Omega)$ and $\mu \in \mathfrak{M}_+^b(\Omega)$. Assume that u is a nonnegative renormalized solution to equation (2.13). If $\mu \in L^{-p,\frac{q}{p-1}}$ for some $q > p-1$, then $u \in L^q(\Omega)$ and*

$$||u||_{L^q(\Omega)} \leq C\left(||\mu||_{L^{-p,\frac{q}{p-1}}(\mathbb{R}^N)} + ||f||^{\frac{1}{p-1}}_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}\right) \tag{2.11}$$

for some a positive constant C depending only on $N, p, q, \varepsilon, diam(\Omega)$.
Conversely, if $u \in L^q(\Omega)$, then for any compact set $K \subset \Omega$, there exists a positive constant C_K depending only on $N, p, q, \varepsilon, diam(\Omega)$ and $dist(K, \partial\Omega)$ such that $\chi_K \mu \in L^{-p,\frac{q}{p-1}}(\mathbb{R}^N)$ and

$$||\chi_K \mu||_{L^{-p,\frac{q}{p-1}}(\mathbb{R}^N)} \leq C_K\left(||u||_{L^q(\Omega)} + ||f||^{\frac{1}{p-1}}_{\mathcal{M}^{\frac{N}{p-\varepsilon}}(\Omega)}\right). \tag{2.12}$$

In particular, for any Borel set $E \subset \Omega$,

$$C_{p,\frac{q}{q+1-p}}(E) = 0 \Longrightarrow \mu(E) = 0. \tag{2.13}$$

We recall [2, Theorem 3.8].

Theorem 2.8 *There exists a positive constant c_1 such that if u is a renormalized solution to $-\Delta_p u = \mu$ in Ω and $u = 0$ on $\partial\Omega$, then for any $x \in \Omega$*

$$|u(x)| \leq c_1 W_{1,p}^{2diam(\Omega)}[\mu](x).$$

The next statement is proved in [2, Theorem 2.4], and in [5] for a variant.

Theorem 2.9 *There exist positive constants c_2, c_3 such that*

$$\int_{2B} exp(c_2 W_{1,p}^R[\mu_B]) \leq c_3 r^N,$$

for all $B = B_r(x_0) \subset \mathbb{R}^N$, $2B = B_{2r}(x_0)$, $R > 0$ such that $||\mathbf{M}_{p,R}[\mu]||_{L^\infty(\mathbb{R}^N)} \leq 1$.

3 Estimates from below

If G is any domain in $\mathbb{R}^N$ with a compact boundary and g is nondecreasing, $g(0) = g^{-1}(0) = 0$ and satisfies (1.7), there always exists a maximal solution to (1.4) in G. It is constructed as the limit, when $n \to \infty$, of the solutions of

$$\begin{array}{ll} -\Delta_p u_n + g(u_n) = 0 & \text{in } G_n := \{x \in \mathbb{R}^N : \text{dist}\,(x, G^c) > \frac{1}{n}\} \\ \lim\limits_{\rho_n(x)\to 0} u_n(x) = \infty & \\ \lim\limits_{|x|\to\infty} u_n(x) = 0 & \text{if } G \text{ is unbounded,} \end{array} \tag{3.1}$$

where $\rho_n(x) := \text{dist}\,(x, \partial\Omega_n)$. Our main estimates are the following.

Theorem 3.1 *Let $K \subset B_{1/4}\backslash\{0\}$ be a compact set and let $U_j \in C^1_{loc}(K^c)$, $j = 1, 2$, be the maximal solutions of*

$$-\Delta_p u + e^u - 1 = 0 \qquad \text{in } K^c \tag{3.2}$$

for U_1 *and*

$$-\Delta_p u + u^q = 0 \qquad \text{in } K^c \tag{3.3}$$

for U_2*, where* $p-1 < q < \frac{pq_1}{N}$*. Then there exist constants* C_k*,* $k=1,2,3,4$*, depending on* N*,* p *and* q *such that*

$$U_1(0) \geq -C_1 + C_2 \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K\cap B_r)}{r^{N-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} \tag{3.4}$$

and

$$U_2(0) \geq -C_3 + C_4 \int_0^1 \left(\frac{C_{p,\frac{q_1}{q_1-p+1}}(K\cap B_r)}{r^{N-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r}. \tag{3.5}$$

Proof. Step 1. For $j\in\mathbb{Z}$ define $r_j = 2^{-j}$ and $S_j = \{x : r_j \leq |x| \leq r_{j-1}\}$, $B_j = B_{r_j}$. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K\cap S_j$ for $j=3,...,J$. By [13, Theorem 3.4.27], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that $supp(\mu_j) \subset K\cap S_j$,

$$c^{-1}\mathcal{H}_1^{N-p}(K\cap S_j) \leq \mu_j(\mathbb{R}^N) \leq c\mathcal{H}_1^{N-p}(K\cap S_j)\ \forall j$$

and

$$||\mathbf{M}_{p,1}[\mu_j]||_{L^\infty(\mathbb{R}^N)} = 1.$$

Now, we will show that for ε small enough, there holds,

$$\int_{B_1} \exp\left(\frac{2N}{p} c_1 \mathbf{W}_{1,p}^1\left[\sum_{k=3}^J \varepsilon\mu_k\right](x)\right) dx \leq C, \tag{3.6}$$

where c_1 is the constant in Theorem 2.8, and C does not depend on J.
Indeed, we have

$$A := \int_{B_1} \exp\left(\frac{2N}{p} c_1 \mathbf{W}_{1,p}^1\left[\sum_{k=3}^J \varepsilon\mu_k\right](x)\right) dx = \sum_{j=1}^{\infty}\int_{S_j} \exp\left(\frac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1\left[\sum_{k=3}^J \mu_k\right](x)\right) dx.$$

Since

$$\mathbf{W}_{1,p}^1\left[\sum_{k=3}^J \mu_k\right](x) \leq \max\{1, 5^{\frac{2-p}{p-1}}\}\left(\mathbf{W}_{1,p}^1\left[\sum_{k\geq j+2}\mu_k\right](x) + \mathbf{W}_{1,p}^1\left[\sum_{k\leq j-2}\mu_k\right](x) + \sum_{k=\max\{j-1,3\}}^{j+1}\mathbf{W}_{1,p}^1[\mu_k](x)\right)$$

and

$$\exp(\sum_{i=1}^5 a_i) \leq \sum_{i=1}^5 \exp(5a_i) \qquad \forall a_i.$$

Thus,

$$\begin{aligned}
A &\leq \sum_{j=1}^{\infty}\int_{S_j} \exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}\tfrac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1\left[\sum_{k\geq j+2}\mu_k\right](x)\right) dx \\
&+ \sum_{j=1}^{\infty}\int_{S_j} \exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}\tfrac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1\left[\sum_{k\leq j-2}\mu_k\right](x)\right) dx \\
&+ \sum_{j=1}^{\infty}\sum_{k=\max(j-1,3)}^{j+1}\int_{S_j} \exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}\tfrac{2N}{p} c_1 \varepsilon^{\frac{1}{p-1}} \mathbf{W}_{1,p}^1[\mu_k](x)\right) dx \\
&:= A_1 + A_2 + A_3.
\end{aligned}$$

Estimate of A_3: We apply Theorem 2.9 for $\mu = \mu_k$ and $B = B_{k-1}$,

$$\int_{2B_{k-1}} exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}\frac{2N}{p}c_1\varepsilon^{\frac{1}{p-1}}\mathbf{W}^1_{1,p}[\mu_k](x)\right)dx \le c_3 r_k^N$$

with $5\max\{1, 5^{\frac{2-p}{p-1}}\}\frac{2N}{p}K\varepsilon^{\frac{1}{p-1}} \in (0, c_2]$. In particular,

$$\int_{S_j} exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}K\varepsilon^{\frac{1}{p-1}}\mathbf{W}^1_{1,p}[\mu_k](x)\right)dx \le 4c_3 r_k^N \;\; k = j-1, j, j+1.$$

Which implies

$$A_3 \le c_4\sum_{j=1}^{+\infty} 4c_3 r_j^N = c_5 < \infty. \tag{3.7}$$

Estimate of A_1: Since

$$\sum_{k\ge j+2}\mu_k\left(B_t(x)\right) = 0 \qquad \forall x \in S_j, t \in (0, r_{j+1}),$$

thus,

$$\begin{aligned} A_1 &= \sum_{j=1}^{\infty}\int_{S_j}\exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}\frac{2N}{p}c_1\varepsilon^{\frac{1}{p-1}}\int\limits_{r_{j+1}}^{1}\left(\frac{\sum\limits_{k\ge j+2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\right)dx \\ &\le \sum_{j=1}^{\infty}\int_{S_j}\exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}\frac{2N}{p}c_1\varepsilon^{\frac{1}{p-1}}\frac{p-1}{N-p}\left(\sum_{k\ge j+2}\mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}}\right)dx. \end{aligned}$$

Note that $\mu_k(S_k) \le \mu_k(B_{r_{k-1}}(0)) \le r_{k-1}^{N-p}$, which leads to

$$\left(\sum_{k\ge j+2}\mu_k(S_k)\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} \le \left(\sum_{k\ge j+2} r_{k-1}^{N-p}\right)^{\frac{1}{p-1}} r_{j+1}^{-\frac{N-p}{p-1}} = \left(\sum_{k\ge 0} r_k^{N-p}\right)^{\frac{1}{p-1}} = \left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}.$$

Therefore

$$\begin{aligned} A_1 &\le \sum_{j=1}^{\infty}\int_{S_j}\exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}\frac{2N}{p}c_1\varepsilon^{\frac{1}{p-1}}\frac{p-1}{N-p}\left(\frac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right)dx \\ &= \exp\left(5\max\{1, 5^{\frac{2-p}{p-1}}\}\tfrac{2N}{p}c_1\varepsilon^{\frac{1}{p-1}}\tfrac{p-1}{N-p}\left(\tfrac{1}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}\right)|B_1| = c_6 \end{aligned}$$

Estimate of A_2: for $x \in S_j$,

$$\mathbf{W}^1_{1,p}\left[\sum_{k\le j-2}\mu_k\right](x) = \int\limits_{r_{j-1}}^{1}\left(\frac{\sum\limits_{k\le j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} = \sum_{i=1}^{j-1}\int\limits_{r_i}^{r_{i-1}}\left(\frac{\sum\limits_{k\le j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}$$

Since $r_i < t < r_{i-1}$, $\sum\limits_{k\le i-2}\mu_k(B_t(x)) = 0, \forall i = 1, ..., j-1$, thus

$$\begin{aligned}
\mathbf{W}^1_{1,p}\left[\sum_{k\le j-2}\mu_k\right](x) &= \sum_{i=1}^{j-1}\int_{r_i}^{r_{i-1}}\left(\frac{\sum\limits_{i-1\le k\le j-2}\mu_k(B_t(x))}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\\
&\le \sum_{i=1}^{j-1}\int_{r_i}^{r_{i-1}}\left(\frac{\sum\limits_{i-1\le k\le j-2}\mu_k(S_k)}{t^{N-p}}\right)^{\frac{1}{p-1}}\frac{dt}{t}\\
&\le \sum_{i=1}^{j-1}\left(\sum_{i-1\le k\le j-2}\mu_k(S_k)\right)^{\frac{1}{p-1}}\frac{p-1}{N-p}r_i^{-\frac{N-p}{p-1}}\\
&\le \sum_{i=1}^{j-1}\left(\sum_{i-1\le k\le j-2}r_{k-1}^{N-p}\right)^{\frac{1}{p-1}}\frac{p-1}{N-p}r_i^{-\frac{N-p}{p-1}}\\
&\le \frac{p-1}{N-p}\sum_{i=1}^{j-1}\left(\sum_{i-1\le k\le j-2}r_{k-1-i}^{N-p}\right)^{\frac{1}{p-1}}\\
&\le \frac{p-1}{N-p}\sum_{i=1}^{j-1}\left(\sum_{k\ge i-1}r_{k-1-i}^{N-p}\right)^{\frac{1}{p-1}}\\
&= \frac{p-1}{N-p}\left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}(j-1)\\
&\le \frac{p-1}{N-p}\left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 &\le \sum_{j=1}^{\infty}\int_{S_j}\exp\left(5\max\{1,5^{\frac{2-p}{p-1}}\}\frac{2N}{p}c_1\varepsilon^{\frac{1}{p-1}}\frac{p-1}{N-p}\left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}j\right)dx\\
&= \sum_{j=1}^{\infty}c_7 r_j^N\exp\left(5\max\{1,5^{\frac{2-p}{p-1}}\}\frac{2N}{p}c_1\varepsilon^{\frac{1}{p-1}}\frac{p-1}{N-p}\left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}j\right)\\
&= \sum_{j=1}^{\infty}c_7\exp\left(\left(5\max\{1,5^{\frac{2-p}{p-1}}\}\frac{2N}{p}c_1\varepsilon^{\frac{1}{p-1}}\frac{p-1}{N-p}\left(\frac{4^{N-p}}{1-2^{-(N-p)}}\right)^{\frac{1}{p-1}}-N\log(2)\right)j\right)\\
&= c_8 \quad \text{for } \varepsilon \text{ small enough.}
\end{aligned}$$

Consequently, $A \le C := c_6 + c_8 + c_5$ for ε small enough. This implies

$$\left\|\exp\left(c_1\mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J}\varepsilon\mu_k\right]\right)\right\|_{\mathcal{M}^{\frac{2N}{p}}(B_1(0))} \le c_{10}\left(\int_{B_1(0)}\exp\left(\frac{2N}{p}c_1\mathbf{W}^1_{1,p}\left[\sum_{k=3}^{J}\varepsilon\mu_k\right](x)\right)dx\right)^{\frac{p}{2N}} \le c_{11} \tag{3.8}$$

where the constant c_{11} does not depend on J. Set $B = B_{\frac{1}{4}}$. For ε small enough, it follows from [2], 3.6 and Theorem 2.8, that there exists a renormalized solution u to equation

$$\begin{array}{ll} -\Delta_p u + e^u - 1 = \varepsilon \sum_{j=3}^{J} \mu_j & \text{in } B \\ u = 0 & \text{in } \partial B. \end{array} \tag{3.9}$$

By standard regularity theory, $u \in C^{1,\alpha}_{loc}(B\backslash K)$. From Corollary 2.5 and estimate (3.8), we have

$$u(0) \geq -c_{12} + c_{13} W_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^{J} \mu_j \right](0).$$

Therefore

$$\begin{aligned}
u(0) &\geq -c_{12} + c_{13} \mathbf{W}_{1,p}^{\frac{1}{4}} \left[\sum_{j=3}^{J} \mu_j \right](0) = -c_{12} + c_{13} \sum_{i=2}^{\infty} \int_{r_{i+1}}^{r_i} \left(\frac{\sum_{j=3}^{J} \mu_j(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\geq -c_{12} + c_{13} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} = -c_{12} + c_{13} \sum_{i=2}^{J-2} \int_{r_{i+1}}^{r_i} \left(\frac{\mu_{i+2}(S_{i+2})}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\geq -c_{12} + c_{14} \sum_{i=2}^{J-2} (\mu_{i+2}(S_{i+2}))^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} = -c_{12} + c_{14} \sum_{i=2}^{J-2} (\mu_{i+2}(S_{i+2}))^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{12} + c_{15} \sum_{i=2}^{J-2} \left(\mathcal{H}_1^{N-p}(K \cap S_{i+2}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} = -c_{12} + c_{16} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}}.
\end{aligned}$$

Note that

$$\left(\mathcal{H}_1^{N-p}(K \cap S_i) \right)^{\frac{1}{p-1}} \geq \frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \quad \forall i.$$

Therefore,

$$\begin{aligned}
u(0) &\geq -c_{12} + c_{16} \sum_{i=4}^{\infty} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} - \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} \right) r_i^{-\frac{N-p}{p-1}} \\
&= -c_{12} + c_{16} \left(\frac{1}{\max(1, 2^{\frac{2-p}{p-1}})} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} - \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \right) \\
&= -c_{12} + c_{16} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_{i-1}) \right)^{\frac{1}{p-1}} r_{i-1}^{-\frac{N-p}{p-1}} - \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \right) \\
&\geq -c_{12} + c_{16} \left(\frac{2^{\frac{N-p}{p-1}}}{\max(1, 2^{\frac{2-p}{p-1}})} - 1 \right) \sum_{i=4}^{\infty} \left(\mathcal{H}_1^{N-p}(K \cap B_i) \right)^{\frac{1}{p-1}} r_i^{-\frac{N-p}{p-1}} \\
&\geq -c_{17} + c_{18} \int_0^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_t(0))}{t^{N-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.
\end{aligned}$$

Since U_1 is the maximal solution in K^c, u satisfies the same equation in $B\backslash K$ and $U_1 \geq u = 0$ on ∂B, it follows that U_1 dominates u in $B\backslash K$. Then $U_1(0) \geq u(0)$ and we derive (3.4).

Step 2. Fix a positive integer J such that $K \subset \{x : r_J \leq |x| < 1/8\}$. Consider the sets $K \cap S_j$ for $j = 3, ..., J$. By [13, Theorem 2.5.3], there exists $\mu_j \in \mathfrak{M}^+(\mathbb{R}^N)$ such that

$$\mu_j(K \cap S_j) = \int\limits_{\mathbb{R}^N} (G_p[\mu_j](x))^{q_1} dx = C_{p,\frac{q_1}{q_1-p+1}}(K \cap S_j).$$

We have, for any $a_k \geq 0$,

$$\left(\sum_{k=0}^{\infty} a_k\right)^r \leq \sum_{k=0}^{\infty} \theta_{k,r} a_k^r$$

where $\theta_{k,r}$ has the following expression with $\theta > 0$,

$$\theta_{k,r} = \begin{cases} 1 & \text{if } r \in (0,1] \\ \left(\frac{\theta+1}{\theta}\right)^{r-1}(\theta+1)^{kr} & \text{if } r > 1. \end{cases}$$

Thus,

$$\begin{aligned}
\int\limits_{B_1(0)} \left(\mathbf{W}_{1,p}^1\left[\sum_{k=3}^{J}\mu_k\right](x)\right)^{q_1} dx &\leq \sum_{k=3}^{J} \theta_{k,\frac{1}{p-1}}^{q_1} \theta_{k,q_1} \int\limits_{B_1(0)} \left(\mathbf{W}_{1,p}^1[\mu_k](x)\right)^{q_1} dx \\
&\leq \sum_{k=3}^{J} \theta_{k,\frac{1}{p-1}}^{q_1} \theta_{k,q_1} \int\limits_{R} \left(\mathbf{W}_{1,p}^1[\mu_k](x)\right)^{q_1} dx \\
&\leq c_{19} \sum_{k=3}^{J} \theta_{k,\frac{1}{p-1}}^{q_1} \theta_{k,q_1} \int\limits_{R} (G_p[\mu_k](x))^{q_1} dx \\
&= c_{19} \sum_{k=3}^{J} \theta_{k,\frac{1}{p-1}}^{q_1} \theta_{k,q_1} C_{p,\frac{q_1}{q_1-p+1}}(K \cap S_k) \\
&\leq c_{20} \sum_{k=3}^{J} \theta_{k,\frac{1}{p-1}}^{q_1} \theta_{k,q_1} 2^{-k\left(N-\frac{pq_1}{q_1-p+1}\right)} \\
&\leq c_{21} \quad \text{for } \theta \text{ small enough,}
\end{aligned}$$

where the constant c_{21} does not depend on J. Hence,

$$\left\|\left(\mathbf{W}_{1,p}^1\left[\sum_{k=3}^{J}\mu_k\right]\right)^q\right\|_{\mathcal{M}^{\frac{q_1}{q}}(B_1(0))} \leq c_{22}\left\|\mathbf{W}_{1,p}^1\left[\sum_{k=3}^{J}\mu_k\right]\right\|_{L^{q_1}(B_1(0))} \leq c_{23} \tag{3.10}$$

where c_{23} is independent of J. Take $B = B_{\frac{1}{4}}$. Note that $\frac{q_1}{q} > \frac{N}{p}$. By [2], 3.10, Corollary 2.5 and Theorem 2.8, there exists a renormalized solution u to equation

$$\begin{array}{ll} -\Delta_p u + u^q = \sum_{j=3}^{J} \mu_j & \text{in } B \\ u = 0 & \text{on } \partial B. \end{array} \tag{3.11}$$

It belongs to $C^{1,\alpha}_{loc}(B \backslash K)$ and

$$u(0) \geq -c_{24} + c_{25} W_{1,p}^{\frac{1}{4}}\left[\sum_{j=3}^{J}\mu_j\right](0).$$

As above, we also get that

$$u(0) \geq -c_{26} + c_{27} \int_0^1 \left(\frac{C_{p, \frac{q_1}{q_1 - p + 1}}(K \cap B_r)}{r^{N-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$

Since U_2 is the maximal solution in U_2 in $B \backslash K$, it dominates the solution u in $B \backslash K$, and thus $U_2(0) \geq u(0)$. Therefore, we get (3.5). □

4 Proof of the main results

Proof of Theorem 1. Let u be the maximal solution of

$$-\Delta_p u + e^u - 1 = 0 \qquad \text{in } \Omega \tag{4.1}$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B}_\delta \cap \Omega$. Set $K = \Omega^c \cap \overline{B_{1/4}(z_0)}$. Let $U_1 \in C^1(K^c)$ be the maximal solution of (3.2). We have $u \geq U_1$ in Ω. By Theorem 3.1,

$$\begin{aligned} U_1(z_0) &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \quad (\text{since } B_{r-|z_0|} \subset B_r(z_0))) \\ &\geq -C_1 + C_2 \int_{2\delta}^1 \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\ &\geq -C_1 + C_2^{'} \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \end{aligned}$$

Therefore,

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U_1 \geq -C_1 + C_2^{'} \int_\delta^{1/2} \left(\frac{\mathcal{H}_1^{N-p}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \to \infty \quad \text{as } \delta \to 0.$$

□

Proof of Theorem 2. Let u be the maximal solution to

$$-\Delta_p u + u^q = 0 \qquad \text{in } \Omega. \tag{4.2}$$

Fix $x_0 \in \partial\Omega$. We can assume that $x_0 = 0$. Let $\delta \in (0, 1/12)$. For $z_0 \in \overline{B}_\delta \cap \Omega$. Set $K = \Omega^c \cap \overline{B_{1/4}(z_0)}$. Let $U_2 \in C^1(K^c)$ be the maximal solution of (3.3). We have $u \geq U_2$

in Ω. By Theorem 3.1,

$$\begin{aligned}
U_1(z_0) &\geq -C_1 + C_2 \int_\delta^1 \left(\frac{C_{p,\frac{q_1}{q_1-p+1}}(K \cap B_r(z_0))}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\
&\geq -C_1 + C_2 \int_\delta^1 \left(\frac{C_{p,\frac{q_1}{q_1-p+1}}(K \cap B_{r-|z_0|})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} (\text{ since } B_{r-|z_0|} \subset B_r(z_0))) \\
&\geq -C_1 + C_2 \int_{2\delta}^1 \left(\frac{C_{p,\frac{q_1}{q_1-p+1}}(K \cap B_{\frac{r}{2}})}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \\
&\geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{C_{p,\frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.
\end{aligned}$$

Therefore,

$$\inf_{B_\delta \cap \Omega} u \geq \inf_{B_\delta \cap \Omega} U \geq -C_1 + C_2' \int_\delta^{1/2} \left(\frac{C_{p,\frac{q_1}{q_1-p+1}}(K \cap B_r)}{r^{N-2}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \to \infty \quad \text{as } \delta \to 0.$$

□

5 Large solutions of quasilinear Hamilton-Jacobi equations

In this section we used our previous results to give sufficient conditions for existence of solutions to the problem

$$\begin{aligned}
-\Delta_p u + |\nabla u|^q + bu^{p-1} &= 0 \qquad \text{in } \Omega \\
\lim_{\rho(x)\to 0} u(x) &= \infty,
\end{aligned} \tag{5.1}$$

where b is a real number and $p-1 < q < p$.

Lemma 5.1 *The maximal solution of (4.1) is a large solution if and only if for any $a > 0$ and $b < b_a := \theta_1 a^{1-p}$ the maximal solution of*

$$-\Delta_p v + e^{av} + b = 0 \qquad \text{in } \Omega, \tag{5.2}$$

is a large solution, where θ_1 is a positive constant depending on N, p and Ω.

Proof. Since monotonicity and Vazquez' condition (1.4) hold, it is sufficient to exhibit a large subsolution (i.e. tending to infinity on the boundary) in order to conclude on the existence of a large solution to (5.2).
Assume $u := u_{1,-1}$ is a large solution of (4.1), then for any $\Lambda \geq 1$

$$-\Delta_p u_{1,-1} + e^{u_{1,-1}} - \Lambda = 1 - \Lambda \leq 0 \qquad \text{in } \Omega,$$

thus $u_{1,-1}$ is a subsolution of the corresponding solution and there exists a larger solution which is necessarily a large solution $u_{1,-\Lambda}$ of

$$-\Delta_p u + e^u - \Lambda = 0 \qquad \text{in } \Omega. \tag{5.3}$$

Set $\min\{u_{1,1}(x) : x \in \Omega\} = \theta > 0$. then, for any $c \in (0,1)$ and $d \geq 0$ there holds

$$e^{u_{1,-1}} - 1 \geq m_\theta e^{cu_{1,-1}} \geq m_\theta e^{cu_{1,-1}} - d \quad \text{ on } [\theta, \infty)$$

with $m_\theta = e^{(1-c)\theta} - e^{-c\theta}$. This implies that $-\Delta_p u_{1,-1} + m_\theta e^{cu_{1,-1}} - d \le 0$, therefore $v := u_{1,-1} + c^{-1}\ln m_\theta$ satisfies $-\Delta_p u_{1,-1} + e^{cu_{1,-1}} - d \le 0$. Therefore there exists a large solution $u_{c,-d}$ to

$$-\Delta_p u + e^{cu} - d = 0 \qquad \text{in } \Omega. \tag{5.4}$$

For $\alpha > 0$ and $\beta \in \mathbb{R}$, set $u_{c,-d} = \alpha w + \beta$, then $-\Delta_p w + \alpha^{1-p} e^{\beta c} e^{\alpha c w} - d\alpha^{1-p} = 0$. If we take $\beta = \frac{p-1}{c}\ln\alpha$, then

$$-\Delta_p w + e^{\alpha c w} - d\alpha^{1-p} = 0 \qquad \text{in } \Omega. \tag{5.5}$$

Since $\alpha > 0$ and $d \ge 0$ are arbitrary, we see that for any $a > 0$ and $b \ge 0$, there exists a large solution $u = u_{a,-b}$ to

$$-\Delta_p u + e^{au} - b = 0 \qquad \text{in } \Omega. \tag{5.6}$$

We can notice that since $u_{a,0} = a^{-1}u_{1,0} + (1-p)a^{-1}\ln a$, the minimum $\theta := \theta_a$ of $u_{a,0}$ satisfies $\theta_a = a^{-1}\theta_1 + (1-p)a^{-1}\ln a$. For $\epsilon > 0$, there holds

$$e^{au_{a,0}} - \epsilon e^{au_{a,0}} = (1-\epsilon)e^{au_{a,0}} \ge (1-\epsilon)\theta_1 a^{1-p}.$$

Therefore $-\Delta_p u_{a,0} + \epsilon e^{au_{a,0}} + (1-\epsilon)\theta_1 a^{1-p} \le 0$. Thus $v = u_{a,0} - a^{-1}\ln\epsilon$ satisfies

$$-\Delta_p v + e^{av} + (1-\epsilon)\theta_1 a^{1-p} \le 0$$

which implies that there exists a large solution to the corresponding equation. Since ϵ is arbitrary, it follows that for any $b < b_a := \theta_1 a^{1-p}$, there exists a large solution $u = u_{a,b}$ to (5.2). □

Theorem 5.2 *Assume $p-1 < q < p$ and (1.12) holds. Then there exists $b^* = b^*(p,q,N,\Omega) > 0$ such that for any $b \in (-\infty, b^*)$, problem (5.1) admits a solution.*

Proof. If (1.12) holds, for any $a > 0$ and $b < b_a$, there exists a large solution u to (5.2). We set $u = \alpha \ln w$ with $\alpha > 0$, then

$$-\Delta_p w + (p-1)\frac{|\nabla w|^p}{w} + \alpha^{1-p} w^{\alpha a + p - 1} + b\alpha^{1-p} w^{p-1} = 0 \qquad \text{in } \Omega. \tag{5.7}$$

By Hölder's inequality

$$(p-1)\frac{|\nabla w|^p}{w} \ge |\nabla w|^q - \frac{p-q}{p}\left(\frac{q}{p(p-1)}\right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}},$$

therefore

$$-\Delta_p w + |\nabla w|^q + \alpha^{1-p} w^{\alpha a + p - 1} + b\alpha^{1-p} w^{p-1} - \frac{p-q}{p}\left(\frac{q}{p(p-1)}\right)^{\frac{q}{p-q}} w^{\frac{q}{p-q}} \le 0.$$

Since $q > p-1$, $\dfrac{q}{p-q} > p-1$. We choose α and a such that

$$\alpha a + p - 1 = \frac{q}{p-q} \quad \text{and} \quad \alpha^{1-p} = \frac{p-q}{p}\left(\frac{q}{p(p-1)}\right)^{\frac{q}{p-q}}.$$

Therefore w satisfies

$$-\Delta_p w + |\nabla w|^q + b\alpha^{1-p} w^{p-1} \le 0.$$

This implies that there exists a large solution to (5.1). □

Theorem 5.3 *Let $q > 1$ and assume that there exists a solution to*

$$\begin{aligned} -\Delta u + |\nabla u|^2 + |u|^{q-1} u = 0 \qquad & \textit{in } \Omega \\ \lim_{\rho(x) \to 0} u(x) = \infty. \end{aligned} \tag{5.8}$$

Then for any $s > 1$ there holds

$$\int_0^1 \frac{C_{2,s}(B_r(x) \cap \Omega^c)}{r^{N-2}} \frac{dr}{r} = \infty \qquad \forall x \in \partial\Omega. \tag{5.9}$$

Proof. For $\delta > 0$ set $\Omega_\delta := \{x \in \Omega : \rho(x) < \delta\}$. There exists $\delta_0 > 0$ such that $u(x) > 1$ in Ω_{δ_0}. For $\sigma > 0$ we set $u = v^\sigma$, therefore

$$-\Delta v - (\sigma - 1)\frac{|\nabla v|^2}{v} + \sigma v^{\sigma-1} |\nabla v|^2 + \frac{1}{\sigma} v^{(q-1)\sigma+1} = 0.$$

Since $v > 1$ in Ω_{δ_0}, it follows

$$-\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} = \sigma \frac{|\nabla v|^2}{v} \left(1 - \frac{1}{\sigma} - v^\sigma\right) \leq 0 \qquad \text{in } \Omega_{\delta_0}.$$

For $0 < \delta < \delta_0$ and $m > \inf\{u^{\frac{1}{\sigma}}(x) : x \in \partial\Omega_{\delta_0}\}$ we denote by $v_{m,\delta}$ the solution of

$$\begin{aligned} -\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} = 0 \qquad & \text{in } \Omega_{\delta,\delta_0} := \{x \in \Omega : \delta < \rho(x) < \delta_0\} \\ v = m \qquad & \text{in } \partial\Omega_{\delta_0} \\ \lim_{\text{dist}\,(x,\partial\Omega_\delta) \to 0} v(x) = \infty. \end{aligned} \tag{5.10}$$

Then $v_\delta \geq u^{\frac{1}{s}}$ and $v_{\delta'} < v_\delta$ if $0 < \delta' < \delta$. Thus $v_m = \lim_{\delta \to 0} v_{m,\delta}$ satisfies

$$\begin{aligned} -\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} = 0 \qquad & \text{in } \Omega_{\delta_0} \\ v = m \qquad & \text{in } \partial\Omega_{\delta_0} \\ \lim_{\rho(x)) \to 0} v(x) = \infty. \end{aligned} \tag{5.11}$$

Notice that, since $\partial\Omega_{\delta_0}$ is Lipschitz, the boundary data is preserved in the approximation process. Letting $m \to \infty$ and using the monotonicity of $\{v_m\}$, it implies that that there exists a large solution to

$$-\Delta v + \frac{1}{\sigma} v^{(q-1)\sigma+1} = 0 \qquad \text{in } \Omega_{\delta_0}.$$

By Labutin's result it implies in particular (5.9) with $s = \dfrac{(q-1)\sigma + 1}{(q-1)\sigma}$ and $s > 1$ is arbitrary. □

Remark. If we set $v = e^u$ in (5.8), then v satisfies

$$-\Delta v + e^{(q+1)v} = |\nabla v|^2 (1 - e^v) \qquad \text{in } \Omega. \tag{5.12}$$

From this, we can construct a large solution of

$$-\Delta v + e^{(q+1)v} = 0 \qquad \text{in } \Omega_{\delta_0} \setminus \Omega_\delta. \tag{5.13}$$

It would be interesting to see what Wiener type criterion the existence a such a large solution implies. We conjecture that this condition is

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(B_r(x)\cap\Omega^c)}{r^{N-2}}\frac{dr}{r} = \infty \qquad \forall x \in \partial\Omega. \tag{5.14}$$

Theorem 5.4 *Assume that for some $p > 0$ there exists a function $u \in C(\overline{\Omega})$ satisfying*

$$\begin{aligned} -\Delta u + u^{-p} &= 0 \qquad && \text{in } \Omega \\ u &= 0 \qquad && \text{on } \partial\Omega. \end{aligned} \tag{5.15}$$

Then for any $s > 1$ (5.9) holds.

Proof. We set $v = e^{-v}$, then v is a large solution of

$$-\Delta v + |\nabla v|^2 + e^{(p+1)v} = 0 \qquad \text{in } \Omega \tag{5.16}$$

and we conclude using the preceding theorem (and the remark hereafter). □

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